WIDTH OF THREE-PION RESONANCES (*)

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(presented by W. R. Frazer)

We have estimated the width of $I = 0$ three pion resonances on the basis of a dispersion-theoretic calculation of the three-pion scattering amplitude, finding for the ω-meson a full width of the order of 10-20 MeV. We employ the angular momentum expansion for three relativistic particles worked out by Wick [1], who has generalized the scheme introduced by Dalitz in the study of τ-meson decay [2]. First, two of the three pions are combined to have angular momentum l, invariant mass squared σ, and helicity λ; then the two pions are combined with the third to form a state of total angular momentum J. Since an $I = 0$ state of three pions is totally antisymmetric, only odd values of l occur. We consider only $l = 1$, thus neglecting F and higher waves in the pion-pion system. The amplitude must, of course, be symmetrized, but let us defer this complication for a moment. Moreover, we consider only the two quantum-number assignments which have been discussed in connection with three-pion resonances of negative G-parity, namely, 1^- and 0^-.

If we treat the dependence on the variable σ by factoring out initial- and final-state pion-pion interactions in the $l = I = 1$ state, and assume that the remaining matrix element M does not vary rapidly with σ, then we obtain the following approximate form of the unitarity condition:

$$\operatorname{Im} M(s) \approx \kappa(s) M(s+i\varepsilon) M(s-i\varepsilon) \tag{1}$$

where

$$\kappa(s) = \frac{1}{\pi} \int_{4m_\pi^2}^{(\sqrt{s}-m_\pi)^2} d\sigma \frac{q^3(s,\sigma)}{\sqrt{s}} \left[\frac{(\sigma-4)^3}{\sigma}\right]^{\frac{1}{2}} \frac{\gamma}{(m_\rho^2-\sigma)^2+\gamma^2\frac{(\sigma-4)^3}{\sigma}}. \tag{2}$$

Here we designate the square of the invariant mass of the three-pion system by s, and $q(s, \sigma)$ denotes the momentum of the third pion in the overall CM system. The quantity γ is proportional to the ρ-meson width ($\gamma \approx 0.2$ in pion units). Our procedure up to this point has been parallel to that used by Ball, Frazer, and Nauenberg in treating the state $\pi+\pi+N$ [3].

Note that Eq. (1) is identical in form to the partial wave unitarity condition for two-body scattering. The properties of the unstable ρ-meson are contained in the generalized phase-space factor $\kappa(s)$. In the limit $\gamma \to 0$, $\kappa(s)$ reduces to the two-body P-wave phase space $q^3(s, m_\rho^2)/\sqrt{s}$.

We shall show later that the effect of symmetrization is to modify the function $\kappa(s)$. Before taking up this question let us show how the formalism we have developed can be used to estimate the width of three-pion resonances. In order to do this one must somehow evaluate the interaction between the pions, then solve the N/D equations. The resonance should appear as a zero of the D function. As a first rough attempt in this direction we have represented the inter-

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action by a pole (i.e., we have used an effective-range formula) whose residue is adjusted to fit the position of the resonance. We then find that

$$M(s) = \frac{1/\bar{\kappa}(s)}{s_r - s - i\dfrac{\kappa(s)}{\bar{\kappa}(s)}}, \tag{3}$$

where

$$\bar{\kappa}(s) \equiv \frac{s-s_0}{\pi} \cdot \mathscr{P}\int_{(3m_\pi)^2}^{\infty} ds' \frac{\kappa(s')}{(s'-s)(s'-s_0)(s'-s_r)}, \tag{4}$$

where s_r is the position of the resonance and s_0 is the position of the interaction pole, taken below the three-pion threshold. For a narrow resonance we then have a width of $\Gamma \approx \kappa(s_r)\bar{\kappa}(s_r)$. We shall see below that the width is not very sensitive to the position of s_0. In fact, for large negative s_0 the dependence is only logarithmic.

Since the width Γ is proportional to $\kappa(s_r)$, we can see from Eq. (2) why a three-pion $I = 0$ resonance with energy well below $m_\rho + m_\pi$ should be narrow. The mass squared of the two-pion system cannot be large enough to lie in the region of the ρ-meson peak, and the decay occurs via the tail of the ρ-meson distribution. The existence of a second, lower-lying pion-pion resonance in the $J = I = 1$ state would, of course, invalidate the present treatment [4]. It is interesting to note that $\kappa(s)$ is essentially equal to the decay probability corresponding to Fig. 1a.

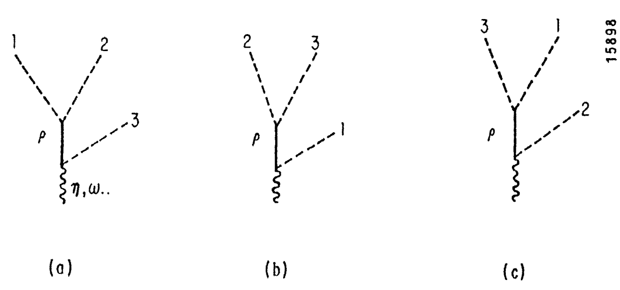

Fig. 1.

Finally, let us consider the effect of symmetrization on $\kappa(s)$. We used an expansion in states of the form $(12)3\rangle$, where pions 1 and 2 are combined to have angular momentum $l = 1$. The following state will then have the proper symmetry: $(12)3\rangle + (23)1\rangle + (31)2\rangle$. The effect of introducing such a state is to change $\kappa(s)$ to be essentially equal to the decay matrix element calculated from the sum of the three diagrams of Fig. 1. This matrix element has been written down for both 0^- and 1^- by Shaw and Wong [5], who pointed out that the symmetrization produces a tremendous suppression in $\kappa(s)$ for small s in the 0^- case. The 1^- matrix element calculated from any one term in Fig. 1 is already totally antisymmetric except for the ρ-meson propagators (the denominators in Eq. (3)). Nevertheless the symmetrization affects the calculation of Γ. For low values of s the three propagators are essentially constant and add coherently. For high values of s the result is approximately the sum of the squares, so that $\kappa(s_r)/\bar{\kappa}(s_r)$ is raised by a factor of about 3 for small s_r, as compared to the unsymmetrized calculation.

The results we find for the ω (1^- assignment assumed) are summarized in the table below:

s_0	Full width of ω
4	20.4 MeV
0	18.4 MeV
− 100	7.0 MeV

The disagreement between our result and that of Gell-Mann, Sharp, and Wagner [6] can probably be understood as a violation of unitarity symmetry, which these authors assume in relating π^0 decay to the width of the ω. Current calculations of nucleon-nucleon scattering by Scotti and Wong indicate that the ω-nucelon coupling is stronger by a factor of about three than the ρ-nucleon coupling. This factor will appear squared in the ω width, bringing our results into rough agreement.

LIST OF REFERENCES

1. G. C. Wick, Annals of Physics *18*, 65 (1962). See also L. F. Cook, and B. W. Lee, Phys. Rev. (to be published).
2. R. H. Dalitz, Phys. Rev. *94*, 1046 (1954).
3. J. S. Ball, W. R. Frazer, and M. Nauenberg, University of California, La Jolla, preprint (1962).
4. M. Nauenberg and A. Pais, Phys. Rev. Letters *8*, 82 (1962), have pointed out that the Dalitz plot of the ω decay indicates that this decay does not proceed via a low-lying pion-pion resonance.
5. G. L. Shaw and D. Y. Wong, Phys. Rev. Letters *8*, 336 (1962).
6. M. Gell-Mann, D. Sharp and W. G. Wagner, Phys. Rev. Letters *8*, 261 (1962).

DISCUSSION

MANDELSTAM: I want to ask a simple question to Frazer on the remark he just made: the Berkeley group did put up that number, then they retracted and said less than 15 MeV. Is your remark based on the position after that retraction or before?

FRAZER: I did not know about that retraction at all.

MANDELSTAM: Personnally, I am a bit worried about using an isobar approximation like this, where you have three particles such as pions, all three of which can interact simultaneously. It might be, as occurred in certain model calculations by Schiff, that the fact that 3 particles interact simultaneously would give a much bigger binding than if one just took a resonance and a particle. This is of course apart from any symmetry corrections which Frazer and Wong included. My second question is: from our experience with the ϱ, one generally finds that for P-wave amplitudes one needs two poles of opposite sign on the left. This is partly a consequence of the centrifugal barrier. Now this could be equivalent, depending on the magnitudes of these two poles, to one pole very far out on the left. So I think it is quite reasonable that the value they get could be much greater than 100, and one should allow for a narrower width than the numbers they quote.

FRAZER: I agree.

CHEW: I want to draw attention to some recent work by Zwanziger which has to do with this problem of combining resonances with particles in multi-particle states. He has been able to make it extremely plausible, by studying the analytic continuation of the unitary condition, that you can actually define S-matrix elements for these unstable particles in a precise way, and that then there are cuts on unphysical sheets associated with them. These cuts have been inferred for quite a long time, probably first by Blankenbecler and his collaborators at Princeton, but the new remark is that the rule for the discontinuity across the cut is a direct and simple generalization of the ordinary rule for discontinuities across physical cuts. In fact the whole formalism for unstable particles is similar to that for stable, provided you are willing to work on the unphysical sheets. It seems from this that one should not perhaps make a strong distinction in one's thinking between the physical and the unphysical sheets. One should include all sheets in the same framework.

MANDELSTAM: I'd just like to comment that Gunson seems to have arrived at the same conclusion.

TREIMAN: Your ω decays to a $\varrho+\pi$. Does the Dalitz plot for ω decay support this?

FRAZER: Actually, this has been investigated in a note added in proof to the original paper by the Alvarez group on the ω. They investigated whether the decay into a ϱ and a π, (of course the ϱ has a rather large width) would modify the Dalitz plot, and they concluded that the modification was small and not observable with present statistics.

BLANKENBECLER: I would like to ask Frazer: if you symmetrize the matrix element the way you have indicated on the board, it does not seem to me that you satisfy unitarity in all three possible pairs in the final state. Namely the matrix element will not have the phase of the 2, 3 pair, simply because the first term has only the phase of the 1, 2 pair.

FRAZER: I have nothing to add to that. I think it certainly deserves more study, but what we have written down does not seem to contradict anything too obvious.