

# One-loop amplitudes of charged fermions in constant homogeneous electromagnetic fields

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**Abstract.** The correlator of vector and tensor fermionic currents is considered as the concrete example of the two-point one-loop amplitudes modified by a constant homogeneous magnetic field. The crossed-field limit of this correlator is found. The tensor current is a fermionic part of the Pauli Lagrangian relevant for the electromagnetic interaction of fermions through the anomalous magnetic moment. Under assumption that this interaction enters the effective QED Lagrangian, the contribution to the photon polarization operator linear in AMM is calculated.

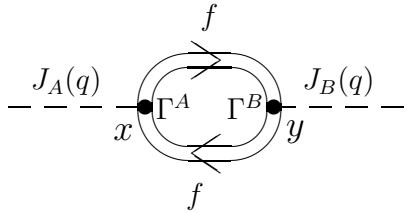
## 1. Introduction

Electromagnetic fields are everywhere around us and they are of importance in many cases and circumstances. In the Universe, at mega scales, the strength of the magnetic field is varying in the range from  $10^{-16}$  Gauss in the intergalactic medium [1] up to  $10^{15}$  Gauss in strongly magnetized neutron stars called magnetars [2]. In terrestrial conditions, it is also possible to produce strong fields at present, even with a larger strength than in magnetars. According to theoretical estimates, extremely strong electromagnetic fields [3] can exist in the quark-gluon plasma produced in the heavy-ion collisions [4] which are studying at the Relativistic Heavy-Ion Collider (RHIC) and Large Hadron Collider (LHC). Both in magnetars and heavy-ion collisions, an existence of background electromagnetic field can influence substantially quantum processes governing underlying physics in these systems and properties of interacting quantum fields. The later ones can be changed as the result of an interaction with a non-trivial vacuum, in particular, filled by the magnetic field. These effects can be account perturbatively in the form of radiative corrections, after relevant loop contributions are calculated [5–9]. The simplest and very intensively studied diagrams are the two-point one-loop Feynman graphs of which the photon polarization operator and electron self-energy are well-known examples.

The general case of the two-point one-loop fermionic amplitudes modified by a constant homogeneous magnetic field was studied in [10]. The Lagrangian density of the local fermion interaction can be presented in the form [9, 10]:

$$\mathcal{L}_{\text{int}}(x) = \left[ \bar{f}(x) \Gamma^A f(x) \right] J_A(x) \equiv j^A(x) J_A(x), \quad (1)$$

where  $f(x)$  is the quantum field of a charged fermion,  $\Gamma_A$  is any of the  $\gamma$ -matrices from the standard set  $\{1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5, \sigma_{\mu\nu}\}$  [11], and  $J_A(x)$  is a generalized electrically neutral current



**Figure 1.** Feynman diagram describing the two-point one-loop fermionic amplitude. Double lines indicate that effects of an external electromagnetic field are taken into account exactly in the fermion propagators.

which can be the photon field, neutrino current, axion field or its derivative, etc. [in these notations, the corresponding coupling constant is included into  $J_A(x)$ ]. After the generalized currents  $J_A$  and  $J_B$  are cut, the two-point one-loop amplitude presented in figure 1 is reduced to the correlator of two fermionic currents  $j_A(x)$  and  $j_B(y)$  which can be written as follows [9, 10]:

$$\Pi_{AB} = \int d^4X e^{-i(qX)} \text{Sp} \{ S_F(-X) \Gamma_A S_F(X) \Gamma_B \}, \quad (2)$$

where  $q^\mu$  is the four-momentum carried by the generalized current and  $S_F(X)$  is the Lorentz-invariant part of an exact fermion propagator calculated in an external field background [12]. We assume the constant homogeneous magnetic field configuration for the external field. Among the existing representations of  $S_F(X)$  in this field, we accept the so-called Fock-Schwinger one [12, 13] in which the fermion propagator has an explicitly Lorentz-covariant form. Effects of the constant homogeneous magnetic field on correlations among the scalar, pseudoscalar, vector and axial-vector currents in the Fock-Schwinger formalism were already studied [9, 10] but correlators of the tensor current  $j_{\mu\nu}(x) \equiv [\bar{f}(x)\sigma_{\mu\nu}f(x)]$  with all the other currents  $j_B(y)$  (see figure 1) were not considered in this approach, except the pseudoscalar-tensor correlator [14].

In this paper, we present the propagator of a charged fermion in the constant homogeneous magnetic field in the Fock-Schwinger representation, show the results for the two-point correlation function with the vector and tensor vertices, find its crossed-field limit, calculate the contribution of the electron anomalous magnetic moment into the photon polarization operator, and finalize with conclusions.

## 2. Propagator in Constant Homogeneous Magnetic Field

The general form of the charged fermion propagator in the Fock-Schwinger representation is well known [12, 13]:

$$G_F(x, y) = e^{i\Omega(x, y)} S_F(x - y), \quad (3)$$

where  $\Omega(x, y)$  is the Lorentz non-invariant phase. In the two-point one-loop amplitude shown in figure 1, the phase factors of the two propagators cancel each other:  $\Omega(x, y) + \Omega(y, x) = 0$ , and the Lorentz-invariant parts  $S_F(x - y)$  and  $S_F(y - x)$  of the fermion propagators are required only [see equation (2)].

Let us choose the frame in which the constant homogeneous magnetic field is directed along the third axis,  $\mathbf{B} = (0, 0, B)$ . For the constant homogeneous magnetic field, the four-potential can be taken in the exactly Lorentz-covariant form,  $A_\mu(x) = -F_{\mu\nu}x^\nu/2$ . Minkowski space filled with this field can be divided into two subspaces: the Euclidean one with the metric tensor  $\Lambda_{\mu\nu} = \varphi_{\mu\rho}\varphi^\rho_\nu \equiv (\varphi\varphi)_{\mu\nu}$ , which is nothing else but the two-dimensional plane orthogonal to the field direction, and pseudo-Euclidean two-dimensional one with the metric tensor  $\tilde{\Lambda}_{\mu\nu} = (\tilde{\varphi}\tilde{\varphi})_{\mu\nu}$ . The metric tensor of the Minkowski space is their difference,  $g_{\mu\nu} = \tilde{\Lambda}_{\mu\nu} - \Lambda_{\mu\nu}$ . Introduced above antisymmetric tensors  $\varphi_{\mu\nu} = F_{\mu\nu}/B$  and  $\tilde{\varphi}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\varphi^{\rho\sigma}/2$  are the dimensionless tensor of the external magnetic field and its dual, respectively, and  $\epsilon_{\mu\nu\rho\sigma}$  is the antisymmetric Levi-Civita symbol of the Minkowski space with the definition  $\epsilon^{0123} = 1$  [11]. These tensors are the Levi-Civita symbols in the Euclidean and pseudo-Euclidean subspaces.

After these notations are introduced, the Lorentz-invariant part of the fermion propagator (3) is as follows [9]:

$$S_F(X) = -\frac{i\beta}{2(4\pi)^2} \int_0^\infty \frac{ds}{s^2} \exp \left( -i \left[ m_f^2 s + \frac{1}{4s} (X \tilde{\Lambda} X) - \frac{\beta \cot(\beta s)}{4} (X \Lambda X) \right] \right) \times \left\{ (X \tilde{\Lambda} \gamma) \cot(\beta s) - i(X \tilde{\varphi} \gamma) \gamma_5 - \frac{\beta s}{\sin^2(\beta s)} (X \Lambda \gamma) + m_f s [2 \cot(\beta s) + (\gamma \varphi \gamma)] \right\}, \quad (4)$$

where  $\beta = eB|Q_f|$ ,  $e > 0$  is the elementary charge, and  $m_f$  and  $Q_f$  are the mass and relative charge of the fermion, respectively.

### 3. Orthogonal Basis in Magnetic Field Background

Correlators with a non-zero rank can be decomposed in some set of four independent vectors. In the magnetic field background one can determine the following orthogonal basis [9]:

$$b_\mu^{(1)} = (q\varphi)_\mu, \quad b_\mu^{(2)} = (q\tilde{\varphi})_\mu, \quad b_\mu^{(3)} = q^2 (\Lambda q)_\mu - (q\Lambda q) q_\mu, \quad b_\mu^{(4)} = q_\mu. \quad (5)$$

As we study the correlator of the vector and tensor currents, it is the Lorentz third-rank tensor  $T_{\mu\nu\rho}$  satisfying the decomposition:

$$T_{\mu\nu\rho} = \sum_{i,j,k=1}^4 T_{ijk} \frac{b_\mu^{(i)} b_\nu^{(j)} b_\rho^{(k)}}{(b^{(i)} b^{(i)}) (b^{(j)} b^{(j)}) (b^{(k)} b^{(k)})}, \quad T_{ijk} = T^{\mu\nu\rho} b_\mu^{(i)} b_\nu^{(j)} b_\rho^{(k)}. \quad (6)$$

Similar expressions for the vector and second-rank tensor are presented in [14] and an extension to the higher rank tensors is obvious.

### 4. Correlator of Vector and Tensor Currents in Magnetic Field

The correlators constructed from the fermionic tensor current and a current of other Lorentz structure are the second-, third- and fourth-rank tensors. If we restrict ourselves by the vector-tensor correlator, the decomposition (6) for the third-rank tensor should be used. There are 64 matrix elements in general, but the antisymmetry in a pair of indices results in 24 independent matrix elements. One should also take into account the orthogonality of the vector current to its four-momentum which further reduces the number of independent matrix elements to 18. Their explicit values are a matter of calculations. From the 18 coefficients in the basis decomposition (6) corresponding to the 18 independent matrix elements, only 4 are non-trivial. It is convenient to use the following double-integral representation for these coefficients:

$$\begin{aligned} \Pi_{ijk}(q^2, q_\perp^2, \beta) &= \frac{1}{4\pi^2} \int_0^\infty \frac{dt}{t} \int_0^1 du Y_{ijk}(q^2, q_\perp^2, \beta; t, u) \times \\ &\times \exp \left\{ -i \left[ m_f^2 t - \frac{q_\perp^2}{4} t (1 - u^2) + q_\perp^2 \frac{\cos(\beta t u) - \cos(\beta t)}{2\beta \sin(\beta t)} \right] \right\}, \end{aligned} \quad (7)$$

where the integration variables  $t = s_1 + s_2$  and  $u = (s_1 - s_2)/(s_1 + s_2)$  are the combinations of two proper-time variables  $s_1$  and  $s_2$  entering the Lorentz-invariant parts (4) of the fermionic propagators. Note also the relation between the momenta squared:  $q_\parallel^2 = q^2 + q_\perp^2$ .

Calculations result the following integrands in the vector-tensor correlator:

$$Y_{114}^{(\text{VT})}(q^2, q_\perp^2, \beta; t, u) = -m_f q_\perp^2 q^2 \frac{\beta t \cos(\beta t u)}{\sin(\beta t)}, \quad (8)$$

$$Y_{223}^{(\text{VT})}(q^2, q_\perp^2, \beta; t, u) = m_f q_\perp^2 (q_\parallel^2)^2 \frac{\beta t}{\sin(\beta t)} [\cos(\beta t) - \cos(\beta t u)], \quad (9)$$

$$Y_{224}^{(\text{VT})}(q^2, q_\perp^2, \beta; t, u) = m_f q_\parallel^2 \frac{\beta t}{\sin(\beta t)} [q_\perp^2 \cos(\beta t) - q_\parallel^2 \cos(\beta t u)], \quad (10)$$

$$Y_{334}^{(\text{VT})}(q^2, q_\perp^2, \beta; t, u) = -m_f q_\perp^2 q_\parallel^2 (q^2)^2 \frac{\beta t \cos(\beta t u)}{\sin(\beta t)}. \quad (11)$$

In the basis (5), vanishing of  $Y_{4jk}^{(\text{VT})}$  explicitly demonstrates the vector current conservation.

## 5. Vector-Tensor Correlator in the Crossed-Field Limit

Correlators in electromagnetic crossed fields can be obtained from the ones calculated in the magnetic field after the pure field parameter  $\beta_F^2 = e^2 Q_f^2 F_{\mu\nu} F^{\mu\nu} / 4$  is neglected. Quantities obtained in the crossed fields are dependent on the field strength through the dynamical parameter  $\chi_f^2 = e^2 Q_f^2 (qFFq) = \beta^2 q_\perp^2$ . The crossed-field limit is valid for an ultrarelativistic particle moving in the direction transverse to the field strength in a relatively weak magnetic field. As basic vectors, it is convenient to use the following normalized orthogonal set:

$$b_\mu^{(1)} = \frac{eQ_f}{\chi_f} (qF)_\mu, \quad b_\mu^{(2)} = \frac{eQ_f}{\chi_f} (q\tilde{F})_\mu, \quad b_\mu^{(3)} = \frac{e^2 Q_f^2}{\chi_f^2 \sqrt{q^2}} [q^2 (FFq)_\mu - (qFFq) q_\mu], \quad b_\mu^{(4)} = \frac{q_\mu}{\sqrt{q^2}}. \quad (12)$$

As above, the coefficients in the tensor decomposition can be presented as double integrals:

$$\Pi_{ijk}(q^2, \chi_f) = \frac{1}{4\pi^2} \int_0^\infty \frac{dt}{t} \int_0^1 du Y_{ijk}(q^2, \chi_f; t, u) \exp \left\{ -i \left[ \left( m_f^2 - \frac{q^2}{4} (1-u^2) \right) t + \frac{1}{48} \chi_f^2 (1-u^2)^2 t^3 \right] \right\}. \quad (13)$$

The integrands in the vector-tensor correlation function in the crossed-field Limit are as follows:

$$\begin{aligned} Y_{114}^{(\text{VT})} &= -m_f \sqrt{q^2}, & Y_{223}^{(\text{VT})} &= \frac{m_f \chi_f^2 t^2}{2\sqrt{q^2}} (1-u^2), \\ Y_{224}^{(\text{VT})} &= -m_f \sqrt{q^2} \left[ 1 + \frac{\chi_f^2 t^2}{2q^2} (1-u^2) \right], & Y_{334}^{(\text{VT})} &= -m_f \sqrt{q^2}. \end{aligned} \quad (14)$$

## 6. Electron AMM Contribution to Polarization Operator of Photon

Searches for physics beyond the Standard Model is the main stream of modern theoretical and experimental physics [15]. In particular, the existing discrepancy between the theoretical prediction for the muon anomalous magnetic moment (AMM) within the Standard Model and its precision measurements at Brookhaven National Laboratory [16] can be explained by contributions of “New Physics”. Models beyond the Standard Model can effectively modify the QED Lagrangian

$$\mathcal{L}_{\text{int}}(x) = -eQ_f [\bar{f}(x) \gamma_\mu f(x)] A^\mu(x), \quad (15)$$

where  $Q_f$  is the electric charge of the fermion in units of the elementary charge  $e$ , with an extra term called the Pauli Lagrangian density [17–19]:

$$\mathcal{L}_{\text{AMM}}(x) = \frac{eQ_f}{4m_f} a_f [\bar{f}(x) \sigma_{\mu\nu} f(x)] F^{\mu\nu}(x), \quad (16)$$

where  $a_f$  is the anomalous magnetic moment (AMM) of the fermion. The existence of (16) in the effective QED Lagrangian modifies the photon polarization operator which is nothing else but the correlator of two vector currents in the conventional QED. From the definition of the photon polarization operator in terms of the  $\gamma \rightarrow \gamma$  transition amplitude:

$$\mathcal{M}^{(VV)} = -i\varepsilon'_\mu(q) \mathcal{P}^{\mu\nu}(q) \varepsilon_\nu(q), \quad (17)$$

it is obvious that the photon polarization tensor  $\mathcal{P}^{\mu\nu}(q)$  in vacuum is transverse to the four-momentum  $q$  only. Accounting the background magnetic field results in some complications and in the basis (5) tensor  $\mathcal{P}^{\mu\nu}(q)$  can be presented in the form:

$$\mathcal{P}_{\mu\nu}(q) = \sum_{\lambda=1}^3 \frac{b_\mu^{(\lambda)} b_\nu^{(\lambda)}}{(b^{(\lambda)})^2} \Pi^{(\lambda)}(q), \quad (18)$$

where  $\Pi^{(\lambda)}(q)$  are its eigenvalues. From three possible polarizations, two eigenstates only are physical:

$$\varepsilon_\mu^{(1)} = b_\mu^{(1)} / \sqrt{q_\perp^2}, \quad \varepsilon_\mu^{(2)} = b_\mu^{(2)} / \sqrt{q_\parallel^2}. \quad (19)$$

Corresponding photon dispersion relations are solutions of the dispersion equations:

$$q^2 - \Pi^{(\lambda)}(q) = 0. \quad (20)$$

In an external magnetic field, the field-induced part of the  $\gamma \rightarrow \gamma$  transition amplitude is well known (see, for example, [8, 9] and references therein):

$$\Delta \mathcal{M}^{(VV)} = \frac{e^2}{4\pi^2} \left[ \frac{(f\varphi)(f'\varphi)}{4q_\perp^2} Y_{VV}^{(1)} + \frac{(f\tilde{\varphi})(f'\tilde{\varphi})}{4q_\parallel^2} Y_{VV}^{(2)} + \frac{(q\varphi\varphi f q)(q\varphi\varphi f' q)}{q^2 q_\perp^2 q_\parallel^2} Y_{VV}^{(3)} \right], \quad (21)$$

where  $f_{\mu\nu}^{(\prime)} = q_\mu \varepsilon_\nu^{(\prime)}(q) - q_\nu \varepsilon_\mu^{(\prime)}(q)$ . It should be noted that the loop amplitude above is originated by the electron only. The field-induced contribution (21) modifies the eigenvalues of the polarization operator as follows [8, 9]:

$$\Pi^{(\lambda)}(q) = -i\mathcal{P}(q^2) - \frac{\alpha}{\pi} Y_{VV}^{(\lambda)}, \quad (22)$$

where  $\mathcal{P}(q^2)$  is the standard vacuum part [11, 12] and  $\alpha = e^2/(4\pi)$  is the fine structure constant. The last term can be presented in the form of the double integral [8]:

$$Y_{VV}^{(\lambda)} = \int_0^\infty \frac{dt}{t} \int_0^1 du \left\{ \frac{\beta t}{\sin(\beta t)} y_{VV}^{(\lambda)} e^{-i\Omega} - \frac{1}{2} q^2 (1 - u^2) e^{-i\Omega_0} \right\}. \quad (23)$$

The integrands  $y_{VV}^{(\lambda)}$  are presented by the expressions [8]:

$$y_{VV}^{(1)} = \frac{1}{2} q_\parallel^2 [\cos(\beta t u) - u \cot(\beta t) \sin(\beta t u)] - q_\perp^2 \frac{\cos(\beta t u) - \cos(\beta t)}{\sin^2(\beta t)}, \quad (24)$$

$$y_{VV}^{(2)} = \frac{1}{2} q_\parallel^2 (1 - u^2) \cos(\beta t) - \frac{1}{2} q_\perp^2 [\cos(\beta t u) - u \cot(\beta t) \sin(\beta t u)], \quad (25)$$

$$y_{VV}^{(3)} = \frac{1}{2} q^2 [\cos(\beta t u) - u \cot(\beta t) \sin(\beta t u)]. \quad (26)$$

As mentioned earlier, the Pauli term (16) in the Lagrangian produces an additional contribution into the photon polarization operator. If we restrict ourselves with a contribution

linear in the electron AMM, the correlator of the vector and tensor fermionic currents,  $\Pi_{\mu\nu\rho}^{(VT)}(q)$ , gives the required contribution. The photon polarization operator is diagonal in the same basis (5) and the eigenvalues  $\Pi^{(\lambda)}(q)$  are modified as follows:

$$\Pi^{(\lambda)}(q) = -i \mathcal{P}(q^2) - \frac{\alpha}{\pi} Y_{VV}^{(\lambda)} + \frac{\alpha}{\pi} a_e Y_{VT}^{(\lambda)}. \quad (27)$$

The double-integral representation (23) can be accepted for  $Y_{VT}^{(\lambda)}$  with the integrands:

$$y_{VT}^{(1)} = y_{VT}^{(3)} = q^2 \cos(\beta t u), \quad y_{VT}^{(2)} = q_{\parallel}^2 \cos(\beta t u) - q_{\perp}^2 \cos(\beta t). \quad (28)$$

Note that  $y_{VT}^{(3)} \sim q^2$ , the same as  $y_{VV}^{(3)}$  (26), and this leaves the third mode  $\varepsilon_{\mu}^{(3)} \sim b_{\mu}^{(3)}$  unphysical.

In QED, the electron AMM  $a_e = \alpha/(2\pi)$  [11, 12] to leading order and the AMM correction in  $\Pi^{(\lambda)}(q)$  has a small impact. One should not expect substantial contribution to  $a_e$  from “New Physics” as supported by experimental data [15] but this is not true for neutrino.

## 7. Conclusions

The two-point one-loop fermionic amplitude with the vector and tensor vertices is considered. The influence of a constant homogeneous external magnetic field is taken into account exactly by using the exact propagators of a charged fermion in the Fock-Schwinger representation. The limit of the external electromagnetic crossed fields is obtained. The contribution of this amplitude to the photon polarization operator is presented and its impact is discussed.

## Acknowledgements

The work of A. D. and I. K. on calculating the vector-tensor correlator in the background magnetic field and its impact into the photon polarization operator is supported by the Russian Science Foundation (Grant No. 18-72-10070). A. P. is partially supported by the *YSU Initiative Scientific Research Activity* (Project No. AAAA-A16-116070610023-3).

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