LOCAL AND GLOBAL EQUIVALENCE OF PROJECTIVE REPRESENTATIONS

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1. Introduction

From the classical work of Wigner and Bargmann [1,2] it is well known that a physical system, which is invariant under a symmetry group of transformations of space-time, is described in quantum mechanics by an (anti) unitary projective representation of that group in a Hilbert space. Classifying irreducible unitary projective representations of symmetry groups is almost a routine procedure in theoretical physics nowadays and as a result the very concept "elementary particle" has eventually been associated by theoreticians with the abstract notion of equivalence class of irreducible unitary projective representations of the symmetry group of the universe (space-time) in which the particle exists.

Long before this abstract description of quantum mechanical particles, there existed already the usual description of particle states by means of wave functions obeying appropriate equations of motion.
It appeared that the solutions of the wave equation for a free particle span a Hilbert space in which the projective representation corresponding to that particle can be realised in a natural way. Thus, fortunately, the two approaches fit well together and the wave equations got a firm group theoretical basis after having been derived from classical mechanics by heuristic arguments.

Nevertheless, these two approaches are not physically equivalent, although this seems to be partly forgotten now. If we read Bargmann and Wigner [3] we see that there are two problems: projective representations in the same equivalence class may possibly

a) not look like each other (even if they are physically equivalent)
b) not be physically equivalent (even if they look like each other).

Here we are mainly concerned with the second problem which in fact says that it is not sufficient to determine all equivalence classes of irreducible unitary projective representations of a symmetry group. There is more work to be done, viz. to distinguish the physically inequivalent representations within each class.

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A substantial part of this work has been done elsewhere [4] and we refer to that paper for more detailed information. We give here the main concepts and results.

One of the results is that for the Poincaré and Galilei groups equivalence of projective representations indeed implies physical equivalence, so it is quite natural to forget the whole problem. However, there exist other symmetry groups of different universes for which projective equivalence does not imply physical equivalence. To be specific: we will give examples of projective representations, describing a free particle resp. a particle in a non-zero external field, which are projectively equivalent.

An analogous situation occurs in classical mechanics, where projectively equivalent representations of a symmetry group may give rise to inequivalent gauge functions, hence to inequivalent Lagrangians. The work of Lévy-Leblond [5] in this field has acted as a guide line for our investigations and especially the mathematical concept "superequivalence of group exponents" which is essential for our work, has been introduced by him. For the notions of unitary continuous projective representation (UCPR) and group exponents we refer to Bargmann [2] .

2. Locally operating UCPRs and gauge matrices

Here we introduce the special kind of UCPRs that we are interested in.

- Definition: A locally operating UCPR of a symmetry group G is a UCPR that has the following two characteristics:
  a) the Hilbert space in which it operates is spanned by complex (spinor) wave functions $\psi_\sigma(x)$ on space-time X, obeying a given equation of motion.
  b) the unitary operators U(g) work locally on these wave functions, i.e. there exists a nonsingular matrix function $A(g;x)$ on $G \times X$ such that

$$(U(g)\psi)_\sigma(g\circ x) = \sum_{\sigma'=1}^{n} A_{\sigma\sigma'}(g;x)\,\psi_{\sigma'}(x) \qquad (1)$$

Condition b) says that the value of the transformed function $U(g)\psi$ in the transformed event $g\circ x$ is related to the value of the original function ψ in the original event x, under linear mixing of the components, multiplied by (gauge) functions which in general may be different for all components, such that $U(g)\psi$ obeys the same equation of motion as ψ does.

The matrix $A(g;x)$ is called the gauge matrix of the locally operating UCPR. The "usual" wave functions in quantum mechanics have a transformation character (1) with a gauge matrix of the form

$$A_{\sigma\sigma'}(g;x) = e^{i\,\theta(g;x)} D_{\sigma\sigma'}(\gamma(g)) \qquad (2)$$

where $\theta(g;x)$ is an x-dependent real gauge function and $D_{\sigma\sigma'}(\gamma)$ a matrix representation of the homogeneous part Γ of the symmetry group, which leaves the origin x_o of X invariant.
However, we do not know any a priori reason why the gauge matrix should be restricted to this particular form. On the contrary, trying to prove a posteriori that any gauge matrix can be chosen in the form (2), we get the result that this is possible only for symmetry groups which have a special property (see (9)), including the Poincaré and Galilei groups.

Of course, the wave functions and the gauge matrices must have sufficient continuity, differentiability and integrability properties, but we do not specify them explicitely, because we do not want to assume specific features of the inner product and of the equation of motion.

It is not claimed that any UCPR can be realised by (1). There may exist projective equivalence classes of UCPRs which do not contain a locally operating representative. Such classes of UCPRs are not considered in this paper. From now on we implicitely mean by UCPR a locally operating UCPR.

The properties of the gauge matrix A can be divided in two classes:

1) specific properties, depending on the explicit form of the equation of motion and the inner product, to ensure that they are invariant under U(g). These properties do <u>not</u> necessarily imply unitarity of the matrix A(g;x) (although the operator U(g) is unitary). We will not use these specific properties in the sequel.
2) general properties, related with the projective nature of U(g), especially related with the group exponents. Our theory will be based on these general properties.

The basic property of the gauge matrix follows from the multiplication rule of the operators U(g):

$$A(g';\, g\circ x)\, A(g;x)\, A^{-1}(g'g;x) = e^{i\,\xi(g',g)} \,,\quad \forall g',g \in G,\ \forall x \in X \qquad (3)$$

where ξ is the exponent of the UCPR with gauge matrix A (or shortly: ξ is the exponent of A). Remark that the left hand side of (3) must be independent of x. From a convention follows

$$A(e;x) = \mathbb{1} \qquad \forall x \in X \quad \text{(e unit element of G)}, \qquad (4)$$

From (3) follows by substitution $g' = \gamma'$, $g = \gamma$ and $x = x_o$ (origin of X)

$$A(\gamma';x_0)A(\gamma;x_0) = e^{i\xi(\gamma',\gamma)}A(\gamma'\gamma;x_0) \quad , \forall \gamma',\gamma \in \Gamma , \qquad (5)$$

hence the matrices $\{A(\gamma ; x_0)\}_{\gamma \in \Gamma}$ form a continuous projective matrix representation (CPMR) of Γ , not necessarily unitary, but its factor system still has modulus one. Equivalence between CPMRs is defined in the usual way.

We use property (5) to define a new equivalence relation between UCPRs of the symmetry group.

- Definition: Two UCPRs of G are called globally equivalent if they are projectively equivalent and if their gauge matrices give rise to equivalent CPMRs of Γ .

The physical idea behind this definition is that two globally equivalent UCPRs operate on wave functions of the same "spinor type" with respect to the homogeneous part Γ of the group G. In the terminology of the introduction: globally equivalent UCPRs "look like each other".

It still may occur that two globally equivalent UCPRs describe systems which are not physically equivalent, because no restriction is put on their local properties. Hence the probability density, energy density etc. need not be related in the two systems.

Therefore we need a more restrictive equivalence concept, demanding more equivalence between gauge matrices.

- Definition: Two gauge matrices A and A' of G are called equivalent if they have the same dimension and if there exists a real continuous function ζ on G and a nonsingular matrix function S on X such that

$$A'(g;x) = e^{i\zeta(g)}S(g\circ x)A(g;x)\, S^{-1}(x) \quad , \forall g \in G, \forall x \in X \qquad (6)$$

- Definition: Two UCPRs are called locally equivalent if they are projectively equivalent and if their gauge matrices are equivalent.

It is clear that local equivalence implies global equivalence.

Two UCPRs which are projectively equivalent with an equivalence transformation S, that operates as multiplication with a matrix S(x), are obviously locally equivalent. Such an equivalence transformation is in fact a generalised gauge transformation of the second (i.e. local) kind. Now we have good reasons to expect that within a given global equivalence class, local equivalence of UCPRs coincides with equivalence from the physical point of view. On the other hand it still remains possible that one physical system can be described by wave functions with different numbers of components, so by UCPRs that are globally inequivalent (hence, locally inequivalent). This is problem a) of the introduction, but we do not go into that point.

The aim of this work is: to find all local equivalence classes of UCPRs contained in a given global equivalence class. (see § 4).

3. Centered gauge matrices

We try here to obtain gauge matrices in a simple form.

- Definition: A gauge matrix A is called centered if it has the property:

$$A(g;x_o) = A(\gamma(g);x_o) \quad , \quad \forall g \in G \tag{7}$$

where $\gamma(g)$ is the homogeneous part of the transformation g. We choose $\forall x \in X$ a transformation h_x which transforms the origin x_o into x (convention $h_{x_o} = e$).

- For each centered gauge matrix A holds

$$A(g;x) = e^{i\xi(g;h_x)} A(\gamma(gh_x);x_o) \quad , \quad \forall g \in G, \ \forall x \in X \tag{8}$$

which follows from (3) by substitution of (g,h_x,x_o) for (g',g,x), using (4).

- Proposition: Each equivalence class of gauge matrices contains a centered representative. (For proof see [4]).

We consider now symmetry groups which have the property that $\gamma(gh_x)$ is independent of x, hence

$$\gamma(gh_x) = \gamma(g) \quad , \quad \forall g \in G, \forall x \in X \tag{9}$$

From (8) it is clear that for such groups any centered gauge matrix takes the form

$$A(g;x) = e^{i\xi(g;h_x)} A(\gamma(g); x_o) \tag{10}$$

This is precisely the usual form (2), where the gauge function $\xi(g;h_x)$ contains all x-dependence of the gauge matrix and where $A(\gamma(g); x_o)$ is a projective (!) matrix representation of the homogeneous part Γ of G. All symmetry groups that are a semidirect product of the translations with the homogeneous transformations (especially the Galilei and Poincaré groups) have property (9), so for these groups any gauge matrix is equivalent to the usual form (2), which now has been justified a posteriori for symmetry groups of this special type.

For groups that do not have property (9) (e.g. the Newton and de Sitter groups) there may exist gauge matrices A where the x-dependence occurs also in an overall gauge function only:

$$A(g;x) = e^{i\theta(g;x)} D(g) \tag{11}$$

but in general these gauge matrices are not centered, whereas the matrices D(g) are not a projective representation of Γ .

We are convinced that it can not be proved from the general properties of gauge matrices that for any symmetry group each gauge matrix is equivalent to (11), although this might follow from the specific properties in each individual case. Consequently we are bound to keep working with the concept of gauge matrix as a generalisation of gauge function.

4. Main result and example

In order to find all local classes of UCPRs contained in a given global class, special kinds of exponents and gauge matrices have to be defined. We refer to chapter 4 of reference [4] for the details. Fortunately we can state the results without all these technicalities.

A global equivalence class of UCPRs determines an (equivalence class of) exponent of G, say $\xi(g',g)$, and an (eq. cl. of) CPMR of Γ, say $D^{(n,\alpha)}(\gamma)$, where n is the dimension of the matrices and α an additional label.
The local equivalence classes contained in this global class are now given by the following set of gauge matrices:

$$A(g;x) = e^{i\xi(g,h_x) - i\zeta(\gamma(gh_x))}\, D^{(n,\alpha)}(\gamma(gh_x)) \qquad (12)$$

where ζ runs through a space of one-dimensional real continuous representations of Γ that can not be extended to such a representation of the whole group G.

When the symmetry group is a semi-direct product of the translations with the homogeneous transformations, then any representation of Γ can be extended to G and the global class contains only one local class, given by (12) with $\zeta \equiv 0$. Hence, for the "usual" symmetry groups (Poincaré, Galileï) physical equivalence coincides with projective equivalence. That this is not generally true for other groups shows the next counterexample.

The Newton group in a one-dimensional oscillating universe. (see reference [6]). This "non-relativistic" cosmological group operates on space-time as follows.

$$(x,t) \in X, \quad g = (b,a,v) \in G, \quad g \circ (x,t) = (g\circ x,\, g\circ t) \quad \text{with}$$

$$\left.\begin{aligned} g\circ x &= x + \frac{v}{\omega}\sin\omega t + a\cos\omega t \\ g\circ t &= t + b \end{aligned}\right\} \qquad (13)$$

Here ω is a "constant of nature": the frequency of the oscillating universe. In the limit $\omega \to 0$ we get the (inhomogeneous) Galilei group. The subgroup Γ of homogeneous transformations, leaving the origin $x_o = (0,0) \in X$ invariant, contains all elements $\gamma(g) = (0,0,v)$ and the transformations that transform (0,0) to an arbitrary point (x,t) are $h_{(x,t)} = (t,x,0)$. The multiplication rule can be calculated easily from (13) and from this we get

$$\gamma(gh_{(x,t)}) = (0,\ 0,\ v\cos\omega t - a\omega\sin\omega t) \qquad (14)$$

This group has a one-dimensional space of (equivalence classes of)exponents $\{ m\ \xi_1(g',g) \mid m \in \mathbb{R}\}$. Any UCPR can be written as $U(g) = e^{ibH}e^{-iaP}e^{ivN}$ and the generators obey

$$[H,P] = i\omega^2 N\ , \quad [H,N] = -iP\ , \quad [P,N] = -im\ . \qquad (15)$$

The Casimir operator is $2mH - P^2 - \omega^2 N^2$.

A free particle with mass $m > 0$ in this universe is described by the UCPR:

$$(U(g)\psi)(g \circ (x,t)) = e^{im\xi_1(g,h_{(x,t)})}\psi(x,t) \qquad (16)$$

with generators

$$\left.\begin{aligned} H &= i\partial_t \\ P &= -i\cos\omega t\,\partial_x + m\omega x\sin\omega t \\ N &= mx\cos\omega t + \frac{\sin\omega t}{\omega}\,i\partial_x \end{aligned}\right\} \qquad (17)$$

(The generators follow from the explicit form of $\xi_1(g',g)$, given in reference [4]). From the Casimir operator follows the equation of motion:

$$i\partial_t\psi(x,t) = \left\{ \frac{(-i\partial_x)^2}{2m} + \tfrac{1}{2}m\omega^2 x^2 \right\}\psi(x,t) \qquad (18)$$

Hence a free particle in this oscillating universe behaves like a particle on a spring in a Galilean universe. The gauge matrix of this UCPR is one-dimensional. The one-dimensional real continuous representations of Γ are $\zeta(\gamma) = \frac{f}{\omega^2}v$, $(f \in \mathbb{R})$ and they can not be extended to representations of G. (For convenience we took the arbitrary constant equal to f/ω^2) From (12) and (14) we get (one-dimensional) gauge matrices of UCPRs (one for each f) that are globally equivalent but locally inequivalent with (16):

$$A(g;x) = e^{im\xi_1(g',g) - i\frac{f}{\omega^2}(v\cos\omega t - a\omega\sin\omega t)} \qquad (19)$$

The generators in these UCPRs have extra terms, compared to (17): P gets an extra term $-\frac{f}{\omega}\sin\omega t$, N gets extra $-\frac{f}{\omega^2}\cos\omega t$ and H stays invariant. With these extra terms the generators still obey (15). The Casimir operator gives again the equation of motion, which now has the extra term $-fx\psi(x,t)$ compared to (18). This term is just a linear potential term in the Schrödinger equation, arising from a uniform external (one-dimensional) free field f.

So the new UCPRs, that we get from (19), describe particles with the same mass in uniform force fields in this oscillating universe. They are all locally inequivalent, but globally equivalent, hence projectively equivalent.

The de Sitter group also has the same features (see reference [4] for a one-dimensional model of this group).

References

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[2] V. Bargmann, On unitary ray representations of continuous groups, Ann of Math. 59 (1954) 1 - 46.

[3] See ref [1] , at least from p. 212 (5th line from bottom) to p. 213 (3rd line from above).

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