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Calculus of Variations

Blow up solutions for Sinh-Gordon equation with residual mass

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Abstract
We are concerned with the Sinh-Gordon equation in bounded domains. We construct blow up solutions with residual mass exhibiting either partial or asymmetric blow up, i.e. where both the positive and negative part of the solution blow up. This is the first result concerning residual mass for the Sinh-Gordon equation showing in particular that the concentration-compactness theory with vanishing residuals of Brezis-Merle can not be extended to this class of problems.

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1 Introduction

We are concerned with the following Sinh-Gordon equation

$$\begin{cases} \Delta u + \rho^+ \dfrac{e^u}{\int_\Omega e^u dx} - \rho^- \dfrac{e^{-u}}{\int_\Omega e^{-u} dx} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is smooth and bounded and ρ^+, ρ^- are two positive parameters. The latter problem arises as a mean field equation in the study of the equilibrium turbulence [21, 24]. Moreover, it is also related to constant mean curvature surfaces [20, 29]. Observe that for $\rho^- = 0$, (1.1) reduces to the standard Liouville equation which has been extensively studied in the literature. Therefore, many efforts have been done to study existence [3, 12–15] and blow up phenomena [1, 10, 16, 17, 20, 23, 25, 26, 28] for this class of problems.

In the present paper we further explore the blow up phenomenon of (1.1). Let u_n be a sequence of solutions to (1.1) corresponding to $\rho^\pm = \rho_n^\pm \le C$. Define the positive and negative blow up set as

$$S_\pm := \left\{ x \in \bar{\Omega} : \ \exists x_n \to x \ s.t. \ \pm u_n(x_n) - \log \int_\Omega e^{\pm u_n} dx + \log \rho_n^\pm \to +\infty \ as \ n \to \infty \right\}.$$

We have $S_\pm \cap \partial\Omega = \emptyset$ by [1], and $S_\pm \subset \Omega$ is finite by the argument of [5]. For $p \in S_\pm$ the local mass is defined by

$$m_\pm(p) = \lim_{r\to 0} \lim_{n\to\infty} \frac{\rho_n^\pm \int_{B_r(p)} e^{\pm u_n} dx}{\int_\Omega e^{\pm u_n} dx}.$$

By [16, 20] we know that $m_\pm(p)$ satisfy a quantization property, i.e. $m_\pm(p) \in 8\pi\mathbb{N}$. Moreover, in view of the relation

$$(m_+(p) - m_-(p))^2 = 8\pi(m_+(p) + m_-(p)),$$

see for example [23], the couple (m_+, m_-), up to the order, takes the value in the set

$$\Sigma := \left\{ 8\pi \left(\frac{k(k-1)}{2}, \frac{k(k+1)}{2} \right), \ k \in \mathbb{N} \setminus \{0\} \right\}, \tag{1.2}$$

see [16, 20]. Finally, by standard analysis [23], one has, for $n \to +\infty$,

$$\rho_n^\pm \frac{e^{\pm u_n}}{\int_\Omega e^{\pm u_n} dx} \rightharpoonup \sum_{p \in S_\pm} m_\pm(p)\delta_p + r_\pm,$$

in the sense of measures, where $r_\pm \in L^1(\Omega)$ are residual terms. From the above convergence, $\rho^\pm$ will be called global masses of the blow up solutions. Observe that both the local masses and the residual terms affect the global masses. In striking contrast with the concentration-compactness theory of Brezis-Merle [5], the latter residuals may not be zero a priori. This fact has important effects in the blow up analysis, variational analysis and Leray-Schauder degree theory of (1.1). One of the goals of the present paper is to provide the first explicit example of blow up solutions exhibiting residual terms, thus confirming that the concentration-compactness theory can not be extended to this class of problems.

1.1 Partial blow up

We start here with a related problem, that is partial blow up with prescribed global mass. More precisely, we look for blowing up solutions $-u_n$ with $\rho_n^- \to 8\pi k$, $k \in \mathbb{N}$, such that u_n have prescribed global mass $\rho_n^+ = \rho^+ \in (0, 8\pi)$. To this end we introduce

$$\mathcal{F}_k\Omega := \Big\{ \boldsymbol{\xi} := (\xi_1, \cdots, \xi_k) \in \Omega^k \ : \ \xi_i \neq \xi_j \text{ for } i \neq j \Big\} \tag{1.3}$$

and consider the following singular (at $\xi_i \in \Omega$) mean field equation:

$$\begin{cases} \Delta z(x, \boldsymbol{\xi}) + \rho^+ \dfrac{h(x, \boldsymbol{\xi}) e^{z(x,\boldsymbol{\xi})}}{\int_\Omega h(x, \boldsymbol{\xi}) e^{z(x,\boldsymbol{\xi})} dx} = 0 & \text{in } \Omega, \\ z(x, \boldsymbol{\xi}) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

where $\boldsymbol{\xi} \in \mathcal{F}_k\Omega$ and $h(x, \boldsymbol{\xi}) = e^{-8\pi \sum_{i=1}^k G(x,\xi_i)}$. Here $G(x, y)$ is the Green function of the Laplace operator in Ω with Dirichlet boundary condition and we denote its regular part by $H(x, y)$. Equation (1.4) is the Euler-Lagrange equation of the functional

$$I_{\boldsymbol{\xi}}(z) := \frac{1}{2} \int_\Omega |\nabla z|^2 dx - \rho^+ \log \left(\int_\Omega h(x, \boldsymbol{\xi}) e^z dx \right).$$

To the latter functional and (a combination of) the Green functions we associate the following map:

$$\Lambda(\boldsymbol{\xi}) := \frac{1}{2} I_{\boldsymbol{\xi}}(z(\cdot, \boldsymbol{\xi})) - 32\pi^2 \Big(\sum_{i=1}^k H(\xi_i, \xi_i) + \sum_{j \neq i} G(\xi_i, \xi_j) \Big). \tag{1.5}$$

It is known by [2] that if Ω is simply connected and $\rho^+ \in (0, 8\pi)$, then for any $\boldsymbol{\xi} \in \mathcal{F}_k\Omega$ there exists a unique solution to (1.4) and the solution is non-degenerate, in the sense that the linearized problem admits only the trivial solution. Then, by making use of the implicit function theorem it is not difficult to show that the function Λ is smooth, see for example [8]. Finally, as in [22], a compact set $\mathcal{K} \subset \mathcal{F}_k\Omega$ of critical points of Λ is said to be C^1-stable if, fixed a neighborhood $\mathcal{U}$ of $\mathcal{K}$, any map $\Phi : \mathcal{U} \to \mathbb{R}$ sufficiently close to Λ in C^1-sense has a critical point in $\mathcal{U}$.

The first result of this paper is the following.

Theorem 1.1 *Let Ω be simply connected, $\rho^+ \in (0, 8\pi)$ and let $\mathcal{K} \subset \mathcal{F}_k\Omega$, $k \in \mathbb{N}$, be a C^1-stable set of critical points of Λ. Then, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ there exists a solution u_λ of (1.1) with $\rho^\pm = \rho_\lambda^\pm$ such that the following two properties hold:*

1. $\rho_\lambda^+ = \rho^+$, $\rho_\lambda^- \to 8k\pi$ *as* $\lambda \to 0$.
2. *There exist* $\boldsymbol{\xi}(\lambda) \in \mathcal{F}_k\Omega$ *and* $\delta_i(\lambda) > 0$ *such that* $d(\boldsymbol{\xi}, \mathcal{K}) \to 0$, $\delta_i \to 0$ *and*

$$u_\lambda(x) \to z(x, \boldsymbol{\xi}) - \sum_{i=1}^k \left(\log \frac{1}{(\delta_i^2 + |x - \xi_i|^2)^2} + 8\pi H(x, \xi_i) \right) \ \text{ in } H_0^1(\Omega),$$

as $\lambda \to 0$, *where* z *solves (1.4).*

Some comments are in order. The assumptions that Ω is simply connected and $\rho^+ \in (0, 8\pi)$ guarantee the existence of a unique non-degenerate solution to (1.4): in general, the above result holds true whenever such solution exists. For example, one can drop the condition on Ω by assuming ρ^+ to be sufficiently small, see for example [8].

On the other hand, if Ω is simply connected and $\rho^+ \in (0, 8\pi)$ it is not difficult to show that for $k = 1$ the minimum of Λ is a C^1-stable set of critical points of Λ, see for example [8]. Moreover, for non-simply connected domains the function Λ always admits a C^1-stable set of critical points [7].

Therefore, the conclusion of Theorem 1.1 holds true if either Ω is simply connected, $\rho^+ \in (0, 8\pi)$ and $k = 1$, or Ω is multiply connected, ρ^+ sufficiently small and $k \geq 1$. Finally, the location of the blow up set can be determined by using the following expression, which can be derived similarly as in [8]:

$$\partial_{\xi_j} \Lambda(\boldsymbol{\xi}) = 8\pi \frac{\partial z}{\partial x}(\xi_j, \boldsymbol{\xi}) - 32\pi^2 \Big(\frac{\partial H}{\partial x}(\xi_j, \xi_j) + \sum_{i \neq j} \frac{\partial G}{\partial x}(\xi_i, \xi_j) \Big). \tag{1.6}$$

1.2 Asymmetric blow up

We next construct blow up solutions with residual mass exhibiting the asymmetric blow up, i.e. where both the positive and negative part of the solution blow up. Since the local masses (m_+, m_-) belong to the set Σ defined in (1.2), for $k \geq 2$ we look for blowing up solution u_n with $\rho_n^- \to 4\pi k(k+1)$ and $\rho_n^+ = \rho^+ = 4\pi k(k-1) + \rho_0$, where $\rho_0 \in (0, 8\pi)$ is a fixed residual mass. For simplicity of presentation we assume that k is odd, the case of k even being similar. We consider here l-symmetric domains Ω with $l \geq 2$ even, i.e. if $x \in \Omega$ then $\mathcal{R}_l \cdot x \in \Omega$, where

$$\mathcal{R}_l := \begin{pmatrix} \cos \frac{2\pi}{l} & \sin \frac{2\pi}{l} \\ -\sin \frac{2\pi}{l} & \cos \frac{2\pi}{l} \end{pmatrix}, \quad l \geq 2 \text{ even}. \tag{1.7}$$

Consider then the following singular (at $x = 0$) mean field equation:

$$\begin{cases} \Delta z(x) + \rho_0 \dfrac{e^{z(x) - 8k\pi G(x,0)}}{\int_\Omega e^{z(x) - 8k\pi G(x,0)} dx} = 0 & \text{in } \Omega, \\ z(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Again by [2] we know that if Ω is simply connected and $\rho^0 \in (0, 8\pi)$, then there exists a unique non-degenerate solution to (1.8).

The second result of this paper is the following.

Theorem 1.2 *Let Ω be a simply connected l-symmetric domain according to (1.7) and $\rho^+ = 4\pi k(k-1) + \rho_0$ with $k \in \mathbb{N}$ odd, $l \geq 2$ even and $\rho_0 \in (0, 8\pi)$. Then, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, there exists a solution u_λ of (1.1) with $\rho^\pm = \rho_\lambda^\pm$ such that the following two properties hold:*

1. $\rho_\lambda^+ = \rho^+, \ \rho_\lambda^- \to 4\pi k(k+1)$ *as* $\lambda \to 0$.
2. *There exists* $\delta_i(\lambda) \to 0$ *(defined in (4.2)) such that*

$$u_\lambda(x) \to z(x) + \sum_{i=1}^{k} (-1)^i \Big(\log \frac{1}{(\delta_i(\lambda)^{\alpha_i} + |x|^{\alpha_i})^2} + 4\pi\alpha_i H(x, 0) \Big) \ \ in\ H_0^1(\Omega), \quad \alpha_i = 4i - 2,$$

as $\lambda \to 0$, *where* z *solves (1.8)*.

Observe that the assumption that Ω is simply connected and $\rho_0 \in (0, 8\pi)$ is used only to ensure the existence of a non-degenerate solution to (1.8): in general, the above result holds true whenever such solution exists. On the other hand, the symmetry condition of the domain is imposed to rule out the degeneracy of the singular Liouville equation.

The argument follows the strategy introduced in [8, 9] for the Toda system, that is a system of Liouville-type equations, and it is based on the perturbation method starting from an approximate solution and studying the invertibility of the linearized problem. The main difficulty is due to the coupling of the local and global nature of the problem since we are prescribing both the local and global masses. In particular, blow up solutions of (1.1) with local masses $(4\pi k(k-1), 4\pi k(k+1))$ have been constructed in [11] by superposing k different bubbles with alternating sign. Gluing the solution of (1.8) to the latter blow up solutions we are able to construct blow up solutions with residual mass, that is with $\rho_n^+ = \rho^+ = 4\pi k(k-1) + \rho_0$ and $\rho_n^- \to 4\pi k(k+1)$ for any $k \geq 2$. In this generality the latter construction is quite delicate and technically more difficult compared to the one in [9, 11], since we have more sign-changing singular bubbles, and since one need to consider all the interactions of different bubbles and also the interaction with the global solution $z(x)$ which makes the linear theory more complicated, see the linear theory in Sect. 4.3. We remark that the same strategy can be carried out for more general asymmetric Sinh-Gordon equations, for example for the Tzitzéica equation [18].

The paper is organized as follows. Sect. 2 contains some notation and preliminary results which will be used in the paper. Sect. 3 is devoted to the proof of Theorem 1.1 while the proof of Theorem 1.2 is derived in Sect. 4.

2 Preliminaries

In this section we collect some notation and useful information that we will use in this paper. We shall write

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac{1}{2}} \quad \text{and} \quad \|u\|_p = \Big(\int_\Omega u^p dx\Big)^{\frac{1}{p}}$$

to denote the norm in $H_0^1(\Omega)$ and in $L^p(\Omega)$, respectively, for $1 \leq p \leq +\infty$. For $\alpha \geq 2$, let us define the Hilbert spaces:

$$L_\alpha(\mathbb{R}^2) := L^2\Big(\mathbb{R}^2, \frac{|y|^{\alpha-2}}{(1+|y|^\alpha)^2} dy\Big),$$
$$H_\alpha(\mathbb{R}^2) := \{u \in W^{1,2}_{loc}(\mathbb{R}^2) \cap L_\alpha(\mathbb{R}^2) : \|\nabla u\|_{L^2(\mathbb{R}^2)} < \infty\},$$

with $\|u\|_{L_\alpha}$ and $\|u\|_{H_\alpha} := (\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \|u\|^2_{L_\alpha})^{\frac{1}{2}}$ denoting their norms, respectively. For simplicity, we will denote L_2 and H_2 by L and H, respectively. Let us recall that the embedding $H_\alpha(\mathbb{R}^2) \to L_\alpha(\mathbb{R}^2)$ is compact [11]. For any $p > 1$, let $i_p^* : L^p(\Omega) \to H_0^1(\Omega)$ be the adjoint operator of the embedding $i_p : H_0^1(\Omega) \to L^{\frac{p}{p-1}}(\Omega)$, i.e. for $v \in L^p(\Omega)$, $u = i_p^*(v)$ if and only if in the weak sense

$$-\Delta u = v \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

Then one has $\|i_p^*(v)\|_{H_0^1(\Omega)} \leq c_p \|v\|_p$ for some constant $c_p > 0$ depending only on Ω and $p > 1$.

The symbol $B_r(p)$ will stand for the open metric ball of radius r and center p. To simplify the notation we will write B_r for balls which are centered at 0. Throughout the whole paper

c, C will stand for constants which are allowed to vary among different formulas or even within the same line.

3 Partial blow up

3.1 Approximate solutions

In order to prove Theorem 1.1 we introduce the associated equation

$$\begin{cases} \Delta u + \rho^+ \dfrac{e^u}{\int_\Omega e^u} - \lambda e^{-u} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $\lambda > 0$ will be suitably chosen small. By the definition of i_p^*, problem (3.1) is equivalent to the following:

$$u = i_p^*(F(u)),\ u \in H_0^1(\Omega) \tag{3.2}$$

where $F(u) = \rho^+ g(u) - \lambda f(u)$ and

$$g(u) = \frac{e^u}{\int_\Omega e^u dx},\ f(u) = e^{-u}. \tag{3.3}$$

First let us introduce the approximate solutions we will use. Recall that solutions of the following regular Liouville equation [6]:

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w dx < \infty,$$

are given by

$$w_{\delta,\xi}(x) = \log \frac{8\delta^2}{(\delta^2 + |x - \xi|^2)^2}$$

for $\delta > 0,\ \xi \in \mathbb{R}^2$ and we set

$$w(x) = \log \frac{8}{(1 + |x|^2)^2}.$$

Since we are considering Dirichlet boundary condition, let us introduce the projection:

$$\Delta Pu = \Delta u \quad \text{in } \Omega, \qquad Pu = 0 \quad \text{on } \partial\Omega.$$

By the maximum principle,

$$Pw_{\delta,\xi}(x) = w_{\delta,\xi}(x) - \log 8\delta^2 + 8\pi H(x, \xi) + O(\delta^2) \quad \text{in } C^1\text{-sense}, \tag{3.4}$$

where $H(x, y)$ is the regular part of the Green's function of the Dirichlet Laplacian in Ω, $G(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + H(x, y)$, see [11] for the derivation of (3.4)

Let $k \geq 1$, fix $\boldsymbol{\xi} \in \mathcal{F}_k\Omega$ and consider $z(x, \boldsymbol{\xi})$ which is the unique solution to (1.4). The approximate solutions we will use are given by

$$W = z(x, \boldsymbol{\xi}) - \sum_{i=1}^{k} Pw_i(x), \quad w_i(x) = w_{\delta_i,\xi_i}(x), \tag{3.5}$$

where the parameters δ_i are suitably chosen such that

$$8\delta_i^2 = \lambda d_i(\boldsymbol{\xi}), \quad d_i(\boldsymbol{\xi}) = \exp\Big[8\pi(H(\xi_i,\xi_i) + \sum_{j\neq i} G(\xi_i,\xi_j)) - z(\xi_i,\boldsymbol{\xi})\Big]. \tag{3.6}$$

Our aim is to find a solution u to (3.1) of the form $u = W + \phi$ where ϕ is small in some sense. Before we go further, let us first collect some useful well-known facts.

As it is shown in [4], any solution $\psi \in H$ of

$$\Delta\psi + e^{w_{\delta,\xi}}\psi = 0 \quad \text{in} \quad \mathbb{R}^2,$$

can be expressed as a linear combination of

$$Z^0_{\delta,\xi}(x) = \frac{\delta^2 - |x-\xi|^2}{\delta^2 + |x-\xi|^2}, \quad Z^i_{\delta,\xi}(x) = \frac{x_i - \xi_i}{\delta^2 + |x-\xi|^2}, \ i = 1, 2.$$

Moreover, the projections of $Z^i_{\delta,\xi}$ have the following expansion:

$$\begin{aligned} PZ^0_{\delta,\xi}(x) &= Z^0_{\delta,\xi}(x) + 1 + O(\delta^2), \\ PZ^i_{\delta,\xi}(x) &= Z^i_{\delta,\xi}(x) + O(1), \ i = 1, 2 \quad \text{in } C^1\text{-sense}. \end{aligned} \tag{3.7}$$

Finally, by straightforward computations and taking into account the choice of λ in (3.6) the following estimates hold true [8]:

Lemma 3.1 *For any $\mathcal{C} \subset \mathcal{F}_k\Omega$ compact and $\boldsymbol{\xi} \in \mathcal{C}$, one has*

$$\begin{aligned} \|Pw_i\| &= O(|\log\lambda|^{\frac{1}{2}}), \quad \|\nabla_{\boldsymbol{\xi}} Pw_i\| = O(\lambda^{-\frac{1}{2}}), \\ \|W\| &= O(|\log\lambda|^{\frac{1}{2}}), \quad \|\nabla_{\boldsymbol{\xi}} W\| = O(\lambda^{-\frac{1}{2}}), \end{aligned}$$

and there exists some $a > 0$ such that for any $i = 1, \cdots, k$ and $j = 1, 2$, it holds that

$$\|PZ_i^j\| = a\lambda^{-\frac{1}{2}}(1 + o(1)), \quad \|\nabla_{\boldsymbol{\xi}} PZ_i^j\| = O\left(\frac{1}{\lambda}\right), \tag{3.8}$$

and

$$\langle PZ_i^j, PZ_l^k\rangle = o\left(\frac{1}{\lambda}\right) \quad \text{if} \quad i \neq l \text{ or } j \neq k, \tag{3.9}$$

where $Z_i^j = Z^j_{\delta_i,\xi_i}$ and $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$.

In the section, we set

$$K = \text{Span}\{PZ_i^j, i = 1, \cdots, k \ j = 1, 2\} \tag{3.10}$$

and

$$K^\perp = \{\phi \in H_0^1(\Omega), \int_\Omega \nabla\phi \cdot \nabla PZ_i^j dx = 0, \ i = 1, \cdots, k, \ j = 1, 2\}. \tag{3.11}$$

Denote by

$$\Pi : H_0^1(\Omega) \to K, \ \Pi^\perp : H_0^1(\Omega) \to K^\perp$$

be the corresponding projections. To solve (3.1), it is equivalent to solve the following system:

$$\Pi(u - i_p^*(F(u))) = 0, \ \Pi^\perp(u - i_p^*(F(u))) = 0. \tag{3.12}$$

3.2 Estimate of the error

We next estimate the error of the approximate solution:

$$R = \Delta W + \rho^+ \frac{e^W}{\int_\Omega e^W} - \lambda e^{-W} \text{ and } \mathcal{R} = i_p^*(R).$$

Lemma 3.2 *For any $p \geq 1$ we have, for $\boldsymbol{\xi} \in \mathcal{C} \subset \mathcal{F}_k\Omega$, $\mathcal{C}$ compact,*

$$\|R\|_p = O(\lambda^{\frac{2-p}{2p}}), \quad \|\partial_{\boldsymbol{\xi}} R\|_p = O(\lambda^{\frac{1-p}{p}}).$$

Moreover, $\|\mathcal{R}\| \leq c_p \|R\|_p$ for some $c_p > 0$ depending on p, Ω.

Proof By the definition of W,

$$\begin{aligned} R &= \Delta W + \rho^+ \frac{e^W}{\int_\Omega e^W} - \lambda e^{-W} \\ &= \Delta(z(x,\boldsymbol{\xi}) - \sum_i P w_i) + \rho^+ \frac{e^{z(x,\boldsymbol{\xi}) - \sum_i P w_i}}{\int_\Omega e^{z(x,\boldsymbol{\xi}) - \sum_i P w_i}} - \lambda e^{\sum_i P w_i - z(x,\boldsymbol{\xi})} \\ &= \Big(\sum_i e^{w_i} - \lambda e^{\sum_i P w_i - z(x,\boldsymbol{\xi})} \Big) + \Big(\Delta z(x,\boldsymbol{\xi}) + \rho^+ \frac{e^{z(x,\boldsymbol{\xi}) - \sum_i P w_i}}{\int_\Omega e^{z(x,\boldsymbol{\xi}) - \sum_i P w_i}} \Big) \\ &:= E_1(x) + E_2(x). \end{aligned}$$

Estimate of $E_1 = \Big(\sum_i e^{w_i} - \lambda e^{\sum_i P w_i - z(x,\boldsymbol{\xi})} \Big)$. Take $\eta > 0$ such that $|\xi_i - \xi_j| \geq 2\eta$ and $d(\xi_i, \partial\Omega) \geq 2\eta$. First, using (3.4), we have

$$W = z(x,\boldsymbol{\xi}) - \sum_i P w_i = z(x,\boldsymbol{\xi}) - \sum_i \Big[\log \frac{1}{(\delta_i^2 + |x - \xi_i|^2)^2} + 8\pi H(x, \xi_i) \Big] + O(\lambda).$$

Hence, on $B_\eta(\xi_i)$, writing $x = \xi_i + \delta_i y$, one has

$$\begin{aligned} e^{-W(x)} &= e^{\sum_{i=1}^k \Big[\log \frac{1}{(\delta_i^2 + |x - \xi_i|^2)^2} + 8\pi H(x,\xi_i) \Big] - z(x,\boldsymbol{\xi})} (1 + O(\lambda)) \\ &= e^{w(y)} \cdot \exp \Big(8\pi H(\xi_i, \xi_i) + \sum_{j \neq i} 8\pi G(\xi_i, \xi_j) \\ &\quad - 4 \log \delta_i - \log 8 - z(\xi_i, \boldsymbol{\xi}) \Big) (1 + O(\lambda) + O(\delta_i |y|)) \\ &= \frac{d_i(\boldsymbol{\xi})}{8\delta_i^4} e^{w(y)} (1 + O(\lambda) + O(\delta_i |y|)). \end{aligned}$$

Thus

$$\begin{aligned} e^{w_i} - \lambda e^{-W(x)} &= \frac{8}{\delta_i^2 (1 + |y|^2)^2} \Big[1 - \frac{\lambda}{8\delta_i^2} d_i(\boldsymbol{\xi}) + O(\lambda) + O(\delta_i |y|) \Big] \\ &= O\Big(\frac{1}{(1 + |y|^2)^2} \Big) + O\Big(\frac{|y|}{\lambda^{\frac{1}{2}} (1 + |y|^2)^2} \Big). \end{aligned} \tag{3.13}$$

It follows that

$$\|e^{w_i} - \lambda e^{-W(x)}\|_{L^p(B(\xi_i,\eta))} = O(\lambda^{\frac{2-p}{2p}}) \text{ for any } p \geq 1.$$

Moreover,

$$\|e^{w_j}\|_{L^\infty(B(\xi_i,\eta))} = O(\lambda) \text{ for } j \neq i \text{ and } \|e^{w_i} - \lambda e^{-W(x)}\|_{L^\infty(\Omega\setminus\cup_i B(\xi_i,\eta))} = O(\lambda).$$

Combining the above estimates,

$$\|E_1\|_p = O(\lambda^{\frac{2-p}{2p}}) \text{ for } p \geq 1. \tag{3.14}$$

Estimate of $E_2 = \left(\Delta z(x, \boldsymbol{\xi}) + \rho^+ \frac{e^{z(x,\boldsymbol{\xi})-\sum_i Pw_i}}{\int_\Omega e^{z(x,\boldsymbol{\xi})-\sum_i Pw_i}}\right)$. First of all,

$$\begin{aligned} W &= z(x, \boldsymbol{\xi}) - \sum_i Pw_i \\ &= z(x, \boldsymbol{\xi}) + 2\sum_i \log(\delta_i^2 + |x - \xi_i|^2) - 8\pi \sum_i H(x, \xi_i) + O(\lambda) \\ &= \log h(x, \boldsymbol{\xi}) + z(x, \boldsymbol{\xi}) + 2\sum_i \log \frac{\delta_i^2 + |x - \xi_i|^2}{|x - \xi_i|^2} + O(\lambda), \end{aligned} \tag{3.15}$$

where

$$h(x, \boldsymbol{\xi}) = \prod_{i=1}^k |x - \xi_i|^4 \exp[-8\pi H(x, \xi_i)] = \prod_{i=1}^k \exp\Big(- 8\pi G(x, \xi_i)\Big).$$

So

$$e^W = h(x, \boldsymbol{\xi}) e^{z(x,\boldsymbol{\xi})} + O(\lambda). \tag{3.16}$$

One has

$$E_2 = \Delta z(x, \boldsymbol{\xi}) + \rho^+ \frac{e^W}{\int_\Omega e^W} = \Delta z(x, \boldsymbol{\xi}) + \rho^+ \frac{h(x, \boldsymbol{\xi}) e^{z(x,\boldsymbol{\xi})}}{\int_\Omega h(x, \boldsymbol{\xi}) e^{z(x,\boldsymbol{\xi})}} + O(\lambda) = O(\lambda),$$

since $z(x, \boldsymbol{\xi})$ is a solution of (1.4). Thus

$$\|E_2\|_\infty = O(\lambda). \tag{3.17}$$

Estimate of $\partial_{\boldsymbol{\xi}} E_1$. Next we consider the derivatives. By straightforward computations we get

$$\begin{aligned} \partial_{\xi_i^j} E_1 &= \sum_\ell e^{w_\ell} \partial_{\xi_i^j} w_\ell + \lambda e^{-W} \partial_{\xi_i^j} W \\ &= \lambda e^{-W} \partial_{\xi_i^j} z(x, \boldsymbol{\xi}) + \left(\sum_i e^{w_i} - \lambda e^{-W}\right) \sum_{\ell=1}^k \partial_{\xi_i^j} Pw_\ell \\ &\quad - \sum_\ell e^{w_\ell} \partial_{\xi_i^j} (Pw_\ell - w_\ell) - \sum_{\ell\neq i} e^{w^i} \partial_{\xi_i^j} Pw_\ell \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

It is then not difficult to show that

$$\begin{aligned}
&\|I_1\|_p \le \|E_1\|_p + \sum_i \|e^{w_i}\|_p = O(\lambda^{\frac{1-p}{p}}),\\
&\|I_2\|_p \le \|E_1\|_p \|\partial_{\boldsymbol\xi} P w_j\|_\infty = O(\lambda^{\frac{1-p}{p}}),\\
&\|I_3\|_p \le \|e^{w_i}\|_p \|\partial_{\boldsymbol\xi}(P w_j - w_j)\|_\infty = O(\lambda^{\frac{1-p}{p}}),\\
&\|I_4\|_p = 0.
\end{aligned}$$

Combining all the above estimates,

$$\|\partial_{\boldsymbol\xi} E_1\|_p = O(\lambda^{\frac{1-p}{p}}). \tag{3.18}$$

Estimate of $\partial_{\boldsymbol\xi} E_2$. The estimate of the derivative of E_2 is analogous. Using the equation satisfied by $z(x, \boldsymbol\xi)$ in (1.4) and (3.15),

$$\begin{aligned}
\frac{1}{\rho^+}\partial_{\xi_i^j} E_2 &= -\frac{(\partial_{\xi_i^j} z(x,\boldsymbol\xi)h + \partial_{\xi_i^j} h)e^{z(x,\boldsymbol\xi)}}{\int_\Omega h e^{z(x,\boldsymbol\xi)}dx} + \frac{h e^{z(x,\boldsymbol\xi)}\int_\Omega(\partial_{\xi_i^j} z(x,\boldsymbol\xi)h + \partial_{\xi_i^j} h)e^{z(x,\boldsymbol\xi)}}{(\int_\Omega h e^{z(x,\boldsymbol\xi)}dx)^2}\\
&\quad + \frac{e^W \partial_{\xi_i^j} W}{\int_\Omega e^W} - \frac{e^W \int_\Omega e^W \partial_{\xi_i^j} W dx}{(\int_\Omega e^W)^2}\\
&= O(\lambda).
\end{aligned}$$

Thus we have

$$\|\partial_{\boldsymbol\xi} E_2\|_\infty = O(\lambda). \tag{3.19}$$

Finally, combining the estimates for E_1 and E_2, we have

$$\|R\|_p = O(\lambda^{\frac{2-p}{2p}}), \quad \|\partial_{\boldsymbol\xi} R\|_p = O(\lambda^{\frac{1-p}{p}}).$$

Once we get the estimate for R, the estimate for $\mathcal{R}$ follows directly. □

3.3 The linear operator

In this subsection, we consider the following problem: given $h \in H_0^1(\Omega)$ we look for a function $\phi \in H_0^1(\Omega)$ and c_{ij} such that

$$\begin{cases}
\Delta\phi + \rho^+\left(\frac{e^W\phi}{\int_\Omega e^W dx} - \frac{e^W\int_\Omega e^W\phi dx}{(\int_\Omega e^W dx)^2}\right) + \sum_{i=1}^k e^{w_i}\phi = \Delta h + \sum_{i,j} c_{ij} e^{w_i} Z_i^j,\\
\int_\Omega \nabla\phi\nabla P Z_i^j dx = 0, \ j = 1, 2, \ i = 1, \cdots, k.
\end{cases} \tag{3.20}$$

It is equivalent to

$$\phi - i_p^*(M(W)[\phi]) = h - \sum_{ij} c_{ij} P Z_i^j, \ \phi \in K^\perp \tag{3.21}$$

where

$$M(W)[\phi] = \rho^+\left(\frac{e^W\phi}{\int_\Omega e^W dx} - \frac{e^W\int_\Omega e^W\phi dx}{(\int_\Omega e^W dx)^2}\right) + \sum_{i=1}^k e^{w_i}\phi.$$

Let $L : K^{\perp} \to K^{\perp}$ be the linear operator defined by

$$L(\phi) = \phi - \Pi^{\perp}(i_p^*(M(W)[\phi])),$$

then the problem is equivalent to first solving ϕ for

$$L(\phi) = \Pi^{\perp}(h) \tag{3.22}$$

and then finding c_{ij} for

$$\Pi(i_p^*(M(W)[\phi])) = \Pi(h) - \sum_{ij} c_{ij} P Z_i^j. \tag{3.23}$$

First we have the following apriori estimate:

Lemma 3.3 *Let $\mathcal{C} \subset \mathcal{F}_k\Omega$ be a fixed compact set. Then, there exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0, \lambda_0)$, $\boldsymbol{\xi} \in \mathcal{C}$ and $h \in H_0^1(\Omega)$, any solution $\phi \in H_0^1(\Omega)$ of*

$$\begin{cases} \Delta\phi + \rho^+ \left(\dfrac{e^W \phi}{\int_\Omega e^W dx} - \dfrac{e^W \int_\Omega e^W \phi dx}{(\int_\Omega e^W dx)^2} \right) + \sum_{i=1}^k e^{w_i}\phi = \Delta h, \\ \int_\Omega \nabla\phi \cdot \nabla P Z_i^j dx = 0, \;\; j = 1, 2, \;\; i = 1, \cdots, k, \end{cases} \tag{3.24}$$

satisfies

$$\|\phi\| \leq C|\log\lambda| \|h\|.$$

Proof We prove it by contradiction. Assume there exist $\lambda_n \to 0$, $\boldsymbol{\xi}_n \to \boldsymbol{\xi}^* \in \mathcal{F}_k\Omega$, $h_n \in H_0^1(\Omega)$ and $\phi_n \in H_0^1(\Omega)$ which solves (3.24) with

$$\|\phi_n\| = 1, \;\; |\log\lambda_n| \|h_n\| \to 0 \text{ as } n \to \infty.$$

For $i = 1, \cdots, k$, define $\tilde{\phi}_i(y)$ as

$$\tilde{\phi}_i(y) = \begin{cases} \phi_i(\delta_i y + \xi_i), & y \in \tilde{\Omega}_i = \frac{\Omega - \xi_i}{\delta_i}, \\ 0, & y \in \mathbb{R}^2 \setminus \tilde{\Omega}_i. \end{cases}$$

In the following, we omit the index n for simplicity.

Step 1. We claim that

$$\tilde{\phi}_i(y) \to \gamma_i \frac{1 - |y|^2}{1 + |y|^2} \text{ weakly in } H(\mathbb{R}^2) \text{ and strongly in } L(\mathbb{R}^2), \tag{3.25}$$

and

$$\phi \to 0 \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^q(\Omega) \text{ for } q \geq 2. \tag{3.26}$$

Let $\psi \in C_0^\infty(\Omega \setminus \{\xi_1^*, \cdots, \xi_k^*\})$, multiply equation (3.24) by ψ and integrate, then

$$-\int_\Omega \nabla\psi \cdot \nabla\phi + \sum_{i=1}^k \int_\Omega e^{w_i}\phi\psi dx + \rho^+ \left(\frac{\int_\Omega e^W \phi\psi dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W \phi dx \int_\Omega e^W \psi dx}{(\int_\Omega e^W dx)^2} \right)$$
$$= \int_\Omega \Delta h \psi dx.$$

By the assumption on ϕ, using the fact that in $\Omega \setminus \{\xi_1^*, \cdots, \xi_k^*\}$, $e^{w_i} = O(\lambda)$ and $e^W = h(x, \boldsymbol{\xi}) e^{z(x,\boldsymbol{\xi})} + O(\lambda)$, one has

$$\phi \to \phi^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^q(\Omega) \text{ for } q \geq 2,$$

which gives

$$-\int_\Omega \nabla\phi^* \cdot \nabla\psi dx + \rho^+ \left(\frac{\int_\Omega he^z \phi^* \psi dx}{\int_\Omega he^z dx} - \frac{\int_\Omega he^z \psi dx \int_\Omega he^z \phi^* dx}{(\int_\Omega he^z dx)^2} \right) = 0.$$

So $\|\phi^*\|_{H_0^1(\Omega)} \leq 1$ and it solves

$$\Delta\phi^* + \rho^+ \left(\frac{he^z\phi^*}{\int_\Omega he^z dx} - \frac{he^z \int_\Omega he^z \phi^* dx}{(\int_\Omega he^z dx)^2} \right) = 0.$$

By the non-degeneracy of $z(x, \boldsymbol{\xi})$, we can get that $\phi^* = 0$. Thus (3.26) is proved.

Now let us prove (3.25). Multiplying (3.24) again by ϕ and integrating,

$$\int_\Omega |\nabla\phi|^2 dx - \sum_{i=1}^k \int_\Omega e^{w_i}\phi^2 dx - \rho^+ \left(\frac{\int_\Omega e^W \phi^2 dx}{\int_\Omega e^W dx} - \frac{(\int_\Omega e^W \phi dx)^2}{(\int_\Omega e^W dx)^2} \right) = \int_\Omega \nabla h \cdot \nabla\phi dx.$$

From the above equation, one can get that

$$\begin{aligned} \int_{\tilde{\Omega}_i} e^w \tilde{\phi}_i^2 dx &= \int_\Omega e^{w_i}\phi^2 dx \\ &\leq \int_\Omega |\nabla\phi|^2 dx - \rho^+ \left(\frac{\int_\Omega e^W \phi^2 dx}{\int_\Omega e^W dx} - \frac{(\int_\Omega e^W \phi dx)^2}{(\int_\Omega e^W dx)^2} \right) - \int_\Omega \nabla h \cdot \nabla\phi dx \\ &\leq 1 + o(1) + \|h\| = O(1) \end{aligned}$$

where we used (3.26). So we get that $\tilde{\phi}_i$ is bounded in $H(\mathbb{R}^2)$. There exists $\tilde{\phi}_0$ such that

$$\tilde{\phi}_i \to \tilde{\phi}_0 \text{ weakly in } H(\mathbb{R}^2) \text{ and strongly in } L(\mathbb{R}^2).$$

Let $\tilde{\psi} \in C_0^\infty(\mathbb{R}^2)$ and define $\psi_i = \tilde{\psi}(\frac{x-\xi_i}{\delta_i})$. Multiplying (3.24) by ψ_i and integrating over Ω,

$$\begin{aligned} \int_\Omega \nabla\phi \cdot \nabla\psi_i dx - \textstyle\sum_j \int_\Omega e^{w_j}\phi\psi_i dx - \rho^+ \left(\frac{\int_\Omega e^W \phi\psi_i dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W \phi dx \int_\Omega e^W \psi_i dx}{(\int_\Omega e^W dx)^2} \right) \\ = \int_\Omega \nabla h \cdot \nabla\psi_i dx. \end{aligned} \tag{3.27}$$

Since $\psi_i(x) = 0$ if $|x - \xi_i| \geq R\delta_i$ for some $R > 0$, we have

$$\int_\Omega e^{w_j}\phi\psi_i dx = O(\delta_j^2) \quad \text{for } j \neq i.$$

Passing to the limit in (3.27), we have

$$\int_{\mathbb{R}^2} \nabla\tilde{\phi}_0 \cdot \nabla\tilde{\psi} dx - \int_{\mathbb{R}^2} e^w \tilde{\phi}_0 \tilde{\psi} dx = 0.$$

Moreover, by the orthogonality condition in (3.24), we have

$$\int_{\mathbb{R}^2} \tilde{\phi}_0 e^w \frac{y_j}{1+|y|^2} dy = 0, \ j = 1, 2.$$

So we deduce that

$$\tilde{\phi}_0 = \gamma_i \frac{1-|y|^2}{1+|y|^2}.$$

Step 2. We claim that $\gamma_i = 0$ for $i = 1, \cdots, k$. Multiplying equation (3.24) by PZ_i^0 and integrate over Ω,

$$\int_\Omega \nabla\phi \cdot \nabla PZ_i^0 dx - \sum_j \int_\Omega e^{w_j} \phi PZ_i^0 dx - \rho^+ \left(\frac{\int_\Omega e^W \phi PZ_i^0 dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W \phi dx \int_\Omega e^W PZ_i^0 dx}{(\int e^W dx)^2} \right)$$
$$= \int_\Omega \nabla h \cdot \nabla PZ_i^0 dx. \tag{3.28}$$

Since

$$\int_\Omega \nabla\phi \cdot \nabla PZ_i^0 dx = \int_\Omega e^{w_i} \phi Z_i^0 dx = \int_{\tilde{\Omega}_i} e^w Z^0 \tilde{\phi}_i dy$$

where $Z^0 = \frac{1-|y|^2}{1+|y|^2}$ and by (3.7),

$$\begin{aligned}\sum_j \int_\Omega e^{w_j} \phi PZ_i^0 dx &= \int_\Omega e^{w_i} \phi PZ_i^0 dx + \sum_{j\neq i} \int_\Omega e^{w_j} \phi PZ_i^0 dx \\ &= \int_{\tilde{\Omega}_i} e^w \tilde{\phi}_i (1 + Z^0(y) + O(\delta_i^2)) dy + \sum_{j\neq i} \int_\Omega e^{w_j} \phi PZ_i^0 dx \\ &= \int_{\tilde{\Omega}_i} e^w \tilde{\phi}_i (1 + Z^0(y)) dy + O(\lambda^{\frac{1}{p}}),\end{aligned}$$

for some $p > 1$, by Hölder inequality. Moreover, by (3.26), (3.7) and (3.15), one has

$$\rho^+ \left(\frac{\int_\Omega e^W \phi PZ_i^0 dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W \phi dx \int_\Omega e^W PZ_i^0 dx}{(\int_\Omega e^W dx)^2} \right) = O(\lambda).$$

From (3.28) and the above estimates, one has

$$\lim_{\lambda\to 0} |\log\lambda| \int_{\tilde{\Omega}_i} e^w \tilde{\phi}_i dy = 0. \tag{3.29}$$

Next we multiply equation (3.24) by Pw_i and integrate over Ω,

$$\int_\Omega \nabla\phi \cdot \nabla Pw_i dx - \sum_j \int_\Omega e^{w_j} \phi Pw_i dx - \rho^+ \left(\frac{\int_\Omega e^W \phi Pw_i dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W \phi dx \int_\Omega e^W Pw_i dx}{(\int_\Omega e^W dx)^2} \right)$$
$$= \int_\Omega \nabla h \cdot \nabla Pw_i dx.$$

Now we estimate the above equation term by term.

$$\int_\Omega \nabla\phi \cdot \nabla Pw_i dx = \int_\Omega e^{w_i} \phi dx = \int_{\tilde{\Omega}_i} e^w \tilde{\phi}_i dy = o(1)$$

by (3.25) and the fact that

$$\int_{\mathbb{R}^2} e^w \frac{1-|y|^2}{1+|y|^2} dy = 0.$$

By the expansion of Pw_i,

$$\begin{aligned}\sum_j \int_\Omega e^{w_j}\phi P w_i dx &= \int_\Omega e^{w_i}\phi P w_i dx + \sum_{j\neq i}\int_\Omega e^{w_j}\phi P w_i dx\\ &= \int_{\tilde\Omega_i} e^w \tilde\phi_i \Big(-4\log\delta_i - 2\log(1+|y|^2) + 8\pi H(\xi_i,\xi_i) + O(\delta_i|y| + \delta_i^2)\Big)dy\\ &\quad + \sum_{j\neq i}\int_{\tilde\Omega_j} e^w\tilde\phi_j(8\pi G(\xi_i,\xi_j) + O(\delta_j|y| + \delta_j^2))dy\\ &= \gamma_i \int_{\mathbb{R}^2} e^w \frac{1-|y|^2}{1+|y|^2}[-2\log(1+|y|^2)]dy + o(1).\end{aligned}$$

Moreover,

$$\rho^+\left(\frac{\int_\Omega e^W \phi P w_i dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W \phi dx \int_\Omega e^W P w_i dx}{(\int_\Omega e^W dx)^2}\right) = o(1)$$

and

$$\int_\Omega \nabla h\cdot\nabla P w_i dx = O(\|h\|_p\|P w_i\|) = O(\log\lambda)^{\frac12}\|h\| = o(1).$$

Combining all the above estimates, we have

$$\gamma_i\int_{\mathbb{R}^2} e^w\frac{1-|y|^2}{1+|y|^2}[-2\log(1+|y|^2)]dy = 0,$$

which implies that $\gamma_i = 0$ since

$$\int_{\mathbb{R}^2} e^w\frac{1-|y|^2}{1+|y|^2}[-2\log(1+|y|^2)]dy \neq 0.$$

Step 3. Finally, we derive a contradiction.

Multiply equation (3.24) by ϕ and integrate:

$$\int_\Omega|\nabla\phi|^2dx - \sum_i\int_\Omega e^{w_i}\phi^2 dx - \rho^+\left(\frac{\int_\Omega e^W\phi^2dx}{\int_\Omega e^W dx} - \frac{(\int_\Omega e^W\phi dx)^2}{(\int_\Omega e^W dx)^2}\right) = \int_\Omega\nabla h\cdot\nabla\phi dx.$$

From the estimates in step 1-2 and the assumptions on ϕ and h, it is not difficult to show that the left hand side of the above equation tends to 1, while the right hand side has limit 0. This is a contradiction which concludes the proof. □

Now we can derive a priori estimates for problem (3.20).

Proposition 3.4 *Let $\mathcal{C}\subset\mathcal{F}_k\Omega$ be a compact set. Then, there exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda\in(0,\lambda_0)$, $\boldsymbol{\xi}\in\mathcal{C}$ and $h\in H_0^1(\Omega)$, if (ϕ, c_{ij}) is a solution of (3.20), we have*

$$\|\phi\| \le C|\log\lambda|\|h\|.$$

Proof By Lemma 3.3 and (3.8) , we know that

$$\|\phi\| \le C|\log\lambda|\left(\|h\| + \sum_{ij}|c_{ij}|\|PZ_i^j\|\right) \le C|\log\lambda|\left(\|h\| + \sum_{ij}\frac{1}{\sqrt\lambda}|c_{ij}|\right).$$

In order to estimate c_{ij}, multiply the equation (3.20) by PZ_i^j and integrating over Ω,

$$\int_\Omega \phi e^{w_i}(PZ_i^j - Z_i^j)dx + \sum_{\ell\neq i}\int_\Omega e^{w_\ell}\phi PZ_i^j dx + O\left(\int_\Omega |\phi||PZ_i^j|dx + \int_\Omega |\phi|\int_\Omega |PZ_i^j|dx\right)$$
$$= \int_\Omega \nabla h\cdot\nabla PZ_i^j + c_{ij}\int_\Omega e^{w_i}Z_i^j PZ_i^j dx + \sum_{k\neq i,\ell\neq j} o\left(\frac{|c_{k\ell}|}{\lambda}\right),$$

where in the last line we use (3.9). Since for any $q\geq 1$,

$$\int_\Omega \phi e^{w_i}(PZ_i^j - Z_i^j)dx + \sum_{\ell\neq i}\int_\Omega e^{w_\ell}\phi PZ_i^j dx = O(\|\phi\|(\|e^{w_i}\|_q + \|e^{w_\ell}PZ_i^j\|_q)) = O\left(\lambda^{\frac{1-q}{q}}\|\phi\|\right),$$
$$O\left(\int_\Omega |\phi||PZ_i^j| + \int_\Omega |\phi|\int_\Omega |PZ_i^j|\right) = O(\|PZ_i^j\|_2\|\phi\|) = O\left(|\log\lambda|^{\frac{1}{2}}\|\phi\|\right),$$
$$\int_\Omega \nabla h\cdot\nabla PZ_i^j = O(\|h\|\|PZ_i^j\|) = O\left(\frac{1}{\sqrt{\lambda}}\|h\|\right),$$

we have

$$|c_{ij}| + o\left(\sum_{k\neq j,\ell\neq i}|c_{k,\ell}|\right) = O\left(\lambda^{\frac{1}{q}}\|\phi\| + \lambda|\log\lambda|^{\frac{1}{2}}\|\phi\| + \lambda^{\frac{1}{2}}\|h\|\right).$$

Summing all $|c_{ij}|$ up and choosing suitable $q\in(1,2)$, we can get that

$$\|\phi\|\leq C|\log\lambda|\|h\|.$$

□

From the above a priori estimate and the Fredholm alternative it is then standard to derive the following existence result.

Proposition 3.5 *Let $\mathcal{C}\subset\mathcal{F}_k\Omega$ be a compact set. Then, there exist $\lambda_0>0$ and $C>0$ such that for any $\lambda\in(0,\lambda_0)$, $\boldsymbol{\xi}\in\mathcal{C}$ and $h\in H_0^1(\Omega)$, there exists a unique solution (ϕ, c_{ij}) of (3.20), which satisfies*

$$\|\phi\|\leq C|\log\lambda|\|h\|.$$

Proof For the first equation (3.22), since $\phi\to\Pi^\perp(i_p^*(M(W)[\phi]))$ is the compact operator in $K^\perp$, the existence and uniqueness of ϕ follows from the Fredholm alternative and the above a priori estimate, and then c_{ij} are determined by (3.23). Once existence of (ϕ, c_{ij}) is obtained, the estimate follows from Proposition 3.4. □

3.4 Nonlinear problem

The aim of this subsection is to find $(\phi, \{c_{ij}\})$ such that $u = W_\xi + \phi_\xi$ solves

$$\begin{cases}\Delta u + \rho^+\dfrac{e^u}{\int_\Omega e^u dx} - \lambda e^{-u} = \sum_{ij} c_{ij}e^{w_i}Z_i^j,\\ \int_\Omega \nabla\phi\nabla PZ_i^j dx = 0, \quad j=1,2,\ i=1,\cdots,k.\end{cases}$$

For this purpose, we shall find a solution ϕ of

$$\begin{cases} \Delta\phi + \rho^+\Big(\frac{e^W\phi}{\int_\Omega e^W dx} - \frac{e^W\int_\Omega e^W\phi dx}{(\int_\Omega e^W dx)^2}\Big) + \sum_{i=1}^k e^{w_i}\phi = -\Big(R + S(\phi) + N(\phi)\Big) \\ \quad + \sum_{ij} c_{ij} e^{w_i} Z_i^j, \\ \\ \int_\Omega \nabla\phi\cdot\nabla PZ_i^j dx = 0, \quad j = 1, 2,\ i = 1, \cdots, k. \end{cases} \tag{3.30}$$

where R is the error term defined in Sect. 3.2,

$$N(\phi) = -\lambda\Big(f(W+\phi) - f(W) - f'(W)\phi\Big) + \rho^+\Big(g(W+\phi) - g(W) - g'(W)\phi\Big),$$

$$S(\phi) = -\left(\sum_{i=1}^k e^{w_i} + \lambda f'(W)\right)\phi,$$

$$f(W) = e^{-W},\ g(W) = \frac{e^W}{\int_\Omega e^W dx}.$$

From the above linear theory, the existence of a solution to the nonlinear problem (3.30) follows a standard strategy using contraction mapping.

Proposition 3.6 *Let $\mathcal{C} \subset \mathcal{F}_k\Omega$ be compact set. For any $\epsilon > 0$ sufficiently small, there exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0, \lambda_0)$ and $\boldsymbol{\xi} \in \mathcal{C}$, there exists a unique (ϕ, c_{ij}) solution of (3.30) satisfying the estimates:*

$$\|\phi\| \le C\lambda^{\frac{1}{2}-\epsilon},\ \|\partial_{\xi_i^j}\phi\| \le C\lambda^{-\epsilon},\ |c_{ij}| \le C\lambda. \tag{3.31}$$

Proof Denote the solution to (3.20) by $\phi := T(h)$. Then (3.30) is equivalent to

$$\phi = T(i_p^*(R + N(\phi) + S(\phi))) =: \mathcal{T}(\phi).$$

The solution ϕ can be obtained through contraction mapping. Define

$$\mathcal{B} = \{\phi \in K^\perp, \|\phi\| \le \Lambda|\log\lambda|\lambda^{\frac{2-p}{2p}}\}$$

for Λ large and λ small and p close to 1.

From Proposition 3.5 and the error estimate for R, for $\phi, \phi_1, \phi_2 \in \mathcal{B}$, similarly to the estimate in Proposition 4.10 in [8], one has

$$\begin{aligned} \|\mathcal{T}\phi\| &\le C_p|\log\lambda|\|i_p^*(R + N(\phi) + S(\phi))\| \\ &\le C_p|\log\lambda|(\|R\|_p + \|N(\phi)\|_p + \|S(\phi)\|_p) \\ &\le \Lambda|\log\lambda|\lambda^{\frac{2-p}{2p}}, \\ &\text{and} \\ \|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\| &\le C_p(\|N(\phi_1) - N(\phi_2)\|_p + \|S(\phi_1) - S(\phi_2)\|_p) \\ &\le \frac{1}{2}\|\phi_1 - \phi_2\|. \end{aligned}$$

So $\mathcal{T}$ maps $\mathcal{B}$ into itself and it is a contraction mapping. For ϵ small, we can choose p sufficiently close to 1, such that $\mathcal{B} \subset \{\phi,\ \|\phi\| \le C\lambda^{\frac{1}{2}-\epsilon}\}\}$. Since $\mathcal{T}$ is a contraction mapping in $\mathcal{B}$, we can also get that the fixed point in $\mathcal{B}$ is unique, i.e. the solution ϕ is unique in $\mathcal{B}$. The estimate for ϕ follows from the above estimates. The estimates for $\partial_{\xi_i}\phi$ are obtained similarly to Proposition 4.10 in [8] and the estimate for c_{ij} follows from Proposition 3.4. □

3.5 The reduced problem

We introduce here the finite-dimensional reduction. In the previous subsection we have found a solution $u = W + \phi$ to the problem

$$\begin{cases} \Delta u + \rho^+ \dfrac{e^u}{\int_\Omega e^u dx} - \lambda e^{-u} = \sum_{ij} c_{ij} e^{w_i} Z_i^j \\ \int_\Omega \nabla\phi \cdot \nabla P Z_i^j dx = 0, \ j = 1, 2, \ i = 1, \cdots, k. \end{cases}$$

Consider now the associated energy functional:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \rho^+ \log \int_\Omega e^u dx - \lambda \int_\Omega e^{-u} dx \tag{3.32}$$

and let $\tilde{J}(\boldsymbol{\xi}) = J(W_{\boldsymbol{\xi}} + \phi_{\boldsymbol{\xi}})$.

Lemma 3.7 *Let $\boldsymbol{\xi} \in \mathcal{F}_k\Omega$ be a critical point of $\tilde{J}$, then for λ small, $u = W_{\boldsymbol{\xi}} + \phi_{\boldsymbol{\xi}}$ is a solution of (3.1).*

Proof If $\boldsymbol{\xi}$ is a critical point of $\tilde{J}(\boldsymbol{\xi})$, then one has

$$\langle J'(u), \partial_{\boldsymbol{\xi}}(W_{\boldsymbol{\xi}} + \phi_{\boldsymbol{\xi}})\rangle = 0,$$

which is equivalent to

$$\langle \sum_{ij} c_{ij} e^{w_i} Z_i^j, \partial_{\xi_\ell^s}(W_{\boldsymbol{\xi}} + \phi_{\boldsymbol{\xi}})\rangle = 0 \quad \text{for } \ell = 1, \cdots, k, \ s = 1, 2. \tag{3.33}$$

Let us fix $q > 1$. Since

$$\|e^{w_i} Z_i^j\|_q = O(\lambda^{\frac{2-3q}{2q}}), \tag{3.34}$$

combining the estimate (3.31), one has

$$\int_\Omega e^{w_i} Z_i^j \partial_{\xi_\ell^s} \phi_{\boldsymbol{\xi}} dx = O(\|e^{w_i} Z_i^j\|_q \|\partial_\xi \phi\|) = O\left(\lambda^{\frac{2-3q}{2q} - \epsilon}\right) = o\left(\frac{1}{\lambda}\right), \tag{3.35}$$

$$\int_\Omega e^{w_i} Z_i^j \partial_{\xi_\ell^s} W_{\boldsymbol{\xi}} dx = -\int_\Omega P Z_\ell^s e^{w_i} Z_i^j dx + O\left(\frac{1}{\sqrt{\lambda}}\right) = \frac{a}{\lambda}\delta_{i\ell}\delta_{js} + o\left(\frac{1}{\lambda}\right), \tag{3.36}$$

Combining the estimates (3.35) and (3.36), we conclude that

$$c_{ij} + o(1) \sum_{\ell \neq i, s \neq j} c_{\ell s} = 0,$$

which implies that all c_{ij} are zero. So the corresponding u is a solution of (3.1) as desired. □

Recall the definition of Λ in (1.5). We next consider the expansion of the energy.

Proposition 3.8 *It holds*

$$J(W) = \Lambda(\boldsymbol{\xi}) - 8\pi k \log \lambda - (16\pi - 24\pi \log 2)k + o(1),$$

C^1 uniformly in $\boldsymbol{\xi}$ in compact sets of Ω.

Proof By the definition of $J(W)$ and W, one has

$$J(W) = \frac{1}{2}\int_\Omega \left(|\nabla z|^2 + \sum_{i=1}^k |\nabla P w_i|^2 - 2\sum_{i=1}^k \nabla P w_i \cdot \nabla z + 2\sum_{i\neq j} \nabla P w_i \cdot \nabla P w_j \right) dx$$
$$- \rho^+ \log \int_\Omega e^W dx - \lambda \int_\Omega e^{-W} dx.$$

Using (3.16),

$$\frac{1}{2}\int_\Omega |\nabla z|^2 dx - \rho^+ \log \int_\Omega e^W dx = \frac{1}{2}\int_\Omega |\nabla z|^2 dx - \rho^+ \log \int_\Omega h(x, \boldsymbol{\xi}) e^{z(x,\boldsymbol{\xi})} dx + O(\lambda).$$

While using (3.13) and the estimate for E_1,

$$\lambda \int_\Omega e^{-W} dx = \sum_{i=1}^k \int_\Omega e^{w_i} dx + o(1) = 8k\pi + o(1),$$
$$\int_\Omega \nabla P w_i \cdot \nabla z dx = \int_\Omega e^{w_i} z(x, \boldsymbol{\xi}) dx = \int_{\tilde{\Omega}_i} \frac{8}{(1+|y|^2)^2} z(\delta_i y + \xi_i, \boldsymbol{\xi}) dy = 8\pi z(\xi_i, \boldsymbol{\xi}) + o(1),$$

where $\tilde{\Omega}_i = (\Omega - \xi_i)/\delta_i$. Moreover, using the expansion (3.4)

$$\begin{aligned}
\int_\Omega |\nabla P w_i|^2 dx &= \int_\Omega e^{w_i} P w_i dx \\
&= \int_\Omega e^{w_i} \Big(\log \frac{1}{(\delta_i^2 + |x-\xi_i|^2)^2} + 8\pi H(x, \xi_i) + O(\lambda) \Big) dx \\
&= 64\pi^2 H(\xi_i, \xi_i) - 2\int_{\tilde{\Omega}_i} e^{w_i} (\log \delta_i^2 + \log(1+|y|^2)) dy + o(1) \\
&= 64\pi^2 H(\xi_i, \xi_i) - 16\pi \log \delta_i^2 - 16\pi + o(1) \\
&= 64\pi^2 H(\xi_i, \xi_i) - 16\pi \log \frac{\lambda d_i(\boldsymbol{\xi})}{8} - 16\pi + o(1) \\
&= -64\pi^2 H(\xi_i, \xi_i) - 128\pi^2 \sum_{j\neq i} G(\xi_i, \xi_j) + 16\pi z(\xi_i, \boldsymbol{\xi}) \\
&\quad - 16\pi \log \lambda - 16\pi + 48\pi \log 2 + o(1),
\end{aligned}$$

and for $i \neq j$,

$$\begin{aligned}
\int_\Omega \nabla P w_i \cdot \nabla P w_j dx &= \int_\Omega e^{w_i} \Big(\log \frac{1}{(\delta_j^2 + |x-\xi_j|^2)^2} + 8\pi H(x, \xi_j) + O(\lambda) \Big) dx \\
&= 64\pi^2 G(\xi_i, \xi_j) + o(1).
\end{aligned}$$

Combining all the above estimates, we have

$$\begin{aligned}
J(W) &= \frac{1}{2}\int_\Omega |\nabla z|^2 dx - \rho^+ \log \int_\Omega h(x, \boldsymbol{\xi}) e^{z(x,\boldsymbol{\xi})} dx - 8\pi k \log \lambda \\
&\quad - 32\pi^2 \sum_{i=1}^k \left(H(\xi_i, \xi_i) + \sum_{j\neq i} G(\xi_i, \xi_j) \right) - (16\pi - 24\pi \log 2)k + o(1) \\
&= \Lambda(\boldsymbol{\xi}) - 8\pi k \log \lambda - (16\pi - 24\pi \log 2)k + o(1).
\end{aligned}$$

Next, we consider the derivative of $J(W)$.

$$
\begin{aligned}
\partial_{\xi_i^j} J(W) &= \int_\Omega \left(-\Delta W - \rho^+ \frac{e^W}{\int_\Omega e^W dx} + \lambda e^{-W} \right) \partial_{\xi_i^j} W dx = -\int_\Omega (E_1(x) + E_2(x)) \partial_{\xi_i^j} W dx \\
&= 4 \int_\Omega E_1(z) Z_i^j dx + o(1) = 4 \int_\Omega \left(\sum_\ell e^{w_\ell} - \lambda e^{-W} \right) Z_i^j dx,
\end{aligned}
$$

where E_1, E_2 were introduced in Lemma 3.2 and where we used

$$
\partial_{\xi_i^j} W = -4 P Z_i^j + O(1).
$$

Using the definition of w_i and Z_i^j, for $\ell \neq i$

$$
\int_\Omega e^{w_\ell} Z_i^j dx = \int_\Omega \frac{8\delta_\ell^2}{(\delta_\ell^2 + |x - \xi_\ell|^2)^2} \frac{x_j - \xi_i^j}{\delta_i^2 + |x - \xi_i|^2} dx = 8\pi \frac{\xi_\ell^j - \xi_i^j}{|\xi_\ell - \xi_i|^2} + o(1).
$$

Moreover, taking $\eta > 0$ such that $|\xi_i - \xi_j| \geq 2\eta$ and $d(\xi_i, \partial\Omega) \geq 2\eta$, we have

$$
\begin{aligned}
\int_{B(\xi_\ell,\eta)} \lambda e^{-W} Z_i^j dx &= \lambda \int_{B(\xi_\ell,\eta)} \exp\Big[8\pi \sum_i H(x, \xi_i) \\
&\quad - z(x, \boldsymbol{\xi}) + O(\lambda) \Big] \frac{x_j - \xi_i^j}{\delta_i^2 + |x - \xi_i|^2} \prod_{i=1}^k \frac{1}{(\delta_i^2 + |x - \xi_i|^2)^2} dx \\
&= \frac{\lambda}{\delta_\ell^2} \int_{\tilde{\Omega}_\ell} \exp\Big[8\pi H(\xi_\ell, \xi_\ell) \\
&\quad + 8\pi \sum_{j \neq \ell} G(\xi_\ell, \xi_j) - z(\xi_\ell, \boldsymbol{\xi}) \Big] \frac{1}{(1 + |y|^2)^2} \frac{\xi_\ell^j - \xi_i^j}{|\xi_\ell - \xi_i|^2} dx + o(1) \\
&= 8\pi \frac{\xi_\ell^j - \xi_i^j}{|\xi_\ell - \xi_i|^2} + o(1).
\end{aligned}
$$

Let

$$
\gamma(x, \boldsymbol{\xi}) = 8\pi H(x, \xi_i) + 8\pi \sum_{j \neq i} G(x, \xi_j) - z(x, \boldsymbol{\xi}).
$$

Then,

$$
\begin{aligned}
\int_{B(\xi_i,\eta)} \lambda e^{-W} Z_i^j \, dx &= \lambda \int_{B(\xi_i,\eta)} \exp\Big[8\pi \sum_i H(x,\xi_i) - z(x,\boldsymbol{\xi}) + O(\lambda)\Big] \\
&\quad \frac{x_j - \xi_i^j}{\delta_i^2 + |x-\xi_i|^2} \prod_{i=1}^{k} \frac{1}{(\delta_i^2 + |x-\xi_i|^2)^2} dx \\
&= \frac{\lambda}{\delta_i^3} \int_{\tilde{\Omega}_i} \frac{1}{(1+|y|^2)^2} \frac{y_j}{1+|y|^2} \\
&\quad \exp\Big[8\pi H(\xi_i + \delta_i y, \xi_i) + 8\pi \sum_{j\neq i} G(\xi_i + \delta_i y, \xi_j) \\
&\quad - z(\xi_i + \delta_i y, \boldsymbol{\xi})\Big] dy + o(1) \\
&= \frac{8}{\delta_i} \int_{B(0,\frac{\eta}{\delta_i})} \frac{y_j}{(1+|y|^2)^3} exp[\gamma(\xi_i + \delta_i y, \boldsymbol{\xi}) - \gamma(\xi_i, \boldsymbol{\xi})] dy + o(1) \\
&= \frac{8}{\delta_i} \int_{\mathbb{R}^2} \frac{y_j}{(1+|y|^2)^3} \frac{\partial \gamma}{\partial x}(\xi_i, \boldsymbol{\xi}) \cdot \delta_i y \, dy + o(1) \\
&= 2\pi \frac{\partial \gamma}{\partial x}(\xi_i, \boldsymbol{\xi}) + o(1).
\end{aligned}
$$

Finally,

$$
\Big| \int_{\Omega \setminus \bigcup_i B(\xi_i,\eta)} \lambda e^{-W} Z_i^j \, dx \Big| \leq C\lambda \int_{\Omega \setminus \bigcup_i B(\xi_i,\eta)} e^{\sum_\ell P w_\ell} |Z_i^j| dx \leq C\lambda = o(1).
$$

Combining the above estimates, we have

$$
\partial_{\xi_i^j} J(W) = -8\pi \frac{\partial \gamma}{\partial x}(\xi_i, \boldsymbol{\xi}) + o(1) = \partial_{\xi_i^j} \Lambda(\boldsymbol{\xi}) + o(1),
$$

as desired, where we used (1.6). □

Finally, we have the following expansion of the reduced energy.

Proposition 3.9 *It holds*

$$
\tilde{J}(\boldsymbol{\xi}) := J(W_{\boldsymbol{\xi}} + \phi_{\boldsymbol{\xi}}) = J(W_{\boldsymbol{\xi}}) + o(1),
$$

$\mathcal{C}^1$ *uniformly in* $\boldsymbol{\xi}$ *in compact sets of* $\mathcal{F}_k\Omega$.

Proof To simplify the notation, we shall drop the sub-index $\boldsymbol{\xi}$ in the proof. It is not difficult to show that

$$\begin{aligned} J(W+\phi)-J(W) &= \frac{1}{2}\int_\Omega |\nabla\phi|^2dx + \int_\Omega \nabla W\cdot\nabla\phi dx + \lambda\int_\Omega e^{-W}(1-e^{-\phi})dx \\ &\quad + \rho^+\Big(\log\int_\Omega e^W dx - \log\int_\Omega e^{W+\phi}\Big)dx \\ &= -\int_\Omega \Delta z(x,\boldsymbol{\xi})\phi dx - \rho^+\int_\Omega \frac{h(x,\boldsymbol{\xi})e^{z(x,\boldsymbol{\xi})}\phi}{\int_\Omega h(x,\boldsymbol{\xi})e^{z(x,\boldsymbol{\xi})}dx}dx \\ &\quad + \int_\Omega \sum_i e^{w_i}\phi dx - \lambda\int_\Omega e^{-W}\phi dx \\ &\quad + \rho^+\Big(\log\int_\Omega e^W dx - \log\int_\Omega e^{W+\phi}dx \\ &\quad + \int_\Omega \frac{h(x,\boldsymbol{\xi})e^{z(x,\boldsymbol{\xi})}\phi}{\int_\Omega h(x,\boldsymbol{\xi})e^{z(x,\boldsymbol{\xi})}dx}dx\Big) \\ &\quad + \lambda\int_\Omega e^{-W}(1-e^{-\phi}+\phi)dx + \|\phi\|^2 = o(1). \end{aligned}$$

Next we consider the derivatives.

$$\begin{aligned} \partial_{\xi_i^j}[J(W+\phi)-J(W)] &= -\int_\Omega \left(\Delta(W+\phi) + \rho^+\frac{e^{W+\phi}}{\int_\Omega e^{W+\phi}dx} - \lambda e^{-(W+\phi)}\right)\partial_{\xi_i^j}\phi dx \\ &\quad -\int_\Omega \Bigg[\Delta\phi + \rho^+\left(\frac{e^{W+\phi}}{\int_\Omega e^{W+\phi}dx} - \frac{e^W}{\int_\Omega e^W dx}\right) \\ &\quad -\lambda(e^{-(W+\phi)} - e^{-W})\Bigg]\partial_{\xi_i^j}W dx \\ &= \sum_{i,j}\int_\Omega c_{ij}e^{w_i}Z_i^j\partial_{\xi_i^j}\phi dx - \int_\Omega \Delta\phi\partial_{\xi_i^j}W dx - \int_\Omega \lambda e^{-W}\phi\partial_{\xi_i^j}W dx \\ &\quad + \int_\Omega \lambda(e^{-(W+\phi)} - e^{-W} + e^{-W}\phi)\partial_{\xi_i^j}W dx \\ &\quad + \rho^+\int_\Omega\left(\frac{e^{W+\phi}}{\int e^{W+\phi}} - \frac{e^W}{\int e^W}\right)\partial_{\xi_i^j}W dx. \end{aligned}$$

Using the estimate for c_{ij} in Proposition 3.6 and (3.34), we have

$$\sum_{i,j}\int_\Omega c_{ij}e^{w_i}Z_i^j\partial_{\xi_i^j}\phi dx = O\left(\sum_{i,j}|c_{ij}|\|\partial_{\xi_i^j}\phi\|\cdot\|e^{w_i}Z_i^j\|_q\right) = O(\lambda^{\frac{2-3q}{2q}+1-\epsilon}) = o(1),$$

provided q is sufficiently close to 1. Recalling the definitions of f, g in (3.3) we exploit now the estimates in [8, Lemma 4.7]. For some $\theta\in(0,1)$ and p sufficiently close to 1 we have

$$\begin{aligned} \int_\Omega \lambda(e^{-(W+\phi)} - e^{-W} + e^{-W}\phi)\partial_{\xi_i^j}W dx &= \int_\Omega \lambda f''(W+\theta\phi)\phi^2\partial_{\xi_i^j}W dx \\ &= O(\|\lambda f''(W+\theta\phi)\phi^2\|_p\|\partial_{\xi_i^j}W\|_q) \\ &= O(\lambda^{\frac{1-pq}{pq}-\frac{1}{2}+1-2\epsilon}) = o(1). \end{aligned}$$

Moreover, for some $\tilde{\theta} \in (0, 1)$ and suitable p, q

$$\begin{aligned}\rho^+ \int_\Omega \left(\frac{e^{W+\phi}}{\int e^{W+\phi}} - \frac{e^W}{\int e^W} \right) \partial_{\xi_i^j} W \, dx &= \rho^+ \int_\Omega g'(W + \tilde{\theta}\phi)\phi \partial_{\xi_i^j} W \, dx \\ &= O(\|g'(W + \tilde{\theta}\phi)\phi\|_p \|\partial_{\xi_i^j} W\|_q) \\ &= O(\lambda^{\frac{1}{2}-\epsilon}) = o(1).\end{aligned}$$

Recall that

$$\lambda e^{-W} = \sum_{i=1}^k e^{w_i} + O(\lambda) \quad \text{and} \quad \partial_{\xi_i^j} W = -4PZ_i^j + O(1),$$

for $\boldsymbol{\xi}$ in compact sets of $\mathcal{F}_k\Omega$. Then

$$\begin{aligned}\lambda \int_\Omega e^{-W} \phi \partial_{\xi_i^j} W dx &= -4 \sum_{\ell=1}^k \int_\Omega e^{w_\ell} \phi P Z_i^j dx + o(1) \\ &= -4 \int_\Omega e^{w_i} Z_i^j \phi dx - 4 \sum_{\ell \neq i} \int_\Omega e^{w_\ell} \phi Z_i^j dx + o(1) \\ &= -4 \int_\Omega \nabla\phi \cdot \nabla P Z_i^j dx + o(1) = o(1)\end{aligned}$$

by the orthogonality condition satisfied by ϕ. Moreover, again by the orthogonality condition we have

$$\begin{aligned}\int_\Omega \Delta\phi \partial_{\xi_i^j} W dx = - \int_\Omega \nabla\phi \cdot \nabla \partial_{\xi_i^j} W dx &= -4 \int_\Omega \nabla\phi \cdot (\nabla P Z_i^j + O(1)) dx \\ = O(1) \int_\Omega |\nabla\phi| dx = o(1).\end{aligned}$$

Combining the above estimates, we have

$$\partial_{\xi_i^j} \tilde{J}(\boldsymbol{\xi}) = \partial_{\xi_i^j} J(W) + o(1), \tag{3.37}$$

as desired. □

Proof of Theorem 1.1 Let $\mathcal{K} \subset \mathcal{F}_k\Omega$ be a C^1-stable set of critical points of Λ. Then, by Propositions 3.8-3.9, for $\lambda > 0$ small, there exists $\boldsymbol{\xi}_\lambda$ critical point of $\tilde{J}$ and $d(\boldsymbol{\xi}_\lambda, \mathcal{K}) \to 0$ as $\lambda \to 0$. By Lemma 3.7, $u_\lambda = W_{\xi_\lambda} + \phi_{\xi_\lambda}$ is a solution of (3.1). It follows that u_λ solves the original problem (1.1) with $\rho_\lambda^+ = \rho^+$ and

$$\rho_\lambda^- = \lambda \int_\Omega e^{-u_\lambda} dx = \lambda \int_\Omega e^{-W_{\xi_\lambda}} dx + o(1) = 8k\pi + o(1).$$

Moreover, from the definition of u_λ, and using (3.4), (3.5) and (3.31), we can derive the second property in Theorem 1.1. □

4 Asymmetric blow up

4.1 Approximate solutions

In this section we will derive the proof of Theorem 1.2. To this end we will always assume that Ω is $l-$symmetric for $l \geq 2$ even according to (4.1) below. Therefore, we will consider symmetric functions such that

$$u(x) = u(\mathcal{R}_l \cdot x), \tag{4.1}$$

recall (1.7), and define

$$\mathcal{H}_l := \left\{ u \in H_0^1(\Omega),\ u \text{ satisfies } (4.1) \right\}.$$

Consider problem (3.1) and let $k \geq 2$ be an odd integer. In order to construct blow up solutions with local masses $(4\pi k(k-1), 4\pi k(k+1))$, we need to consider the following singular Liouville equation. Let $\alpha \geq 2$. It is known [27] that

$$w_\delta^\alpha(x) = \log \frac{2\alpha^2\delta^\alpha}{(\delta^\alpha + |x|^\alpha)^2}, \quad \delta > 0,$$

solves the problem

$$\Delta w + |x|^{\alpha-2}e^w = 0 \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{\alpha-2}e^w dx < \infty,$$

and

$$\int_{\mathbb{R}^2} |x|^{\alpha-2}e^w dx = 4\pi\alpha.$$

Similarly to the previous section, let Pu be the projection of the function u into $H_0^1(\Omega)$. We look here for a sign changing solution of the form

$$u = W + \phi(x), \quad W(x) = z(x) + \sum_{i=1}^{k}(-1)^i Pw_i(x),$$

where ϕ is a small error term, $z(x)$ is the unique solution of (1.8) and $Pw_i = Pw_{\delta_i}^{\alpha_i}$ with

$$\alpha_i = 4i - 2, \quad \delta_i = d_i\lambda^{\frac{k-i+1}{4i-2}}, \ d_i > 0, \quad i = 1, \cdots, k. \tag{4.2}$$

The latter parameters are chosen such that the interaction of different bubbles is small. More precisely, the following functions will play an important role in the interaction estimate:

$$\begin{aligned} \Theta_i(y) &= Pw_i(\delta_i y) - w_i(\delta_i y) - (\alpha_i - 2)\log|\delta_i y| \\ &\quad + \sum_{j\neq i}(-1)^{j-i}Pw_j - z(\delta_i y) + \log\lambda, \quad \text{i odd,} \end{aligned} \tag{4.3}$$

$$\begin{aligned} T_i(y) &= Pw_i(\delta_i y) - w_i(\delta_i y) - (\alpha_i - 2)\log|\delta_i y| \\ &\quad + \sum_{j\neq i}(-1)^{j-i}Pw_j + z(\delta_i y) - \log Q, \quad \text{i even,} \end{aligned} \tag{4.4}$$

where

$$Q = \rho_0^{-1}\int_\Omega e^{z-8k\pi G(x,0)}dx. \tag{4.5}$$

As we will see in the sequel, in order to make these two functions small, we will need to choose δ_i and α_i such that

$$(\alpha_i - 2) + \sum_{j<i}(-1)^{j-i}2\alpha_j = 0, \quad i = 1, \cdots, k, \tag{4.6}$$

and

$$-\alpha_i \log\delta_i - \log(2\alpha_i^2) - 2\sum_{j>i}(-1)^{j-i}\alpha_j\log\delta_j - z(0) + \sum_{j=1}^{k}(-1)^{j-i}h_j(0) + \log\lambda = 0, \quad \text{i odd}, \tag{4.7}$$

$$-\alpha_i \log\delta_i - \log(2\alpha_i^2) - 2\sum_{j>i}(-1)^{j-i}\alpha_j\log\delta_j + z(0) + \sum_{j=1}^{k}(-1)^{j-i}h_j(0) - \log Q = 0, \quad \text{i even}, \tag{4.8}$$

where $h_i(x) = 4\pi\alpha_i H(x,0)$. From (4.6) we deduce that $\alpha_1 = 2$ and $\alpha_i = \alpha_{i-1} + 4$ for $i \geq 2$ which implies the choice of α_i in (4.2). On the other hand, from (4.7) and (4.8) one easily deduces that

$$\delta_k^{\alpha_k} = \lambda e^{\sum_j(-1)^{j-k}h_j(0) - z(0) - \log(2\alpha_k^2)} = \lambda e^{8k\pi H(0,0) - z(0) - \log(2\alpha_k^2)},$$

and

$$\delta_{i-1}^{\alpha_{i-1}} = \frac{\delta_i^{\alpha_i}}{4\alpha_i^2\alpha_{i-1}^2 Q}\lambda.$$

From the above identities, one can get that

$$\delta_i = d_i\lambda^{\frac{k-i+1}{4i-2}},$$

for some $d_i > 0$, which implies (4.2).

We estimate now Θ_i and T_i. First, using the maximum principle it is not difficult to see that

$$\begin{aligned} Pw_i(x) &= w_i(x) - \log(2\alpha_i^2\delta_i^{\alpha_i}) + h_i(x) + O(\delta_i^{\alpha_i}) \\ &= -2\log(\delta_i^{\alpha_i} + |x|^{\alpha_i}) + h_i(x) + O(\delta_i^{\alpha_i}) \end{aligned} \tag{4.9}$$

and for $i, j = 1, \cdots, k$,

$$Pw_i(\delta_j y) = \begin{cases} -2\alpha_i\log(\delta_j|y|) + h_i(0) + O\left(\frac{1}{|y|^{\alpha_i}}\left(\frac{\delta_i}{\delta_j}\right)^{\alpha_i}\right) \\ +O(\delta_j|y|) + O(\delta_i^{\alpha_i}) \quad \text{if } i < j \\ \\ -2\alpha_i\log\delta_i - 2\log(1+|y|^{\alpha_i}) + h_i(0) \\ +O(\delta_i(y)) + O(\delta_i^{\alpha_i}) \quad \text{if } i = j \\ \\ -2\alpha_i\log\delta_i + h_i(0) + O\left(|y|^{\alpha_i}\left(\frac{\delta_j}{\delta_i}\right)^{\alpha_i}\right) \\ +O(\delta_j|y|) + O(\delta_i^{\alpha_i}) \quad \text{if } i > j. \end{cases} \tag{4.10}$$

where $h_i(x) = 4\pi\alpha_i H(x, 0)$.

Remark 4.1 From the above expansion, one can get that for $|x| \geq \delta_0$ for $\delta_0 > 0$ small, the following expansion holds:

$$\sum_{i=1}^{k}(-1)^i Pw_i(x) = 4\pi\sum_i(-1)^i\alpha_i H(x,0) - 2\sum_i(-1)^i\alpha_i\log|x| + O(\delta_k^{\alpha_k})$$

From the definition of α_i we have $\sum_{i=1}^k(-1)^i\alpha_i = (-1)^k 2k$ and hence, for k odd it holds

$$\sum_{i=1}^{k}(-1)^i Pw_i(x) = -8kG(x,0) + O(\lambda).$$

We next introduce the following shrinking annulus

$$A_j = \left\{x \in \Omega, \sqrt{\delta_{j-1}\delta_j} \leq |x| \leq \sqrt{\delta_j\delta_{j+1}}\right\}, \quad j = 1, \cdots, k, \tag{4.11}$$

where $\delta_0 := 0$ and $\delta_{k+1} := +\infty$.

Lemma 4.2 *For any $y \in \frac{A_i}{\delta_i}$, the following estimates hold:*

$$\Theta_i(y) = O(\delta_i|y| + \lambda), \quad i \text{ odd}, \tag{4.12}$$

$$T_i(y) = O(\delta_i|y| + \lambda), \quad i \text{ even}. \tag{4.13}$$

In particular,

$$\sup_{y\in\frac{A_i}{\delta_i}}|\Theta_i(y)| + \sup_{y\in\frac{A_i}{\delta_i}}|T_i(y)| = O(1). \tag{4.14}$$

Proof Consider $y \in \frac{A_i}{\delta_i}$. From (4.10), and using (4.6) and (4.7), for i odd,

$$\begin{aligned}
\Theta_i(y) &= -\alpha_i\log\delta_i - \log(2\alpha_i^2) + h_i(0) - (\alpha_i - 2)\log|\delta_i y| + O(\delta_i|y| + \delta_i^{\alpha_i})\\
&\quad + \sum_{j<i}(-1)^{i-j}\Big[-2\alpha_j\log(\delta_i|y|) + h_j(0) + O\Big(\frac{1}{|y|^{\alpha_j}}\Big(\frac{\delta_j}{\delta_i}\Big)^{\alpha_j}\Big) + O(\delta_i|y| + \delta_j^{\alpha_j})\Big]\\
&\quad + \sum_{j>i}(-1)^{j-i}\Big[-2\alpha_j\log\delta_j + h_j(0) + O\Big(|y|^{\alpha_j}\Big(\frac{\delta_i}{\delta_j}\Big)^{\alpha_j}\Big) + O(\delta_i|y| + \delta_j^{\alpha_j})\Big]\\
&\quad - z(0) + \log\lambda + O(\delta_i|y|)\\
&= \Big[\sum_{j=1}^{k}(-1)^{j-i}h_j(0) - \alpha_i\log\delta_i - \log(2\alpha_i^2) - 2\sum_{j>i}(-1)^{j-i}\alpha_j\log\delta_j - z(0) + \log\lambda\Big]\\
&\quad (= 0 \text{ because of } (4.7))\\
&\quad - \log|\delta_i|y||\Big[(\alpha_i - 2) + \sum_{j<i}(-1)^{i-j}2\alpha_j\Big]\\
&\quad (= 0 \text{ because of } (4.6))\\
&\quad + O(\delta_i|y|) + \sum_j\delta_j^{\alpha_j} + \sum_{j>i}O\Big(|y|^{\alpha_j}\Big(\frac{\delta_i}{\delta_j}\Big)^{\alpha_j}\Big) + \sum_{j<i}O\Big(\frac{1}{|y|^{\alpha_j}}\Big(\frac{\delta_j}{\delta_i}\Big)^{\alpha_j}\Big)\\
&= O(\delta_i|y|) + \sum_j\delta_j^{\alpha_j} + \sum_{j>i}O\Big(|y|^{\alpha_j}\Big(\frac{\delta_i}{\delta_j}\Big)^{\alpha_j}\Big) + \sum_{j<i}O\Big(\frac{1}{|y|^{\alpha_j}}\Big(\frac{\delta_j}{\delta_i}\Big)^{\alpha_j}\Big)\\
&= O(\delta_i|y| + \lambda).
\end{aligned}$$

Similarly, for i even,

$$
\begin{aligned}
T_i(y) = &\Big[\sum_{j=1}^{k}(-1)^{j-i}h_j(0) - \alpha_i\log\delta_i - \log(2\alpha_i^2) - 2\sum_{j>i}(-1)^{j-i}\alpha_j\log\delta_j + z(0) - \log Q\Big]\\
&(= 0 \text{ because of } (4.8))\\
&-\log|\delta_i|y||\Big[(\alpha_i - 2) + \sum_{j<i}(-1)^{i-j}2\alpha_j\Big]\\
&(= 0 \text{ because of } (4.6))\\
&+ O(\delta_i|y|) + \sum_j \delta_j^{\alpha_j} + \sum_{j>i} O\Big(|y|^{\alpha_j}\Big(\frac{\delta_i}{\delta_j}\Big)^{\alpha_j}\Big) + \sum_{j<i} O\Big(\frac{1}{|y|^{\alpha_j}}\Big(\frac{\delta_j}{\delta_i}\Big)^{\alpha_j}\Big)\\
= &\, O(\delta_i|y|) + \sum_j \delta_j^{\alpha_j} + \sum_{j>i} O\Big(|y|^{\alpha_j}\Big(\frac{\delta_i}{\delta_j}\Big)^{\alpha_j}\Big) + \sum_{j<i} O\Big(\frac{1}{|y|^{\alpha_j}}\Big(\frac{\delta_j}{\delta_i}\Big)^{\alpha_j}\Big)\\
= &\, O(\delta_i|y| + \lambda).
\end{aligned}
$$

Finally, (4.14) follows from the above two estimates since $\delta_i|y| = O(1)$ when $y \in \frac{A_i}{\delta_i}$. □

Finally, we will need the following non-degeneracy result for entire singular Liouville equations which was derived in [11, Theorem 6.1] for $l = 2$ and which can be extended to any $l \geq 2$ even.

Proposition 4.3 *Assume $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfying (4.1) is a solutions of*

$$
\Delta\phi + 2\alpha^2\frac{|y|^{\alpha-2}}{(1+|y|^\alpha)^2}\phi = 0 \quad in \quad \mathbb{R}^2, \qquad \int_{\mathbb{R}^2}|\nabla\phi|^2 dy < \infty,
$$

with $\alpha \geq 2$ and $\frac{\alpha}{2}$ odd. Then,

$$
\phi(y) = \gamma\frac{1-|y|^\alpha}{1+|y|^\alpha}, \quad \textit{for some } \gamma \in \mathbb{R}.
$$

4.2 Estimate of the error term

In this subsection we estimate the error of the approximate solution. To this end, set

$$
\begin{aligned}
E_1 &= \rho^+\frac{e^W}{\int_\Omega e^W dx} - \sum_{i\ even}|x|^{\alpha_i-2}e^{w_i} - \rho_0\frac{e^{z-8k\pi G(x,0)}}{\int_\Omega e^{z-8k\pi G(x,0)}dx},\\
E_2 &= \lambda e^{-W} - \sum_{i\ odd}|x|^{\alpha_i-2}e^{w_i}.
\end{aligned}
$$

Lemma 4.4 *For any $q \geq 1$ sufficiently close to 1, the following holds:*

$$
\|E_1\|_q = O\Big(\lambda^{\frac{2-q}{2q(2k-1)}}\Big), \quad \|E_2\|_q = O\Big(\lambda^{\frac{2-q}{2q(2k-1)}}\Big).
$$

Proof First we consider E_2. Recall the definition of the annulus A_i in (4.11).

$$
\int_\Omega E_2^q dx = \sum_{i=1}^{k}\int_{A_i}E_2^q dx = \sum_{i\ odd}\int_{A_i}E_2^q dx + \sum_{i\ even}\int_{A_i}E_2^q dx = I_1 + I_2.
$$

One has

$$
\begin{aligned}
I_1 &= \sum_{i\ odd} \int_{A_i} E_2^q dx = \sum_{i\ odd} \int_{A_i} |\lambda e^{\sum_{l\ odd} Pw_l - \sum_{l\ even} Pw_l - z} - \sum_{j\ odd} |x|^{\alpha_j - 2} e^{w_j}|^q dx \\
&\le C \sum_{i\ odd} \int_{A_i} ||x|^{\alpha_i - 2} e^{w_i} - \lambda e^{\sum_{l\ odd} Pw_l - \sum_{l\ even} Pw_l - z}|^q dx \\
&\quad + C \sum_{i,j\ odd,\ i \neq j} \int_{A_i} ||x|^{\alpha_j - 2} e^{w_j}|^q dx \\
&= I_{11} + I_{12}.
\end{aligned}
$$

Let us estimate I_{11}. For fixed i odd,

$$
\begin{aligned}
&\int_{A_i} ||x|^{\alpha_i - 2} e^{w_i} - \lambda e^{\sum_{l\ odd} Pw_l - \sum_{l\ even} Pw_l - z}|^q dx \\
&= \int_{A_i} |x|^{q(\alpha_i - 2)} e^{q w_i} |1 - e^{Pw_i - w_i - (\alpha_i - 2)\log|x| + \sum_{j \neq i\ odd} Pw_j - \sum_{l\ even} Pw_l - z + \log\lambda}|^q dx \\
&= C\delta_i^{2-2q} \int_{\frac{A_i}{\delta_i}} \frac{|y|^{q(\alpha_i - 2)}}{(1 + |y|^{\alpha_i})^{2q}} |1 - e^{\Theta_i(y)}|^q dy = C\delta_i^{2-2q} \int_{\frac{A_i}{\delta_i}} \frac{|y|^{q(\alpha_i - 2)}}{(1 + |y|^{\alpha_i})^{2q}} |\Theta_i(y)|^q dy \\
&\text{(using (4.12))} \\
&= O\Big(\delta_i^{2-2q} \int_{\frac{A_i}{\delta_i}} \frac{|y|^{q(\alpha_i - 2)}}{(1 + |y|^{\alpha_i})^{2q}} |\delta_i|y| + \lambda|^q dy\Big) = O(\delta_i^{2-2q}\lambda^q + \delta_i^{2-q}) = O(\delta_1^{2-2q}\lambda^q + \delta_k^{2-q}) \\
&= O(\lambda^{q + k(1-q)} + \lambda^{\frac{2-q}{2(2k-1)}}) = O(\lambda^{\frac{2-q}{2(2k-1)}}),
\end{aligned}
$$

provided that q is close to 1. Therefore, we get $I_{11} = O(\lambda^{\frac{2-q}{2(2k-1)}})$.

For I_{12}, fix $j \neq i$ odd,

$$
\begin{aligned}
&\int_{A_i} ||x|^{\alpha_j - 2} e^{w_j}|^q dx = C \int_{A_i} \left(\frac{|x|^{\alpha_j - 2} \delta_j^{\alpha_j}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \right)^q dx \\
&= C\delta_j^{2-2q} \int_{\frac{\sqrt{\delta_{i-1}\delta_i}}{\delta_j} \le |y| \le \frac{\sqrt{\delta_i \delta_{i+1}}}{\delta_j}} \frac{|y|^{q(\alpha_j - 2)}}{(1 + |y|^{\alpha_j})^{2q}} dy \\
&= \begin{cases} O\left(\delta_j^{2-2q} \left(\frac{\sqrt{\delta_i \delta_{i+1}}}{\delta_j} \right)^{(\alpha_j - 2)q + 2} \right) & \text{for } j > i \\[2ex] O\left(\delta_j^{2-2q} \left(\frac{\sqrt{\delta_i \delta_{i-1}}}{\delta_j} \right)^{-(\alpha_j + 2)q + 2} \right) & \text{for } j < i \end{cases} \\
&= \begin{cases} O\Big(\delta_3^{2-2q} (\frac{\delta_{k-1}}{\delta_k})^{(\alpha_k - 2)q + 2}\Big) = O\left(\lambda^{\frac{(k-2)(1-q)}{5} + \frac{(2k+1)(2(k-1)q+1)}{4(k-1)^2 - 1}} \right) \\[2ex] O\Big(\delta_1^{2-2q} (\frac{\delta_{k-2}}{\delta_{k-1}})^{q(2 + \alpha_{k-2}) - 2}\Big) = O\left(\lambda^{k(1-q) + \frac{(2k+1)(2(k-2)q-1)}{4(k-2)^2 - 1}} \right) \end{cases} \\
&= O\Big(\lambda^{\frac{2-q}{2(2k-1)}}\Big).
\end{aligned} \tag{4.15}
$$

provided that q is close to 1. Therefore, $\|I_1\|_q = O\Big(\lambda^{\frac{2-q}{2q(2k-1)}}\Big)$.

Next, let us estimate I_2. For l even fixed,

$$\int_{A_l} E_2^q dx \le C\int_{A_l} |\lambda e^{-W}|^q dx + C\sum_{i\ odd}\int_{A_l} ||x|^{\alpha_i-2}e^{w_i}|^q dx = I_{21} + I_{22}.$$

We have,

$$\begin{aligned}
I_{21} &= C\int_{A_l} |\lambda e^{-Pw_l-\sum_{j\neq l\ even}Pw_j-z+\sum_{i\ odd}Pw_i}|^q dx\\
&= C\lambda^q\delta_l^2\int_{\frac{A_l}{\delta_l}} |e^{-w_l(\delta_l y)-(\alpha_l-2)\log|\delta_l y|-T_l(y)-\log Q}|^q dy\\
&\text{(using (4.13))}\\
&= O\Big(\delta_l^{2+2q}\lambda^q\int_{\sqrt{\frac{\delta_{l-1}}{\delta_l}}\le|y|\le\sqrt{\frac{\delta_{l+1}}{\delta_l}}}\frac{(1+|y|^{\alpha_l})^{2q}}{|y|^{(\alpha_l-2)q}}(1+\delta_l|y|+\lambda)^q dy\Big)\qquad(4.16)\\
&= O\Big(\delta_l^{2+2q}\lambda^q\Big[\Big(\frac{\delta_{l+1}}{\delta_l}\Big)^{\frac{(\alpha_l+2)q}{2}+1}+\Big(\frac{\delta_l}{\delta_{l-1}}\Big)^{\frac{(\alpha_l-2)q}{2}-1}\Big]\Big)\\
&= O\Big(\delta_2^{2+2q}\lambda^q\Big[\Big(\frac{\delta_3}{\delta_2}\Big)^{\frac{(\alpha_2+2)q}{2}+1}+\Big(\frac{\delta_2}{\delta_1}\Big)^{\frac{(\alpha_2-2)q}{2}-1}\Big]\Big)\\
&= O\Big(\lambda^{q+\frac{(k-1)(1+q)}{3}-\frac{(2k+1)(2q-1)}{6}}\Big) = O\Big(\lambda^{\frac{2-q}{2(2k-1)}}\Big),
\end{aligned}$$

if q is close to 1. Moreover, similarly to the estimate of I_{12}, one can also get that $I_{22} = O\Big(\lambda^{\frac{2-q}{2(2k-1)}}\Big)$.

Combining all the above estimates, one has

$$\int_\Omega E_2^q dx = O(\lambda^{\frac{2-q}{2(2k-1)}}). \qquad (4.17)$$

Next we consider E_1. First we need to estimate $\int_\Omega e^W dx$. For i even fixed,

$$\begin{aligned}
\int_{A_i} e^W dx &= \int_{A_i} e^{Pw_i-w_i+z+\sum_{j\neq i}(-1)^{j-i}Pw_j-(\alpha_i-2)\log|x|}|x|^{\alpha_i-2}e^{w_i}dx\\
&= \int_{\frac{A_i}{\delta_i}} e^{T_i(y)+\log Q}|\delta_i y|^{\alpha_i-2}e^{w_i(\delta_i y)}\delta_i^2 dy\\
&= \int_{\frac{A_i}{\delta_i}} e^{\log Q+O(\delta_i|y|+\lambda)}|\delta_i y|^{\alpha_i-2}e^{w_i(\delta_i y)}\delta_i^2 dy = 4\pi\alpha_i Q + O(\lambda^{\frac{1}{2(2k-1)}}),
\end{aligned}$$

where we have used Lemma 4.2 for the estimate of $T_i(y)$ and the fact that

$$\int_{\mathbb{R}^2} \frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}dy = 4\pi\alpha_i.$$

For $i < k$ odd and fixed, reasoning as in (4.16) with $q = 1$, one has

$$\int_{A_i} e^W dx = \int_{A_i} e^{-Pw_i-\sum_{j\neq i}(-1)^{j-i}Pw_j+z}dx = O(\lambda^{\frac{2k-5}{6}}).$$

Finally for $i = k$ which is odd, using Remark 4.1,

$$\begin{aligned}\int_{A_k} e^W dx &= \int_{A_k} e^z e^{-Pw_k-\sum_{j\neq k}(-1)^{j-k}Pw_j}dx \\ &= \int_{|x|>\sqrt{\delta_{k-1}\delta_k}} e^{z-8k\pi G(x,0)}dx + O(\delta_k^{\alpha_k}) + O(\lambda^{\frac{1}{2(2k-1)}}) \\ &= \int_\Omega e^{z-8k\pi G(x,0)}dx + O(\lambda^{\frac{1}{2(2k-1)}}).\end{aligned}$$

In conclusion, one has

$$\begin{aligned}\int_\Omega e^W dx &= \int_\Omega e^{z-8k\pi G(x,0)}dx + \sum_{i\ even} 4\pi\alpha_i Q + O(\lambda^{\frac{1}{2(2k-1)}}) \\ &= \frac{\rho^+}{\rho_0}\int_\Omega e^{z-8k\pi G(x,0)}dx + O(\lambda^{\frac{1}{2(2k-1)}}),\end{aligned} \tag{4.18}$$

where we used the definition of Q in (4.5) and the fact that

$$\sum_{i\ even} 4\pi\alpha_i Q = \frac{\rho^+-\rho_0}{\rho_0}\int_\Omega e^{z-8k\pi G(x,0)}dx,$$

since $\sum_{i\ even} 4\pi\alpha_i = 4\pi k(k-1) = \rho^+ - \rho_0$.

With the estimate for $\int_\Omega e^W dx$ in hand, we now consider E_1.

$$\int_\Omega E_1^q dx = \sum_{i\ even}\int_{A_i} E_1^q dx + \sum_{l\ odd}\int_{A_l} E_1^q dx = J_1 + J_2.$$

First for i even fixed,

$$
\begin{aligned}
\int_{A_i} E_1^q dx &= \int_{A_i} |\rho^+ \frac{e^W}{\int_\Omega e^W ex} - \rho_0 \frac{e^{z-8k\pi G(x,0)}}{\int_\Omega e^{z-8k\pi G(x,0)} dx} \\
&\quad - |x|^{\alpha_i - 2} e^{w_i} - \sum_{j \neq i \ even} |x|^{\alpha_j - 2} e^{w_j}|^q dx \\
&\leq C \int_{A_i} |\rho^+ \frac{e^W}{\int_\Omega e^W ex} - |x|^{\alpha_i-2} e^{w_i}|^q dx + C \int_{A_i} |\rho_0 \frac{e^{z-8k\pi G(x,0)}}{\int_\Omega e^{z-8k\pi G(x,0)} dx}|^q dx \\
&\quad + C \sum_{j \neq i \ even} \int_{A_i} ||x|^{\alpha_j - 2} e^{w_j}|^q dx \\
&= C \int_{A_i} |\rho^+ \frac{e^W}{\int_\Omega e^W dx} - |x|^{\alpha_i-2} e^{w_i}|^q dx + O(\lambda^{\frac{2-q}{2(2k-1)}}) + O(\delta_{i+1}^{4kq+2}) \\
&= C \delta_i^{2-2q} \int_{\frac{A_i}{\delta_i}} \frac{|y|^{(\alpha_i-2)q}}{(1+|y|^{\alpha_i})^{2q}} \\
&\quad \left| 1 - e^{P w_i(\delta_i y) - w_i(\delta_i y) - (\alpha_i - 2)\log|\delta_i y| + \sum_{j \neq i}(-1)^{j-i} P w_j + z + \log \frac{\rho^+}{\int_\Omega e^W dx}} \right|^q dx \\
&\quad (\text{ by } (4.13)) \\
&= C \delta_i^{2-2q} \int_{\frac{A_i}{\delta_i}} \frac{|y|^{(\alpha_i-2)q}}{(1+|y|^{\alpha_i})^{2q}} \\
&\quad \left| 1 - e^{T_i(y) + \log Q + \log \frac{\rho_0}{\int_\Omega e^{z-8k\pi G(x,0)} dx} + O(\lambda^{\frac{1}{2(2k-1)}})} \right|^q dx \\
&= C \delta_i^{2-2q} \int_{\frac{A_i}{\delta_i}} \frac{|y|^{(\alpha_i-2)q}}{(1+|y|^{\alpha_i})^{2q}} |\delta_i |y| + O(\lambda^{\frac{1}{2(2k-1)}})|^q dy = O(\lambda^{\frac{2-q}{2(2k-1)}}).
\end{aligned}
$$

So we have

$$
J_1 = O\left(\lambda^{\frac{2-q}{2(2k-1)}}\right). \tag{4.19}
$$

Next, consider J_2. For $l < k$ odd and fixed, similarly to the estimates in (4.16), (4.15) and using (4.18)

$$
\begin{aligned}
\int_{A_l} |E_1|^q dx &= O(1) \Big(\int_{A_l} |e^{-P w_l - \sum_{j \neq l}(-1)^{j-l} P w_j + z}|^q dx + \int_{A_l} \left| \frac{e^{z-8k\pi G(x,0)}}{\int_\Omega e^{z-8k\pi G(x,0)} dx} \right|^q dx \\
&\quad + \sum_{j \ even} \int_{A_l} ||x|^{\alpha_j-2} e^{w_j}|^q dx \Big) = O(\lambda^{\frac{2-q}{2(2k-1)}}).
\end{aligned}
$$

Finally, we consider the case $l = k$ which is odd: using (4.18) and (4.15)

$$\begin{aligned}\int_{A_k} E_1^q dx &\le C\int_{A_k}\left|\rho^+\frac{e^{z+\sum_i(-1)^i Pw_i}}{\int_\Omega e^W dx}-\rho_0\frac{e^{z-8k\pi G(x,0)}}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\right|^q dx\\ &\quad + C\sum_{i\ even}\int_{A_k}|x|^{(\alpha_i-2)q}e^{qw_i}dx\\ &= C\int_{A_k}\left|\rho_0\frac{e^{z+\sum_i(-1)^i Pw_i}}{\int_\Omega e^{z-8k\pi G(x,0)}dx}-\rho_0\frac{e^{z-8k\pi G(x,0)}}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\right|^q dx + O(\lambda^{\frac{2-q}{2(2k-1)}})\\ &= O(\delta_k^{2q})+O(\lambda^{\frac{2-q}{2(2k-1)}})=O(\lambda^{\frac{2-q}{2(2k-1)}}).\end{aligned}$$

In conclusion, one has

$$\|E_1\|_q = O\left(\lambda^{\frac{2-q}{2q(2k-1)}}\right).$$

□

4.3 The linear theory

In this subsection, we consider the linear problem: given $h\in\mathcal{H}_l$, we look for $\phi\in\mathcal{H}_l$ such that

$$\Delta\phi+\rho^+\left(\frac{e^W\phi}{\int_\Omega e^W dx}-\frac{e^W\int_\Omega e^W\phi dx}{(\int_\Omega e^W dx)^2}\right)+\lambda e^{-W}\phi=\Delta h \quad \text{in } \Omega. \tag{4.20}$$

First we have the following apriori estimate:

Lemma 4.5 *There exist $\lambda_0>0$ and $C>0$ such that for any $\lambda\in(0,\lambda_0)$, $h\in\mathcal{H}_l$ and $\phi\in\mathcal{H}_l$ solution of (4.20) we have*

$$\|\phi\|\le C|\log\lambda|\|h\|.$$

We start by listing some straightforward integrals which will be useful in the proof of Lemma 4.5.

Lemma 4.6 *The following hold:*

$$\int_{\mathbb{R}^2}\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}dy=0, \tag{4.21}$$

$$\int_{\mathbb{R}^2}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}\log(1+|y|^{\alpha_i})^2dy=-4\pi\alpha_i, \tag{4.22}$$

$$\int_{\mathbb{R}^2}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}\log|y|dy=-4\pi. \tag{4.23}$$

Proof of Lemma 4.5 We prove it by contradiction. Assume there exist $\lambda_n\to 0$, $h_n\in\mathcal{H}_l$ and $\phi_n\in\mathcal{H}_l$ which solves (4.20) such that

$$\|\phi_n\| = 1, \quad |\log \lambda_n| \|h_n\| \to 0 \quad \text{as } n \to \infty.$$

In the following, we omit the index n for simplicity. For $i = 1, \cdots, k$, define $\tilde{\phi}_i(y)$ as

$$\tilde{\phi}_i(y) = \begin{cases} \phi_i(\delta_i y), & y \in \tilde{\Omega}_i = \frac{\Omega}{\delta_i}, \\ 0, & y \in \mathbb{R}^2 \setminus \tilde{\Omega}_i. \end{cases}$$

Step 1. We claim that

$$\phi \to 0 \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^q(\Omega) \text{ for } q \geq 2. \tag{4.24}$$

and

$$\tilde{\phi}_i \text{ is bounded in } H_{\alpha_i}(\mathbb{R}^2)$$

Letting $\psi \in C_0^\infty(\Omega \setminus \{0\})$ and multiplying equation (4.20) by ψ and integrating, one has

$$\begin{aligned} &-\int_\Omega \nabla\psi \cdot \nabla\phi dx + \int_\Omega \lambda e^{-W} \phi\psi dx \\ &+ \rho^+ \left(\frac{\int_\Omega e^W \phi\psi dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W \phi dx \int_\Omega e^W \psi dx}{(\int_\Omega e^W dx)^2} \right) = \int_\Omega \Delta h \psi dx. \end{aligned} \tag{4.25}$$

By the assumption on ϕ, using the fact that in compact sets of $\Omega \setminus \{0\}$,

$$e^W = e^{z(x) - 8k\pi G(x,0)} + O(\lambda) \quad \text{and} \quad \lambda e^{-W} = O(\lambda),$$

one has

$$\phi \to \phi^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^q(\Omega) \text{ for } q \geq 2$$

where

$$\begin{aligned} &-\int_\Omega \nabla\phi^* \cdot \nabla\psi dx + \rho^+ \frac{\int_\Omega e^{z-8k\pi G(x,0)} \phi^* \psi dx}{\int_\Omega e^{z-8k\pi G(x,0)}} \\ &- \rho^+ \frac{\int_\Omega e^{z-8k\pi G(x,0)} \psi dx \int_\Omega e^{z-8k\pi G(x,0)} \phi^* dx}{(\int_\Omega e^{z-8k\pi G(x,0)} dx)^2} = 0. \end{aligned}$$

So $\|\phi^*\|_{H_0^1(\Omega)} \leq 1$ and it solves

$$\Delta\phi^* + \rho^+ \left(\frac{e^{z-8k\pi G(x,0)} \phi^*}{\int_\Omega e^{z-8k\pi G(x,0)} dx} - \frac{e^{z-8k\pi G(x,0)} \int_\Omega e^{z-8k\pi G(x,0)} \phi^* dx}{(\int_\Omega e^{z-8k\pi G(x,0)} dx)^2} \right) = 0.$$

By the non-degeneracy of $z(x)$ we get $\phi^* = 0$. Thus (4.24) is proved.

Now we prove that $\tilde{\phi}_i$ is bounded in $H_{\alpha_i}(\mathbb{R}^2)$. First it is easy to check that

$$\int_{\mathbb{R}^2} |\nabla\tilde{\phi}_i|^2 dy = \int_\Omega |\nabla\phi_i|^2 dx \leq 1 \quad \text{for } i = 1, \cdots, k. \tag{4.26}$$

We multiply (4.20) again by ϕ and integrate,

$$\int_\Omega |\nabla\phi|^2 dx - \int_\Omega \lambda e^{-W}\phi^2 dx \\ -\rho^+\Big(\frac{\int_\Omega e^W\phi^2 dx}{\int_\Omega e^W dx} - \frac{(\int_\Omega e^W \phi dx)^2}{(\int_\Omega e^W dx)^2}\Big) = \int_\Omega \nabla h\cdot\nabla\phi dx. \tag{4.27}$$

From the above equation, one can get that,

$$\int_\Omega \lambda e^{-W}\phi_i^2 dx \le \int_\Omega |\nabla\phi|^2 dx - \rho^+\Big(\frac{\int_\Omega e^W\phi^2 dx}{\int_\Omega e^W dx} - \frac{(\int_\Omega e^W \phi dx)^2}{(\int_\Omega e^W dx)^2}\Big) - \int_\Omega \nabla h\cdot\nabla\phi dx \\ \le 1 + o(1) + \|h\| = O(1)$$

where we used (4.24). Let i be odd. Lemma 4.4 gives

$$\int_\Omega |x|^{\alpha_i - 2} e^{w_i}\phi^2 dx \le C,$$

or equivalently

$$\int_{\mathbb{R}^2} \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde\phi_i^2 dy \le C.$$

Combined with (4.26), we deduce that $\tilde\phi_i$ is bounded in $H_{\alpha_i}(\mathbb{R}^2)$ when i is odd.

We consider now the case for i even. From (4.18), $e^W = e^{z-8k\pi G(x,0)} + O(\lambda)$ uniformly on compact sets of $\Omega\setminus\{0\}$ and recalling (4.24), we get that

$$\int_\Omega e^W\phi dx = O(1). \tag{4.28}$$

Moreover, by (4.27) one can get that

$$\rho^+\left(\frac{\int_\Omega e^W\phi^2 dx}{\int_\Omega e^W dx} - \frac{(\int_\Omega e^W \phi dx)^2}{(\int_\Omega e^W dx)^2}\right) = O(1). \tag{4.29}$$

Combining (4.28) and (4.29), we have

$$\int_\Omega e^W\phi^2 dx = O(1). \tag{4.30}$$

By Lemma 4.4, (4.24) and (4.30), $\int_\Omega |x|^{\alpha_i-2}e^{w_i}\phi^2 dx = O(1)$ for i even, which implies that

$$\int_{\mathbb{R}^2} \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde\phi_i^2 dy = O(1).$$

So we get that also for i even, $\tilde\phi_i$ is bounded in $H_{\alpha_i}(\mathbb{R}^2)$.

Step 2. We claim that

$$\tilde\phi_i(y) \to \gamma_i\frac{1-|y|^2}{1+|y|^2} \text{ weakly in } H_{\alpha_i}(\mathbb{R}^2) \text{ and strongly in } L_{\alpha_i}(\mathbb{R}^2),\ \gamma_i\in\mathbb{R}. \tag{4.31}$$

From Step 1, we know that $\tilde\phi_i \to \tilde\phi_i^*$ weakly in $H_{\alpha_i}(\mathbb{R}^2)$ and strongly in $L_{\alpha_i}(\mathbb{R}^2)$. Consider $\tilde\psi \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ and let $\mathcal{K}$ be its support. For n large, one has

$$\mathcal{K}\subset\frac{A_i}{\delta_i} = \left\{y\in\tilde\Omega_i, \sqrt{\frac{\delta_{i-1}}{\delta_i}} \le |y| \le \sqrt{\frac{\delta_{i+1}}{\delta_i}}\right\}.$$

Define $\psi_i = \tilde{\psi}(\frac{x}{\delta_i})$. Multiplying (4.20) by ψ_i and integrating over Ω,

$$\int_\Omega \nabla\phi \cdot \nabla\psi_i dx - \rho^+ \left(\frac{\int_\Omega e^W \phi\psi_i dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W \phi dx \int_\Omega e^W \psi_i dx}{(\int_\Omega e^W dx)^2} \right) - \int_\Omega \lambda e^{-W} \phi\psi_i dx = \int_\Omega \nabla h \cdot \nabla\psi_i dx. \tag{4.32}$$

Consider first i even. According to Lemma 4.4, one has

$$\begin{aligned} \rho^+ \frac{\int_\Omega e^W \phi dx}{\int_\Omega e^W dx} &= \sum_{j\ even} \int_\Omega |x|^{\alpha_j - 2} e^{w_j} \phi dx + \rho_0 \frac{\int_\Omega e^{z - 8k\pi G(x,0)} \phi dx}{\int_\Omega e^{z - 8k\pi G(x,0)} dx} + o(1) \\ &= \sum_{j\ even} \int_{\mathbb{R}^2} \frac{2\alpha_j^2 |y|^{\alpha_j - 2} \tilde{\phi}_j}{(1 + |y|^{\alpha_i})^2} dy + \rho_0 \frac{\int_\Omega e^{z - 8k\pi G(x,0)} \phi dx}{\int_\Omega e^{z - 8k\pi G(x,0)} dx} + o(1) \\ &= \sum_{j\ even} \int_{\mathbb{R}^2} \frac{2\alpha_j^2 |y|^{\alpha_j - 2} \tilde{\phi}_j^*}{(1 + |y|^{\alpha_i})^2} dy + o(1) \end{aligned}$$

where in the last line we used (4.24). Similarly, one has

$$\begin{aligned} \rho^+ \frac{\int_\Omega e^W \phi\psi_i dx}{\int_\Omega e^W dx} &= \int_{\mathbb{R}^2} \frac{2\alpha_i^2 |y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} \tilde{\psi} \tilde{\phi}_i^* dy + o(1), \\ \rho^+ \frac{\int_\Omega e^W \psi_i dx}{\int_\Omega e^W dx} &= \int_{\mathbb{R}^2} \frac{2\alpha_i^2 |y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} \tilde{\psi} dy + o(1), \\ \lambda \int_\Omega e^{-W} \phi\psi_i dx &= \sum_{j\ odd} \int_\Omega |x|^{\alpha_j - 2} e^{w_j} \phi\psi_i dx + o(1) = o(1). \end{aligned}$$

Thus, $\tilde{\phi}_i^*$ satisfies

$$\begin{aligned} &\int_{\mathbb{R}^2} \nabla\tilde{\phi}_i^* \cdot \nabla\tilde{\psi}_i dy - \int_{\mathbb{R}^2} \frac{2\alpha_i^2 |y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} \tilde{\phi}_i^* \tilde{\psi} dy \\ &\quad = -\frac{1}{\rho^+} \Big(\int_{\mathbb{R}^2} \frac{2\alpha_i^2 |y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} \tilde{\psi} dy \Big) \Big(\int_{\mathbb{R}^2} \frac{2\alpha_i^2 |y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} \tilde{\phi}_i^* dy \Big). \end{aligned}$$

From this we deduce that the function

$$\tilde{\phi}_i^* - \frac{1}{\rho^+} \int_{\mathbb{R}^2} \frac{2\alpha_i^2 |y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} \tilde{\phi}_i^* dy \in H_{\alpha_i}(\mathbb{R}^2)$$

is a solution of

$$\Delta\phi + \frac{2\alpha_i^2 |y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} \phi = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}. \tag{4.33}$$

Since $\int |\nabla \tilde{\phi}_i^*|^2 dy \le 1$, $\tilde{\phi}_i^*$ is a solution in the whole space $\mathbb{R}^2$. By Proposition 4.3, we get that $\tilde{\phi}_i^* - \frac{1}{\rho^+}\int_{\mathbb{R}^2} \frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i^* dy = \gamma_i \frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}$ for some γ_i. By(4.21) one has

$$\int_{\mathbb{R}^2} \frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i^* dy = \frac{1}{\rho^+}\int_{\mathbb{R}^2} \frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} dy \int_{\mathbb{R}^2} \frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i^* dy$$

which implies that

$$\left(\frac{4\pi\alpha_i}{\rho^+} - 1\right)\int_{\mathbb{R}^2} \frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i^* dy = 0.$$

Since $\rho^+ \neq 4\pi\alpha_i$ we deduce that

$$\tilde{\phi}_i^* = \gamma_i \frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}.$$

Hence, (4.31) is proved for i even.

We next turn to i odd. In this case, we consider (4.32) with i odd and estimate each term separately,

$$\int_\Omega e^W \psi_i dx = o(1), \quad \int_\Omega e^W \phi\psi_i dx = o(1),$$

and

$$\lambda \int_\Omega e^{-W}\phi\psi_i dx = \int_\Omega |x|^{\alpha_i-2}e^{w_i}\phi\psi_i dx + o(1) = \int_{\mathbb{R}^2} \frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i\tilde{\psi} dy + o(1).$$

Hence, $\tilde{\phi}_i^*$ satisfies

$$\int_{\mathbb{R}^2} \nabla\tilde{\phi}_i^* \cdot \nabla\tilde{\psi} dy - \int_{\mathbb{R}^2} \frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i^*\tilde{\psi} dy = 0,$$

namely $\tilde{\phi}_i^*$ is a solution of

$$\Delta\phi + \frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\phi = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\},$$

and again we conclude by using Proposition 4.3.

Step 3. In this step, we will prove some estimates on the speed of convergence. We set

$$\sigma_i(\lambda) := |\log\lambda| \int_{\mathbb{R}^2} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i dy. \tag{4.34}$$

We will show that

$$\begin{cases} \sigma_i(\lambda) = o(1) & \text{for i odd} \\ \sigma_i(\lambda) - \dfrac{4\pi\alpha_i}{\rho^+}\left(\sum_{j\ even}\sigma_j(\lambda) + |\log\lambda|\rho_0 \dfrac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)} dx}\right) = o(1) & \text{for i even.} \end{cases}$$

Set $Z_i^0 = \frac{\delta_i^{\alpha_i} - |x|^{\alpha_i}}{\delta_i^{\alpha_i} + |x|^{\alpha_i}}$, we know that Z_i^0 is a solution of

$$\Delta Z + |x|^{\alpha_i-2}e^{w_i}Z = 0 \quad \text{in} \quad \mathbb{R}^2.$$

Let PZ_i^0 be its the projection onto $H_0^1(\Omega)$, that is

$$\Delta PZ_i^0 + |x|^{\alpha_i-2}e^{w_i}Z_i^0 = 0 \text{ in } \Omega, \quad PZ_i^0 = 0 \text{ on } \partial\Omega.$$

By maximum principle one can show

$$PZ_i^0 = Z_i + 1 + O(\delta_i^{\alpha_i}) = \frac{2\delta_i^{\alpha_i}}{\delta_i^{\alpha_i} + |x|^{\alpha_i}} + O(\delta_i^{\alpha_i}), \tag{4.35}$$

which implies

$$PZ_i^0(\delta_j y) = \begin{cases} O\left(\frac{1}{|y|^{\alpha_i}}(\frac{\delta_i}{\delta_j})^{\alpha_i}\right) + O(\delta_i^{\alpha_i}) & \text{for } i < j, \\ \frac{2}{1+|y|^{\alpha_i}} + O(\delta_i^{\alpha_i}), & \text{for } i = j, \\ 2 + O\left(|y|^{\alpha_i}(\frac{\delta_j}{\delta_i})^{\alpha_i}\right) + O(\delta_i^{\alpha_i}) & \text{for } i > j, \end{cases} \tag{4.36}$$

and

$$\|PZ_i^0\|_q^q = O(\delta_i^2),\ q > 1. \tag{4.37}$$

First we consider i even. Multiply (4.20) by PZ_i^0 and integrate over Ω,

$$\begin{aligned} &\int_\Omega \nabla\phi \cdot \nabla PZ_i^0 dx - \rho^+\left(\frac{\int_\Omega e^W \phi PZ_i^0 dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W \phi dx \int_\Omega e^W PZ_i^0 dx}{(\int_\Omega e^W dx)^2}\right) \\ &- \int_\Omega \lambda e^{-W}\phi PZ_i^0 dx = -\int_\Omega \nabla h \cdot \nabla PZ_i^0 dx. \end{aligned} \tag{4.38}$$

For the first term,

$$\int_\Omega \nabla\phi \cdot \nabla PZ_i^0 dx = -\int_\Omega \phi\Delta PZ_i^0 dx = \int_\Omega |x|^{\alpha_i-2}e^{w_i}\phi Z_i^0 dx. \tag{4.39}$$

By Lemma 4.4, (4.24), (4.35) and (4.37),

$$\begin{aligned} \frac{\int_\Omega e^W \phi PZ_i^0 dx}{\int_\Omega e^W dx} &= \sum_{j\ even} \int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi PZ_i^0 dx \\ &\quad + \rho_0 \frac{\int_\Omega e^{z-8k\pi G(x,0)} PZ_i^0 \phi dx}{\int_\Omega e^{z-8k\pi G(x,0)} dx} + o\Big(\frac{1}{|\log\lambda|}\Big) \\ &= \int_\Omega |x|^{\alpha_i-2}e^{w_i}\phi dx + \int_\Omega |x|^{\alpha_i-2}e^{w_i}\phi Z_i^0 dx \\ &\quad + \sum_{j\neq i\ even} \int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi PZ_i^0 dx + o\Big(\frac{1}{|\log\lambda|}\Big) \end{aligned} \tag{4.40}$$

For $j \neq i$,

$$\int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi PZ_i^0 dx = \int_{\tilde{\Omega}_j} \frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j PZ_i^0(\delta_j y)dy$$
$$= \begin{cases} \int_{\mathbb{R}^2} \frac{4\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j dy + O\Big(\int_{\tilde{\Omega}_j}\Big(|y|^{\alpha_i}(\frac{\delta_j}{\delta_i})^{\alpha_i}+\delta_i^{\alpha_i}\Big)\frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j\Big)dy, & \text{for } i > j, \\ O\Big(\int_{\tilde{\Omega}_j}\Big(\frac{1}{|y|^{\alpha_i}}(\frac{\delta_i}{\delta_j})^{\alpha_i}+\delta_i^{\alpha_i}\Big)\frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j\Big)dy, & \text{for } i < j, \end{cases}$$
$$= \begin{cases} \frac{2\sigma_j(\lambda)}{|\log\lambda|} + o\big(\frac{1}{|\log\lambda|}\big), & \text{for } i > j, \\ o\big(\frac{1}{|\log\lambda|}\big), & \text{for } i < j, \end{cases} \tag{4.41}$$

where we used (4.36).

Replace ϕ by 1 in the estimate of (4.41) , one has for $j \neq i$,

$$\int_\Omega |x|^{\alpha_j-2}e^{w_j} PZ_i^0 dx = \int_{\tilde{\Omega}_j} \frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} PZ_i^0(\delta_j y)dy$$
$$= \begin{cases} \int_{\mathbb{R}^2} \frac{4\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} dy + O\Big(\int_{\tilde{\Omega}_j}\Big(|y|^{\alpha_i}(\frac{\delta_j}{\delta_i})^{\alpha_i}+\delta_i^{\alpha_i}\Big)\frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\Big)dy, & \text{for } i > j, \\ O\Big(\int_{\tilde{\Omega}_j}\Big(\frac{1}{|y|^{\alpha_i}}(\frac{\delta_i}{\delta_j})^{\alpha_i}+\delta_i^{\alpha_i}\Big)\frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\Big)dy, & \text{for } i < j, \end{cases}$$
$$= \begin{cases} 8\pi\alpha_j + o\big(\frac{1}{|\log\lambda|}\big), & \text{for } i > j, \\ o\big(\frac{1}{|\log\lambda|}\big), & \text{for } i < j, \end{cases} \tag{4.42}$$

Using Lemma 4.4 , (4.42) and (4.37),

$$\begin{aligned} \rho^+\frac{\int_\Omega e^W PZ_i^0 dx}{\int_\Omega e^W dx} &= \sum_{j\ even}\int_\Omega |x|^{\alpha_j-2}e^{w_j}PZ_i^0 dx + \rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}PZ_i^0 dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx} + o\Big(\frac{1}{|\log\lambda|}\Big) \\ &= \int_\Omega |x|^{\alpha_i-2}e^{w_i}PZ_i^0 dx + \sum_{j\neq i\ even}\int_\Omega |x|^{\alpha_j-2}e^{w_j}PZ_i^0 dx + o\Big(\frac{1}{|\log\lambda|}\Big) \\ &= \int_\Omega |x|^{\alpha_i-2}e^{w_i}dx + \int_\Omega |x|^{\alpha_i-2}e^{w_i}Z_i^0 dx \\ &\quad + \sum_{j\neq i\ even}\int_\Omega |x|^{\alpha_j-2}e^{w_j}PZ_i^0 dx + o\Big(\frac{1}{|\log\lambda|}\Big) \\ &\text{(using (4.21))} \\ &= 4\pi\alpha_i + \sum_{j<i\ even} 8\pi\alpha_j + o\Big(\frac{1}{|\log\lambda|}\Big). \end{aligned} \tag{4.43}$$

Moreover,

$$\rho^+ \frac{\int_\Omega e^W \phi dx}{\int_\Omega e^W dx} = \sum_{i\ even} \int_\Omega |x|^{\alpha_i - 2} e^{w_i} \phi dx + \rho_0 \frac{\int_\Omega e^{z-8k\pi G(x,0)} \phi dx}{\int_\Omega e^{z-8k\pi G(x,0)} dx} + o\Big(\frac{1}{|\log \lambda|}\Big) \tag{4.44}$$

and again by Lemma 4.4 and (4.41)

$$\begin{aligned} \lambda \int_\Omega e^{-W} \phi P Z_i^0 dx &= \sum_{j\ odd} \int_\Omega |x|^{\alpha_j - 2} e^{w_j} \phi P Z_i^0 dx + o\Big(\frac{1}{|\log \lambda|}\Big) \\ &= \sum_{j<i\ odd} \frac{2\sigma_j(\lambda)}{|\log \lambda|} + o\Big(\frac{1}{|\log \lambda|}\Big). \end{aligned} \tag{4.45}$$

Finally, for the last term,

$$\int_\Omega \nabla h \cdot \nabla P Z_i^0 dx = O(\|h\| \|P Z_i^0\|) = o\Big(\frac{1}{|\log \lambda|}\Big). \tag{4.46}$$

Combining (4.38), (4.41), (4.39), (4.40), (4.43), (4.44), (4.45) and (4.46), we deduce that for i even,

$$\begin{aligned} &\frac{4\pi(\alpha_i + \sum_{j<i\ even} 2\alpha_j)}{\rho^+} \Big(\sum_{j\ even} \frac{\sigma_j(\lambda)}{|\log \lambda|} + \rho_0 \frac{\int_\Omega e^{z-8k\pi G(x,0)} \phi dx}{\int_\Omega e^{z-8k\pi G(x,0)} dx} \Big) \\ &- \frac{1}{|\log \lambda|} (\sigma_i(\lambda) + \sum_{j<i} 2\sigma_j(\lambda)) = o\Big(\frac{1}{|\log \lambda|}\Big). \end{aligned} \tag{4.47}$$

Next we consider (4.38) for i odd. In this case, again we estimate (4.38) term by term. Similarly to the estimate for i even, first by Lemma 4.4, (4.37) and (4.41), one has

$$\begin{aligned} \rho^+ \frac{\int_\Omega e^W \phi P Z_i^0 dx}{\int_\Omega e^W dx} &= \sum_{j\ even} \int_\Omega |x|^{\alpha_j - 2} e^{w_j} \phi P Z_i^0 dx \\ &\quad + \rho_0 \frac{\int_\Omega e^{z-8k\pi G(x,0)} \phi P Z_i^0 dx}{\int_\Omega e^{z-8k\pi G(x,0)} dx} + o\Big(\frac{1}{|\log \lambda|}\Big) \\ &= \sum_{j<i\ even} \frac{2\sigma_j(\lambda)}{|\log \lambda|} + o\Big(\frac{1}{|\log \lambda|}\Big), \\ \rho^+ \frac{\int_\Omega e^W P Z_i^0 dx}{\int_\Omega e^W dx} &= \sum_{j\ even} \int_\Omega |x|^{\alpha_j - 2} e^{w_j} P Z_i^0 dx \\ &\quad + \rho_0 \frac{\int_\Omega e^{z-8k\pi G(x,0)} P Z_i^0 dx}{\int_\Omega e^{z-8k\pi G(x,0)} dx} + o\Big(\frac{1}{|\log \lambda|}\Big) \\ &= \sum_{j<i\ even} 8\pi\alpha_j + o\Big(\frac{1}{|\log \lambda|}\Big), \qquad (4.48) \\ \rho^+ \frac{\int_\Omega e^W \phi dx}{\int_\Omega e^W dx} &= \sum_{j\ even} \int |x|^{\alpha_j - 2} e^{w_j} \phi dx \\ &\quad + \rho_0 \frac{\int_\Omega e^{z-8k\pi G(x,0)} \phi dx}{\int_\Omega e^{z-8k\pi G(x,0)} dx} + o\Big(\frac{1}{|\log \lambda|}\Big) \end{aligned}$$

$$= \sum_{j\ even} \frac{\sigma_j(\lambda)}{|\log\lambda|} + \rho_0 \frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx} + o\Big(\frac{1}{|\log\lambda|}\Big), \tag{4.49}$$

and

$$\begin{aligned}\lambda\int_\Omega e^{-W}\phi PZ_i^0 dx &= \sum_{j\ odd}\int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi PZ_i^0 dx + o\Big(\frac{1}{|\log\lambda|}\Big)\\ &= \int_\Omega |x|^{\alpha_i-2}e^{w_i}\phi Z_i^0 dx + \frac{\sigma_i(\lambda)}{|\log\lambda|} + \sum_{j<i\ odd}\frac{2\sigma_j(\lambda)}{|\log\lambda|} + o\Big(\frac{1}{|\log\lambda|}\Big).\end{aligned}$$

Combining all these terms, one can get that for i odd,

$$\begin{aligned}&\frac{8\pi\sum_{j<i\ even}2\alpha_j}{\rho^+}\Big(\sum_{j\ even}\frac{\sigma_j(\lambda)}{|\log\lambda|} + \rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\Big)\\ &-\frac{1}{|\log\lambda|}(\sigma_i(\lambda) + \sum_{j<i}2\sigma_j(\lambda)) = o\Big(\frac{1}{|\log\lambda|}\Big).\end{aligned} \tag{4.50}$$

By considering the difference of (4.47) and (4.50), one has the following:

$$\begin{cases}\frac{4\pi\alpha_{i+1}}{\rho^+}\Big(\sum_{j\ even}\sigma_j(\lambda) + |\log\lambda|\rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\Big) - \sigma_{i+1} - \sigma_i = o(1) & \text{for i odd,}\\ \frac{4\pi\alpha_i}{\rho^+}\Big(\sum_{j\ even}\sigma_j(\lambda) + |\log\lambda|\rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\Big) - \sigma_{i+1} - \sigma_i = o(1) & \text{for i even.}\end{cases} \tag{4.51}$$

From (4.50), we first have $\sigma_1(\lambda) = o(1)$. From (4.51), we have

$$\begin{cases}\sigma_i(\lambda) = o(1) & \text{for i odd}\\ \sigma_i(\lambda) - \frac{4\pi\alpha_i}{\rho^+}\Big(\sum_{j\ even}\sigma_j(\lambda) + |\log\lambda|\rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\Big) = o(1) & \text{for i even.}\end{cases} \tag{4.52}$$

Step 4. We claim that $\gamma_i = 0$ for $i = 1, \cdots, k$.

When i is even, multiplying equation (4.20) by Pw_i and integrating over Ω,

$$
\begin{aligned}
&\int_\Omega \nabla\phi\cdot\nabla Pw_i dx - \rho^+\Big(\frac{\int_\Omega e^W\phi Pw_i dx}{\int_\Omega e^W dx} - \frac{\int_\Omega e^W\phi dx\int_\Omega e^W Pw_i dx}{(\int_\Omega e^W dx)^2}\Big)\\
&\quad - \lambda\int_\Omega e^{-W}\phi Pw_i dx = \int_\Omega \nabla h\cdot\nabla Pw_i dx.
\end{aligned}
\tag{4.53}
$$

Now we estimate the above equation term by term. For the first term, we have

$$
\int_\Omega \nabla\phi\cdot\nabla Pw_i dx = \int_\Omega |x|^{\alpha_i-2}e^{w_i}\phi dx = \int_{\mathbb{R}^2}|y|^{\alpha_i-2}e^{w_i(\delta_i y)}\tilde{\phi}_i dy = o(1) \tag{4.54}
$$

by (4.31) and (4.21).

To estimate the second term, by Lemma 4.4 and (4.24), we have

$$
\begin{aligned}
\rho^+\frac{\int_\Omega e^W\phi Pw_i dx}{\int_\Omega e^W dx} &= \sum_{j\ even}\int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi Pw_i dx + \rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi Pw_i}{\int_\Omega e^{z-8k\pi G(x,0)}dx} + o\Big(\frac{1}{|\log\lambda|}\Big)\\
&= \sum_{j\ even}\int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi Pw_i dx + o(1).
\end{aligned}
\tag{4.55}
$$

By (4.10) and (4.2), we have

$$
\int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi Pw_i dx = \int_{\tilde{\Omega}_j}\frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j Pw_i(\delta_j y)dy
$$

$$
= \begin{cases}
\int_{\tilde{\Omega}_j}\frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j(-2\alpha_i\log\delta_i + h_i(0))dy \\
+O\Big(\int_{\tilde{\Omega}_j}\frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j(|y|^{\alpha_j}(\frac{\delta_j}{\delta_i})^{\alpha_i} + \delta_j|y| + \delta_i^{\alpha_i})dy\Big) & \text{for } j<i\\
\int_{\tilde{\Omega}_i}\frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i(-2\alpha_i\log\delta_i - 2\log(1+|y|^{\alpha_i}) + h_i(0))dy\\
+O\Big(\int_{\tilde{\Omega}_i}\frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i(\delta_i|y| + \delta_i^{\alpha_i})dy\Big) & \text{for } j=i\\
\int_{\tilde{\Omega}_j}\frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j(-2\alpha_i\log(\delta_j|y|) + h_i(0))dy\\
+O\Big(\int_{\tilde{\Omega}_j}\frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j\Big(\frac{1}{|y|^{\alpha_i}}(\frac{\delta_i}{\delta_j})^{\alpha_i} + \delta_j|y| + \delta_i^{\alpha_i}\Big)dy\Big) & \text{for } j>i
\end{cases}
$$

$$
= \begin{cases}
\int_{\tilde{\Omega}_j}\frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j[-2\alpha_i\log d_i - 2(k-i+1)\log\lambda + h_i(0)]dy + o(1) & \text{for } j<i\\
\int_{\tilde{\Omega}_i}\frac{2\alpha_i^2|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_i[-2\alpha_i\log d_i - 2(k-i+1)\log\lambda - 2\log(1+|y|^{\alpha_i}) + h_i(0)]dy + o(1) & \text{for } j=i\\
\int_{\tilde{\Omega}_j}\frac{2\alpha_j^2|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2}\tilde{\phi}_j[-2\alpha_i\log d_j - 2(k-j+1)\frac{2i-1}{2j-1}\log\lambda - 2\alpha_i\log|y| + h_i(0)]dy + o(1) & \text{for } j>i.
\end{cases}
\tag{4.56}
$$

Based on (4.56), by the definition of $\sigma_j(\lambda)$ (see (4.34)), Lemma 4.6 and (4.31), we get

$$\int_{\Omega} |x|^{\alpha_j-2} e^{w_j} \phi P w_i dx$$

$$= \begin{cases} 2(k-i+1)\sigma_j(\lambda) + o(1) & \text{for } j < i \\ 2(k-i+1)\sigma_i(\lambda) + \int_{\tilde{\Omega}_i} \frac{2\alpha_i^2 |y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \tilde{\phi}_i [-2\log(1+|y|^{\alpha_i})] dy + o(1) & \text{for } j = i \\ 2(k-j+1)\frac{2i-1}{2j-1}\sigma_j(\lambda) + \int_{\tilde{\Omega}_j} \frac{2\alpha_j^2 |y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j [-2\alpha_i \log|y|] dy + o(1) & \text{for } j > i \end{cases}$$

$$= \begin{cases} 2(k-i+1)\sigma_j(\lambda) + o(1) & \text{for } j < i \\ 2(k-i+1)\sigma_i(\lambda) + 4\pi\alpha_i\gamma_i + o(1) & \text{for } j = i \\ 2(k-j+1)\frac{2i-1}{2j-1}\sigma_j(\lambda) + 8\pi\alpha_i\gamma_j + o(1) & \text{for } j > i, \end{cases} \tag{4.57}$$

where we used [11, (4.18)-(4.20)].

Then by (4.57) and (4.55), one has

$$\begin{aligned} \rho^+ \frac{\int_\Omega e^W \phi P w_i dx}{\int_\Omega e^W dx} &= \sum_{j \ even} \int_\Omega |x|^{\alpha_j-2} e^{w_j} \phi P w_i dx + \rho_0 \frac{\int_\Omega e^{z-8k\pi G(x,0)} \phi P w_i}{\int_\Omega e^{z-8k\pi G(x,0)} dx} + o\Big(\frac{1}{|\log\lambda|}\Big) \\ &= \sum_{j \ even} \int_\Omega |x|^{\alpha_j-2} e^{w_j} \phi P w_i dx + o(1) \\ &= 4\pi\alpha_i\Big(\gamma_i + \sum_{j>i \ even} 2\gamma_j\Big) + 2(k-i+1)\Big(\sigma_i(\lambda) + \sum_{j<i \ even} \sigma_j(\lambda)\Big) \\ &\quad + \sum_{j>i \ even} 2(k-j+1)\frac{2i-1}{2j-1}\sigma_j(\lambda) + o(1). \end{aligned} \tag{4.58}$$

Similarly, by replacing ϕ by 1 in (4.57), one can deduce that

$$\begin{aligned} \rho^+ \frac{\int_\Omega e^W P w_i dx}{\int_\Omega e^W dx} = 8\pi|\log\lambda| \Bigg(& \sum_{j \le i \ even} (k-i+1)\alpha_j \\ & + \sum_{j>i \ even} (k-j+1)\frac{2i-1}{2j-1}\alpha_j \Bigg) + O(1), \end{aligned} \tag{4.59}$$

and the estimate for $\rho^+ \frac{\int_\Omega e^W \phi dx}{\int_\Omega e^W dx}$ has been obtained in (4.48).

Moreover,

$$\begin{aligned}\lambda\int_\Omega e^{-W}\phi Pw_i dx &= \sum_{j\ odd}\int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi Pw_i dx+o(1)\\ &=8\pi\alpha_i\sum_{j>i\ odd}\gamma_j+\sum_{j<i\ odd}2(k-i+1)\sigma_j(\lambda)\\ &\quad+\sum_{j>i\ odd}2(k-j+1)\frac{2i-1}{2j-1}\sigma_j(\lambda)+o(1),\end{aligned}\tag{4.60}$$

and

$$\int_\Omega \nabla h\cdot\nabla Pw_i dx=O(\|h\|_p\|Pw_i\|)=O(\log\lambda)^{\frac{1}{2}}\|h\|=o(1).\tag{4.61}$$

Putting all the estimates in (4.54), (4.58), (4.59), (4.48), (4.60) and (4.61) into (4.53), we get that for i even,

$$\begin{aligned}&4\pi\alpha_i(\gamma_i+\sum_{j>i}2\gamma_j)+\sum_{j\le i}2(k-i+1)\sigma_j+\sum_{j>i}2(k-j+1)\frac{2i-1}{2j-1}\sigma_j\\ &-\frac{8\pi}{\rho^+}\sum_{j\le i\ even}(k-i+1)\alpha_j\Big(\sum_{l\ even}\sigma_l(\lambda)+|\log\lambda|\rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\Big)\\ &-\frac{8\pi}{\rho^+}\sum_{j>i\ even}(k-j+1)\frac{2i-1}{2j-1}\alpha_j\Big(\sum_{l\ even}\sigma_l(\lambda)+|\log\lambda|\rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\Big)=o(1).\end{aligned}\tag{4.62}$$

Next we consider i odd. Similarly to the previous estimates, one has

$$\begin{aligned}\rho^+\frac{\int_\Omega e^W\phi Pw_i dx}{\int_\Omega e^W dx}&=\sum_{j\ even}\int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi Pw_i dx+\rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi Pw_i}{\int_\Omega e^{z-8k\pi G(x,0)}dx}+o\Big(\frac{1}{|\log\lambda|}\Big)\\ &=8\pi\alpha_i\sum_{j>i\ even}\gamma_j+\sum_{j<i\ even}2(k-i+1)\sigma_j(\lambda)\\ &\quad+\sum_{j>i\ even}2(k-j+1)\frac{2i-1}{2j-1}\sigma_j(\lambda)+o(1)\end{aligned}\tag{4.63}$$

$$\begin{aligned}\rho^+\frac{\int_\Omega e^W Pw_i dx}{\int_\Omega e^W dx}&=8\pi|\log\lambda|\Big(\sum_{j<i\ even}(k-i+1)\alpha_j\\ &\quad+\sum_{j>i\ even}(k-j+1)\frac{2i-1}{2j-1}\alpha_j\Big)+O(1),\end{aligned}\tag{4.64}$$

and

$$\begin{aligned}\lambda\int_\Omega e^{-W}\phi Pw_i dx&=\sum_{j\ odd}\int_\Omega |x|^{\alpha_j-2}e^{w_j}\phi Pw_i dx+o(1)\\ &=4\pi\alpha_i\gamma_i+8\pi\alpha_i\sum_{j>i\ odd}\gamma_j+\sum_{j\le i\ odd}2(k-i+1)\sigma_j(\lambda)\\ &\quad+\sum_{j>i\ odd}2(k-j+1)\frac{2i-1}{2j-1}\sigma_j(\lambda)+o(1).\end{aligned}\tag{4.65}$$

Putting all the estimates in (4.54), (4.63), (4.64), (4.48), (4.65) and (4.61) into (4.53), we have for i odd,

$$
\begin{aligned}
&4\pi\alpha_i(\gamma_i+\sum_{j>i}2\gamma_j)+\sum_{j\le i}2(k-i+1)\sigma_j+\sum_{j>i}2(k-j+1)\frac{2i-1}{2j-1}\sigma_j\\
&-\frac{8\pi}{\rho^+}\sum_{j<i\ even}(k-i+1)\alpha_j\Big(\sum_{l\ even}\sigma_l(\lambda)+|\log\lambda|\rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\Big)\\
&-\frac{8\pi}{\rho^+}\sum_{j>i\ even}(k-j+1)\frac{2i-1}{2j-1}\alpha_j\Big(\sum_{l\ even}\sigma_l(\lambda)+|\log\lambda|\rho_0\frac{\int_\Omega e^{z-8k\pi G(x,0)}\phi dx}{\int_\Omega e^{z-8k\pi G(x,0)}dx}\Big)=o(1).
\end{aligned}
\tag{4.66}
$$

Combining (4.52), (4.62) and (4.66), we have

$$4\pi\alpha_i\Big(\gamma_i+\sum_{j>i}2\gamma_j\Big)=o(1),$$

from which we deduce that $\gamma_i=0$ for $i=1,\cdots,k$.
Step 5. Finally, we derive a contradiction.
Multiplying equation (4.20) by ϕ and integrating, we get

$$\int_\Omega|\nabla\phi|^2dx-\lambda\int_\Omega e^{-W}\phi^2dx-\rho^+\Big(\frac{\int_\Omega e^W\phi^2dx}{\int_\Omega e^Wdx}-\frac{(\int_\Omega e^W\phi dx)^2}{(\int_\Omega e^Wdx)^2}\Big)=\int_\Omega\nabla h\cdot\nabla\phi dx.$$

From (4.24) and the assumptions on ϕ and h, we have that the left hand side of the above equation tends to 1 while the right hand side is of order $o(1)$. This yields a contradiction. □

Using the a priori estimates in Lemma 4.5 and the Fredholm alternative, we have the following existence result similarly to the proof of Proposition 3.5:

Proposition 4.7 *There exist $\lambda_0>0$ and $C>0$ such that for any $\lambda\in(0,\lambda_0)$, $h\in\mathcal{H}_l$ there exists a unique solution $\phi\in\mathcal{H}_l$ solution of (4.20) satisfying*

$$\|\phi\|\le C|\log\lambda|\|h\|.$$

4.4 Conclusion

By exploiting the linear theory developed in the previous subsection it is then standard to derive an existence result for the nonlinear problem (4.67) based on the contraction mapping, similarly to Proposition 3.6. We here give the sketch proof.

Proposition 4.8 *For any $\epsilon>0$ sufficiently small, there exist $\lambda_0>0$ and $C>0$ such that for any $\lambda\in(0,\lambda_0)$, there exists a unique $\phi\in\mathcal{H}_l$ solution of*

$$\Delta(W+\phi)+\rho^+\frac{e^{W+\phi}}{\int_\Omega e^{W+\phi}dx}-\lambda e^{-W-\phi}=0\quad in\quad\Omega \tag{4.67}$$

satisfying

$$\|\phi\|\le C\lambda^{\frac{1}{2(2k-1)}-\epsilon}. \tag{4.68}$$

Proof The proof is similar to Proposition 3.6 in the last section. The nonlinear problem (4.67) is equivalent to

$$\Delta\phi + \rho^+ \left(\frac{e^W \phi}{\int_\Omega e^W dx} - \frac{e^W \int_\Omega e^W \phi dx}{(\int_\Omega e^W dx)^2} \right) + \lambda e^{-W}\phi = -(\bar{E} + \bar{N}(\phi)) \tag{4.69}$$

where

$$\bar{E} = \Delta W + \rho^+ \frac{e^W}{\int_\Omega e^W dx} - \lambda e^{-W},$$
$$\bar{N}(\phi) = \rho^+[g(W+\phi) - g(W) - g'(W)\phi] - \lambda[f(W+\phi) - f(W) - f'(W)\phi],$$
$$f(W) = e^{-W} \text{ and } g(W) = \frac{e^W}{\int_\Omega e^W dx}.$$

Denote the solution to (4.20) by $\phi := \bar{T}(h)$, then (4.69) is equivalent to

$$\phi = \bar{T}(i_p^*(\bar{E} + \bar{N}(\phi))) =: \bar{\mathcal{T}}(\phi).$$

Define

$$\bar{\mathcal{B}} = \{\phi \in \mathcal{H}_l, \|\phi\| \le \Lambda |\log\lambda| \lambda^{\frac{2-p}{2p(2k-1)}}\}$$

for Λ large and λ small. If we choose p sufficiently close to 1, one can see that $\bar{\mathcal{B}} \subset \{\phi, \|\phi\| \le C\lambda^{\frac{1}{2(2k-1)}-\epsilon}\}\}$.

From Proposition 4.7 and the error estimate for $\bar{E}$, for $\phi, \phi_1, \phi_2 \in \bar{\mathcal{B}}$, similarly to the estimate in Proposition 5.4 in [9], one has

$$\begin{aligned} \|\bar{\mathcal{T}}\phi\| &\le C_p |\log\lambda| \|i_p^*(\bar{E} + \bar{N}(\phi))\| \\ &\le C_p |\log\lambda| (\|\bar{E}\|_p + \|\bar{N}(\phi)\|_p \\ &\le \Lambda |\log\lambda| \lambda^{\frac{2-p}{2p(2k-1)}}, \end{aligned}$$

and

$$\begin{aligned} \|\bar{\mathcal{T}}(\phi_1) - \bar{\mathcal{T}}(\phi_2)\| &\le C_p \|\bar{N}(\phi_1) - \bar{N}(\phi_2)\|_p \\ &\le \frac{1}{2}\|\phi_1 - \phi_2\|. \end{aligned}$$

So $\bar{\mathcal{T}}$ maps $\bar{\mathcal{B}}$ into itself and it is a contraction mapping, we can get that the solution ϕ is unique in $\bar{\mathcal{B}}$. The estimate for ϕ follows from the above estimate. □

Proof of Theorem 1.2 By Proposition 4.8, $u_\lambda = W_\lambda + \phi_\lambda$ is a solution to the original problem (1.1) with $\rho_\lambda^+ = \rho^+ = 4\pi k(k-1) + \rho_0$ and $\rho_\lambda^- = \lambda \int_\Omega e^{-u_\lambda} dx$. Then by Lemma 4.4 and (4.68)

$$\begin{aligned} \rho_\lambda^- &= \lambda \int_\Omega e^{-u_\lambda} dx = \lambda \int_\Omega e^{-W_\lambda} dx + o(1) = \sum_{i\ odd} \int_\Omega |x|^{\alpha_i - 2} e^{w_i} dx + o(1) \\ &= \sum_{i\ odd} 4\pi\alpha_i + o(1) = 4\pi k(k+1) + o(1). \end{aligned}$$

Moreover, from the definition of u_λ and properties (3.37), (4.68) and (4.9), the second property in Theorem 1.2 can be derived easily. □

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