Quantum Backreaction Effects in the Late Stage of Black Hole Evaporation

Quanten-Rückwirkungseffekte am Ende der Verdampfung schwarzer Löcher

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Jonas Neuser

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Gutachter/in:
Prof. Dr. Thomas Thiemann
Prof. Dr. Beatriz Elizaga Navascues

Abstract

Black holes are described by their gravitational properties and this makes them fascinating objects to study and test theories of gravity. Our current description of the gravitational interaction using general relativity leaves some open questions: In the case of black holes, general relativity predicts singularities, points where the theory breaks down and physical quantities become infinitely large. In addition, the seminal work of Stephen Hawking states that black holes emit radiation following a thermal spectrum and suggests that black holes lose mass and slowly evaporate. The latter is in conflict with classical general relativity, where black holes can only grow but never decrease in mass. It is believed that a theory of quantum gravity, where general relativity is quantized, will avoid singularities and give a detailed description for the evaporation of black holes. In the present thesis, we address black hole evaporation, also known as the backreaction problem, using tools of (perturbative) quantum gravity.

In the literature, backreaction is not fully understood from a theoretical point of view. Based on heuristic arguments using the Stefan-Boltzmann law, one finds a finite lifetime for black holes which scales as M^3, where M is the black hole mass. Ideally, one would perform a full non-perturbative quantum gravity computation and find the spectrum of the emitted particles, the lifetime of the black hole, and determine the fate of the black hole after evaporation. As a first step, one uses quantum field theory, the mathematical framework to describe matter inside the Standard Model of particle physics, to study matter around the black hole. Quantum field theory assumes a fixed spacetime and can only be formulated when fixing the black hole spacetime. Then, using a semi-classical version of the field equations of general relativity, one determines the change of the classical metric. This is only an approximation because for evaporating black holes the mass of the black hole changes and the spacetime is fully dynamical. Thus, for a realistic picture one has to avoid references to any background spacetime and attempt a background independent formulation.

In this thesis, we take an intermediate step between semi-classical gravity and full non-perturbative quantum gravity, known as the hybrid approach. We work in the Hamiltonian formulation with the full phase space of general relativity and the matter sector. In contrast to the semi-classical methods based on quantum field theory on black hole spacetimes, the framework is background independent and it includes backreaction effects.

In our approach, we face two challenges: (i) In its Hamiltonian formulation, general relativity is fully constrained, i.e. the theoretical description includes redundancies and we have to single out the observable degrees of freedom. (ii) The equations of general relativity are non-linear and mathematically complex. Exact solutions are only known for very special cases usually using symmetry assumptions. To address both obstacles, we employ the reduced phase space approach which completely solves all of the redundancies. To the best of our knowledge, this approach has not been implemented in the current form before. The idea is to take the symmetry group of some class of exact solutions of general relativity and to define the symmetric and non-symmetric variables with respect to the symmetry group. Then, we further split the symmetric and non-symmetric variables into observable and non-observable degrees of freedom. The observable degrees of freedom are defined non-perturbatively on the full phase space and their dynamics is fully described by the reduced Hamiltonian, a Hamiltonian involving the observable degrees of freedom only. We find an implicit, non-perturbative expression for the reduced Hamiltonian and expand it to second order in the perturbations.

After the general outline of the formalism, we apply it to the case of perturbations around spherically symmetric spacetimes. Classically, the solution of the equations of general relativity assuming spherical symmetry is the non-rotating Schwarzschild black hole. We explicitly compute the reduced Hamiltonian describing the dynamics of the perturbations to second order. As a non-trivial consistency check, we derive the Hamiltonian equations of motion and compare them with the Regge-Wheeler-Zerilli equations for perturbations on the Schwarzschild spacetime. Ignoring backreaction effects, we find perfect agreement and it assures us that the framework produces correct results.

Finally, we briefly discuss how we plan to use the formalism to analyse evaporating black holes. We introduce the notion of apparent horizons, describing the boundary of the black hole region as seen by an observer. The area of the apparent horizon is an interesting quantity because it is expected to decrease due to black hole evaporation. It can be used to link the theoretical findings to current and future experiments.

Zusammenfassung

Schwarze Löcher werden durch ihre Gravitationseigenschaften beschrieben, was sie zu faszinierenden Objekten für die Untersuchung und den Vergleich verschiedener Gravitationstheorien macht. Unsere derzeitige Beschreibung der Gravitation beruht auf der allgemeinen Relativitätstheorie und lässt einige Fragen unbeantwortet: Im Fall von schwarzen Löchern sagt die allgemeine Relativitätstheorie Singularitäten voraus, das sind Punkte, an denen die Theorie zusammenbricht und physikalische Größen unendlich groß werden. Darüber hinaus zeigt die bahnbrechende Arbeit von Stephen Hawking, dass schwarze Löcher Strahlung aussenden, die einem thermischen Spektrum folgt. Das legt nahe, dass schwarze Löcher Masse verlieren und langsam verdampfen. Letzteres steht im Widerspruch zur klassischen allgemeinen Relativitätstheorie, nach der schwarze Löcher nur wachsen, aber niemals an Masse verlieren können. Es wird angenommen, dass eine Theorie der Quantengravitation, in der die allgemeine Relativitätstheorie quantisiert wird, Singularitäten vermeiden und eine detaillierte Beschreibung der Verdampfung schwarzer Löcher liefern wird. In der vorliegenden Dissertation befassen wir uns mit der Verdampfung schwarzer Löcher, auch bekannt als Rückwirkungsproblem, unter Verwendung von Werkzeugen der (perturbativen) Quantengravitation.

In der Literatur ist die Rückwirkung aus theoretischer Sicht nicht vollständig verstanden. Auf der Grundlage heuristischer Argumente unter Verwendung des Stefan-Boltzmann-Gesetzes findet man eine endliche Lebensdauer für schwarze Löcher, die mit M^3 skaliert, wobei M die Masse des schwarzen Lochs ist. Idealerweise würde man eine vollständige nicht-perturbative Rechnung in einer Quantengravitationstheorie durchführen und das Spektrum der emittierten Teilchen, die Lebensdauer des schwarzen Lochs und das Ergebnis des Vermpfungsprozesses bestimmen. In einem ersten Schritt beschreibt man die Materie um das schwarze Loch mithilfe der Quantenfeldtheorie, dem mathematischen Rahmen für die Beschreibung der Materie im Standardmodell der Teilchenphysik. Die Quantenfeldtheorie nimmt eine feste Raumzeit an und kann nur formuliert werden, wenn die Raumzeit des schwarzen Lochs unveränderlich ist. Mit Hilfe einer semi-klassischen Version der Feldgleichungen der allgemeinen Relativitätstheorie wird dann die Änderung der klassischen Metrik bestimmt. Dies ist jedoch nur eine Näherung, da sich bei verdampfenden schwarzen Löchern die Masse des schwarzen Lochs ändert und die Raumzeit völlig dynamisch ist. Um ein realistischeres Bild zu erhalten, müssen daher Bezüge zu eine Hintergrund-Raumzeit vermieden werden und es muss versucht werden, eine vom Hintergrund unabhängige Formulierung zu finden.

In dieser Arbeit verwenden wir einen Zwischenschritt zwischen der semi-klassischen Gravitation und der vollständigen nicht-perturbativen Quantengravitation, der als hybrider Ansatz bezeichnet wird. Wir arbeiten in der Hamilton'schen Formulierung mit dem vollen Phasenraum der Allgemeinen Relativitätstheorie und dem Materiesektor. Im Gegensatz zu den semiklassischen Methoden, die auf der Quantenfeldtheorie beruhen, ist die Vorgehensweise hintergrundunabhängig und schließt Rückwirkungseffekte mit ein.

Bei unserem Ansatz stehen wir vor zwei wesentlichen Herausforderungen: (i) Die Hamilton'schen Formulierung der allgemeinen Relativitätstheorie enthält Zwangsbedingungen, d. h. die theoretische Beschreibung ist redundant und wir müssen die beobachtbaren Freiheitsgrade bestimmen. (ii) Die Gleichungen der allgemeinen Relativitätstheorie sind nichtlinear und mathematisch komplex. Exakte Lösungen sind nur für sehr spezielle Fälle bekannt, die in der Regel auf Symmetrieannahmen beruhen. Um beiden Hindernissen zu begegnen, verwenden wir eine Herangehensweise basierend auf dem reduzierten Phasenraum, welcher die Zwangsbedingungen vollständig löst. Nach unserem besten Wissen wurde diese Vorgehensweise in unserer Form noch nicht verwendet. Die Idee ist, die Symmetriegruppe einer Klasse von exakten Lösungen der allgemeinen Relativitätstheorie dazu zu verwenden, die Variablen in symmetrische und nicht-symmetrische Variablen aufzuteilen. Anschließend werden die symmetrischen und nicht-symmetrischen Variablen in beobachtbare und nicht-beobachtbare Freiheitsgrade unterteilt. Die beobachtbaren Freiheitsgrade sind nicht-perturbativ auf dem vollen Phasenraum definiert und ihre Dynamik ist vollständig durch den reduzierten Hamiltonian, einem Hamiltonian, der nur die beobachtbaren Freiheitsgrade umfasst, beschrieben. Wir finden einen impliziten, nicht-perturbativen Ausdruck für den reduzierten Hamiltonian und entwickeln ihn bis zur zweiten Ordnung in den Störungen.

Nach dem allgemeinen Überblick über den Formalismus wenden wir ihn auf den Fall von Störungen um sphärisch symmetrische Raumzeiten an. In der klassischen Theorie ist die Lösung der Gleichungen der allgemeinen Relativitätstheorie unter der Annahme sphärischer Symmetrie das nicht rotierende schwarze

Loch. Wir bestimmen explizit den reduzierten Hamiltonian, der die Dynamik der Störungen bis zweite Ordnung beschreibt. Als nicht-triviale Konsistenzprüfung leiten wir die Hamilton'schen Bewegungsgleichungen für die Störungen her und vergleichen sie mit den Regge-Wheeler-Zerilli-Gleichungen. Wenn wir die Rückwirkungseffekte ignorieren, finden wir eine perfekte Übereinstimmung, die uns garantiert, dass unsere Herangehensweise korrekte Ergebnisse liefert.

Am Ende der Arbeit beschreiben wir kurz, wie wir den Formalismus zur Analyse verdampfender schwarzer Löcher verwenden wollen. Wir führen den Begriff des "apparent horizon" ein, der den Rand der Region des schwarzen Lochs aus der Sicht eines Beobachters beschreibt. Die Fläche des apparent horizon ist eine interessante Größe, da erwartet wird, dass sie durch die Verdampfung des schwarzen Lochs abnimmt. In Zukunft, planen wir sie zu verwenden, um die theoretischen Erkenntnisse mit aktuellen und zukünftigen Experimenten zu verknüpfen.

Declaration

The work in this thesis is based on the following publications of the author:

- "Quantum Field Theory of black holes with backreaction. Part II. Spherically symmetric 2nd order Einstein sector", Journal of Cosmology and Astroparticle Physics **2025**.01-001; together with Thomas Thiemann [1], published under the Creative Commons Attribution 4.0 Licence
- "Quantum Field Theory of black holes with backreaction. Part III. Spherically symmetric 2nd order Maxwell sector", Journal of Cosmology and Astroparticle Physics **2025**.01-002; together with Thomas Thiemann [2], published under the Creative Commons Attribution 4.0 Licence
- "Quantum Field Theory of black holes with backreaction IV. Spherically symmetric 2nd order Einstein-Maxwell sector in generalised gauges", General Relativity and Gravitation **57**,89 (2025), published under the Creative Commons Attribution 4.0 License [3]

Chapter 5, the central part of the thesis, covers the content of these papers. In sections 5.2.1 to 5.2.4, we dicuss the Einstein and Maxwell theory based on [1, 2] and in section 5.2.5, we cover [3]. The text and equations of the publications has been partly reused and this is clearly indicated in the introductory remarks of each section.

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Contents

1 Introduction

Black holes, as solutions to the equations of general relativity, are a key to our understanding of gravity using both theoretical and experimental methods. In recent years, black holes have gained new attention: On the experimental side, mergers of binary black holes give new insights into the physics of black holes and verify general relativity in the regime of strong gravitational fields [4]. Supermassive black holes, such as the one located in the centre of our galaxy or in the galaxy M87 were directly detected using radio astronomy by the event horizon telescope [5, 6].

On the theoretical side, several theorems provide information about black holes in a mathematically rigorous way. For instance, the **singularity theorem** by Roger Penrose (Nobel prize in 2020) shows that general relativity predicts the existence of points in spacetime where physical quantities get infinitely large (singularities) [7]. This demonstrates that new theories beyond general relativity need to be developed. The analogy between properties of black holes and thermodynamics hints at a deep connection between entropy and the area of the event horizon and between temperature and mass (**black hole thermodynamics**, see [8]). Applying tools of quantum field theory (i.e. the mathematical framework of the Standard Model of particle physics) to matter on black hole spacetimes, Stephen Hawking provided an explanation for the temperature of black holes. He showed that black holes emit radiation as an almost perfect black body with a characteristic temperature (**Hawking radiation**, see [9, 10]). This relation between general relativity and quantum mechanics leads to speculations about the quantum nature of the gravitational field. It is generally believed that a quantization of general relativity is needed to fully resolve the theoretical challenges in black hole physics [11].

The study of Hawking radiation through quantum field theory on classical spacetimes is flawed because the (semi-classical) Einstein equations are violated. In general relativity, the Einstein equations relate the curvature of spacetime to the energy-momentum tensor, that captures the energy and momentum density of the matter. Black holes are a solution of the equations of general relativity without matter, i.e. for a vanishing energy-momentum tensor. However, the emission of Hawking radiation leads to a non-vanishing expectation value for the energy-momentum tensor [12, 13]. This shows that the semi-classical Einstein equations, which relate the classical curvature of spacetime to the expectation value of the energy-momentum tensor are violated. To fix this issue, we have to consider the change in the gravitational field due to the presence of the radiation. This problem known as **backreaction** has not been successfully resolved in full generality.

The complete evaporation of black holes raises the following issue: Consider the formation of a black hole through gravitational collapse of a star, for example due to a supernova explosion, and its evaporation due to Hawking radiation. Before the star collapses into a black hole, we consider a quantum field on spacetime which is in a pure state. Then, the star collapses and evaporates due to the Hawking effect. The black hole uniqueness theorem suggests that the Hawking radiation only carries information about mass, charge and angular momentum of the black hole, obtained by the lowest orders in a multipole expansion of the gravitational field. Any higher multipole moments of the matter that fell into the black hole will be lost. Therefore, not all information about the matter inside the black hole is accessible after the complete evaporation. Since not all the information is available, the state after evaporation is not pure but a mixed state and we have a conflict with unitary evolution of states in quantum mechanics, where one cannot evolve from a pure to a mixed state. This is the famous **information paradox** of black holes [14, 15].

The discussion above shows that classical general relativity not only breaks down at the singularity of the black hole but is also in strong tension with quantum field theory [8, 13, 16, 17]. It is believed that satisfying answers can only be provided once a quantum theory of gravity is available. A successful theory of quantum gravity should be able to explain the fate and possible resolution of the singularity and resolve the black hole information paradox. Additionally, quantum gravity should provide a first principle explanation for Hawking radiation, the evaporation of black holes and the result of this process. It should also explain the entropy of black holes from a microscopic level.

1.1 Quantum Gravity and Black Holes

In the literature, many approaches to quantum gravity are discussed, of which string theory [18], asymptotic safety [19], causal dynamical triangulations [20], causal sets [21, 22] and canonical quantum gravity (e.g. loop quantum gravity and spin foams) [23–28] are the most prominent ones. The candidate theory needs to be rigorously developed such that it provides a framework to tackle the questions raised above.

We saw before that the entropy of a black hole is proportional to the area rather than its volume. The event horizon is associated to the area of the black hole while the interior is related to its volume. This leads to speculations whether the information in the interior is fully encoded on its boundary (event horizon). This is a realisation of the holographic principle studied in **string theory** [29, 30]. In general, it states that the information inside a region of spacetime is fully encoded on its boundary. The most important realization of the holographic principle is the AdS/CFT conjecture [31]. AdS stands for the anti-de Sitter spacetime, a maximally symmetric spacetime with a constant negative curvature. CFT denotes conformal field theory, a special type of quantum field theory which is invariant under conformal transformations, i.e. transformations that include dilations and preserve angles. According to the AdS/CFT conjecture, there is a dictionary between a gravity theory formulated on an AdS spacetime in the bulk and a conformal field theory on its boundary.

Black holes are one of the main applications of the holographic principle [32]. For instance, in string theory, there is significant work on explaining the microscopic origin of the black hole entropy [33]. There are also proposals about the resolution of the black hole information paradox (see [34–36]). Two suggestions are the following:

In the **black hole complementarity** proposal, an observer falling into the black hole would notice nothing special at the event horizon, while an observer outside the black hole would see the information being spread out across the “stretched horizon”, a thin quantum mechanical membrane located just outside the event horizon. This membrane “stores” the information about the infalling particles making the information accessible for an outside observer and resolving the paradox. Since no observer can see both copies of the information at the same time the theory is consistent. However, there is no first-principle derivation of the existence of the stretched horizon and black hole complementarity is subject to ongoing debates.

For instance in [35], the authors argue that black hole complementarity is in contradiction with entanglement in quantum mechanics. They propose a so-called **firewall** located outside the event horizon, a high-energy barrier that destroys all the matter falling into the black hole and hence erases the information before it crosses the horizon. The existence of a firewall violates the equivalence principle in general relativity, that a free falling observer should locally see the same physics as in flat spacetime. In addition, the existence of the firewall is not derived from any fundamental theory and therefore remains rather speculative.

In **loop quantum gravity**, a microscopic explanation for the black hole entropy has also been derived based on semi-classical methods of isolated horizons, a generalization of the event horizon [37, 38]. It was shown in spherically symmetric models for the interior of the black hole, where only symmetric degrees of freedom were considered, that the singularity is avoided. It is replaced by a region where quantum effects are non-negligible [39–46]. In these models, the infalling matter moves in, reaches a maximum density, bounces back and moves out again. This behaviour has been seen in approaches based on the treatment of the interior of the black hole as a Kantowski-Sachs spacetime, which is homogeneous (spacetime is the same everywhere) and anisotropic (spacetime looks different in different directions). In [47–49], the authors studied purely spherically symmetric models where both the interior and exterior are accessible by means of numerically simulating spherically symmetric matter distributions in the form of dust (non-interacting particles) and perfect fluids. They also find singularity avoidance. In a different approach, the transition of a black hole to a white hole, a region of spacetime that matter cannot enter, is proposed. This transition is due to quantum gravity effects and a transition amplitude can be computed numerically. [50–52].

The study of black holes in two dimensional toy models, such as the **CGHS black hole** give insights into the evaporation process [53–55]. In these models, the semi-classical equations are exactly solvable and one obtains a detailed description for the evaporation of the two dimensional black hole. It is however unclear how to translate the specific tools for two dimensions to the realistic case of four dimensional spacetimes and how to generalise to quantum gravity with no semi-classical approximations. In four dimensions, general relativity with collapsing null dust (non-interacting massless particles) as matter content is exactly solvable (**Vaidya spacetime**) [56, 57]. Since null dust is not a fundamental matter content of the universe, one could instead consider a Higgs - like scalar field. The coupling of a spherically symmetric scalar field to general relativity has been studied classically [58–60].

In the present thesis, we investigate the problem of backreaction in black hole evaporation. In a heuristic argument, the power of the emitted radiation can be estimated using the Stefan-Boltzmann law where the area is given by the area of the event horizon. Assuming that the power is given by a change of the mass of the black hole, we obtain the time evolution of the black hole mass. For a black hole with mass M, we find a finite lifetime which scales as M^3. Using more elaborate techniques of semi-classical gravity, a more detailed spectrum of the emitted Hawking quanta and black hole evaporation was computed [61–63]. However, this approximation is expected to fail at the final stage of Hawking evaporation because the assumption of a fixed spacetime in the derivation of the Hawking effect is strongly violated.

A full quantum gravity calculation of Hawking evaporation based on first principles faces several challenges: First of all, the choice of coordinates in general relativity is arbitrary and the theoretical description contains additional unphysical degrees of freedom. This is evident for instance in the Hamiltonian formulation based on the ADM variables, where not all phase space variables are independent [64]. They have to satisfy certain constraints, i.e. relations between the variables. Thus, the description is redundant and we have to find observable quantities that are independent of the arbitrariness [65, 66]. Additionally, the Hamiltonian is a sum of constraints and vanishes if the constraints are satisfied. Therefore, it is not generating any time evolution (**problem of time**). To address both challenges, we work in the relational formalism [67–70]. It removes redundancies by defining observables with respect to some reference fields. Time evolution of the observables can then be recovered and it is described by a new Hamiltonian usually called the physical Hamiltonian.

1.2 Black Hole Perturbation Theory

Ideally, one would like to perform the analysis described above in a non-perturbative setting. In the presence of dust as a reference field this program is under active construction (see [70]). Instead of using dust or other matter fields as a reference, we could use variables within the gravitational sector instead. As it turns out, this is highly non-trivial due to the complexity of the constraints. Until we have a full non-perturbative theory of quantum gravity available, we use perturbation theory to develop an approximate description. The idea is to split the degrees of freedom into symmetric variables defined by some symmetry assumptions and the non-symmetric variables (see [71]). We assume that the model restricted to the symmetric variables can be solved exactly. Then, we expand the constraints to some order in the perturbations. In the case of black holes, we take the symmetric variables to have either spherical symmetry (non-rotating black hole) or axial symmetry (rotating black hole). Since semi-classical computations suggest that black holes emit angular momentum faster than mass [62], we consider spherical symmetry in this manuscript. The extension of the program to the case of axial symmetry is possible using similar tools and left for future investigations.

Perturbation theory around spherically symmetric spacetimes and in particular the Schwarzschild black hole (non-rotating black hole) is well established in the literature. It was pioneered by Regge, Wheeler and Zerilli who first showed that the perturbations satisfy two differential master equations [72, 73]. They are wave equations with potentials which are known as the Regge-Wheeler and Zerilli potentials (see [74] for a more recent treatment). These works are constrained to the black hole exterior region. For a complete picture of black hole formation through gravitational collapse, one has to take both the exterior and interior of the black hole into account. Perturbations in the exterior region were also investigated in the Hamiltonian formulation for the case of non-rotating black holes [75–79] and later extended to arbitrary spherically symmetric spacetimes in [80, 81]. In all of these approaches the spherically symmetric background is fixed and non-dynamical. Any influence of the perturbations on the dynamical background is ignored and hence these approaches are not suitable to investigate black hole evaporation where backreaction plays a key role.

1.2.1 Hybrid Approach

In the present thesis, we take backreaction into account and do not assume any fixed background keeping it fully dynamical. This procedure was first explored in the context of cosmology where the symmetric variables are defined to be homogeneous and isotropic [82, 83]. In the seminal work [84], the authors introduce the **hybrid approach** to quantum cosmology which was later extended in [85–92]. The idea is to investigate perturbations in cosmology by first switching off the dynamics of the background and defining observables for the perturbations. These observables are introduced through canonical transformations depending on the homogeneous and isotropic degrees of freedom. Then, one can show that there exists a transformation for the background variables, depending up to second order on the perturbations, that is canonical on the full phase space only up to second order. The result is a Hamiltonian constraint, expanded to second order in the perturbations. It encodes the dynamics of the system which can then be analysed in the quantum theory. In a complementary work [93–96], backreaction effects were taken systematically into account using **space adiabatic perturbation theory**, a generalization of the Born-Oppenheimer approximation for molecular systems.

In the case of perturbations around spherically symmetric spacetimes, the hybrid approach was used to study perturbations in the interior of black holes in the Kantowski-Sachs formulation [97]. In the computations, the authors of [97] did not fully resolve the redundancies and are left with some remnant constraints. This is fine for the cosmological model because the resulting Hamiltonian constraint trivially Poisson commutes with itself. Thus, the commutator in the quantum theory is well-defined. However for perturbations around black holes, we have more spherically symmetric constraints. In the pairwise computation of Poisson brackets between two constraints, we should obtain a linear combination of constraints. As shown in [71], in the hybrid approach this is only satisfied up to higher orders. This will translate to the quantum theory and obstruct its construction. To avoid these issues, in our work, we fully resolve the redundancies. In [71], it is demonstrated that the hybrid approach is equivalent to our formulation, when one restricts the treatment to second order and does not solve the Hamiltonian constraint (reduction in stages).

1.2.2 Reduced Phase Space Approach

In [71], a different strategy to perform Hamiltonian perturbation theory based on the **reduced phase space** approach is introduced. It solves all redundancies by fixing the reference fields to some value leading to a physical Hamiltonian describing the dynamics of the observables. The key advantage is that it defines observables before performing perturbative computations and it gives an explicit formula for the computation of the physical Hamiltonian to arbitrary orders. The formalism is based on a splitting of the variables into four distinct sets. On the one hand we have the symmetric and non-symmetric variables defined by the symmetry assumption. On the other hand we split the variables into observable (true) and non-observable (redundant) degrees of freedom. Then, the redundant degrees of freedom are used as reference fields to define the observable degrees of freedom. Following the computations we find a physical Hamiltonian [1–3, 98].

The applications to the question of backreaction and Hawking evaporation is now as follows: We start with the full phase space of general relativity coupled to matter. In the case of non-rotating black holes, we use spherical symmetry and split the variables into spherically symmetric and non-spherically symmetric degrees of freedom. The spherically symmetric sector will be treated exactly and chosen as the background, whereas the perturbations are given by the non-symmetric degrees of freedom. Then, we need to further distinguish between the true and the redundant variables. This step is arbitrary but we can motivate our choice as follows: For the application of the formalism to evaporating black holes, we want to treat both the exterior and interior of the spacetime and the description should be regular across the horizon. By handling both the interior and exterior in a consistent way, we do not lose information about matter falling into the black hole. This might be important for addressing the black hole information paradox. The redundancies are removed using the **Gullstrand-Painlevé coordinates** [99, 100], associated to an observer freely falling into the black hole from infinity. The usage of these coordinate system does not fix the background, rather it selects a class of spacetimes and the spherically symmetric degrees of freedom are still dynamical. The Gullstrand-Painlevé coordinates are suitable to study gravitational collapse and cover either a black hole or a white hole spacetime. Hence, this opens up the possibility to explore the black hole – white hole transition proposal in the quantum theory [98].

To recapitulate and clarify some key aspects of our approach, we highlight some of its important properties and advantages in the following:

First of all, we treat both the spherically symmetric and the non-symmetric variables on equal footing. The resulting physical Hamiltonian describes the dynamics of the full phase space including the interactions between the non-symmetric and the symmetric variables. The black hole mass is part of the spherically symmetric variables, while the Hawking radiation is part of the non-symmetric degrees of freedom. In order to completely understand black hole evaporation, we have to determine how the radiation is changing the dynamics of the background. This backreaction is completely neglected in the treatments by Moncrief, where the black hole mass is fixed. Also in the more general analysis by Brizuela and Martín-García, the background is general but is not influenced by the perturbations.

In the formalism, we reverse the usual procedure in Hamiltonian perturbation theory applied in the hybrid approach, where one first perturbs the constraints and then defines observables that are independent of the redundancy. In contrast to that, we first introduce the observables on the full phase space without approximations. Then, we derive a non-perturbative physical Hamiltonian describing the full dynamics of general relativity coupled to matter without the need to employ any perturbative strategy. Since the physical Hamiltonian is only known in an implicit form, we have to use perturbation theory in order to get an explicit expression. In other words, we disentangle the solution of the redundancy from perturbation theory. This clear separation, allows us to unambiguously extend the analysis to higher orders. In the standard approach, one has to reconsider the construction of observables at every order in perturbation theory and beyond second order, to the author's knowledge, there is no consensus in the literature (see [71] and references therein). To summarize, the framework allows for a consistent and well-defined Hamiltonian perturbation theory applicable to all orders.

In contrast to other treatments by Moncrief who only considers the exterior of the Schwarzschild spacetime and [97] who only studied the interior of the black hole, we study both the interior and exterior. Working in Gullstrand-Painlevé coordinates, we have access to both the interior and exterior simultaneously. This is crucial to analyse the formation of black holes through gravitational collapse and their subsequent evaporation. It might also play an important role when addressing the black hole information paradox.

At the current state of our framework, we do not refer to any of the previous results in loop quantum gravity about the resolution of the black hole singularity. In the approaches based on the Kantowski-Sachs formulation, one finds a resolution of the black hole singularity by a well-motivated substitution of some of the classical variables by bounded functions of them. In [43], the analysis in the black hole interior was extended to all of spacetime. The authors demonstrated that the full spacetime is free of any singularities. In our approach, we do not modify the classical Hamiltonian of the system. We leave it open whether the black hole singularity is avoided through different methods, for example due to backreaction effects. If this is not the case and the presence of the singularity causes technical issues when formulating the quantum theory, we can refer to the aforementioned works.

The difference of our approach to the hybrid approach is that we completely solve the all the redundancies in the theory. In [71], it is shown that for a partial solution of the redundancies with respect to the perturbations only, one recovers the results in the hybrid approach. We argued above that for a partial reduction, there are anomalies in the quantum theory which can be avoided in our formulation. Furthermore, in contrast to the hybrid approach, we define the observables before applying perturbation theory which allows for an immediate generalization to higher orders.

The backreaction problem of black hole evaporation can be studied in increasing levels of complexity and mathematical rigour. On the one side of the spectrum, there is the heuristic arguments based on a back of the envelope calculation using the Stefan Boltzmann law. On the other side, we have the ultimate goal to perform a full non-perturbative quantum gravity calculation, where both matter and gravity are quantized. Between these two extremes there are several intermediate steps. One step beyond the heuristic arguments are the semi-classical Einstein equations. In this approach, gravity is treated classically, while matter is treated quantum mechanically. The expectation value of the energy momentum tensor of matter gives the input to compute changes of the classical gravitational field. In this category fall many of the approaches mentioned above such as the CGHS model and the Vaidya spacetime. In this theses we go beyond these semi-classical works and also consider general relativity as a quantum theory. Since a full non-perturbative quantum gravity calculation still seems out of reach using tools available today, we take a perturbative formulation, where we only treat the observable symmetric degrees of freedom exactly and use perturbation theory for the observable non-symmetric variables.

In this thesis, we are mostly concerned with the computations for the physical Hamiltonian describing the dynamics of the theory to second order. As matter content, we investigate the electromagnetic field because photons are expected to be an important messenger in the spectrum of Hawking radiation. In the future, we plan to extend this to include also other matter fields such as neutrinos. A careful classical analysis is essential for constructing a well defined quantum theory. In the quantum theory, we plan to apply the hybrid quantization framework, where we combine a non-perturbative quantization for the symmetric degrees of freedom with a perturbative Fock quantization for the perturbations.

In order to check the validity of our approach, we verify whether the physical Hamiltonian is consistent with the literature when neglecting backreaction and restricting to the black hole exterior. This is a non-trivial step because, as we explained above, the setup is quite different to the usual approaches. We reverse the definition of observables and the application of perturbation theory and this might not lead to the same results. Additionally, we included backreaction effects and there could be new terms in the equations of motion. Performing several non-standard steps, we can indeed show that our approach leads to the same results in the regime where both theories apply.

1.3 Outline of the thesis

The thesis is structured in two parts. In sections 2 to 4, we introduce basic concepts about constrained Hamiltonian systems, general relativity and the Hawking effect. Then, in the second part in chapter 5, we present original results on the Hamiltonian perturbation theory around spherically symmetric spacetimes based on the publications [1–3, 98]. Some longer equations in chapter 5 were moved to the appendices and they also contain original work. In more detail, this manuscript is organized as follows:

In section 2, we review the theory of **constrained Hamiltonian systems**. First, we explain the Dirac algorithm for singular Lagrangians and discuss the separation into first and second class constraints. Then, we explain the construction of observables using the relational framework. We show that in the classical theory this framework is completely equivalent to a complementary viewpoint where the redundancies are fixed completely. In the end, the formalism is extended to field theories and we comment on some aspects that require special care due to the infinite number of degrees of freedom and boundary terms.

In part 3, we define basic notions in general relativity and derive the Einstein equations from the **Einstein-Hilbert action**. Assuming spherical symmetry, we find the Schwarzschild solution for non-rotating black holes and the Reissner-Nordstrøm solution for charged black holes. We also briefly comment on rotating black holes which are solutions of general relativity for axial symmetry. Using a foliation of spacetime, we obtain the Hamiltonian formulation of general relativity based on the **ADM variables**. With the techniques for constrained Hamiltonian systems developed in section 2 we solve the Hamiltonian theory and obtain the reduced Hamiltonian for spherically symmetric spacetimes. We conclude this section with a brief review of classical theorems about black holes.

Then, chapter 4 discusses **quantum field theory** on curved spacetimes. We discuss basic concepts of scalar quantum field theory on curved spacetimes and apply the tools to the derivation of the **Hawking effect**. Then, we comment on some known results about the evaporation of black holes based on semi-classical computations.

The **perturbation theory** around spherically symmetric spacetimes in general relativity is discussed in part 5. First, we perform the computation in the Lagrangian setup based on a linearization of the Einstein equations. The presentation relies on a modern reformulation of the works by Regge-Wheeler and Zerilli due to [74]. We show that the differential equations for the perturbations reduce to two master equations. Then, we discuss how the Hamiltonian framework derived in section 2 is applied to general relativity. First we perform the computation in the Gullstrand-Painlevé coordinates based on the author's original work in [2, 3]. We explicitly perform the computation and obtain a **physical Hamiltonian** describing the dynamics of the perturbations. As a consistency check, we show that the Hamilton equations of motion match the ones obtained in the Lagrangian case. For charged black holes we find agreement with the results in [101], Then, we extend the formulation to a generalization of the Gullstrand-Painlevé coordinates first discussed in [1].

Ideas for our plans to study the physics of evaporating black holes are presented in section 6. We sketch a way to introduce a Fock quantization for the perturbations and mention some technical challenges that might arise. In the second part, we provide a small outlook into current and future astrophysical experiments to detect evaporating black holes and how our formalism relates to them. The considerations in the present work appear to be entirely classical. However, as just mentioned, once we gain sufficient mathematical control of the mode functions, we have at our disposal a well defined Fock representation for the physical perturbations. All classical observables that we can compute in classical perturbation theory then have an in principle straightforward quantisation by expanding in annihilation and creation operators followed by normal ordering. This defines at least a (densely defined) quadratic form (i.e. matrix elements exist but not necessarily of its formal square), e.g. for the physical Hamiltonian expanded to 3rd order which includes self-interactions of the perturbations. Accordingly, all the steps performed in this thesis are directly relevant for the (hybrid) quantisation of black hole perturbation theory.

In section 7, we summarise our findings and conclude our investigation. Possible future avenues for research are outlined and briefly discussed.

In the appendices we also present original work. In appendices A and B, we list some lengthy formulas for the computations in section 5 in order not clutter the main text.

2

Constrained Hamiltonian Systems and the Reduced Phase Space Formalism

In general relativity or the Standard model of particle physics, theories are usually formulated in the language of gauge theories, where the theoretical description contains redundancies. For instance in general relativity, we have the freedom to choose arbitrary coordinate systems and the Lagrangian is invariant under changes of coordinates. The Hamiltonian description for systems with gauge symmetries is not straight forward because the Legendre transformation will be singular. Since for a canonical quantization program, the Hamiltonian of the system is a crucial ingredient, we have to generalize the Hamiltonian theory.

In [65, 66], a method to obtain a Hamiltonian formulation for Lagrangians with singular Legendre transformation is presented. The Hamiltonian theory now involves constraints, equations on phase space that have to be satisfied for physical solutions of the equations of motion. In the first part of this chapter, we review the theory of constrained Hamiltonian systems.

We find that there are two kinds of constraints: first and second class constraints. The first class constraints generate gauge transformations and the physical degrees of freedom are given by gauge invariant variables that commute with all the first-class constraints. In contrast to that, the second class constraints are directly used to find a reduced description on the hypersurface in phase space where the constraints are satisfied using the Dirac bracket. In the case of first class constraints, gauge invariant variables can be constructed explicitly using the notion of relational observables [67, 70]. The idea is that we consider the value of a given variable once another variable takes a certain value. Another approach is to fix some of the degrees of freedom to certain values (gauge fixing) and work on the reduced phase space where the gauge fixing and the constraints are satisfied. In this text, we will introduce both approaches and show that they are completely equivalent.

Finally, we extend the discussion to the case of field theories. Instead of a discrete index, we now have a continuous label for the point in space and the constraints may contain differential operators. In the calculation of variations and Poisson brackets these differential operators lead to boundary terms that can spoil the differentiability. As we will see, boundary terms need to be added to the constraints in order to obtain a well-defined Hamiltonian theory.

Due to a lack of mathematical symbols, we will use a different index notation from the rest of the manuscript in this chapter.

2.1 Singular Legendre Transformations, Constrained Hamiltonian Systems and the Dirac Algorithm

Consider a classical Lagrangian $L(q^a, \dot{q}^a)$ without explicit time dependence, where a labels the degrees of freedom ($a = 1, \ldots, f$). For simplicity of the discussion, we are only considering a finite range for the index a. The Lagrangian L defines the corresponding action principle by studying the functional $S[q]$ defined by

$$S[q] := \int_{t_0}^{t_1} \mathrm{d}t \, L(q^a, \dot{q}^a) \,. \tag{2.1.1}$$

The integral depends on the trajectory $q(t)$ and its time derivative and is taken over the time interval $[t_0, t_1]$. For the dynamics, we vary the action with respect to $q(t)$. The stationary points of the action give the Euler-Lagrange equations determining the classical evolution of the system:

$$\frac{\partial L}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0 \,. \tag{2.1.2}$$

Notice, that it is a second order differential equation in the time t. However, in general, it is not possible to solve the system equations for the accelerations $\ddot{q}^a$. In fact, this is only possible locally, provided the matrix

$$M_{ab} := \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b} \,. \tag{2.1.3}$$

is invertible, i.e. the determinant $\det(M)$ is non-vanishing. In the case that the M is singular, the acceleration is not uniquely determined by the position and velocities.

For the study of the quantum properties of a theory, it is beneficial to investigate the corresponding Hamiltonian formulation as well. The aim is a change in perspective where instead of studying the theory in terms of the tangent bundle coordinatised by the configuration variables and the velocities, we use the cotangent bundle coordinatised by the configuration variables and conjugate momenta. The momenta p_a conjugate to the configuration variables are introduced by the derivative of the Lagrangian with respect to the velocities $\dot{q}^a$:

$$p_a := \frac{\partial L}{\partial \dot{q}^a} \,. \tag{2.1.4}$$

In order to fully remove the velocities $\dot{q}^a$, we have to solve the relations (**2.1.4**) for the velocities as functions of the momenta and configuration variables. If the matrix M defined in (**2.1.3**) is invertible, the inverse function theorem guarantees, that we can locally solve for $\dot{q}^a$ in terms of p_a and q^a. However, if the rank of M is some number $(f - r) > 0$, then we are only able to solve $f - r$ equations for $f - r$ velocities. Let us without loss of generality solve for the first $f - r$ velocities and obtain relations of the form

$$\dot{q}^A = v^A(q^a, p_A, \dot{q}^i) \,, \tag{2.1.5}$$

where $A = 1, \ldots, f - r$, $a = 1, \ldots, f$ and $i = f - r + 1, \ldots, f$. The remaining momenta lead to constraints of the form

$$\phi_i(q, p) := p_i - \frac{\partial L}{\partial \dot{q}^i}\Big|_{\dot{q}^A = v^A(q^a, p_A, \dot{q}^j)} \overset{!}{=} 0 \,. \tag{2.1.6}$$

These relations are called **primary constraints**.

The Hamiltonian H' corresponding to a Lagrangian L is called the **primary Hamiltonian** and is defined by

$$H' = p_a \dot{q}^a - L(q^a, \dot{q}^a)\Big|_{\dot{q}^A = v^A(q^a, p_A, \dot{q}^j)} . \tag{2.1.7}$$

Here, we are only able to replace a subset of the velocities for the momenta and we leave $\dot{q}^j$ ($j = f - r + 1, \ldots, f$). It is easy to show that the primary Hamiltonian is of the form

$$H'(q^a, p_a, \dot{q}^i) = \dot{q}^i \phi_i + \tilde{H}(q^a, p_a) . \tag{2.1.8}$$

Inside H', the velocities $\dot{q}^i$ act as Lagrange multipliers enforcing the constraints ϕ_i. The $\dot{q}^i$ are not dynamical but arbitrary functions.

In summary, we are left with a phase space coordinatized by q^a, p_a and $v^i := \dot{q}^i$ equipped with the symplectic structure $\{\cdot, \cdot\}$ and Hamiltonian H'. The only non-vanishing Poisson bracket is $\{q^a, p_b\} = \delta^a_b$. The Hamiltonian equations of motion are given by

$$\dot{q}^a = \{q^a, H'\}, \quad \dot{p}_a = \{p_a, H'\}, \quad \phi_i = 0 . \tag{2.1.9}$$

It is not difficult to see that they are fully equivalent to the Euler-Lagrange equations.

As it is defined right now, the Hamiltonian theory is however not consistent, because the primary constraints might not be consistent with the dynamics. We have to ensure that they are preserved under time evolution:

$$\dot{\phi}_i\Big|_{\phi=0} = \{\phi_i, H'\}\Big|_{\phi=0} = 0 , \tag{2.1.10}$$

where $\phi = 0$ means that this relation holds on the constraint surface where $\phi_i = 0$ for all $i = 1, \ldots, r$. There are three possible outcomes for the left-hand side:

1. $\dot{\phi}_i\big|_{\phi=0} = 0$ for some $i = 1, \ldots, \alpha$:

   In this case the constraints ϕ_i are already consistent with the dynamics and nothing has to be done.

2. $\dot{\phi}_i\big|_{\phi=0} \neq 0$ and $\{\phi_i, \phi_j\}\big|_{\phi=0} = 0$ for $i = \alpha + 1, \ldots, \beta$ and $j = 1, \ldots, r$:

   In this case we add the $\dot{\phi}_i$ to the list of constraints and iterate the algorithm by studying the time derivative of the newly added constraints. The Hamiltonian generating time evolution is still H' which only contains the original list of constraints ϕ_i, $i = 1, \ldots, r$. All the additional constraints added to the list of constraints are called **secondary constraints**.

3. $\dot{\phi}_m\big|_{\phi=0} \neq 0$ for general v^i and $\{\phi_i, \phi_j\}\big|_{\phi=0} \neq 0$ with $i = \beta + 1, \ldots, r$ and $j = 1, \ldots, r$:

The algorithm is guaranteed to terminate after at most $2m - r$ steps because at each occurrence of case 2, the number of independent constraints increases by 1 and more than $2m$ constraints would fully constrain the phase space.

After the algorithm terminated we are left with r' constraints of which r are primary and $r' - r$ are secondary. For $i = 1, \ldots, \alpha$, the ϕ_i are of case 1 and for $i = \alpha + 1, \ldots, r'$, the ϕ_i are of case 3. In case 3, we introduce the matrix $\{\phi_i, \phi_j\}$ for $i = \alpha + 1, \ldots, r'$, $j = 1, \ldots, r$. There are two possibilities:

In the first one, the rank of the matrix is not maximal and smaller than $r' - \alpha$. Then, the theory is inconsistent because we are not able to choose the v^i such that the constraints are consistent with the dynamics. We have to rule out such theories and will not consider them further.

If the rank of the matrix is maximal, i.e. equal to $r' - \alpha$, we are able to solve the system of equations

$$\left\{\tilde{H}, \phi_i\right\}\Big|_{\phi=0} + v^j\{\phi_i, \phi_j\} = 0\,, \tag{2.1.11}$$

with $i = \alpha, \ldots, r'$ and $j = 1, \ldots, r$. The solution is of the form

$$v^i = v_0^i + \lambda^\mu v_\mu^i\,, \tag{2.1.12}$$

where v_0^i is a special solution of the inhomogeneous equation and v_μ^i, $\mu = 1, \ldots, r - (r' - \alpha)$ is a basis for the general solution of the homogeneous system. Then, we define

$$H := \tilde{H} + v_0^j\phi_j, \quad \phi_\mu := v_\mu^j\phi_j \tag{2.1.13}$$

The ϕ_μ are constraints that commute with all the other constraints ϕ_j, $j = 1, \ldots, r'$.

There are two different kinds of constraints. Either they commute with all the other constraints or they have non-trivial Poisson brackets. This is captured by the following definition:

Definition 2.1: Constraints can be classified into two types:

1. A constraint is called **first-class**, if it weakly commutes with all constraints.

2. A set of constraints $\{\phi_i\}_{i\in I}$ for some index set I is **second class**, if the matrix $\{\phi_i, \phi_j\}$, $i, j \in I$ has maximal rank.

Given a set of constraints $\{\phi_i\}_{i\in I}$, we might wonder if the above definition exhausts all the possibilities. In fact, this is the case, as we show in the following lemma:

Lemma 2.2: Given a collection of constraints $\{\phi_i\}_{i\in I}$ we can define new constraints $\tilde{\phi}_i$ through linear combinations such that $\tilde{\phi}_i$ is first class for $i \in I_1$ and $\tilde{\phi}_i$ is second class for $i \in I_2$.

Proof:
Some of the constraints might already be first class and we are done. Thus, consider the set of all constraints $\{\phi_i\}_{i\in I'}$ which are not first class. If the matrix $M_{ij} = \{\phi_i, \phi_j\}$ is non-singular, the constraints $\{\phi_i\}_{i\in I'}$ are already second-class. If the matrix M_{ij} is singular, we have $\det(M) = 0$ and we can find a non-trivial solution v^i of the equation

$$v^i\{\phi_i, \phi_j\} = 0\,. \tag{2.1.14}$$

for all $j \in I'$. Since by definition $v^i\phi_i$ commutes with the first class constraints and it commutes with the second-class constraints, $v^i\phi_i$ is first-class. Iterating until M is non-singular, we succesfully decomposed the set of constraints $\{\phi_i\}_{i\in I}$. □

The decomposition of a given set of constraints into first and second-class is not unique. We can for example add a linear combination of first-class constraints to a second-class constraint without destroying the second-class property. It is even possible to add a square of a second class constraint to a first-class constraint, mainting the first-class property.

In the following we will study both the second and first-class constraints separately.

2.1.1 First-Class Constraints and Gauge Transformations

In this subsection we will ignore the second-class constraints and focus on the first-class constraints. We call a general function f on phase space of first-class, if it commutes with all constraints, i.e. $\{f, \phi_i\} = 0$ for all i. By construction, the first class constraints and the Hamiltonian H defined in (**2.1.13**) are first class functions.

The first-class constraints generate gauge transformations and we define the **extended Hamiltonian** as

$$H_\lambda := H + \lambda^i \phi_i \,. \tag{2.1.15}$$

In the definition of the extended Hamiltonian we include both the primary and secondary first-class constraints. The presence of the primary first class constraints follows directly from the Hamiltonian H' in (**2.1.8**). However, the inclusion of all the secondary first class constraints does not strictly follow from the formalism. It turned out to be the correct prescription in various examples and physical situations.

The time evolution of physical observables should be independent of the choice of λ^i. This is the case if the observables commute with the constraints and we define

Definition 2.3: A function f on phase space is called **weak Dirac observable**, if

$$\{f, \phi_I\}\Big|_{\phi=0} = 0, \forall I \,. \tag{2.1.16}$$

It is called a strong Dirac observable if the equality is strong (i.e. without imposing the constraints).

The physics of a Hamiltonian theory with first class constraints is then obtained by finding the Dirac observables and computing their time evolution. The first class constraints generate gauge transformations and requiring that that Dirac observables commute with the constraints means that they are gauge invariant.

2.1.2 Second-Class Constraints and the Dirac bracket

Consider a set of second class constraints $\{\phi_i\}_{i\in I}$ together with the matrix $M_{ij} := \{\phi_i, \phi_j\}$. It follows that i runs over an even number of indices because M is non-singular and anti-symmetric. For second class constraints we have to proceed differently to first class constraints. The goal is to understand the dynamics on the constraint surface, i.e. the hypersurface in spacetime where the second class constraints are satisfied. There are no gauge transformations for second class constraints and we simply have to find a way to compute Poisson brackets on the constraint surface.

We define a map $\{\cdot, \cdot\}_D$ on phase space called the **Dirac bracket**

$$\{f, g\}_D := \{f, g\} - \{f, \phi_i\} \left(M^{-1}\right)^{ij} \{\phi_j, g\} \,, \tag{2.1.17}$$

with the inverse matrix $(M^{-1})^{ij}$: $(M^{-1})^{ij} M_{jk} = \delta^i_k$. The following lemma shows that the Dirac bracket has similar properties as the Poisson bracket and has a nice behaviour when we apply it to the constraints:

Lemma 2.4: The Dirac bracket $\{\cdot, \cdot\}_D$ satisfies for any phase space functions f, g, h:

1. Antisymmetry: $\{f, g\}_D = -\{g, f\}_D$
2. Linearity: $\{f + g, h\} = \{f, h\} + \{g, h\}$

3. Leibniz rule: $\{f, gh\}_D = g\{f, h\}_D + \{f, g\}_D h$

4. Jacobi identity: $\{f, \{g, h\}_D\}_D + \{g, \{h, f\}_D\}_D + \{h, \{f, g\}_D\}_D = 0$

5. $\{\phi_i, f\}_D = 0$

6. $\{f, g\}_D\big|_{\phi=0} = \{f, g\}\big|_{\phi=0}$ for g first-class and f arbitrary

Proof:
The first three properties follow easily from the properties of the regular Poisson bracket. A longer and more tedious calculation shows that the Dirac bracket satisfies the Jacobi identity. The fifth point follows from

$$\{\phi_i, f\}_D = \{\phi_i, f\} - \{\phi_i, \phi_j\}\,(M^{-1})^{jk}\{\phi_k, f\} = \{\phi_i, f\} - M_{ij}(M^{-1})^{jk}\{\phi_k, f\} = 0\,. \tag{2.1.18}$$

For the last point notice that $\{\phi_i, g\}\big|_{\phi=0}$ is zero for g of first-class and the Dirac bracket reduces to the Poisson bracket. □

Point five is the key property that allows us to set the second-class constraints to zero before or after evaluating the Dirac bracket. Using the Dirac bracket we can hence restrict the dynamics to the constraint surface.

2.1.3 Gauge Fixings

We saw that first-class constraints generate gauge transformations. In other words, there are multiple points in phase space which correspond to the same physical situation. One idea is to reduce the redundancy by imposing gauge fixing conditions $G_i(q, p) = 0$ which single out a unique point in each gauge orbit. Such a gauge fixing condition should have two properties:

1. Accessibility of the gauge:

   Given any value for the canonical variables q, p with $G_i(q, p) \neq 0$ there needs to be a gauge transformation $q \to q'$, $p \to p'$ such that $G_i(q', p') = 0$.

2. Complete gauge fixing:

   The gauge fixing condition needs to fix the gauge completely and there are no residual gauge transformations. In other words the matrix $\{G_i, \phi_j\}$ has to be invertible.

The second requirement shows that adding gauge fixing conditions as new "constraints" to the first-class constraints, the combined set is of second-class. Therefore, fully fixing the gauge, we moved from a system of first-class constraints to a system of second-class constraints. Then, we define the dynamics on the constraint surface, where the first-class constraints and gauge fixings are satisfied using the Dirac bracket formalism in the previous section.

2.2 Relational Observables and the Reduction Point of View

In the previous sections we showed how to treat Hamiltonian theories in the presence of constraints. We introduced the notion of constraints of first and second class and highlighted the difference in their treatment. In this section, we present more advanced concepts to handle theories with first class constraints by explicitly constructing Dirac observables. For this, we introduce reference degrees of freedom and study the dynamics with respect to them. This concept is known as “relational observables” and was first introduced by Rovelli in [67]. The concept was extended and more formally studied in [68, 102]. In the context of general relativity it gained some interest because the observables in full general relativity could be constructed in the presence of a dust field as reference [69, 70, 103]. In this section, we follow the notation and treatment in [71] and focus on the finite-dimensional case and leave the extension to field theories for the next section.

We consider a list of first-class constraints $\{\phi'_i\}_{i\in I}$ and note that the number of canonical pairs must be larger than the number of constraints. Thus, we can divide the variables into two sets: the gauge (redundant) degrees of freedom (u^i, v_i) with $i \in I$ and the remaining true (observable) degrees of freedom (r^a, s_a) with $a \in A$. The split is arbitrary but often motivated by the physical situation. Next, we assume that we can locally solve the constraints ϕ'_i for the momenta v as $v_i = -h_i(r, s, u)$. Then, we can pass to equivalent constraints of the form $\phi_i = v_i + h_i(r, s, u)$. A solution like this is usually not unique whenever the constraint depends on higher than linear order contributions of v. In this case several “branches” of the solution exist and we have to provide further physical input to restrict to one of these branches.

The advantage of the new constraints ϕ_i is that they define the same constraint surface but have the following property:

Lemma 2.5: The constraints $\phi_i = v_i + h_i(r, s, u)$ are of first class and Abelian.

Proof:
First note that the constraints $\{\phi_i\}_{i\in I}$ and $\{\phi'_i\}_{i\in I}$ span the same constraint surface. This implies that the constraints are related by $\phi_i = f_i^j \phi'_j$ with possibly very complicated phase space dependent functions f_i^j. It follows that ϕ_i are of first-class, because

$$\Big\{\phi_i, \phi'_j\Big\}\Big|_{\phi'=0} = f_i^k \Big\{\phi'_k, \phi'_j\Big\}\Big|_{\phi'=0} + \phi'_k \Big\{f_i^k, \phi'_j\Big\}\Big|_{\phi'=0} = 0\,, \tag{2.2.1}$$

where we used the first-class property of the constraints $\{\phi'_i\}_{i\in I}$. The Poisson bracket between two new constraints can thus be written as

$$\{\phi_i, \phi_j\} = \kappa_{ij}{}^k \phi_k\,, \tag{2.2.2}$$

for some phase space functions $\kappa_{ij}{}^k$. From the definition of the Poisson bracket, it follows that the left-hand side is independent of v. Thus, the right-hand side of equation (**2.2.2**) has to be independent of v as well and $\kappa_{ij}{}^k$ must vanish. This proves that the $\{\phi_i\}_{i\in I}$ form an Abelian algebra. □

The physical interpretation of the framework is as follows: The presence of first-class constraints in the system indicates that we have redundancies in the theoretical description, i.e. not all variables correspond to physically measurable quantities. To resolve this redundancy, we promote a subset of the variables to reference degrees of freedom with respect to which we measure the other variables. In our situation, we take as references the variables u^i and describe the values of (r, s) when u^i takes a certain value τ^i. This is the idea of relational observables that we will introduce in the next subsection.

A complementary viewpoint is the gauge fixing prescription, where we fix the value of u^i to a fixed number τ^i. Then, we restrict the dynamics to the constraint surface and the gauge $u^i = \tau^i$. This will be discussed afterwards and the equivalence of the two prescriptions will be shown.

2.2.1 Relational Observables

In this section, we assume the above setup, where we define the canonical variables $(u^i, v_i)_{i\in I}$ and $(r^a, s_a)_{a\in A}$ and a set of abelianized first class constraints of the form $\phi_i = v_i + h_i(r, s, u)$. Let $G^i = u^i - \tau^i$ be the gauge fixing condition where τ^i are phase space independent. The Poisson algebra of the G^i and ϕ_i is

$$\{\phi_i, \phi_j\} = \{G^i, G^j\} = 0, \quad \{\phi_i, G^j\} = -\delta_i^j \, . \tag{2.2.3}$$

The following map is of interest

Definition 2.6: Let f be any function on phase space. Then, the **relational observable** associated to f when the u^i take the values τ^i is defined as

$$O_f(\tau) := \sum_{n=0}^{\infty} \frac{1}{n!} G^{i_1} \cdots G^{i_n} \{\phi_{i_1}, \cdots, \phi_{i_n}, f\}, \tag{2.2.4}$$

where for any phase space functions $f_1, \ldots, f_n$ we defined $\{f_1, \cdots, f_n\} := \{f_1, \{f_2, \ldots, f_n\}\}$ and $\{f\} := f$.

Physically, the relational observable is interpreted as the value the function f takes, when the references u^i take the value τ^i. As it turns out, the notion of relational observables is convenient for the construction of Dirac observables. In the the following theorem we prove several useful properties of relational observables.

Theorem 2.7: Let f be a phase space function. Then the relational observable $O_f(\tau)$ satisfies

1. Dirac observable: $O_f(\tau)$ is a Dirac observable
2. Compatibility with the pointwise product: $O_{fg}(\tau) = O_f(\tau) O_g(\tau)$
3. For $f(q, p)$ any analytic function of q, p, we have $O_f(\tau) = f(O_q(\tau), O_p(\tau))$
4. Dirac bracket homomorphism: We have $\{O_f(\tau), O_{f'}(\tau)\} = O_{\{f,f'\}_D}(\tau)$, where $\{\cdot, \cdot\}_D$ is the Dirac bracket given by
$$\{f, g\}_D = \{f, g\} - \{f, G^i\}\{\phi_i, g\} + \{f, \phi_i\}\{G^i, g\} \tag{2.2.5}$$

Proof:

1. First, we show that for all n, we have
$$\{\phi_1, \ldots, \phi_n, f\} = \left\{\phi_{\sigma(1)}, \ldots, \phi_{\sigma(n)}, f\right\} \tag{2.2.6}$$
for all permutations $\sigma \in S_n$. We prove this by induction. The case $n = 1$ is trivial and the case $n = 2$ is easy to verify using the Jacobi identity. Assume that the equation holds for all values smaller

or equal to n and consider the case $n+1$. There are now two options: If the permutation fixes the number 1, we use the induction hypothesis. If 1 and 2 are swapped we have

$$\begin{aligned}\{\phi_1,\ldots,\phi_n,f\} &= \{\phi_1,\{\phi_2,\{\phi_3,\ldots,\phi_n,f\}\}\} \\ &= -\{\phi_2,\{\{\phi_3,\ldots,\phi_n,f\},\phi_1\}\} - \{\{\phi_3,\ldots,\phi_n,f\},\{\phi_1,\phi_2\}\} \\ &= \{\phi_2,\phi_1,\phi_3,\ldots\phi_n,f\}\end{aligned} \tag{2.2.7}$$

and the rest follows from the induction hypothesis for $n-1$.

Let us now calculate

$$\begin{aligned}\{\phi_i,O_f(\tau)\} = &-\sum_{n=1}^{\infty}\frac{1}{(n-1)!}G^{i_1}\cdots G^{i_{n-1}}\{\phi_i,\phi_{i_1},\cdots,\phi_{i_{n-1}},F\} \\ &+\sum_{n=0}^{\infty}\frac{1}{n!}G^{i_1}\cdots G^{i_n}\{\phi_i,\phi_{i_1},\cdots,\phi_{i_n},F\} = 0\end{aligned} \tag{2.2.8}$$

2. We have

$$\begin{aligned}O_{fg}(\tau) &= \sum_{n=0}^{\infty}\frac{1}{n!}G^{i_1}\cdots G^{i_n}\{\phi_{i_1},\cdots,\phi_{i_n},fg\} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\frac{1}{n!}G^{i_1}\cdots G^{i_n}\{\phi_{i_1},\cdots,\phi_{i_l},f\}\{\phi_{i_{l+1}},\ldots,\phi_{i_n},g\} \\ &= O_f(\tau)O_g(\tau)\end{aligned} \tag{2.2.9}$$

In the first step, we used a variant of the generalised Leibniz rule. We need to show that

$$\Big\{\phi_{(i_1},\ldots,\phi_{i_n)},fg\Big\} = \sum_{l=0}^{n}\binom{n}{l}\Big\{\phi_{(i_1},\ldots,\phi_{i_l},f\Big\}\Big\{\phi_{i_{l+1}},\ldots,\phi_{i_n)},g\Big\} \tag{2.2.10}$$

where we used total symmetrisation of the indices $i_1,\ldots,i_n$.

The base case ($n=1$) follows from the Leibniz rule of the Poisson bracket. We assume that the claim holds for n and consider the case $n+1$:

$$\begin{aligned}\Big\{\phi_{(i_1},\ldots,\phi_{i_n+1)},fg\Big\} &= \sum_{l=1}^{n+1}\binom{n}{l-1}\Big\{\phi_{(i_1},\{\phi_{i_2},\ldots,\phi_{i_l},f\}\Big\{\phi_{i_{l+1}},\ldots,\phi_{i_{n+1})},g\Big\}\Big\} \\ &= \sum_{l=1}^{n+1}\binom{n}{l-1}\Big\{\phi_{(i_1},\ldots,\phi_{i_l},f\Big\}\Big\{\phi_{i_{l+1}},\ldots,\phi_{i_{n+1})},g\Big\} + \Big\{\phi_{(i_2},\ldots,\phi_{i_l},f\Big\}\Big\{\phi_{i_1},\phi_{i_{l+1}},\ldots,\phi_{i_{n+1})},g\Big\} \\ &= \sum_{l=1}^{n+1}\binom{n}{l-1}\Big\{\phi_{(i_1},\ldots,\phi_{i_l},f\Big\}\Big\{\phi_{i_{l+1}},\ldots,\phi_{i_{n+1})},g\Big\} + \Big\{\phi_{(i_1},\ldots,\phi_{i_{l-1}},f\Big\}\Big\{\phi_{i_l},\ldots,\phi_{i_{n+1})},g\Big\} \\ &= \sum_{l=1}^{n+1}\left[\left(\binom{n}{l-1}+\binom{n}{l}\right)\Big\{\phi_{(i_1},\ldots,\phi_{i_l},f\Big\}\Big\{\phi_{i_{l+1}},\ldots,\phi_{i_{n+1})},g\Big\}\right] + f\Big\{\phi_{(i_1},\ldots,\phi_{i_{n+1})},g\Big\} \\ &= \sum_{l=0}^{n+1}\binom{n+1}{l}\Big\{\phi_{(i_1},\ldots,\phi_{i_l},f\Big\}\Big\{\phi_{i_{l+1}},\ldots,\phi_{i_{n+1})},g\Big\}\end{aligned}$$

3. We expand $f(q,p)$ as a power series in q,p and then use that $O_f(\tau)$ is linear and compatible with the pointwise product.

4. The Poisson bracket between two relational observables is given by

$$\begin{aligned}\{O_f(\tau), O_g(\tau)\} = \sum_{n,m=0}^{\infty} \frac{1}{n!m!}\Big[&nG^{i_1}\dots G^{i_{n-1}}G^{j_1}\dots G^{j_m}\{\phi_{i_1},\dots,\phi_{i_n},f\}\Big\{\phi_{j_1},\dots,\phi_{j_m},G^{i_n},g\Big\}\\ &- mG^{i_1}\dots G^{i_n}G^{j_1}\dots G^{j_{m-1}}\Big\{\phi_{i_1},\dots,\phi_{i_n},G^{j_m},f\Big\}\{\phi_{j_1},\dots,\phi_{j_m},g\}\\ &+ G^{i_1}\dots G^{i_n}G^{j_1}\dots G^{j_m}\{\{\phi_{i_1},\dots,\phi_{i_n},f\},\{\phi_{j_1},\dots,\phi_{j_m},g\}\}\Big]\end{aligned}$$

where we used that $\{G^i, \phi_{i_1}, \dots, \phi_{i_n}, f\} = \{\phi_{i_1}, \dots, \phi_{i_n}, G^i, f\}$. After reindexing and changing the double summation, we obtain

$$\begin{aligned}\{O_f(\tau), O_g(\tau)\} = \sum_{n=0}^{\infty}\frac{1}{n!}G^{i_1}\dots G^{i_n}\sum_{l=0}^{n}\binom{n}{l}\Big[&\{\phi_{i_1},\dots\phi_{i_l},\phi_j,f\}\Big\{\phi_{i_{l+1}},\dots,\phi_{i_n},G^j,g\Big\}\\ &-\Big\{\phi_{i_1},\dots,\phi_{i_l},G^j,f\Big\}\{\phi_{i_{l+1}},\dots,\phi_{i_n},\phi_j,g\}\\ &+\{\{\phi_{i_1},\dots,\phi_{i_l},f\},\{\phi_{i_{l+1}},\dots,\phi_{i_n},g\}\}\Big]\end{aligned} \tag{2.2.11}$$

Next, we use induction to show that

$$\{\phi_{(i_1},\cdots,\phi_{i_n)},\{f,g\}\} = \sum_{l=0}^{n}\binom{n}{l}\{\{\phi_{(i_1},\cdots,\phi_{i_l},f\},\{\phi_{i_{l+1}},\cdots,\phi_{i_n)},g\}\}. \tag{2.2.12}$$

The case $n = 1$ is clear. Assume that the relation holds for n and consider the case $n + 1$:

$$\begin{aligned}&\{\phi_{(i_1},\cdots,\phi_{i_n+1)},\{f,g\}\} = \Big\{\phi_{(i_1},\Big\{\phi_{i_2},\dots,\phi_{i_{n+1})},\{f,g\}\Big\}\Big\}\\ &= \sum_{l=1}^{n+1}\binom{n}{l-1}\{\{\phi_{(i_1},\cdots,\phi_{i_l},f\},\{\phi_{i_{l+1}},\cdots,\phi_{i_{n+1})},g\}\} + \{\{\phi_{(i_2},\cdots,\phi_{i_l},f\},\{\phi_{i_1},\phi_{i_{l+1}},\cdots,\phi_{i_{n+1})},g\}\}\end{aligned}$$

The rest of the calculation is analogous to the calculation in item 2.

Finally, combining the results above we have shown that

$$\{O_f(\tau), O_g(\tau)\} = \sum_{n=0}^{\infty}\frac{1}{n!}G^{i_1}\dots G^{i_n}\Big\{\phi_{i_1},\dots,\phi_{i_n},\{f,g\} - \{f,G^j\}\{\phi_j,g\} + \{f,\phi_j\}\{G^j,g\}\Big\} \tag{2.2.13}$$

We recognise the Dirac bracket of f and g, Which finishes the proof.

□

For fully constrained Hamiltonian theories such as general relativity, the Hamiltonian of the system vanishes, leading to the "problem of time". The system appears to be "frozen" and no dynamics is possible at all. The relational framework provides a tool to resolve this issue, using the relational observables by introducing a so-called multi-fingered time evolution. It is possible to study the evolution of the observables, through a variation of the gauge fixing condition induced by a change in τ^i. The easiest situation is a 1-parameter family of variations $\tau^i(t)$ which leads to the notion of physical Hamiltonian:

Definition 2.8: Consider a 1-parameter family of gauge fixings $G^i = u^i - \tau^i(t)$. Let f be a function on phase space depending only on the true degrees of freedom (r, s). The **physical Hamiltonian** $H(t)$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}O_f(\tau(t)) =: \{O_f(t), H(t)\}\,. \tag{2.2.14}$$

Theorem 2.9: The physical Hamiltonian is given by $H(t) = O_{\dot\tau(t)^i h_i}(\tau(t))$.

Proof:
We calculate the time derivative of the relational observable and find

$$\frac{\mathrm{d}}{\mathrm{d}t}O_f(\tau(t)) = -\dot\tau^j \sum_{n=0}^{\infty} \frac{1}{n!} G^{i_1} \dots G^{i_n} \{\phi_j, \phi_{i_1}, \dots, \phi_{i_n}, f\} \tag{2.2.15}$$

On the other hand, we have

$$\begin{aligned}\left\{O_f(\tau), O_{\dot\tau(t)^i h_i}(\tau(t))\right\} &= O_{\{f(r,s), \dot\tau(t)^i h_i(r,s,u)\}_D}(\tau(t)) = O_{\{f(r,s), \dot\tau(t)^i h_i(r,s,u)\}}(\tau(t)) \\ &= O_{\{f(r,s), \dot\tau(t)^i \phi_i\}}(\tau(t)) = -\dot\tau(t)^j \sum_{n=0}^{\infty} \frac{1}{n!} G^{i_1} \dots G^{i_n} \{\phi_{i_1}, \dots, \phi_{i_n}, \phi_j, f\}\end{aligned} \tag{2.2.16}$$

In the first step, we used the compatibility with the Poisson bracket. Then, we notice that the arguments of the Poisson bracket are independent of v. Next, we can replace h_i by ϕ_i because f is independent of u. Finally, we see that both equations coincide. □

Using the properties of the relational observable, we rewrite the physical Hamiltonian purely in terms of Dirac observables and an explicit time dependence $H(t) = \dot\tau^i h_i(O_r(\tau(t)), O_s(\tau(t)), \tau(t))$.

2.2.2 Gauge Fixing Viewpoint

Consider now the different point of view, where we fix the gauge, i.e. we impose the condition $G^i = u^i - \tau^i \stackrel{!}{=} 0$. In order that the gauge is preserved under time evolution, we must have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}G = \dot G^i + \{G^i, \lambda^i \phi_i\}\,. \tag{2.2.17}$$

This equation is equal to $\dot\tau^i = \{u^i, \lambda^j v_j\}$ and we have the solution $\lambda^i_* = \dot\tau^i$. The dynamics of the system is described using the reduced Hamiltonian.

Definition 2.10: Let f be any function on phase space, depending only on the degrees of freedom (r, s). The reduced Hamiltonian H is defined by

$$\{f, H\} = \left\{f, \lambda^i \phi_i\right\}_{\phi=0, G=0, \lambda=\lambda_*} \tag{2.2.18}$$

A straight-forward computation shows that the right hand side of (**2.2.18**) is equal to $\lambda^i_*\{f, h_i(r, s, \tau(t))\}$. Hence, the physical Hamiltonian is equal to $H(t) = \dot\tau^i h_i(r, s, \tau(t))$.

We conclude the discussion by relating the reduced and the relational observable formulation. Fixing the gauge $G = 0$ in the relational formalism, the physical Hamiltonian $H = \dot\tau^i h_i(O_r(\tau), O_s(\tau), \tau(t))$ simplifies to the reduced Hamiltonian because $O_r(\tau)\big|_{G=0} = r$ and similarly for $O_s(\tau)$. For the other way, we simply apply the map defining the relational observable to the reduced Hamiltonian H to recover the physical Hamiltonian.

To end this section and before considering field theories, let us discuss the example of a free relativistic particle:

Example 2.11 (Free Relativistic Particle)**:** Consider a free, relativistic particle in a spacetime (M, g). The worldline of the particle is a curve $\gamma : \mathbb{R} \to M$ with the tangent vectors $u^\mu(\tau)$. The action functional is given by

$$S[\gamma] = -m \int_{\mathbb{R}} \mathrm{d}\tau \sqrt{-g_{\mu\nu}(\gamma(\tau)) u^\mu(\tau) u^\nu(\tau)}\,. \tag{2.2.19}$$

The conjugate momentum is defined as the functional derivative of the action with respect to the velocity u^μ. We have

$$p_\mu(\tau) = \frac{\delta S}{\delta u^\mu(\tau)} = m \frac{g_{\mu\nu}(\gamma(\tau)) u^\nu}{\sqrt{-g_{\mu\nu}(\gamma(\tau)) u^\mu u^\nu}}\,. \tag{2.2.20}$$

It is easy to see that the momenta p_μ satisfy the constraint $\phi = p^\mu p_\mu + m^2 \stackrel{!}{=} 0$. Performing the Legendre transformation, we see that the Hamiltonian vanishes and we have a completely constrained system. Hence, the Hamiltonian is a sum of constraints given by

$$H = \lambda (p_\mu p^\mu + m^2)\,. \tag{2.2.21}$$

The constraint does not generate any additional, secondary constraints. Let us choose the spatial components (x^i, p_i) as the physical degrees of freedom and (x^0, p_0) as the gauge degrees of freedom. Solving the constraint ϕ, we obtain

$$\tilde{\phi} = p_0 + \sqrt{p^i p_i + m^2}\,, \tag{2.2.22}$$

where we chose the sign of the square root in order for the particle to have positive energy. For the gauge fixing we choose $G = x^0 - \tau$. In the relational framework, we obtain the Dirac observables

$$O_{x^0}(\tau) = \tau \qquad\qquad O_{p_0}(\tau) = p_0 \tag{2.2.23}$$

$$O_{x^i}(\tau) = x^i + (\tau - x^0) \frac{p^i}{\sqrt{p_j p^j + m^2}} \qquad\qquad O_{p_i}(\tau) = p_i \tag{2.2.24}$$

The physical Hamiltonian for the degrees of freedom $O_{x^i}(\tau)$ and $O_{p_i}(\tau)$ is given by

$$H = \sqrt{p_j p^j + m^2} \tag{2.2.25}$$

which agrees with the investigations above.

On the other hand consider the gauge fixed approach where we impose $G = x^0 - \tau$. Evaluating the stability condition for the gauge fixing, we find the equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau} G = -1 + \left\{ x^0, \lambda \tilde{\phi} \right\} = -1 + \lambda \tag{2.2.26}$$

The solution is $\lambda^* = 1$. Let us consider a function $f(x^i, p_i)$ of the physical degrees of freedom. Then,

$$\left\{ f, \lambda \tilde{\phi} \right\}_{\phi=0, G=0, \lambda=\lambda^*} = \frac{p^i}{\sqrt{p_j p^j + m^2}} \frac{\partial f}{\partial x^i} \tag{2.2.27}$$

As expected, the physical Hamiltonian is given by

$$H = \sqrt{p_j p^j + m^2} \tag{2.2.28}$$

2.3 Extension to Field Theories

In this section we extend the formalism introduced in the chapters before to the case of field theories. Most of the concepts directly apply in a straight forward way. Before the indices ran over a finite number of values which needs to be generalised to allow for an infinite number. The canonical variables are now labelled by discrete indices and a continuous index labeling the point x in space.

The derivation of the Hamiltonian theory from the Lagrangian works similarly. We obtain a list of constraints $\phi'_i(x)$ which depend both on a discrete label i and a continuous label x. For the Dirac algorithm, we have to be careful because for arbitrary field configurations, the symplectic structure and the constraints are not guaranteed to be finite. Additionally, it is not clear whether the constraints are functionally differentiable. To resolve the first problem, we provide boundary conditions on the fields such that both the constraints and the symplectic structure remain finite. Furthermore, for a set of constraints $\phi'_i(x)$ we use test functions $f^i(x)$ to smear the constraints

$$\phi'[f] := \int \mathrm{d}x\, f^i(x)\phi'_i(x)\,. \tag{2.3.1}$$

In order that $\phi'[f]$ remains finite, we have to impose boundary conditions on $f^i(x)$ as well.

In order to cure the possible issue of functional differentiability of the smeared constraints, we need to introduce boundary terms. The idea is to cancel the problematic terms of the variation occurring at the boundary with boundary terms that are added to the constraints. These terms are carefully chosen so that the contributions form the variation and the boundary term cancel out and the combination has a well defined variational derivative. In terms of the constraints $\phi'[f]$, we add a boundary term $B[f]$ and define

$$H[f] := \phi'[f] + B[f]\,. \tag{2.3.2}$$

The functions $H[f]$ have now well-defined functional derivatives. Proper gauge transformations correspond to those functions $f^i(x)$ such that $B[f] = 0$. Transformations with non-trivial $B[f]$ are called symmetry transformations.

In the remaining part, we discuss the relational formalism for field theories as a generalisation of the previous discussions. Consider a list of first class constraints $\{\phi'_i(x)\}_{i\in I}$ and canonical variables $(u^i(x), v_i(x))_{i\in I}$ and $(r^a(x), s_a(x))_{a\in A}$. Proceeding in the same way as before, we solve the constraints $\phi'_i(x) = 0$ for $v_i(x)$ as $v_i(x) = v_i^*(x) := -h_i(r, s, u; V)$ where $\{V_A\}_{A\in\mathscr{A}}$ are integration constants from solving differential equations. These constants appear as “new” degrees of freedom and have the interpretation of global degrees of freedom located at infinity. Certainly, the V_A are functions of the original variables v_i and do not contain any new information when what was already present before. Thus, without loss of generality we can consider canonical variables such that $\mathscr{A} \subset I$.

The boundary degrees of freedom v_A can be complemented by conjugate variables $\{U^A\}_{A\in\mathscr{A}}$ constructed from the u^i. Gauge fixing all of the u^i would be too restrictive, because the U^A are dynamical boundary degrees of freedom. We therefore only restrict the u^i which are not part of the U^A to fixed values with gauge fixing conditions of the form $G^i(x) = u^i(x) - \tau^i(x) = 0$ for all the $i \notin \mathscr{A}$. Then, similar to the case before, the Lagrange multipliers are determined by requiring that

$$\Big\{G^i(x), H[f]\Big\} = 0 \tag{2.3.3}$$

Here we do not consider a one-parameter family of gauge fixing conditions. The solution seems trivial at first, but there are in fact non-trivial solutions to the equation in field theory, because $H[f]$ involves differential operators. Therefore, we have a solution $f^i = f^i_*(r, s; V, \lambda)$ which depends on some integration constants λ^A.

Let $F(r, s; U, V)$ be any function of the true degrees of freedom. As before, the function F should have the same Poisson bracket with reduced Hamiltonian H and with the $H[f]$ when we restrict to the constraint surface ($v = v_*$), gauge cut ($u = u_*$) and solution of the stability condition ($f = f_*$). Explicitly,

$$\{F, H\} = \{F, H[f]\}_{u=u_*, v=v_*, f=f_*} \tag{2.3.4}$$

Before deriving the reduced Hamiltonian, we study the boundary term. Using suitable integration by parts, we can write it in the form

$$B[f] = \int_{\text{boundary}} f^A(x) j_A(x)\,, \tag{2.3.5}$$

with some boundary currents $j_A(x)$. The reduced Hamiltonian is determined by the following theorem (see Proposition in section 4.5.1 of [98]):

Theorem 2.12: Let $\chi[j]$ be a functional such that $f^A_* = \frac{\delta\chi[j]}{\delta j_A}\Big|_{j=j_*}$. Then, the physical Hamiltonian is given by $H = \chi[j_*]$.

Proof:
We start from the defining equation of the physical Hamiltonian and calculate for any function $F(r, s, U, V)$

$$\begin{aligned}
&\{F, H[f]\}_{u=u_*, v=v_*, f=f_*} = \{F, H[f]_{u=u_*, v=v_*}\}_{f=f_*} - \int \mathrm{d}x \left[\frac{\delta H[f]}{\delta u^i(x)}\Big\{F, u^i_*(x)\Big\} + \frac{\delta H[f]}{\delta v_i(x)}\{F, v^*_i(x)\}\right] \\
&= \{F, B[f]_{u=u_*, v=v_*}\}_{f=f_*} + \int \mathrm{d}x \left[\Big\{F, u^i_*(x)\Big\}\{v_i(x), H[f]\} - \{F, v^*_i(x)\}\Big\{u^i(x), H[f]\Big\}\right] \\
&= \int \mathrm{d}x\, f^A_*(x)\{F, j^*_A(x)\}
\end{aligned} \tag{2.3.6}$$

In the first step we brought the restriction to the constraint surface and to the gauge cut inside the Poisson bracket. The last two terms come from the subtraction of additional terms compensating the implicit dependence of $H[f]$ on (r, s) due to u_* and v_*. In the next step, the functional derivatives of $H[f]$ are rewritten using Poisson brackets. The second term vanishes because $u^i_*(x) = \tau^i(x)$ is a constant on phase space and the last term vanishes because of the stability condition of the gauge fixing $G^i(x) = 0$. We now replace $f^A_*(x)$ by $\frac{\delta\chi[j]}{\delta j_A(x)}\big|_{j=j_*}$ and obtain

$$\{F, H[f]\}_{u=u_*, v=v_*, f=f_*} = \int \mathrm{d}x\, \frac{\delta\chi[j]}{\delta j_A(x)}\bigg|_{j=j_*} \{F, j^*_A(x)\} = \{F, \chi[j_*]\} \tag{2.3.7}$$

We read off the reduced Hamiltonian $H = \chi[j_*]$. □

To summarize, for field theorise, the reduced Hamiltonian can be defined without considering one-parameter families of gauge fixings due to the presence of boundary terms. In this picture, the reduced Hamiltonian does not have any explicit time dependence and the system is conservative.

3

Review of Black Holes in Classical General Relativity

Black holes are regions of spacetime from which nothing can escape. In the vicinity of black holes the gravitational field becomes so strong that not even light can escape the black hole region. It is therefore of fundamental importance for our understanding of black holes to have a theory of gravity and our current theory is general relativity developed by Albert Einstein in 1915. In this section we will review basic concepts of general relativity and discuss black holes as the solutions of the symmetry reduced Einstein equations. More information on various topics, touched upon in this section, can be found in the standard textbooks on the subject [7, 104–106].

In the first part, we introduce basic concepts of differential geometry that are needed for the subsequent discussions. Then, we present the Einstein-Hilbert action for general relativity and obtain the field equations for the gravitational fields. We reduce the equations to spherical symmetry and obtain the famous Schwarzschild and Reissner-Nordstrøm solution for non-rotating black holes. We also shortly comment on the case of rotating black holes and the Kerr-Newman family of axial symmetric solutions of the Einstein equations.

Then, we discuss the Hamiltonian formulation of general relativity based on a foliation of spacetime into three dimensional hypersurfaces, first studied by Arnowitt, Deser and Misner [64]. Due to the invariance of the Einstein-Hilbert action under changes of coordinates we obtain a constrained Hamiltonian system and use the tools developed in section 2. Solving the dynamics for spherical symmetry, we obtain the Hamiltonian counterpart to the solutions obtained from the equations of motion.

Finally, we present important theorems and properties concerning classical black holes. We mention the famous singularity theorems by Hawking and Penrose showing that black holes with singularities in their centre are a consequence of generic gravitational collapse and a prediction of general relativity. In other words, singularities form after the formation of black holes not only in highly symmetric models, which are never exactly realised in nature, but also for generic initial data for the collapsing matter. We also discuss the area theorem stating that the area of the event horizon of a black hole can never decrease. We end with a surprising similarity between black holes and thermodynamics which remains rather mysterious in classical general relativity.

3.1 Basic Notions in Differential Geometry

In order to understand general relativity, we first have to provide some basic definitions about topology and differential geometry. This also sets the notation used throughout the rest of this thesis.

We begin by recalling some basic definitions of topological spaces which we are necessary for understanding the theory of manifolds and differential geometry:

Definition 3.1: A **topological space** $(X, \mathcal{O})$ is a set X together with a collection $\mathcal{O}$ of subsets of X. The sets in $\mathcal{O}$ are called open and they satisfy

1. $\emptyset, X \in \mathcal{O}$
2. For $A_1, \ldots, A_n \in \mathcal{O}$ we have $\cap_{i=1}^{n} A_i \in \mathcal{O}$
3. For $(A_i)_{i \in \mathcal{I}}$ with any index set $\mathcal{I}$ we have $\cup_{i \in \mathcal{I}} A_i \in \mathcal{O}$

Definition 3.2: Let $(X, \mathcal{O})$ be a topological space.

1. A subset N of X is called a **neighbourhood** of a point $x \in X$, if there exists an open set $A \in \mathcal{O}$ such that $x \in A \subset N$
2. $(X, \mathcal{O})$ is called **Hausdorff**, if for $x, x' \in X$ there exist neighbourhoods A of x and A' of x' such that $A \cap A' = \emptyset$

Given two topological spaces, we can investigate maps between them:

Definition 3.3: Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces.

1. A map $f : X \to Y$ is called **continuous**, if and only if $f^{-1}(A) \in X$ for $A \in Y$.
2. If f^{-1} exists and both f and f^{-1} are continuous then f is called a **homeomorphism**.

Finally, we need the notion of paracompactness. This is necessary such that we can define partitions of unity and such that the integration theory on manifolds is well defined.

Definition 3.4: Let $(X, \mathcal{O})$ be a topological space

1. An open cover $(A_i)_{i \in \mathcal{I}}$ of $(X, \mathcal{O})$ is a collection of open sets $(A_i \in \mathcal{O})$ such that $\cup_{i \in \mathcal{I}} A_i \supseteq X$
2. An open cover $(B_j)_{j \in \mathcal{J}}$ is called a refinement of an open cover $(A_i)_{i \in \mathcal{I}}$, if and only if for any B_j there exists an A_i such that $B_j \subset A_i$.
3. An open cover is locally finite, if and only if for any point $x \in X$ there exists an open neighbourhood N of x such that $N \cap A_i \neq \emptyset$ for only finitely many $i \in \mathcal{I}$
4. $(X, \mathcal{O})$ is called **paracompact**, if every open cover of X has a locally finite refinement.

A smooth manifold is a topological space which locally looks like $\mathbb{R}^n$. This is made mathematically precise with the introduction of an atlas in the following definition:

Definition 3.5: Let $\mathcal{M}$ be a paracompact, Hausdorff and finite-dimensional topological space. $\mathcal{M}$ is called a **smooth differentiable manifold**, if it is equipped with an atlas $(U_I, x_I)_{I \in \mathscr{I}}$, where

1. $U_I \subset \mathcal{M}$ open sets covering $\mathcal{M}$
2. $x_I : U_I \to \mathbb{R}^m$ homeomorphism called the coordinates, where m is called the dimension of $\mathcal{M}$

3. The maps $\varphi_{IJ} = x_J \circ x_I^{-1} : x_I(U_I \cap U_J) \to x_J(U_I \cap U_J)$ are smooth maps between subsets of $\mathbb{R}^m$.

The concept of a manifold allows for the introduction of a rich mathematical structure built upon the notion of smooth functions from the manifold to the complex numbers:

Definition 3.6: Let $\mathcal{M}$ be a smooth manifold with atlas $(U_I, x_I)_{I\in\mathscr{I}}$. We define

1. A smooth **function** f on $\mathcal{M}$ is a map $f : \mathcal{M} \to \mathbb{C}$ such that $f \circ x_I^{-1} : x_I(U_I) \to \mathbb{C}$ is smooth in the usual sense for all $I \in \mathscr{I}$. Notation: $f \in C^\infty(\mathcal{M})$

2. A smooth **vector field** v on $\mathcal{M}$ is a map $v : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, $f \mapsto v[f]$ such that the Leibniz rule is satisfied ($v[fg] = v[f]g + fv[g]$ for $f, g \in C^\infty(\mathcal{M})$) and constants are annihilated ($v[\lambda] = 0$ for $\lambda \in \mathbb{C}$). Notation: $v \in T^1_0(\mathcal{M})$

3. A smooth **1-form** ω on $\mathcal{M}$ is a map $\omega : T^1_0(\mathcal{M}) \to C^\infty(\mathcal{M})$, $v \mapsto \omega[v]$ that satisfies linearity ($\omega[fu + gv] = f\omega[u] + g\omega[v]$ with $f, g \in C^\infty(\mathcal{M})$ and $u, v \in T^1_0(\mathcal{M})$. Notation $\omega \in T^0_1(\mathcal{M})$

4. A smooth **tensor field** t on $\mathcal{M}$ is a map $t : (\Pi^a_{r=1} T^0_1(\mathcal{M})) \times (\Pi^b_{s=1} T^1_0(\mathcal{M})) \to C^\infty(\mathcal{M})$ defined by $((\omega_1, \dots, \omega_a), (v_1, \dots v_b)) \mapsto t(\omega_1, \dots, \omega_a, v_1, \dots, v_b)$. Notation: $t \in T^a_b(\mathcal{M})$

There is an interesting operation on the space of vector fields $T^1_0(\mathcal{M})$. Given two vector fields $v, w \in T^1_0(\mathcal{M})$ we define their **commutator** as $[u, v](f) := u[v[f]] - v[u[f]]$. One easily checks the Leibniz property and sees that this defines a new vector field $[u, v] \in T^1_0(\mathcal{M})$.

The definitions above are quite abstract and it is often useful to work in a chart where one has access to the maps $x_I : U_I \to \mathbb{R}^m$. This defines m functions on the manifold $\mathcal{M}$: $x^i_I : U_I \to \mathbb{R}$. Given these functions we define coordinate vector fields ∂^I_i by $\partial^I_j(x^i_I) = \delta^i_j$. Then, one shows that any vector field can be expanded in terms of the coordinate vector fields as $v = v^i\partial^I_i$, where v^i are the components of the vector field.

Given the coordinate vector fields ∂^I_i we define the coordinate 1-forms by $\mathrm{d}x^i_I : T^1_0(\mathcal{M}) \to \mathbb{C}$ by $\mathrm{d}x^i_I(\partial^I_j) = \delta^i_j$. Every one form ω is then uniquely expanded in terms of the coordinate one forms as $\omega = \omega_i \mathrm{d}x^i_I$. Using the basis for vector fields and 1-forms we can also expand tensor fields $t \in T^a_b(\mathcal{M})$ as $t = t^{i_1\dots i_a}{}_{j_1\dots j_b}\partial^I_{i_1} \otimes \dots \otimes \partial^I_{i_a} \otimes \mathrm{d}x^{j_1}_I \otimes \dots \otimes \mathrm{d}x^{j_b}_I$.

The construction above is performed with the reference to a given chart I in the atlas. However, the same construction can be performed in all the charts. For computations it is convenient to drop the label I in the above formulas and manipulate vector fields, 1-forms and tensors simultaneously in all charts. In addition, it is often useful to work with the components $t^{i_1\dots i_a}{}_{j_1\dots j_b}$ directly. This is called the **abstract index notation** and will be used throughout the computations in this thesis.

A useful concept in differential geometry are differential forms:

Definition 3.7: Let $\mathcal{M}$ be a smooth manifold with atlas (U_I, x_I). A smooth n-form ω is a tensor field $\omega \in T^0_n(\mathcal{M})$ which is completely anti-symmetric, i.e. for any permutation $\pi \in S_n$ and vector fields $v_1, \dots v_n \in T^1_0(\mathcal{M})$, we have

$$\omega[v_1, \dots, v_n] = \mathrm{sgn}(\pi)\omega[v_{\pi(1)}, \dots, v_{\pi(n)}] , \tag{3.1.1}$$

where $\mathrm{sgn}(\pi)$ is the sign of the permutation π. Notation: $\omega \in \Lambda_n(\mathcal{M})$.

On the space of n-forms we define the operations

1. Exterior Product: Let $\omega_1 \in \Lambda_{n_1}(\mathcal{M})$ and $\omega_2 \in \Lambda_{n_2}(\mathcal{M})$ then, we define a map $\wedge : \Lambda_{n_1} \times \Lambda_{n_2} \to \Lambda_{n_1+n_2}$ by

$$(\omega_1 \wedge \omega_2)[v_1, \ldots, v_{n_1+n_2}] = \frac{1}{n_1! n_2!} \sum_{\pi \in S_{n_1+n_2}} \omega_1[v_{\pi(1)}, \ldots, v_{\pi(n_1)}]\omega_2[v_{\pi(n_1+1)}, \ldots, v_{\pi(n_1+n_2)}] \,. \tag{3.1.2}$$

2. Exterior Derivative: Let $\omega \in \Lambda_n(\mathcal{M})$. The exterior derivative is a map $\mathrm{d} : \Lambda_n(\mathcal{M}) \to \Lambda_{n+1}(\mathcal{M})$ given by

$$(\mathrm{d}\omega)[v_0, \ldots v_n] = \sum_{k=0}^{n} (-1)^k v_k[\omega(v_0, \ldots, \hat{v}_k, \ldots, v_n)] + \sum_{0 \leq k < l \leq n} (-1)^{k+l} \omega([v_k, v_l], v_0, \ldots, \hat{v}_k, \ldots, \hat{v}_l, \ldots v_n) \,. \tag{3.1.3}$$

$\hat{v}^k$ means that the vector field v^k is omitted.

A diffeomorphism from $\mathcal{M}$ to $\mathcal{M}$ is a smooth map $\varphi : \mathcal{M} \to \mathcal{M}$ such that its inverse φ^{-1} exists and both φ, φ^{-1} are smooth. Every such diffeomorphism defines the pull-back and push-forward of functions, vector fields, 1-forms and tensor fields. First, the pull back of a function $f \in C^\infty(\mathcal{M})$ is defined by $\varphi^* f(p) = f(\varphi(p))$, $p \in \mathcal{M}$. Then, the push-forward of a vector field is defined by $(\varphi_* v)[f] = v[\varphi^* f]$ and the pull-back of a 1-form is defined by $\varphi^* \omega[v] = \omega[\varphi_* v]$. Since φ is a diffeomorphism, there exists an inverse φ^{-1} and we can also define the push-forward of functions and 1-forms ($\varphi_* f = (\varphi^{-1})^* f)$) and the pull-back of vector fields $\varphi^* v = (\varphi^{-1})_* v$. This allows us to define the pull-back and push-forward of general tensor fields $t \in T^a_b(\mathcal{M})$:

$$\begin{aligned} \varphi^* t(\omega_1, \ldots, \omega_a, v^1, \ldots v^b) &= t((\varphi^{-1})^* \omega_1, \ldots, (\varphi^{-1})^* \omega_a, \varphi_* v^1, \ldots \varphi_* v^b) \\ \varphi_* t(\omega_1, \ldots, \omega_a, v^1, \ldots v^b) &= t(\varphi^* \omega_1, \ldots, \varphi^* \omega_a, (\varphi^{-1})_* v^1, \ldots (\varphi^{-1})_* v^b) \end{aligned} \tag{3.1.4}$$

In coordinates, we have

$$(\varphi^* t)^{i_1 \ldots i_a}{}_{j_1 \ldots j_b} = t^{i'_1 \ldots i'_a}{}_{j'_1 \ldots j'_b} \frac{\partial (\varphi^{-1})^{i_1}}{\partial x^{i'_1}} \cdots \frac{\partial (\varphi^{-1})^{i_a}}{\partial x^{i'_a}} \frac{\partial \varphi^{j'_1}}{\partial x^{j_a}} \cdots \frac{\partial \varphi^{j'_b}}{\partial x^{j_b}} \tag{3.1.5}$$

Given a vector field $v \in T^1_0(\mathcal{M})$ we define its integral curve. This is a curve $c^v_p : I \subset \mathbb{R} \to \mathcal{M}$ such that $c^v_p(0) = p$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} x^i(c^v_p(t)) = v^i[c^v_p(t)] \tag{3.1.6}$$

The integral curves define a flow $\Phi^v_t : \mathcal{M} \to \mathcal{M}$ by $\Phi^v_t(p) := c^v_p(t)$. The **Lie derivative** of a tensor field $t \in T^a_b(\mathcal{M})$ is defined by

$$\mathcal{L}_v(t) = \lim_{\epsilon \to 0} \frac{(\Phi^v_\epsilon)^* t - t}{\epsilon} \tag{3.1.7}$$

In coordinates one finds

$$\begin{aligned} \mathcal{L}_v t = v^i \partial_i t^{i_1 \ldots i_a}{}_{j_1 \ldots j_b} - \partial_i v^{i_1} t^{i i_2 \ldots i_a}{}_{j_1 \ldots j_b} - \cdots - \partial_i v^{i_a} t^{i_1 \ldots i_{a-1} i}{}_{j_1 \ldots j_b} \\ + \partial_{j_1} v^i t^{i_1 \ldots i_a}{}_{i j_2 \ldots j_b} + \cdots + \partial_{j_b} v^i t^{i_1 \ldots i_a}{}_{j_1 \ldots j_{b-1} i} \end{aligned} \tag{3.1.8}$$

Definition 3.8: Let $\mathcal{M}$ be a smooth manifold with atlas $(U_I, x_I)_{I \in \mathscr{I}}$. A tensor field $g \in T^0_2(\mathcal{M})$ which is symmetric ($g[v, w] = g[w, v]$) and non-degenerate ($g[v, w] = 0$ for all w implies $v = 0$) is called a **pseudo-Riemannian metric**.

In abstract index notation the metric is denoted by g_{ij}. Symmetry implies that $g_{ij} = g_{ji}$ and the non-degeneracy allows for the definition of the inverse metric g^{ij} with $g^{ij}g_{jk} = \delta^i_k$.

3.2 Connections, Torsion and Curvature

So far, the only structures that allow for taking derivatives on manifolds are the exterior derivative for n-forms and the Lie derivative $\mathcal{L}_v$ that depends on a vector field v. For general tensor fields, there exists the notion of a connection that we will discuss in the following:

Definition 3.9: A **connection** is a map $\nabla : T^a_b(\mathcal{M}) \to T^a_{b+1}(\mathcal{M})$ satisfying for all $t_1, t_2 \in T^a_b(\mathcal{M})$, $\lambda_1, \lambda_2 \in \mathbb{C}$ and $f \in C^\infty(\mathcal{M})$

1. Linearity:
$$\nabla(\lambda_1 t_1 + \lambda_2 t_2) = \lambda_1 \nabla t_1 + \lambda_2 \nabla t_2 \tag{3.2.1}$$

2. Leibniz rule:
$$\nabla(t_1 \otimes t_2) = \nabla t_1 \otimes t_2 + t_1 \otimes \nabla t_2 \tag{3.2.2}$$

3. Consistency with action on functions: $\nabla f = \mathrm{d}f$

4. Commutes with contractions

It is useful to extend the abstract index notation to the covariant derivative. For any vector field $v \in T^1_0(\mathcal{M})$ and tensor $t \in T^a_b(\mathcal{M})$, we use the vector field to contract the extra index coming from the covariant derivative and we define
$$(\nabla_v t)^{j_1 \dots j_a}{}_{k_1 \dots k_b} := v^i (\nabla t)_i{}^{j_1,\dots j_a}{}_{k_1 \dots k_b} \tag{3.2.3}$$

Then, we evaluate the covariant derivative on the the basis ∂_i and define $\nabla_i := \nabla_{\partial_i}$. We introduce the connection coefficients Γ^k_{ij} by
$$\nabla_i \partial_j := \Gamma^k_{ij} \partial_k \tag{3.2.4}$$

On one forms, we have $\nabla_i \mathrm{d}x^j = -\Gamma^j_{ik} \mathrm{d}x^k$ which follows from the compatibility of the covariant derivative with contractions. In total for any tensor $t \in T^a_b(\mathcal{M})$ we have

$$\nabla_i t^{j_1 \dots j_a}{}_{k_1 \dots k_b} = \partial_i t^{j_1 \dots j_a}{}_{k_1 \dots k_b} + \sum_{l=1}^{a} \Gamma^{j_l}_{ik} t^{j_1 \dots k \hat{j}_l \dots j_a}{}_{k_1 \dots k_b} - \sum_{l=1}^{b} \Gamma^k_{ik_l} t^{j_1 \dots j_a}{}_{k_1 \dots k \hat{k}_l \dots k_b} \tag{3.2.5}$$

where the indices omitted with the hat are replaced by the index k.

Definition 3.10: The **torsion** $T \in T^1_2(\mathcal{M})$ of a connection ∇ is defined as

$$T[\omega, u, v] := \omega[\nabla_u v - \nabla_v u - [u, v]] \tag{3.2.6}$$

An explicit evaluation of this expression gives the coefficients

$$T^i{}_{jk} := T[\mathrm{d}x^i, \partial_j, \partial_k] := \Gamma^i_{jk} - \Gamma^i_{kj} \tag{3.2.7}$$

Definition 3.11: The **curvature** $R \in T^1_3(\mathcal{M})$ of a connection ∇ is defined as

$$R[\omega, u, v, w] := \omega[\nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w] \tag{3.2.8}$$

The components of the curvature tensor in terms of the connection coefficients are

$$R^i{}_{jkl} := R[\mathrm{d}x^i, \partial_k, \partial_l, \partial_j] = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{mk}\Gamma^m_{lj} - \Gamma^i_{ml}\Gamma^m_{kj} \tag{3.2.9}$$

For a vector field $v \in T^1_0(\mathcal{M})$, we have the identity

$$\nabla_i \nabla_j v^k - \nabla_j \nabla_i v^k = R^k{}_{lij} v^l \tag{3.2.10}$$

So far, the connection was arbitrary and we wish to constrain it using physical input. As it turns out, using the metric g and requiring the connection to be torsion-free there is a unique choice:

Theorem 3.12: Let $(\mathcal{M}, g)$ be a Pseudo-Riemannian manifold. Then, there exists a unique, torsion-free connection which is compatible with the metric ($\nabla g = 0$), the **Levi-Civita connection**.

An explicit evaluation of the metric compatibility condition gives an explicit formula for the connection coefficients:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}) \, . \tag{3.2.11}$$

This defines the unique Levi-Civita connection associated to the metric g.

Definition 3.13: Let $c : I \subset \mathbb{R} \to \mathcal{M}$, $t \mapsto c(t)$ be a curve on $\mathcal{M}$. For any function $f \in C^\infty(\mathcal{M})$, the tangent vector $\dot{c}$ of the curve c is defined by

$$\dot{c}[f] := \frac{\mathrm{d}}{\mathrm{d}t} f[c(t)] \, . \tag{3.2.12}$$

The curve is called a **geodesic**, if the tangent vector $\dot{c}$ satisfies $\nabla_{\dot{c}}(\dot{c}) = 0$.

3.3 General Relativity in Lagrangian Formulation

We introduced all the necessary notions of differential geometry to introduce Einstein's theory of general relativity based on the Lagrangian formulation. In this section, we define the Einstein-Hilbert action and derive the corresponding equations of motion using the variational principle.

In general relativity, the core quantity is the metric g in spacetime and the uniquely defined, torsion-free Levi-Civita connection. The dynamics is summarized by the **Einstein-Hilbert action**:

$$S[g, \phi] = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-\det g} (R[g] - 2\Lambda) + S_{\text{matter}}[g, \phi] \tag{3.3.1}$$

Here ϕ stands for any collection of matter fields and S_{matter} is the matter action depending on the metric and the matter fields. $\kappa = 16\pi$ is the gravitational coupling constant in units $G = c = 1$ and Λ is the cosmological constant. For the discussion of black holes in this thesis, we ignore this constant and set it to zero.

The equations of motion are derived by a variation of the action with respect to the inverse metric. Let us first study the variation of the metric determinant and the Ricci scalar. For the determinant we obtain

$$\delta\sqrt{-\det g} = -\frac{1}{2}\sqrt{-\det g}g_{\mu\nu}\delta g^{\mu\nu} \tag{3.3.2}$$

For the Ricci scalar, we use its definition in terms of the Ricci tensor and the Christoffel symbols

$$\begin{aligned}\delta R &= \delta g^{ij}R_{ij} + g^{ij}\delta R_{ij}\\ &= \delta g^{ij}R_{ij} + g^{ij}\delta R^{k}{}_{ikj}\\ &= \delta g^{ij}R_{ij} + g^{ij}\left(\nabla_k \delta\Gamma^k_{ij} - \nabla_j \delta\Gamma^k_{ki}\right)\\ &= \delta g^{ij}R_{ij} + \nabla_k\left(g^{ij}\delta\Gamma^k_{ij} - g^{ik}\delta\Gamma^j_{ji}\right)\end{aligned} \tag{3.3.3}$$

Inside the action, the second term is a total derivative and leads to a boundary term. We ignore this boundary term for now and only consider the first term. We have

$$\delta S[g,\phi] = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-\det g}\left(R_{ij} - \frac{1}{2}g_{ij}R + g_{ij}\Lambda - \kappa T_{ij}\right)\delta g^{ij} \tag{3.3.4}$$

In this equation, we introduced the energy-momentum tensor T_{ij} which arises from the variation of the matter action with respect to the metric and is defined as

$$T_{ij} := -\frac{2}{\sqrt{-\det(g)}}\frac{\delta S_{\text{matter}}}{\delta g^{ij}}\,. \tag{3.3.5}$$

Thus, we successfully derived the Einstein equations in their Lagrangian form:

$$R_{ij} - \frac{1}{2}g_{ij}R + \Lambda g_{ij} = \kappa T_{ij}\,, \tag{3.3.6}$$

In the following sections we study the equations in more detail. A general solution is not known and we simplify the equations with the help of symmetry assumptions.

3.4 Exact Solutions for Black Holes

Black holes are solutions to the Einstein equations imposing rotational and axial symmetry. For rotational symmetry we find the non-rotating, uncharged Schwarzschild black hole and the non-rotating and charged Reissner-Nordstrøm black hole. Imposing axial symmetry the solution is given by the uncharged Kerr black hole and the charged Kerr-Newman black hole. In these section we closely follow the notation in [74].

In this manuscript we mainly focus on rotational symmetry. An isometry of a spacetime $(\mathcal{M}, g)$ is a map $\psi : \mathcal{M} \to \mathcal{M}$ such that $\psi^* g = g$. That is ψ preservers the metric and thus distances and angles. The set of all isometries together with composition form a group. A spacetime $(\mathcal{M}, g)$ is called **spherically symmetric**, if its group of isometries contains the rotation group $SO(3)$. Furthermore, given a point $p \in \mathcal{M}$ the orbit of this point should be isomorphic to the sphere S^2.

Therefore, the spacetime factorises as $M \times S^2$, where M is an arbitrary 2-dimensional manifold and S^2 is the factor coming from the orbits of the isometries. On M we choose local coordinates x^a and on S^2 we choose $x^A = (\theta, \phi)$ as the usual polar coordinates. The general ansatz for the metric compatible with spherical symmetry is:

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b + \gamma^2(x^a)\Omega_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B\,, \tag{3.4.1}$$

where Ω_{AB} is the metric of the two-sphere S^2 and $\gamma(x^a)$ is a scalar function on M. In the following we raise and lower indices $a, b, c, \ldots$ with the metric g_{ab} and indices $A, B, C, \ldots$ with the metric Ω_{AB}. We define two covariant derivatives D and ∇ which are compatible with the metric Ω_{AB} on the sphere S^2 and the metric g_{ab} on M respectively. We would like to construct a metric compatible covariant derivative on $M \times S^2$ constructed from both D and ∇. It can be shown that the only non-vanishing Christoffel symbols with mixed indices are given by

$$\Gamma^a_{AB} = -\gamma^a\gamma\Omega_{AB} \tag{3.4.2}$$

$$\Gamma^A_{Ba} = \frac{\gamma_a}{\gamma}\delta^A_B, \tag{3.4.3}$$

where $\gamma_a := \nabla_a\gamma$.

The expression for the metric and the Christoffel symbols can be used to explicitly compute the curvature tensors. In fact, the only non-vanishing components of these tensors are:

$${}^{(4)}R_{ab} = {}^{(2)}R_{ab} - \frac{2}{\gamma}\nabla_a\nabla_b\gamma\,, \tag{3.4.4}$$

$${}^{(4)}R_{AB} = \Omega_{AB}(1 - \gamma_a\gamma^a - \gamma\Box\gamma)\,, \tag{3.4.5}$$

$${}^{(4)}R = {}^{(2)}R + \frac{2}{\gamma^2} - 2\frac{\gamma_a\gamma^a}{\gamma^2} - \frac{4}{\gamma}\Box\gamma\,. \tag{3.4.6}$$

In these equations $\Box = g^{ab}\nabla_a\nabla_b$ and ${}^{(2)}R_{ab}$, ${}^{(2)}R$ are the Ricci tensor and Ricci scalar of the manifold M respectively, determined by the metric g_{ab}.

Let us consider the gravity theory coupled to some matter content. We assume that it is provided to us in the form of an energy momentum tensor T_{ij}. For consistency, we require T_{ij} to be spherically symmetric. In general, it will have the following form

$$T_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = T_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b + T\gamma^2\Omega_{AB}\mathrm{d}x^A \otimes \mathrm{d}x^B\,. \tag{3.4.7}$$

This allows the formulation of Einstein's equations. We have

$$\begin{aligned} \kappa T_{ab} &= G_{ab} = -\frac{2}{\gamma}\nabla_a\nabla_b\gamma - g_{ab}\left(\frac{1}{\gamma^2} - \frac{\gamma_a\gamma^a}{\gamma^2} - \frac{2}{\gamma}\Box\gamma\right) \\ \kappa T &= G^A_A = \frac{\Box\gamma}{\gamma} - \frac{1}{2}{}^{(2)}R \end{aligned} \tag{3.4.8}$$

using that in two dimensions ${}^{(2)}R_{ab} = \frac{1}{2}g_{ab}{}^{(2)}R$. We have the following Bianchi identity:

$$\nabla_b T^{ba} + \frac{2}{\gamma}\gamma_b T^{ba} - \frac{2}{\gamma}\gamma^a T = 0\,. \tag{3.4.9}$$

In the following two subsections, we study solutions of the symmetry reduced Einstein equations. In the simplest case, we neglect the energy momentum tensor and study the vacuum solutions. We then extend the treatment by analysing the electromagnetic field as matter content.

3.4.1 Vacuum Case and the Schwarzschild Solution

For this section, we ignore any matter contributions and study the pure vacuum case, $T_{ij} = 0$. For the solution of the equations we consider the one-forms $\gamma_a \,\mathrm{d}x^a$, where $\gamma_a := \nabla_a \gamma$ and $t_a \,\mathrm{d}x^a$ where $t_a = \epsilon_{ab}\gamma^a$. We observe that they are orthogonal $g^{ab}\gamma_a t_b = 0$ and we have $\gamma^a\gamma_a = -t^a t_a$. Defining $f := \gamma^a\gamma_a$, we can write the metric as $g_{ab} = \frac{1}{f}(\gamma_a\gamma_b - t_a t_b)$ and the Levi-Civita symbol as $\epsilon_{ab} = -\frac{1}{f}(t_a\gamma_b - \gamma_a t_b)$.

Consider now the Einstein equations. Taking the trace of the first equation in (**3.4.8**), we have $\gamma\Box\gamma = 1-f$. Inserting this again into the first equation, we have

$$\nabla_a\gamma_b = \frac{1}{2\gamma} g_{ab}(1-f)\,. \tag{3.4.10}$$

Consider the quantity $\gamma(1-f)$. Taking the covariant derivative, we have

$$\nabla_a(\gamma(1-f)) = \gamma_a(1-f) - 2\gamma\gamma^b\nabla_a\gamma_b = \gamma_a(1-f) - \gamma_a(1-f) = 0, \tag{3.4.11}$$

and we have the solution $f = 1 - r_s/\gamma$.

Let's consider the Schwarzschild coordinates (t, r) and we require $\gamma = r$ and $g_{tr} = 0$. Then, we observe that $\gamma_a \mathrm{d}x^a = \mathrm{d}r$ and $t_a \mathrm{d}x^a = \epsilon_{ab}g^{bc}\gamma_c \mathrm{d}x^a = g^{rr}\mathrm{d}t$. Therefore, we have $g = -\frac{(g^{rr})^2}{f}\mathrm{d}t^2 + \frac{1}{f}\mathrm{d}r^2$ and using the explicit form of the inverse we have $g^{rr} = f$ and $g = -f\mathrm{d}t^2 + \frac{1}{f}\mathrm{d}r^2$.

Another interesting coordinate system are the Gullstrand-Painlevé (GP) coordinates (τ, r), where we impose $\gamma = r$ and $g_{rr} = 1$. Then, we have again $\gamma_a \mathrm{d}x^a = \mathrm{d}r$ and $t_a \mathrm{d}x^a = g^{rr}\mathrm{d}\tau - g^{tr}\mathrm{d}r$. Then, the metric is

$$g = -\frac{(g^{rr})^2}{f}\mathrm{d}\tau^2 + 2\frac{g^{rr}g^{tr}}{f}\,\mathrm{d}\tau\,\mathrm{d}r + \frac{1-(g^{tr})^2}{f}\,\mathrm{d}r^2 \tag{3.4.12}$$

From the requirement $g_{rr} = 1$, we have that $(g^{tr})^2 = 1 - f = r_s/r$. Inserting this into the equation for the metric we have

$$g = -\frac{(g^{rr})^2}{f}\mathrm{d}\tau^2 + 2\frac{g^{rr}g^{tr}}{f}\,\mathrm{d}\tau\,\mathrm{d}r + \mathrm{d}r^2 \tag{3.4.13}$$

Inverting the metric

$$g^{-1} = \begin{pmatrix} -\frac{f^2}{(g^{rr})^2} & -\frac{fg^{tr}}{g^{rr}} \\ -\frac{fg^{tr}}{g^{rr}} & f \end{pmatrix} \tag{3.4.14}$$

This equation has a consistent solution given by $g^{rr} = f$ and we have the metric in Gullstrand-Painlevé form:

$$g = -f\mathrm{d}\tau^2 \pm 2\sqrt{\frac{r_s}{r}}\mathrm{d}\tau\mathrm{d}r + \mathrm{d}r^2 \tag{3.4.15}$$

The sign depends on the sign we chose when taking the square root. It corresponds to the out- and ingoing GP coordinates.

3.4.2 Electromagnetic Matter and the Reissner-Nordstrøm solution

In spherical symmetry, there exists an exact solution for general relativity coupled to electromagnetic matter, called the Reissner-Nordstrøm solution. The electromagnetic field is described by the field strength tensor F_{ij}, an anti-symmetric tensor of rank 2. The equations of motion can be derived from the Maxwell-Lagrangian given by

$$S_{\mathrm{em}} = -\frac{\epsilon_0}{4}\int_{\mathcal{M}} \sqrt{-\det g}\, g^{ik}g^{jl}F_{ij}F_{kl} \tag{3.4.16}$$

From the definition of the energy-momentum tensor, we obtain

$$T_{ij} = \epsilon_0 \left(F_{ik} F_j{}^k - \frac{1}{4} g_{ij} F_{kl} F^{kl} \right) . \tag{3.4.17}$$

In spherical symmetry, the field strength tensor F_{ij} is parametrized by two functions $\alpha(t,r)$ and $\beta(t,r)$: $F_{ab} = \eta_{ab}\alpha(t,r)$ and $F_{AB} = \eta_{AB}\beta(t,r)$ with the antisymmetric tensor $\eta_{ab} = \sqrt{\det(g)}\epsilon_{ab}$ and similar for η_{AB}. In standard electromagnetism, without magnetic monopoles, the function β will not show up because there needs to exist a vector potential A such that $F = \mathrm{d}A$. Requiring this, it is not difficult to see that β has to vanish.

From the parametrization of the field strength tensor in terms of α, we obtain

$$T_{ab} = -\frac{\epsilon_0}{2} g_{ab} \alpha^2 , \tag{3.4.18}$$
$$T_{AB} = \frac{\epsilon_0}{2} \gamma^2 \Omega_{AB} \alpha^2 . \tag{3.4.19}$$

The next step is the solution of the equations of motion of the electric field. This is equivalent to solving the conservation equation of the energy momentum tensor $\nabla_i T^{ij} = 0$. We have the following equation:

$$\nabla_a \alpha^2 + 4 \frac{\gamma_a}{\gamma} \alpha^2 = 0 . \tag{3.4.20}$$

The solution to this equation is straight forward and given by $\alpha^2 = Q^2 \gamma^{-4}$, with the integration constant Q. This constant will be related to the electric charge of the black hole.

Consider now the Einstein equations. We take the trace of the first equation in (**3.4.8**) and get

$$g^{ab} \kappa T_{ab} = -\kappa \epsilon_0 \alpha^2 = \frac{2}{\gamma} \Box \gamma - 2 \frac{1-f}{\gamma^2} \tag{3.4.21}$$

This gives $\gamma \Box \gamma = 1 - f - \frac{\kappa}{2} \gamma^2 \alpha^2$. From the first equation it follows that

$$\nabla_a \gamma_b = \frac{1}{2\gamma} g_{ab} \left(1 - f - \frac{1}{2} \kappa \epsilon_0 \gamma^2 \alpha^2 \right) \tag{3.4.22}$$

Then, consider the covariant derivative of $\gamma(1-f)$:

$$\nabla_a (\gamma(1-f)) = \gamma_a (1-f) - 2\gamma \gamma^b \nabla_a \gamma_b = \gamma_a (1-f) - \gamma_a \left(1 - f - \frac{1}{2} \kappa \gamma^2 \alpha^2 \right) = \frac{1}{2} \kappa \epsilon_0 \gamma^2 \alpha^2 \gamma_a \tag{3.4.23}$$

Using the explicit expression for α in terms of γ and Q, the solution is $\gamma(1-f) = -\frac{1}{2\gamma} \kappa \epsilon_0 Q^2 + r_s$ where r_s is an integration constant. The full solution is then

$$f(\gamma) = 1 - \frac{r_s}{r} + \frac{\kappa \epsilon_0}{2\gamma^2} Q^2 \tag{3.4.24}$$

The electric charge of the black hole is given by $(4\pi\epsilon_0)Q$.

Choosing Schwarzschild like coordinates (t, r) the computations in the previous section give the metric

$$g = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 \tag{3.4.25}$$

In Gullstrand-Painlevé coordinates (τ, r), we have

$$g = -f(r)\,\mathrm{d}\tau^2 \pm 2\sqrt{1-f(r)}\mathrm{d}\tau\mathrm{d}r + \mathrm{d}r^2 \tag{3.4.26}$$

3.4.3 Rotating Black Holes and Axial Symmetry

In the previous sections, we assumed the solution of the Einstein equations to be spherically symmetric. We found the Schwarzschild and Reissner-Nordstrøm solutions which both admit a timelike Killing vector ∂_t. A Killing vector v is a vector field such that $\mathcal{L}_v(g)_{ij} = 0$ and a vector field v is called timelike, if $g(v,v) < 0$. Recall the following definition of static and stationary spacetimes:

Definition 3.14: A spacetime $(\mathcal{M}, g)$ is called **stationary**, if and only if there exists a timelike Killing vector field T. The spacetime is called **static**, if it is stationary and T is orthogonal to a family of hypersurfaces.

In the previous section, the Einstein equations imply the metric to be static (Birkhoff's theorem). However, spherical symmetry is a strong restriction because in the universe black holes are expected to rotate and thus break spherical symmetry. The symmetry generalises to axial symmetry around the axis of rotation and we have to find new solutions to the Einstein equations. The situation for rotating black holes is much more involved and there is no equivalent theorem to Birkhoff's theorem.

The most general static black hole solution known until today is characterised by charge Q, mass M and angular momentum J is the **Kerr-Newman black hole** [104, 105, 107]. In Boyer-Lindquist coordinates (t, r, θ, ϕ) the metric reads

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t)^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2\,. \tag{3.4.27}$$

For the metric, we defined $\Delta := r^2 - 2Mr + a^2 + Q^2$ and $\rho^2 := r^2 + a^2\cos^2\theta$. The parameter a is related to the angular momentum of the black hole by $a = J/M$. The non-vanishing components of the vector potential are given by $A_t = -rQ/\rho^2$ and $A_\phi = arQ\sin^2\theta/\rho^2$. Taking the limit $a \to 0$, we recover the Reissner-Nordstrøm solution for charged non-rotating black holes. The Kerr-Newman metric is therefore a generalisation of the Reissner-Nordstrøm metric to include rotation of the black hole.

The Kerr-Newman black hole also admits a rich symmetry structure: The metric admits two Killing vector fields ∂_t and ∂_ϕ since the metric coefficients are independent of the coordinates t and ϕ. The presence of the timelike Killing vector ∂_t shows that the metric is stationary. However, due to the rotation of the black hole the metric is not static. Additionally, the metric admits a Killing tensor $K_{\mu\nu}$ (see [108], i.e. a symmetric tensor satisfying $\nabla_{(\mu}K_{\nu\rho)} = 0$). It is due to this additional symmetry, that the geodesic equations for rotating black holes are completely integrable.

In section 5, we consider corrections to the highly symmetric black hole solutions for spherically symmetric spacetimes. This analysis for the case of rotating black holes is more challenging and has first been executed for the Kerr-Newman black holes by Teukolsky in [109, 110]. The calculation is based on the Newman Penrose formalism and two equations for the two physical degrees of freedom of the gravitational field were derived. Due to the increased complexity of axial symmetric black hole solutions, we will restrict to spherically symmetric spacetimes in this thesis.

3.5 General Relativity in Hamiltonian Formulation

In this section we present the Hamiltonian formulation of general relativity based on the ADM formulation first introduced by Arnowitt, Deser and Misner in [64]. A review, on which this section is based can be found in [24]. The idea is to use a foliation of spacetime by three dimensional hypersurfaces so that we can define a Hamiltonian generating time evolution from one hypersurface to the next.

We recall the following definition of Cauchy surfaces and globally hyperbolicity necessary to introduce foliations of spacetime:

Definition 3.15: Let $(\mathcal{M}, g)$ be a spacetime. Then

1. A curve $c : I \subset \mathbb{R} \to \mathcal{M}$ is called **causal** if its tangent vectors $\dot{c}$ are timelike or null $g(\dot{c}, \dot{c}) \leq 0$.

2. A spacelike hypersurface Σ inside a spacetime $(\mathcal{M}, g)$ is called **Cauchy surface**, if and only if every inextendible causal curve intersects Σ exactly once.

3. A spacetime $(\mathcal{M}, g)$ with a Cauchy surface Σ is called **globally hyperbolic**.

In the following, we assume $(\mathcal{M}, g)$ to be globally hyperbolic. Then a theorem due to Geroch [104], implies that we can foliate $\mathcal{M}$ into non-intersecting hypersurfaces $\Sigma_t \cong \Sigma$. The hypersurfaces Σ_t are Cauchy surfaces and isomorphic to Σ. Thus, the topology of spacetime is $\mathcal{M} = \mathbb{R} \times \Sigma$ and we choose local coordinates x^μ on Σ. Consider now an embedding $X : \mathbb{R} \times \Sigma \to \mathcal{M}$, $(t, x) \to X_t(x)$, where $X_t(\Sigma) = \Sigma_t$.

The surfaces Σ_t have timelike normals $n^i \partial_i$ with $g_{ij} n^i n^j = -1$ and the vector fields $X^i_{,\mu} \partial_i$ are tangential to Σ_t. The vector field $T^i \partial_i := \partial_t X^i \partial_i$ can be decomposed into its tangential and normal components:

$$T^i = N n^i + N^\mu X^i_{,\mu} \,. \tag{3.5.1}$$

N is called the **lapse function** and N^μ is the the **shift vector field**. With this information we can pull back the metric to $\mathbb{R} \times \Sigma$ and obtain

$$\begin{aligned} \mathrm{d}^2 s &= g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j \\ &= g_{ij}[(N n^i + N^\mu X^i_{,\mu})\, \mathrm{d}t + X^i_{,\mu}\, \mathrm{d}x^\mu][(N n^j + N^\nu X^j_{,\nu})\, \mathrm{d}t + X^j_{,\nu}\, \mathrm{d}x^\nu] \\ &= -(N^2 - N^\mu N^\nu g_{ij} X^i_{,\mu} X^j_{,\nu})\, \mathrm{d}t^2 + 2 N N^\mu g_{ij} X^i_{,\mu} X^j_{,\nu}\, \mathrm{d}t\, \mathrm{d}x^\nu + g_{ij} X^i_{,\mu} X^j_{,\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \end{aligned} \tag{3.5.2}$$

In the computation we used that $g_{ij} n^i n^j = -1$ and that $g_{ij} n^i X^j_{,\mu} = 0$.

It is convenient to introduce the first and second fundamental form for hypersurfaces:

Definition 3.16: The **first fundamental form** (induced metric) on the hypersurfaces Σ is defined by

$$m_{ij} = g_{ij} + n_i n_j \,. \tag{3.5.3}$$

The **second fundamental form** is defined by

$$K_{ij} = m_i^k m_j^l \nabla_k n_l \tag{3.5.4}$$

The first fundamental form has the property that $m_{ij} n^i = 0$ using the normalisation of the normal n^i and similarly in the second argument. This shows that m is purely spatial and we pull it back onto the hypersurfaces. We define $m_{\mu\nu} := m_{ij} X^i_{,\mu} X^j_{,\nu}$ and the metric simplifies to

$$\mathrm{d}s^2 = -(N^2 + m_{\mu\nu} N^\mu N^\nu)\mathrm{d}t^2 + 2NN^\mu m_{\mu\nu} \mathrm{d}t \mathrm{d}x^\nu + m_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{3.5.5}$$

For the formulation of general relativity we also need to split the curvature tensor into the part defined by the metric m within the hypersurface and the extrinsic components describing how the hypersurfaces bend inside the surrounding four-dimensional spacetime. The induced metric $m_{\mu\nu}$ defines a unique torsion-free connection $\overline{\nabla}$ within the surfaces Σ. This connection defines a Riemann curvature tensor and we have to understand its relation to the four-dimensional Riemann curvature tensor.

Theorem 3.17 (Gauß-Codazzi equations [106])**:** The projections of the four-dimensional Riemann tensor in tangential and normal directions are the **Gauß-Codazzi equations**:

$$m_i^{i'} m_j^{j'} m_k^{k'} m_l^{l'}\, {}^{(4)}R_{i'j'k'l'} = {}^{(3)}R_{ijkl} + K_{il} K_{jk} - K_{ik} K_{jl} \tag{3.5.6}$$

$$m_i^{i'} m_j^{j'} m_k^{k'}\, {}^{(4)}R_{i'j'k'l} n^l = (\overline{\nabla}_i K_{jk} - \overline{\nabla}_j K_{ik}) \tag{3.5.7}$$

These two projections of the Riemann tensor are the ones that can be written in terms of the intrinsic and extrinsic geometry of the hypersurfaces alone. For the remaining projection along the two normal directions, we additionally need the acceleration of the normal defined by $a_i = \nabla_n n_i$. We observe that a_i is a spatial quantity because $n^i a_i = n^i \nabla_n n_i = \frac{1}{2} \nabla_n (n^i n_i) = 0$. One shows that $\nabla_i n_j = K_{ij} - n_i a_j$ and can derive an expression for the projection of the curvature into the two normal directions (see [111]). We will not present it here as it is not necessary for the further discussions. Due to the symmetry of the Riemann tensor, this exhausts the full list of projections of the Riemann tensor for codimension 1 hypersurfaces.

Let's consider the projection of the Ricci scalar along the two normal directions:

$$\begin{aligned} n^i n^j\, {}^{(4)}R_{ij} &= {}^{(4)}R_{kilj} n^i n^j g^{kl} = n^i [\nabla_j, \nabla_i] n^j \\ &= \nabla_i n^i \nabla_k n^k - \nabla_k n^i \nabla_i n^k + \nabla_i (a^i - n^i \nabla_k n^k) \\ &= (K_i^i)^2 - K^{ij} K_{ij} + \nabla_i (a^i - n^i K) \end{aligned} \tag{3.5.8}$$

The Ricci scalar is obtained by contracting indices of the Riemann tensor:

$$\begin{aligned} {}^{(4)}R = {}^{(4)}R_{ijkl} g^{ik} g^{jl} &= {}^{(4)}R_{ijkl} m^{ik} m^{jl} - 2\, {}^{(4)}R_{ijkl} m^{ik} n^j n^l \\ &= {}^{(3)}R + K^{ij} K_{ij} - (K_i^i)^2 - 2\nabla_i (a^i - n^i K) \end{aligned} \tag{3.5.9}$$

The total derivative will be neglected for now as it only contributes a boundary term to the action. Pulling the tensors back to the hypersurfaces, we obtain

$${}^{(4)}R = {}^{(3)}R + K^{\mu\nu} K_{\mu\nu} - (m^{\mu\nu} K_{\mu\nu})^2 - 2\nabla_\mu (a^\mu - n^\mu K) \tag{3.5.10}$$

Inserting everything into the Einstein-Hilbert action and dropping boundary terms, we obtain the **ADM action**:

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}x\, |N| \sqrt{\det m} \Big(R + K_{\mu\nu} K^{\mu\nu} - (m^{\mu\nu} K_{\mu\nu})^2 \Big) \tag{3.5.11}$$

From now on, we assume future oriented foliations with $N > 0$ so that the absolute value of N can be replaced by N itself.

The Hamiltonian formulation of general relativity based on the ADM action is derived using a Legendre transformation of (3.5.11). We introduce the momenta $W^{\mu\nu}$, Π and Π_μ by

$$\begin{aligned} W^{\mu\nu} &:= \frac{\delta S}{\delta \dot{q}_{\mu\nu}} = \frac{\sqrt{\det m}}{\kappa}(K^{\mu\nu} - m^{\mu\nu} m^{\rho\sigma} K_{\rho\sigma}) \\ \Pi &:= \frac{\delta S}{\delta \dot{N}} = 0 \\ \Pi_\mu &:= \frac{\delta S}{\delta \dot{N}^\mu} = 0 \end{aligned} \tag{3.5.12}$$

We obtain two primary constraints $\Pi = 0$ and $\Pi_\mu = 0$. The first equation can be used to determine the velocities $\dot{m}_{\mu\nu}$ in terms of $W^{\mu\nu}$. From the first equation we have that

$$m_{\mu\nu} W^{\mu\nu} = -2\frac{\sqrt{\det m}}{\kappa} m^{\mu\nu} K_{\mu\nu} \tag{3.5.13}$$

Then, we can solve for $K_{\mu\nu}$ and obtain

$$K^{\mu\nu} = \frac{\kappa}{\sqrt{\det m}}\left(W^{\mu\nu} - \frac{1}{2} m^{\mu\nu} m_{\rho\sigma} W^{\rho\sigma}\right) \tag{3.5.14}$$

We find the primary Hamiltonian

$$\begin{aligned} H &= \int_\Sigma \mathrm{d}x\, \dot{m}_{\mu\nu} W^{\mu\nu} + \dot{N}\Pi + \dot{N}^\mu \Pi_\mu - \frac{\sqrt{\det m} N}{\kappa}\left(R + K_{\mu\nu}K^{\mu\nu} - (m^{\mu\nu}K_{\mu\nu})^2\right) \\ &= \int_\Sigma \mathrm{d}x\, \dot{N}\Pi + \dot{N}^\mu \Pi_\mu + 2\overline{\nabla}_\mu N_\nu W^{\mu\nu} + \frac{N}{\kappa}\left[\frac{\kappa^2}{\sqrt{\det m}}\left(W_{\mu\nu}W^{\mu\nu} - \frac{1}{2}(m^{\mu\nu}W_{\mu\nu})^2\right) - \sqrt{\det m} R\right] \end{aligned} \tag{3.5.15}$$

The third term is integrated by parts and we have

$$H = \int_\Sigma \mathrm{d}x\, \dot{N}\Pi + \dot{N}^\mu \Pi_\mu - 2N_\mu \overline{\nabla}_\nu W^{\mu\nu} + \frac{N}{\kappa}\left[\frac{\kappa^2}{\sqrt{\det m}}\left(W_{\mu\nu}W^{\mu\nu} - \frac{1}{2}(m^{\mu\nu}W_{\mu\nu})^2\right) - \sqrt{\det m} R\right] \tag{3.5.16}$$

On the phase space, the only non-vanishing Poisson brackets are

$$\{m_{\mu\nu}(t,x), W^{\rho\sigma}(t,y)\} = \kappa \delta^\rho_{(\mu}\delta^\sigma_{\nu)}\delta(x,y), \quad \{N(t,x), \Pi(t,y)\} = \delta(x,y), \quad \{N^\mu(t,x), \Pi_\nu(t,y)\} = \delta^\mu_\nu \delta(x,y) \tag{3.5.17}$$

Thus, we successfully derived the primary Hamiltonian H and the constraints $\Pi \overset{!}{=} 0$ and $\Pi_\mu \overset{!}{=} 0$. Following Dirac's analysis outlined in section 2, we need to impose the stability of the primary constraints. Taking Poisson brackets with the Hamiltonian we find

$$\{\Pi(x), H\} = V_0(x), \quad \{\Pi_\mu(x), H\} = V_\mu(x) \tag{3.5.18}$$

where we defined the secondary constraints

$$\begin{aligned} V_0(x) &= \frac{\kappa}{\sqrt{\det m}}\left(W_{\mu\nu}W^{\mu\nu} - \frac{1}{2}(m^{\mu\nu}W_{\mu\nu})^2\right) - \frac{1}{\kappa}\sqrt{\det m} R \\ V_\mu(x) &= -2 m_{\mu\rho} D_\nu W^{\nu\rho} \end{aligned} \tag{3.5.19}$$

We have to add these constraints to the list of constraints and check their stability again. For that it is useful to study the algebra of V_0 and V_μ. The computation is most transparent by introducing "smeared" constraints. We define $V_0[f] := \int_\Sigma \mathrm{d}x\, f(x) V_0(x)$ and $\vec{V}[\vec{f}] = \int_\Sigma f^\mu(x) V_\mu(x)$. We obtain the hypersurface deformation algebra

$$\begin{aligned}\left\{\vec{V}[\vec{f}], \vec{V}[\vec{g}]\right\} &= \kappa \vec{V}[[\vec{f}, \vec{g}]] \\ \left\{\vec{V}[\vec{f}], V_0[g]\right\} &= \kappa V_0[\vec{f}[g]] \\ \{V_0[f], V_0[g]\} &= \kappa \vec{V}[m^{-1}(f\mathrm{d}g - g\mathrm{d}f)]\end{aligned} \tag{3.5.20}$$

Using this algebra it is easy to see that the constraints V, V_μ do not generate further secondary constraints.

The lapse function N and shift vector N^μ are completely arbitrary functions. This allows for a simplification of the canonical framework by treating N, N^μ as Lagrange multipliers and we move to the reduced action (**canonical ADM action**):

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}x \left[\dot{m}_{\mu\nu} W^{\mu\nu} - N^\mu V_\mu - N V_0\right] \tag{3.5.21}$$

In section 2, we saw that in the case of field theories (such as general relativity) we have to be careful for spacetimes with boundaries. For a well-defined variational principle boundary terms have to be added to the constraints in order to make the variational principle well-defined. Additionally, the constraints and symplectic structure need to remain finite and we have to specify suitable fall-off behaviours for the canonical variables.

In the ADM action, we have the canonical variables $m_{\mu\nu}$ and $W^{\mu\nu}$. We impose asymptotic flat boundary conditions on them which in Cartesian coordinates are given by

$$\begin{aligned} m_{\mu\nu} &= \delta_{\mu\nu} + \frac{1}{r} f^+_{\mu\nu}(\Omega) + \frac{1}{r^2} f^-_{\mu\nu}(\Omega) \\ W^{\mu\nu} &= \frac{1}{r^2} F^{\mu\nu}_-(\Omega) + \frac{1}{r^3} F^{\mu\nu}_+(\Omega)\end{aligned} \tag{3.5.22}$$

The fall-off condition depends on the parity, where $(+)$ is positive and $(-)$ is negative parity. Let P be the parity operator defined by $Px = -x$. Then, $f^\pm_{\mu\nu}(P\Omega) = \pm f^\pm_{\mu\nu}(\Omega)$ and similarly for $F^{\mu\nu}_\pm$.

In the following we would like to perform a careful boundary term analysis of the constraints. Adapted to the spherical symmetry of the Schwarzschild and Reissner-Nordstrøm black holes, we transform the asymptotically flat boundary conditions to spherical coordinates. Additionally, we would like to work in Gullstrand-Painlevé coordinates for the spherically symmetric sector. This is not compatible with the asymptotic flat boundary conditions we defined above and we need to modify the fall-off conditions of the spherically symmetric sector. The discussion in the following paragraphs is based on [98] where more details can be found.

Transforming to spherical coordinates we find the following asymptotic behaviour

$$\begin{aligned} m_{33} &= 1 + \frac{1}{r} f^+_{33}(\Omega) + \frac{1}{r^2} f^-_{33}(\Omega) \\ m_{3A} &= 0 + f^+_{3A}(\Omega) + \frac{1}{r} f^-_{3A}(\Omega) \\ m_{AB} &= r^2 \Omega_{AB} + r f^+_{AB}(\Omega) + f^-_{AB}(\Omega)\end{aligned} \tag{3.5.23}$$

$$W^{33} = \sqrt{\Omega}F^{33}_-(\Omega) + \sqrt{\Omega}\frac{1}{r}F^{33}_+(\Omega)$$
$$W^{3A} = \sqrt{\Omega}\frac{1}{r}F^{3A}_-(\Omega) + \sqrt{\Omega}\frac{1}{r^2}F^{3A}_+(\Omega)$$
$$W^{AB} = \sqrt{\Omega}\frac{1}{r^2}F^{AB}_-(\Omega) + \sqrt{\Omega}\frac{1}{r^3}F^{AB}_+(\Omega)$$

In this manuscript we would like to work in Gullstrand-Painlevé type coordinates. Recall the Schwarzschild metric in Gullstrand-Painlevé coordinates from (**3.4.15**):

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}\tau^2 + 2\sqrt{\frac{r_s}{r}}\,\mathrm{d}\tau\,\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{3.5.24}$$

We need to match this metric with the metric in ADM variables in equation (**3.5.5**). We observe that we recover the Gullstrand-Painlevé metric for $N = 1$, $N^3 = \sqrt{r_s/r}$, $N^A = 0$, $m_{33} = 1$, $m_{3A} = 0$ and $m_{AB} = r^2\Omega_{AB}$. Note that $m_{\mu\nu}$ is compatible with the choice for the asymptotics of $m_{\mu\nu}$ in spherical coordinates. For the momenta, we compute

$$\begin{aligned} K_{33} &= -\frac{1}{2}(N^3\partial_3 m_{33} + 2m_{33}\partial_r N^3) = \frac{1}{2}\sqrt{\frac{r_s}{r^3}} \\ K_{AB} &= -\frac{1}{2}N^3\partial_3 m_{AB} = -\sqrt{r r_s}\Omega_{AB} \end{aligned} \tag{3.5.25}$$

where we used

$$K_{ij} = \frac{1}{2N}\left(\dot{m}_{ij} - N^k m_{ij,k} - N^k{}_{,i}m_{kj} - N^k{}_{,j}m_{ik}\right) \tag{3.5.26}$$

Then, the only non-vanishing momenta are:

$$W^{33} = 2\sqrt{r r_s}, \quad W^{AB} = \frac{1}{2r^2}\sqrt{r r_s}\Omega^{AB} \tag{3.5.27}$$

where we used formula (**3.5.12**) and $g^{ij}K_{ij} = -\frac{3}{2}\sqrt{\frac{r_s}{r^3}}$.

Let's assume we have performed the transformation from Schwarzschild to Gullstrand-Painlevé coordinates. We have to suitably adapt the fall-off conditions in (**3.5.23**) to this new coordinate system. Since the transformation only affects the spherically symmetric sector, we distinguish in the following between the spherically symmetric degrees of freedom and the non-symmetric ones. We use the following notation:

$$\begin{aligned} m_{33} = e^{2\mu} + x^v, \quad m_{3A} = 0 + x_A, \quad m_{AB} = e^{2\lambda}\Omega_{AB} + x^h\Omega_{AB} + X_{AB} \\ W^{33} = \frac{1}{2}e^{-2\mu}\pi_\mu + y_v, \quad W^{3A} = 0 + \frac{1}{2}y^A, \quad W^{AB} = \frac{1}{4}e^{-2\lambda}\pi_\lambda + \frac{1}{2}y_h\Omega^{AB} + Y^{AB} \end{aligned} \tag{3.5.28}$$

The pairs (μ, π_μ) and (λ, π_λ) are the spherically symmetric part of m and W. The variables $(x^{v/h/e/o}, y_{v/h/e/o})$ and $(X^{e/o}, Y_{e/o})$ are non-spherically symmetric.

For the non-symmetric variables, we immediately obtain the fall-off conditions:

$$x^v = \frac{1}{r}(x^v_\infty)^+ + \frac{1}{r^2}(x^v_\infty)^-, \quad y^v = (y^v_\infty)^- + \frac{1}{r}(y^v_\infty)^+$$
$$x^A = (x^A_\infty)^+ + \frac{1}{r}(x^A_\infty)^-, \quad y^A = \frac{1}{r}(y^A_\infty)^- + \frac{1}{r^2}(y^A_\infty)^+$$
$$x^h = r(x^v_\infty)^+ + (x^v_\infty)^-, \quad y^h = \frac{1}{r^2}(y^h_\infty)^- + \frac{1}{r^3}(y^h_\infty)^+$$

$$X^{AB} = r(X_\infty^{AB})^+ + (X_\infty^{AB})^-, \quad Y^{AB} = \frac{1}{r^2}(Y_\infty^{AB})^- + \frac{1}{r^3}(Y_\infty^{AB})^+$$

Based on the Gullstrand-Painlevé metric and the corresponding momenta $W^{\mu\nu}$, we assume the following fall-off conditions on the spherically symmetric degrees of freedom

$$\mu = 0 + \frac{1}{r^{5/2}}\mu_\infty, \quad \lambda = \log(r) + \frac{1}{r^{5/2}}\lambda_\infty, \quad \pi_\mu = \sqrt{\Omega}\pi_\mu^\infty\sqrt{r}, \quad \pi_\lambda = \sqrt{\Omega}\pi_\lambda^\infty\sqrt{r} \tag{3.5.29}$$

The fall-off conditions of the symmetric and non-symmetric variables make the symplectic term of the action well defined:

$$\int \mathrm{d}\Sigma W^{\mu\nu}\dot{m}_{\mu\nu} \tag{3.5.30}$$

It converges as r tends to infinity because it is of order r^{-2}. The terms of order r^{-1} vanish because they correspond to an integral over the sphere of an integrand with odd parity.

Let us now consider the smeared constraints $\vec{V}[\vec{f}]$ and $V_0[f]$. For the variation of the diffeomorphism constraint, we obtain

$$\begin{aligned} \delta\vec{V}[\vec{f}] &= \int \mathrm{d}\Sigma\, f^\mu \delta[m_{\nu\rho,\mu}W^{\nu\rho} - 2\partial_\rho(m_{\mu\nu}P^{\nu\rho})] \\ &= \int \mathrm{d}\Sigma \left[\delta W^{\mu\nu}(\mathcal{L}_{\vec{f}}m)_{\mu\nu} - (\mathcal{L}_{\vec{f}}W)^{\mu\nu}\delta m_{\mu\nu}\right] + \int_{\partial\Sigma} \mathrm{d}\Sigma_\rho[f^\rho W^{\mu\nu}\delta m_{\mu\nu} - 2f^\mu\delta(m_{\mu\nu}W^{\nu\rho})] \end{aligned} \tag{3.5.31}$$

where we used the Lie derivative $\mathcal{L}_{\vec{f}}$ with respect to the vector field f^i. The boundary integrals are evaluated over an asymptotic sphere at $r = \infty$ and the outward normal is pointing in the radial direction $\mathrm{d}\Sigma_\rho = \delta_\rho^3\,\mathrm{d}\Omega$. The boundary terms are given by

$$\int_{\partial\Sigma} \mathrm{d}\Omega\left[f^3W^{\mu\nu}\delta m_{\mu\nu} - 2f^\mu\delta\left(m_{\mu\nu}W^{3\nu}\right)\right] \tag{3.5.32}$$

In the first boundary term, the symmetric part of $W^{ij}\delta m_{ij}$ behaves as r^{-2} and the even non-symmetric part behaves as r^{-2} and for the odd non-symmetric part we have r^{-1}. Therefore, we have to choose $f^3 = r(f_\infty^3)^+ + (f_\infty^3)^-$ in order for the boundary term to vanish in the limit. Then, provided that the second contribution to the boundary term is finite, we have written the boundary term as a total variation.

In the second boundary term, the spherically symmetric contribution of $\delta(m_{\mu\nu}W^{3\nu})$ grows like the $r^{1/2}$ and for a finite result we require a stronger fall-off behaviour for the spherically symmetric contribution f^3_{sym}. We choose $f^3_{\mathrm{sym}} = O(r^{-1/2})$. For the nonsymmetric contributions to $\delta(m_{\mu\nu}W^{3\nu})$ we find that for $\mu = 3$ the term behaves as constant for the odd and like r^{-1} for the even contributions. For the fall-off of f^3 as above it gives a finite limit as $r \to \infty$. For $\mu = A$, the term behaves as r for the odd parity and as a constant for the even parity. Therefore, we have to choose $f^A = (f_\infty^A)^+ + r^{-1}(f_\infty^A)^-$ to obtain a finite limit.

Thus, we rewrote the boundary term in terms of a total derivative:

$$-2\delta\int_{\partial\Sigma} \mathrm{d}\Omega f^\mu m_{\mu\nu}W^{3\nu} \tag{3.5.33}$$

We define the improved generator of gauge transformations $\vec{H}[\vec{f}] = \vec{V}[\vec{f}] + \vec{B}[\vec{f}]$, where

$$\vec{B}[\vec{f}] = 2\int_{\partial\Sigma} \mathrm{d}\Omega\left[f^\mu m_{\mu\nu}W^{3\nu}\right]. \tag{3.5.34}$$

Let's consider the variation of the Hamiltonian constraint. We need the variation of the Ricci scalar, which was derived in terms of the variation of the Christoffel symbols in equation (3.3.3). It remains to compute the variation of the Christoffel symbols in terms of the metric:

$$\delta\Gamma^i_{jk} = \frac{1}{2} g^{il}(\nabla_j \delta g_{lk} + \nabla_k \delta g_{jl} - \nabla_l g_{jk}) \tag{3.5.35}$$

Combining it with (3.3.3), we obtain

$$\begin{aligned} \delta R &= \delta g^{ij} R_{ij} + \frac{1}{2}\nabla_k\Big(g^{kl}g^{ij}(2\nabla_i \delta g_{lj} - \nabla_l g_{ij}) - g^{jl}g^{ik}\nabla_i \delta g_{jl}\Big) \\ &= \delta g^{ij} R_{ij} + \nabla^i\nabla^j \delta g_{ij} - \nabla_k \nabla^k (g^{ij}\delta g_{ij}) \end{aligned} \tag{3.5.36}$$

Recall the smeared Hamiltonian constraint

$$V_0[f] = \int \mathrm{d}\Sigma\, f \Big[\frac{\kappa}{\sqrt{\det m}}\Big(W_{\mu\nu}W^{\mu\nu} - \frac{1}{2}(m^{\mu\nu}W_{\mu\nu})^2\Big) - \frac{1}{\kappa}\sqrt{\det m} R\Big] \tag{3.5.37}$$

We observe, that the only dependence on derivatives of canonical variables comes from the Ricci scalar term. A variation leads to the boundary term

$$\begin{aligned} &\int \mathrm{d}\Sigma_k \frac{\sqrt{\det m}}{\kappa}\Big[\Big(m^{ik}m^{jl} - m^{ij}m^{kl}\Big) f\nabla_l \delta m_{ij} + \Big(m^{kl}m^{ij} - m^{ik}m^{jl}\Big)\nabla_l f \delta m_{ij}\Big] \\ = &\int \mathrm{d}\Sigma_k \frac{\sqrt{\det m}}{\kappa}\Big[\Big(m^{ik}m^{jl} - m^{ij}m^{kl}\Big)(f\nabla_l \delta m_{ij} - \nabla_l f \delta m_{ij})\Big] \end{aligned} \tag{3.5.38}$$

Similarly to the diffeomorphism constraint, we would like to rewrite the boundary term as a total differential. For this, notice that inside the variation we can subtract the non-dynamical part of the metric: $\delta m_{ij} = \delta(m_{ij} - m^{\mathrm{ND}}_{ij})$. We pull the variation outside of the integral and obtain

$$\delta \int \mathrm{d}\Sigma_k \frac{\sqrt{\det m}}{\kappa}\Big[\Big(m^{ik}m^{jl} - m^{ij}m^{kl}\Big)\Big(f\nabla_l(m_{ij} - m^{\mathrm{ND}}_{ij}) - \nabla_l f(m_{ij} - m^{\mathrm{ND}}_{ij})\Big)\Big] \tag{3.5.39}$$

We now show that using the fall-off conditions the correction terms could be neglected. For $\sqrt{\det m}$ and m^{-1} putting the variation reduces the power of r in the asymptotic expansion by one. For $m - m^{\mathrm{ND}}$ the power of r is not increasing. Therefore, the correction terms have a power of r less compared to the expression above and they vanish in the limit $r \to \infty$ provided (3.5.39) is finite,

It remains to show that (3.5.39) is finite for suitably chosen f. The fall-off conditions on m are not modified by the transformation to Gullstrand-Painlevé coordinates. Additionally, working in Cartesian coordinates it is easy to see that $\sqrt{\det m}$ and m^{-1} are of order 1 and that $m - m^{\mathrm{ND}}$ is of order $O(r^{-1})$ even and $O(r^{-2})$ odd. Hence, choosing $f = f^+_\infty + r f^-_\infty$ the integral over the sphere gives a finite result.

The improved generator for the Hamiltonian constraint is given by $H[f] = V_0[f] + B[f]$, where

$$B[f] := -\int \mathrm{d}\Sigma_k \frac{\sqrt{\det m}}{\kappa}\Big[\Big(m^{ik}m^{jl} - m^{ij}m^{kl}\Big)\Big(f\nabla_l(m_{ij} - m^{\mathrm{ND}}_{ij}) - \nabla_l f(m_{ij} - m^{\mathrm{ND}}_{ij})\Big)\Big] \tag{3.5.40}$$

3.6 Black Hole Solutions in Hamiltonian Formulation

We saw that non-rotating black holes are solutions of the spherically symmetric Einstein equations. In the following, we derive the Schwarzschild and Reissner-Nordstrøm black hole from the Hamiltonian theory based on the ADM formulation. A treatment of the coupled Einstein - Maxwell system in the Hamiltonian form can also be found in [112] and we use a similar notation for the canonical variables. In the analysis, we split the canonical variables into spherically symmetric and non-symmetric degrees of freedom as in the previous section and neglect the non-symmetric ones for now.

For the computations we apply the tools we developed for handling constrained Hamiltonian systems using the reduced phase space approach (see section 2). Recall the expansion of the metric and its conjugate momentum in terms of the symmetric canonical variables

$$\begin{aligned} m_{33} &= e^{2\mu}, \quad m_{3A} = 0, \quad m_{AB} = e^{2\lambda}\Omega_{AB} \\ W^{33} &= \sqrt{\Omega}\frac{1}{2}e^{-2\mu}\pi_\mu, \quad W^{3A} = 0, \quad W^{AB} = \frac{1}{4}e^{-2\lambda}\pi_\lambda\Omega^{AB} \end{aligned} \tag{3.6.1}$$

In addition to the gravitational variables, we introduce electromagnetic matter. The electromagnetic field is described by the vector potential A_i. The matter action is written in terms of the Faraday tensor $F_{ij} = \partial_i A_j - \partial_j A_i$:

$$S_{\text{em}} = -\frac{1}{4g^2}\int \mathrm{d}x\sqrt{-g}g^{ik}g^{jl}F_{ij}F_{kl} \tag{3.6.2}$$

In the action we introduced the coupling constant $g := \epsilon_0^{-1/2}$.

We now split the four dimensional metric into its 3+1 form. The inverse metric in this decomposition is given by

$$g^{tt} = -\frac{1}{N^2}, \quad g^{t\mu} = \frac{N^\mu}{N^2}, \quad g^{\mu\nu} = m^{\mu\nu} - \frac{N^\mu N^\nu}{N^2} \tag{3.6.3}$$

This gives

$$S = \frac{1}{4g^2}\int \mathrm{d}x\sqrt{m}N\left(\frac{2}{N^2}m^{\mu\nu}F_{0\mu}F_{0\nu} - 4\frac{N^\mu}{N^2}m^{\nu\rho}F_{\mu\nu}F_{0\rho} - m^{\mu\nu}m^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} + 2\frac{N^\mu N^\rho}{N^2}m^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right) \tag{3.6.4}$$

We introduce the momenta E^i conjugate to A_i. A_0 appears in the action without time derivative and we obtain the primary constraint $E^0 = 0$. For the momentum conjugate to A_μ we find

$$E^\mu = \frac{\sqrt{m}}{g^2 N}(m^{\mu\nu}F_{0\nu} - N^\rho m^{\mu\sigma}F_{\rho\sigma}) \tag{3.6.5}$$

$$\partial_0 A_\mu = \frac{g^2 N}{\sqrt{m}}\left(m_{\mu\nu}E^\nu + \sqrt{m}\frac{N^\rho}{g^2 N}F_{\rho\mu}\right) + \partial_\mu A_0 \tag{3.6.6}$$

Inserting this into the action, we obtain the action in its Hamiltonian form

$$S_{\text{em}} = \int \mathrm{d}\Sigma \mathrm{d}t \dot{A}_i E^i + \dot{A}_0 E^0 + \lambda E^0 + N\left(\frac{g^2}{2\sqrt{m}}m_{\mu\nu}E^\mu E^\nu + \frac{\sqrt{m}}{4g^2}F_{\mu\nu}F^{\mu\nu}\right) + N^\rho F_{\rho\mu}E^\mu - A_0\partial_\mu E^\mu \tag{3.6.7}$$

Imposing the consistency of the primary constraint $E^0 = 0$, we find the Gauß constraint $\partial_\mu E^\mu$. Additionally, we read off the electromagnetic contributions to the Hamiltonian and diffeomorphism constraints

$$\begin{aligned} V_0^{\text{em}} &= \frac{g^2}{2\sqrt{m}} m_{\mu\nu} E^\mu E^\nu + \frac{\sqrt{m}}{4g^2} F_{\mu\nu} F^{\mu\nu} \\ V_\mu^{\text{em}} &= F_{\mu\nu} E^\nu \end{aligned} \tag{3.6.8}$$

Let us specialize to spherical symmetry. For a spherically symmetric vector field, we only have the contributions E^3 and A_3. The Gauß constraint implies that $\partial_3 E^3 = 0$ and therefore E^3 has to be a radial constant. We take $E^3 = \sqrt{\Omega}\xi$.

Because the action depends on A_μ only through the tensor $F_{\mu\nu}$ which contains antisymmetric derivatives of A_μ, there is no contribution from A_3. This immediately implies that ξ is constant in time due to the Hamiltonian equations for E_3. We fix the gauge of the vector potential to $A_3 = 0$ and consistency of the gauge fixing condition gives a differential equation for A_0, which can be solved explicitly.

Combining the gravitational and electromagnetic contributions to the electromagnetic Hamiltonian and reducing to spherical symmetry, we obtain the spherically symmetric Hamiltonian and the spherically symmetric radial diffeomorphism constraint. In terms of the spherically symmetric variables (μ, π_μ) and (λ, π_λ) defined in (**3.6.1**) and the electric charge ξ we find

$$\begin{aligned} C_v &= 4\pi \left[e^{-\mu-2\lambda} \left(\frac{\pi_\mu^2}{8} - \frac{\pi_\mu \pi_\lambda}{4} + 2e^{4\lambda} \left(2\lambda'' + 3(\lambda')^2 - 2\lambda'\mu' - e^{2(\mu-\lambda)} \right) \right) + \frac{g^2}{4} e^{\mu-2\lambda} \xi^2 \right] \\ C_h &= 4\pi \left[\mu' \pi_\mu + \lambda' \pi_\lambda - \pi_\mu' \right] \end{aligned} \tag{3.6.9}$$

where the prime corresponds to radial derivatives. We changed the variable for the constraints from V to C in order to agree with the notation used in the section on perturbation theory. The subscript v stands for "vertical" and is the Hamiltonian constraint, the subscript h stands for "horizontal" and denotes the radial diffeomorphism constraint.

The strategy is to obtain a solution of these constraints for π_μ and π_λ. We start with the diffeomorphism constraint and obtain

$$\pi_\lambda = \frac{1}{\lambda'} \left(\pi_\mu' - \mu' \pi_\mu \right). \tag{3.6.10}$$

We insert this into the Hamiltonian constraint and obtain a differential equation for π_μ:

$$-\partial_r \pi_\mu^2 + (\lambda' + 2\mu')\pi_\mu^2 + 16e^{4\lambda}\lambda' \left(2\lambda'' + 3(\lambda')^2 - 2\lambda'\mu' - e^{2(\mu-\lambda)} \right) + 2g^2 \lambda' e^{2\mu} \xi^2 = 0 \tag{3.6.11}$$

We can simplify the first two terms by multiplying the expression by $e^{-\lambda-2\mu}$. We also rewrite the terms independent of π_μ in terms of a total derivative

$$-\partial_r \left(e^{-\lambda-2\mu} \pi_\mu^2 \right) + \partial_r \left(16(\lambda')^2 e^{-2\mu+3\lambda} - 16e^\lambda - 2g^2 e^{-\lambda} \xi^2 \right) = 0 \tag{3.6.12}$$

The integration of the differential equation is straight forward. We obtain the solution

$$\pi_\mu^2 = e^{2\mu+\lambda} \left[16r_s + 16 \left((\lambda')^2 e^{-2\mu+3\lambda} - e^\lambda - \frac{g^2}{4} e^{-\lambda} \xi^2 \right) \right] \tag{3.6.13}$$

where r_s is an integration constant. The solution for π_λ is obtained by inserting the solution for π_μ into the expression for the diffeomorphism constraint. We obtain

$$\pi_\lambda = \frac{1}{2\pi_\mu}\Big(\pi_\mu^2 + 16e^{4\lambda}(2\lambda'' + 3(\lambda')^2 - 2\lambda'\mu' - e^{2(\mu-\lambda)}) + 4g^2e^{2\mu}\xi^2\Big) \tag{3.6.14}$$

The next step in the program requires a gauge fixing condition. For this we consider the Gullstrand-Painlevé gauge given by the conditions $\lambda - \log(r) = 0$ and $\mu = 0$. This gauge is preserved under time evolution provided that the following equations are satisfied

$$\begin{aligned} 0 = \dot{\mu} &= 4\pi\left[\frac{1}{4}e^{-\mu-2\lambda}(\pi_\mu - \pi_\lambda)N + \partial_r N^3 + \mu' N^3\right] \\ 0 = \dot{\lambda} &= 4\pi\left[-\frac{1}{4}e^{-\mu-2\lambda}\pi_\mu N + \lambda' N^3\right] \end{aligned} \tag{3.6.15}$$

This is a system of differential equations for the lapse function N and the radial shift N^3. For the solution, we start by solving the second equation for N:

$$N = \frac{4}{\pi_\mu}e^{\mu+2\lambda}\lambda' N^3 \tag{3.6.16}$$

This solution is used in the first equation to eliminate N and to obtain a differential equation for N^3.

$$\left(1 - \frac{\pi_\lambda}{\pi_\mu}\right)\lambda' N^3 + \mu' N^3 + \partial_r N^3 = 0 \tag{3.6.17}$$

In this equation, we use our knowledge from the spherically symmetric constraints. We have that $\pi_\lambda = \frac{1}{\lambda'}\left(\pi_\mu' - \mu'\pi_\mu\right)$ and get the differential equation

$$\left(2\mu'\pi_\mu + \lambda'\pi_\mu - \pi_\mu'\right)N^3 + \pi_\mu \partial_r N^3 = 0 \tag{3.6.18}$$

We multiply this equation by $e^{-2\mu-\lambda}$ and divide it by π_μ^2. Then, the differential equation simplifies to a total derivative

$$\partial_r\left(\frac{1}{\pi_\mu}e^{2\mu+\lambda}N^3\right) = 0 \tag{3.6.19}$$

The solution for N^3 is straight forward and inserting it into (**3.6.16**) we have the solution

$$N = Ce^{-\mu+\lambda}\lambda', \quad N^3 = \frac{1}{4}Ce^{-2\mu-\lambda}\pi_\mu \tag{3.6.20}$$

The integration introduced an integration constant C. It can be absorbed into a reparametrisation of the time coordinate. In GP gauge we have $N = C$ and $N^3 = C\sqrt{r_s/r}$. Thus, we set $C = 1$ to make contact with the Schwarzschild metric in GP coordinates that we discussed before.

Let us now evaluate the boundary terms from the previous section

$$\vec{B}[\vec{f}_*] = 2\int_{\partial\Sigma} \mathrm{d}\Omega\, f_*^3 W^{33} = \int_{\partial\Sigma} \mathrm{d}\Omega\, f_*^3 \pi_\mu = \lim_{r\to\infty} \pi e^{-\lambda}\pi_\mu^2 \tag{3.6.21}$$

In the calculation we only kept the first non-trivial orders for $r \to \infty$. f_*^3 was substituted by the solution of the stability condition we found above.

The reduced Hamiltonian is available through Theorem 2.12 by finding the function $\chi[j]$, where $j = \pi_\mu$.

We have to solve the functional differential equation

$$\lim_{r\to\infty} \pi e^{-\lambda}\pi_\mu = \frac{\delta\chi[\pi_\mu]}{\delta\pi_\mu(r)} \tag{3.6.22}$$

The solution is given by $\chi[\pi_\mu] = \lim_{r\to\infty} \frac{1}{2}\pi e^{-\lambda}\pi_\mu^2$ and the physical Hamiltonian is

$$H = \frac{1}{\kappa}\chi[\pi_\mu] = \lim_{r\to\infty} \frac{\pi}{2\kappa r}\pi_\mu^2 \tag{3.6.23}$$

In this equation we reinserted the gravitational coupling constant κ which is present in front of the gravitational action (see (**3.5.21**)). In the units $G = c = 1$, we have $\kappa = 16\pi$. Inserting the solution for π_μ^2 in (**3.6.13**), we have

$$H = \frac{1}{\kappa}\chi[\pi_\mu] = \lim_{r\to\infty} \frac{\pi}{2\kappa}[16 r_s] = M \tag{3.6.24}$$

where we used that $r_s = 2M$. The reduced Hamiltonian just corresponds to the ADM mass of the black hole.

In the solution of the constraint we found an integration constant $r_s = 2M$ related to the mass of the black hole. The discussion in section 2 suggests that there is a canonical pair (M, P_M) corresponding to a pair of global, observable degrees of freedom located at infinity. The momentum conjugate to the mass can be constructed explicitly (see [113] and appendix A in [98]). Introducing (M, P_M) using canonical transformations leads to a new constraint of the form $M' = 0$. In perturbation theory the momentum P_M might play an important role and the mass of the black hole M might actually be dynamical. In this thesis, we will not study this further and leave this alternative approach for future work.

3.7 Classical Theorems on Black Holes

There are several general results concerning black holes which can be proven rigorously in classical general relativity. The most famous ones are the singularity theorems showing that singularities are a generic prediction of general relativity and the black hole area theorem that the area of the black hole event horizon can only grow with time and never decreases. In this section, we would like to recall some of these theorems and refer the reader to the literature for detailed proofs of the statements.

In the previous sections, we discussed the solutions of general relativity for spherical symmetry in the Lagrangian and Hamiltonian formulation. We also mentioned the extension of the results to rotating black holes and axial symmetry. In all of these spacetimes, the metric is not regular and possesses divergences at certain points in spacetime. For the Schwarzschild solution in Schwarzschild coordinates, the metric is divergent at $r = 0$ and $r = 2M$. In contrast to that, the same metric in Gullstrand-Painlevé coordinates is only singular at $r = 0$ while $r = 2M$ is perfectly regular. Hence, in contrast to the initial believe, the point $r = 2M$ is not singular and the irregularity in the Schwarzschild coordinates is just an artifact of the choice of coordinates.

To avoid coordinate artifacts, the study of scalar quantities such as the Ricci scalar R and Kretschmann scalar $R_{ijkl}R^{ijkl}$ are better suited to investigate singularities in spacetime. In fact, the Kretschmann scalar is perfectly regular at the horizon but diverges at $r = 0$ showing that there might be a singularity. However, it could still be possible that the Kretschmann scalar is perfectly regular but R_{ijkl} is still pathological in some coordinate system.

A possibility to circumvent the problems of coordinate dependence and using the curvature tensor as a measure for the presence of singularities in spacetimes, are geodesics. If the geodesic describing the motion of a freely falling observer in spacetime terminates in finite affine time, we call the spacetime singular.

For the understanding of the singularity theorems, we need to introduce the notion of null geodesic congruences and review some of their properties. More details on null geodesic congruences can be found in [104]. Here we will follow closely the notation in [98]. We work in the Hamiltonian formulation of general relativity and assume (M, g) to be globally hyperbolic and foliated into three-dimensional spatial hypersurfaces isomorphic to Σ. As before, let n be the timelike normal to Σ ($g(n, n) = -1$).

Inside the Cauchy surface Σ, we consider a closed, oriented 2-surface S without boundary. This surface comes equiped with a spacelike normal s which is tangent to Σ ($g(n, s) = 0$) and we choose it to be normalized $g(s, s) = 1$. Like in section 3.5, we define the induced metric on the Cauchy surface Σ by $m = g + n \otimes n$. Similarly, S is equipped with the metric $h = m - s \otimes s$ by pulling back the metric m on Σ to S.

At each point $y^A \in S$ ($A = 1, 2$), we define null geodesics $c_{y^A}(\lambda)$ with initial tangents $l_\pm = n \pm s$ and affine parameter λ. This defines two null geodesic congruences $C_S^\pm$ through S with tangent vectors

$$\partial_\lambda^\pm = \frac{\partial c^\mu_{y,\pm}(\lambda)}{\partial \lambda} \partial_\mu =: l_\pm, \quad \partial_A^\pm = \frac{\partial c^\mu_{y,\pm}(s)}{\partial y^A} =: e_{A,\pm} \tag{3.7.1}$$

Here $\partial_\lambda^\pm$ is the parallel transport of $l_\pm$ from S along the geodesic and $\nabla_{l_\pm} l_\pm = 0$. The vector field $e_{A,\pm}$ is the deviation vector field which describes the deviation of nearby geodesics.

We would like to describe how the geodesic congruence behaves as we move along the geodesics. This motivates to investigate the covariant derivative $\nabla_{l_\pm} e_A$. Using the fact that $[\partial_\lambda^\pm, \partial_A^\pm] = 0$, this is equivalent to $\nabla_{e_A} l_\pm$. Since $g(l_\pm, \nabla_{e_A} l_\pm) = 0$, the only interesting projection is captured by the tensor

$$\kappa^\pm_{AB} = g(e_{A,\pm}, \nabla_{e_B,\pm} l_\pm) \tag{3.7.2}$$

Additionally, on $C_S^\pm$ we introduce

$$h^\pm_{AB} = g(e_{A,\pm}, e_{B,\pm})\,. \tag{3.7.3}$$

and its inverse $h_\pm^{AB}$ defined by $h_\pm^{AB} h^\pm_{BC} = \delta^A_C$.

Then, we decompose $\kappa^\pm_{AB}$ into expansion, shear and rotation:

$$\theta_\pm := h_\pm^{AB} \kappa^\pm_{AB}, \quad \sigma^\pm_{AB} := \kappa^\pm_{(AB)} - \frac{1}{2}\theta_\pm h^\pm_{AB}, \quad \omega^\pm_{AB} := \kappa^\pm_{[AB]} \tag{3.7.4}$$

The quantities describe the expansion, shear and rotation of nearby geodesics in the congruence.

Taking the derivative of $\kappa^\pm_{AB}$ in the direction $l_\pm$, we find the equation

$$\nabla_{l_\pm} \kappa^\pm_{AB} = -\kappa^\pm_{AC} \kappa^\pm_{DB} h_\pm^{CD} - R(e_A, l_\pm, e_B, l_\pm) \tag{3.7.5}$$

and using the decomposition of $\kappa^\pm_{AB}$ we find Raychaudhuri's equation:

$$\nabla_{l_\pm} \theta_\pm = -\frac{1}{2}\theta_\pm^2 - \sigma^\pm_{AB}\sigma_\pm^{AB} + \omega^\pm_{AB}\omega_\pm^{AB} - R(l_\pm, l_\pm)\,. \tag{3.7.6}$$

It describes the change of the expansion along the geodesic congruence and is central to the proof of the singularity theorems.

The expansion is a central object in this section and later in section 6 we show a strategy to construct an explicit, perturbative formula for it. In terms of the variables $(m_{\mu\nu}, W^{\mu\nu})$, we have

$$\begin{aligned} \theta_\pm &= h_\pm^{\mu\nu}\nabla_\mu l_{\pm\nu} = (m^{\mu\nu} - s^\mu s^\nu)\nabla_\mu(n_\nu \pm s_\nu) \\ &= K - s^\mu s^\nu K_{\mu\nu} \pm m^{\mu\nu}\nabla_\mu s_\nu \\ &= -s^\mu s^\nu \frac{W_{\mu\nu}}{\sqrt{m}} \pm D_\mu s^\mu \end{aligned} \tag{3.7.7}$$

In order to avoid explicitly evaluating covariant derivatives, we can multiply the expression by the square root of m and use the following expression

$$\sqrt{m}\theta_\pm = -s_\mu s_\nu W^{\mu\nu} \pm \partial_i(\sqrt{m} m^{\mu\nu} s_\nu) \tag{3.7.8}$$

For the study of black holes, the following definition of trapped surfaces turns out to be useful. A trapped region is defined as a region of spacetime such that both the in- and out-going expansion are smaller than zero. This coincides with the physical picture of a black hole, where the null rays in a trapped region inside the black hole need to converge towards the singularity.

Definition 3.18: Consider a globally hyperbolic spacetime $(\mathcal{M}, g)$ and a Cauchy surface Σ in it.

1. A closed, orientable 2-surface $S \subset \Sigma$ without boundary $\partial_\Sigma S = \emptyset$ is called **trapped** if $\theta_+, \theta_- < 0$.
2. A trapped region in Σ is a closed subset $T \subset \Sigma$ such that $S := \partial_\Sigma T$ (boundary within Σ) is trapped.
3. The trapped surface in Σ defined by the total trapped region (closure of union of all trapped regions) is called the **apparent horizon** A_Σ of Σ.

The above definition is within one Cauchy surface Σ. Using the foliation of spacetime, we extend the trapped surfaces / apparent horizon to all of spacetime. This gives the trapping horizon / apparent horizon respectively

Definition 3.19: Consider a globally hyperbolic spacetime $(\mathcal{M}, g)$ and a foliation $\mathcal{F} = \bigcup_{\tau\in\mathbb{R}} \Sigma_\tau$ of M by Cauchy surfaces Σ_τ.

1. If $\tau \mapsto S_\tau \subset \Sigma_\tau$ is a one parameter family of trapped surfaces then $\mathcal{S} := \bigcup_{\tau\in\mathbb{R}} S_\tau$ is called a trapping horizon.
2. Let $A_\tau := A_{\Sigma_\tau}$ be the apparent horizon of Σ_τ. Then $A_\mathcal{F} := \bigcup_{\tau\in\mathbb{R}} A_\tau$ is called the apparent horizon of $\mathcal{F}$.

For the visualization of the causal structure of spacetimes it is useful to study conformal completions of spacetime. The idea is to embed the spacetime $(\mathcal{M}, g)$ into a larger spacetime $(\overline{\mathcal{M}}, \overline{g})$ such that the metrics are conformally related $\overline{g} = \Omega^2 g$. The conformal factor Ω goes to zero at infinity. This brings the the points infinitely far away to a finite distance but the metric still describes the same causal structure.

For a visual representation, we can study a two-dimensional surface in the conformally compactified spacetime. The resulting picture, known as a Penrose diagram is a two-dimensional representation of the causal structure of the full spacetime in a finite plot. The Penrose diagram of the Schwarzschild black hole forming from gravitational collapse is displayed in figure 3.1.

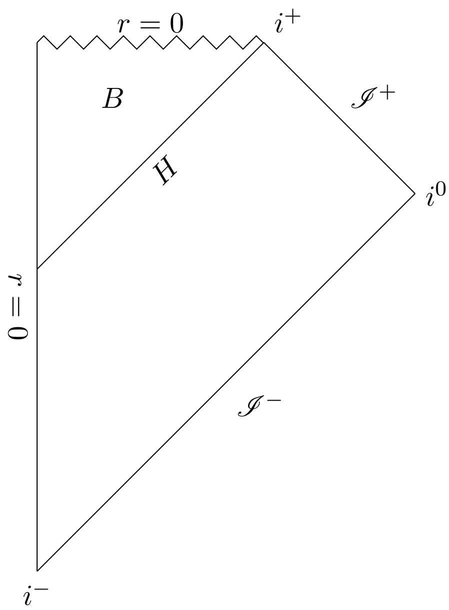

Figure 3.1: Conformal Diagram of a Schwarzschild black hole

In this picture we marked future and past timelike infinity i^+ and i^- which corresponds to the points where timelike geodesics end and begin. A massive particle or timelike observer will start in i^- and end up in i^+. The lines $\mathscr{I}^+$ and $\mathscr{I}^-$ are called future and past null infinity. Ingoing light rays begin at $\mathscr{I}^-$ and end up at $\mathscr{I}^+$. Finally, i^0 is spacelike infinity and it is the end point of spacelike geodesics. It represents the point at infinity, infinitely far away.

The Penrose diagram in figure 3.1 of a Schwarzschild black hole has additional features. There is a region B which is not in the past of $\mathscr{I}^+$ and therefore, no signal can reach $\mathscr{I}^+$ from inside B. This region is called the black hole region and the boundary $H = \partial B$ is the event horizon of the black hole. The zigzag line is the black hole singularity and we see that observers falling into the black hole will eventually end up in the singularity.

In contrast to the apparent horizon, the event horizon is a "teleological" notion because we need to know the full future evolution of the metric in order to determine the location of the black hole region B and event horizon H. In contrast to that, the apparent horizon is defined quasi-locally in each Cauchy surface. However, the definition of apparent horizons depends on the particular foliation we chose for its definition. We can think of this choice of foliation as being associated to an observer looking at the black hole in his own frame of reference. For instance, in the Gullstrand-Painlevé gauge and the corresponding GP foliation, we naturally singled out an observer freely falling in the black hole spacetime. The apparent horizon then gets a direct operational meaning as the region of spacetime this observer would consider as the black hole region. In contrast to that, the observer could already be inside the event horizon because of some gravitational collapse in the future without being aware of it. This shows that it is more natural to use the apparent horizons if we would like to make contact to actual experiments.

The singularity theorem due to Penrose is

Theorem 3.20 (Penrose 1965)**:** Let $(\mathcal{M}, g)$ be a globally hyperbolic and connected spacetime with a non-compact Cauchy surface Σ. Suppose that $R_{\mu\nu}l^\mu l^\nu \geq 0$ for all l null which is satisfied if the Einstein equations hold and the matter satisfies the strong energy condition. In addition, suppose that M contains a trapped surface S and let θ_0 be the maximum value of the expansion of both in- and out-going null geodesic congruences. Then at least one inextendible future directed null geodesic orthogonal to S has affine length smaller than $2/|\theta_0|$.

Later it was generalised by Hawking by weakening some of the assumptions. For a proof of this theorem see [104].

In the Schwarzschild spacetime, all the spheres with constant radius $r < 2M$ are trapped surfaces. There also exist trapped surfaces for perturbations of the Schwarzschild black hole (see section 5): By continuity, there is a value of r sufficiently small such that the expansion of the sphere around $r = 0$ is smaller than zero. Thus, the singularity inside a black hole is not an artefact of treating the black holes with exact spherical symmetry but is a generic prediction of general relativity.

Another important theorem about black holes in classical general relativity will be discussed next. For its understanding, we need one preliminary definition:

Definition 3.21: A spacetime $(\mathcal{M}, g)$ with conformal completion $(\tilde{\mathcal{M}}, \tilde{g})$, $\psi : \mathcal{M} \to \tilde{\mathcal{M}}$ is called **strongly future asymptotically predictable**, if there exists an open region $\tilde{O} \subset \tilde{\mathcal{M}}$ such that $\overline{\psi(\mathcal{M}) \cap J^-(\mathscr{I}^+)} \subset \tilde{O}$ (the closure in $\tilde{\mathcal{M}}$ of the intersection between $\psi(\mathcal{M})$ and the causal past of future null infinity). Additional, $(\tilde{O}, \tilde{g})$ has to be globally hyperbolic.

Then, Stephen Hawking proved the black hole area theorem:

Theorem 3.22 (Black Hole Area Theorem; Hawking 1971)**:** Let $(\mathcal{M}, g)$ be strongly asymptotically predictable, $R(l, l) \geq 0$ for all l null. Let $\tilde{\Sigma}_0$ and $\tilde{\Sigma}_1$ be Cauchy surfaces for $\tilde{O}$ and $\tilde{\Sigma}_1 \subset I^+(\tilde{\Sigma}_0)$ (timelike future of $\tilde{\Sigma}_0$). Let $\mathcal{H}_j = H \cap \Sigma_j$ $(j = 0, 1)$, where $H = \partial B$ is the event horizon of the black hole region B. Then, the area of $\mathcal{H}_1$ is larger than or equal to the area of $\mathcal{H}_2$, where we define the area functional of $\mathcal{H}_i$ as

$$\mathrm{Ar}[\mathcal{H}_i] = \int_Y \mathrm{d}^2 y \sqrt{\det\left(X^*_{\mathcal{H}_i} g\right)}, \tag{3.7.9}$$

where $X_{\mathcal{H}_i} : Y \subset \mathbb{R}^2 \to \mathcal{H}_i \subset M$ is the embedding into M describing $\mathcal{H}_i$.

For a proof of this theorem see [104]. The theorem implies that under reasonable assumptions, the area of the event horizon can never decrease. Therefore, in classical general relativity black holes can only grow by absorbing matter or by merging with other black holes.

The black hole solutions we presented in section 3.4 are the most general ones in four dimensions according to the following theorem:

Theorem 3.23 (black hole uniqueness (no hair) theorem)**:** The Kerr-Newman black hole solution is the only asymptotically stationary, electrovacuum and strongly asymptotically predictable spacetime.

As stated the theorem has not yet been proven. For further technical assumptions, it is possible to prove weaker versions of the theorem (see [7]).

We conclude this section with an intriguing analogy between black holes and thermodynamics. Consider the Kerr-Newman family of black holes characterised by mass M, charge Q and angular momentum $J = aM$. The spacetime has horizons at $r_\pm = m \pm \sqrt{m^2 - a^2 - Q^2}$. Let us study the outer horizon located at r_+. This horizon has area equal to

$$A = 4\pi(r_+^2 + a^2) = 16\pi M_{\mathrm{irr}} \tag{3.7.10}$$

where $M_{\text{irr}} = m^2 + \sqrt{m^4 - J^2}$ is the irreducible mass of the black hole. Calculating the variation δA, we find the relation

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \nu \delta Q \tag{3.7.11}$$

with $\kappa = \frac{r_+ - m}{r_+^2 + a^2}$, $\Omega_H = \frac{a}{r_+^2 + a^2}$ and $\nu = \frac{r_+ Q}{r_+^2 + a^2}$. In analogy with the first law of thermodynamics ($\delta E = T\delta S - p\delta V$) this suggests the identification $E \to M$, $T \to \kappa$ and $S \to A$.

This suggests an analogy between black holes and classical thermodynamics: The zeroth law of thermodynamics that $T =$ const. translates to the condition that $\kappa =$ const. on the horizon which can be proven explicitly. The second law of thermodynamics states that the entropy S of a system is never decreasing. This corresponds to the statement that the area of the event horizon A is not decreasing which is the statement of the black hole area theorem. The third law of thermodynamics states that $T = 0$ can never be reached in physical processes. A black hole with $\kappa = 0$ corresponds to an extremal black hole $m^2 = a^2 + Q^2$ and we expect it to be impossible to reach such black holes in physical processes.

At this point, this analogy is on a formal level because there is no physical reason to assign a temperature or entropy to a black hole. Black holes are not emitting any radiation so assigning a non-zero temperature to them seems impossible. A partial answer to this issue is given in the next section. In the presence of quantum fields on a black hole spacetime we can show that black holes emit radiation following a thermal spectrum. The characteristic temperature of this black body radiation will be proportional to κ.

4

Quantum Field Theory on Curved Spacetimes and the Hawking Effect

In the discussion of classical properties of black holes in the previous section, we encountered an unexpected similarity between classical thermodynamics and black hole physics. There is an analogy between certain relations satisfied by the parameters of the Kerr-Newman spacetime and the laws of classical thermodynamics. In particular we made the correspondence between temperature T and surface gravity κ and between entropy S and the area of the event horizon A. However, this correspondence is very formal because classical black holes are perfect absorbers and have zero temperature.

In the following, we shed some more light onto this equivalence by considering quantum field theory on black hole spacetimes. As we will see, due to quantum effects black holes emit particles following a thermal spectrum with a characteristic temperature. This provides an exact expression for the temperature we assign to a black hole and fixes the proportionality constants between temperature and surface gravity, and between entropy and area of the horizon.

The calculation assumes a fixed spacetime, ignoring any backreaction effects of the radiation on the black hole spacetime. Since, the radiation carries away energy from the black hole we expect the black hole to lose some of its mass and slowly evaporate. In the end, we show a simple heuristic argument for how this black hole evaporation process takes place.

This section is structured as follows: First, we introduce quantum field theory on curved spacetime. For simplicity we only consider a free, massive scalar field and discuss some general properties and new features if we generalize from Minkowski to arbitrary spacetimes. Then, we apply the formalism to the Schwarzschild black hole and derive the famous result that black holes emit radiation. We conclude with a short outlook on the problem of backreaction and the evaporation of black holes.

4.1 Scalar Quantum Field Theory in Curved Spacetime

The subject of quantum field theory in curved spacetime is discussed in by many authors in the literature, see [8, 13, 16, 17]. In this thesis we only consider the quantization of scalar fields for simplicity. The discussion is easily generalized to general fields and with a bit more work also to the case of fermions.

Consider a spacetime $(\mathcal{M}, g)$ and a Klein Gordon scalar field ϕ minimally coupled to general relativity. The action for this system is given by

$$S = -\int_M \mathrm{d}^4x \, \frac{1}{2}(\nabla_\mu\phi\nabla^\mu\phi + m^2\phi^2)\,. \tag{4.1.1}$$

Here, ∇_μ is the unique torsion free covariant derivative associated to the metric g and m is the mass of the scalar field. A variation of the action with respect to the scalar field gives the Klein-Gordon equation

$$(\Box - m^2)\phi = 0\,, \tag{4.1.2}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. We call the set of solutions of the Klein-Gordon equation $\boldsymbol{Sol}$ and we denote the complexification of $\boldsymbol{Sol}$ by $\boldsymbol{Sol}_\mathbb{C}$. On the space of complexified solutions $\boldsymbol{Sol}_\mathbb{C}$, we define a sesquilinear form given by

$$\langle\psi,\phi\rangle = -i\int_\Sigma \mathrm{d}S_\mu\,(\psi^*\nabla^\mu\phi - (\nabla^\mu\psi)^*\phi)\,. \tag{4.1.3}$$

This map is well-defined since one can show that it is independent of the choice of Cauchy surface. Let Σ_1 and Σ_2 be two Cauchy surfaces which are the boundary of some spacetime region R. Using Stokes theorem and the fact that ψ,ϕ satisfy the Klein-Gordon equation, we have

$$\langle\psi,\phi\rangle_{\Sigma_1} - \langle\psi,\phi\rangle_{\Sigma_2} = \int_R \mathrm{d}^4x\,\sqrt{-\det g}(\psi^*\Box\phi - \Box\psi^*\phi) = 0 \tag{4.1.4}$$

$\langle\cdot,\cdot\rangle$ is a possible candidate for an inner product on the space of solutions. However, it fails to be positive definite and the only way out is to find a subspace where it is positive definite. Consider a decomposition $\boldsymbol{Sol}_\mathbb{C} = \boldsymbol{Sol}^+_\mathbb{C}\oplus\boldsymbol{Sol}^-_\mathbb{C}$, such that $\langle\cdot,\cdot\rangle$ is positive definite on $\boldsymbol{Sol}^+_\mathbb{C}$ and $\boldsymbol{Sol}^-_\mathbb{C} = (\boldsymbol{Sol}^+_\mathbb{C})^*$. Then, for $\psi,\phi\in\boldsymbol{Sol}^-_\mathbb{C}$ we find elements ψ^+,ϕ^+ in $\boldsymbol{Sol}^+_\mathbb{C}$ such that $\psi = (\psi^+)^*$ and $\phi = (\phi^+)^*$. Then

$$\langle\psi,\phi\rangle = \langle(\psi^+)^*,(\phi^+)^*\rangle = -\langle\psi^+,\phi^+\rangle \leq 0 \tag{4.1.5}$$

Hence, $\langle\cdot,\cdot\rangle$ is negative definite on $\boldsymbol{Sol}^-_\mathbb{C}$. Additionally, by definition $\langle\psi,\phi\rangle = 0$ for $\psi\in\boldsymbol{Sol}^+_\mathbb{C}$ and $\phi\in\boldsymbol{Sol}^-_\mathbb{C}$.

The one particle Hilbert space $\mathcal{H}$ is given by the completion of $\boldsymbol{Sol}^+_\mathbb{C}$ with respect to the inner product $\langle\cdot,\cdot\rangle$. Based on this Hilbert space $\mathcal{H}$ we construct the symmetric Fock space $\mathcal{F}_S(\mathcal{H})$ as

$$\mathcal{F}_S(\mathcal{H}) = \bigoplus_{n=0}^{\infty}\mathcal{H}^{\odot n}\,, \tag{4.1.6}$$

where $\odot$ denotes the symmetrized tensor product of the one-particle Hilbert spaces. The inner product on the Fock space is given by the sum of the inner products on all the n-particle Hilbert spaces.

On the Fock space we define creation and annihilation operators. They act on a n-particle state as

$$a(f)\psi_1\odot\cdots\odot\psi_n = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\langle f,\psi_k\rangle\psi_1\odot\cdots\odot\hat{\psi}_k\odot\cdots\odot\psi_n \tag{4.1.7}$$

$$a^\dagger(f)\psi_1\odot\cdots\odot\psi_n = \sqrt{n+1}f\odot\psi_1\odot\cdots\odot\psi_n\,, \tag{4.1.8}$$

where $f\in\mathcal{H}$ and $\hat{\psi}_k$ means that ψ_k is omitted. The operators a and $a^\dagger$ satisfy the commutation relations

$$[a(f),a(g)] = 0,\quad [a^\dagger(f),a^\dagger(g)] = 0,\quad [a(f),a^\dagger(g)] = \langle f,g\rangle 1_\mathcal{H}\,. \tag{4.1.9}$$

The construction heavily depends on the choice of a positive subspace of $\boldsymbol{Sol}_\mathbb{C}$. For different choices we will obtain different Hilbert spaces and consequently a different Fock space. Consider two different choices of the positive subspace and the two one-particle Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Now, the question is under which circumstances the two spaces are unitarily equivalent, i.e. there exists a unitary map $U:\mathcal{H}_1\to\mathcal{H}_2$.

In [8], Wald proves a theorem which shows under which circumstances this is the case. Consider the decomposition $\boldsymbol{Sol}_{\mathbb{C}} = \boldsymbol{Sol}_{\mathbb{C}}^{+} \oplus \boldsymbol{Sol}_{\mathbb{C}}^{-}$ and the projectors P_+, P_- onto $\boldsymbol{Sol}_{\mathbb{C}}^{\pm}$ respectively. Then

$$(\psi, \phi) := \langle P_+\psi, P_+\phi\rangle \tag{4.1.10}$$

defines a positive bilinear form on $\boldsymbol{Sol}$. Then, we decompose (ψ, ϕ) into real and imaginary parts

$$(\psi, \phi) = g(\psi, \phi) - \frac{i}{2}\omega(\psi, \phi) \tag{4.1.11}$$

where both $g(\psi, \phi)$ and $\omega(\psi, \phi)$ are real. From the properties of $\langle\cdot,\cdot\rangle$ it follows that g is symmetric and ω anti-symmetric. For $\psi, \phi \in \boldsymbol{Sol}$, ω is determined by

$$\mathrm{Im}(\psi, \phi) = \frac{1}{2i}\Big(\langle P_+\psi, P_+\phi\rangle - \overline{\langle P_+\psi, P_+\phi\rangle}\Big) = \frac{1}{2i}(\langle P_+\psi, P_+\phi\rangle + \langle P_-\psi, P_-\phi\rangle) = \frac{1}{2i}\langle\psi, \phi\rangle \tag{4.1.12}$$

where we used that $\psi = P_+\psi + P_-\psi$ and that with respect to $\langle\cdot,\cdot\rangle$ the positive and negative subspaces are orthogonal. Therefore, ω is given by

$$\omega(\psi, \phi) = \int_\Sigma \mathrm{d}S_\mu \left(\psi\nabla^\mu\phi - \nabla^\mu\psi\phi\right) \tag{4.1.13}$$

which is the symplectic form.

For the discussion of the unitary equivalence we need the notion of Hilbert-Schmidt operators and the Riesz lemma (see [114]):

Definition 4.1: A bounded operator $A \in \mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$ is called **Hilbert-Schmidt** operator, if A has finite Hilbert-Schmidt norm:

$$\|A\|_{\mathrm{HS}} := \sum_i \|Ae_i\|^2\,, \tag{4.1.14}$$

where e_i is an orthonormal basis of $\mathcal{H}$.

Lemma 4.2 (Riesz lemma)**:** Let $\mathcal{H}$ be a Hilbert space. For each element $l \in \mathcal{H}^*$ there is a unique $y \in \mathcal{H}$ such that $l(x) = \langle y, x\rangle$ for all $x \in \mathcal{H}$.

The map g plays a crucial role in the following theorem:

Theorem 4.3: Consider two different decompositions of the space of complexified solutions $\boldsymbol{Sol}_{\mathbb{C}}$ and let g_1, g_2 be defined as above. Then, the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are unitarily equivalent if

1. g_1 and g_2 induce equivalent norms
2. If case 1 holds, then the Riesz lemma implies the existence of an operator $Q : \boldsymbol{Sol} \to \boldsymbol{Sol}$ such that

$$g_1(\psi, Q\phi) = g_2(\psi, \phi) - g_1(\psi, \phi)\,, \tag{4.1.15}$$

   This operator has to be a Hilbert Schmidt operator

For a proof of this statement see Theorem 4.4.1 in [8].

Let us see some features of different vacua in quantum field theories more explicitly. Consider two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with orthonormal basis $v_I^{(1)}$ and $v_I^{(2)}$ respectively. Then, we define the annihilation operators $a_I^{(1)} = a(v_I^{(1)})$ and similarly for $\mathcal{H}_2$. These give us access to two different notions of particles in the two Hilbert spaces. The creation and annihilation operators are related through a Bogoliubov transformation

$$\begin{aligned} a_I^{(2)} &= \sum_J A_{IJ} a_J^{(1)} + B_{IJ} (a_J^{(1)})^\dagger \\ (a_I^{(2)})^\dagger &= \sum_J (A_{IJ})^* (a_J^{(1)})^\dagger + (B_{IJ})^* (a_J^{(1)}) \end{aligned} \tag{4.1.16}$$

The coefficients A and B are given by $A_{IJ} = \langle v_I^{(2)}, v_J^{(1)} \rangle$ and $B_{IJ} = \langle v_I^{(2)}, (v_J^{(1)})^* \rangle$. Imposing that $a_I^{(2)}$ satisfies canonical commutation relations, we find $A_{IK} {B_J}^K = B_{IK} {A_J}^K$ and that $A_{IK} ({A_J}^K)^* - B_{IK} ({B_J}^K)^* = \delta_{IJ}$.

For the comparison of the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we introduce the number operators. On $\mathcal{H}_1$, it is defined by $N^{(1)} = \sum_I (a_I^{(1)})^\dagger a_I^{(1)}$ and similarly for $N^{(2)}$. The vacuum is the state in the Hilbert space which is annihilated by the annihilation operator, i.e. $a_I^{(1)} |0\rangle_{(1)} = 0$ for all I. Then, we observe that if B is non-zero, even if the expected number of particles in one vacuum is zero there might be an infinite number of particles in the other vacuum. For instance, for the vacuum $|0\rangle_{(1)}$ we have

$$\begin{aligned} \langle 0|_{(1)} N^{(2)} |0\rangle_{(1)} &= \sum_{IJK} (B_{IJ})^* B_{IK} \langle 0|_{(1)} a_J^{(1)} (a_K^{(1)})^\dagger |0\rangle_{(1)} \\ &= \sum_{IJK} (B_{IJ})^* B_{IK} \langle 0|_{(1)} [a_J^{(1)}, (a_K^{(1)})^\dagger] |0\rangle_{(1)} \\ &= \sum_{IJ} (B_{IJ})^* {B_I}^J = \mathrm{Tr}\Big(B^T B\Big) \end{aligned} \tag{4.1.17}$$

Not only is it non-zero in general but it might even be divergent. For general spacetimes, there is no way to decide which of the representations are preferred. Using additional physical input, we can reduce the ambiguity in the choice of vacuum. In the presence of symmetries, we can require the vacuum to be invariant under the action of the symmetry.

Another important assumption is the Hadamard condition. Consider the two point function of a scalar field theory. We require the singularity structure of the two point function to resemble the one in Minkowski space, i.e.

$$\langle 0| \phi(x)\phi(y) |0\rangle = \frac{u(x,y)}{\sigma(x,y)} + v(x,y) \log(\sigma(x,y)) + w(x,y) \tag{4.1.18}$$

where $u(x,x) = (2\pi)^{-2}$ and $\sigma(x,y)$ is the squared geodesic distance between x and y (see [115, 116]). The functions v, w are expanded in powers of $\sigma(x,y)$ as $v(x,y) = \sum_{n=0}^{\infty} v_n(x,y)\sigma^n(x,y)$ and $w(x,y) = \sum_{n=0}^{\infty} w_n(x,y)\sigma^n(x,y)$. Imposing $(\Box_x - m^2) \langle 0| \phi(x)\phi(y) |0\rangle$, we find a recursion relation for the functions $v_n(x,y)$ and $w_n(x,y)$ [117]. This fixes all the v_n and the w_n except for w_0. This corresponds to the fact that a Green's function is uniquely determined up to adding a singularity free solution of the Klein-Gordon equation. Hadamard proved that the series for v and w are uniformly converging inside a region where $\sigma(x,y)$ is single valued and the metric is analytic.

In the following we will use the Hadamard condition as a selection criterion for representations of the canonical commutation and adjointness relations. However, even then, different representations are not necessarily unitarily equivalent in spatially non-compact spacetimes.

4.2 Hawking Effect for Black Holes Formed by Gravitational Collapse

We now apply the tools we discussed to quantum fields on a Schwarzschild black hole spacetime. For a detailed review of the Hawking effect see [10, 13]. Black holes in the universe are forming through gravitational collapse and we need to generalise from the case of eternal black holes discussed so far. We assume that we have a portion of spacetime in the past, which is well described by a Minkowski spacetime. Then, in an intermediate region, a star forms and collapses to a black hole. After enough time passes, the black hole settles to a final state which is well described by the Schwarzschild black hole spacetime. At $\mathscr{I}^-$ the vacuum state of the quantum field is well described by the Minkowski vacuum, while at $\mathscr{I}^+$ the vacuum is well described by the vacuum in a Schwarzschild spacetime.

There is no realistic and exact collapse solution for the intermediate region and we have to use approximations motivated by physical arguments in order to compare the two vacua. For the computations we assume a massless scalar field minimally coupled to gravity.

The solutions of the Klein-Gordon equation in Minkowski spacetime are well known. The linearity of the wave equation allows us to construct the general solution of the wave equation as linear combinations of modes. A mode is a solution of the wave equation for pure frequency ω and near past null infinity $\mathscr{I}^-$ we have the mode expansion

$$\phi\Big|_{\mathscr{I}^-} = \sum_{lm} \int_{\mathbb{R}^+} \frac{\mathrm{d}\omega}{\sqrt{2\pi}\sqrt{2\omega}} \left(a^M_{lm\omega} \frac{1}{r} e^{i\omega v} Y_{lm} + \text{c.c.} \right), \tag{4.2.1}$$

where $v = t + r$ and $u = t - r$ is constant on $\mathscr{I}^+$.

In the Schwarzschild part of spacetime we expand the scalar field in terms of spherical harmonics $\phi = \sum_{lm} b_{lm} Y_{lm}$ The wave equation for ϕ reduces to a two dimensional wave equation for b_{lm} which reads

$$(-\partial_t^2 + \partial_{r^*}^2) b_{lm} - \left(1 - \frac{r_s}{r}\right) \left[\frac{l(l+1)}{r^2} + \frac{r_s}{r^3}\right] b_{lm} = 0 \tag{4.2.2}$$

where $r^* = r + r_s \log(r/r_s - 1)$ is the tortoise coordinate. At the horizon $r \to r_s$ we have $r^* \to -\infty$ and for $r \to \infty$ we have $r^* \to \infty$. The potential vanishes both for $r^* \to -\infty$ and $r^* \to \infty$. At $\mathscr{I}^+$, we can find the solution of the wave equation behaves as

$$\phi\Big|_{\mathscr{I}^+} = \sum_{lm} \int_{\mathbb{R}^+} \frac{\mathrm{d}\omega}{\sqrt{2\pi}\sqrt{2\omega}} \left(a^S_{lm\omega} \frac{1}{r} e^{i\omega U} Y_{lm} + \text{c.c.} \right) \tag{4.2.3}$$

where $U = t - r^*$ and $V = t + r^*$ is constant at $\mathscr{I}^+$.

Past null infinity defines a Cauchy surface $\Sigma_- := \mathscr{I}^-$, but future null infinity $\mathscr{I}^+$ does not. Worldlines falling into the black hole will hit the singularity and are not seen in $\mathscr{I}^+$. We need to add the horizon of the black hole H, in order to make it a complete Cauchy surface $\Sigma_+ := H \cup \mathscr{I}^+$.

We would now like to compute the Klein-Gordon inner product at Σ_- to determine the Bogoliubov coefficients. For this we have to propagate modes at $\mathscr{I}^+$ back through the collapsing star to $\mathscr{I}^-$. Let ϕ be a solution of the Klein-Gordon equation $\Box\phi = 0$ of the form $\phi = Ae^{iS}$ where A is varying much slower than S. Defining $l = \nabla S$, the Klein-Gordon equation implies that $l^i l_i = 0$. Additionally, we have $(\nabla_l l)_i = l^j \nabla_j l_i = l^j \nabla_i \nabla_j S = l^j \nabla_i l_j = 0$. Therefore, in this approximation, we have that l is null and tangent to a null geodesic. In order to propagate the field from $\mathscr{I}^+$ to $\mathscr{I}^-$, we cover the spacetime with a grid of constant $u = -2r_s e^{-U/(2r_s)}$ and $v = 2r_s e^{V/(2r_s)}$ (Kruskal coordinates). The null ray connecting $\mathscr{I}^+$

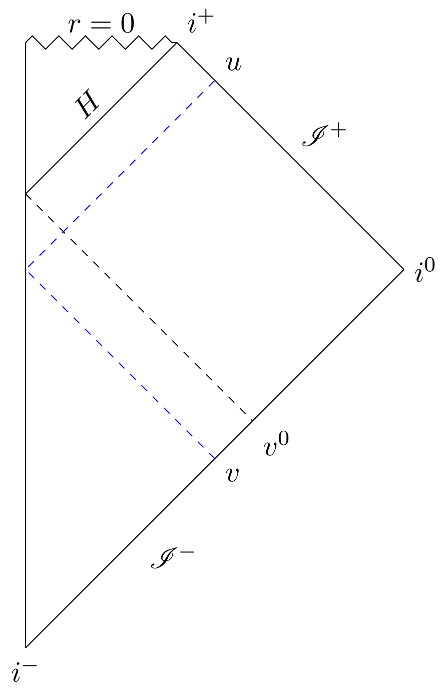

Figure 4.1: Penrose diagram for the computation of Hawking radiation. The dashed black line is the last null ray from past null infinity $\mathscr{I}^-$ reaching an observer at $\mathscr{I}^+$. All the null rays for $v > v_0$ will fall into the singularity. The dashed blue line corresponds to the path of a null ray from $\mathscr{I}^-$ to $\mathscr{I}^+$ assuming the geometric optics approximation.

to $\mathscr{I}^-$ has two parts (see figure 4.1): Starting from $\mathscr{I}^+$ at a point u, we follow a path with constant u until we reach $r = 0$. Then, the null ray transitions to an ingoing null ray and it goes along a path with constant v towards $\mathscr{I}^-$. Along this full null ray, we set the phase S to be equal to the phase S that we have at $\mathscr{I}^+$. This approximation is called the geometric optics approximation.

From the Penrose diagram we see that $v = v^0 + u$ and using the relation between u and U we have

$$v = v^0 - 2r_s e^{-\frac{U}{2r_s}} \qquad \Longrightarrow \qquad U = -2r_s \log\left(\frac{v^0 - v}{2r_s}\right) \tag{4.2.4}$$

Since v^0 is the last ray reaching $\mathscr{I}^+$, we have

$$f^S_{\omega lm}(v) = \begin{cases} \frac{1}{\sqrt{2\pi}\sqrt{2\omega}r} e^{i\omega U(v)} Y_{lm} & \text{for } v < v^0 \\ 0 & \text{for } v > v^0 \end{cases} \tag{4.2.5}$$

Then, we have to compute the Klein-Gordon inner product between $f^S_{\omega lm}$ and the Minkowski space modes $f^M_{\omega lm} = \frac{1}{\sqrt{2\pi}\sqrt{2\omega}} \frac{1}{r} e^{i\omega v} Y_{lm}$. We find

$$\begin{aligned} \langle f^S_{\omega lm}, f^M_{\omega' l' m'} \rangle &= \frac{i}{2\pi\sqrt{\omega\omega'}} \delta_{ll'} \delta_{mm'} \int_{-\infty}^{v_0} \mathrm{d}v (i\omega' + i\omega \partial_v U(v)) e^{i\omega' v - i\omega U(v)} \\ &= \frac{-1}{2\pi\sqrt{\omega\omega'}} \delta_{ll'} \delta_{mm'} \int_{-\infty}^{v_0} \mathrm{d}v \left(\omega' + \frac{2r_s\omega}{v^0 - v}\right) e^{i\omega' v - i\omega U(v)} \end{aligned} \tag{4.2.6}$$

Without loss of generality we consider $v^0 = 0$ and introduce the variable $t = -v\omega'$. The inner product reads

$$\langle f^S_{\omega lm}, f^M_{\omega' l' m'} \rangle = \frac{-1}{2\pi\sqrt{\omega\omega'}} \delta_{ll'} \delta_{mm'} (2r_s\omega')^{-2ir_s\omega} \int_0^\infty \mathrm{d}t \left(1 + \frac{2r_s\omega}{t}\right) t^{2ir_s\omega} e^{-it} \tag{4.2.7}$$

We realize the similarity of the integrand with the definition of the gamma function $\Gamma(z)$ defined by

$$\Gamma(z) = \int_0^\infty \frac{\mathrm{d}t}{t} t^z e^{-t} \tag{4.2.8}$$

and using contour techniques we find that

$$\Gamma(z) = \pm e^{\pm i\pi z/2} \int_0^\infty \frac{\mathrm{d}s}{s} s^z e^{\pm is} \tag{4.2.9}$$

The integral in (**4.2.7**) involves the gamma functions for $z = 2ir_s\omega + 1$ and $z = 2ir_s\omega$ and we have

$$\langle f^S_{\omega lm}, f^M_{\omega' l' m'} \rangle = \frac{i}{2\pi\sqrt{\omega\omega'}} \delta_{ll'} \delta_{mm'} (2r_s\omega')^{-2ir_s\omega} e^{\pi r_s\omega} (\Gamma(2ir_s\omega + 1) + 2ir_s\omega\Gamma(2ir_s\omega)) \tag{4.2.10}$$

Then, using the identity $\Gamma(z+1) = z\Gamma(z)$ we have

$$\langle f^S_{\omega lm}, f^M_{\omega' l' m'} \rangle = \frac{i}{\pi\sqrt{\omega\omega'}} \delta_{ll'} \delta_{mm'} (2r_s\omega')^{-2ir_s\omega} e^{\pi r_s\omega} \Gamma(2ir_s\omega + 1) \tag{4.2.11}$$

Using similar steps, we find for the other Bogoliubov coefficient

$$\langle f^S_{\omega lm}, (f^M_{\omega' l' m'})^* \rangle = \frac{-i}{\pi\sqrt{\omega\omega'}} \delta_{ll'} \delta_{mm'} (2r_s\omega')^{-2ir_s\omega} e^{-\pi r_s\omega} \Gamma(2ir_s\omega + 1) \tag{4.2.12}$$

We obtain the relation $|\langle f^S_{\omega lm}, (f^M_{\omega' l' m'})^* \rangle| = e^{-2\pi r_s\omega} |\langle f^S_{\omega lm}, f^M_{\omega' l' m'} \rangle|$

In the previous section, we saw that the number of particles in the mode is equal to

$$\langle N^S_{\omega lm} \rangle_M = {}_M\langle 0| (a^S_{\omega lm})^\dagger a^S_{\omega lm} |0\rangle_M = \sum_{l'm'} \int_{\mathbb{R}+} \mathrm{d}\omega' \, |\langle f^S_{\omega lm}, (f^M_{\omega' l' m'})^* \rangle|^2 \tag{4.2.13}$$

We define the greybody factor Γ_ω as

$$\Gamma_{\omega lm} = \sum_{l'm'} \int_{\mathbb{R}+} \mathrm{d}\omega' |\langle f^S_{\omega lm}, f^M_{\omega' l' m'} \rangle|^2 - |\langle f^S_{\omega lm}, (f^M_{\omega' l' m'})^* \rangle|^2 \tag{4.2.14}$$

In the previous section, we showed that a Bogoliubov transformation with coefficients A_{IJ}, B_{IJ} has to satisfy the relation $A_{IK}(A_J{}^K)^* - B_{IK}(B_J{}^K)^* = \delta_{IJ}$. Rewriting A and B in terms of the inner products of the modes $f^{S/M}_{\omega lm}$ implies that the integrand for the greybody factor is infinite. It involves a delta distribution evaluated at zero which arises because we used plane waves for the computation. For properly normalized wave packets the greybody factor will be finite.

In the propagation of the scalar field from $\mathscr{I}^+$ to $\mathscr{I}^-$ we completely ignored any backscattering effects due to the curvature of spacetime. Consider the situation in the other direction, i.e. a scalar field at $\mathscr{I}^-$ which propagates towards $\mathscr{I}^+$. Then, due to the potential in (**4.2.2**), there will be a certain fraction $1 - \Gamma_{\omega lm}$ which is backscattered towards $\mathscr{I}^+$ without getting close to the event horizon (reflection at the potential barrier). This mode does not contribute to Hawking radiation. The remaining fraction Γ_ω is responsible for

Hawking radiation as it passes close to the event horizon. For large ω we expect the wave to pass through the potential barrier and we have $\Gamma_{\omega lm} \approx 1$. For small ω we expect $\Gamma_{\omega lm} \ll 1$ because a significant portion of the wave is backscattered.

Combining the definition of the greybody factor $\Gamma_{\omega lm}$ in (**4.2.14**) with equation (**4.2.13**), we find

$$\langle N^S_{\omega lm} \rangle_M = \frac{\Gamma_{\omega lm}}{e^{\beta\omega} - 1} \tag{4.2.15}$$

where $\beta = 4\pi r_s$. Except for the greybody factor, this is a Bose distribution for the number of particles with inverse temperature β. The corresponding temperature is called Hawking temperature T_H and is defined by

$$T_H = \frac{1}{4\pi r_s} = \frac{1}{8\pi}\frac{\hbar c^3}{G k_B}\frac{1}{M} \sim 6.17 \times 10^{-8} K \left(\frac{M_\odot}{M}\right) \tag{4.2.16}$$

where we restored units and computed the temperature for a solar mass black hole. The temperature is very small for black holes of solar mass and up but the expression diverges for $M \to 0$.

4.3 Hawking Effect for Eternal Black Holes

In the derivation above we considered a black hole which formed from gravitational collapse. For the derivation of the Hawking effect, we had to make some assumptions about the dynamics of the collapse in order to propagate the field from $\mathscr{I}^+$ to $\mathscr{I}^-$. A more formal and in-depth analysis is possible for the maximally extended Schwarzschild spacetime in Kruskal coordinates (see figure 4.2).

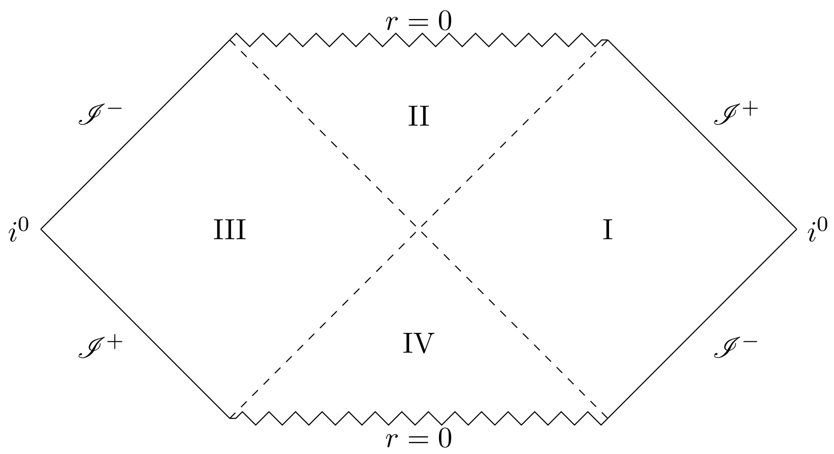

Figure 4.2: Maximally (Kruskal) extension of the Schwarzschild spacetime. The regions are: the exterior regions I and III, the black hole region II and the white hole region IV. The diagonal lines are the event horizons of the black and white hole region.

In this spacetime, we have a black hole and a white hole region (regions II and IV in figure 4.2). The white hole region is not present in the situation of gravitational collapse. On this spacetime, we can consider a scalar quantum field theory as explained before. We saw that every representation of the algebra of creation and annihilation operators corresponds to different vacuum states $|0\rangle$. Such a vacuum state $|0\rangle$ is called quasi-free, if all of its n-point functions can be reduced to 1 and 2-point functions. There are three main proposals for quasi-free, vacuum states on this spacetime [105]: The first one is the Hartle-Hawking vacuum which is a regular Hadamard state everywhere and is invariant under the Schwarzschild Killing vector. It is the unique state with this property. The second state is called the Unruh vacuum which is singular on the white hole horizon but is regular on the black hole horizon and at future infinity. The third vacuum is the Boulware vacuum which is singular on both horizons. In this state there is no Hawking radiation, i.e. no flux of radiation at future and past null infinity.

A first step to compute the backreaction is to understand the expectation value of the energy momentum tensor in the black hole background. This quantity is in general divergent because the definition of the energy momentum tensor involves products of operator valued distributions evaluated at the same spacetime point. Since, the product of operator valued distributions is ill-defined, we have to choose suitable regularization strategies. One proposal is due to Wald [8], where the points where the operator valued distributions are evaluated are separated into two different points x, y. Afterwards the coincidence limit $x \to y$ is taken. If the states are of Hadamard form, the construction is guaranteed to work.

For the Hartle-Hawking vacuum the renormalized energy momentum tensor has successfully been constructed [118]. The energy momentum tensor for the Unruh vacuum is studied by taking the difference between the Unruh and the Hartle-Hawking vacuum [119]. In these calculations, detailed expressions for the flux of radiation and the total power can be obtained. One finds that the power of the radiation is inversely proportional to the square of the mass. This agrees with the heuristic argument based on the Stefan-Boltzmann law mentioned in the introduction.

The next step of the computation would be to plug the expressions for the expectation values of the energy momentum tensor into the Einstein equations. However, such a calculation is highly non-trivial. Instead, let us assume that the energy which is radiated away decreases the mass of the black hole, i.e. $\dot{M} \propto -M^{-2}$. Then an integration of this relation gives a lifetime for the black hole which scales as M^3. This equation for the change of the black hole mass has also been found in numerical simulations based on the Vaidya spacetime [120, 121].

For the computation of the black hole lifetime as explained above, two important assumptions were made: First, the derivation of the power of the Hawking radiation depends on the assumption that the black hole mass is fixed. Second, the curvature at the horizon are small compared to the Planck curvature, i.e. $M \gg M_{\text{Planck}}$. Both of these assumptions will not be satisfied towards the end of the black hole lifetime, when M approaches the Planck mass. At this point, quantum gravity effects will play an important role and might completely change the picture. Therefore, these semi-classical techniques will not be able to provide an answer to the fate of evaporating black holes.

5

Black Hole Perturbation Theory in General Relativity

This part is the main part of this thesis where we apply the tools developed in sections 2 to 4 to investigate perturbation theory around spherically symmetric spacetimes. Perturbation theory is a very important concept within general relativity because the Einstein equations are highly non-linear and exact solutions are only known under special symmetry assumptions. In section 3 we saw that we obtain the black hole solutions under the assumptions of spherical and axial symmetry. Another important class of solutions are cosmological models for which we impose isotropy (the universe looks the same in every direction) and homogeneity (the universe is the same at every point).

In the first part of this chapter, we review black hole perturbation theory in the Regge-Wheeler-Zerilli approach using linearized equations of motion in a modern notation following [74]. We define gauge invariant variables and reduce the equation of motions to two master equations for two master variables.

In the second part, we apply the new Hamiltonian framework to perturbations around spherically symmetric spacetimes. The discussion is based on the publications [1–3, 98]. The first paper [98] discusses the general setup and how the reduced phase space programme is applied to black hole perturbation theory. It discusses the definition of the true and gauge degrees of freedom and how to compute the reduced Hamiltonian. In [2], the formalism is applied to second order perturbation theory in the pure gravity case. The explicit solution of the constraint equations are constructed and the physical Hamiltonian is simplified using several canonical transformations. For a consistency check, the equations of motion of the reduced Hamiltonian are compared with the Lagrangian approach to black hole perturbation theory when backreaction is neglected. This is a non-trivial step because the setup is conceptually very different from previous treatments.

The analysis was generalised in [3] to include the electromagnetic field as matter. Finally, the publication [1] considers generalised gauges for the spherically symmetric degrees of freedom with the same asymptotic behaviour as the GP gauge condition. The paper shows that the same arguments lead to a generalized reduced Hamiltonian as for the strict GP gauge and it reduces to the physical Hamiltonian in [3], if we enforce the strict GP gauge.

The real virtue of our approach is that it clearly defines the meaning of gauge invariance independent of perturbation theory. We define the true and gauge degrees of freedom on the full phase space and obtain a non-perturbative expression for the physical Hamiltonian. Only in a second step, we perturbatively expand the reduced Hamiltonian in terms of the true non-symmetric degrees of freedom. This calculation can be cast into the language of standard perturbation theory of constraints with respect to all degrees of freedom including gauge degrees of freedom with precise coefficients [71].

Another advantage of the non-perturbative definition of the true degrees of freedom is the possibility to generalize to higher orders. In the standard approach to Hamiltonian perturbation theory, we need to define gauge invariant variables order by order in perturbation theory. There is no consensus in the literature how to define gauge invariants beyond second order.

Furthermore, working in GP gauge, we have access both to the interior and exterior of the black hole. This is important for a complete picture of black holes, where one has to keep track of matter falling into the black hole. In the literature, one is usually concerned with the exterior region of the black hole spacetime studying the dynamics of gravitational waves on a black hole background.

Additionally, we incorporate backreaction effects which arise due to interactions between the symmetric and non-symmetric true degrees of freedom. In the literature one usually fixes the background spacetime and treats it as independent of the perturbations. In [75], Moncrief considers a fixed Schwarzschild spacetime and in [80, 81], the background is general but interactions between the background and the perturbations are ignored. The backreaction in this thesis looks rather trivial because we only study general relativity coupled to electromagnetic matter and the symmetric true degrees of freedom are the black hole mass M and the electric charge ξ. Since both of them arise as integration constants from solving the constraints, they are fixed and will not change dynamically. In [98], a scalar field as matter is discussed and there are dynamical, symmetric, true degrees of freedom. In this case and for other matter like massive bosons and fermions there will be non-trivial backreaction.

5.1 Lagrangian Perturbation Theory

In the following we review the Lagrangian approach to black hole perturbation theory based on a linearization of the Einstein equations (**3.3.6**). The study of linear perturbations around the Schwarzschild black hole was first initiated by Regge, Wheeler and Zerilli [72, 73]. Our exposition is based on the more modern reformulation in [74] and also follows appendix E of [2]. The idea for the Lagrangian perturbation theory is as follows:

Assume we know an explicit solutions of the equations for some metric g_{ij} and energy-momentum tensor T_{ij}. This solution is usually obtained by assuming highly symmetric field configurations. Then, we consider small deviations from the exact solution in the metric h_{ij} and the energy-momentum tensor t_{ij}. The deviations are assumed to be small corrections so that it is valid to expand the Einstein equations around the exact solutions. To first order, we will obtain a linear system of equations for the perturbations.

However, the equations of general relativity are generally covariant, i.e. they transform under changes of coordinates. The definition, of what is a small correction depends on the system of coordinates. Additionally, the perturbations we are studying are in general not corresponding to physical observables because they are depending on the choice of coordinates.

The way out of this is the introduction of combinations of the perturbations which are invariant under changes of coordinates. Of course this cannot be achieved exactly and we have to restrict to coordinate transformations up to first order in the perturbations. Consider a vector field $\xi^i(x)$ on spacetime, which generates diffeomorphisms via its flow $\Phi^\xi_\epsilon : \mathcal{M} \to \mathcal{M}$. For small ϵ a tensor T changes by

$$(\varphi^\epsilon_\xi)^* T = T + \epsilon \mathcal{L}_\xi T + O(\epsilon^2) \tag{5.1.1}$$

In the equation we used the definition of the Lie derivative $\mathcal{L}_\xi$. Therefore, to first order a tensor is gauge invariant if $\mathcal{L}_\xi T = 0$.

The strategy for a gauge invariant perturbation theory is as follows: From the perturbations of the metric h_{ij} and the energy-momentum tensor t_{ij}, we construct quantities which are invariant under gauge transformations. Then, we insert these quantities into the equations of motion and derive the dynamics for the gauge invariants. The gauge invariants will correspond to physical observables because they will not change under changes of coordiantes (to first order).

As discussed before, the background metric for spherical symmetry factorizes as $M \times S^2$ into a part involving the metric on the sphere S^2 and a part for a two dimensional manifold M. This suggests a decomposition of the perturbations into spherical scalar, vector and tensor harmonics [122–124]. We show a brief review of their theory here and derive some useful formulae.

For a function $f : S^2 \to \mathbb{C}$ consider the equation $\Omega^{AB} D_A D_B f = -l(l+1)f$ with $l = 0, 1, 2, \ldots$. A convenient and complete set of solutions are the **spherical harmonics** usually denoted by $Y_{lm} : S^2 \to \mathbb{C}$. They are defined by

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P^l_m(\cos\theta) e^{im\phi} \tag{5.1.2}$$

where $P^l_m(\cdot)$ are the associated Legendre polynomials. Under complex conjugation, the spherical harmonics transform as $(Y_{lm})^* = (-1)^m Y_{l,-m}$. The spherical harmonics are square integrable functions on the sphere and form an orthonormal basis with respect to the measure on the sphere.

$$\int_{S^2} \mathrm{d}\Omega \sqrt{\Omega} (Y_{lm})^* Y_{l'm'} = \delta_{ll'} \delta_{mm'} \tag{5.1.3}$$

where $\sqrt{\Omega} := \sqrt{\det \Omega}$. For real-valued functions it is convenient to introduce real valued spherical harmonics. They are defined by

$$L_{lm} := \begin{cases} \frac{1}{\sqrt{2}}((Y_{lm})^* + Y_{lm}), & m > 0 \\ Y_{l,0}, & m = 0 \\ \frac{i}{\sqrt{2}}((Y_{lm})^* - Y_{lm}), & m < 0 \end{cases} \tag{5.1.4}$$

Then, every real valued function on the sphere can be expanded in terms of the real-valued spherical harmonics with real coefficients. Let $f : S^2 \to \mathbb{R}$ be a function on the sphere. Then we have

$$f = \sum_{lm} f_{lm} L_{lm} \,, \tag{5.1.5}$$

with the coefficients $f_{lm} := \int \mathrm{d}\Omega \sqrt{\Omega} f L_{lm}$ The convergence in (**5.1.5**) is with respect to the L^2 norm on the sphere.

For vector and tensor fields on the sphere, we have to derive a similar decomposition. The **vector spherical harmonics** are defined by

$$[L^e_{lm}]_A := \frac{1}{\sqrt{l(l+1)}} D_A L_{lm}, \quad [L^o_{lm}]_A := \frac{1}{\sqrt{l(l+1)}} \eta_{AB} \Omega^{BC} D_C L_{lm} \,, \tag{5.1.6}$$

where $\eta_{AB} = \sqrt{\Omega} \epsilon_{AB}$ which satisfies $\eta_{AB}\eta^{AC} = \delta^C_B$. These functions are orthogonal with respect to the inner product

$$\int_{S^2} \Omega^{AB} [L^I_{lm}]_A [L^{I'}_{l'm'}]_B \sqrt{\Omega} \mathrm{d}\Omega = \delta^{II'} \delta_{ll'} \delta_{mm'} \,. \tag{5.1.7}$$

For the proof, we use integration by parts where boundary terms can be neglected because the sphere has no boundary. If $I \neq I'$ we obtain the combination of derivatives $\eta^{AB} D_A D_B$ acting on a scalar function L_{lm}. This vanishes due to anti-symmetry and torsion freeness of the derivative D_A. For the two even harmonics we have $\Omega^{AB} D_A D_B L_{lm} = -l(l+1) L_{lm}$. The orthonormality of the scalar spherical harmonics and the normalisation factor gives the result. Consider now two odd spherical harmonics. We have the combination $\Omega^{AB} \eta_A{}^C \eta_B{}^D = \Omega^{CD}$ and it reduces to the case of even spherical harmonics.

The **tensor spherical harmonics** are defined by

$$
\begin{aligned}
[L^{\mathrm{tr}}_{lm}]_{AB} &:= \frac{1}{\sqrt{2}} \Omega_{AB} L_{lm} \\
[L^{e}_{lm}]_{AB} &:= \sqrt{\frac{(l-2)!}{2(l+2)!}} (D_A D_B + \frac{1}{2} l(l+1) \Omega_{AB}) L_{lm} \\
[L^{o}_{lm}]_{AB} &:= \sqrt{\frac{(l-2)!}{2(l+2)!}} D_{(A}\Big(\eta_{B)}{}^C D_C L_{lm}\Big)
\end{aligned}
\tag{5.1.8}
$$

The functions are orthonormal, i.e.

$$
\int_{S^2} \mathrm{d}\Omega \, \sqrt{\Omega} \Omega^{AC} \Omega^{BD} [L^{I}_{lm}]_{AB} [L^{I'}_{l'm'}]_{CD} = \delta^{II'} \delta_{ll'} \delta_{mm'}
\tag{5.1.9}
$$

where $I = \mathrm{tr}, e, o$. By construction, the harmonic $[L^{\mathrm{tr}}_{lm}]_{AB}$ is orthogonal to the other two and L^{tr} is properly normalized because $\Omega_{AB} \Omega^{AB} = 2$ and L_{lm} are orthonormal.

For the other calculations, we show that

$$
D^A [L^{I}_{lm}]_{AB} = -\frac{1}{2} \sqrt{\frac{(l-1)(l+2)}{2}} [L^{I}_{lm}]_B
\tag{5.1.10}
$$

The calculation is based on the observation that for a function f we have $D^A D_B D_A f = R^A{}_{BAC} D^C f + D_B D^A D_A f = D_B (1 + D^A D_A) f$. This relation is based on the connection of the Riemann tensor and the commutator of covariant derivatives.

We prove equation (**5.1.10**) separately for even and odd parity. For the even parity we calculate

$$
\begin{aligned}
\sqrt{\frac{2(l+2)!}{(l-2)!}} D^A [L^{e}_{lm}]_{AB} &= \Big(D^A D_A D_B + \frac{1}{2} l(l+1) D_B\Big) L_{lm} \\
&= \Big(R^A{}_{BAC} D^C - \frac{1}{2} l(l+1) D_B\Big) L_{lm} \\
&= -\frac{(l+2)(l-1)}{2} D_B L_{lm}
\end{aligned}
\tag{5.1.11}
$$

where we used $R_{ABCD} = \Omega_{AC} \Omega_{BD} - \Omega_{AD} \Omega_{BC}$. For the odd parity equations we use that for any function f we have $D_A D_B D_C f = R_{ABCD} D^D f + D_B D_A D_C f$ and obtain

$$
\begin{aligned}
\sqrt{\frac{2(l+2)!}{(l-2)!}} D^A [L^{o}_{lm}]_{AB} &= \frac{1}{2} D^A D_A \eta_B{}^C D_C L_{lm} + \frac{1}{2} \eta^{AC} D_A D_B D_C L_{lm} \\
&= \frac{1}{2} \eta_B{}^C D_C ((1 - l(l+1)) L_{lm}) - \frac{1}{2} \eta^{AC} (\Omega_{AC} \Omega_{BD} - \Omega_{AD} \Omega_{BC}) D^D L_{lm} \\
&= -\frac{(l+2)(l-1)}{2} \eta_B{}^C D_C L_{lm}
\end{aligned}
\tag{5.1.12}
$$

Using these expressions, we can simplify the integrals by integration by parts where the boundary terms vanish. The integrals reduce to integrals over the vector harmonics and we can use their orthonormality properties.

Finally, we list some identities for later use:

$$
\begin{aligned}
D_A D_B [L^o_{lm}]^A &= [L^o_{lm}]_B \\
D_A D_B [L^e_{lm}]^A &= (1 - l(l+1))[L^e_{lm}]_B \\
D_C D^{(A} L^{B)C}_{o,lm} &= \frac{1}{2}(6 - l(l+1)) L^{AB}_{o,lm} \\
D_C D^{(A} L^{B)C}_{e,lm} &= \frac{1}{2}(6 - l(l+1)) L^{AB}_{e,lm} + \frac{1}{2}\sqrt{\frac{(l-1)l(l+1)(l+2)}{2}}\Omega^{AB} L_{lm} \\
D^B D_B [L^I_{lm}]_A &= (1 - l(l+1))[L^I_{lm}]_A \\
D^C D_C [L^I_{lm}]_{AB} &= (4 - l(l+1))[L^I_{lm}]_{AB}
\end{aligned}
\tag{5.1.13}
$$

The identities are proven using similar tools as shown before.

5.1.1 Linearized Einstein Equations

In section 3.4 we studied solutions of the Einstein equations (**3.3.6**) for spherical symmetry. For the vacuum case we found the Schwarzschild solution describing non-rotating, uncharged black holes. Then, we added the electromagnetic field and obtained the Reissner-Nordstrøm solution for non-rotating charged black holes.

In the following, we go beyond spherical symmetry by introducing non-symmetric degrees of freedom. In this case, an analytic solution of the Einstein equations has not been found and we use perturbation theory to find approximate solutions. For the computations, we split the metric into a spherically symmetric background and non symmetric perturbations. We decompose the metric as

$$
{}^{(4)}g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j = (g_{ab} + h_{ab})\mathrm{d}x^a \otimes \mathrm{d}x^b + h_{aB}(\mathrm{d}x^a \otimes \mathrm{d}x^B + \mathrm{d}x^B \otimes \mathrm{d}x^a) + (\gamma^2 \Omega_{AB} + h_{AB})\mathrm{d}x^A \otimes \mathrm{d}x^B \,. \tag{5.1.14}
$$

Recall the notation of section 3.4: The spherically symmetric metric was of product form $M \times S^2$. On M we had indices $a, b, \ldots$ running over $t, 3$ with local coordinates $x^a = (t, r)$ and a spherically symmetric metric g_{ab}. On S^2 we used indices $A, B, \ldots$ running over the angular directions θ, ϕ with local coordinates $x^A = (\theta, \phi)$ and the metric on the sphere Ω_{AB}. γ is a function of x^a. The unique torsion free connection compatible with g_{ab} is denoted by ∇ and the one associated to Ω_{AB} is D_A. The non-symmetric corrections are h_{ab}, h_{aB} and h_{AB}.

Similarly, we split the full energy-momentum tensor into background and perturbations:

$$
{}^{(4)}T_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = (T_{ab} + t_{ab}) \otimes \mathrm{d}x^a \otimes \mathrm{d}x^b + t_{aB}(\mathrm{d}x^a \otimes \mathrm{d}x^B + \mathrm{d}x^B \otimes \mathrm{d}x^a) + (\gamma^2 T \Omega_{AB} + t_{AB})\mathrm{d}x^A \otimes \mathrm{d}x^B \tag{5.1.15}
$$

T_{ab} and T are the spherically symmetric background contributions and t_{ab}, t_{aB} and t_{AB} are the non-symmetric perturbations. For both the metric and the energy-momentum tensor, we assumed that all the symmetric contributions are put into the background variables.

Assume, that we successfully solved the background Einstein equations derived already in section 3.4:

$$
\kappa T_{ab} = \left(-\frac{2}{\gamma}\nabla_a \nabla_b \gamma - g_{ab}\left(\frac{1}{\gamma^2} - \frac{\gamma_a \gamma^a}{\gamma^2} - \frac{2}{\gamma}\Box\gamma\right)\right) \tag{5.1.16}
$$

$$\kappa T = \frac{1}{\gamma^2}\left(\gamma\Box\gamma - \frac{1}{2}{}^2R\gamma^2\right) \tag{5.1.17}$$

Recall that $\gamma_a := \nabla_a\gamma$ and $\Box := g^{ab}\nabla_a\nabla_b$ is the d'Alembertian of the metric g_{ab}.

The linearized Einstein equations are obtained by computing the linearized Einstein tensor which is constructed from the linearized Ricci tensor. First, we derive the linearized Christoffel symbols. For simplicity of notation, we use the metric g_{ab} to raise and lower the indices $a, b, c, \ldots$ and Ω_{AB} to raise and lower the indices $A, B, C, \ldots$. The first order corrections to the Christoffel symbols are given by

$$\delta\Gamma^a_{bc} = \frac{1}{2}(\nabla_c h^a{}_b + \nabla_b h^a{}_c - \nabla^a h_{bc}) \tag{5.1.18}$$

$$\delta\Gamma^a_{bC} = \frac{1}{2}(D_C h^a{}_b + \nabla_b h^a{}_C - \nabla^a h_{bC}) - \frac{\gamma_b}{\gamma}h^a{}_C \tag{5.1.19}$$

$$\delta\Gamma^a_{BC} = \frac{1}{2}(D_B h^a{}_C + D_C h^a{}_B - \nabla^a h_{BC}) + \gamma\gamma_d\Omega_{BC}h^{ad} \tag{5.1.20}$$

$$\delta\Gamma^A_{bc} = \frac{1}{2\gamma^2}\left(\nabla_b h^A{}_c + \nabla_c h^A{}_b - D^A h_{bc}\right) \tag{5.1.21}$$

$$\delta\Gamma^A_{bC} = \frac{1}{2\gamma^2}\left(D_C h^A{}_b - D^A h_{bC} + \nabla_b h^A{}_C\right) - \frac{\gamma_b}{\gamma^3}h^A{}_C \tag{5.1.22}$$

$$\delta\Gamma^A_{BC} = \frac{1}{2\gamma^2}\left(D_C h^A{}_B + D_B h^A{}_C - D^A h_{BC}\right) + \frac{\gamma_d}{\gamma}\Omega_{BC}h^{dA} \tag{5.1.23}$$

From the perturbed Christoffel symbols we compute the linear perturbations of the Ricci tensor

$$\begin{aligned}\delta R_{ab} = {} & \nabla_c(\nabla_a h^c{}_b + \nabla_b h^c{}_a - \nabla^c h_{ab}) + \frac{\gamma^c}{\gamma}(\nabla_a h_{cb} + \nabla_b h_{ca} - \nabla_c h_{ab}) - \frac{1}{2}\nabla_a\nabla_b h^c{}_c \\ & - \frac{1}{2\gamma^2}D^C D_C h_{ab} + \frac{1}{2\gamma^2}D^C\left(\nabla_a h_{Cb} + \nabla_b h_{Ca}\right) - \frac{1}{2\gamma^2}\nabla_a\nabla_b h^C{}_C \\ & + \frac{1}{2\gamma^3}\left(\gamma_a\nabla_b h^C{}_C + \gamma_b\nabla_a h^C{}_C\right) - \frac{1}{\gamma^4}\left(\gamma_a\gamma_b - \gamma\nabla_a\nabla_b\gamma\right)h^C{}_C\end{aligned} \tag{5.1.24}$$

$$\begin{aligned}\delta R_{aB} = {} & \frac{1}{2}D_B\left(\nabla_c h^c{}_a - \nabla_a h^c{}_c + \frac{1}{\gamma}\gamma_a h^c{}_c\right) - \frac{1}{2}\left(\Box h_{aB} - \nabla_c\nabla_a h^c{}_B\right) \\ & - \frac{1}{\gamma}\left(\gamma_a\nabla_c h^c{}_B - \gamma_c\nabla_a h^c{}_B\right) - \frac{1}{\gamma^2}\left(\gamma_a\gamma_c + \gamma\nabla_a\nabla_c\gamma\right)h^c{}_B + \frac{1}{2\gamma^2}D^C\left(D_B h_{aC} - D_C h_{aB}\right) \\ & + \frac{1}{2\gamma^2}\nabla_a\left(D_C h^C{}_B - D_B h^C{}_C\right) - \frac{1}{\gamma^3}\gamma_a\left(D_C h^C{}_B - D_B h^C{}_C\right)\end{aligned} \tag{5.1.25}$$

$$\begin{aligned}\delta R_{AB} = {} & \Omega_{AB}\left[\gamma\gamma_c\nabla_d\left(h^{cd} - \frac{1}{2}g^{cd}h^e{}_e\right) + \left(\gamma_c\gamma_d + \gamma\nabla_c\nabla_d\gamma\right)h^{cd}\right] - \frac{1}{2}D_A D_B h^c{}_c + \frac{1}{\gamma}\gamma_d\Omega_{AB}D_C h^{dC} \\ & + \frac{1}{2}\nabla_c\left(D_A h^c{}_B + D_B h^c{}_A\right) - \frac{1}{2}\Box h_{AB} + \frac{1}{2\gamma^2}D_C(D_A h^C{}_B + D_B h^C{}_A - D^C h_{AB}) \\ & - \frac{1}{2\gamma^2}D_A D_B h^C{}_C + \frac{1}{\gamma}\gamma^c\nabla_c\left(h_{AB} - \frac{1}{2}\Omega_{AB}h^C{}_C\right) - \frac{2}{\gamma^2}\gamma^c\gamma_c\left(h_{AB} - \frac{1}{2}\Omega_{AB}h^D{}_D\right)\end{aligned} \tag{5.1.26}$$

Before computing the linearized Ricci scalar and computing the Einstein tensor as the sum of the linearized Ricci scalar and Ricci tensor, we study the gauge transformations. As it turns out, the linearized Einstein equations simplify drastically when presented in gauge invariant variables. Additionally, the discussion is further simplified exploiting the spherical symmetry of the background spacetime. The perturbations of the metric and the energy-momentum tensor are expanded into scalar, vector, and tensor spherical harmonics. To linearized order, the modes with different l, m decouple and can be treated separately. We use the convention

$$\begin{aligned}
h_{ab} &= \sum_{l,m} h^{lm}_{ab} L_{lm}, \quad t_{ab} = \sum_{l,m} t^{lm}_{ab} L_{lm} \\
h_{aB} &= \sum_{l,m} \sum_{I\in\{e,o\}} h^{I,lm}_{a} [L^{lm}_I]_B, \quad t_{aB} = \sum_{l,m} \sum_{I\in\{e,o\}} t^{I,lm}_{a} [L^{lm}_I]_B \\
h_{AB} &= \gamma^2 \sum_{l.m} \left(h^{tr,lm} L_{lm} \Omega_{AB} + \sum_{I\in\{e,o\}} h^{I,lm} [L^{lm}_I]_{AB} \right) \\
t_{AB} &= \gamma^2 \sum_{l,m} \left(t^{tr,lm} \Omega_{AB} L_{lm} + \sum_{I\in\{e,o\}} t^{I,lm} [L^{I}_{lm}]_{AB} \right)
\end{aligned} \tag{5.1.27}$$

Here, tr stands for the trace part of the tensor perturbation and $I = e, o$ denotes the even and odd parity harmonics. The sum over l starts at $l = 1$ except for the tensor harmonics where it starts at $l = 2$. The $l = 0$ contributions are spherically symmetric and we assume them to be part of the background variables.

The gauge invariant variables are found by studying gauge transformations of the metric, i.e. changes of coordinates generated by a vector field and by constructing combinations of the variables independent of this transformation (to linear order). In the preliminaries we saw how tensor fields transform under gauge transformations generated by a vector field ξ^i in terms of the Lie derivative. Adapting to the spherically symmetric background and the decomposition of the perturbations, we split the vector field into the components $\xi^i = (\xi^a, \xi^A)$. Then, we find the transformations

$$\begin{aligned}
h_{ab} &\to h_{ab} - \nabla_a \xi_b - \nabla_b \xi_a, \quad h_{aB} \to h_{aB} - \nabla_a \xi_B - D_B \xi_a + \frac{2\gamma_a}{\gamma} \xi_B \\
h_{AB} &\to h_{AB} - D_A \xi_B - D_B \xi_A - 2\gamma\gamma^a \xi_a \Omega_{AB} \\
t_{ab} &\to t_{ab} - \xi^c \nabla_c T_{ab} - \nabla_a \xi^c T_{cb} - \nabla_b \xi^c T_{ac}, \quad t_{Ab} \to t_{Ab} - D_A \xi^c T_{cb} - \nabla_b \xi_A T + 2\frac{\gamma_b}{\gamma} \xi_A T \\
t_{AB} &\to t_{AB} - \gamma^2 \xi^c \nabla_c T \Omega_{AB} - 2\gamma\gamma_a \xi^a T \Omega_{AB} - D_A \xi_B T - D_B \xi_A T
\end{aligned} \tag{5.1.28}$$

In order to determine the transformation of the modes in the expansion into scalar, vector and tensor spherical harmonics we decompose the components of the vector field ξ^i according to $\xi^a = \sum_{lm} \xi^a_{lm} L_{lm}$ and $\xi^A = \sum_{I,l,m} \xi^I_{lm} [L^I_{lm}]^A$. This gives the following transformation of the metric perturbations

$$\begin{aligned}
h^{lm}_{ab} &\to h^{lm}_{ab} - \nabla_a \xi^{lm}_b - \nabla_b \xi^{lm}_a \\
h^{e,lm}_a &\to h^{e,lm}_a - \sqrt{l(l+1)} \xi^{lm}_a - \nabla_a \xi^e_{lm} + \frac{2}{\gamma} \gamma_a \xi^e_{lm} \\
h^{o,lm}_a &\to h^{o,lm}_a - \nabla_a \xi^o_{lm} + \frac{2}{\gamma} \gamma_a \xi^o_{lm} \\
h^{tr}_{lm} &\to h^{tr}_{lm} + \frac{\sqrt{l(l+1)}}{\gamma^2} \xi^e_{lm} - \frac{2}{\gamma} \gamma^a \xi^{lm}_a \\
h^I_{lm} &\to h^I_{lm} - \frac{2}{\gamma^2} \sqrt{\frac{(l+2)(l-1)}{2}} \xi^I_{lm}
\end{aligned} \tag{5.1.29}$$

For the transformations of the energy-momentum tensor, we find

$$
\begin{aligned}
t_{ab}^{lm} &\to t_{ab}^{lm} - \xi_{lm}^{c}\nabla_c T_{ab} - \nabla_a \xi_{lm}^{c} T_{cb} - \nabla_b \xi_{lm}^{c} T_{ac} \\
t_a^{e,lm} &\to t_a^{e,lm} - \sqrt{l(l+1)}\xi_{lm}^{b} T_{ab} - \nabla_a \xi_{lm}^{e} T + 2\frac{\gamma_a}{\gamma}\xi_{lm}^{e} T \\
t_a^{o,lm} &\to t_a^{o,lm} - \nabla_a \xi_{lm}^{o} T + 2\frac{\gamma_a}{\gamma}\xi_{lm}^{o} T \\
t_{lm}^{tr} &\to t_{lm}^{tr} - \xi_{lm}^{a}\nabla_a T - \frac{2}{\gamma}\gamma_a \xi_{lm}^{a} T + \frac{\sqrt{l(l+1)}}{\gamma^2}\xi_{lm}^{e} \\
t_{lm}^{I} &\to t_{lm}^{I} - \frac{2}{\gamma^2}\sqrt{\frac{(l+2)(l-1)}{2}}\xi_{lm}^{I}
\end{aligned}
\tag{5.1.30}
$$

We search for combinations of the variables that transform trivially under gauge transformations. The number of independent variables are determined by counting degrees of freedom: The metric and the energy-momentum tensor in four dimensions have both ten degrees of freedom. The vector field which generates the gauge transformations has four components. We choose the gauge invariant variables such that we obtain six gauge invariant variables for the metric and ten gauge invariant variables for the energy-momentum tensor. For the metric we have

$$
\begin{aligned}
\tilde{h}_{ab}^{lm} &= h_{ab}^{lm} - \frac{1}{\sqrt{l(l+1)}}\nabla_a \epsilon_b^{lm} - \frac{1}{\sqrt{l(l+1)}}\nabla_b \epsilon_a^{lm} \\
\tilde{h}_a^{lm} &= h_a^{o,lm} - \sqrt{\frac{2}{(l+2)(l-1)}}\frac{\gamma^2}{2}\nabla_a h^{o,lm} \\
K_{lm} &= h_{lm}^{tr} + \frac{1}{2}\sqrt{\frac{2l(l+1)}{(l+2)(l-1)}}h_{lm}^{e} - \frac{2}{\sqrt{l(l+1)}\gamma}\gamma^a \epsilon_a^{lm}
\end{aligned}
\tag{5.1.31}
$$

We introduced the quantity ϵ_a^{lm} defined by

$$
\epsilon_a^{lm} := h_a^{e,lm} - \frac{1}{2}\sqrt{\frac{2}{(l+2)(l-1)}}\gamma^2 \nabla_a h_{lm}^{e}\,, \tag{5.1.32}
$$

which transforms as $\epsilon_a^{lm} \to \epsilon_a^{lm} - \sqrt{l(l+1)}\xi_a^{lm}$ under gauge transformations.

For the components of the energy-momentum tensor, we find the following gauge invariant quantities

$$
\begin{aligned}
\tilde{t}_{ab}^{lm} &= t_{ab}^{lm} - \frac{1}{\sqrt{l(l+1)}}\nabla_c T_{ab}\epsilon_{lm}^{c} - \frac{2}{\sqrt{l(l+1)}}(T_{ac}\nabla_b \epsilon_{lm}^{c} + T_{bc}\nabla_a \epsilon_{lm}^{c}) \\
\tilde{t}_a^{o,lm} &= t_a^{o,lm} - T h_a^{o,lm} \\
\tilde{t}_a^{e,lm} &= t_a^{e,lm} - \frac{1}{\sqrt{l(l+1)}}T_{ab}\epsilon_{lm}^{b} - \sqrt{\frac{2}{(l+2)(l-1)}}\frac{\gamma^2}{2}T\nabla_a h_{lm}^{e} \\
\tilde{t}_{lm}^{I} &= t_{lm}^{I} - T h_{lm}^{I} \\
\tilde{t}_{lm}^{tr} &= t_{lm}^{tr} - \frac{1}{\sqrt{l(l+1)}}\epsilon_{lm}^{a}\nabla_a T + T h_{lm}^{tr}
\end{aligned}
\tag{5.1.33}
$$

Next, we insert the gauge invariant variables into the perturbed Einstein equations. The linearized Einstein equations split into two independent sets of equations, for the perturbations with even and odd parity. In the following, we will discuss the two cases separately.

5.1.2 The Odd Parity Master Equation

After inserting the gauge invariant variables, the Einstein equations for the odd parity perturbations are given by two differential equations. The equations for different l and m variables are independent and we can solve them separately. For simplicity of notation, we do not display the labels l, m of the perturbations from now on. The odd parity Einstein equations read

$$\nabla_a \tilde{h}^a = \sqrt{\frac{2}{(l+2)(l-1)}} \kappa \gamma^2 \tilde{t}^o \tag{5.1.34}$$

$$\begin{aligned} &\frac{1}{2}\Big(\nabla_b \nabla_a \tilde{h}^b - \Box \tilde{h}_a\Big) - \frac{1}{\gamma}(\gamma_a \nabla_b - \gamma_b \nabla_a)\tilde{h}^b - \frac{1}{\gamma^2}(\gamma_a \gamma_b + \gamma \nabla_a \nabla_b \gamma)\tilde{h}^b \\ &+ \frac{(l+2)(l-1)}{2\gamma^2}\tilde{h}_a + \frac{\Box\gamma}{\gamma}\tilde{h}_a + \frac{\gamma^b \gamma_b}{\gamma^2}\tilde{h}^a = \kappa \tilde{t}^o_a \end{aligned} \tag{5.1.35}$$

The odd parity master equation was first found by Regge-Wheeler in [72]. Later in [125], the odd parity master equation was derived in the presence of matter. The derivation is based on the observation, that the second equation can be rewritten in the form (see [125] for more details)

$$\nabla^c(\gamma^4 \nabla_{[a}(\gamma^{-2}\tilde{h}_{c]})) + \frac{1}{2}(l+2)(l-1)\tilde{h}_a = \gamma^2 \kappa \tilde{t}^o_a \tag{5.1.36}$$

$$\frac{1}{2}\epsilon_{ac}\nabla^c(\gamma^4 \epsilon^{de} \nabla_d(\gamma^{-2} h_e)) + \frac{1}{2}(l+2)(l-1)\tilde{h}_a = \gamma^2 \kappa \tilde{t}^o_a \tag{5.1.37}$$

The first term, involving the differential operators suggests to introduce the variable ψ defined by

$$\psi^o = \gamma^3 \epsilon^{ab} \nabla_a \Big(\gamma^{-2}\tilde{h}_b\Big). \tag{5.1.38}$$

Applying the operator $\epsilon^{ba}\nabla_b(\gamma^{-2}\cdot)$ to the equation and introducing the variable ψ^o, we observe that we obtain a master equation for the variable ψ^o:

$$-\frac{1}{2}\gamma \nabla_a \Big(\gamma^{-2}\nabla^a(\gamma \psi^o)\Big) + \frac{(l+2)(l-1)}{2\gamma^2}\psi^o = \gamma \epsilon^{ab} \nabla_a \tilde{t}^o_b. \tag{5.1.39}$$

After expanding the derivatives, we obtain the Regge-Wheeler equation for the odd parity perturbations

$$\Box \psi^o + \left(-\frac{l(l+1)}{\gamma^2} + \frac{1}{\gamma^2}(2 - 2\gamma_a \gamma^a + \gamma \Box \gamma)\right)\psi^o = -2\gamma \epsilon^{ab} \nabla_a \tilde{t}^o_b\,. \tag{5.1.40}$$

Note that the calculation does not involve the background equations in any way. There is no background energy-momentum tensor showing up. The only dependence on the background matter is through the terms involving γ. This will be different for the even parity perturbations.

Consider the vacuum Schwarzschild solution, $T_{ab} = T = 0$. In section 3.4 we found that $\gamma_a \gamma^a = f$ and $\gamma \Box \gamma = 1 - f$, where $f = 1 - r_s/\gamma$. With these results, the master equation reduces to

$$\Box \psi^o + \left(-\frac{l(l+1)}{\gamma^2} + 3\frac{r_s}{\gamma^3}\right)\psi^o = 0\,. \tag{5.1.41}$$

We also found similar expressions for f and $\Box\gamma$ in the presence of spherically symmetric electromagnetic matter. We have $\gamma \Box \gamma = 1 - f - \frac{\kappa \epsilon_0}{2\gamma^2}Q^2$ with $f = 1 - r_s/\gamma + \frac{\kappa \epsilon_0}{2\gamma^2}Q^2$. In this scenario the master equation becomes

$$\Box\psi^o + \left(-\frac{l(l+1)}{\gamma^2} + 3\frac{r_s}{\gamma^3} - 2\frac{\kappa\epsilon_0}{\gamma^4}Q^2\right)\psi^o = -2\gamma\epsilon^{ab}\nabla_a\tilde{t}^o_b\,. \tag{5.1.42}$$

5.1.3 The Even Parity Master Equation

In the odd parity section, we successfully derived a master equation, which was of the form of a wave equation for the master variable ψ^o. This master variable was a scalar quantity defined as a linear combination of gravitational variables. In the following we perform a similar construction for the even parity sector. The four equations that the perturbations have to satisfy are given by

$$\begin{aligned}\kappa\tilde{t}_{ab} =& \frac{1}{2}\left(\nabla_c\nabla_a\tilde{h}^c_b + \nabla_c\nabla_b\tilde{h}^c_a - \Box\tilde{h}_{ab}\right) + \frac{\gamma_c}{\gamma}\left(\nabla_a\tilde{h}^c_b + \nabla_b\tilde{h}^c_a - \nabla^c\tilde{h}_{ab}\right) - \frac{1}{2}\nabla_a\nabla_b\tilde{h}^c_c \\ &+ \frac{l(l+1)}{2\gamma^2}\tilde{h}_{ab} - \nabla_a\nabla_b K - \frac{1}{\gamma}(\gamma_a\nabla_b + \gamma_b\nabla_a)K + \left(-\frac{{}^2R}{2} - \frac{1}{\gamma^2} + \frac{\gamma^c\gamma_c}{\gamma^2} + \frac{2\Box\gamma}{\gamma}\right)\tilde{h}_{ab} \\ &- \frac{1}{2}g_{ab}\Big(\nabla_c\nabla_d\tilde{h}^{cd} - \Box\tilde{h}^c_c + 2\frac{\gamma_c}{\gamma}\left(2\nabla_d\tilde{h}^{cd} - \nabla^c\tilde{h}^d_d\right) + \frac{2}{\gamma^2}(\gamma_c\gamma_d + 2\gamma\nabla_c\nabla_d\gamma)\tilde{h}^{cd} \\ &+ \frac{l(l+1)}{\gamma^2}\tilde{h}^c_c - 2\Box K - 6\frac{\gamma^c}{\gamma}\nabla_c K + \frac{(l+2)(l-1)}{\gamma^2}K - {}^2R_{cd}\tilde{h}^{cd}\Big)\end{aligned} \tag{5.1.43}$$

$$\begin{aligned}\kappa\gamma^2\tilde{t}^{tr} =& \frac{\gamma}{2}\left(\nabla^a(\gamma\nabla_a\tilde{h}^b_b) - \gamma\nabla_a\nabla_b\tilde{h}^{ab} - 2\gamma_a\nabla_b\tilde{h}^{ab} - 2\nabla_a\nabla_b\gamma\tilde{h}^{ab}\right) - \frac{l(l+1)}{4}\tilde{h}^a_a \\ &+ {}^2R\frac{\gamma^2}{4}\tilde{h}^a_a + \frac{\gamma^2}{2}\Box K + \gamma\gamma^a\nabla_a K\end{aligned} \tag{5.1.44}$$

$$\kappa\tilde{t}^e_a = \frac{\sqrt{l(l+1)}}{2}\left(\nabla_b\tilde{h}^b_a - \nabla_a\tilde{h}^b_b + \frac{\gamma_a}{\gamma}\tilde{h}^b_b - \nabla_a K\right) \tag{5.1.45}$$

$$\kappa\gamma^2\tilde{t}^e = -\frac{1}{2}\sqrt{\frac{(l+2)(l-1)}{2}}\tilde{h}^a_a \tag{5.1.46}$$

The first equation can be simplified using the fact that the Einstein tensor for any two-dimensional manifold vanishes automatically. Therefore, consider $g_{ab} + h_{ab}$ as a two dimensional metric, then to linear order the Einstein tensor is given by

$$\nabla^c\nabla_a h_{bc} + \nabla^c\nabla_b h_{ac} - \Box h_{ab} - \nabla_a\nabla_b h^c_c + g_{ab}(\Box h^c_c - \nabla^c\nabla^d h_{cd}) + \frac{1}{2}{}^{(2)}R(g_{ab}h^c_c - 2h_{ab}) = 0 \tag{5.1.47}$$

Inserting this identity into the first equation, all the second derivatives of h_{ab} are removed from the equation (see [125]).

Furthermore, note that the last equation fixes the trace part of $\tilde{h}_{ab}$. We split the tensor $\tilde{h}_{ab}$ into its trace $h = g^{ab}\tilde{h}_{ab}$ and trace-free components $k_{ab} = \tilde{h}_{ab} - \frac{1}{2}g_{ab}h$. Inserting identity (**5.1.47**) and splitting $\tilde{h}_{ab}$ into k_{ab} and h we find

$$\begin{aligned}\kappa\tilde{t}_{ab} =& \frac{\gamma^c}{\gamma}(\nabla_a k_{cb} + \nabla_b k_{ca} - \nabla_c k_{ab}) + \frac{l(l+1)}{2\gamma^2}k_{ab} - \nabla_a\nabla_b K + \left(-\frac{1}{\gamma^2} + \frac{\gamma^c\gamma_c}{\gamma^2} + \frac{2\Box\gamma}{\gamma}\right)k_{ab} \\ &- \frac{1}{\gamma}(\gamma_a\nabla_b + \gamma_b\nabla_a)K + \frac{1}{2\gamma}(\gamma_a\nabla_b h + \gamma_b\nabla_a h) - \frac{1}{2\gamma}g_{ab}\gamma^c\nabla_c h + \frac{1}{4}g_{ab}\left(\frac{2}{\gamma^2} + \frac{l(l+1)}{\gamma^2}\right)h \\ &- \frac{1}{2}g_{ab}\left(4\frac{\gamma^c}{\gamma}\nabla^d k_{cd} + \frac{2}{\gamma^2}(\gamma_c\gamma_d + 2\gamma\nabla_c\nabla_d\gamma)k^{cd} - 2\Box K - 6\frac{\gamma_c}{\gamma}\nabla^c K + \frac{(l+2)(l-1)}{\gamma^2}K\right)\end{aligned} \tag{5.1.48}$$

$$\kappa\gamma^2\tilde{t}^{tr} = \left(\frac{\gamma^2}{4}\Box h - \frac{\gamma}{2}\Box\gamma h\right) - \frac{\gamma}{2}\left(\gamma\nabla_a\nabla_b k^{ab} + 2\gamma_a\nabla_b k^{ab} + 2\nabla_a\nabla_b\gamma k^{ab}\right) - \frac{l(l+1)}{4}h + {}^2R\frac{\gamma^2}{4}h + \frac{\gamma^2}{2}\Box K + \gamma\gamma^a\nabla_a K \tag{5.1.49}$$

$$\kappa\tilde{t}^e_a = \frac{\sqrt{l(l+1)}}{2}\left(\nabla_m k^m_a - \frac{1}{2}\nabla_a h + \frac{\gamma_a}{\gamma}h - \nabla_a K\right) \tag{5.1.50}$$

$$\kappa\gamma^2\tilde{t}^e = -\frac{1}{2}\sqrt{\frac{(l+2)(l-1)}{2}}h \tag{5.1.51}$$

Let us prepare for the derivation of the master equation with two observations. First, the third equation relates the gradient of K to the divergence of k_{ab}:

$$\nabla_a K = \nabla_b k^b_a + \frac{\gamma_a}{\gamma}h - \frac{1}{2}\nabla_a h - \frac{2}{\sqrt{l(l+1)}}\kappa\tilde{t}^e_a \tag{5.1.52}$$

The second relation is found by plugging the relation into the second equation. We find an expression for the contraction $\nabla_a\nabla_b\gamma k^{ab}$:

$$\gamma\nabla_a\nabla_b\gamma k^{ab} = -\kappa\gamma^2\tilde{t}^{tr} - \left(\frac{l(l+1)}{4} - \frac{\gamma^a\gamma_a}{2} - \frac{\gamma^2}{4}{}^{(2)}R\right)h - \frac{\kappa}{\sqrt{l(l+1)}}\nabla^a(\gamma^2 t^e_a) \tag{5.1.53}$$

In the case of pure gravity, this relation follows from the trace freeness of k_{ab} and the fact that the Einstein equations imply that $\nabla_a\nabla_b\gamma$ is proportional to g_{ab}. In the presence of matter the situation is more complicated and the relation is related to the conservation of the energy-momentum tensor.

For the derivation of the master equation, we need three intermediate equations. First, we take the trace of the first equation. It determines the value of $\Box K$.

$$\Box K - \frac{\lambda}{\gamma^2}K - \frac{2}{\gamma^3}\gamma^a Z_a = S_1\,, \tag{5.1.54}$$

where we defined $Z_a := \gamma(\gamma^b k_{ab} - \gamma\nabla_a K)$ and $\lambda := (l+2)(l-1)$. The source term S_1 is

$$S_1 = \kappa\tilde{t}^a_a - h\left(\frac{1}{\gamma^2} + \frac{l(l+1)}{2\gamma^2}\right) + \frac{4}{\gamma}\nabla_a\nabla_b\gamma k^{ab} + 2\frac{\gamma^a}{\gamma}(\nabla^b k_{ab} - \nabla_a K) \tag{5.1.55}$$

$$= \kappa\tilde{t}^a_a + \frac{\gamma^a}{\gamma}\nabla_a h - \left(\frac{3l(l+1)+2}{2\gamma^2} - {}^{(2)}R\right)h - 4\kappa t^{tr} - \frac{\kappa}{\sqrt{l(l+1)}}\frac{4}{\gamma}\nabla^a(\gamma t^e_a)\,. \tag{5.1.56}$$

For the second equation we contract the first equation with γ^a. Then, we use the equation for S_1 to remove the term $\Box K$. We obtain a differential equation that determines the derivative of the quantity $\gamma^a Z_a$:

$$\nabla_b(\gamma^a Z_a) + \kappa\gamma T_b{}^a Z_a + \frac{l(l+1)}{2\gamma}Z_b + \frac{1}{2}\gamma^3\kappa T_b{}^a\nabla_a K + \frac{\gamma}{2}\left(3(1-\gamma_a\gamma^a) + \gamma^2\kappa T^a_a\right)\nabla_b K + \frac{1}{2}\lambda\nabla_b(\gamma K) = S^2_b\,, \tag{5.1.57}$$

In this equation we obtain another source term S^2_b defined as

$$S^2_b = \kappa\gamma^2\gamma^a t_{ab} - \frac{1}{2}\gamma^2\gamma^a\gamma_a\nabla_b h - \frac{1}{4}\gamma_b h(2 + l(l+1)) + 2\gamma\gamma_b\nabla_c\nabla_d\gamma k^{cd} + 2\gamma\gamma_b\gamma^a(\nabla^c k_{ca} - \nabla_a K) - \gamma^2\gamma_b S_1$$

$$= \kappa\gamma^2\gamma^a t_{ab} - \frac{1}{2}\gamma^2\gamma^a\gamma_a\nabla_b h - \frac{1}{2}\gamma^2\kappa\tilde{t}^a_a - \frac{1}{2}\gamma^2\gamma_b S_1 \tag{5.1.58}$$

The third and last equation is a direct consequence of the original third equation contracted with γ^a. It reads

$$\nabla^a Z_a - \frac{\gamma^a}{\gamma} Z_a + \gamma^2 \Box K = S_3\,, \tag{5.1.59}$$

with the source term S_3 given by

$$S_3 = \frac{2\gamma\kappa\gamma^a}{\sqrt{l(l+1)}} t_a - \gamma_a\gamma^a h + \frac{1}{2}\gamma\gamma^a\nabla_a h + \gamma\nabla_a\nabla_b\gamma k^{ab}\,. \tag{5.1.60}$$

The derivation of the master equation in the presence of background matter is non-trivial in the general setup presented in this section. In the case of electromagnetic matter, the differential equations simplify because T_{ij} is diagonal. A derivation of the master equation in Schwarzschild coordinates can be found in [101]. In contrast to the odd parity case, the master variable in the even parity will depend non-trivially on T_{ij} and to the author's knowledge, the master equation in full generality has not yet been constructed. We therefore restrict to the case $T_{ij} = 0$ here.

We apply the differential operator ∇^b on the equation S_b^2. Then, using $T_{ab} = 0$ and the Schwarzschild solution for the background, we obtain the following equation

$$\Box(\gamma^a Z_a) + \frac{l(l+1)}{2\gamma}\left(\nabla^a Z_a - \frac{\gamma^a}{\gamma} Z_a\right) + \frac{3\gamma}{2}(1-\gamma_a\gamma^a)\Box K + \frac{1}{2}\lambda\Box(\gamma K) = \nabla^b S_b^2 \tag{5.1.61}$$

In this equation, we replace $\nabla^a Z_a$ using the equation for S_3 and $\Box K$ using the equation for S_1. Additionally, we introduce the variable $\psi = 2\gamma^a Z_a + \lambda\gamma K$ and obtain

$$\Box\psi + \left[-\frac{l(l+1)}{\gamma^2} + \frac{3}{\gamma^2}(1-\gamma_a\gamma^a)\right]\psi = 2\nabla^b S_b^2 - 3\gamma(1-\gamma^a\gamma_a)S_1 - \frac{l(l+1)}{\gamma}(S_3 - \gamma^2 S_1) \tag{5.1.62}$$

Therefore, we have a master equation for the master variable ψ with a potential that, after inserting the Schwarzschild solution, is the same as in the odd parity sector.

In the literature, usually a different master equation and master variable based on the original work by Zerilli [73] are discussed. The covariant form of the Zerilli master variable is

$$\psi^e = \frac{1}{\lambda+2}\left(\gamma K + \frac{1}{\Lambda}\gamma^a Z_a\right), \tag{5.1.63}$$

where $\Lambda = \frac{1}{2}\lambda + \frac{3M}{\gamma}$. The Zerilli master variable is related to the master variable we found through a differential relation (see [101]):

$$\psi = -6r_s\gamma^a\nabla_a\psi^e + \left(\lambda(\lambda+2) + \frac{18r_s^2}{\gamma^2\Lambda}\gamma^a\gamma_a\right)\psi^e \tag{5.1.64}$$

For pure gravity, the Zerilli master variable satisfies a wave equation for a different potential:

$$\Box\psi^e - \frac{1}{4\Lambda^2\gamma^2}\left(\lambda^2(\lambda+2) + 3\lambda^2\frac{r_s}{\gamma} + 9\lambda\frac{r_s^2}{\gamma^2} + 9\frac{r_s^3}{\gamma^3}\right)\psi_Z = 0 \tag{5.1.65}$$

5.2 Hamiltonian Perturbation Theory

In the Lagrangian approach, we reduced the linearized Einstein equations to two wave equations for master variables $\psi^{e/o}$. We obtained the Regge-Wheeler equation for the odd parity perturbations and the Zerilli equation for the even parity. In this section, we switch to the Hamiltonian perspective, apply the framework of constrained Hamiltonian systems developed in section 2 and derive a reduced Hamiltonian describing the dynamics of the perturbations. The formulation incorporates backreaction effects and we expand the physical Hamiltonian to second order in the perturbations. As a consistency check, we compare the Hamiltonian equations of motion to the Regge-Wheeler-Zerilli equations in their common domain of validity.

The outline of the strategy is as follows: First, we explain the details about the applications of the formalism to the case of spherically symmetric black holes based on [98]. We explain how we split the variables into the observable and non-observable degrees of freedom. Then, we find an asymptotic solution of the stability condition of the GP gauge fixing and explicitly evaluate the boundary terms. With this knowledge, we use theorem 2.12 to determine the physical Hamiltonian describing the dynamics of the perturbations.

Then, in the next part, we explicitly apply the formalism to compute the physical Hamiltonian to second order. After perturbing the perturbations of the constraints to second order, we solve the constraints for the momenta of the gauge degrees of freedom. With the help of the previous part, we find the physical Hamiltonian. In the computation we use the GP gauge and closely follow [2, 3]. In the final section, we generalize the gauge for the background degrees of freedom and compute the reduced Hamiltonian in this case. This discussion is based on [1].

5.2.1 Preliminaries

In the following, we study some necessary preliminary concepts and computations to understand the later calculations. We define the decomposition into spherically symmetric / non-symmetric variables and choose the gauge / true degrees of freedom. Then, we derive an implicit expression for the physical Hamiltonian which will be used in later sections. This part is based on [98].

In section 3.5 we introduced the ADM formulation of general relativity. The phase space is coordinatized by the induced metric $m_{\mu\nu}$ and its conjugate momentum $W^{\mu\nu}$. In this section, we also performed a detailed boundary term analysis and split the variable into spherically symmetric and non-symmetric degrees of freedom. We defined

$$\begin{aligned} m_{33} &= e^{2\mu} + x^v, \quad m_{3A} = 0 + x_A, \quad m_{AB} = e^{2\lambda}\Omega_{AB} + x^h\Omega_{AB} + X_{AB} \\ W^{33} &= \frac{1}{2}e^{-2\mu}\pi_\mu + y_v, \quad W^{3A} = 0 + \frac{1}{2}y^A, \quad W^{AB} = \frac{1}{4}e^{4\lambda}\pi_\lambda + \frac{1}{2}y_h\Omega^{AB} + Y^{AB} \end{aligned} \tag{5.2.1}$$

For the perturbations we have four constraints: one perturbed Hamiltonian constraint and three perturbed diffeomorphism constraints. Therefore, we choose four of the canonical pairs as gauge degrees of freedom and obtain two pairs of observable degrees of freedom. The pairs (x^v, y_v), (x_A, y^A) and (x^h, y_h) will be the gauge degrees of freedom and the pairs (X_{AB}, Y^{AB}) will be the observable degrees of freedom.

Similarly to the Lagrangian case, it is convenient for the calculations to expand the variables into scalar, vector and tensor spherical harmonics. We use the convention

$$\begin{aligned} x^{v/h} &= \sum_{l\geq 1,m} x^{v/h}_{lm} L_{lm}, \quad y_{v/h} = \sum_{l\geq 1,m} y^{lm}_{v/h} L_{lm} \\ x_A &= \sum_{l\geq 1,m,I} x^I_{lm} [L^I_{lm}]_A, \quad y^A = \sum_{l\geq 1,m,I} y^{lm}_I [L^I_{lm}]^A, \\ X_{AB} &= \sum_{l\geq 2,m,I} X^I_{lm} [L^I_{lm}]_A, \quad Y^{AB} = \sum_{l\geq 2,m,I} Y^{lm}_I [L^I_{lm}]^A, \end{aligned} \tag{5.2.2}$$

For the calculation of the reduced Hamiltonian, we need to choose a gauge fixing condition, restricting the value of μ, λ and of x^v, x_A, x^h. The choice of admissible gauge fixings is limited by physical requirements. First of all, it needs to be compatible with the fall-off conditions we imposed on the variables. Additionally, in some coordinate systems such as the standard Schwarzschild coordinates, the black hole solution is singular across the black hole horizon. For a complete picture of black holes, we have to cover both the interior and exterior of the black hole spacetime. Otherwise, we could not determine the future evolution of matter inside the black hole. In some models of quantum gravity on black hole spacetimes, the matter bounces at the central singularity [47, 126] and this could be an important part for our understanding of the evolution of black holes. Based on the discussion of spherically symmetric black holes, we impose the Gullstrand-Painlevé (GP) gauge. The condition is $m_{33} = 1$, $m_{3A} = 0$ and $\Omega^{AB} m_{AB} = 2r^2$, which implies $\mu = 0$, $\lambda = \log r$ for the background and $x^v = x_A = x^h = 0$ for the perturbations. Note that this gauge does not fix the background to be the Schwarzschild solution in Gullstrand-Painlevé gauge. Rather the name GP gauge means that the gauge condition is motivated by a certain class of Schwarzschild solutions. Adding matter fields such as a scalar field, the background metric will look drastically different from the Schwarzschild metric in Gullstrand-Painlevé coordinates even when presented in GP gauge.

5.2.1.1 Asymptotic Solution of the Stability Conditions

The GP gauge fixing conditions are not preserved under time evolution in general and we have to assure this by fixing the Lagrange multipliers. The exact solution for the Lagrange multipliers in all of spacetime is a very challenging task. However, as it turns out, this is not necessary because for the derivation of the physical Hamiltonian we only need the boundary terms of the constraints. These boundary terms are evaluated at infinity and consequently we only need the asymptotic value of the Lagrange multipliers.

For the solution of the stability conditions, recall the gravitational contributions to the constraints of the full theory. It is enough to consider these, since the matter contributions are independent of the gravitational momenta. In section 3.5, we derived

$$V_0^{\text{grav}} = \frac{1}{\sqrt{m}} \Big(m_{\mu\rho} m_{\nu\sigma} - \frac{1}{2} m_{\mu\nu} m_{\rho\sigma} \Big) W^{\mu\nu} W^{\rho\sigma} - \sqrt{m}^{(3)} R \tag{5.2.3}$$

$$V_\mu^{\text{grav}} = -2 m_{\mu\nu} \nabla_\rho W^{\nu\rho} \tag{5.2.4}$$

In this section, we also considered the smeared versions of the constraints $V_0[f]$ and $\vec{V}[\vec{f}]$. The stability condition for the gauge fixing $m_{33} = 1$, $m_{3A} = 0$ and $m_{AB}\Omega^{AB} = 2r^2$ is given by the calculation of the Poisson brackets with the Hamiltonian:

$$\Big\{ m_{33}, V_0[f] + \vec{V}[\vec{f}] \Big\} = \frac{2f}{\sqrt{m}} \Big(m_{3\mu} m_{3\nu} - \frac{1}{2} m_{33} m_{\mu\nu} \Big) W^{\mu\nu} + 2 m_{\mu 3} \nabla_3 f^\mu = 0 \tag{5.2.5}$$

$$\left\{m_{3A}, V_0[f] + \vec{V}[\vec{f}]\right\} = \frac{2f}{\sqrt{m}}\left(m_{3\mu}m_{A\nu} - \frac{1}{2}m_{3A}m_{\mu\nu}\right)W^{\mu\nu} + 2m_{\mu 3}\nabla_A(f^\mu) + 2m_{\mu A}\nabla_3 f^\mu = 0 \tag{5.2.6}$$

$$\left\{\Omega^{AB}m_{AB}, V_0[f] + \vec{V}[\vec{f}]\right\} = \frac{2f}{\sqrt{m}}\left(m_{A\mu}m_{B\nu} - \frac{1}{2}m_{AB}m_{\mu\nu}\right)\Omega^{AB}W^{\mu\nu} + 2\Omega^{AB}m_{\mu A}\nabla_B f^\mu = 0 \tag{5.2.7}$$

For the asymptotic solution of these equations, we only keep the leading order terms in the limit $r \to \infty$ and drop all the sub-leading terms. Asymptotically, we have to solve the following equations for $r \to \infty$:

$$\frac{f}{4\sqrt{m}}\left(\pi^\infty_\mu - \pi^\infty_\lambda\right)\sqrt{r} + m_{3\mu}\partial_3 f^\mu + \Gamma_{33\nu}f^\nu = 0 \tag{5.2.8}$$

$$\frac{f}{4\sqrt{m}}\pi^\infty_\mu\sqrt{r} - \frac{1}{2r^2}\Omega^{AB}m_{\mu A}\partial_B f^\mu - \frac{1}{2r^2}\Omega^{AB}\Gamma_{AB\nu}f^\nu = 0 \tag{5.2.9}$$

$$\frac{f}{\sqrt{m}}r^2 y_A + m_{\mu 3}\partial_A f^\mu + \Gamma_{3A\mu}f^\mu + m_{\mu A}\partial_3 f^\mu + \Gamma_{A3\mu}f^\mu = 0 \tag{5.2.10}$$

We split the covariant derivative of f^μ into the partial derivative and the Christoffel symbols. The leading-order asymptotic expressions for the Christoffel symbols are

$$\begin{aligned}
\Gamma_{333} &\sim -\frac{1}{2r^2}(x^v_\infty)^+ \\
\Gamma_{33A} &\sim \frac{1}{2r}\partial_A(x^v_\infty)^+ \\
\Gamma_{3AB} &\sim \frac{1}{2}\Big(\partial_B(x^\infty_A)^+ + \partial_A(x^\infty_B)^+ - 2r\Omega_{AB}\Big) \sim -r\Omega_{AB} \\
\Gamma_{A33} &\sim -\frac{1}{2r}\partial_A(x^v_\infty)^+ \\
\Gamma_{AB3} &\sim \frac{1}{2}\Big(\partial_B(x^\infty_A)^+ - \partial_A(x^\infty_B)^+ + 2r\Omega_{AB}\Big) \sim r\Omega_{AB} \\
\Gamma_{ABC} &\sim \frac{r^2}{2}\Gamma^\Omega_{ABC}
\end{aligned} \tag{5.2.11}$$

Here, Γ^Ω_{ABC} is the standard Christoffel symbol on the sphere calculated from the metric Ω_{AB}. Additionally, we need the expression for the determinant of the metric and to leading order in r we find $\sqrt{m} \sim r^2\sqrt{\Omega}$. Inserting the results for the asymptotic Christoffel symbols, the leading order contributions to the last equation are

$$\partial_A f^3 + r^2\partial_3 f_A = 0 \tag{5.2.12}$$

Then, assuming that S^A behaves as r^{-2}, the equation implies that the non-symmetric part of S^3 behaves as r^{-1}. Using this, the first two equations are

$$\frac{f}{4r^{3/2}}\left(\pi^\infty_\mu - \pi^\infty_\lambda\right) + \partial_3 f^3_{\text{sym}} = 0 \tag{5.2.13}$$

$$\frac{f}{4r^{3/2}}\pi^\infty_\mu - \frac{1}{r}f^3_{\text{sym}} = 0 \tag{5.2.14}$$

This is the same equation as in section 3.6 and we solve it using the same strategy. From the second equation, we get a relation between f and f^3_{sym}:

$$f = \frac{4\sqrt{r}}{\pi^\infty_\mu}f^3_{\text{sym}} \tag{5.2.15}$$

Then, the first equation gives

$$\partial_3 f^3_{\rm sym} + \frac{1}{r}\left(1 - \frac{\pi^\infty_\lambda}{\pi^\infty_\mu}\right) f^3_{\rm sym} = 0 \tag{5.2.16}$$

For the solution of the differential equation, we have to provide the relation between the constants π^∞_μ and π^∞_λ. The diffeomorphism constraint in terms of the partial derivative and Christoffel symbol reads

$$V_\mu = -2m_{\mu\nu}\partial_\rho W^{\nu\rho} - 2\Gamma_{\mu\nu\rho}W^{\nu\rho}. \tag{5.2.17}$$

Taking the radial component, we find that the leading order contribution behaves as $r^{-1/2}$ and it implies that $\pi^\infty_\mu = 2\pi^\infty_\lambda$. With this relation, the differential equation for $f^3_{\rm sym}$ becomes

$$\partial_3 f^3_{\rm sym} + \frac{1}{2r} f^3_{\rm sym} = 0\,. \tag{5.2.18}$$

The solution of the differential equation is straight forward. In terms of a suitably chosen integration constant C, we have

$$f^3_{\rm sym} \sim \frac{C\pi^\infty_\mu}{4\sqrt{r}} \tag{5.2.19}$$

Then, the lapse function f is asymptotically equal to $f \sim C$. As in the spherically symmetric case in section 3.6, the lapse goes to 1 for $C = 1$. In terms of the variable π_μ, we can equivalently write $f^3_{\rm sym} \sim \pi_\mu/(4\sqrt{\Omega}r)$ because by definition, it has the same asymptotic behaviour.

5.2.1.2 Evaluation of the Boundary Terms

For the physical Hamiltonian, we need to evaluate the boundary terms and the stability conditions for the gauge fixings. The non-vanishing contributions to the boundary term of the diffeomorphism constraint is

$$\vec{B}[\vec{f}] = 2\int_{\partial\Sigma} \mathrm{d}\Omega \left[\frac{1}{2} f^3\pi_\mu + f^3 y_v + r^2 f^A y_A\right]. \tag{5.2.20}$$

Using the fact that the perturbative contribution to f^3 vanishes like r^{-1} and the one to f^A like r^{-2} only the first term survives. Using the explicit expression for f^3 we found above, we are left with

$$\vec{B}[\vec{f}] = \int_{\partial\Sigma} \mathrm{d}\Omega\,\sqrt{\Omega}\frac{\pi^2_\mu}{4r} = \lim_{r\to\infty} \pi\frac{\pi^2_\mu}{r} \tag{5.2.21}$$

Thus, the value of the boundary term of the diffeomorphism constraint only depends on the asymptotic value of π_μ at infinity.

The boundary term of the Hamiltonian constraint gives

$$B[f] := -\int \mathrm{d}\Sigma_\rho \frac{\sqrt{\det m}}{\kappa}\Big[\big(m^{\mu\rho}m^{\nu\sigma} - m^{\mu\nu}m^{\rho\sigma}\big)\Big(f\nabla_\sigma(m_{\mu\nu} - m^{\rm ND}_{\mu\nu}) - \nabla_\sigma f(m_{\mu\nu} - m^{\rm ND}_{\mu\nu})\Big)\Big] \tag{5.2.22}$$

Let us insert the asymptotic expansion for $m_{\mu\nu}$. For the square root of the determinant of m, the dominant contribution comes from the non-dynamical part of the metric. Additionally, we use that in GP gauge, the metric takes the form $m_{33} = 1$, $m_{3A} = 0$ and $m_{AB}\Omega^{AB} = 2r^2$. The inverse metric is then $m^{33} = 1$, $m^{3A} = 0$

and m^{AB} is the inverse of m_{AB} as a 2×2 matrix. We find

$$\begin{aligned} B[f] :=& -\int_{\partial\Sigma} \mathrm{d}\Omega \frac{r^2\sqrt{\det\Omega}}{\kappa}\Big[m^{\nu\sigma}\Big(f\nabla_\sigma(m_{3\nu}-m^{\mathrm{ND}}_{3\nu})-\nabla_\sigma f(m_{3\nu}-m^{\mathrm{ND}}_{3\nu})\Big) \\ &- m^{\mu\nu}\Big(f\nabla_3(m_{\mu\nu}-m^{\mathrm{ND}}_{\mu\nu})-\nabla_3 f(m_{\mu\nu}-m^{\mathrm{ND}}_{\mu\nu})\Big)\Big] \\ =& \int_{\partial\Sigma} \mathrm{d}\Omega \frac{r^2\sqrt{\det\Omega}}{\kappa}\Big[m^{AB}\Big(f\nabla_3(m_{AB}-m^{\mathrm{ND}}_{AB})-\nabla_3 f(m_{AB}-m^{\mathrm{ND}}_{AB})\Big)\Big] \end{aligned} \tag{5.2.23}$$

Next, we bring the inverse metric m^{AB} inside the bracket and have

$$= \int_{\partial\Sigma} \mathrm{d}\Omega \frac{r^2\sqrt{\det\Omega}}{\kappa}\Big[f\nabla_3(2-m^{AB}_{\mathrm{ND}}m_{AB})-\nabla_3 f(2-m^{AB}_{\mathrm{ND}}m_{AB})\Big] \tag{5.2.24}$$

However the bracket term vanishes because $m^{AB}_{\mathrm{ND}}m_{AB}=2$ in Gullstrand-Painlevé gauge. Since this is the dominant, finite order at infinity and it vanishes, all the higher orders have to vanish because they fall off faster. In [98], a more careful analysis reveals that $B[f]$ behaves as $O(r^{-1})$.

The reduced Hamiltonian is then derived using theorem 2.12. Since the boundary term is exactly the same as in section 3.6, we have the reduced Hamiltonian

$$H = \lim_{r\to\infty}\frac{\pi}{2\kappa r}\pi_\mu^2 = \lim_{r\to\infty}\frac{\pi}{2\kappa r}\Big((\pi_\mu^{(0)})^2 + 2\pi_\mu^{(0)}\pi_\mu^{(2)} + O(3)\Big) \tag{5.2.25}$$

We expanded the spherically symmetric, gauge momentum π_μ to second order in the perturbations where $\pi_\mu^{(0)}$ is the zeroth order solution of the symmetric constraints and $\pi_\mu^{(2)}$ the second order solution. Note that as in section 3.6, the reduced Hamiltonian is not just the boundary term as one would naively expect. There is a non-trivial factor of $1/2$ due to the non-trivial relation between the reduced Hamiltonian and the boundary term (see theorem 2.12).

5.2.1.3 The Electromagnetic Field

In addition to the gravitational degrees of freedom, we study the electromagnetic field. In section 3.6, we derived the Hamiltonian formulation in terms of the electric field E^μ and the vector potential A_μ. We found that for the spherically symmetric part, the only non-vanishing component was $E^3=\sqrt{\Omega}\xi$, where ξ is a constant related to the electric charge. The full electric field and vector potential are expanded into scalar and vector spherical harmonics according to

$$\begin{aligned} A_3 &= \sum_{l\geq 1,m} x^{lm}_M L_{lm} \\ A_B &= \sum_{l\geq 1,m,I} X^{I,lm}_M [L_{I,lm}]_B \\ E^3 &= \sqrt{\Omega}\xi + \sqrt{\Omega}\sum_{l\geq 1,m} y^M_{lm} L_{lm} \\ E^B &= \sqrt{\Omega}\sum_{l\geq 1,m,I} Y^M_{I,lm}[L_{I,lm}]^B \,. \end{aligned} \tag{5.2.26}$$

For the electromagnetic field we also have to choose suitable fall-off conditions for the variables (A_μ, E^μ).

According to [98], a good choice for a well-defined Hamiltonian theory is

$$\begin{aligned} x_M &\sim (x_M)^+ r^{-1} + (x_M)^- r^{-2} \\ X^I_M &\sim (X^I_M)^+ + (X^I_M)^- r^{-1} \\ y^M &\sim (y_M)^- + (y_M)^+ r^{-1} \\ Y^M_I &\sim (Y^M_I)^- r^{-1} + (Y^M_I)^+ r^{-2} \,. \end{aligned} \tag{5.2.27}$$

where we suppressed the label l, m for simplicity. As for the gravitational perturbations, the subscript $+/-$ stands for odd and even parity with respect to the parity operator P introduced in section 3.5. This should not be confused with the decomposition into even and odd vector spherical harmonics. In the literature, one also uses the terms polar and axial for the vector spherical harmonics in order to better distinguish between the two concepts.

5.2.1.4 Outline of the Next Steps in The Computation

For the computation of the reduced phase space and the reduced Hamiltonian, we have to solve the constraints V_0 and V_μ for the symmetric degrees of freedom π_μ, π_λ and for the non-symmetric degrees of freedom y_v, y^A, y_h. We split the full constraints into the symmetric contributions C and the non-symmetric contributions Z, i.e. the modes of V with $l \geq 1$. The symmetric constraints are found by averaging the constraints over the sphere:

$$C_v := \int_{S^2} V_0 \mathrm{d}\Omega, \quad C_h := \int_{S^2} V_3 \mathrm{d}\Omega \tag{5.2.28}$$

The modes of the non-symmetric constraints ($l \geq 1$) are defined by

$$Z^v_{lm} := \int_{S^2} L_{lm} V_0 \, \mathrm{d}\Omega \,, \quad Z^h_{lm} := \int_{S^2} L_{lm} V_3 \, \mathrm{d}\Omega \,, \quad Z^I_{lm} := \int_{S^2} [L^I_{lm}] V_A \, \mathrm{d}\Omega \tag{5.2.29}$$

Note that all of these constraints involve both symmetric and non-symmetric degrees of freedom. Consider an expansion of the smearing functions $f = f_{\mathrm{sym}} + \sum_{l\geq 1,m} f_{lm} L_{lm}$, $f^3 = f^3_{\mathrm{sym}} + \sum_{l\geq 1,m} f^3_{lm} L_{lm}$ and $f^A = \sum_{I,l\geq 1,m} f^I_{lm} [L^I_{lm}]^A$. Then, we have

$$V_0[f] + \vec{V}[\vec{f}] = \int \mathrm{d}r \left[f_{\mathrm{sym}} C_v + f^3_{\mathrm{sym}} C_h + \sum_{l\geq 1,m} \left(Z^v_{lm} f_{lm} + Z^h_{lm} f^3_{lm} + \sum_I Z^I_{lm} f^I_{lm} \right) \right] \tag{5.2.30}$$

We determine the reduced phase space by solving the symmetric constraints C for π_μ, π_λ and the non-symmetric constraints Z for y_v, y^A, y_h. Due to the complexity of the constraints of the full theory, such a solution is very difficult using the non-perturbative constraints C and Z. Therefore, we truncate the constraints after some order in the perturbations:

$$\begin{aligned} C_{v/h} &= {}^{(0)}C_{v/h} + {}^{(1)}C_{v/h} + {}^{(2)}C_{v/h} + O(3) \\ Z^{v/h/I}_{lm} &= {}^{(0)}Z^{v/h/I}_{lm} + {}^{(1)}Z^{v/h/I}_{lm} + {}^{(2)}Z^{v/h/I}_{lm} + O(3) \end{aligned} \tag{5.2.31}$$

The subscript i in ${}^{(i)}C$ and ${}^{(i)}Z$ stands for the order in the perturbations (x, y) and (X, Y). Using the definition of the constraints, we have ${}^{(1)}C_{v/h} = 0$ because this contribution is linear in the perturbations and the average of the perturbations over the sphere vanishes. By construction of the non-symmetric constraints Z, the zeroth order contributions ${}^{(0)}Z^{v/h/I}_{lm}$ vanish. In addition to the constraints, we also expand the gauge momenta into orders of the perturbations.

$$\begin{aligned} \pi_{\mu/\lambda} &= \pi^{(0)}_{\mu/\lambda} + \pi^{(1)}_{\mu/\lambda} + \pi^{(2)}_{\mu/\lambda} + O(3) \\ y &= y^{(0)} + y^{(1)} + y^{(2)} + O(r) \end{aligned} \tag{5.2.32}$$

We insert this expansion of the variables into the expansion of the constraints and collect terms of equal order. The solution is then obtained by solving the constraints order by order. Since ${}^{(1)}C_{v/h} = {}^{(0)}Z^{v/h/I} = 0$, we set $y^{(0)} = 0$ and $\pi^{(1)}_{\mu/\lambda} = 0$.

The zeroth order equations are ${}^{(0)}C_{v/h} = 0$ and this determines $\pi^{(0)}_{\mu}, \pi^{(0)}_{\lambda}$ as we saw in section 3.5. To first order we have to solve the non-symmetric constraints ${}^{(1)}Z^{v/h/I}_{lm} = 0$ for $y^{(1)}$ where we use the solution $\pi^{(0)}_{\mu/\lambda}$ of the zeroth order symmetric constraints. To second order, we have both symmetric and non-symmetric constraints. Since in this thesis we are only interested in the reduced Hamiltonian to second order, it is sufficient to know $\pi^{(2)}_{\mu}$. This function is determined by solving the equations

$$\int \mathrm{d}r' \left[\frac{\delta^{(0)} C_{v/h}}{\delta \pi_\mu(r')} \pi^{(2)}_\mu(r') + \frac{\delta^{(0)} C_{v/h}}{\delta \pi_\lambda(r')} \pi^{(2)}_\lambda(r') \right] + {}^{(2)}C_{v/h} = 0 \tag{5.2.33}$$

In this equation we have to use the zeroth and first order solutions that we already determined. In [71, 98], it was shown that this procedure to iteratively solve the constraints generalizes to higher orders. Thus, we obtain a perturbative solution of the constraints for the gauge momenta and can determine the physical Hamiltonian to arbitrary order using equation (**5.2.25**).

5.2.2 The Perturbed Constraints

After all the necessary preparatory steps, we begin the explicit computations for the reduced Hamiltonian to second order in this section. For the calculation of the perturbed constraints, we do not fix any gauge and derive all the formulas in full generality for later use. Although the perturbative calculation of the constraints is possible in any gauge, for the formalism described in this thesis to apply, we have to restrict to gauges which are compatible with the asymptotic behaviour of the Gullstrand-Painlevé gauge in section 3.5. This is necessary so that the boundary term analysis is still valid and we can find the reduced Hamiltonian as in section 5.2.1. The exposition of the calculation in this section is based on appendix A of [1] and appendix B of [2].

First, we calculate various quantities of the spherically symmetric metric. The Christoffel symbols are given by

$$\begin{aligned} \Gamma^3_{33} &= \mu' \\ \Gamma^3_{AB} &= -\lambda' e^{2(\lambda-\mu)} \Omega_{AB} \\ \Gamma^A_{3B} &= \lambda' \delta^A_B, \end{aligned} \tag{5.2.34}$$

The Christoffel symbol Γ^A_{BC} is just the usual Christoffel symbol on the 2-sphere S^2. All other components of the Christoffel symbol vanish. The Ricci tensor, scalar curvature, and Einstein tensor are computed by taking derivatives of the Christoffel symbols. The non-vanishing components are

$$\begin{aligned} R_{33} &= -2(\lambda'' + \lambda'(\lambda' - \mu')) \\ R_{AB} &= \Omega_{AB}\Big(1 - (\lambda')^2 e^{-2\mu+2\lambda} - (\lambda'' + \lambda'(\lambda' - \mu'))e^{2(\lambda-\mu)}\Big) \\ R &= 2e^{-2\lambda} - 2(\lambda')^2 e^{-2\mu} - 4(\lambda'' + \lambda'(\lambda' - \mu'))e^{-2\mu} \\ G_{33} &= (\lambda')^2 - e^{2(\mu-\lambda)} \\ G_{AB} &= \Omega_{AB}(\lambda'' + \lambda'(\lambda' - \mu'))e^{2(\lambda-\mu)} \end{aligned} \tag{5.2.35}$$

Next, we turn to the computations of the first and second order corrections of the diffeomorphism and Hamiltonian constraints. We split the induced metric and its conjugate momentum into the spherically symmetric background and the perturbations. We use the notation $m_{\mu\nu} = \overline{m}_{\mu\nu} + \delta m_{\mu\nu}$ and $W^{\mu\nu} = \overline{W}^{\mu\nu} + \delta W^{\mu\nu}$. The variables with a bar are the background expanded in terms of μ, π_μ and λ, π_λ. The degrees of freedom with δ are the perturbations which we assume not to have any spherically symmetric contribution. In the following calculations, we raise and lower indices with the background metric $\overline{m}_{\mu\nu}$.

Before calculating the linear and quadratic perturbations of the constraints, we recall useful formulas for the first and second order perturbations of the inverse metric, the Christoffel symbols, the Ricci tensor and the Ricci scalar. We have up to terms of higher order

$$\begin{aligned} m^{\mu\nu} &= \overline{m}^{\mu\nu} - \delta m^{\mu\nu} + \delta m^{\mu\rho}\delta m^{\nu\sigma}\overline{m}_{\rho\sigma} \\ \sqrt{m} &= \sqrt{\overline{m}}\Big(1 + \frac{1}{2}\delta m^\mu{}_\mu + \frac{1}{8}\Big((\delta m^\mu{}_\mu)^2 - 2\delta m^{\mu\nu}\delta m_{\mu\nu}\Big)\Big) \\ \sqrt{m}^{-1} &= \sqrt{\overline{m}}^{-1}\Big(1 - \frac{1}{2}\delta m^\mu{}_\mu + \frac{1}{8}\Big((\delta m^\mu{}_\mu)^2 + 2\delta m^{\mu\nu}\delta m_{\mu\nu}\Big)\Big) \\ \Gamma^\mu_{\nu\rho} &= \overline{\Gamma}^\mu_{\nu\rho} + \frac{1}{2}\overline{m}^{\mu\sigma}(\nabla_\nu\delta m_{\rho\sigma} + \nabla_\rho\delta m_{\nu\sigma} - \nabla_\sigma\delta m_{\nu\rho}) - \frac{1}{2}\delta m^{\mu\sigma}(\nabla_\nu\delta m_{\rho\sigma} + \nabla_\rho\delta m_{\nu\sigma} - \nabla_\sigma\delta m_{\nu\rho}) \\ R_{\mu\nu} &= \frac{1}{2}\left(2\overline{m}^{\rho\sigma}\nabla_\rho\nabla_{(\mu}\delta m_{\nu)\sigma} - \Box\delta m_{\mu\nu} - \nabla_\mu\nabla_\nu\delta m^\rho{}_\rho\right) - \frac{1}{2}\nabla_\rho\Big(\delta m^{\rho\sigma}\left(2\nabla_{(\mu}\delta m_{\nu)\sigma} - \nabla_\sigma\delta m_{\mu\nu}\right)\Big) \\ &\quad + \frac{1}{2}\nabla_\nu(\delta m^{\rho\sigma}\nabla_\mu\delta m_{\rho\sigma}) + \frac{1}{4}\nabla^\sigma\delta m^\rho{}_\rho\left(2\nabla_{(\mu}\delta m_{\nu)\sigma} - \nabla_\sigma\delta m_{\mu\nu}\right) \\ &\quad - \frac{1}{4}\left(2\nabla_{(\rho}\delta m_{\nu)\sigma} - \nabla_\sigma\delta m_{\mu\rho}\right)\left(2\nabla_{(\nu}\delta m_{\alpha)\beta} - \nabla_\beta\delta m_{\alpha\nu}\right)\overline{m}^{\alpha\sigma}\overline{m}^{\rho\beta} \\ R &= \nabla_\mu\nabla_\nu\delta m^{\mu\nu} - \Box\delta m^\mu{}_\mu \\ &\quad + \delta m^{\mu\nu}\left(\Box\delta m_{\mu\nu} + \nabla_\mu\nabla_\nu\delta m^\rho{}_\rho - 2\overline{m}^{\rho\sigma}\nabla_{(\rho}\nabla_{\mu)}\delta m_{\nu\sigma}\right) - \nabla_\rho\delta m^{\rho\sigma}\nabla^\mu\delta m_{\mu\sigma} \\ &\quad + \nabla_\rho\delta m^{\rho\sigma}\nabla_\sigma\delta m^\mu{}_\mu + \frac{3}{4}\nabla^\mu\delta m^{\rho\sigma}\nabla_\mu\delta m_{\rho\sigma} - \frac{1}{4}\nabla^\sigma\delta m^\rho{}_\rho\nabla_\sigma\delta m^\mu{}_\mu - \frac{1}{2}\nabla^\rho\delta m_{\mu\sigma}\nabla^\sigma\delta m^\mu{}_\rho \end{aligned} \tag{5.2.36}$$

We begin with the computations for the gravitational contributions to the diffeomorphism constraint. To first order we have

$$^{(1)}V^{\text{grav}}_\mu = -2\nabla_\rho\Big(\delta m_{\mu\nu}\overline{W}^{\nu\rho} + \overline{m}_{\mu\nu}\delta W^{\nu\rho}\Big) + \overline{W}^{\nu\rho}\nabla_\mu\delta m_{\nu\rho}\,. \tag{5.2.37}$$

We decompose the indices into the radial 3 and angular $A, B, \ldots$ components. This gives two sets of constraints, the radial and angular diffeomorphism constraints. We find

$$\begin{aligned} ^{(1)}V^{\text{grav}}_3 =& -2e^\mu\partial_r\Big(e^\mu\delta W^{33}\Big) - 2e^{2\mu}D_A\delta W^{3A} + 2e^{2\lambda}\lambda'\Omega_{AB}\delta W^{AB} - \sqrt{\Omega}\partial_r(\pi_\mu e^{-2\mu})\delta m_{33} \\ & - \sqrt{\Omega}\frac{\pi_\mu}{2}e^{-2\mu}\partial_r\delta m_{33} - \sqrt{\Omega}\frac{\pi_\lambda}{2}e^{-2\lambda}D^A\delta m_{3A} + \sqrt{\Omega}\frac{\pi_\lambda}{4}e^{-2\lambda}\Omega^{AB}\partial_r\delta m_{AB}\,, \end{aligned} \tag{5.2.38}$$

$$\begin{aligned}{}^{(1)}V_A^{\text{grav}} =& -2\Omega_{AB}\partial_r\Big(e^{2\lambda}\delta W^{B3}\Big) - 2e^{2\lambda}\Omega_{AB}D_C\delta W^{BC} + \frac{\sqrt{\Omega}\pi_\mu}{2}e^{-2\mu}D_A\delta m_{33} \\ & - \sqrt{\Omega}\partial_r\Big(\pi_\mu e^{-2\mu}\delta m_{3A}\Big) + \frac{\sqrt{\Omega}\pi_\lambda}{4}e^{-2\lambda}\Big(D_A\Omega^{CD}\delta m_{CD} - 2D^B\delta m_{AB}\Big)\,.\end{aligned} \tag{5.2.39}$$

The second order perturbations of the full diffeomorphism constraint takes the form

$${}^{(2)}V_\mu^{\text{grav}} = -2\nabla_\rho(\delta m_{\mu\nu}\delta W^{\nu\rho}) + \delta W^{\nu\rho}\nabla_\mu\delta m_{\nu\rho}\,. \tag{5.2.40}$$

For the calculation, we do not need the non-symmetric constraints at second order. It is sufficient to only compute the radial component $\mu = 3$ because the angular components have no spherically symmetric part. We obtain

$$\begin{aligned}{}^{(2)}V_3^{\text{grav}} =& \delta W^{33}\partial_r\delta m_{33} - 2\partial_r\Big(\delta W^{33}\delta m_{33}\Big) - 2\delta m_{3A}\partial_r\delta W^{3A} \\ & - 2D_A\Big(\delta m_{33}\delta W^{3A}\Big) + \delta W^{AB}\partial_r\delta m_{AB} - 2D_A\Big(\delta m_{3B}\delta W^{AB}\Big)\,.\end{aligned} \tag{5.2.41}$$

Next, we investigate the gravitational contributions to the Hamiltonian constraint. The first order corrections to the Hamiltonian constraint are given by

$$\begin{aligned}{}^{(1)}V_0^{\text{grav}} =& \frac{1}{\sqrt{\overline{m}}}\Big(\overline{W}^{\rho\sigma}\overline{m}_{\mu\rho}\overline{m}_{\nu\sigma} - \frac{1}{2}\overline{m}_{\mu\nu}\overline{W}^{\rho\sigma}\overline{m}_{\rho\sigma}\Big)\Big(2\delta W^{\mu\nu} + 2\overline{m}^{\mu\alpha}\delta m_{\alpha\beta}\overline{W}^{\beta\nu} - \frac{1}{2}\overline{m}^{\alpha\beta}\delta m_{\alpha\beta}\overline{W}^{\mu\nu}\Big) \\ & + \sqrt{\overline{m}}\Big(\overline{G}_{\mu\nu}\delta m_{\rho\sigma}\overline{m}^{\mu\rho}\overline{m}^{\nu\sigma} - \nabla_\mu\nabla_\nu\delta m_{\rho\sigma}\overline{m}^{\mu\rho}\overline{m}^{\nu\sigma} + \Box\delta m_{\mu\nu}\overline{m}^{\mu\nu}\Big)\,.\end{aligned} \tag{5.2.42}$$

The indices $\mu, \nu, \ldots$ are split into the radial and angular components. We find

$$\begin{aligned}{}^{(1)}V_0^{\text{grav}} =& \frac{1}{2}(\pi_\mu - \pi_\lambda)e^{\mu-2\lambda}\delta W^{33} - \frac{1}{2}\pi_\mu e^{-\mu}\delta W^{AB}\Omega_{AB} - \frac{1}{16}\sqrt{\Omega}\pi_\mu^2 e^{-\mu-4\lambda}\Omega^{AB}\delta m_{AB} \\ & + \frac{3}{16}\sqrt{\Omega}\pi_\mu^2 e^{-3\mu-2\lambda}\delta m_{33} - \frac{1}{8}\sqrt{\Omega}\pi_\mu\pi_\lambda e^{-3\mu-2\lambda}\delta m_{33} \\ & + e^{\mu+2\lambda}\Big[e^{-2(\mu+\lambda)}(D_A D^A\delta m_{33} - \delta m_{33}) + e^{-2(\mu+\lambda)}\Big(\partial_r^2 - \mu'\partial_r - \lambda'\partial_r + (\lambda')^2\Big)\Omega^{AB}\delta m_{AB} \\ & + e^{-4\mu}\Big(-2\lambda'\partial_r + 6\lambda'\mu' - 3(\lambda')^2 - 2\lambda''\Big)\delta m_{33} - 2e^{-2(\mu+\lambda)}(\partial_r - \mu' + \lambda')D^A\delta m_{3A} \\ & + e^{-4\lambda}\Big(D_A D^A\delta m_{CD}\Omega^{CD} - D^A D^B\delta m_{AB}\Big)\Big]\,.\end{aligned} \tag{5.2.43}$$

The second order contributions to the Hamiltonian constraint are more involved due to the presence of the Ricci scalar and the square root of the metric. Expanding everything carefully, we obtain the following second order perturbation of the Hamiltonian constraint

$$\begin{aligned}{}^{(2)}V_0^{\text{grav}} =& \frac{1}{\sqrt{\overline{m}}}\Big[\delta W^{\mu\nu}\delta W^{\rho\sigma}\Big(\overline{m}_{\mu\rho}\overline{m}_{\nu\sigma} - \frac{1}{2}\overline{m}_{\mu\nu}\overline{m}_{\rho\sigma}\Big) + \delta W^{\mu\nu}\overline{W}^{\rho\sigma}(4\delta m_{\mu\rho}\overline{m}_{\nu\sigma} - (\delta m_{\mu\nu}\overline{m}_{\rho\sigma} + \delta m_{\rho\sigma}\overline{m}_{\mu\nu})) \\ & + \overline{W}^{\mu\nu}\overline{W}^{\rho\sigma}\Big(\delta m_{\mu\rho}\delta m_{\nu\sigma} - \frac{1}{2}\delta m_{\mu\nu}\delta m_{\rho\sigma}\Big) + \frac{1}{8}\Big(2\delta m^{\mu\nu}\delta m_{\mu\nu} + (\delta m^\mu{}_\mu)^2\Big)\Big(\overline{W}^{\rho\sigma}\overline{W}_{\rho\sigma} - \frac{1}{2}(\overline{W}^\rho{}_\rho)^2\Big) \\ & + \frac{1}{2}\delta m^\rho_\rho(\delta W^\mu{}_\mu W^\nu{}_\nu + W^\sigma{}_\sigma W^{\mu\nu}\delta m_{\mu\nu} - 2W^{\mu\nu}W^{\sigma\tau}\delta m_{\mu\sigma}m_{\nu\tau} - 2\delta W^{\mu\nu}W_{\mu\nu})\Big] \\ & - \sqrt{\overline{m}}\Big[\delta m^\mu{}_\rho\delta m^{\rho\nu}\overline{R}_{\mu\nu} - \frac{1}{2}\delta m^\rho{}_\rho\delta m^{\mu\nu}\overline{R}_{\mu\nu} + \frac{1}{8}\Big((\delta m^\rho{}_\rho)^2 - 2\delta m^{\mu\nu}\delta m_{\mu\nu}\Big)\overline{R} \\ & + \nabla_\mu\delta m_{\nu\rho}\nabla_\sigma\delta m_{\alpha\beta}\overline{m}_{(1)}^{\mu\nu\rho\sigma\alpha\beta} + \delta m_{\mu\nu}\nabla_\rho\nabla_\sigma\delta m_{\alpha\beta}\overline{m}_{(2)}^{\mu\nu\rho\sigma\alpha\beta}\Big]\,,\end{aligned} \tag{5.2.44}$$

In the last row, we abbreviated the contractions with the metric by some tensors $m_{(1)}$ and $m_{(2)}$. They are linear combinations of the products of three inverse metrics defined by

$$\overline{m}_{(1)}^{\mu\nu\rho\sigma\alpha\beta} := -\overline{m}^{\mu\nu}\overline{m}^{\rho\beta}\overline{m}^{\sigma\alpha} + \overline{m}^{\mu\nu}\overline{m}^{\rho\sigma}\overline{m}^{\alpha\beta} + \frac{3}{4}\overline{m}^{\mu\sigma}\overline{m}^{\nu\alpha}\overline{m}^{\rho\beta} - \frac{1}{4}\overline{m}^{\mu\sigma}\overline{m}^{\nu\rho}\overline{m}^{\alpha\beta} - \frac{1}{2}\overline{m}^{\mu\beta}\overline{m}^{\nu\alpha}\overline{m}^{\rho\sigma} \tag{5.2.45}$$

$$\begin{aligned}\overline{m}_{(2)}^{\mu\nu\rho\sigma\alpha\beta} :=& \frac{1}{2}\overline{m}^{\mu\nu}\overline{m}^{\rho\alpha}\overline{m}^{\sigma\beta} - \frac{1}{2}\overline{m}^{\mu\nu}\overline{m}^{\rho\sigma}\overline{m}^{\alpha\beta} + \overline{m}^{\mu\alpha}\overline{m}^{\nu\beta}\overline{m}^{\rho\sigma} + \overline{m}^{\mu\rho}\overline{m}^{\nu\sigma}\overline{m}^{\alpha\beta} - \overline{m}^{\mu\sigma}\overline{m}^{\nu\alpha}\overline{m}^{\rho\beta} \\ &- \overline{m}^{\mu\rho}\overline{m}^{\nu\alpha}\overline{m}^{\sigma\beta}\,.\end{aligned} \tag{5.2.46}$$

The computations for the first four lines of the second order Hamiltonian constraint are straight forward. In the last line, we have first and second covariant derivatives of the perturbed metric. To simplify the computations, we explicitly compute the decomposition of the second derivative of the perturbed metric into radial and angular components. It is convenient to introduce the following derivative operator:

$$D^{(a,b)} := \partial_r - a\mu' - b\lambda', \tag{5.2.47}$$

Then, the second covariant derivatives of $\delta m_{\mu\nu}$ in radial and angular components are

$$\begin{aligned}
\nabla_r\nabla_r\delta m_{33} &= D^{(3,0)}D^{(2,0)}\delta m_{33} \\
\nabla_r\nabla_r\delta m_{3A} &= D^{(2,1)}D^{(1,1)}\delta m_{3A} \\
\nabla_r\nabla_r\delta m_{AB} &= D^{(1,2)}D^{(0,2)}\delta m_{AB} \\
\nabla_r\nabla_A\delta m_{33} &= D^{(2,1)}(D_A\delta m_{3B} - 2\lambda'\delta m_{3A}) \\
\nabla_r\nabla_A\delta m_{3B} &= D^{(1,2)}\Big(D_A\delta m_{3B} - \lambda'\delta m_{AB} + \lambda' e^{2(\lambda-\mu)}\Omega_{AB}\delta m_{33}\Big) \\
\nabla_r\nabla_A\delta m_{BC} &= D^{(0,3)}\Big(D_A\delta m_{BC} + \lambda' e^{2(\lambda-\mu)}(\Omega_{AB}\delta m_{3C} + \Omega_{AC}\delta m_{3B})\Big) \\
\nabla_A\nabla_r\delta m_{33} &= D_A D^{(2,1)}\delta m_{33} - 2\lambda' D^{(1,2)}\delta m_{3A} \\
\nabla_A\nabla_r\delta m_{3B} &= \lambda' e^{2(\lambda-\mu)}\Omega_{AB}D^{(2,1)}\delta m_{33} + D_A D^{(1,2)}\delta m_{3B} - \lambda' D^{(0,3)}\delta m_{AB} \\
\nabla_A\nabla_r\delta m_{BC} &= D_A D^{(0,3)}\delta m_{BC} + \lambda' e^{2(\lambda-\mu)}\Big(\Omega_{AB}D^{(1,2)}\delta m_{3C} + \Omega_{AC}D^{(1,2)}\delta m_{3B}\Big) \\
\nabla_A\nabla_B\delta m_{33} &= D_A D_B\delta m_{33} + \lambda' e^{2(\lambda-\mu)}\Omega_{AB}D^{(2,2)}\delta m_{33} - 4\lambda' D_{(A}\delta m_{B)r} + 2(\lambda')^2\delta m_{AB} \\
\nabla_A\nabla_B\delta m_{3C} &= 2\lambda' e^{2(\lambda-\mu)}\Omega_{C(A}D_{B)}\delta m_{33} + D_A D_B\delta m_{3C} + \lambda' e^{2(\lambda-\mu)}\Omega_{AB}D^{(1,2)}\delta m_{3C} \\
&\quad - 2(\lambda')^2 e^{2(\lambda-\mu)}\Omega_{AC}\delta m_{3B} - (\lambda')^2 e^{2(\lambda-\mu)}\Omega_{BC}\delta m_{3A} - 2\lambda' D_{(A}\delta m_{B)C} \\
\nabla_A\nabla_B\delta m_{CD} &= (\lambda')^2 e^{4(\lambda-\mu)}(\Omega_{BC}\Omega_{AD} + \Omega_{BD}\Omega_{AC})\delta m_{33} + 2\lambda' e^{2(\lambda-\mu)}\Big(\Omega_{C(A}D_{B)}\delta m_{3D} + \Omega_{D(A}D_{B)}\delta m_{3C}\Big) \\
&\quad + D_A D_B\delta m_{CD} + \lambda' e^{2(\lambda-\mu)}\Big(\Omega_{AB}D^{(0,2)}\delta m_{CD} - 2\lambda'\Omega_{A(C}\delta m_{D)B}\Big)
\end{aligned} \tag{5.2.48}$$

The first three lines of (**5.2.44**) involve the gravitational momenta. Their splitting into radial and angular components is

$$\begin{aligned}
&\frac{1}{2}e^{4\mu}\delta W^{33}\delta W^{33} + 2e^{2(\mu+\lambda)}\Omega_{AB}\delta W^{3A}\delta W^{3B} + e^{4\lambda}\Big(\delta W^{AB}\delta W_{AB} - \frac{1}{2}(\delta W^A{}_A)^2\Big) \\
&- \Omega_{AB}e^{2(\mu+\lambda)}\delta W^{33}\delta W^{AB} + \frac{1}{4}(3\pi_\mu - \pi_\lambda)\delta W^{33}\delta m_{33} - \delta W^{33}\Omega^{AB}\delta m_{AB}\frac{\pi_\mu}{4}e^{2(\mu-\lambda)} + \pi_\mu\delta W^{3A}\delta m_{3A} \\
&- \Omega_{AB}\delta W^{AB}\delta m_{33}\frac{\pi_\mu}{4}e^{2(\lambda-\mu)} - \frac{1}{2}\delta W^{AB}\delta m_{AB}(\pi_\mu - \pi_\lambda) + \frac{1}{4}(\pi_\mu - \pi_\lambda)\Omega_{AB}\Omega^{CD}\delta W^{AB}\delta m_{CD} \\
&+ \Big(\frac{3}{64}\pi_\mu^2 + \frac{1}{32}\pi_\mu\pi_\lambda\Big)e^{-4\mu}(\delta m_{33})^2 + \Big(\frac{1}{16}\pi_\mu^2 + \frac{1}{8}\pi_\mu\pi_\lambda\Big)e^{-2(\mu+\lambda)}\Omega^{AB}\delta m_{3A}\delta m_{3B}
\end{aligned} \tag{5.2.49}$$

$$-\frac{3}{32}\pi_\mu^2 e^{-2(\mu+\lambda)}\delta m_{33}\Omega^{AB}\delta m_{AB} + \left(\frac{\pi_\mu^2}{32} - \frac{\pi_\mu\pi_\lambda}{16} + \frac{\pi_\lambda^2}{16}\right)e^{-4\lambda}\Omega^{AB}\Omega^{CD}\delta m_{AC}\delta m_{BD}$$
$$+\left(\frac{\pi_\mu^2}{64} + \frac{\pi_\mu\pi_\lambda}{32} - \frac{\pi_\lambda^2}{32}\right)e^{-4\lambda}\Omega^{AB}\Omega^{CD}\delta m_{AB}\delta m_{CD}\,.$$

The fourth line in (**5.2.44**) contains the background curvature tensors and gives

$$\begin{aligned}
&-\frac{1}{4}\Big(1+\big((\lambda')^2+2\lambda''-2\lambda'\mu'\big)\Big)e^{2(\lambda-\mu)}e^{-4\mu-2\lambda}(\delta m_{33})^2\\
&+\frac{1}{2}(\lambda''+\lambda'(\lambda'-\mu'))e^{-2\lambda-4\mu}\delta m_{33}\Omega^{AB}\delta m_{AB} - (\lambda''+\lambda'(\lambda'-\mu'))e^{-2\lambda-4\mu}\Omega^{AB}\delta m_{3A}\delta m_{3B}\\
&+\frac{1}{2}\Big(1-(\lambda')^2e^{2(\lambda-\mu)}\Big)e^{-6\lambda}\delta m^{AB}\delta m_{AB} - \frac{1}{4}\Big(1-(\lambda')^2e^{2(\lambda-\mu)}\Big)e^{-6\lambda}\Big(\delta m_{AB}\Omega^{AB}\Big)^2
\end{aligned} \tag{5.2.50}$$

Then, we study the first term in the last line of (**5.2.44**) involving the first covariant derivative of the metric perturbations. The expansion of this term in terms of radial and angular components is

$$\begin{aligned}
&-\sqrt{\overline{m}}\nabla_\mu\delta m_{\nu\rho}\nabla_\sigma\delta m_{\alpha\beta}\overline{m}_{(1)}^{\mu\nu\rho\sigma\alpha\beta} = \frac{1}{2}e^{-4\mu-2\lambda}(\partial_r-2\mu')\delta m_{33}(\partial_r-2\lambda')\delta m_{AB}\Omega^{AB}\\
&-e^{-4\mu-2\lambda}(\partial_r-2\mu')\delta m_{33}\Big(D^A\delta m_{3A}-\lambda'\Omega^{AB}\delta m_{AB}+2\lambda' e^{2(\lambda-\mu)}\delta m_{33}\Big)\\
&+\frac{1}{2}e^{-4\mu-2\lambda}(D_A\delta m_{33}-2\lambda'\delta m_{3A})(D_B\delta m_{33}-2\lambda'\delta m_{3B})\Omega^{AB}\\
&+e^{-2\mu-4\lambda}\Big(D_A\delta m_{BC}+\lambda' e^{2(\lambda-\mu)}(\Omega_{AB}\delta m_{3C}+\Omega_{AC}\delta m_{3B})\Big)(D_D\delta m_{33}-2\lambda'\delta m_{3D})\Big(\Omega^{AB}\Omega^{CD}-\frac{1}{2}\Omega^{AD}\Omega^{BC}\Big)\\
&+e^{-2\mu-4\lambda}\Big(D_A\delta m_{BC}+\lambda' e^{2(\lambda-\mu)}(\Omega_{AB}\delta m_{3C}+\Omega_{AC}\delta m_{3B})\Big)(\partial_r-\mu'-\lambda')\delta m_{3D}(\Omega^{AD}\Omega^{BC}-2\Omega^{AB}\Omega^{CD})\\
&+e^{-2\mu-4\lambda}\Big(D_A\delta m_{3B}-\lambda'\delta m_{AB}+\lambda' e^{2(\lambda-\mu)}\Omega_{AB}\delta m_{33}\Big)\Big(D_C\delta m_{3D}-\lambda'\delta m_{CD}+\lambda' e^{2(\lambda-\mu)}\Omega_{CD}\delta m_{33}\Big)\times\\
&\qquad\qquad\times\Big(\frac{3}{2}\Omega^{AC}\Omega^{BD}-\frac{1}{2}\Omega^{AD}\Omega^{BC}-\Omega^{AB}\Omega^{CD}\Big)\\
&+e^{-2\mu-4\lambda}(\partial_r-2\lambda')\delta m_{AB}\Big(D_C\delta m_{3D}-\lambda'\delta m_{CD}+\lambda' e^{2(\lambda-\mu)}\Omega_{CD}\delta m_{33}\Big)(\Omega^{AB}\Omega^{CD}-\Omega^{AD}\Omega^{BC})\\
&+\frac{1}{4}e^{-2\mu-4\lambda}(\partial_r-2\lambda')\delta m_{AB}(\partial_r-2\lambda')\delta m_{CD}(3\Omega^{AC}\Omega^{BD}-\Omega^{AB}\Omega^{CD})\\
&+e^{-6\lambda}\Big(D_A\delta m_{BC}+\lambda' e^{2(\lambda-\mu)}(\Omega_{AB}\delta m_{3C}+\Omega_{AC}\delta m_{3B})\Big)\Big(D_D\delta m_{EF}+\lambda' e^{2(\lambda-\mu)}(\Omega_{DE}\delta m_{3F}+\Omega_{DF}\delta m_{3E})\Big)\overline{m}_{(1)}^{ABCDEF}
\end{aligned} \tag{5.2.51}$$

Finally, the last term consists of the second derivative of the metric perturbation. Its splitting into radial and angular contributions is given by

$$\begin{aligned}
&-\sqrt{\overline{m}}\delta m_{\mu\nu}\nabla_\rho\nabla_\sigma\delta m_{\alpha\beta}\overline{m}_{(2)}^{\mu\nu\rho\sigma\alpha\beta}\\
&=e^{-4\mu-2\lambda}\Big[\frac{1}{2}\delta m_{33}\nabla_r\nabla_r\delta m_{AB}\Omega^{AB}-\frac{1}{2}\delta m_{33}\nabla_r\nabla_A\delta m_{3B}\Omega^{AB}-\frac{1}{2}\delta m_{33}\nabla_A\nabla_r\delta m_{3B}\Omega^{AB}+\frac{1}{2}\delta m_{33}\nabla_A\nabla_B\delta m_{33}\Omega^{AB}\Big]\\
&+e^{-2\mu-4\lambda}\Big[\frac{1}{2}\delta m_{33}\nabla_A\nabla_B\delta m_{CD}(\Omega^{AC}\Omega^{BD}-\Omega^{AB}\Omega^{CD})+\delta m_{3A}\nabla_r\nabla_B\delta m_{CD}(\Omega^{AB}\Omega^{CD}-\Omega^{AC}\Omega^{BD})\\
&+\delta m_{3A}\nabla_B\nabla_C\delta m_{3D}(2\Omega^{AD}\Omega^{BC}-\Omega^{AC}\Omega^{BD}-\Omega^{AB}\Omega^{CD})+\delta m_{3A}\nabla_B\nabla_r\delta m_{CD}(\Omega^{AB}\Omega^{CD}-\Omega^{AC}\Omega^{BD}\\
&+\delta m_{AB}\nabla_r\nabla_r\delta m_{CD}\Big(\Omega^{AC}\Omega^{BD}-\frac{1}{2}\Omega^{AB}\Omega^{CD}\Big)+\delta m_{AB}\nabla_r\nabla_C\delta m_{3D}\Big(\frac{1}{2}\Omega^{AB}\Omega^{CD}-\Omega^{AC}\Omega^{BD}\Big)\\
&+\delta m_{AB}\nabla_C\nabla_r\delta m_{3D}\Big(\frac{1}{2}\Omega^{AB}\Omega^{CD}-\Omega^{AC}\Omega^{BD}\Big)+\delta m_{AB}\nabla_C\nabla_D\delta m_{33}\Big(\Omega^{AC}\Omega^{BD}-\frac{1}{2}\Omega^{AB}\Omega^{CD}\Big)\Big]\\
&+e^{-6\lambda}\delta m_{AB}\nabla_C\nabla_D\delta m_{EF}\overline{m}_{(2)}^{ABCDEF}
\end{aligned} \tag{5.2.52}$$

In the last step, we insert the explicit form of the background metric and background momenta in terms of (μ, π_μ) and (λ, π_λ). The perturbations are expanded into the variables (x^v, y_v), (x^h, y_h), (x_A, y^A) and (X_{AB}, Y^{AB}) and then further decomposed into the (l, m)-dependent modes.

At first order, the constraints do not contain any spherically symmetric contribution because the average of a spherical harmonic with $l > 0$ over the sphere vanishes. The non-spherically symmetric contributions to the constraints are decomposed into scalar and vector spherical harmonics. For simplicity of the notation, we will not write the (l, m)-dependence of the coefficents in the mode expansion in the following equations. For the first order constraints, we obtain the non-symmetric constraints using the integrals in (**5.2.29**)

$$\begin{aligned}{}^{(1)}Z^h_{lm} &= -2e^{\mu}\partial_r(e^{\mu}y_v) + e^{2\mu}\sqrt{l(l+1)}y_e + 2\lambda' e^{2\lambda}y_h - \partial_r(\pi_\mu e^{-2\mu})x^v - \frac{1}{2}\pi_\mu e^{-2\mu}\partial_r x^v \\ &\quad + \frac{\pi_\lambda}{2}e^{-2\lambda}\sqrt{l(l+1)}x^e + \frac{\pi_\lambda}{2}e^{-2\lambda}\partial_r x^h\end{aligned} \tag{5.2.53}$$

$$\begin{aligned}{}^{(1)}Z^e_{lm} &= \sqrt{2(l+2)(l-1)}\left(e^{2\lambda}Y_e + \frac{\pi_\lambda}{4}e^{-2\lambda}X^e\right) - \partial_r\left(e^{2\lambda}y_e + \pi_\mu e^{-2\mu}x^e\right) - \sqrt{l(l+1)}e^{2\lambda}y_h \\ &\quad + \frac{1}{2}\sqrt{l(l+1)}\pi_\mu e^{-2\mu}x^v\end{aligned} \tag{5.2.54}$$

$${}^{(1)}Z^o_{lm} = \sqrt{2(l+2)(l-1)}\left(e^{2\lambda}Y_o + \frac{\pi_\lambda}{4}e^{-2\lambda}X^o\right) - \partial_r\left(e^{2\lambda}y_o + \pi_\mu e^{-2\mu}x^o\right) \tag{5.2.55}$$

$$\begin{aligned}{}^{(1)}Z^v_{lm} &= \frac{1}{2}(\pi_\mu - \pi_\lambda)e^{\mu-2\lambda}y_v - \frac{1}{2}\pi_\mu e^{-\mu}y_h - \frac{1}{8}\pi_\mu^2 e^{-\mu-4\lambda}x^h + \frac{1}{16}(3\pi_\mu^2 - 2\pi_\mu\pi_\lambda)e^{-3\mu-2\lambda}x^v \\ &\quad + e^{-3\mu+2\lambda}(-2\lambda'\partial_r + 6\lambda'\mu' - 3(\lambda')^2 - 2\lambda'' - e^{2\mu-2\lambda}(l(l+1)+1))x^v \\ &\quad + 2e^{-\mu}\left(\partial_r^2 - \mu'\partial_r - \lambda'\partial_r + (\lambda')^2 - \frac{1}{2}e^{2\mu-2\lambda}l(l+1)\right)x^h + 2e^{-\mu}\sqrt{l(l+1)}(\partial_r - \mu' + \lambda')x^e \\ &\quad - e^{\mu-2\lambda}\sqrt{\frac{(l+2)(l+1)l(l-1)}{2}}X^e\end{aligned} \tag{5.2.56}$$

At second order, we are only interested in the symmetric contributions of the constraints. We obtain them, by averaging the radial diffeomorphism and Hamiltonian constraints over the sphere. The results are rather lengthy and we use a short-hand notation to avoid the indices (l, m). For two sets of coefficients in a mode expansion $\{f_{lm}\}_{lm}$ and $\{g_{lm}\}_{lm}$, we define

$$f \cdot g := \sum_{lm} f_{lm}g_{lm}\,. \tag{5.2.57}$$

In this notation, the gravitational contributions to the second order symmetric constraints are using equation (**5.2.28**)

$${}^{(2)}C_h = -x^o \cdot \partial_r y_o + Y_o \cdot \partial_r X^o + y_v \cdot \partial_r x^v - 2\partial_r(x^v \cdot y_v) - x^e\partial_r \cdot y_e + Y_e \cdot \partial_r X^e + y_h \cdot \partial_r x^h \tag{5.2.58}$$

$$\begin{aligned}{}^{(2)}C_v &= \frac{e^{\mu}}{2}y_o \cdot y_o + e^{-\mu+2\lambda}Y_o \cdot Y_o + \frac{1}{2}e^{-\mu-2\lambda}\pi_\mu x^o \cdot y_o - \frac{1}{2}e^{-\mu-2\lambda}(\pi_\mu - \pi_\lambda)Y_o \cdot X^o \\ &\quad + \left(\frac{\pi_\mu^2}{16} + \frac{\pi_\mu\pi_\lambda}{8}\right)e^{-3\mu-4\lambda}x^o \cdot x^o - e^{-3\mu}x^o \cdot \left(-4\lambda'\partial_r - 2\lambda'' + (\lambda')^2 + 6\lambda'\mu' - \frac{1}{2}e^{2(\mu-\lambda)}l(l+1)\right)x^o \\ &\quad - e^{-\mu-2\lambda}\sqrt{\frac{(l+2)(l-1)}{2}}x^o \cdot (\partial_r - 2\lambda')X^o + \left(\frac{\pi_\mu^2}{32} - \frac{\pi_\mu\pi_\lambda}{16} + \frac{\pi_\lambda^2}{16}\right)e^{-\mu-6\lambda}X^o \cdot X^o \\ &\quad - e^{-2\lambda-\mu}X^o \cdot \left(\partial_r^2 - 4\lambda'\partial_r - \mu'\partial_r + \frac{5}{2}(\lambda')^2 + \lambda'\mu' - \lambda''\right)X^o - \frac{3}{4}e^{-2\lambda-\mu}\partial_r X^o \cdot \partial_r X^o \\ &\quad + \frac{1}{2}e^{3\mu-2\lambda}y_v \cdot y_v + \frac{1}{4}e^{-\mu-2\lambda}x^v \cdot y_v(3\pi_\mu - \pi_\lambda) + e^{-5\mu-2\lambda}x^v \cdot x^v\left(\frac{3}{64}\pi_\mu^2 + \frac{1}{32}\pi_\mu\pi_\lambda\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}e^{-5\mu+2\lambda}x^v\cdot\Big(e^{2(\mu-\lambda)}+12\lambda'\partial_r+6\lambda''-30\lambda'\mu'+9(\lambda')^2\Big)x^v+\frac{1}{2}e^{\mu}y_e\cdot y_e+\frac{1}{2}\pi_\mu e^{-\mu-2\lambda}x^e\cdot y^e\\
&+\left(\frac{1}{16}\pi_\mu^2+\frac{1}{8}\pi_\mu\pi_\lambda\right)e^{-3\mu-4\lambda}x^e\cdot x^e+e^{-3\mu}x^e\cdot\Big(4\lambda'\partial_r+2\lambda''-6\lambda'\mu'-(\lambda')^2\Big)x^e+\frac{1}{8}e^{-\mu-6\lambda}\pi_\mu^2x^h\cdot x^h\\
&-\frac{1}{2}e^{-\mu-2\lambda}\Big(4(\lambda')^2x^h\cdot x^h-4\lambda'x^h\cdot\partial_rx^h+\partial_rx^h\cdot\partial_rx^h\Big)\\
&+e^{-\mu+2\lambda}Y_e\cdot Y_e-\frac{1}{2}e^{-\mu-2\lambda}(\pi_\mu-\pi_\lambda)X^e\cdot Y_e+e^{-\mu-6\lambda}\left(\frac{1}{32}\pi_\mu^2-\frac{1}{16}\pi_\mu\pi_\lambda+\frac{1}{16}\pi_\lambda^2\right)X^e\cdot X^e\\
&+e^{\mu-2\lambda}\frac{1}{4}\Big(-3\partial_rX^e\cdot\partial_rX^e+X^e\cdot(-4\partial_r^2+4\mu'\partial_r+16\lambda'\partial_r-4\mu'\lambda'-10(\lambda')^2+4\lambda'')X^e\Big)\\
&-e^{\mu}y_h\cdot y_v-\frac{\pi_\mu}{2}e^{\mu-4\lambda}y_v\cdot x^h-\frac{1}{4}\pi_\mu e^{-3\mu}x^v\cdot y_h-\frac{3}{16}\pi_\mu^2e^{-3\mu-4\lambda}x^v\cdot x^h\\
&-e^{-3\mu}x^v\cdot\Big(\frac{1}{2}l(l+1)e^{2(\mu-\lambda)}x^h+\partial_r^2x^h-3\mu'\partial_rx^h+(\lambda')^2x^h-\lambda'\partial_rx^h\Big)-e^{-3\mu}\partial_rx^h\cdot\partial_rx^v\\
&-\sqrt{\frac{(l+2)(l+1)l(l-1)}{8}}e^{-\mu-2\lambda}x^v\cdot X^e-e^{-3\mu}\sqrt{l(l+1)}x^v\cdot\partial_rx^e\\
&-e^{-3\mu}\sqrt{l(l+1)}x^e\cdot(-3\mu'x^v-\lambda'x^v+\partial_rx^v)+e^{-\mu-2\lambda}\sqrt{l(l+1)}x^e\cdot\Big(2\lambda'x^h-\partial_rx^h\Big)\\
&+\sqrt{\frac{(l+2)(l-1)}{2}}e^{-\mu-2\lambda}x^e\cdot(2\lambda'X^e-\partial_rX^e)
\end{aligned}
\tag{5.2.59}
$$

The results have been double checked with the symbolic computation features of the computer algebra system Mathematica and the package xAct [127].

As matter content we study electromagnetic radiation. In section 3.6 the Hamiltonian formulation for the electromagnetic field was introduced. It gives rise to matter contributions to the diffeomorphism C_μ^{em} and Hamiltonian constraints C_0^{em}. Additionally, the Gauß constraint V_G has to be satisfied by the electromagnetic field.

In the section 3.6 on the Hamiltonian treatment of spherically symmetric spacetimes, we found that the background electromagnetic field is fully described by a non-trivial radial electric field $\overline{E}^3=\sqrt{\Omega}\xi$. We found that the background Gauß constraint implies that ξ is a constant which we related to the electric charge. In this section we use the notation $E^\mu=\overline{E}^\mu+\delta E^\mu$ and $A_\mu=\overline{A}_\mu+\delta A_\mu$, where the quantities with δ are considered as small non-symmetric perturbations.

Since the Gauß constraint V_G is linear in the electric field, we can treat it exactly. We find

$$
V_G=\partial_\mu E^\mu=\partial_r\delta E^3+D_A\delta E^A\,, \tag{5.2.60}
$$

where we used the expression $\overline{E}^3=\sqrt{\Omega}\xi$. The index μ is split into its radial and angular directions. Additionally, we notice that E^μ is a vector density, so that the covariant divergence and the partial divergence are the same. In terms of the mode decomposition, we have

$$
(V_G)_{lm}=\partial_r y_{lm}^M-\sqrt{l(l+1)}Y_{e,lm}^M \tag{5.2.61}
$$

The electromagnetic part of the diffeomorphism constraint is given by

$$
V_\mu^{\text{em}}=F_{\mu\nu}E^\nu\,, \tag{5.2.62}
$$

The calculations are simpler if we introduce the magnetic field $B^\mu:=1/2\,\epsilon^{\mu\nu\rho}F_{\nu\rho}$ with the Levi-Civita symbol $\epsilon^{\mu\nu\rho}$. The relation between $F_{\mu\nu}$ and B^μ is easily inverted and we have $F_{\mu\nu} = \epsilon_{\mu\nu\rho}B^\rho$. In terms of the perturbed vector potential, the magnetic field is

$$B^3 = \sum_{l\geq 1,m} \sqrt{l(l+1)} X_M^{o,lm} L_{lm} \tag{5.2.63}$$

$$B^A = \sum_{l\geq 1,m} \left(\partial_r X_M^{o,lm} L^A_{e,lm} + (\sqrt{l(l+1)} x_M^{lm} - \partial_r X_M^{e,lm}) L^A_{o,lm} \right) \tag{5.2.64}$$

In terms of the magnetic field, the diffeomorphism constraint reads

$$V_\mu^{\text{em, no G}} = F_{\mu\nu}E^\nu = \epsilon_{\mu\nu\rho}E^\nu B^\rho \tag{5.2.65}$$

For the calculation of the first order perturbations, we notice that the background magnetic field vanishes. Therefore, the only non-vanishing combination is of the form $\overline{E}^3 \delta B^\mu$. Due to the antisymmetry of the Levi-Civita symbol only the angular components of the constraint are non-vanishing and we have

$$V_A^{\text{em, no G}} = -\sqrt{\Omega}\xi\epsilon_{AC}B^C = \sqrt{\Omega}\xi \sum_{l\geq 1,m} \left(-\partial_r X_M^{o,lm} L^o_{lm} + (\sqrt{l(l+1)} x_M^{lm} - \partial_r X_M^{e,lm}) L^e_{lm} \right) \tag{5.2.66}$$

where $\epsilon_{AC} = \epsilon_{3AC}$ is the Levi-Civita symbol on the sphere.

To second order, we only need the spherically symmetric radial diffeomorphism constraint. To second order it is given by

$$\begin{aligned} {}^{(2)}C_h &= \int_{S^2} \epsilon_{AC}\delta E^A \delta B^C \mathrm{d}\Omega \\ &= Y_o^M \cdot \partial_r X_M^o + Y_e^M \cdot \partial_r X_M^e - \sqrt{l(l+1)} Y_e^M \cdot x_M \,. \end{aligned} \tag{5.2.67}$$

where we used the abbreviation $(\cdot)$ for the sum over the product of two coefficients in the mode expansion.

The electromagnetic contributions to the Hamiltonian constraint V_0^{em} are also conveniently rewritten using the magnetic field:

$$V_0^{\text{em}} = \frac{1}{2} m_{\mu\nu} \left(\frac{g^2}{\sqrt{m}} E^\mu E^\nu + \frac{\sqrt{m}}{g^2} B^\mu B^\nu \right) . \tag{5.2.68}$$

To first order, only the electric terms contribute because the background magnetic field vanishes. We have

$${}^{(1)}V_0^{\text{em}} = \frac{g^2}{2\sqrt{\overline{m}}} (\delta m_{33} - \frac{1}{2} \overline{m}_{33} \delta m_{\mu\nu} \overline{m}^{\mu\nu}) (\overline{E}^3)^2 + \frac{g^2}{\sqrt{\overline{m}}} \overline{m}_{33} \overline{E}^3 \delta E^3 \,. \tag{5.2.69}$$

The first order correction to the electromagnetic Hamiltonian constraint does not have any symmetric components. We insert the explicit form of the perturbed metric and the electric field and obtain

$$({}^{(1)}Z_v^{\text{em}})_{lm} = \sqrt{\Omega} \frac{g^2}{2} e^{-\mu - 2\lambda} \left(\left(\frac{1}{2} x^v_{lm} - x^h_{lm} e^{2(\mu-\lambda)} \right) \xi^2 + 2e^{2\mu} \xi y^M_{lm} \right) . \tag{5.2.70}$$

The second order corrections of the electromagnetic Hamiltonian constraint are

$$
\begin{aligned}
V_0^{\mathrm{em}} = &\frac{g^2}{2\sqrt{\overline{m}}}\Big(\frac{1}{8}\overline{m}_{33}((\delta m^{\mu}{}_{\mu})^2 + 2\delta m_{\mu\nu}\delta m^{\mu\nu}) - \frac{1}{2}\delta m_{33}\delta m^{\mu}{}_{\mu}\Big)(\overline{E}^3)^2 \\
&+ \frac{g^2}{\sqrt{\overline{m}}}(\delta m_{3\mu}\delta E^{\mu}\overline{E}^3 - \frac{1}{2}\overline{m}_{33}\delta m_{\mu\nu}\overline{m}^{\mu\nu}\overline{E}^3\delta E^3) + \frac{g^2}{2\sqrt{\overline{m}}}\overline{m}_{\mu\nu}\delta E^{\mu}\delta E^{\nu} + \frac{\sqrt{\overline{m}}}{2g^2}\overline{m}_{\mu\nu}\delta B^{\mu}\delta B^{\nu})\,.
\end{aligned}
\tag{5.2.71}
$$

We split the indices into radial and angular components and use the decomposition of the perturbed metric and the electromagnetic field. We find the second order symmetric contribution

$$
\begin{aligned}
{}^{(2)}C_v^{\mathrm{em}} = &\frac{g^2}{2}e^{-\mu-2\lambda}\Big[\frac{1}{8}\big(- e^{-4\mu}x^v\cdot x^v - 4e^{-2\mu-2\lambda}x^v\cdot x^h + 4e^{-2\mu-2\lambda}(x^o\cdot x^o + x^e\cdot x^e) + 8e^{-4\lambda}x^h\cdot x^h \\
&+ 2e^{-4\lambda}(X^e\cdot X^e + X^o\cdot X^o)\big)e^{2\mu}\xi^2 + \xi(x^v - 2e^{2\mu-2\lambda}x^h)\cdot y^M \\
&+ 2(x^oY_o^M + x^eY_e^M)\xi + e^{2\mu}y^M\cdot y^M + e^{2\lambda}Y_e^M\cdot Y_e^M + e^{2\lambda}Y_o^M\cdot Y_o^M\Big] \\
&+ \frac{1}{2g^2}e^{-\mu-2\lambda}\Big[e^{2\mu}l(l+1)X_M^o\cdot X_M^o + e^{2\lambda}\partial_r X_M^o\cdot\partial_r X_M^o \\
&+ e^{2\lambda}(\sqrt{l(l+1)}x_m - \partial_r X_e^M)\cdot(\sqrt{l(l+1)}x_m - \partial_r X_e^M)\Big]\,.
\end{aligned}
\tag{5.2.72}
$$

5.2.3 Solution of the Perturbed Constraints in Gullstrand-Painlevé Gauge

The next step in the reduced phase space formulation is the solution of the constraints for the momenta of the gauge degrees of freedom. In the previous section we derived the first order non-symmetric constraints which have to be solved for the perturbative gauge momenta and the second order symmetric constraints which give rise to second order corrections to the symmetric gauge momenta.

In this section, we specialise to GP gauge and study both the gravitational and electromagnetic contributions to the constraints. We closely follow [2, 3]. Before presenting the constraints in this gauge, we evaluate the Gauß constraint. It fixes the variable y_M in terms of Y_e^M:

$$
y_M^{lm} := \sqrt{l(l+1)}\int Y_{e,lm}^M\,\mathrm{d}r
\tag{5.2.73}
$$

We gauge fix the conjugate variable $x_{lm}^M = 0$.

The task is to solve the constraints ${}^{(0)}C + {}^{(2)}C = 0$ for π_μ and π_λ and ${}^{(1)}Z = 0$ for $y_{v/h}$ and y_I. For the derivation of the second order physical Hamiltonian, we only need to compute the solutions of the constraints up to second order in the perturbations. For the momenta y, we will only get first order solutions $y^{(1)}$. The solutions for π_μ and π_λ will have first and second order contributions. We therefore split them into the zeroth order parts $\pi_\mu^{(0)}$, $\pi_\lambda^{(0)}$ and second order parts $\pi_\mu^{(2)}$, $\pi_\lambda^{(2)}$. The zeroth order solution $\pi_\mu^{(0)}$, $\pi_\lambda^{(0)}$ was already determined in section 3.6.

Imposing the GP gauge on the background degrees of freedom ($\mu = 0$, $\lambda = \log r$), the first order non-symmetric constraints are

$$
{}^{(1)}Z_{lm}^h = -2\partial_r y_v + \sqrt{l(l+1)}y_e + 2ry_h - \partial_r\pi_\mu x^v - \frac{1}{2}\pi_\mu\partial_r x^v + \frac{\pi_\lambda}{2r^2}\sqrt{l(l+1)}x^e + \frac{\pi_\lambda}{2r^2}\partial_r x^h
\tag{5.2.74}
$$

$$
{}^{(1)}Z_{lm}^e = \sqrt{2(l+2)(l-1)}\Big(r^2Y_e + \frac{\pi_\lambda}{4r^2}X^e\Big) - \partial_r(r^2y_e + \pi_\mu x^e) + \sqrt{l(l+1)}\Big(\frac{\pi_\mu}{2}x^v - r^2y_h\Big) - \xi\partial_r X_M^e
\tag{5.2.75}
$$

$$
{}^{(1)}Z^o_{lm} = \sqrt{2(l+2)(l-1)}\Big(r^2 Y_o + \frac{\pi_\lambda}{4r^2}X^o\Big) - \partial_r\Big(r^2 y_o + \pi_\mu x^o\Big) - \xi\partial_r X^o_M \tag{5.2.76}
$$

$$
\begin{aligned}
{}^{(1)}Z^v_{lm} &= \frac{1}{2r^2}(\pi_\mu - \pi_\lambda)y_v - \Big(2r\partial_r + l(l+1) + 2 - 2\frac{r_s}{r}\Big)x^v - \frac{1}{2}\pi_\mu y_h - \frac{1}{r^2}\sqrt{\frac{(l+2)(l+1)l(l-1)}{2}}X^e \\
&\quad + 2\Big(\partial_r^2 - \frac{1}{r}\partial_r - \frac{(l+2)(l-1)}{2r^2} - \frac{r_s}{r^3}\Big)x^h + 2\sqrt{l(l+1)}\Big(\partial_r + \frac{1}{r}\Big)x^e + \frac{g^2}{r^2}\xi\sqrt{l(l+1)}\int Y^M_{lm}\,\mathrm{d}r
\end{aligned} \tag{5.2.77}
$$

In the equations we replaced the quadratic terms in π_μ and π_λ by their background values which is consistent to the order we are working in.

In the GP gauge for the background, the second order symmetric constraints read

$$
\begin{aligned}
{}^{(2)}C_h &= -x^o\cdot\partial_r y_o + Y_o\cdot\partial_r X^o + y_v\cdot\partial_r x^v - 2\partial_r(x^v\cdot y_v) - x^e\cdot\partial_r y_e + Y_e\cdot\partial_r X_e + y_h\cdot\partial_r x^h \\
&\quad + Y^M_e\cdot(X^e_M)' + Y^M_o\cdot(X^o_M)'
\end{aligned} \tag{5.2.78}
$$

$$
\begin{aligned}
{}^{(2)}C_v &= \frac{1}{2}y_o\cdot y_o + \frac{1}{2r^2}\pi_\mu x^o\cdot y_o + \frac{1}{r^2}x^o\cdot\Big(4r\partial_r - 3 + \frac{l(l+1)}{2} + 2\frac{r_s}{r}\Big)x^o + r^2 Y_o\cdot Y_o \\
&\quad - \frac{1}{2r^2}(\pi_\mu - \pi_\lambda)Y_o\cdot X^o - \frac{1}{r^4}X^o\cdot\Big(r^2\partial_r^2 - 4r\partial_r + \frac{7}{2} + \frac{r_s^2}{g^2\xi^2 - 4rr_s}\Big)X^o - \frac{3}{4r^2}\partial_r X^o\cdot\partial_r X^o \\
&\quad - \frac{1}{r^3}\sqrt{\frac{(l+2)(l-1)}{2}}x^o\cdot(r\partial_r - 2)X^o \\
&\quad + \frac{1}{2r^2}y_v\cdot y_v + \frac{1}{4r^2}(3\pi_\mu - \pi_\lambda)x^v\cdot y_v + x^v\cdot\Big(3r\partial_r + 1 + \frac{r_s}{r} - \frac{g^2\xi^2}{4r^2}\Big)x^v + \frac{1}{2}y_e\cdot y_e + \frac{1}{2r^2}\pi_\mu x^e\cdot y_e \\
&\quad + \frac{1}{r^2}x^e\cdot\Big(4r\partial_r - 3 + 2\frac{r_s}{r}\Big)x^e - \frac{1}{2r^4}\Big(4\Big(1 - \frac{r_s}{r}\Big)x^h\cdot x^h - 4rx^h\cdot\partial_r x^h + r^2\partial_r x^h\cdot\partial_r x^h\Big) \\
&\quad + r^2 Y_e\cdot Y_e - \frac{1}{2r^2}(\pi_\mu - \pi_\lambda)X^e\cdot Y_e - \frac{1}{r^4}X^e\cdot\Big(r^2\partial_r^2 - 4r\partial_r + \frac{7}{2} + \frac{r_s^2}{g^2\xi^2 - 4rr_s}\Big)X^e \\
&\quad - \frac{3}{4r^2}\partial_r X^e\cdot\partial_r X^e - y_h\cdot y_v - \frac{\pi_\mu}{2r^4}y_v\cdot x^h - \frac{1}{4}\pi_\mu x^v\cdot y_h - \partial_r x^h\cdot\partial_r x^v \\
&\quad - \frac{1}{r^2}x^v\cdot\Big(r^2\partial_r^2 x^h - r\partial_r x^h + \frac{l(l+1)+2}{2}x^h + 3\frac{r_s}{r}x^h - \frac{g^2\xi^2}{2r^2}x^h\Big) - \sqrt{l(l+1)}x^v\cdot\partial_r x^e \\
&\quad - \frac{1}{2r^2}\sqrt{\frac{(l+2)(l+1)l(l-1)}{2}}x^v\cdot X^e - \sqrt{l(l+1)}\frac{1}{r}x^e\cdot(-x^v + r\partial_r x^v) \\
&\quad + \frac{1}{r^3}\sqrt{l(l+1)}x^e\cdot\Big(2x^h - r\partial_r x^h\Big) + \sqrt{\frac{(l+2)(l-1)}{2}}\frac{1}{r^3}x^e\cdot(2X^e - r\partial_r X^e) \\
&\quad + \frac{g^2}{2r^2}\Bigg[\Big(2x^o\cdot Y^M_o + \sqrt{l(l+1)}\Big(x^v - \frac{2}{r^2}x^h\Big)\cdot\int Y^M_e\,\mathrm{d}r + 2x^e\cdot Y^M_e\Big)\xi \\
&\qquad + l(l+1)\int Y^M_e\,\mathrm{d}r\cdot\int Y^M_e\,\mathrm{d}r + r^2(Y^M_e\cdot Y^M_e + Y^M_o\cdot Y^M_o)\Bigg] \\
&\quad + \frac{1}{2g^2r^2}\Big[l(l+1)X^o_M\cdot X^o_M + r^2({X^o_M}'\cdot{X^o_M}' + {X^e_M}'\cdot{X^e_M}')\Big]
\end{aligned} \tag{5.2.79}
$$

In the following, we perform the calculations for the solutions of the constraints in two steps. First, we solve the second order symmetric constraints to obtain the second order corrections to the background momenta $\pi^{(2)}_\mu$, $\pi^{(2)}_\lambda$. In this step, we assume that we successfully solved the non-symmetric first order constraints for y. Then, we consider the first order constraints and solve them for y and insert the results into the solution we obtained for the second order symmetric constraints. Finally, we simplify the results of the computations using canonical transformations.

5.2.3.1 Solution of the Second Order Constraints

As explained above, the symmetric constraints are solved for the momenta π_μ and π_λ. We consider the ansatz $\pi_\mu = \pi_\mu^{(0)} + \pi_\mu^{(2)}$ and $\pi_\lambda = \pi_\lambda^{(0)} + \pi_\lambda^{(2)}$. Inserting this into the spherically symmetric Hamiltonian and diffeomorphism constraints, we obtain to second order

$$C_v \sim \frac{4\pi}{4r^2}\Big(\pi_\mu^{(0)}\pi_\mu^{(2)} - \pi_\mu^{(0)}\pi_\lambda^{(2)} - \pi_\mu^{(2)}\pi_\lambda^{(0)}\Big) + {}^{(2)}C_v = 0\,, \tag{5.2.80}$$

$$C_h \sim 4\pi\bigg(\frac{1}{r}\pi_\lambda^{(2)} - (\pi_\mu^{(2)})' + {}^{(2)}C_h\bigg) = 0\,. \tag{5.2.81}$$

In these equations, ${}^{(2)}C_v$ and ${}^{(2)}C_h$ are the expressions quadratic in the perturbations (x, y) and (X, Y) defined in (**5.2.58**) and (**5.2.59**). The exact form of them depends on the solution of the first order constraints. As we will see, the exact expression will not be relevant for the solution of the second order constraints that we discuss in this part. By construction, ${}^{(2)}C_v$ and ${}^{(2)}C_h$ will not depend on $\pi_\mu^{(2)}, \pi_\lambda^{(2)}$. Setting $x^{v/h/e/o} = 0$ as required by the full GP gauge, ${}^{(2)}C_v$ and ${}^{(2)}C_h$ simplify drastically and we have

$${}^{(2)}C_h = Y_o \cdot \partial_r X^o + Y_e \cdot \partial_r X_e + Y_e^M \partial_r X_M^e + Y_o^M \partial_r X_M^o \tag{5.2.82}$$

$$\begin{aligned}
{}^{(2)}C_v = {} & r^2 Y_o \cdot Y_o - \frac{1}{2r^2}({\pi_\mu}^{(0)} - {\pi_\lambda}^{(0)}) Y_o \cdot X^o - \frac{1}{r^4} X^o \cdot \left(r^2\partial_r^2 - 4r\partial_r + \frac{7}{2} + \frac{r_s^2}{g^2\xi^2 - 4rr_s}\right) X^o \\
& + r^2 Y_e \cdot Y_e - \frac{1}{2r^2}({\pi_\mu}^{(0)} - {\pi_\lambda}^{(0)}) X^e \cdot Y_e - \frac{1}{r^4} X^e \cdot \left(r^2\partial_r^2 - 4r\partial_r + \frac{7}{2} + \frac{r_s^2}{g^2\xi^2 - 4rr_s}\right) X^e \\
& - \frac{3}{4r^2}(\partial_r X^e \cdot \partial_r X^e + \partial_r X^o \cdot \partial_r X^o) + \frac{1}{2}(y_o^{(1)} \cdot y_o^{(1)} + y_e^{(1)} \cdot y_e^{(1)}) + \frac{1}{2r^2} y_v^{(1)} \cdot y_v^{(1)} - y_h^{(1)} \cdot y_h^{(1)} \\
& + \frac{g^2}{2r^2}\left[l(l+1)\int Y_e^M \,\mathrm{d}r \cdot \int Y_e^M \,\mathrm{d}r + r^2(Y_e^M \cdot Y_e^M + Y_o^M \cdot Y_o^M)\right] \\
& + \frac{1}{2g^2r^2}\left[l(l+1) X_M^o \cdot X_M^o + r^2(\partial_r X_M^o \cdot \partial_r X_M^o + \partial_r X_M^e \cdot \partial_r X_M^e)\right]
\end{aligned} \tag{5.2.83}$$

In the above equations, we have to substitute the solution of the first order constraints for $y_{v/h/e/o}^{(1)}$.

The quantity $\pi_\lambda^{(2)}$ is related to $(\pi_\mu^{(2)})'$ through the second equation $C_h = 0$ and will be used to eliminate $\pi_\lambda^{(2)}$ in the first equation. We have

$$\pi_\lambda^{(2)} = r\bigg((\pi_\mu^{(2)})' - \frac{1}{4\pi}{}^{(2)}C_h\bigg)\,. \tag{5.2.84}$$

This implies using the first equation $C_v = 0$ that

$$\pi_\mu^{(0)}\left(1 - \frac{\pi_\lambda^{(0)}}{\pi_\mu^{(0)}}\right)\pi_\mu^{(2)} - r\pi_\mu^{(0)}(\pi_\mu^{(2)})' + \frac{r\pi_\mu^{(0)}}{4\pi}{}^{(2)}C_h + \frac{4r^2}{4\pi}{}^{(2)}C_v = 0\,. \tag{5.2.85}$$

The zeroth order symmetric constraints imply that $\pi_\lambda^{(0)}$ satisfies $\pi_\lambda^{(0)} = r(\pi_\mu^{(0)})'$ (see (**3.6.10**)) This simplifies the differential equation for $\pi_\mu^{(2)}$ and we have after dividing by r^2

$$\left(\frac{1}{r}\pi_\mu^{(0)}\pi_\mu^{(2)}\right)' - \frac{1}{4\pi r}\pi_\mu^{(0)}{}^{(2)}C_h - \frac{4}{4\pi}{}^{(2)}C_v = 0\,. \tag{5.2.86}$$

The solution of this differential equation is straight forward using an integration over r:

$$\pi_\mu^{(2)} = \frac{4r}{4\pi\pi_\mu^{(0)}}\int \mathrm{d}r\left[\frac{\pi_\mu^{(0)}}{4r}{}^{(2)}C_h + {}^{(2)}C_v\right]. \tag{5.2.87}$$

The solution for $\pi_\lambda^{(2)}$ is determined by solving C_v for $\pi_\lambda^{(2)}$ in terms of $\pi_\mu^{(2)}$:

$$\pi_\lambda^{(2)} = \left(1 - \frac{\pi_\lambda^{(0)}}{\pi_\mu^{(0)}}\right)\pi_\mu^{(2)} + \frac{4r^2}{4\pi\pi_\mu^{(0)}}{}^{(2)}C_v \tag{5.2.88}$$

Since the physical Hamiltonian depends only on π_μ, we provide the following explicit expression

$$\pi_\mu \sim \pi_\mu^{(0)} + \pi_\mu^{(2)} = \pi_\mu^{(0)}\left[1 + \frac{1}{4\pi(4r_s - g^2\xi^2/r)}\int \mathrm{d}r\left(\sqrt{\frac{r_s}{r} - \frac{g^2\xi^2}{4r^2}}{}^{(2)}C_h + {}^{(2)}C_v\right)\right] \tag{5.2.89}$$

5.2.3.2 The Dipole Perturbations ($l = 1$)

After completing the second order constraints we now focus on the non-symmetric first-order constraints. The dipole perturbations with $l = 1$ need to be treated separate from the rest of the perturbations, due to the absence of the tensor spherical harmonics for $l = 1$. There are no gravitational observables left, only the electromagnetic dipole perturbations $X_M^{e/o}, Y_{e/o}^M$ will contribute to the physical Hamiltonian.

The first-order constraints for even and odd parity separate and we can solve them independently. For the odd parity perturbations we have the non-symmetric first-order constraint Z^o. We work in the GP gauge ($x^o = 0$) and find the solution of the differential equation with some integration constants a_m:

$$y_o^{1m} = \frac{1}{r^2}\left(a_m - \xi X_M^{o,1m}\right). \tag{5.2.90}$$

For the interpretation of a_m, we expand the Kerr black hole solution to linear order in the angular momentum parameter a. In Boyer-Lindquist coordinates we have

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{r_s}{r}}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2 + 2\frac{r_s a}{r}\sin^2\theta\mathrm{d}t\mathrm{d}\phi \tag{5.2.91}$$

For the comparison we need to compute the perturbation of the shift vector to the momenta y_o using the stability condition of the gauge fixing $x^o = 0$. This condition reads

$$\dot{x}_{1m}^o = \{x_{1m}^o, N^{(2)}C_0 + N^{3(2)}C_r + r^{-2}\delta N^o \cdot Z^o\}\Big|_{x^o=0,N=1,N^3=\frac{\pi_\mu^{(0)}}{4r}} \tag{5.2.92}$$

$$= y_o^{1m} + r^2\partial_r(r^{-2}\delta N_{1m}^o)\,. \tag{5.2.93}$$

The Poisson bracket is evaluated at the solution of the background stability condition for N and N^3 which is consistent up to the first-perturbative order. The differential equation can be integrated directly and we have

$$\delta N^o_{1m} = \frac{a_m}{3r}\,. \tag{5.2.94}$$

The term associated to the integration constant would grow quadratically as r approaches infinity. For consistency with the fall-off conditions, the integration constant has to vanish to ensure that δN^o decays as r goes to infinity. Comparing with the linearized Kerr metric, we observe that a_0 will be related to the angular momentum parameter a. The three components of a_m stand for the three directions the angular momentum can point to and a_0 is the angular momentum in the z-direction. A more in-depth argument for the relation of the odd parity dipole perturbations to the linearized Kerr black hole, see [74].

We now use the result for y_o^{1m} in (**5.2.87**) for $\pi_\mu^{(2)}$ to find the odd parity dipole contributions to the momentum $\pi_\mu^{(2)}$:

$$\frac{\pi\pi_\mu^{(0)}}{r}\pi_\mu^{(2)}\Big|_{l=1,odd} = \sum_m \int \mathrm{d}r\,\Big[N^3 Y^M_{o,1m}X^M_{o,1m}{}' + N\Big(\frac{g^2}{2}(Y^M_{o,1m})^2 + \frac{1}{2g^2}\bigg((X_M^{o,1m\prime})^2 + \frac{2+g^2\xi^2r^{-2}}{r^2}(X_M^{o,1m})^2 + \frac{a_m^2 + 2\xi a_m X_M^{o,1m}}{2r^4}\bigg)\Big)\Big] \tag{5.2.95}$$

There are three first order constraints in the even parity sector: Z^v, Z^h and Z^e. We solve them for the gauge momenta y_h, y_v and y_e and keep the true degrees of freedom of the electric field $(X^{e,1m}, Y_{e,1m})$. The solution of the constraint equation $Z^v = 0$ for y_h in terms of y_v is

$$y_h^{1m} = \frac{1}{r^2}\bigg(1 - \frac{\pi_\lambda}{\pi_\mu}\bigg)y_v^{1m} + \frac{2\sqrt{2}g^2\xi}{r^2\pi_\mu}\int Y^M_{e,1m}\,\mathrm{d}r\,. \tag{5.2.96}$$

Additionally, we use $Z^h = 0$ to express y_e in terms of y_v:

$$y_e^{1m} = \frac{1}{\sqrt{2}}\bigg(2\partial_r y_v^{1m} - \frac{2}{r}\bigg(1 - \frac{\pi_\lambda}{\pi_\mu}\bigg)y_v^{1m}\bigg) - \frac{4g^2\xi}{r\pi_\mu}\int Y^M_{e,1m}\,\mathrm{d}r \tag{5.2.97}$$

The remaining constraint Z^e together with the solutions for y_h and y_e give a differential equation for y_v:

$$\sqrt{2}r^2\partial_r^2 y_v^{1m} + \frac{\sqrt{2}r(6rr_s - g^2\xi^2)}{4rr_s - g^2\xi^2}\partial_r y_v^{1m} - \frac{2\sqrt{2}g^2\xi^2 rr_s}{(4rr_s - g^2\xi^2)^2}y_v^{1m} = s(r)\,, \tag{5.2.98}$$

where $s(r)$ is a "source" term depending on the electromagnetic field:

$$s(r) = \xi\partial_r X_M^{e,1m} - \frac{4g^2\xi rr_s}{(4rr_s - g^2\xi^2)^{3/2}}\int Y^M_{e,1m}\,\mathrm{d}r + \frac{2g^2\xi r}{\sqrt{4rr_s - g^2\xi^2}}Y^M_{e,1m}\,. \tag{5.2.99}$$

Notice that the source term only contributes for charged black holes ($\xi \neq 0$). For simplicity of the notation we do not display the labels $1, m$ in the following computations.

To fully solve the first-order constraints, we have to find a solution of the second order, linear and inhomogeneous differential equation. The theory for such differential equations implies that the solution is the sum of a particular solution of the inhomogeneous equation and the general solution of the homogeneous equation. The solution of the homogeneous equation is of the form $y_v = C_I y_v^I + C_{II} y_v^{II}$, where C_I and C_{II}

are constants and y_v^I, y_v^{II} are linearly independent solutions given by

$$y_v^I = \frac{1}{\pi_\mu}, \quad y_v^{II} = 1 - \frac{2g\xi}{\pi_\mu} \arctan\left(\frac{\pi_\mu}{2g\xi}\right) \tag{5.2.100}$$

The particular solution of the inhomogeneous equation is obtained from the method of variation of constants. We consider the constants C_I and C_{II} to depend on r and have the ansatz

$$y_v^{\text{part}}(r) = C_I(r) y_v^I(r) + C_{II}(r) y_v^{II}(r) \tag{5.2.101}$$

Inserting this equation into the inhomogeneous differential equation including the source term $s(r)$, we have

$$\sqrt{2}r^2(C_I'' y_v^I + 2C_I' y_v^{I\prime} + C_{II}'' y_v^{II} + 2C_{II}' y_v^{II\prime}) + \frac{\sqrt{2}r(6rr_s - g^2\xi^2)}{4rr_s - g^2\xi^2}(C_I' y_v^I + C_{II}' y_v^{II}) = s(r)\,. \tag{5.2.102}$$

Since y_v^I and y_v^{II} are solutions of the homogeneous differential equation, the terms without derivatives acting on C_I and C_{II} vanish. The second term in the equation motivates us to consider solutions C_I and C_{II} that satisfy $C_I' y_v^I + C_{II}' y_v^{II} = 0$. We take the derivative of this relation and simplify the first term of the differential equation and we obtain

$$\sqrt{2}r^2\left(C_I' y_v^{I\prime} + C_{II}' y_v^{II\prime}\right) = s(r) \tag{5.2.103}$$

In addition, the relation $C_I' y_v^I + C_{II}' y_v^{II} = 0$ is used to eliminate the function C_{II}' from the equation. We have

$$\sqrt{2}r^2 C_I{}' \left(y_v^{I\prime} - \frac{y_v^I}{y_v^{II}} y_v^{II\prime}\right) = s(r) \tag{5.2.104}$$

The solution of this equation is straight forward and we obtain the solutions for C_I and C_{II}:

$$C_I = \int \frac{s(r)}{\sqrt{2}r^2\left(y_v^{I\prime} - \frac{y_v^I}{y_v^{II}} y_v^{II\prime}\right)} \mathrm{d}r \;, \tag{5.2.105}$$

$$C_{II} = \int \frac{s(r)}{\sqrt{2}r^2\left(y_v^{II\prime} - \frac{y_v^{II}}{y_v^I} y_v^{I\prime}\right)} \mathrm{d}r \;. \tag{5.2.106}$$

The integration constant is not important and was neglected because we are only interested in a particular solution of the differential equation. Combining the results, the particular solution reads

$$y_v^{\text{part}} = \int \frac{s(r)}{\sqrt{2}r^2\left(y_v^{I\prime} - \frac{y_v^I}{y_v^{II}} y_v^{II\prime}\right)} \mathrm{d}r\, y_v^I + \int \frac{s(r)}{\sqrt{2}r^2\left(y_v^{II\prime} - \frac{y_v^{II}}{y_v^I} y_v^{I\prime}\right)} \mathrm{d}r\, y_v^{II} \,. \tag{5.2.107}$$

The even parity contributions to $\pi_\mu^{(2)}$ are then given by

$$\begin{aligned} \frac{\pi\pi_\mu^{(0)}}{r}\pi_\mu^{(2)}\Big|_{l=1,\text{even}} = \sum_m \int \mathrm{d}r\, \frac{\pi_\mu}{4r} Y^M_{e,1m}(X^{e,1m}_M)' + \frac{1}{2}(y_e^{1m})^2 - \left(\frac{2\sqrt{2}g^2\xi}{r^2\pi_\mu}\int Y^M_{e,1m}\,\mathrm{d}r\right) y_v + \frac{1}{2g^2}\left[X^{e,1m\prime}_M\right]^2 \\ + \frac{g^2}{2r^2}\left(2\left(\int Y^M_{e,1m}\,\mathrm{d}r\right)^2 + r^2 (Y^M_{e,1m})^2\right), \end{aligned} \tag{5.2.108}$$

In the equation, y_e and y_v are functions of X_M^e and Y_e^M through the solution of the first order constraints.

The equation for $\pi_\mu^{(2)}$ depends in a complicated way on the integrals and derivatives of X_M^e and Y_e^M. We define new variables A^e and Π_A^e as

$$A^{e,1m} = g^2 \int Y^M_{e,1m}\,\mathrm{d}r\,, \quad \Pi_A^{e,1m} = g^{-2}\partial_r X_M^{e,1m}\,. \tag{5.2.109}$$

Inserting the solution for y_e and the new variables A^e, Π_A^e into (**5.2.108**) we get

$$\begin{aligned}\frac{\pi\pi_\mu^{(0)}}{r}\pi_\mu^{(2)}\Big|_{l=1,\text{even}} = \sum_m \int \mathrm{d}r\, &\frac{\pi_\mu}{4r}\Pi_A^{e,1m}A^{e,1m\prime} + \left(y_v' - \frac{1}{r}y_v + \frac{\pi_\lambda}{r\pi_\mu}y_v\right)^2 - \frac{2\sqrt{2}\xi}{r\pi_\mu}\left(y_v' + \frac{\pi_\lambda}{r\pi_\mu}y_v\right)A^{e,1m} \\ &+ \frac{1}{2}\left(g^2(\Pi_A^{e,1m})^2 + \frac{1}{g^2}\left((A^{e,1m\prime})^2 + \frac{1}{r^2\pi_\mu^2}(2\pi_\mu^2 + 16g^2\xi^2)(A^{e,1m})^2\right)\right).\end{aligned} \tag{5.2.110}$$

Because of the complexity of the solution for y_v, inserting it into this equation yields a very complicated expression. It would be desirable to find a canonical transformation simplifying the expressions to reach a more tractable formula for the even dipole perturbations. A further analysis of the dipole perturbations is left for future investigations.

5.2.3.3 Solution of the First Order Constraints - Odd Parity

In this section, we turn to the odd parity perturbations for $l \geq 2$. In the previous chapter on the dipole perturbations, we saw that the solution of the odd parity sector is simpler compared to the even parity one. We expect the same to be true for $l \geq 2$ due to the fact that there is only one constraint equation $Z^o = 0$ that has to be solved. The odd parity sector consists of the gauge degrees of freedom (x^o_{lm}, y_o^{lm}) and the true gravitational degrees of freedom (X^o_{lm}, Y_o^{lm}) as well as the true degrees of freedom of the Maxwell field $(X_M^{o,lm}, Y^M_{o,lm})$. From now on, in this section we will not display the labels l, m.

The constraint equation $Z^o = 0$ is a differential equation for y^o whose solution in GP gauge ($x^o = 0$) is

$$y_o^{(1)} = \frac{1}{r^2}\int \left[\sqrt{2(l+2)(l-1)}\left(r^2 Y_o + \frac{\pi_\lambda^{(0)}}{4r^2}X^o\right) - \xi\partial_r X_M^o\right]\mathrm{d}r\ . \tag{5.2.111}$$

This is already the full solution for the odd parity case and it can be inserted into equation (**5.2.87**) to obtain the odd parity contributions to $\pi_\mu^{(2)}$. The result is still rather long and not very suitable for further investigations. In the following we perform two canonical transformations to simplify the expression.

The canonical transformation leaves the electromagnetic sector invariant. Nevertheless, for a unified treatment of the odd and even parity sectors later, we redefine the variables $A^o = X_M^o$ and $\Pi_A^o = Y_o^M$. The first transformation for the gravitational variables is motivated by the integral appearing in the solution for $y_o^{(1)}$. We introduce a new variable Q for this integral and find a canonically conjugate variable P. Explicitly, we have

$$P := \frac{1}{\sqrt{2}}\partial_r\left(r^{-2}X^o\right) \tag{5.2.112}$$

$$Q := \sqrt{2}\int \mathrm{d}r\left(r^2 Y_o + \frac{\pi_\lambda^{(0)}}{4r^2}X^o\right). \tag{5.2.113}$$

The transformation depends on the electromagnetic field through the implicit dependence of $\pi_\lambda^{(0)}$ on ξ. Let us check that it is indeed a canonical transformation. The Poisson bracket of P with itself vanishes trivially. For the remaining brackets consider smeared versions of the variables, i.e. we define $Q[f] = \int f(r)Q(r)\,\mathrm{d}r$ and similarly $P[f] := \int f(r)P(r)\mathrm{d}r$. Then, we calculate

$$\begin{aligned}\{Q[f], P[g]\} &= \int \mathrm{d}r_1\mathrm{d}r_2\left[\int^{r_1} f(\tilde{r}_1)\mathrm{d}\tilde{r}_1\partial_{r_2}g(r_2)\left\{r_1^2 Y_o(r_1) + \frac{\pi_\lambda^{(0)}(r_1)}{4r_1^2}X^o(r_1), r_2^{-2}X^o(r_2)\right\}\right] \\ &= -\int \mathrm{d}r_1\mathrm{d}r_2\left[\int^{r_1} f(\tilde{r}_1)\mathrm{d}\tilde{r}_1\partial_{r_2}g(r_2)r_1^2\,r_2^{-2}\,\delta(r_1,r_2)\right] = \int \mathrm{d}r f(r)g(r)\end{aligned} \tag{5.2.114}$$

This is the expected result that shows that Q and P are canonically conjugated. Using similar methods one shows explicitly that the Poisson bracket $\{Q[f], Q[g]\}$ vanishes. Therefore, the transformation is canonical.

The solution for $y_o^{(1)}$ and the first canonical transformation are now inserted into equation (**5.2.87**). After imposing the GP gauge ($x^o = 0$) and a simplification using integration by parts, we find

$$\begin{aligned}\frac{4r}{\pi_\mu^{(0)}}\pi_\mu^{(2)}\Big|_{l\geq 2,\text{odd}} &= \int \mathrm{d}r\frac{1}{4r}\pi_\mu^{(0)}(PQ' + \Pi_A^o A^{o\prime}) + \frac{1}{2}\left(r^2P^2 + \frac{1}{r^4}(l+2)(l-1)Q^2 + \frac{1}{r^2}(Q')^2\right) \\ &\quad + \frac{1}{2}\left(g^2(\Pi_A^o)^2 + \frac{1}{g^2}\left(\frac{l(l+1)}{r^2} + \frac{g^2\xi^2}{r^4} + (A^{o\prime})^2\right)\right) - \frac{\sqrt{(l+2)(l-1)}}{r^4}\xi Q A^o\end{aligned} \tag{5.2.115}$$

In the above expression we neglected the boundary term from the integration by parts which is given by

$$\int \mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}r}\left(2r^2P\int P\,\mathrm{d}r + \frac{1}{2}(2r+r_s)\left(\int P\,\mathrm{d}r\right)^2\right). \tag{5.2.116}$$

With the second canonical transformation, we would like to transform $r^2P^2 + r^{-2}(Q')^2$ into new variables (Q^o, P^o) such that the factors of r disappear and we are left with $P^2 + (Q')^2$. Furthermore, this transformation introduces the Regge-Wheeler potential into the Hamiltonian which we found in the analysis of the linearised equations of motion. The transformation rescales Q and P by r and adds a shift to P:

$$Q = rQ^o \tag{5.2.117}$$

$$P = \frac{1}{r}\left(P^o - \frac{\pi_\mu}{4r^2}Q^o\right) \tag{5.2.118}$$

In order to check that the transformation is canonical, we have to calculate three Poisson brackets. The one of Q with itself is trivially vanishing and also the one of P with itself vanishes due to the anti-symmetry of the Poisson bracket. It remains to calculate the one between Q and P and one readily sees that it gives the correct result because Q^o and P^o have the correct prefactors of r.

Applying this transformation in the solution for $\pi_\mu^{(2)}$, we obtain after another integration by parts

$$\begin{aligned}\frac{4r}{\pi_\mu^{(0)}}\pi_\mu^{(2)}\Big|_{l\geq 2,\text{odd}} &= \int \mathrm{d}r\,\Bigg[\frac{1}{4r}\pi_\mu^{(0)}(P^o(Q^o)' + \Pi_A^o(A^o)') - \frac{\sqrt{(l+2)(l-1)}}{r^3}\xi Q^o A^o \\ &\quad + \frac{1}{2}\left((P^o)^2 + (Q^{o\prime})^2 + \frac{1}{r^4}(l(l+1)r^2 - 3rr_s + g^2\xi^2)(Q^o)^2\right) \\ &\quad + \frac{1}{2}\left(g^2(\Pi_A^o)^2 + \frac{1}{g^2}\left((A^{o\prime})^2 + \frac{1}{r^4}\left(l(l+1)r^2 + g^2\xi^2\right)(A^o)^2\right)\right)\Bigg].\end{aligned} \tag{5.2.119}$$

The boundary term arising from the integration by parts was neglected in the above expression. The boundary term is given by

$$-\int \mathrm{d}r\, \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2r^2}\left(r_s - r - \frac{g^2\xi^2}{4r}\right)(Q^o)^2\right). \tag{5.2.120}$$

For a unified treatment of the odd and even parity perturbations later, we define three potentials: the gravitational potential V^o_{grav}, the electromagnetic potential V^o_{em} and the coupling potential V^o_{Coup}:

$$V^o_{\text{grav}} := \frac{1}{r^2}\left(U^o - \frac{3r_s}{2r}W^o\right) \tag{5.2.121}$$

$$V^o_{\text{em}}; = \frac{1}{r^2}\left(U^o + \frac{3r_s}{2r}W^o\right) \tag{5.2.122}$$

$$V^o_{\text{Coup}} := \frac{g\xi}{r^3}W^o\,. \tag{5.2.123}$$

In the definition of the potentials, we further defined two functions U^o and W^o that depend on the Schwarzschild radius r_s and the charge ξ of the background black hole solution:

$$W^o = 1 \tag{5.2.124}$$

$$U^o = l(l+1) - \frac{3r_s}{2r} + \frac{g^2\xi^2}{r^2}\,. \tag{5.2.125}$$

The potentials simplify the solution for $\pi^{(2)}_\mu$ and we have the result of this section including the boundary term that we neglected before

$$\frac{4r}{\pi^{(0)}_\mu}\pi^{(2)}_\mu\Big|_{l\geq 2,\text{odd}} = B_o + \int \mathrm{d}r\, \frac{1}{4r}\pi^{(0)}_\mu\left(P^o(Q^o)' + \Pi^o_A(A^o)'\right) \tag{5.2.126}$$
$$+\frac{1}{2}\left((P^o)^2 + g^2(\Pi^o_A)^2 + (Q^{o\prime})^2 + \frac{1}{g^2}(A^{o\prime})^2 + V^o_{\text{grav}}(Q^o)^2 + \frac{1}{g^2}V^o_{\text{em}}(A^o)^2 + \frac{2}{g}\sqrt{(l+2)(l-1)}V^o_{\text{Coup}}Q^oA^o\right),$$

The boundary term in terms of the original variables X^o and Y_o reads

$$B_o = \sum_{lm}\left[\frac{1}{r^3}\left(1 - \frac{r_s}{r} + \frac{g^2\xi^2}{4r^2}\right)\left(\int \mathrm{d}r\left(r^2 Y^{lm}_o + \frac{\pi^{(0)}_\lambda}{4r^2}X^o_{lm}\right)\right)^2 + \frac{6r - r_s}{4r^4}(X^o_{lm})^2 - \frac{1}{r^2}X^o_{lm}(X^o_{lm})'\right] \tag{5.2.127}$$

At infinity the boundary term vanishes due to the decay behaviour of (X^o, Y_o). The function X^o grows at most linearly and Y_o vanishes quadratically implying that the integral grows linearly with r at infinity. Together, we observe that the boundary term vanishes asymptotically as $O(r^{-1})$.

5.2.3.4 Solution of the First Order Constraints - Even Parity

The remaining first order constraints are in the even parity sector for $l \geq 2$. In the following, we proceed in analogy with the dipole and odd parity first order constraints. There are three first order constraints, the radial and angular diffeomorphism constraints Z^e, Z^h and the Hamiltonian constraint Z^h. In addition, there are three corresponding gauge degrees of freedom (x^v, y_v), (x^h, y_h) and (x^e, y_e) in the gravitational sector. The observable degrees of freedom are (X^e, Y_e) for gravity and (X^e_M, Y^M_e) for the electromagnetic field.

The constraint equation $Z^v = 0$ gives a relation between the momenta y_v and y_h. We solve the constraint for y_h in terms of y_v and obtain

$$y_h = \frac{1}{r^2}\left(1 - \frac{\pi_\lambda^{(0)}}{\pi_\mu^{(0)}}\right) y_v + \frac{\sqrt{2(l-1)l(l+1)(l+2)}}{r^2 \pi_\mu^{(0)}} X^e + \frac{2g^2\xi}{r^2\pi_\mu^{(0)}}\sqrt{l(l+1)} X_M^e \tag{5.2.128}$$

where we imposed the gauge $x^v = x^h = x^e = 0$. From the radial diffeomorphism constraint we obtain a solution for y_e in terms of y_h and y_v:

$$y_e = \frac{1}{\sqrt{l(l+1)}}(-2ry_h + 2\partial_r y_v) \tag{5.2.129}$$

Finally, the angular diffeomorphism constraint in the gauge $x^v = x^h = x^e = 0$ reads

$$\partial_r(r^2 y_e) - \sqrt{l(l+1)} r^2 y_h - \xi \partial_r X_M^e + \sqrt{2(l+2)(l-1)}\left(r^2 Y_e + \frac{\pi_\lambda^{(0)}}{4r^2} X^e\right) = 0\,, \tag{5.2.130}$$

In this equation, we eliminate y_e and y_h using the relations above and obtain a differential equation for y_v. The solution for the spherically symmetric momenta $\pi_\mu^{(0)}$ and $\pi_\lambda^{(0)}$ is inserted and we obtain

$$2r^2\partial_r^2 y_v + \frac{2r(6rr_s - g^2\xi^2)}{4rr_s - g^2\xi^2}\partial_r y_v + \frac{8(l+2)(l-1)r^2 r_s^2 - 2(-4+3l(l+1))rr_s g^2\xi^2 + (l+2)(l-1)g^4\xi^4}{(4rr_s - g^2\xi^2)^2} y_v = s(r)\,, \tag{5.2.131}$$

with a “source” term $s(r)$ depending on the variables (X_M^e, Y_e^M) and (X^e, Y_e).

$$\begin{aligned} s(r) = \sqrt{2(l-1)l(l+1)(l+2)}\Big(&- \frac{2r}{\pi_\mu^{(0)}} X^{e\prime} + r^2 Y_e \\ &+ \frac{4\left(-\left(l^2+l-2\right)g^2\xi^2 r - 2g^2\xi^2 r_s + 4\left(l^2+l-1\right)r^2 r_s + 8rr_s^2\right)}{r(\pi_\mu^{(0)})^3} X^e\Big) \\ + \sqrt{l(l+1)}\left(\xi\Pi_A - \frac{4rg^2\xi}{\pi_\mu^{(0)}}A' - \frac{8g^2\xi\left((l+2)(l-1)g^2\xi^2 - 4\left(l^2+l-1\right)rr_s\right)}{(\pi_\mu^{(0)})^3} A\right) \end{aligned} \tag{5.2.132}$$

It is an inhomogeneous, linear, and second order differential equation with the inhomogeneity $s(r)$. The theory for linear differential equations implies that the solution is the sum of a general solution of the homogeneous equation and a particular solution of the inhomogeneous equation.

So far, a solution has not yet been constructed explicitly in the general case. In [2], we analysed the pure gravity case and obtained an explicit solution. Setting X_M^e, Y_e^M and ξ to zero, we have the equation

$$2r^2\partial_r^2 y_v + 3r\partial_r y_v + \frac{(l+2)(l-1)}{2} y_v = s(r)\,, \tag{5.2.133}$$

where the “source” term $s(r)$ becomes

$$s(r) := \sqrt{2(l+2)(l+1)l(l-1)}\left(r^2 Y^e + \frac{\pi_\lambda}{4r^2}X^e + \frac{l(l+1)-1}{\pi_\mu^{(0)}}X^e - \frac{2r}{\pi_\mu^{(0)}}\partial_r X^e\right). \tag{5.2.134}$$

The homogeneous solution is obtained by setting the source term $s(r)$ to zero. The general solution of the equation is

$$y_v = C_+ r^{\alpha_+} + C_- r^{\alpha_-}\,, \tag{5.2.135}$$

$C_\pm$ are integration constants and $\alpha_\pm$ are defined as

$$\alpha_\pm = -\frac{1}{4} \pm \frac{i}{4}\sqrt{4(l+2)(l-1)-1}\,. \tag{5.2.136}$$

The particular solution of the full equation including the source term is computed using the method of the variation of constants. We find

$$y_v = r^{\alpha_-}\int \mathrm{d}\tilde{r}\left(\frac{2i\tilde{r}^{\alpha_+-\frac{1}{2}}}{\sqrt{4(l+2)(l-1)-1}}s(\tilde{r})\right) - r^{\alpha_+}\int \mathrm{d}\tilde{r}\left(\frac{2i\tilde{r}^{\alpha_--\frac{1}{2}}}{\sqrt{4(l+2)(l-1)-1}}s(\tilde{r})\right) \tag{5.2.137}$$

This expression is quite complicated and inserting it into the expression for $\pi_\mu^{(2)}$ would lead to a very long result. This result would then have to be simplified using canonical transformations to obtain a tractable theory. However, it is not easy to “guess” the required transformations even for the pure gravity setup.

In the following, we take a detour to derive the desired result based on the works by Moncrief [75]. In this formulation we work in a different gauge given by $X^e = x^e = p_2 = 0$, where p_2 is a function of x^v, y_v, x^h, y_h. The advantage of working in this gauge is that the solution of the constraints is possible without solving any differential equations. Additionally, we obtain the canonical transformations by comparing our expressions with the work by Moncrief. Then, we transform the result to the original gauge where (X^e, Y_e) are the true degrees of freedom.

We start with the solution of the diffeomorphism constraints. We work in the gauge $X^e = x^e = 0$ and solve them for y^e and Y^e:

$$y_e^{(1)} = -\frac{1}{\sqrt{l(l+1)}}\Big(-2\partial_r(y_v) + 2ry_h - \partial_r\pi_\mu^{(0)}x^v - \frac{1}{2}\pi_\mu^{(0)}\partial_r x^v + \frac{\pi_\lambda^{(0)}}{2r^2}\partial_r x^h\Big) \tag{5.2.138}$$

$$Y_e^{(1)} = -\frac{1}{r^2\sqrt{2(l+2)(l-1)}}\left(-\partial_r(r^2 y_e^{(1)}) + \sqrt{l(l+1)}\Big(\frac{1}{2}\pi_\mu^{(0)}x^v - r^2 y_h - \xi\partial_r X_M^e\Big)\right), \tag{5.2.139}$$

In the second equation, we have to substitute $y_e^{(1)}$ with the solution of the first equation.

The expressions in this section become rather lengthy and for convenience of notation, we define the following combinations of the background constants r_s and ξ:

$$n := \frac{1}{2}(l+2)(l-1)\,, \tag{5.2.140}$$

$$\Delta := 1 - \frac{r_s}{r} + \frac{g^2\xi^2}{4r^2}\,, \tag{5.2.141}$$

$$\Lambda := n + \frac{3r_s}{2r} - \frac{g^2\xi^2}{2r^2}\,. \tag{5.2.142}$$

Following the analysis of Moncrief in [75], we have to solve the first-order Hamiltonian constraint ${}^{(1)}Z^v_{lm}$ for x^h. We insert the solution of the diffeomorphism constraint above as well as the gauge condition $x^e = X^e = 0$

into this constraint. Then, we find

$$\begin{aligned} {}^{(1)}Z^v_{lm} &= \frac{1}{2r^2}(\pi_\mu^{(0)} - \pi_\lambda^{(0)})y_v - \left(2r\partial_r + l(l+1) + 2 - 2\frac{r_s}{r}\right)x^v - \frac{1}{2}\pi_\mu^{(0)} y_h \\ &\quad + 2\left(\partial_r^2 - \frac{1}{r}\partial_r - \frac{(l+2)(l-1)}{2r^2} - \frac{r_s}{r^3}\right)x^h + \frac{g^2}{r^2}\xi\sqrt{l(l+1)}\int Y^M_{lm}\,\mathrm{d}r \end{aligned} \tag{5.2.143}$$

We observe that Z^v depends on first and second derivatives of x^h. Thus, in order to obtain an explicit expression, we have to solve a second order linear differential equation. This is not straight forward as we saw in the beginning of this subsection. Simplifying the constraint using a canonical transformation, we plan to arrive at a more tractable expression that does not involve derivatives of x^h. We introduce new variables (q_1, p_1) and (q_2, p_2) with a canonical transformation where $q_2 = x^h$. We take the ansatz

$$x^v = q_1 + Bq_2 + C\partial_r q_2 + Dp_1 \tag{5.2.144}$$

$$x^h = q_2 \tag{5.2.145}$$

$$y_v = p_1 + G\partial_r q_2 \tag{5.2.146}$$

$$y_h = p_2 - Bp_1 + \partial_r[(C - DG)p_1] - \partial_r(Gq_1) + Kq_2 - BG\partial_r q_2\,, \tag{5.2.147}$$

The functions C, D, G, B and K will be determined below by imposing conditions on the first order Hamiltonian constraint after the transformation. The transformation is canonical no matter the choice for the functions. Let us calculate some non-trivial Poisson brackets by smearing the variables, i.e. $x^v[f] = \int f(r)x^v(r)\mathrm{d}r$ and similarly for the others. The Poisson bracket of x^v with the other variables are

$$\begin{aligned} \{x^v[f], x^v[g]\} &= \left\{x^v[f], x^h[g]\right\} = 0 \\ \{x^v[f], y_v[g]\} &= \int \mathrm{d}r\, fg \\ \{x^v[f], y_h[g]\} &= \int \mathrm{d}r(-Bfg - (C - DG)f\partial_r g + Bfg - \partial_r(Cf)g - DGf\partial_r g) = 0 \end{aligned} \tag{5.2.148}$$

The Poisson brackets of x^h are

$$\begin{aligned} \left\{x^h[f], x^h[g]\right\} &= \left\{x^h[f], y_v[g]\right\} = 0 \\ \left\{x^h[f], y_h[g]\right\} &= \int \mathrm{d}r\, fg \end{aligned} \tag{5.2.149}$$

The remaining Poisson brackets are

$$\begin{aligned} \{y_v[f], y_v[g]\} &= 0 \\ \{y_v[f], y_h[g]\} &= \int \mathrm{d}r(-\partial_r(Gf)g - Gf\partial_r g) = 0 \\ \{y_h[f], y_h[g]\} &= \int \mathrm{d}r(\partial_r(BGf)g - \partial_r(BGg)f + BG\partial_r fg - BG\partial_r gf) = 0 \end{aligned} \tag{5.2.150}$$

Combining the results, we see that assuming that (q_1, p_1) and (q_2, p_2) are conjugate pairs, we recover the standard Poisson bracket relations for the (x^v, y_v) and (x^h, y_h). Thus, the transformation is canonical.

In order to find the functions C, D, G, B, K, we require the transformed first order Hamiltonian constraint to be independent of the first and second derivative of q_2. Plugging the transformation into the first order

Hamiltonian constraint, the derivative terms of q_2 read

$$2(1-rC)\partial_r^2 q_2 + \Big(\frac{1}{2r^2}(\pi_\mu^{(0)} - \pi_\lambda^{(0)})G - \frac{1}{2}\pi_\mu^{(0)}BG - \frac{2}{r} - 2r\partial_r C - (l(l+1)+2-2\frac{r_s}{r})C - 2rB\Big)\partial_r q_2 \tag{5.2.151}$$

The first bracket implies that $C = r^{-1}$. Using the expression for C and the solution for $\pi_\lambda^{(0)}$ in terms of $\pi_\mu^{(0)}$, we find

$$B = \frac{1}{4r^2 - r\pi_\mu^{(0)}G}\left[\frac{1}{2r}\pi_\mu^{(0)}G + 4\frac{r_s}{r} - \frac{g^2\xi^2}{2r^2} - 2(l(l+1)+2)\right] \tag{5.2.152}$$

As an additional requirement, we would like the first order Hamiltonian constraint to be independent of the first derivative of p_1. The relevant contribution to Z^v is

$$-\Big(2rD + \frac{1}{2}\pi_\mu^{(0)}(C - DG)\Big)\partial_r p_1 \tag{5.2.153}$$

The bracket vanishes provided that

$$D = \frac{\pi_\mu^{(0)}}{r\pi_\mu^{(0)}G - 4r^2} \tag{5.2.154}$$

The function G is determined by looking at the solution for $y_e^{(1)}$ which depends on up to second derivatives of q_2. The contribution to $y_e^{(1)}$ proportional to $\partial_r^2 q_2$ is

$$\frac{1}{\sqrt{l(l+1)}}\Big(2G + \frac{1}{2}\pi_\mu^{(0)}C\Big)\partial_r^2 q_2 \tag{5.2.155}$$

Thus, the solution $y_e^{(1)}$ of the diffeomorphism constraint simplifies provided that

$$G = -\frac{\pi_\mu^{(0)}}{4r} \tag{5.2.156}$$

In summary, we so far determined the following expressions

$$C = \frac{1}{r}, \quad D = \frac{\pi_\mu^{(0)}}{4r^2(\Delta-2)}, \quad G = -\frac{\pi_\mu^{(0)}}{4r}, \quad B = -\frac{1}{2r^2(\Delta-2)}\Big(\frac{r_s}{r} - (l(l+1)+2)\Big). \tag{5.2.157}$$

To find the remaining function K we used the symbolic calculation features of Mathematica. Consider the integral for $\pi_\mu^{(2)}$ in equation (**5.2.87**) and apply the canonical transformation to the variables q and p. It depends on higher derivative terms of the form $q_2 q_2''$, $q_2' q_2''$. The function K can be tuned such that these contributions vanish. We have to take

$$K = \frac{2}{(\Delta-2)^2 r^2 \pi_\mu^{(0)}}\Bigg(\frac{2}{r}\partial_r\Big(r^2\Delta(\Lambda+2\Delta)\Big) - \Lambda l(l+1)\Big(\Delta^2 - 3\Delta + 2\Big) - 2l(l+1)\Big(2\Delta^2 - 5\Delta + 4\Big) - 4\Delta - \frac{r_s}{r}\Big(\Delta^2 - 4\Delta + 2\Big)\Bigg). \tag{5.2.158}$$

We complete the first canonical transformation with the investigation of the electromagnetic sector. The second order Hamiltonian constraint $C_v^{(2)}$ depends on the integral of the momentum Y_e^M. Therefore, we perform a canonical transformation and introduce new canonical variables (A, Π_A) defined by

$$A := -\int Y_e^M \, \mathrm{d}r \,, \quad \Pi_A := -\partial_r X_M^e \,. \tag{5.2.159}$$

The canonical transformations for the gravitational and electromagnetic degrees of freedom are inserted into the first order Hamiltonian constraint. The expression is simplified using Mathematica and we find the following solution for the variable q_2

$$q_2^{(1)} = \frac{1}{r^2 l(l+1)\Lambda}\Big[2r^4(\Lambda + 2\Delta)q_1 - 2r^5((\Delta-2)q_1)' + \sqrt{l(l+1)}g^2\xi A\Big] \,. \tag{5.2.160}$$

In the computation we chose the gauge $p_2 = 0$.

All the first order constraints are successfully solved for the variables y_e, Y_e and $q_2 = x^h$ in the gauge $x^e = X^e = p_2 = 0$. In terms of the variables (q_1, p_1) as well as the solution $q_2^{(1)}$ above, we find

$$\begin{aligned} y_e^{(1)} = & \frac{1}{r^3\left(4r^2+4rr_s-g^2\xi^2\right)\sqrt{l(l+1)\left(4rr_s-g^2\xi^2\right)}}\Big[r^2\left(6rr_s-g^2\xi^2\right)\left(4r(r+r_s)-g^2\xi^2\right)q_1 \\ & + l(l+1)\left(4rr_s-g^2\xi^2\right)\left(r\left(\left(l^2+l+6\right)r+3r_s\right)-g^2\xi^2\right)q_2^{(1)} - 4l(l+1)r^4\sqrt{4rr_s-g^2\xi^2}p_1\Big] \end{aligned} \tag{5.2.161}$$

$$\begin{aligned} Y_e^{(1)} = & \frac{\sqrt{\frac{2l(l+1)}{l^2+l-2}}\left(-g^2\xi^2+\left(l^2+l-2\right)r^2+3rr_s\right)}{r^2\left(4r\left(r_s+r\right)-g^2\xi^2\right)}p_1 + \frac{4\left(l^2+l+3\right)rr_s - g^2\left(l^2+l+2\right)\xi^2}{2\sqrt{2}r\sqrt{(l-1)l(l+1)(l+2)\left(4rr_s-g^2\xi^2\right)}}\partial_r q_1 \\ & + \left(-g^2\xi^2+\left(l^2+l-2\right)r^2+3rr_s\right)\Big[\frac{\sqrt{l(l+1)}}{r^4\sqrt{2(l+2)(l-1)\left(4rr_s-g^2\xi^2\right)}} \\ & - \frac{\sqrt{l(l+1)}\left(-g^2\left(l^2+l+4\right)\xi^2+4\left(l^2+l+5\right)rr_s+8r^2\right)}{2r^4\left(-g^2\xi^2+4r^2+4rr_s\right)\sqrt{2(l+2)(l-1)\left(4rr_s-g^2\xi^2\right)}}\Big]q_2 \\ & + \frac{g^4\left(-3l^2-3l+2\right)\xi^4+2g^2\left(11l^2+11l-14\right)\xi^2 rr_s - 8\left(5l^2+5l-9\right)r^2r_s^2}{2\sqrt{2}\sqrt{l\left(l^3+2l^2-l-2\right)}r^2\left(4rr_s-g^2\xi^2\right)^{3/2}}q_1 \\ & - \frac{\xi}{\sqrt{2(l+2)(l-1)r^2}}\partial_r X_M^e \end{aligned} \tag{5.2.162}$$

The canonical transformation and the solutions $y_e^{(1)}$, $Y_e^{(1)}$ and $q_2^{(1)}$ need to be inserted into the solution of the second order constraints for $\pi_\mu^{(2)}$. Equation (**5.2.87**) will then only involve the true degrees of freedom (q_1, p_1) and (A, Π_A). We will not present the result of the computations with Mathematica due to its length but describe which transformations simplify it further. In analogy to the odd parity sector, the canonical transformations will provide a tractable solution depending on the Hamiltonian version of the Zerilli potential.

The canonical transformation has two parts: For the first one, we would like to remove some of the coupling terms between the gravitational and electromagnetic degrees of freedom. This is achieved by a shift of p_1 by A and Π_A with appropriate transformations of A and Π_A to make the transformation canonical. Additionally, we rescale the variables q_1 and p_1 so that the new variables (Q, P) appear as $1/2(P^2 + (Q')^2)$. We choose the transformation

$$p_1 = \sqrt{\frac{(l+2)(l-1)}{l(l+1)}}\frac{r(\Delta-2)}{\Lambda}\left(P + \frac{\xi}{r\sqrt{(l+2)(l-1)}}\Pi_A + \Gamma A\right) \tag{5.2.163}$$

$$q_1 = \sqrt{\frac{l(l+1)}{(l+2)(l-1)}}\frac{\Lambda}{r(\Delta-2)}Q \tag{5.2.164}$$

$$A = \tilde{A} - \frac{\xi}{r\sqrt{(l+2)(l-1)}}Q \tag{5.2.165}$$

$$\Pi_A = \tilde{\Pi}_A + \Gamma Q \tag{5.2.166}$$

where the function Γ is defined as

$$\begin{aligned}\Gamma = \frac{g^2\xi}{8\sqrt{(l+2)(l-1)}r\Lambda(\Delta-2)\pi_\mu^{(0)}}\Big(r^{-8}\frac{\partial}{\partial r}\Big(16r^9\Lambda(1-\Delta)\Big) - 8l(l+1)\Big(2\Delta^2 - 11\Delta + 9\Big) \\ + 16\Lambda l(l+1)(1-\Delta) + 16\Big(-4\Delta^2 + \Delta + 3\Big)\Big).\end{aligned} \tag{5.2.167}$$

The transformation above removes all the couplings between the gravitational and the electromagnetic field, except for a term proportional to QA.

The second canonical transformation is removing the couplings between the position and momentum variables within the gravitational and electromagnetic sectors. They are removed by shifting the momenta $\tilde{P}$ and $\tilde{\Pi}_A$. The new variables Q^e, P^e and A^e, Π^e_A are defined by

$$Q = Q^e, \quad P = P^e + A_{\text{grav}}Q^e \tag{5.2.168}$$

$$\tilde{A} = \frac{1}{g^2}A^e, \quad \tilde{\Pi}_A = g^2\Pi^e_A - \frac{g^2\xi^2\pi_\mu^{(0)}}{8r^4\Lambda}A^e \tag{5.2.169}$$

The function A_{grav} which shifts the gravitational momentum is

$$\begin{aligned}A_{\text{grav}} =& \frac{1}{2(l-1)(l+2)r^4\Lambda(\pi_\mu^{(0)})^3(\Delta-2)^2}\Big[64(l+2)^2(l-1)^2(3+l(l+1)(12+l(l+1)))r^8r_s^2 + g^8\xi^8(-218+85l(l+1))rr_s \\ &- 5g^{10}\xi^{10}(l+2)(l-1) + 32(l-1)(l+2)r^7r_s(4(1+l(l+1))(18+5l(l+1))r_s^2 - g^2\xi^2(l-1)(l+2)(3+l(l+1)(12+l(l+1)))) \\ &+ 2g^4\xi^4r^3r_s(4(-934+235l(l+1))r_s^2 - g^2\xi^2(252+5l(l+1)(24+17l(l+1)))) + 2g^6\xi^6r^2((914-285l(l+1))r_s^2 \\ &+ 2g^2\xi^2(8+3l(l+1)(2+l(l+1)))) + 8r^5r_s(48(-31+5l(l+1))r_s^4 - 8g^2\xi^2(113+l(l+1)(7+32l(l+1)))r_s^2 \\ &+ g^4\xi^4(l+2)(l-1)(82+17l(l+1)(6+l(l+1)))) + 4r^6(48(35+l(l+1)(-1+3l)(4+3l))r_s^4 \\ &- 8g^2\xi^2(l-1)(l+2)(1+l(l+1))(69+16l(l+1))r_s^2 + g^4\xi^4(l+2)^2(l-1)^2(4+l(l+1)(12+l(l+1)))) \\ &+ 4g^2\xi^2r^4(-8(-469+95l(l+1))r_s^4 + g^2\xi^2(724+l(l+1)(184+223l(l+1)))r_s^2 \\ &- g^4\xi^4(l+2)(l-1)(16+l(l+1)(20+3l(l+1))))\Big].\end{aligned} \tag{5.2.170}$$

The even parity contributions to the solution $\pi_\mu^{(2)}$ of the second order symmetric constraints are then computed explicitly using Mathematica. We perform the first canonical transformation and introduce the new variables (q_1, p_1), (q_2, p_2). Then, we substitute the solutions of the first order constraints $y_e^{(1)}, Y_e^{(1)}$ and $q_2^{(1)}$ that we derived above. We complete the calculation by inserting the last two canonical transformations and simplify the result using integration by parts.

For the presentation of the result, we introduce three potentials: one for the gravitational field V^e_{grav}, one for the electromagnetic field V^e_{em} and one for the coupling term V^e_{coup}. The potentials are given by

$$V^e_{\text{coup}} := \frac{g\xi}{r^3}W^e \tag{5.2.171}$$

$$V^e_{\text{grav}} := \frac{1}{r^2}\Big(U^e - \frac{3r_s}{2r}W^e\Big) \tag{5.2.172}$$

$$V^e_{\rm em} := \frac{1}{r^2}\Big(U^e + \frac{3r_s}{2r}W^e\Big)\,. \tag{5.2.173}$$

These are the same definitions as in the odd parity sector. The difference are the functions W^e and U^e which for the even parity are defined as

$$W^e := \frac{\Delta}{\Lambda^2}\Big(2n + \frac{3r_s}{2r}\Big) + \frac{1}{\Lambda}\Big(n + \frac{r_s}{2r}\Big) \tag{5.2.174}$$

$$U^e := \Big(2n + \frac{3r_s}{2r}\Big)W^e + \Big(\Lambda - n - \frac{r_s}{2r}\Big) - \frac{2n\Delta}{\Lambda} \tag{5.2.175}$$

The solution for $\pi^{(2)}_\mu$ in terms of the potentials is given by

$$\frac{r}{\pi\pi^{(0)}_\mu}\pi^{(2)}_\mu\Big|_{l\geq 2,\text{even}} = \int \mathrm{d}r\,\Big[\frac{1}{4r}\pi_\mu(P^e(Q^e)' + \Pi A') + \frac{1}{2}\Big((P^e)^2 + (Q^{e\prime})^2 + V^e_{\rm grav}(Q^e)^2\Big) \tag{5.2.176}$$

$$+\frac{1}{2}\Big(g^2\Pi^2 + \frac{1}{g^2}(A')^2 + \frac{1}{g^2}V^e_{\rm em}A^2\Big) + \frac{\sqrt{(l+2)(l-1)}}{g}V^e_{\rm coup}AQ^e\Big] \tag{5.2.177}$$

The above expression was simplified using integration by parts and all the boundary terms were neglected. In the appendix A, we present the explicit form of the boundary term. In the following, we show that the boundary term vanishes in the limit as r goes to infinity.

The fall-off conditions on the gravitational and electromagnetic degrees of freedom imply fall-off conditions on the variables introduced in this section using canonical transformations:

$$q_1 \sim q_1^0 r^{-1} \qquad p_1 \sim p_1^0 \qquad q_2 \sim q_2^0 r \qquad A \sim A_0 r \qquad A^e \sim A^e_0 r \qquad Q^e \sim Q^e_0\,. \tag{5.2.178}$$

Here, the sub-/superscript 0 signifies that the quantities are radial constants but could still depend on l, m. In this notation, the first non-trivial order of the boundary term of $\pi^{(2)}_\mu$ is

$$\begin{aligned}
&\frac{1}{\pi\pi^{(0)}_\mu}\Big(-\frac{1}{2}(p_1^0)^2 + \frac{3}{2}(q_1^0)^2 + \frac{(l^2+l+2)}{(l+2)(l+1)l(l-1)}(q_1^0)^2 + 2q_1^0q_2^0 - \frac{3(l^2+l+2)}{2}q_1^0q_2^0 + \frac{1}{2}(q_2^0)^2\\
&-(l^2+l+2)(q_2^0)^2 + \frac{1}{8}(3l^4+6l^3+13l^2+10l+16)(q_2^0)^2 + \frac{g^2(l^2+l+2)\xi}{\sqrt{l(l+1)}(l+2)(l-1)}A_0q_1^0\\
&-\frac{1}{2}g^2\xi\sqrt{l(l+1)}q_2^0A_0 - \frac{1}{2}(Q_0^e)^2 + \frac{g^4\xi^2}{2(l+2)(l-1)}(A_0^e)^2 + 2\frac{g^2\xi}{2\sqrt{(l+2)(l-1)}}A_0^eQ_0^e\Big) + O(r^{-1})
\end{aligned} \tag{5.2.179}$$

Thus, the leading order contributions behaves as $r^{-1/2}$. In the computation for the reduced Hamiltonian, we will see that the contribution due to the boundary term vanishes in the limit $r \to \infty$.

Working in the gauge $x^e = X^e = p_2 = 0$, we reduced the solution for $\pi^{(2)}_\mu$ into a form analogous to the result for the odd parity case. In the rest of this manuscript we would like to work instead in the GP gauge which requires $x^v = x^h = x^e = 0$ so that the physical degrees of freedom are (X^e, Y_e) for the gravitational and (X^e_M, Y^M_e) for the electromagnetic degrees of freedom. In the remaining part of this subsection, we show that we can relate our choice of gauge to the GP gauge. The solution for $\pi^{(2)}_\mu$ stays the same where (Q^e, P^e) and (A^e, Π^e_A) are now functions of (X^e, Y_e) and (X^e_M, Y^M_e).

Let us analyse how we can change the gauge in a simplified example where we have one constraint $C(x, y, X, Y)$ of two conjugate pairs of canonical variables (x, y) and (X, Y). We consider two options to choose the physical degrees of freedom: On the one hand we can choose (x, y) as the physical variables, solve the constraint for Y and gauge fix X. On the other hand, we can select (X, Y) as the physical degrees of freedom, solve the constraint for y and gauge fix x. In the first case, we obtain the equivalent constraint $Z = Y + H(X, x, y)$ and we use the gauge fixing condition $G = X = 0$. In the second case we have $z = y + h(x, X, Y)$ with the gauge fixing $g = x = 0$.

In both cases, the two solutions z and Z define a projector on gauge invariant functions on phase space, i.e. functions that commute with the constraints z and Z respectively. The projectors O and o are defined for any phase space function f. The map in equation (**2.2.4**) evaluated at $\tau = 0$ is equivalently written as

$$O_f := [e^{V_{Z(S)}} \cdot f]_{S=X} \tag{5.2.180}$$

$$o_f := [e^{V_{z(s)}} \cdot f]_{s=x}\,, \tag{5.2.181}$$

where V_A is the Hamiltonian vector field associated to the phase space function A. The Hamiltonian vector field V_A is defined by its action on functions $V_A[f] = \{A, f\}$, where $\{\cdot, \cdot\}$ is the Poisson bracket. The function S are smearing functions that are set to X and x respectively after the evaluation of all the Poisson brackets. Using the properties of the map O that we proved in section 2, we can evaluate O and o for an arbitrary function $F(x, y, X, Y)$ on phase space. We find $[O_F]_{X=0} = F(x, y, 0, -H(0, x, y))$ and $[o_F]_{x=0} = F(0, -h(0, X, Y), X, Y)$ where we used the gauge fixings g, G. Let us introduce the notation $E_{X=0} = [O_F]_{X=0}$ and $e_{x=0} = [o_F]_{x=0}$.

In the process of gauge fixing, we lose all the information about non-gauge invariant contributions and the relation between $E_{X=0}$ and $e_{x=0}$ is not clear. For this reason, we will not implement the gauge fixing and find expressions of the form $E = E(O_x, O_y, X) = F(O_x, O_y, X, -H(X, x, y))$ and $e = e(o_X, o_Y, x) = F(x, -h(x, X, Y), X, Y)$. In this way, we can clearly distinguish the gauge invariant dependence through the maps O, o and the gauge variant contributions due to x, X. After fixing the gauge, we would set these additional gauge variant dependence to zero and there would be no way to recover that dependence from the gauge fixed version. If the original function F were weakly gauge invariant, the functions e, E would be independent of x, X, i.e. $E(O_x, O_y, X) = E(O_x, O_y)$ and similarly for e. In this case, the gauge fixed versions of e, E would contain all the information. Let us therefore assume that F is weakly gauge invariant. Then, we obtain

$$\begin{aligned} E(x, y) &= E(O_x, O_y)_{X=0} = [O_F]_{X=0} = [O_{O_F}]_{X=0} \\ &= [O_{o_F}]_{X=0} = [O_e]_{X=0} = e([O_{o_X}]_{X=0}, [O_{o_Y}]_{X=0}) \\ &= e([o_X]_{X=0, Y=-H}, [o_Y]_{X=0, Y=-H})\,. \end{aligned} \tag{5.2.182}$$

For the third step, we use that O is a projector and in the fourth step we use that $O_F = o_F$ due to the weak gauge invariance of F. Then, we use the definition of e and then pull the projector O inside the argument of e. Finally, we apply the projector O to o_X and o_Y. Similarly we can work in the other direction showing that for weakly gauge invariant F we can work with both gauges and can relate the results afterwards.

This means that for the translation of $\pi_\mu^{(2)}$ between our gauge and the GP gauge we have to first show weak gauge invariance and then perform the explicit computation to relate the expression in the two gauges. Since the gauge $x^e = 0$ is common to both gauge fixings, we only need to consider the change from $X^e = p_2 = 0$ to $x^h = x^v = 0$.

First, we have to define the observable map O to find the gauge invariant extensions of q_1, p_1, A, Π_A. We construct the solution of the first order constraints leaving X^e and p_2 unfixed using Mathematica and have

$$\begin{aligned}q_2^{(1)} &= \frac{1}{2l(l+1)\Lambda}\Big(\sqrt{2(l+2)(l+1)l(l-1)}X^e + 2\Big((l^2+l+2)r^2 - 3rr_s + g^2\xi^2\Big)q_1 \\ &+ r(4r^2+4rrs_{-}g^2\xi^2)q_1' + 2\sqrt{l(l+1)}g^2\xi A + r^2\pi_\mu^{(0)}p_2\Big)\end{aligned} \tag{5.2.183}$$

$$\begin{aligned}y_e^{(1)} &= \frac{\sqrt{l(l+1)}\pi_\mu^{(0)}\left(\left(l^2+l+6\right)r^2+3rr_s-g^2\xi^2\right)}{2r^3\left(4r^2+4rr_s-g^2\xi^2\right)}q_2^{(1)} + \frac{4\sqrt{l(l+1)}r}{4r^2+4rr_s-g^2\xi^2}p_1 \\ &+ \frac{2\left(g^2\xi^2-6rr_s\right)}{\sqrt{l(l+1)}r\pi_\mu^{(0)}}q_1 - \frac{2r}{\sqrt{l(l+1)}}p_2\end{aligned} \tag{5.2.184}$$

$$\begin{aligned}Y_e^{(1)} &= \frac{\left(2r_s\left(-g^2\xi^2+2\left(l^2+l+5\right)r^2+4rr_s\right)-g^2\left(l^2+l+2\right)\xi^2 r\right)}{r^3\pi_\mu^{(0)}\left(g^2\xi^2-4r^2-4rr_s\right)}X^e \\ &- \frac{8}{2\sqrt{2(l+2)(l+1)l(l-1)}r^2(\pi_\mu^{(0)})^3\left(g^2\xi^2-4r^2-4rr_s\right)}\Big[g^6\Big(l^2+l-6\Big)\xi^6 \\ &\qquad - 2g^4l(l+1)\Big(l^2+l+10\Big)\xi^4r^2 + 4rr_s(22-3l(l+1))g^4\xi^4 \\ &\qquad + 16\Big(l(l+1)\Big(l^2+l+10\Big)-2\Big)r^3r_sg^2\xi^2 \\ &\qquad + 16r^2r_s^2\Big(g^2(3l(l+1)-25)\xi^2 - 2\Big(l(l+1)\Big(l^2+l+10\Big)-3\Big)r^2 - 4\Big(l^2+l-9\Big)rr_s\Big)\Big]q_1 \\ &+ \frac{2\sqrt{2l(l+1)}\Lambda}{\sqrt{(l+2)(l-1)}\left(-g^2\xi^2+4r^2+4rr_s\right)}p_1 + \frac{8\left(l^2+l+3\right)rr_s-2\left(l^2+l+2\right)g^2\xi^2}{\sqrt{2(l+2)(l-1)}r^2\left(g^2\xi^2-4r^2-4rr_s\right)\pi_\mu^{(0)}}g^2\xi A \\ &- \frac{\xi}{\sqrt{2(l+2)(l-1)}r^2}\Pi_A + \frac{4\left(2g^2\xi^2+\left(l^2+l-6\right)r^2-9rr_s\right)}{\sqrt{2(l-1)l(l+1)(l+2)}\left(-g^2\xi^2+4r^2+4rr_s\right)}p_2\end{aligned} \tag{5.2.185}$$

Within our approximation we only have to construct the observable map O to first order in the perturbations. It is therefore sufficient to truncate the infinite series in the definition of O at linear order because the solutions $q_2^{(1)}, y_e^{(1)}, Y_e^{(1)}$ are of first order. For any function F, its gauge fixed version defined by the projector O is given by

$$O_F = F + \int \mathrm{d}\tilde{r}\,\Big[p_2(\tilde{r})\{q_2(\tilde{r}) - q_2^{(1)}(\tilde{r}), F\} + x^e(\tilde{r})\{y_e(\tilde{r}) - y_e^{(1)}(\tilde{r}), F(r)\} + X^e(\tilde{r})\{Y_e(\tilde{r}) - Y_e^{(1)}(\tilde{r}), F\}\Big]\,. \tag{5.2.186}$$

For the canonical variables (A, Π_A) and (q_1, p_1), we find the gauge invariant extensions

$$O_A = A - \frac{\xi}{\sqrt{2(l+2)(l-1)}r^2}X^e \tag{5.2.187}$$

$$O_{\Pi_A} = \Pi_A + \frac{8\left(l^2+l+3\right)rr_s-2\left(l^2+l+2\right)g^2\xi^2}{\sqrt{2(l+2)(l-1)}r^2\left(g^2\xi^2-4r^2-4rr_s\right)\pi_\mu^{(0)}}g^2\xi X^e + \frac{g^2\xi}{\sqrt{l(l+1)}\Lambda}p_2 \tag{5.2.188}$$

$$O_{q_1} = q_1 + \sqrt{\frac{2l(l+1)}{(l+2)(l-1)}}\frac{2\Lambda}{\left(-g^2\xi^2+4r^2+4rr_s\right)}X^e \tag{5.2.189}$$

$$\begin{aligned}O_{p_1} &= p_1 - \frac{8}{2\sqrt{2(l+2)(l+1)l(l-1)}r^2(\pi_\mu^{(0)})^3\left(g^2\xi^2-4r^2-4rr_s\right)}\Big[g^6\Big(l^2+l-6\Big)\xi^6 \\ &\qquad - 2g^4l(l+1)\Big(l^2+l+10\Big)\xi^4r^2 + 4rr_s(22-3l(l+1))g^4\xi^4 \\ &\qquad + 16\Big(l(l+1)\Big(l^2+l+10\Big)-2\Big)r^3r_sg^2\xi^2 \\ &\qquad + 16r^2r_s^2\Big(g^2(3l(l+1)-25)\xi^2 - 2\Big(l(l+1)\Big(l^2+l+10\Big)-3\Big)r^2 - 4\Big(l^2+l-9\Big)rr_s\Big)\Big]X^e\end{aligned} \tag{5.2.190}$$

$$+\partial_r\left(\frac{r\left(g^2\xi^2-4r^2-4rr_s\right)}{2l(l+1)\Lambda}p_2\right)-\left(\frac{2r^2}{l(l+1)}+\frac{2r^2}{\Lambda}\right)p_2\,.$$

The gauge-variant contributions to $\pi_\mu^{(2)}$ are now determined as follows: First we start from equation (**5.2.87**) and replace (A,Π_A) and (q_1,p_1) by their gauge invariant extensions and the necessary gauge-variant correction terms. Then, we use the solutions of the first order constraints to eliminate q_2, y_e, Y_e. We will work in the common gauge $x^e=0$ and find

$$\frac{r}{\pi\pi_\mu^{(0)}}\pi_\mu^{(2)}\Big|_{l\geq 2,\text{even}}=\int I(O_{q_1},O_{p_1},O_A,O_{\Pi_A})\,\mathrm{d}r+A_1+A_2+A_3\,. \tag{5.2.191}$$

The integrand $I(O_{q_1},O_{p_1},O_A,O_{\Pi_A})$ is a function of the gauge invariant extensions only and A_1, A_2 and A_3 are gauge variant boundary terms. They depend on X^e,p_2 and q_2, where the dependence on q_2 comes from the fact that we simplified $\pi_\mu^{(2)}$ before inserting the solution for q_2. The explicit expression for the boundary terms is given in appendix A. Surprisingly, the only dependence on the gauge-variant variables is within the boundary term. It is not clear why this is the case and it remains to be seen whether this is true more generally.

In the calculation of the physical Hamiltonian, we are interested in the solution for $\pi_\mu^{(2)}$ in the limit as r goes to infinity. Therefore, we define the leading order contributions to the gravitational and electromagnetic variables and have

$$\begin{aligned}
&q_1\sim q_1^0r^{-1}, && p_1\sim p_1^0, && O_{q_1}\sim\overline{q}_1^0r^{-1}, && O_{p_1}\sim\overline{p}_1^0, && q_2\sim q_2^0r, && p_2\sim p_2^0r^{-2},\\
&X^e\sim X_0^er, && A\sim A_0r, && O_A\sim\overline{A}_0r\,. &&&&&&
\end{aligned} \tag{5.2.192}$$

The variables with sub-/superscript 0 on the right-hand side are independent of r but still depend on l,m. The leading order contribution to the boundary term is given by

$$\begin{aligned}
\sum_{i=1}^{3}A_i=\frac{1}{r}\Big[&\frac{3l^4+6l^3-5l^2-8l+8}{16}(X_0^e)^2-\frac{3}{4}\sqrt{2(l-1)l(l+1)(l+2)}X_0^e\overline{q}_0^1-\frac{g^2\xi l(l+1)}{2\sqrt{2(l+2)(l-1)}}\overline{A}_0\\
&+\frac{2}{(l+2)(l-1)}\overline{p}_1^0p_2^0-\frac{4}{l(l+1)(l-1)^2(l+2)^2}(p_2^0)^2+q_2^0\Big(\frac{1}{4}\sqrt{2(l-1)l(l+1)(l+2)}X_0^e\\
&-\frac{1}{8}(3l(l+1)+2)(l(l+1)q_2^0-4q_1^0)-\frac{1}{2}q_2^0+\frac{1}{2}\sqrt{l(l+1)}g^2\xi A_0\Big)\Big]+O(r^{-3/2})
\end{aligned} \tag{5.2.193}$$

The expression is vanishing as r^{-1} and in the section of the physical Hamiltonian we see that this fall-off behaviour is sufficient to drop the boundary terms. Hence, in the computation of the physical Hamiltonian it is justified to consider the solution for $\pi_\mu^{(2)}$ to be weakly gauge invariant. To complete the transformation between the two physical degrees of freedom, we relate Q^e,P^e,A^e,Π_A^e to the true degrees of freedom X^e,Y^e,A,Π_A. Then, using the appropriate canonical transformation, the final result for $\pi_\mu^{(2)}$ will have the same form as before.

In the computations for our gauge, we defined (A^e,Π_A^e) in terms of the gauge invariant extensions O_A,O_{Π_A}. Thus, it is sufficient to show how O_A and O_{Π_A} are related to the variables (A,Π_A) and (X^e,Y_e) which are the true degrees of freedom in GP gauge. This is achieved by looking at the relations we obtained using the observable map:

$$O_A = A - \frac{\xi}{\sqrt{2(l+2)(l-1)}r^2} X^e \tag{5.2.194}$$

$$O_{\Pi_A} = \Pi_A + \frac{8\left(l^2+l+3\right) r r_s - 2\left(l^2+l+2\right) g^2\xi^2}{\sqrt{2(l+2)(l-1)}r^2\left(g^2\xi^2 - 4r^2 - 4rr_s\right)\pi_\mu^{(0)}} g^2 \xi X^e + \frac{g^2\xi}{\sqrt{l(l+1)}\Lambda} p_2\,, \tag{5.2.195}$$

The function p_2 is given by

$$p_2 = y_h + B y_v - \partial_r (C y_v) \tag{5.2.196}$$

where we have to insert the solutions of the constraints for y_h, y_v that we derived in the beginning of this section. Provided we can solve the differential equation for y_v, we expressed p_2 purely in terms of X^e, Y_e.

In the gravitational sector, the master variables (Q^e, P^e) are defined in terms of O_{q_1}, O_{p_1} with a canonical transformation. It is therefore sufficient to relate X^e, Y_e, A, Π_A to O_{q_1} and O_{p_1}. In the definition of the observable map we had

$$O_{q_1} = D y_v + \sqrt{\frac{2l(l+1)}{(l+2)(l-1)}} \frac{2\Lambda}{4r^2 + 4rr_s - g^2\xi^2} X^e \tag{5.2.197}$$

$$\begin{aligned} O_{p_1} = p_1 &- \frac{8}{2\sqrt{2(l+2)(l+1)l(l-1)}r^2 (\pi_\mu^{(0)})^3 \left(g^2\xi^2 - 4r^2 - 4rr_s\right)} \Big[g^6 \left(l^2+l-6\right)\xi^6 \\ &- 2g^4 l(l+1)\left(l^2+l+10\right)\xi^4 r^2 + 4rr_s(22 - 3l(l+1))g^4\xi^4 \\ &+ 16\left(l(l+1)\left(l^2+l+10\right) - 2\right) r^3 r_s g^2 \xi^2 \\ &+ 16 r^2 r_s^2 \left(g^2(3l(l+1)-25)\xi^2 - 2\left(l(l+1)\left(l^2+l+10\right)-3\right) r^2 - 4\left(l^2+l-9\right) rr_s\right)\Big] X^e \\ + \partial_r &\left(\frac{r\left(g^2\xi^2 - 4r^2 - 4rr_s\right)}{2l(l+1)\Lambda} p_2\right) - \left(\frac{2r^2}{l(l+1)} + \frac{2r^2}{\Lambda}\right) p_2\,. \end{aligned} \tag{5.2.198}$$

Expressing p_2 in terms of X^e, Y_e, A, Π_A as in equation (**5.2.196**), we related the (Q^e, P_e) to the true degrees of freedom in GP gauge.

In this section, we saw that working directly in the GP gauge is more complicated and we decided to use a detour with a different choice of gauge. In this gauge we derived a solution for $\pi_\mu^{(2)}$ which is very similar to the result for the odd parity. Then, we showed that we can go back to the GP gauge and related the master variables (Q^e, P_e) and (A^e, Π^e_A) to the physical degrees of freedom in GP gauge.

For later convenience, we also computed the boundary term for the even parity perturbations in the case of pure gravity. In the GP gauge it is given by

$$\begin{aligned} B_e = \sum_{lm} \Bigg[&\frac{(l+2)(l-1)r^2 + 5(l+2)(l-1)rr_s - (l^2+l-17)r_s^2}{2r^4 r_s \Lambda} X^e_{lm} X^e_{lm} - \frac{1}{r^2} X^e_{lm} (X^e_{lm})' \\ &+ \sqrt{\frac{2(l+2)(l-1)}{l(l+1)}} \frac{r - r_s}{2r\sqrt{rr_s}\Lambda} X^e_{lm} \partial_r y^{(1)}_{v,lm} + \frac{2r(r-r_s)}{2l(l+1)r\Lambda} \partial_r y^{(1)}_{v,lm} \partial_r y^{(1)}_{v,lm} + \frac{2((l^2+l-1)r + r_s)}{2l(l+1)r\Lambda} y^{(1)}_{v,lm} \partial_r y^{(1)}_{v,lm} \\ &+ \sqrt{\frac{(l+2)(l-1)}{2l(l+1)}} \frac{(l-1)(l+2)\left(l^2+l-1\right) r^3 + \left(l(l+1)\left(l^2+l+2\right) - 5\right) r^2 r_s + 3\left(l^2+l-1\right) rr_s^2 + 6r_s^3}{r^{5/2}\sqrt{r_s}\left(\left(l^2+l-2\right) r + 3r_s\right)^2} X^e_{lm} y^{(1)}_{v,lm} \end{aligned} \tag{5.2.199}$$

$$+\frac{\left(l^2+l-2\right)^2 r^3+(l-1)(l+2)\left(l(l+1)\left(l^2+l-3\right)+8\right)r^2 r_s+3\left(l^2+l-11\right)r_s^3-9(2l(l+1)-5)r r_s^2}{8l(l+1)r^4\Lambda^3}(y^{(1)}_{v,lm})^2\Bigg]$$

5.2.3.5 Decoupling of the Equations

In the two subsections before, we obtained the solution for $\pi_\mu^{(2)}$ in the odd and even parity case. With canonical transformations we reduced the expression such that the dependence on the background mass and charge is inside three potentials $V_{\text{grav}}^{(e/o)}$, $V_{\text{em}}^{(e/o)}$ and $V_{\text{coup}}^{(e/o)}$. At second order, it is possible to further simplify the solution for $\pi_\mu^{(2)}$ with a "rotation" of the canonical variables to remove the coupling term.

We chose the naming conventions in the even and odd parity sector so that the solution for $\pi_\mu^{(2)}$ looks identical. The only difference is the definition of the potentials which we do not need to use for the decoupling. We use the following ansatz for the canonical transformation depending on an angle θ:

$$\begin{pmatrix} Q^{o/e} \\ A^{e/o} \end{pmatrix} = \begin{pmatrix} \cos\theta & \frac{1}{g}\sin\theta \\ -g\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} Q_1^{e/o} \\ Q_2^{e/o} \end{pmatrix} \qquad \begin{pmatrix} P^{e/o} \\ \Pi_A^{e/o} \end{pmatrix} = \begin{pmatrix} \cos\theta & g\sin\theta \\ -\frac{1}{g}\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} P_1^{e/o} \\ P_2^{e/o} \end{pmatrix} \tag{5.2.200}$$

This is a canonical transformation provided the parameter θ only depends on the constants l, r_s, g and ξ. Inserting the transformation into the expression for $\pi_\mu^{(2)}$ we obtain for both the even and odd parity

$$\begin{aligned}\frac{r}{\pi\pi_\mu^{(0)}}\pi_\mu^{(2)}\Big|_{l\geq 2} = \int \mathrm{d}r\,&\frac{1}{4r}\pi_\mu^{(0)}(P_1Q_1'+P_2Q_2') \\ &+\frac{1}{2}\Big(P_1^2+g^2P_2^2+(Q_1')^2+\frac{1}{g^2}(Q_2')^2+V_1Q_1^2+\frac{1}{g^2}V_2Q_2^2+\frac{2}{g}\sqrt{(l+2)(l-1)}V_{12}Q_1Q_2\Big)\,.\end{aligned} \tag{5.2.201}$$

The new potentials V_1, V_2 and V_{12} are

$$V_1 := \frac{1}{r^2}\Big(U-\Big(\frac{3r_s}{2r}\cos(2\theta)+\sqrt{(l+2)(l-1)}\frac{g\xi}{r}\sin(2\theta)\Big)W\Big) \tag{5.2.202}$$

$$V_2 := \frac{1}{r^2}\Big(U+\Big(\frac{3r_s}{2r}\cos(2\theta)+\sqrt{(l+2)(l-1)}\frac{g\xi}{r}\sin(2\theta)\Big)W\Big) \tag{5.2.203}$$

$$V_{12} := \frac{1}{r^2}\Big(\sqrt{(l+2)(l-1)}\frac{g\xi}{r}\cos(2\theta)-\frac{3r_s}{2r}\sin(2\theta)\Big)W \tag{5.2.204}$$

The goal of the transformation is to find the angle θ such that the coupling potential V_{12} vanishes. This is achieved if θ satisfies

$$\cos(2\theta)^2 = \frac{9r_s^2}{9r_s^2+4(l+2)(l-1)g^2\xi^2}\,, \tag{5.2.205}$$

$$\sin(2\theta)^2 = \frac{4(l+2)(l-1)g^2\xi^2}{9r_s^2+4(l+2)(l-1)g^2\xi^2}\,. \tag{5.2.206}$$

To find the angle θ we need to take a square root of the equations which could take two values. For consistency with the situation without electromagnetic field, we would like the transformation to reduce to the identity transformation if the electric charge ξ is zero. This condition fixes the sign of both square roots to be positive. This gives the relations for the sine and cosine of 2θ

$$\cos(2\theta) = \frac{3r_s}{\sqrt{9r_s^2+4(l+2)(l-1)g^2\xi^2}}\,, \tag{5.2.207}$$

$$\sin(2\theta) = \frac{2\sqrt{(l+2)(l-1)}g\xi}{\sqrt{9r_s^2 + 4(l+2)(l-1)g^2\xi^2}} \,. \tag{5.2.208}$$

Using this relations the potentials V_1 and V_2 reduce to

$$V_1 = \frac{1}{r^2}\left(U - \frac{1}{2r}\sqrt{9r_s^2 + 4(l+2)(l-1)g^2\xi^2 W}\right), \tag{5.2.209}$$

$$V_2 = \frac{1}{r^2}\left(U + \frac{1}{2r}\sqrt{9r_s^2 + 4(l+2)(l-1)g^2\xi^2 W}\right). \tag{5.2.210}$$

We successfully decoupled the electromagnetic and gravitational degrees of freedom for both the even and odd parity sector. The solution for $\pi_\mu^{(2)}$ then only depends on the two potentials V_1 and V_2.

5.2.4 Reduced Hamiltonian

In section 5.2.1, we derived an implicit expression for the physical Hamiltonian in terms of the asymptotic solution for the spherically symmetric, gravitational variables. The dependence on the matter content of the theory is through the solution of the constraints for $\pi_\mu^{(2)}$. The reduced Hamiltonian is given by the evaluation of a boundary term at infinity and for one asymptotic end we have

$$H = \lim_{r\to\infty} \frac{\pi c}{\kappa r}\pi_\mu^2 = \lim_{r\to\infty} \frac{\pi}{2\kappa r}\left((\pi_\mu^{(0)})^2 + 2\pi_\mu^{(0)}\pi_\mu^{(2)}\right) \tag{5.2.211}$$

$$= M + \frac{1}{\kappa}\int_{\mathbb{R}+} \mathrm{d}r\, \frac{1}{4r}\pi_\mu^{(0)}{}^{(2)}C_h + {}^{(2)}C_v \,. \tag{5.2.212}$$

In terms of the solutions for $\pi_\mu^{(2)}$ we have

$$H = M + H_{l=1} + \frac{1}{\kappa}\sum_{l\geq 2,m,I}\int_{\mathbb{R}+} \mathrm{d}r\, N^3 P^I_{lm}\partial_r Q^I_{lm} + \frac{N}{2}\left((P^I_{lm})^2 + (\partial_r Q^I_{lm})^2 + V_I (Q^I_{lm})^2\right). \tag{5.2.213}$$

Here, $H_{l=1}$ are the contributions from the dipole perturbations. The labels l, m label the coefficients in the expansion into spherical harmonics and I stands for the even and odd parity and the labels 1 and 2 of chapter 5.2.3.5. The functions V_I are the Regge-Wheeler-Zerilli potentials derived in the previous section and we introduced the background solutions of the lapse and shift vector $N = 1, N^3 = 4\pi_\mu^{(0)}/r$.

The dipole contributions $H_{l=1}$ are given by

$$\begin{aligned} H_{l=1} = \frac{1}{\kappa}\sum_m \int \mathrm{d}r \Big[& N^3 Y^M_{o,1m} X^M_{o,1m}{}' + N\Big(\frac{g^2}{2}(Y^M_{o,1m})^2 + \frac{1}{2g^2}\left((X_M^{o,1m}{}')^2 + \frac{2+g^2\xi^2 r^{-2}}{r^2}(X_M^{o,1m})^2\right) \\ & + \frac{a_m^2 + 2\xi a_m X_M^{o,1m}}{2r^4}\Big)\Big] \\ + \sum_m \int \mathrm{d}r \Big[& N^3 \Pi_A^{e,1m} A^{e,1m}{}' + \left(y_v' - \frac{1}{r}y_v + \frac{\pi_\lambda}{r\pi_\mu}y_v\right)^2 - \frac{2\sqrt{2}\xi}{r\pi_\mu}\left(y_v' + \frac{\pi_\lambda}{r\pi_\mu}y_v\right)A^{e,1m} \\ & + \frac{N}{2}\left(g^2(\Pi_A^{e,1m})^2 + \frac{1}{g^2}\left((A^{e,1m}{}')^2 + \frac{1}{r^2\pi_\mu^2}(2\pi_\mu^2 + 16g^2\xi^2)(A^{e,1m})^2\right)\right)\Big] \,. \end{aligned} \tag{5.2.214}$$

We split the integral into two pieces for the odd and even parity contributions.

Setting $a_m = 0$ in the odd parity sector, we obtain

$$\frac{1}{\kappa}\sum_m \int \mathrm{d}r \left[N^3 Y^M_{o,1m} X^M_{o,1m}{}' + N\Big(\frac{g^2}{2}(Y^M_{o,1m})^2 + \frac{1}{2g^2}\bigg((X_M^{o,1m}{}')^2 + \frac{2+g^2\xi^2 r^{-2}}{r^2}(X_M^{o,1m})^2\bigg)\right]. \tag{5.2.215}$$

It has a similar form to the Hamiltonians for $l \geq 2$ for a potential $V^o_{l=1}$ that reads

$$V^o_{l=1} = \frac{2r^2 + g^2\xi^2}{r^4}\,, \tag{5.2.216}$$

This potential is simply the potential V^o_{em} evaluated for $l = 1$.

The even parity contributions to the dipole Hamiltonian still contains the function y_v. Restricting to the case of uncharged black holes we can ignore this contribution because then the source term $s(r)$ vanishes. If we additionally ignore the homogeneous solution for y_v we obtain

$$\sum_m \int \mathrm{d}r \Big[N^3 \Pi_A^{e,1m} A^{e,1m}{}' + \frac{N}{2}\Big(g^2(\Pi_A^{e,1m})^2 + \frac{1}{g^2}\Big((A^{e,1m}{}')^2 + \frac{2}{r^2}(A^{e,1m})^2\Big)\Big)\Big]\,,$$

In this case the potential is very simple and reduces to

$$V^e_{l=1}\Big|_{\xi=0} = \frac{2}{r^2}\,. \tag{5.2.217}$$

This agrees with the restriction of the electromagnetic, even parity potential to the case $\xi = 0$ and $l = 1$.

5.2.5 Generalisation of the Background Gauge Fixing

In [1], we constructed the physical Hamiltonian for perturbations around spherically symmetric spacetimes using a generalized gauge fixing. Instead of using the GP gauge to restrict the value of the background gauge degrees of freedom μ and λ, we consider more general but phase space independent functions. The formula for the physical Hamiltonian in terms of the evaluation of π_μ at infinity is still valid provided that the new, generalized gauge has the same asymptotic behaviour as the GP gauge described in (**3.5.29**). An extension of the treatment to other gauges is of interest for the following reasons:

The generalization of the gauge serves as a proof of concept that the framework in this manuscript can be used for many different scenarios. As we will see in this section, we need to take the same steps as before with suitable generalizations of the canonical transformations. It is non-trivial that this is possible in a straight forward way. In the more general setup, new problems could arise that we did not see when restricting to the Gullstrand-Painlevé gauge. Additional and different methods might be needed to solve the constraints and to bring the physical Hamiltonian into a similar shape to (**5.2.213**). A successful calculation in the generalized setting strengthens the trust into the new framework and serves as a consistency check.

Note that different gauge conditions could a priori yield different physical Hamiltonians which would be neither astonishing nor inconsistent for the following reason: Consider two different gauge fixing conditions G and G', then the Dirac observables corresponding to a phase space function f are $F = O^G_f$ and $F' = O^{G'}_f$ respectively as described in section 2. Unless f is weakly gauge invariant the two functions F and F' are different Dirac observables. It is therefore not obvious a priori that we will obtain the same Dirac observable corresponding to the physical Hamiltonian in the two gauges. In the present case, f should actually be a weak Dirac observable as it is known that the physical Hamiltonian is one of the Poincaré charges when asymptotically flat boundary conditions hold [24]. While we did not explicitly check this, we will independently confirm below that F=F' using that the generalised gauges are asymptotically GP.

The restriction to only one gauge might not be ideal because there could be regions in spacetime where this gauge breaks down. One example is the Schwarzschild gauge where we set $\pi_\mu = 0$ and $\lambda = \log r$. This gauge fixing is perfectly suitable to investigate the exterior of the black hole. Moving towards the event horizon of the black hole, the function μ inside the metric will get infinitely large and the gauge breaks down. This is merely a bad choice of gauge and as we discussed before, moving to the GP gauge we have access to both the interior and exterior of the black hole. In contrast to the Schwarzschild gauge, the GP gauge is perfectly regular across the horizon. Similarly, the GP gauge could have limitations that we are not aware of at the moment and switching to a different gauge might be required in the future.

In the following, we present the derivation of the physical Hamiltonian in generalized gauges based on [1]. We start with the solution of the second order constraints and then describe the solution of the first order constraints. In the computations we do not compute the dipole contributions ($l = 1$) which is left for future explorations. The physical Hamiltonian is simplified using several canonical transformations until it is in a more tractable form.

5.2.5.1 Solution of the Second Order Constraints

The solution of the second order constraints carries through in the same way with slight generalizations. We start from the expansion of the symmetric constraints up to second order in the perturbations. We split the symmetric gauge momenta into terms of zeroth and second order in the perturbations; $\pi_\mu = \pi_\mu^{(0)} + \pi_\mu^{(2)}$. Then we find the equations

$$C_v \sim \pi e^{-\mu-2\lambda}\Big(\pi_\mu^{(0)}\pi_\mu^{(2)} - \pi_\mu^{(0)}\pi_\lambda^{(2)} - \pi_\mu^{(2)}\pi_\lambda^{(0)}\Big) + {}^{(2)}C_v = 0\,, \tag{5.2.218}$$

$$C_h \sim 4\pi\Big[\mu'\pi_\mu^{(2)} + \lambda'\pi_\lambda^{(2)} - (\pi_\mu^{(2)})'\Big] + {}^{(2)}C_h = 0\,. \tag{5.2.219}$$

In the generalized case, we fix $x^{v/h/e/o} = 0$ while keeping λ, μ free. Then the functions ${}^{(2)}C_v$ and ${}^{(2)}C_h$ which are of second order in the perturbations are given by

$${}^{(2)}C_h = Y_o \cdot \partial_r X^o + Y_e \cdot \partial_r X^e + Y_o^M \cdot \partial_r X_M^o + Y_e^M \cdot \partial_r X_M^e \tag{5.2.220}$$

$$\begin{aligned}
{}^{(2)}C_v = {} & e^{-\mu+2\lambda} Y_o \cdot Y_o - \frac{1}{2} e^{-\mu-2\lambda}(\pi_\mu^{(0)} - \pi_\lambda^{(0)}) Y_o \cdot X^o + e^{-\mu-6\lambda}\left(\frac{(\pi_\mu^{(0)})^2}{32} - \frac{\pi_\mu^{(0)}\pi_\lambda^{(0)}}{16} + \frac{(\pi_\lambda^{(0)})^2}{16}\right) X^o \cdot X^o \\
& - e^{-2\lambda-\mu} X^o \cdot \left(\partial_r^2 - 4\lambda'\partial_r - \mu'\partial_r + \frac{5}{2}(\lambda')^2 + \lambda'\mu' - \lambda''\right) X^o - \frac{3}{4} e^{-2\lambda-\mu}\partial_r X^o \cdot \partial_r X^o \\
& + e^{-\mu+2\lambda} Y_e \cdot Y_e - \frac{1}{2} e^{-\mu-2\lambda}(\pi_\mu^{(0)} - \pi_\lambda^{(0)}) X^e \cdot Y_e + e^{-\mu-6\lambda}\left(\frac{(\pi_\mu^{(0)})^2}{32} - \frac{\pi_\mu^{(0)}\pi_\lambda^{(0)}}{16} + \frac{(\pi_\lambda^{(0)})^2}{16}\right) X^e \cdot X^e \\
& + e^{\mu-2\lambda}\frac{1}{4}\Big(-3\partial_r X^e \cdot \partial_r X^e + X^e \cdot (-4\partial_r^2 + 4\mu'\partial_r + 16\lambda'\partial_r - 4\mu'\lambda' - 10(\lambda')^2 + 4\lambda'') X^e\Big) \\
& + \frac{e^\mu}{2} y_o^{(1)} \cdot y_o^{(1)} + \frac{1}{2} e^{3\mu-2\lambda} y_v^{(1)} \cdot y_v^{(1)} + \frac{1}{2} e^\mu y_e^{(1)} \cdot y_e^{(1)} - e^\mu y_h^{(1)} \cdot y_v^{(1)} \\
& + \frac{g^2}{2} e^{-\mu-2\lambda}\Big[\frac{1}{4} e^{2\mu-4\lambda}(X^e \cdot X^e + X^o \cdot X^o)\xi^2 + e^{2\mu} l(l+1) \int Y_e^M \mathrm{d}r \cdot \int Y_e^M \mathrm{d}r \\
& \qquad + e^{2\lambda}(Y_e^M \cdot Y_e^M + Y_o^M \cdot Y_o^M)\Big]
\end{aligned} \tag{5.2.221}$$

$$+\frac{1}{2g^2}e^{-\mu-2\lambda}\Big[e^{2\mu}l(l+1)X^o_M\cdot X^o_M+e^{2\lambda}\partial_r X^o_M\cdot\partial_r X^o_M+e^{2\lambda}\partial_r X^M_e\cdot\partial_r X^M_e\Big]$$

The solution of the differential equations (**5.2.219**) works in the same way as in the GP gauge. The second equation ($C_h=0$) is solved for $\pi_\lambda^{(2)}$

$$\pi_\lambda^{(2)}=\frac{1}{\lambda'}\left((\pi_\mu^{(2)})'-\mu'\pi_\mu^{(2)}-\frac{1}{4\pi}{}^{(2)}C_h\right). \tag{5.2.222}$$

This reduces the first equation to a differential equation for $\pi_\mu^{(2)}$:

$$\frac{\pi_\mu^{(0)}}{\lambda'}\left(\lambda'+\mu'-\lambda'\frac{\pi_\lambda^{(0)}}{\pi_\mu^{(0)}}\right)\pi_\mu^{(2)}-\frac{\pi_\mu^{(0)}}{\lambda'}(\pi_\mu^{(2)})'+\frac{\pi_\mu^{(0)}}{4\pi\lambda'}{}^{(2)}C_h+\frac{e^{\mu+2\lambda}}{\pi}{}^{(2)}C_v=0\,. \tag{5.2.223}$$

Employing the relation between $\pi_\lambda^{(0)}$ and $\pi_\mu^{(0)}$, we eliminate $\pi_\lambda^{(0)}$ and find

$$(e^{-\lambda-2\mu}\pi_\mu^{(0)}\pi_\mu^{(2)})'-\frac{e^{-\lambda-2\mu}}{4\pi}\pi_\mu^{(0)}{}^{(2)}C_h-\frac{e^{-\mu+\lambda}}{\pi}\lambda'{}^{(2)}C_v=0\,. \tag{5.2.224}$$

Integrating this differential equation we find the solution for $\pi_\mu^{(2)}$

$$\pi_\mu^{(2)}=\frac{e^{\lambda+2\mu}}{\pi\pi_\mu^{(0)}}\int\mathrm{d}r\left[\frac{1}{4}e^{-\lambda-2\mu}\pi_\mu^{(0)}{}^{(2)}C_h+e^{-\mu+\lambda}\lambda'{}^{(2)}C_v\right] \tag{5.2.225}$$

The solution for $\pi_\lambda^{(2)}$ is found using the first equation in (**5.2.219**) and we have

$$\pi_\lambda^{(2)}=\left(1-\frac{\pi_\lambda^{(0)}}{\pi_\mu^{(0)}}\right)\pi_\mu^{(2)}+\frac{e^{\mu+2\lambda}}{\pi\pi_\mu^{(0)}}{}^{(2)}C_v\,. \tag{5.2.226}$$

This completes the computations necessary for the second order constraints. There were no significant difficulties in the computations. Restricting μ and λ to their values according to the strict GP gauge ($\mu=0,\lambda=\log r$), we recover the previous expressions.

5.2.5.2 Solution of the First Order Constraints – Odd Parity

We start again with the odd parity sector for $l\geq 2$. The physical degrees of freedom are (X^o,Y_o) for the gravitational and (X^o_M,Y^M_o) for the electromagnetic degrees of freedom. In the generalized Gullstrand-Painlevé gauge, we have to solve one odd parity constraint Z^o for y_o. This constraint reads

$${}^{(1)}Z^o_{lm}=\sqrt{2(l+2)(l-1)}\left(e^{2\lambda}Y_o+\frac{\pi_\lambda}{4}e^{-2\lambda}X^o\right)-\partial_r\left(e^{2\lambda}y_o\right)-\xi\partial_r X^o_M \tag{5.2.227}$$

The solution of this constraint is

$$y_o^{(1)}=e^{-2\lambda}\int\mathrm{d}r\left[\sqrt{2(l+2)(l-1)}\left(e^{2\lambda}Y_o+\frac{\pi_\lambda}{4}e^{-2\lambda}X^o\right)-\xi\partial_r X^o_M\right]. \tag{5.2.228}$$

In terms of the variables (X^o,Y_o), the physical Hamiltonian is rather complicated and not in a very tractable form. In the case of the Gullstrand-Painlevé coordinates, it was simplified significantly using two canonical transformations. We found a generalization of these transformation and the combined transformation introducing the master variables (Q^o, P_o) are:

$$
\begin{aligned}
Q^o &:= \sqrt{2}e^{-\lambda}\int \mathrm{d}r\left(e^{2\lambda}Y_o + \frac{\pi_\lambda^{(0)}}{4}e^{-2\lambda}X^o\right)\\
P^o &:= \frac{e^\lambda}{\sqrt{2}}\partial_r\left(e^{-2\lambda}X^o\right) + \frac{\pi_\mu}{4}e^{-2\lambda}Q^o
\end{aligned}
\tag{5.2.229}
$$

For the electromagnetic variables we define $A^o = X^o_M$ and $\Pi^o_A = Y^M_o$.

The solution of the first order constraints and the canonical transformation are inserted into the solution for $\pi_\mu^{(2)}$. We find

$$
\begin{aligned}
\pi_\mu^{(2)}\Big|_{\text{odd}} = \frac{e^{\lambda+2\mu}}{\pi\pi_\mu^{(0)}}\int \mathrm{d}r\Bigg[&\frac{1}{4}e^{-\lambda-2\mu}\pi_\mu^{(0)}(P^o\cdot(Q^o)' + \Pi^o_A\cdot(A^o)') + \frac{1}{2}e^{-2\mu+\lambda}\lambda'\Big(P^o\cdot P^o + g^2\Pi^o_A\cdot\Pi^o_A\\
&+ Q^{o\prime}\cdot Q^{o\prime} + \frac{1}{g^2}(A^o)'\cdot(A^o)' + V^o_{\text{grav}}Q^o\cdot Q^o + \frac{1}{g^2}V^o_{\text{em}}A^o\cdot A^o + \frac{2}{g}\sqrt{(l+2)(l-1)}V^o_{\text{Coup}}Q^o\cdot A^o\Big)\Bigg]\,,
\end{aligned}
\tag{5.2.230}
$$

The equation above reduces to the form we obtained before in Gullstrand-Painlevé gauge. In the generalized gauge, the potentials are defined by

$$
\begin{aligned}
V^o_{\text{grav}} &:= e^{2\mu-2\lambda}\left(U^o - \frac{3}{2}e^{-\lambda}r_sW^o\right)\\
V^o_{\text{em}} &:= e^{2\mu-2\lambda}\left(U^o + \frac{3}{2}e^{-\lambda}r_sW^o\right)\\
V^o_{\text{Coup}} &:= e^{2\mu-3\lambda}g\xi W^o\,,
\end{aligned}
\tag{5.2.231}
$$

The functions W^o and U^o read

$$
W^o := 1,\quad U^o := l(l+1) - \frac{3}{2}e^{-\lambda}r_s + e^{-2\lambda}g^2\xi^2\,. \tag{5.2.232}
$$

The solution for $\pi_\mu^{(2)}$ was simplified using several integration by parts. The corresponding boundary terms were dropped in the above formula and they are given by

$$
\begin{aligned}
\int\frac{\partial}{\partial r}\Big[&e^{-2\mu+\lambda}\lambda'X^o\cdot\partial_r(e^{-2\lambda}X^o) + \frac{1}{2}\left(\frac{r_s}{2} - e^\lambda + 2e^{3\lambda-2\mu}(\lambda')^2\right)e^{-4\lambda}(X^o)^2\\
&-\frac{e^{-2\lambda}}{2}\left(r_s - e^\lambda - \frac{1}{4}g^2\xi^2e^{-\lambda}\right)(Q^o)^2\Big]
\end{aligned}
\tag{5.2.233}
$$

In the generalized setting, this boundary term also vanishes in the limit $r\to\infty$ as r^{-1}.

In this section, we saw that a similar procedure with appropriate generalizations leads to a very similar structure for the solution of $\pi_\mu^{(2)}$. The results are all compatible with the strict GP gauge when we restrict the background variables to $\mu = 0$ and $\lambda = \log r$.

5.2.5.3 Solution of the First Order Constraints – Even Parity

Let us now turn to the even parity first order constraints for $l \geq 2$. In the case of the strict GP gauge, we found that the attempt to directly solve the first order constraints in the GP gauge leads to very complicated differential equations. The solution of these equations is not straight forward and we avoided explicitly constructing them by working in an intermediate gauge $x^e = X^e = p_2 = 0$. Then, we switched back to the Gullstrand-Painlevé gauge $x^e = x^h = x^v = 0$ and obtained the solution for $\pi_\mu^{(2)}$ in terms of this gauge.

Since the direct approach was very involved already in the strict GP gauge, we assume the same to be true in the generalized setting. Hence, we directly start with the alternative gauge $x^e = X^e = p_2 = 0$. The solution of the first order diffeomorphism constraints in the generalized setup for y_e and Y_e is

$$y_e^{(1)} = -\frac{e^{-2\mu}}{\sqrt{l(l+1)}}\Big(-2e^{\mu}\partial_r(e^{\mu}y_v) + 2\lambda' e^{2\lambda}y_h - \partial_r(\pi_\mu e^{-2\mu})x^v - \frac{1}{2}\pi_\mu e^{-2\mu}\partial_r x^v + \frac{\pi_\lambda}{2}e^{-2\lambda}\partial_r x^h\Big) \tag{5.2.234}$$

$$Y_e^{(1)} = -\frac{e^{-2\lambda}}{\sqrt{2(l+2)(l-1)}}\Big(-\partial_r(e^{2\lambda}y_e^{(1)}) - \sqrt{l(l+1)}e^{2\lambda}y_h + \frac{1}{2}\sqrt{l(l+1)}\pi_\mu e^{-2\mu}x^v - \xi\partial_r X_M^e\Big). \tag{5.2.235}$$

Similarly to the strict GP gauge, we would like to solve the first order Hamiltonian constraint for x^h. The non-trivial dependence of this constraint on derivatives of x^h motivates us to perform a canonical transformation from (x^v, y_v) and (x^h, y_h) to new variables (q_1, p_1) and (q_2, p_2). The ansatz for the transformation is the same as in equation (**5.2.147**). The generalized parameters C, D, G, B and K can be found using a similar strategy as we used before. One finds

$$\begin{aligned} C &= \frac{1}{\lambda'}e^{2\mu-2\lambda}, \quad D = -\frac{4\pi_\mu e^{4\mu}}{\pi_\mu^2 + 16(\lambda')^2 e^{4\lambda}}, \quad G = -\frac{\pi_\mu}{4\lambda'}e^{-2\mu-2\lambda} \\ B &= \frac{8e^{4\mu}}{\pi_\mu^2 + 16e^{4\lambda}(\lambda')^2}\Big(e^{-\lambda}r_s - (l(l+1)+2)\Big). \end{aligned} \tag{5.2.236}$$

The function K is more complicated it is given by

$$\begin{aligned} K = \frac{32e^{-6\lambda}e^{-2\mu}}{M_1^2\pi_\mu(\lambda')^4}\Bigg(& 2e^{4\lambda-2\mu}(\lambda')^5\partial_r\left(\frac{(2\Delta^2 + \Delta\Lambda)\,e^{2\mu}}{(\lambda')^2}\right) - \Lambda l(l+1)\left(\Delta^2 e^{4\mu} - 3\Delta e^{2\lambda+2\mu}(\lambda')^2 + 2e^{4\lambda}(\lambda')^4\right) \\ &- 2l(l+1)\left(2\Delta^2 e^{2\lambda+2\mu}(\lambda')^2 - 5\Delta e^{4\lambda}(\lambda')^4 + 4e^{6\lambda-2\mu}(\lambda')^6\right) - 4\Delta e^{4\lambda}(\lambda')^4 \\ &- r_s e^{-\lambda}\left(\Delta^2 e^{2\lambda+2\mu}(\lambda')^2 - 4\Delta e^{4\lambda}(\lambda')^4 + 2e^{6\lambda-2\mu}(\lambda')^6\right)\Bigg), \end{aligned} \tag{5.2.237}$$

In this expression, we introduced a generalization of the variables n, Δ and Λ that we already encountered in the strict GP gauge. These variables as well as the new variable M_1 are defined by

$$n := \frac{1}{2}(l+2)(l-1) \tag{5.2.238}$$

$$\Delta := 1 - e^{-\lambda}r_s + \frac{1}{4}e^{-2\lambda}g^2\xi^2 \tag{5.2.239}$$

$$\Lambda := n + \frac{3}{2}e^{-\lambda}r_s - \frac{1}{2}e^{-2\lambda}g^2\xi^2 \tag{5.2.240}$$

$$M_1 := \frac{4}{\lambda'}e^{-\mu-\lambda}\Big(\Delta - 2e^{-2\mu+2\lambda}(\lambda')^2\Big). \tag{5.2.241}$$

After this canonical transformation, the first order Hamiltonian constraint is independent of derivatives of the variable q_2. The solution of the first order Hamiltonian constraint Z^v for q_2 is

$$q_2^{(1)} = \frac{(\lambda')^2}{2l(l+1)\Lambda}\Big[4e^{4\lambda-4\mu}(\Lambda + 2\Delta)q_1 - e^{-3\mu+5\lambda}(M_1 q_1)' + 2\sqrt{l(l+1)}g^2\xi A e^{4\mu}\Big]. \tag{5.2.242}$$

where we defined

$$A := -\int Y_e^M \,\mathrm{d}r\,, \quad \Pi_A := -\partial_r X_M^e\,. \tag{5.2.243}$$

Following an analogous strategy, we successfully solved all the first order constraints and can insert the results into the formula for $\pi_\mu^{(2)}$. However, as expected, the result is not very tractable in the generalized gauge and we wish to simplify it using canonical transformations. The transformation involves two steps as in the GP gauge. First, we scale the variables (q_1, p_1), (A, Π_A) and remove coupling terms between gravity and the electromagnetic field. The transformation is

$$p_1 = \sqrt{\frac{(l+2)(l-1)}{l(l+1)}}\frac{e^{-\mu+2\lambda}\lambda' M_1}{4\Lambda}\left(P + \frac{\xi e^{-\lambda}}{\sqrt{(l+2)(l-1)}}\Pi_A + A_3 A\right) \tag{5.2.244}$$

$$q_1 = \sqrt{\frac{l(l+1)}{(l+2)(l-1)}}\frac{4\Lambda}{e^{-\mu+2\lambda}\lambda' M_1}Q \tag{5.2.245}$$

$$A = \tilde{A} - \frac{\xi e^{-\lambda}}{\sqrt{(l+2)(l-1)}}Q \tag{5.2.246}$$

$$\Pi_A = \tilde{\Pi}_A + \Gamma Q \tag{5.2.247}$$

where the generalized form of Γ is

$$\begin{aligned}\Gamma = \frac{g^2\xi}{2e^{3\mu+6\lambda}\lambda'\Lambda\sqrt{(l+2)(l-1)}\pi_\mu M_1}\Big(&\frac{e^{6\lambda+4\mu}}{(\lambda')}\frac{\partial}{\partial r}\left(16\Lambda e^{2\lambda-4\mu}(\lambda')^2\left((\lambda'^2 - \Delta e^{2\mu-2\lambda}\right)\right) \\ &+ 16l(l+1)e^{4(\lambda+\mu)}\left(e^{4\lambda-4\mu}(\lambda')^4 - \Delta^2\right) + 16\Lambda l(l+1)e^{4(\lambda+\mu)}\left(e^{2\lambda-2\mu}(\lambda')^2 - \Delta\right) \\ &- (\lambda')^2\left(2e^{2(\lambda+\mu)}\left(-8g^4\xi^4 - 4e^{2\lambda}\left(8g^2\xi^2 + 21r_s^2\right) + 53g^2\xi^2 r_s e^\lambda + 84r_s e^{3\lambda}\right)\right) \\ &- 8e^{6\lambda}(\lambda')^4\left(8g^2\xi^2 + 4e^{2\lambda} - 21r_s e^\lambda\right) + 32\Delta^2 e^{4(\lambda+\mu)}\Big)\,.\end{aligned} \tag{5.2.248}$$

The second canonical transformation removes the coupling terms QP and $\tilde{A}\tilde{\Pi}_A$. In its generalized form, we have

$$Q = Q^e, \qquad P = P^e + A_{\mathrm{grav}}Q^e \tag{5.2.249}$$

$$\tilde{A} = \frac{1}{g^2}A^e, \qquad \tilde{\Pi}_A = g^2\Pi_A^e - \frac{g^2\xi^2 e^{-4\lambda}\pi_\mu}{8\Lambda}A^e\,, \tag{5.2.250}$$

Since A_{grav} is quite long, we will not display it here explicitly. It can be found in appendix B.

The solutions of the first order constraints and all three canonical transformations are now inserted into the formula for $\pi_\mu^{(2)}$. After integrating by parts and dropping the boundary terms, we obtained using Mathematica

$$\begin{aligned}\pi_\mu^{(2)}\Big|_{\mathrm{even}} = \frac{e^{\lambda+2\mu}}{\pi\pi_\mu^{(0)}}\int \mathrm{d}r\,\Bigg[&\frac{\pi_\mu^{(0)}}{4}e^{-2\mu-\lambda}(P^e\cdot(Q^e)' + \Pi_A^e\cdot(A^e)') + \frac{1}{2}e^{-2\mu+\lambda}\lambda'\left(P^e\dot{P}^e + Q^{e\prime}\cdot Q^{e\prime} + V_{\mathrm{grav}}^e Q^e\cdot Q^e\right) \\ &+ \frac{1}{2}e^{-2\mu+\lambda}\lambda'\left(g^2\Pi_A^e\cdot\Pi_A^e + \frac{1}{g^2}A^{e\prime}\cdot A^{e\prime} + \frac{1}{g^2}V_{\mathrm{em}}^e A^e\cdot A^e\right) + e^{-2\mu+\lambda}\lambda'\frac{1}{g}V_{\mathrm{coup}}^e Q^e\cdot A^e\Bigg]\,.\end{aligned} \tag{5.2.251}$$

The structure of this solution is very similar to the one obtained earlier in GP gauge. The potentials V_{grav}^e,

V^e_{em} and V^e_{coup} are given by

$$V^e_{\text{coup}} := e^{2\mu-3\lambda} g\xi W^e \tag{5.2.252}$$
$$V^e_{\text{grav}} := e^{2\mu-2\lambda}\Big(U^e - \frac{3}{2}e^{-\lambda} r_s W^e\Big) \tag{5.2.253}$$
$$V^e_{\text{em}} := e^{2\mu-2\lambda}\Big(U^e + \frac{3}{2}e^{-\lambda} r_s W^e\Big)\,. \tag{5.2.254}$$

The generalized expressions for W^e and U^e are

$$W^e := \frac{\Delta}{\Lambda^2}\Big(2n + \frac{3}{2}e^{-\lambda} r_s\Big) + \frac{1}{\Lambda}\Big(n + \frac{1}{2}e^{-\lambda} r_s\Big) \tag{5.2.255}$$
$$U^e := \Big(2n + \frac{3}{2}e^{-\lambda} r_s\Big) W + \Big(\Lambda - n - \frac{1}{2}e^{-\lambda} r_s\Big) - \frac{2n\Delta}{\Lambda} \tag{5.2.256}$$

The even parity contributions to the solution $\pi^{(2)}_\mu$ were simplified by canonical transformations and the boundary term was dropped. It is recorded explicitly in appendix B in full detail. Let us analyse the leading r behaviour of the boundary term. As r tends to infinity, the canonical variables behave approximately as

$$q_1 \sim q_1^0 r^{-1} \qquad p_1 \sim p_1^0 \qquad q_2 \sim q_2^0 r \qquad A \sim A_0 r \qquad A^e \sim A^e_0 r \qquad Q^e \sim Q^e_0\,. \tag{5.2.257}$$

In the above relations, the sub-/superscript 0 indicates that the variables are constant with respect to r but still depend on l, m. We also use the asymptotic behaviour of the background degrees of freedom in (**3.5.29**). The leading order contributions to the boundary term of $\pi^{(2)}_\mu$ behaves as

$$\begin{aligned}
&\frac{1}{\pi\pi^\infty_\mu\sqrt{r}}\Big(- \frac{1}{2}(p_1^0)^2 + \frac{3}{2}(q_1^0)^2 + \frac{(l^2+l+2)}{(l+2)(l+1)l(l-1)}(q_1^0)^2 + 2q_1^0 q_2^0 - \frac{3(l^2+l+2)}{2} q_1^0 q_2^0 + \frac{1}{2}(q_2^0)^2 \\
&- (l^2+l+2)(q_2^0)^2 + \frac{1}{8}(3l^4 + 6l^3 + 13l^2 + 10l + 16)(q_2^0)^2 + \frac{g^2(l^2+l+2)\xi}{\sqrt{l(l+1)}(l+2)(l-1)} A_0 q_1^0 \\
&- \frac{1}{2}g^2\xi\sqrt{l(l+1)}q_2^0 A_0 - \frac{1}{2}(Q^e_0)^2 + \frac{g^4\xi^2}{2(l+2)(l-1)}(A^e_0)^2 + 2\frac{g^2\xi}{2\sqrt{(l+2)(l-1)}} A^e_0 Q^e_0\Big) + O(r^{-1})
\end{aligned} \tag{5.2.258}$$

This shows that the leading order behaves as $r^{-1/2}$ and as we will see, it drops out of the physical Hamiltonian.

The above analysis in generalized GP gauge demonstrates that the formalism produces consistent results also for the solution of the even parity constraints. However, we still have to relate the gauge $x^e = X^e = p_2 = 0$ to the original gauge $x^v = x^h = x^e = 0$. For this we would need to prove weak gauge invariance of the solution $\pi^{(2)}_\mu$ and then relate the master variables (Q^e, P^e) in the two gauges. So far this has not yet been achieved due to the complexity of the canonical transformation and the length of the involved expressions. The completion of this part of the even parity analysis is left for future studies.

5.2.5.4 The Physical Hamiltonian

Both the odd and even parity solutions for $\pi^{(2)}_\mu$ can be further simplified by completely decoupling the electromagnetic and gravitational degrees of freedom. In the strict GP gauge in section 5.2.3.5, we found a "rotation" in the sapce of variables $(Q^{e/o}, P^{e/o})$ and $(A^{e/o}, \Pi^{e/o}_A)$ such that the solution for $\pi^{(2)}_\mu$ completely decouples in the transformed variables. The new variables (Q^1, P^1) and (Q^2, P^2) were found exactly in the same way for both the even and odd parity. It is not difficult to see, that the same transformation also works in the case of generalized gauges. Then, we found new potentials V_1, V_2 after the transformation which are defined by

$$V_1^I = e^{2\mu-2\lambda}\Big(U^I - \frac{1}{2}e^{-\lambda}\sqrt{9r_s^2 + 4(l+2)(l-1)g^2\xi^2}W^I\Big) \tag{5.2.259}$$

$$V_2^I = e^{2\mu-2\lambda}\Big(U^I + \frac{1}{2}e^{-\lambda}\sqrt{9r_s^2 + 4(l+2)(l-1)g^2\xi^2}W^I\Big)\,, \tag{5.2.260}$$

The reduced Hamiltonian is found by evaluating the stability condition of the generalized GP gauge and then computing the boundary terms. In [1], it is shown that the physical Hamiltonian is given by the same formula as in the case of GP gauge, In fact, one can check the computation in section 5.2.1 and observe that in the asymptotic evaluation of the boundary term we only needed the asymptotic behaviour of the Gullstrand-Painlevé gauge. We assumed the generalized gauge to have the same asymptotic behaviour and thus the same formula applies. To second order in the perturbations the physical Hamiltonian is

$$H = \lim_{r\to\infty}\frac{\pi}{2\kappa r}\Big((\pi_\mu^{(0)})^2 + 2\pi_\mu^{(0)}\pi_\mu^{(2)} + O(3)\Big) \tag{5.2.261}$$

In terms of the solution of the constraints that we constructed above, we obtain

$$H = M + \frac{1}{\kappa}\sum_{l\geq 2,m,I}\int_{\mathbb{R}+}\mathrm{d}r\,\Big[N_{(0)}^3 P_{lm}^I\partial_r Q_{lm}^I + \frac{N_{(0)}}{2}e^{-\mu}\Big((P_{lm}^I)^2 + (\partial_r Q_{lm}^I)^2 + V_I(Q_{lm}^I)^2\Big)\Big] + O(3)\,, \tag{5.2.262}$$

In this equation, I runs over the labels 1,2 as well as even, odd. (Q^I, P^I) are the decoupled master variables and V_I the corresponding potentials. In the equation we also introduced the zeroth order solution of the stability conditions that we found in section 3.6:

$$N_{(0)} = \lambda' e^{-\mu+\lambda}\,, \tag{5.2.263}$$

$$N_{(0)}^3 = \frac{\pi_\mu}{4}e^{-\lambda-2\mu}\,. \tag{5.2.264}$$

This concludes the discussion of the generalized gauges. We closely followed the computations in the strict GP gauge and generalized the canonical transformations which now depend on the choice for μ and λ. Restricting to the strict GP gauge, the formulas reduce to the ones that we found before. Surprisingly, generalizing the gauge did not produce a completely different expression but it looks quite similar. The structure of the physical Hamiltonian is exactly the same, where now $N_{(0)}$ and $N_{(0)}^3$ have a more complicated form.

The fact that we obtained a similar expression which is consistent with our previous analysis, shows that the method we used can be applied consistently to many different gauges. In the generalization we investigated, we did not face any significant issues and the analysis worked without the need to use a different strategy. In the next section, we will see that the physical Hamiltonian we obtained for generalized gauges is consistent with the Regge-Wheeler-Zerilli equations in the common domain of validity. This non-trivial consistency check shows that our framework is consistent with the literature and self-consistent under the formulation in different gauges. This justifies the application of the formalism to the study of black hole evaporation in the future.

5.2.6 Comparison to Lagrangian Approach

We analysed perturbations around spherically symmetric spacetimes in the Lagrangian and Hamiltonian approach. In the Lagrangian setup, we used a covariant formulation of perturbation theory to derive the covariant form of the Regge-Wheeler-Zerilli wave equation for a master variable. The difference between the odd and even parity perturbations is captured by the potentials acting as an effective mass term for the master variables.

Then, we started from the Hamiltonian formulation of general relativity based on the ADM formulation. We used the Gullstrand-Painlevé gauge fixing to derive a reduced Hamiltonian describing the dynamics of the theory. The result was simplified using several canonical transformations. In the following, we compare the two formalisms by computing the Hamiltonian equations of motion for the reduced Hamiltonian. Finally, we investigate the relation between the master variables in the Hamiltonian and Lagrangian formulation for the pure gravity case.

The relevant second order contribution to the reduced Hamiltonian describing the physics of the perturbations is of the form

$$H = \frac{1}{\kappa}\int \mathrm{d}r\, N^3 P Q' + \frac{N}{2}\Big(P^2 + (Q')^2 + VQ^2\Big)\,. \tag{5.2.265}$$

We will treat all the different fields (even/odd, matter/gravity) simultaneously. The difference between them is the explicit form of the potential function V which needs to be inserted. For simplicity we set $\kappa = 1$.

The Hamilton equations of motion are

$$\dot{Q} = \frac{\delta \tilde{H}}{\delta P} = N^3 Q' + NP\,, \tag{5.2.266}$$

$$\dot{P} = -\frac{\delta \tilde{H}}{\delta Q} = \partial_r(N^3 P) + \partial_r(NQ') - NVQ\,. \tag{5.2.267}$$

The two equations are combined to give a second order partial differential equation for the variable Q. The first equation is used to express P in terms of $\dot{Q}$ and Q'. Then, the second equation becomes

$$-\partial_t\left(\frac{1}{N}\Big(\dot{Q} - N^3 Q'\Big)\right) + \partial_r\left(\frac{N^3}{N}\Big(\dot{Q} - N^3 Q'\Big)\right) + \partial_r(NQ') - NVQ = 0\,. \tag{5.2.268}$$

This equation is not yet in a covariant form using the two dimensional metric g_{ab} on the (t, r) part of spacetime. In GP gauge and in terms of N, N^3 we have

$$g = \begin{pmatrix} -N^2 + (N^3)^2 & N^3 \\ N^3 & 1 \end{pmatrix} \qquad g^{-1} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^3}{N^2} \\ \frac{N^3}{N^2} & 1 - \frac{(N^3)^2}{N^2} \end{pmatrix} \tag{5.2.269}$$

The determinant of the metric is $-N^2$ and we have $\sqrt{-\det(g)} = N$. Dividing the equation of motion for Q by N and introducing components of the metric g, we get

$$\frac{1}{\sqrt{-g}}\partial_t\Big(\sqrt{-g}\Big(g^{tt}\dot{Q} + g^{tr}Q'\Big)\Big) + \frac{1}{\sqrt{-g}}\partial_r\Big(\sqrt{-g}\Big(g^{rt}\dot{Q} + g^{rr}Q'\Big)\Big) = VQ\,. \tag{5.2.270}$$

We recognize the differential operator on the left-hand side as the Laplace operator $\Box = g^{ab}\nabla_a\nabla_b$ associated to g. Thus, the equation reduces to the wave equation

$$\Box Q = VQ\,. \tag{5.2.271}$$

In section 5.1, we derived the wave equation for a master variable $\psi^{e/o}$ in the even and odd parity sectors for pure gravity. In [101], Chandrasekhar generalised the analysis to also include the electromagnetic field working in Schwarzschild coordinates. In the following, we compare the wave equations for the different potentials. First, let us consider the special case of pure gravity and set all the electromagnetic contributions to zero. Then, in the even parity sector we have

$$W^o = 1, \quad U^o = l(l+1) - \frac{3r_s}{2r}, \quad V^o_{\text{grav}} = \frac{1}{r^2}\left(l(l+1) - 3\frac{r_s}{r}\right) \tag{5.2.272}$$

This agrees with the expression for the Regge-Wheeler equation in formula 5.1.41. For the even parity, we have

$$\begin{aligned} W^e &= \frac{1}{\Lambda^2}\left(n^2 + 2n + \frac{3r_s}{2r} - \frac{3r_s^2}{4r^2}\right) \\ V^e_{\text{grav}} &= \frac{1}{r^2}\left[2nW^e + \frac{r_s}{r} - \frac{2n\Delta}{\Lambda}\right] \\ &= \frac{1}{r^2\Lambda^2}\left[2n^3 + 2n^2 + 3n^2\frac{r_s}{r} + \frac{9r_s^2}{2r^2}n + \frac{9r_s^2}{4r^3}\right] \\ &= \frac{1}{4r^2\Lambda^2}\left[\lambda^2(\lambda+2) + 3\lambda^2\frac{r_s}{r} + 9\lambda\frac{r_s^2}{r^2} + 9\frac{r_s^3}{r^3}\right] \end{aligned} \tag{5.2.273}$$

where we used $\lambda = (l+2)(l-1) = 2n$. This potential is the same as the Zerilli potential in the even parity master equation (**5.1.65**). Therefore, in the vacuum case, we have perfect agreement between the equations of motion from the Lagrangian and Hamiltonian framework.

The comparison of the equations of motion including the electromagnetic field is more complicated. In order to show the equivalence of our results with the ones in [101], we have to perform a change of coordinates in the wave equation from Gullstrand-Painlevé to Schwarzschild coordinates. The wave equation in (**5.2.271**) is already in a covariant form and we simply have to express the Laplace operator in the Schwarzschild coordinate system. The metric in Schwarzschild coordinates is given by $g = \text{diag}(-\Delta, \Delta^{-1})$ where Δ is defined in equation (**5.2.141**). In terms of the Schwarzschild coordinates (t, r) we have

$$\Box Q = -\Delta^{-1}\partial_t^2 Q + \partial_r(\Delta\partial_r Q) = \Delta^{-1}\left(-\partial_t^2 + \partial_{r^*}^2\right)Q\,. \tag{5.2.274}$$

The variable r^* is the tortoise coordinate defined by the relation $\Delta\partial_r = \partial_{r^*}$. Then, the wave equation in Schwarzschild coordinates is

$$\left(-\partial_t^2 + \partial_{r^*}^2\right)Q = \Delta VQ\,. \tag{5.2.275}$$

In [101], Chandrasekhar derives the equation

$$\left(-\partial_t^2 + \partial_{r^*}^2\right)Z_i^{(\pm)} = V_i^{(\pm)}Z_i^{(\pm)}\,. \tag{5.2.276}$$

It is a wave equation in Schwarzschild coordinates for the master functions $Z_i^{\pm}$ with the potentials $V_i^{\pm}$. The label ($\pm$) stands for odd ($-$) and even ($+$) parity and $i = 1, 2$ labels the two independent variables within the even and odd parity sector respectively.

The potential in the odd parity sector is

$$V_i^- = \Delta\left[\frac{l(l+1)}{r^2} - \frac{q_j}{r^3}\left(1 + \frac{q_i}{(l-1)(l+2)r}\right)\right] \qquad (i, j = 1, 2, i \neq j)\,, \tag{5.2.277}$$

where q_1 and q_2 are defined by the relations $q_1 + q_2 = 3r_s$ and $-q_1 q_2 = (l+2)(l-1)g^2\xi^2$. Comparing the potential with our solution $V^o_{1/2}$ we find perfect agreement.

For even parity, Chandrasekhar finds the potentials

$$V_1^+ = \frac{\Delta}{r^2}\left[U + \frac{1}{2}(q_1 - q_2)W\right] \tag{5.2.278}$$

$$V_2^+ = \frac{\Delta}{r^2}\left[U - \frac{1}{2}(q_1 - q_2)W\right]\,, \tag{5.2.279}$$

with the same definitions as in the odd parity case for q_1 and q_2. This potential is also exactly the same as $V^e_{1/2}$ of our computations. Therefore for both odd and even parity, the equations of motion of the reduced Hamiltonian match the calculations based on the linearised Einstein equations in the literature.

We conclude this section by relating the master variables of the Hamiltonian and Lagrangian computations. For simplicity, we only study the case of pure gravity for $l \geq 2$ neglecting any electromagnetic contributions. The reduced Hamiltonian generates the dynamics for the gravitational master variables Q^o, Q^e and we would like to know if they agree with the gravitational master variables ψ^o, ψ^e defined in section 5.1.

In the odd parity sector, the variable Q^o is related to the true degrees of freedom (X^o, Y_o) by

$$Q^o = \frac{\sqrt{2}}{r}\int \mathrm{d}r\left(r^2 Y_o + \frac{\pi_\lambda}{4r^2}X^o\right)\,, \tag{5.2.280}$$

In the Lagrangian approach, we defined the master variable

$$\psi^o = r^3\epsilon^{t3}\Big(\partial_t(r^{-2}\tilde{h}_r) - \partial_r(r^{-2}\tilde{h}_t)\Big)\,. \tag{5.2.281}$$

The quantities $\tilde{h}_t$ and $\tilde{h}_r$ are the components of the gauge invariant variable $\tilde{h}_a$ of the odd parity perturbations. In the notation used in the Hamiltonian theory, $\tilde{h}_a$ is related to the metric and angular shift vector by

$$\tilde{h}_r = x^o - \frac{r^2}{\sqrt{2(l+2)(l-1)}}\partial_r(r^{-2}X^e) \tag{5.2.282}$$

$$\tilde{h}_t = h_t^o - \frac{1}{\sqrt{2(l+2)(l-1)}}\partial_t X^e \tag{5.2.283}$$

In the odd parity sector of the Hamiltonian approach only the perturbation of the angular shift vector δN^A is non-vanishing. We expand it into spherical vector harmonics $\delta N^A = \sum_{l,m} r^{-2} h^o_{t,lm} L^A_{o,lm}$. Substituting the gauge invariant variables $\tilde{h}_r$ and $\tilde{h}_t$ in the master variable ψ^o we obtain

$$\psi^o = -r^3\epsilon^{t3}\partial_r(r^{-2}h_t^o)\,. \tag{5.2.284}$$

The stability of the gauge fixing condition $x^o = 0$ fixes the shift vector h_t^o. We have the stability condition

$$\dot{x}^o = \left\{x^o, N^{(2)}C_v + N^{3(2)}C_h + r^{-2}h_t^o \cdot Z^o\right\}_{x^o=0, y_o=y_o^{(1)}, N=1, N^r=\sqrt{r_s/r}} \tag{5.2.285}$$

$$= y_o^{(1)} + r^2\partial_r(r^{-2}h_t)\,. \tag{5.2.286}$$

After calculating the Poisson bracket, we impose the gauge fixing $x^o = 0$ and the solution of the first order constraints $y_o = y_o^{(1)}$. Using the stability condition, the odd parity master variable becomes

$$\psi^o = r\epsilon^{t3}y_o^{(1)}\,. \tag{5.2.287}$$

Inserting the solution of the first order constraints for y_o, we obtain

$$\psi^o = \epsilon^{t3}\frac{\sqrt{2(l+2)(l-1)}}{r}\int \mathrm{d}r\left(r^2Y_o + \frac{\pi_\lambda}{4r^2}X^o\right) \tag{5.2.288}$$

The expression for the Levi-Civita pseudotensor is $\epsilon^{t3} = 1$ and we see that ψ^o agrees with Q^o up to an l-dependent prefactor.

In the even parity case, we arrived at the Zerilli wave equation, satisfied by the master variable ψ^e in the Lagrangian approach and Q^e in the Hamiltonian formulation. We start with the variable ψ^e and show that it corresponds to the variable Q^e up to a constant prefactor. ψ^e is defined as

$$\psi^e = \frac{1}{l(l+1)}\left(\gamma K + \frac{1}{\Lambda}\left(\gamma\gamma^a\gamma^b k_{ab} - \gamma^2\gamma^a\nabla_a K\right)\right). \tag{5.2.289}$$

K and k_{ab} are gauge invariant variables of the even parity sector. In section 5.1, we related them to the metric perturbations as

$$K = r^{-2}x^h - \frac{2}{\sqrt{l(l+1)}r}\gamma^a h_a^e\,, \tag{5.2.290}$$

$$k_{ab} = h_{ab} - \frac{1}{\sqrt{l(l+1)}}(\nabla_a h_b^e + \nabla_b h_a^e)\,. \tag{5.2.291}$$

In the gauge we chose in the Hamiltonian formulation, we set $h_3^e = 0$ and $h^e = 0$. The perturbations of lapse function and shift vector are related to h_{tt}, h_{t3} and h_t^e. Here, we use the notation $N + \sum_{lm}\delta N_{lm}L_{lm}$ for the lapse function and $N^3 + \sum_{lm}\delta N_{lm}^3 L_{lm}$ and $\sum_{lm}\delta N_{lm}^e[L_{e,lm}]^A$ for the shift vector. The metric perturbations are then given by

$$\begin{aligned} h_{tt} &= -2N\delta N + x^v(N^3)^2 + 2N^3\delta N^3\,,\\ h_{t3} &= \delta N^3 + x^v N^3\,,\\ h_{33} &= x^v\,,\\ h_t^e &= r^2\delta N^e\,,\\ h_3^e &= 0\,. \end{aligned} \tag{5.2.292}$$

For ψ^e, we also have to know the vector γ^a which is defined as the derivative of γ. In the GP gauge, $\gamma = r$ and we have $\gamma_a = \partial_a r = (0, 1)$. We raise the index with the inverse metric and get

$$\gamma^t = \sqrt{\frac{r_s}{r}}, \quad \gamma^3 = 1 - \frac{r_s}{r}\,. \tag{5.2.293}$$

Using the results for γ^a, K and k_{ab} the even parity master variable reads

$$\psi^e = \frac{r}{l(l+1)((l+2)(l-1)r+3r_s)}\Big[r^{-2}((l+2)(l-1)r+3r_s)x^h + 2r\gamma^a\gamma^b h_{ab} - 2\sqrt{\frac{r_s}{r}}\partial_t x^h \tag{5.2.294}$$

$$-2\Big(1-\frac{r_s}{r}\Big)\partial_r x^h + \frac{4}{r}\Big(1-\frac{r_s}{r}\Big)x^h + \frac{1}{\sqrt{l(l+1)}}\Big(4r^2\gamma^b\nabla_b(r^{-1}\gamma^a)h^e_a - 4\Lambda\gamma^a h^e_a\Big)\Big]\,. \tag{5.2.295}$$

For the last term involving h^e_a, we have to evaluate the second covariant derivative of γ. In the calculations for the derivation of the Schwarzschild metric, we showed that

$$\nabla_a\nabla_b\gamma = \frac{g_{ab}}{2r^2}r_s\,. \tag{5.2.296}$$

We substitute this result into ψ^e and after some simplification, we find

$$\psi^e = \frac{r}{l(l+1)((l+2)(l-1)r+3r_s)}\Big[r^{-2}\Big((l^2+l+2)r - r_s\Big)x^h + 2r\gamma^a\gamma^b h_{ab} - 2\sqrt{\frac{r_s}{r}}\partial_t x^h \tag{5.2.297}$$

$$-2\Big(1-\frac{r_s}{r}\Big)\partial_r x^h + \frac{4}{r}\Big(1-\frac{r_s}{r}\Big)x^h - 2\sqrt{l(l+1)}\gamma^a h^e_a\Big]\,. \tag{5.2.298}$$

The time derivative of x^h is determined by the Hamilton equations of motion. We obtain them by calculating the Poisson bracket of x^h with the Hamiltonian:

$$\dot{x}^h = \Big\{x^h, \delta N\cdot{}^{(1)}Z^v + \delta N^3\cdot{}^{(1)}Z^h + r^{-2}h^e_t\cdot{}^{(1)}Z^e + N^{(2)}C_v + N^{3(2)}C_h\Big\} \tag{5.2.299}$$

$$= -\frac{1}{2}\pi_\mu\delta N + 2r\delta N^3 - \sqrt{l(l+1)}h^e_t + N^3\partial_r x^h - Ny^v - \frac{1}{4}\pi_\mu x^v\,. \tag{5.2.300}$$

To finish the calculation we have to expand $\gamma^a\gamma^b h_{ab}$ in terms of the explicit expression for γ^a. We have

$$\gamma^a\gamma^b h_{ab} = x^v + \frac{1}{2r}\pi_\mu\delta N^3 - \frac{1}{8r^2}\pi_\mu^2\delta N\,. \tag{5.2.301}$$

Inserting $\dot{x}^h$ and $\gamma^a\gamma^b h_{ab}$ into ψ^e, we find

$$\psi^e = \frac{1}{l(l+1)((l+2)(l-1)r+3r_s)}\Big[\Big(\Big(l^2+l+2\Big) - \frac{r_s}{r}\Big)x^h - 2r\partial_r x^h + \frac{1}{2}\pi_\mu y^v + \frac{1}{8}\Big(\pi_\mu^2 + 16r^2\Big)x^v\Big]\,. \tag{5.2.302}$$

After carefully studying the canonical transformations of the even parity sector, we notice that up to an l-dependent factor, ψ^e is equal to Q^e. Hence, in both the even and odd parity sectors the wave equations and the master variables agree in the Hamiltonian and Lagrangian approach.

5.3 Summary

In the previous chapters, we discussed black hole perturbation theory from the Lagrangian and Hamiltonian perspectives. First, we reviewed the linear perturbation theory based on the Einstein equations in modern notation based on [74, 125]. The treatment separates into the odd and even parity sectors, which can be treated independently. In each sector, we found two master equations which are satisfied by two master variables constructed from the perturbations.

Then, we applied the canonical framework developed in section 2 to general relativity in the Hamiltonian formulation based on ADM variables. We distinguished the canonical variables into symmetric and non-symmetric variables as well as true and gauge degrees of freedom. Then, the reduced phase space is coordinatized by the true symmetric and true non-symmetric degrees of freedom. After constructing the reduced phase space by solving the constraints up to second order, we performed several canonical transformations. The result is a reduced Hamiltonian describing the dynamics of the reduced phase space.

The treatment of Hamiltonian black hole perturbation theory is well established in the literature. However, in order to successfully apply it to the case of black hole evaporation with backreaction, we had to take some non-trivial steps. The advantages of the formulation used above are as follows:

First of all, we used a different formulation for black hole perturbation theory that has not been used in the literature before. In the standard approach, one expands the constraints to second order in the perturbations and then defines gauge invariant variables that commute with the constraints. These gauge invariant variables are only defined up to second order in perturbation theory. When moving to higher orders, one needs to repeat the analysis order by order and define third order gauge invariants, fourth order gauge invariants and so on. On top of that beyond second order, there is no consensus in the literature on how to construct the higher order gauge invariants. In our formulation, we first define the gauge invariants to all orders and then construct the reduced phase space perturbatively. The gauge-invariant (true) degrees of freedom are completely fixed to all orders.

In other treatments, backreaction is not included because the spherically symmetric sector is treated as independent of the non-symmetric degrees of freedom. In the treatment by Moncrief [75], the background is assumed to be the Schwarzschild solution and the perturbations are propagating on this background without affecting it. Later, Brizueal and Martín-García [80, 81] generalized the Hamiltonian analysis to general spherically symmetric backgrounds without backreaction. In the formulation used in this thesis all the backreaction effects are included. For only Maxwell matter the backreaction is trivial because the true symmetric degrees of freedom are integration constants. Including different matter fields (e.g. scalar field, fermions), the backreaction becomes non-trivial.

The GP gauge condition used in the above computations was chosen because it covers both the interior and exterior of the black hole spacetime. This is important for a complete picture of evaporating black holes. For example adding a scalar field, we can investigate the dynamical formation of a black hole and its subsequent evaporation. For this to make sense, we have to track the matter that is inside the black hole. This dynamics of the matter inside the black hole could be important for our understanding of the fate of evaporating black holes.

The computations above can also be regarded as a consistency check of the formalism when applied gravity coupled to Maxwell matter. This consistency check was non-trivial as it involved several non-standard steps. Our formulation applied gauge fixing, while other approaches use gauge invariant variables. We had to translate the master variables between the two pictures in order to verify our results. Additionally, there were several non-trivial canonical transformations involved for the derivation of the reduced Hamiltonian. This is a strong indication that the formalism can be trusted and that an extension to all orders and with more matter content with backreaction is meaningful. Thus, the analysis in this thesis provides the start for future investigations on black holes.

6

Physics of Evaporating Black Holes and Experiments

In section 4.2, we showed that a Schwarzschild black hole emits radiation whose spectrum follows that of a black body with characteristic temperature T_H. We also argued that due to this effect black holes should lose mass and thus evaporate after a finite lifetime which scales as M^3. The argument was very heuristic because in the calculation of the Hawking effect, we assumed the spacetime to remain unchanged. While for very large black holes this seems like a reasonable assumption, for tiny black holes that already evaporated most of their mass this is no longer true.

In this section, we would like to describe how we plan to obtain a more in-depth analysis of Hawking evaporation including backreaction effects. In the first section, we introduce the quasi-local mass as the square root of the area of the apparent horizon and sketch a way to obtain a perturbative expansion of it. Then, we give a brief outlook into the challenges we will have to face when formulating the quantum theory. We conclude with a short discussion of current and future experimental searches for evaporating black holes.

6.1 Backreaction using the Quasi-Local Mass

In section 3.7, we introduced the notion of trapped surfaces and apparent horizons. We argued that the apparent horizon is associated to an observer and that it is the boundary of the black hole as seen by this observer. The area of the apparent horizon is an interesting observable for the study of Hawking evaporation. In the classical theory, we expect the time derivative of the area to be positive in analogy to the black hole area theorem. Going to the quantum theory, the change of the area is not necessarily positive due to black hole evaporation. In quantum field theory, classical energy inequalities, needed for the proof of the area theorem, are not satisfied in general. The backreaction effect would then manifest itself in a decrease of the area of the apparent horizon.

Equivalently to working with the area of the apparent horizon, we can take its square root and define the notion of a quasi-local mass:

Definition 6.1: Given a foliation $\mathcal{F}$ of a globally hyperbolic spacetime $(\mathcal{M}, g)$ by Cauchy surfaces Σ_τ the quasi-local mass at time τ is defined as

$$[M_0]^2 := \frac{\mathrm{Ar}[A_\tau]}{16\pi} \tag{6.1.1}$$

where A_τ is the area of the apparent horizon at time τ.

In [98], it was shown that the shape of the apparent horizon can be computed perturbatively to any order in perturbation theory. Since the rest of the discussion in this thesis is in the Gullstrand-Painlevé

(GP) gauge, we will work in the same gauge here as well. The strategy is as follows: Since the apparent horizon for Schwarzschild black holes is a sphere where the radius is given by the Schwarzschild radius r_s, we assume the perturbed apparent horizon to be of spherical topology. The apparent horizon is a codimension 1 hypersurface and we choose the embedding $Y : \mathbb{R} \times S^2 \to M$ with $Y(\tau, \cdot) : S^2 \to \Sigma_\tau$ to describe it. We assume the embedding to be of the form

$$Y^\tau(\tau, y) = \tau, \quad Y^3(\tau, y) = \rho(\tau, y), \quad Y^A(\tau, y) = y^A\,. \tag{6.1.2}$$

The surface is parametrized by a function $\rho(\tau, y)$. This function is called the **radial profile** and defines the shape of the surface. Then, for each τ we compute the normal s of the surfaces $S_\tau := Y(\tau, S^2)$ tangent to Σ_τ. The radial profile is then determined by evaluating (**3.7.8**) with the normal s, the induced metric m and the conjugate momentum W.

The solution of (**3.7.8**) is calculated using a series expansion of ρ in terms of the perturbations, i.e. $\rho = \rho^{(0)} + \rho^{(1)} + \rho^{(2)} + \ldots$ where $\rho^{(i)}$ is of i-th order in the perturbations. By construction, the zeroth order contribution is just $\rho^{(0)} = r_s$ and then the solution for ρ is constructed order by order. [98] shows that such a perturbative solution can by constructed in an iterative way.

Pulling back the induced metric m onto the surfaces S_τ, we define the area of the apparent horizon. The area will be available in terms of a series in the perturbations $A = A^{(0)} + A^{(2)} + \ldots$. Note that the first order contributions to A vanish because the calculation of the area involves an integral over the sphere and the perturbations are assumed not to contain any symmetric contributions. The zeroth order contribution is just the area of a sphere with the Schwarzschild radius, i.e. $A = 4\pi r_s^2$. Using the definition of the quasi-local mass we find as expected $M_0 = M$, where M is the black hole mass. Therefore, the normalization factor of 16π in the definition of the quasi-local mass was chosen correctly. In the quadratic approximation, the perturbations will lead to quadratic corrections to the quasi-local mass.

After deriving an explicit expression for the quasi-local mass to second order, we quantise it using a Fock representation. This representation is motivated by the shape of the reduced Hamiltonian which we derived in the previous chapter to second order. For a well-defined quantum operator associated to the quasi-local mass, we have to normal order it. In order to construct a rigorous quantum theory for the perturbations, we have to address the following challenges:

The mode functions for the even parity perturbations are solutions of the Regge-Wheeler equation which reads

$$\Box\psi_{l,s} - \frac{l(l+1)r + (1-s^2)r_s}{r^3}\psi_{l,s} = 0\,, \tag{6.1.3}$$

where s corresponds to the spin of the matter field, i.e. $s = 0$ for scalar fields, $s = 1$ for electromagnetic fields and $s = 2$ for the gravitational field. Assuming the modes to have the time dependence $e^{i\omega\tau}$, the equation reduces to an ordinary, second order, linear differential equation. In the mathematics literature a lot is known about these equations and their solutions (see [128]). In [129, 130], the authors show that the solution of the differential equation can be written in terms of confluent Heun functions ([131, 132]). The solution of the radial differential equation are known locally as power series around the points $r = 0$, $2M$ and ∞. However, for a full knowledge of the mode, one has to know the global solution of the Heun equation from $r = 0$ to $r = \infty$. This requires the introduction of appropriate matching conditions of the power series solutions.

Once we have the mode functions under control, we have to address the issue that the constant GP time hypersurfaces in GP coordinates are not Cauchy surfaces. Therefore, the mode functions will not be complete and it is impossible to expand an arbitrary field into the modes. A possible solution to this is outlined in appendix C of [98]. The GP coordinates cover either the ingoing or outgoing Eddington-Finkelstein coordinates. The idea is to glue a white hole spacetime covered by the outgoing GP coordinates to a black hole spacetime covered by the ingoing GP coordinates along the black hole and white hole singularities. In the resulting black hole – white hole spacetime, the constant GP time hypersurfaces are extended through the singularity and are Cauchy surfaces. Defining the mode solutions of the Regge-Wheeler equation on such a Cauchy surface has the potential to be a complete mode system.

6.2 Experimental Tests for Evaporating Black Holes

In this section, we briefly outline the current and future searches for evaporating black holes using astro- and astroparticle experiments. In section 4.2, we briefly sketched that from semi-classical arguments, the spectrum of the emitted Hawking quanta is obtained. However, due to the small Hawking temperature for black holes with a large mass, the radiation is very weak. A more promising source of Hawking radiation are primordial black holes [133–136] that already evaporated away most of their mass and are in the final stage of the evaporation process. These black holes are expected to emit a significant amount of radiation that could be detected in experiments. For these black holes, the assumptions on which the semi-classical backreaction is based are not fully justified and as the black hole mass approaches the Planck mass the description breaks down. In this regime, which is very important for the detection of Hawking radiation, new tools based on quantum gravity need to be developed.

The observational signature highly depends on the fate of evaporating black holes and there are several conflicting proposals in the literature: The black hole could stop evaporating and form a stable remnant [136–139], the black hole could explode in a burst or radiation [140], or it could tunnel from a black hole to a white hole [50, 51, 141]. For observations the proposal of exploding black holes is very interesting because this will lead to stronger signals. The detection of these signals originating from primordial black holes can be used to directly get an estimate for the abundance of primordial black holes. This is of interest because these black holes are a candidate for dark matter in the universe.

The most promising messengers for detecting bursts of radiation from evaporating black holes are neutrinos and gamma rays [61]. For gamma rays, the Fermi Large Area Telescope (Fermi-LAT), a space-based instrument provides bounds on the abundance of evaporating primordial black holes [142, 143]. For ground based detectors, one uses the fact, that gamma rays produce a shower of charged particles in earth's atmosphere. These particles are highly energetic and move faster than the speed of light in air and Cherenkov light is emitted which can be captured by telescopes. The H.E.S.S collaboration is looking for signs of evaporating black holes [144]. With future experiments like the Cherenkov Telescope Array (cta) more data will be available.

In a similar way, neutrino observations put constraints on primordial black holes [145, 146]. For example, the IceCube experiment located at the South Pole in the antarctic ice measures neutrinos through their interaction with water in the ice [147]. They establish bounds on the abundance of primordial black holes. Future neutrino detectors, such as km3Net are expected to provide more data that will improve previous results.

The above, non-exhaustive list of experiments shows that Hawking radiation from primordial black holes is a very active field of research. However, from a theoretical point of view more research in quantum gravity has to be conducted in order to interpret the data of the experiments and to obtain constraints on quantum theories of gravity. Then, the data of current and future experiments might be able to constrain theories of quantum gravity and could help to solve the mystery surrounding the fate of evaporating black holes.

The thesis and the formalism developed in the previous sections aims for addressing these questions. As explained in section 6.1, we define a quantum operator associated to the quasi-local mass. In the quantum theory, we expect the black hole to lose mass and thus the quasi-local mass is expected to decrease with time. This decrease will give a direct insight into the power emitted during the evaporation process.

In the hybrid quantization approach, we go beyond what we discussed in section 6.1 and combine the Fock quantization of the perturbations with a non-perturbative quantisation of the spherically symmetric degrees of freedom. This was first successfully applied in the context of cosmology in the seminal work [84] and then generalized to many other models [85–92]. However, in our situation, there cannot be any backreaction on the black hole mass M because we do not have a Dirac observable conjugate to it. We are left with two options (see [98]): (i) We extend the analysis by introducing a Dirac observable conjugate to M. (ii) We interpret the quasi-local mass M_0 as an extension of the zero mode M to all the modes coming from the perturbations. For additional matter fields such as a scalar field or fermions, we would have explicit backreaction and can systematically compute the backreaction using space adiabatic perturbation theory [148], a generalization of the usual Born-Oppenheimer approximation of molecular physics.

For realistic theory of black holes we have to include additional matter fields. In [98], a scalar field was studied next to the gravitational and electromagnetic degrees of freedom. This scalar field can be interpreted as the Higgs field in the Standard model of particle physics. Using the scalar field we plan to explore realistic black hole formation scenarios through gravitational collapse. Furthermore, we can get insights into the evolution of primordial boson stars. In the future, additional matter fields such as fermions (e.g. neutrinos), which are also important messengers of Hawking radiation, can be added.

7

Conclusion

In this thesis, we discussed a Hamiltonian formulation for theories with constraints (redundancies) that has a direct application to the Hamiltonian formulation of general relativity in terms of ADM variables. For future applications to Hawking radiation and black hole evaporation, we initiated a (perturbative) quantum gravity computation. Non-rotating black holes (Schwarzschild black holes) are spherically symmetric and we treated the spherically symmetric degrees of freedom exactly, whereas the non-symmetric variables are handled perturbatively. The new idea of our approach is to split the degrees of freedom into four sets: We distinguish between the symmetric and non-symmetric degrees of freedom, as well as between redundant and observable variables. The splitting of the variables is performed non-perturbatively on the full phase space of general relativity coupled to matter fields. Then, by fixing the values of the redundant degrees of freedom, we obtain the reduced phase space coordinatized by the observable degrees of freedom. The dynamics of the reduced phase space is determined by the physical Hamiltonian which is given by the evaluation of a boundary term at infinity. In order that the way we fix the redundancy is preserved under time evolution, certain extra stability conditions have to be satisfied. We successfully solved these extra conditions asymptotically at infinity, which is enough to find the non-perturbative expression for the physical Hamiltonian valid to all orders. A non-trivial consistency check demonstrated the validity of our formalism.

7.1 Summary of the Results

Black holes in the universe are uniquely characterised by charge, mass and angular momentum. Semi-classical calculations suggest that due to Hawking radiation angular momentum is radiated away faster than mass. Hence, we restricted to non-rotating black holes and we assume spherical symmetry for the symmetric degrees of freedom. Solving the constraints to first and second order, we explicitly calculated the reduced phase space for both background and perturbations. After suitable canonical transformations, we obtained a tractable physical Hamiltonian to second order in the perturbations. As a consistency check, we computed the Hamilton equations of motion, changed to Schwarzschild coordinates and neglected backreaction. In this common domain of validity, we recover the Regge-Wheeler-Zerilli equations [72, 73, 101] for the perturbations on a black hole spacetime. Therefore, in this limit our formalism agrees with known results in the literature but it goes beyond the Regge-Wheeler-Zerilli equations and Moncrief's Hamiltonian analysis [75] by including backreaction and by working in Gullstrand-Painlevé coordinates.

The real virtue of our approach is the disentangling between the definition of the observables and perturbation theory. In this way, the observables are defined non-perturbatively to all orders of perturbation theory. There is no need to perturbatively resolve the redundancy in the description order by order, which is not well understood in the literature beyond second order (see [71]). The observables are free of redundancies without any approximation and not just up to higher orders in perturbation theory. This makes the formalism very transparent and an extension to higher orders in perturbation theory is possible without technical issues.

In this manuscript, we fully solve all of the constraints on the classical level and the full dynamics are given by the physical Hamiltonian. In contrast to the Hamiltonian in the ADM formulation of general relativity this Hamiltonian generates physical time evolution for the observables. The absence of constraints is also an advantage for the construction of the quantum theory. In the presence of more than one constraint in the classical theory, Poisson brackets between constraints give a linear combination of constraints only up to higher order terms. This translates into the quantum theory, where constraints become constraint operators and their commutator will pick up anomalous terms. These anomalies make the construction of a well-defined quantum theory considerably more complicated.

By construction, our formalism includes backreaction effects. Instead of fixing the spherically symmetric spacetime to the Schwarzschild geometry, like black hole perturbation theory is usually studied in the literature, we left it completely arbitrary and did not remove the interactions between the background and the perturbations. Working on the full phase space of general relativity coupled to matter, we are guaranteed to include all backreaction effects. This is crucial for the application of our approach to black hole evaporation, where the Hawking radiation (perturbations) influences the black hole mass (background). In the present thesis, we only investigated the Einstein-Maxwell sector and observed that backreaction is quite limited because the only spherically symmetric observables are mass and charge of the black hole. The full power of the formalism will unfold when using more general matter for which the symmetric sector is still a (quantum) field theory as exemplified in [98] for (charged) scalar fields.

For fixing the redundancies in the Hamiltonian description, we chose the Gullstrand-Painlevé coordinates that allows the treatment of both the interior and exterior of the black hole spacetime. The GP gauge is associated to an observer freely falling from infinity, which is a good approximation for an observatory around a black hole in free-fall. The GP coordinates are used to describe either a black hole or a white hole spacetime. Glueing the white hole singularity to the black hole singularity, we obtain a black hole – white hole transition scenario (see [98]). In this glued spacetime, the constant GP time surfaces are Cauchy surfaces. A Cauchy surface is a surface in spacetime such that the physics in all of spacetime can be reconstructed from initial data on it. Then, given a complete set of modes, i.e. elementary solutions of the wave equation, on the Cauchy surface, we can uniquely expand any quantum field in terms of the modes. It could be possible to extend the quantum field theory across the singularity from the black hole to the white hole region without technical issues. Then, the singularity would be perfectly regular in the quantum theory without using quantum effects for the symmetric observables.

In the case that such an extension is not possible, we have to apply methods from non-perturbative quantum gravity to the symmetric sector in order to cure the singularity. In this case, there will be a quantum transition region around the singularity where the black hole and white hole spacetimes are glued together. Alternatively, adding a scalar field such as the Higgs to the theory, we can use methods developed for the study of Oppenheimer Snyder collapse toy models.

After establishing the classical Hamiltonian theory, we mentioned some physical applications of the theory in future investigations. The notion of event horizon is very global and not suitable for the study of evaporating black holes because we do not expect an event horizon in spacetimes with evaporating black holes. A better notion are apparent horizons, that correspond to the boundary of the trapped region an observer would measure. The observer dependence of the apparent horizon is not a disadvantage because in a realistic scenario, we will observe the black hole while freely-falling in a black hole spacetime. For instance, an observer on earth will not have any experimental access to event horizons in the universe but can only probe spacetime locally. In section 6, we sketched how we plan to determine a first principle derivation of the quasi-local mass loss. The present thesis will provide the necessary preparations to perform the computations and relate the theory to experimental data. In the end, we also presented some recent experimental searches for burst events of evaporating black holes. In current and future gamma ray and neutrino observatories such events are searched and used to constrain the abundance of primordial black holes.

In summary, in this thesis, we initiated a new approach for the study of quantum backreaction effects in general relativity with a special focus on black hole evaporation. By considering classical perturbation theory in the Hamiltonian framework, we provided the basis for future explorations using canonical quantization techniques. The key novelties of the framework are the clear separation between the definition of observables and perturbation theory as well as the inclusion of backreaction, keeping the spherically symmetric sector dynamical.

7.2 Future Research

The present work offers many opportunities for future research. In the following, we provide a non-exhaustive list of ideas for future projects, applications and extensions of the formalism:

In the computations, we only considered the electromagnetic field as matter and **additional matter fields** such as scalar fields or fermionic particles of the Standard Model are of interest. In order to provide the full spectrum of emitted particles, we have to take all matter fields in the Standard Model of particle physics into account. With regards to current and future astrophysics experiments, neutrinos, photons, and gravitational waves are important messengers [133–136]. For electromagnetic matter, we found that the physical Hamiltonian contains a mass term which only depends on the black hole mass M and the electric charge ξ. In the presence of additional matter fields, we will have more spherically symmetric, observable degrees of freedom. This will lead to additional contributions in terms of the new spherically symmetric degrees of freedom to the mass term (backreaction). For the observation of Hawking radiation, secondary effects from interactions of the emitted particles need to be investigated based on calculations using the Standard Model of particle physics .

In our formulation of perturbation theory in general relativity, we defined the non-perturbative gauge invariants and obtained an implicit expression for the reduced Hamiltonian to all orders. Only in a second step, we explicitly computed the physical Hamiltonian to second order to find the approximate dynamics. In the future, we plan to extend the calculations to **higher orders**. To second order, the perturbations can be considered as free fields with a background-dependent mass term. Higher order corrections will lead to interactions of the perturbations and the interaction of gravitational waves.

In addition, we can extend the formalism to rotating black holes and analyse perturbations around black holes of the Kerr-Newman type. For this, we have to generalise the symmetry group from spherical to axial symmetry. The study of **rotating black holes** is important for a complete understanding of astrophysical black holes forming from stellar collapse. The theory of perturbations around rotating black holes was initiated by Teukolsky in [109, 110]. Rotating black holes will introduce new effects due to the more complex horizon structure. Furthermore, the frame-dragging effect pulls matter close to the black hole horizon in the direction of rotation [105, 106].

The reduced Hamiltonian was found by the careful analysis of the boundary structure of the phase space of general relativity. Essentially it is given by the ADM mass which allows for a systematic expansion with respect to the perturbations to any desired order. In general relativity, one finds a rich structure of boundary degrees of freedom. It is encoded by the **Bondi - Metzner - Sachs (BMS) group** [149–152], an asymptotic symmetry group of general relativity, which is a generalization of the Poincaré group. It gives insights into the radiation at null infinity and we could learn further details about Hawking radiation. In this thesis, the analysis was performed at spacelike infinity and we need to extend it to null infinity [153]. Recently, boundary degrees of freedom gained new interest as a way to study quantum theories of gravity [154, 155].

After establishing the classical theory for the perturbations, the next step is the construction of the **quantum theory**. There are several options for this: The easiest case is to apply a Fock quantisation to the black hole perturbations. In this approach, we have to figure out if the mode functions are well-defined at the horizon and at the singularity. In section 6.1, we briefly discussed some challenges that we have to face when constructing a complete mode system. Only if the mode system is complete, we can uniquely expand any quantum field in terms of modes. In case of the Regge-Wheeler equation, modes are potentially singular at $r = 0$ (signularity), $r = 2M$ (horizon) and at infinity. Hence, it might be necessary to cure and resolve the singularities using suitable regularizations.

A possible way to resolve the singularities is the hybrid approach to quantized perturbed systems in general relativity [84]. The symmetric observables are treated non-perturbatively, while the non-symmetric, true degrees of freedom are quantized using a Fock representation. The non-perturbative quantization for the background is motivated by the non-polynomial nature of the gravitational interaction which is still present in the symmetric sector. The perturbative Fock representation for the perturbations is suggested because truncating the perturbation theory at second order leads to a Hamiltonian which is quadratic in the fields.

The hybrid approach was first successfully used in the cosmological setting [85–92]. In cosmology, the symmetric degrees of freedom are assumed to be homogeneous (independent of the point in spacetime) and isotropic (independent of direction). The results from cosmology can be taken over to the study of black holes by realizing that the interior spacetime of the Schwarzschild black hole can be rewritten in terms of a Kantowski-Sachs cosmology. Similarly, a non-perturbative treatment of dust collapse models leads to a resolution of the singularity inside the black hole. The non-perturbative treatment of the spherically symmetric sector typically leads to the resolution of the singularity inside black holes.

Additionally, a recently found orthonormal basis for singular Schrödinger operators can be used [156, 157]. The elements of this basis have the property that they vanish both at infinity and at 0 such that the operators x and x^{-1} are well defined. Therefore, in terms of this basis, the Coulomb potential for the hydrogen atom, which is proportional to x^{-1} is well-defined at $x = 0$. This could also be useful for the singularity inside the Schwarzschild black hole.

For the derivation of the physical Hamiltonian, we asymptotically evaluated the stability condition of the gauge fixing, guaranteeing that our way to fix the redundancy is preserved under time evolution. In the classical theory, we also solve the consistency conditions perturbatively in the interior of spacetime to some order in the perturbations. Working to second order, we determine lapse and shift in terms of the true degrees of freedom and thus express the full spacetime metric in terms of the observables. Then, we can quantize the expression for the full metric to study its expectation values and fluctuations in the quantum theory. We call the expectation value of the metric the **effective metric** and we plan to investigate its causal structure, i.e. the causal relations between events in spacetime. In general relativity, the causal structure is often visualized in two dimensional pictures, called Penrose diagrams. Comparing the Penrose diagram of classical black holes and of the effective metric, we intend to find changes due to quantum effects. These modifications could be important hints for describing the late stage of black hole evaporation and the result of this process.

Finally, we plan to find **templates** for the experimental investigation of black hole evaporation. Using the notion of quasi-local mass, we aim to compute the spectrum of Hawking radiation in the late stage of the evaporation process. Based on these results, we plan to provide information about the regimes where we expect observable amounts of radiation. This is important for analysing data of current detectors and for designing future experiments. We would like to find out, whether black holes explode in a burst of radiation at the end of their lifetime and if so, we would like to derive templates for these events.

A

Boundary Terms in the Even Parity Computations in Gullstrand-Painlevé Gauge

In the computation for the even parity contributions with $l \geq 2$ to $\pi_\mu^{(2)}$, we solved the first order constraints in an alternative gauge $X^e = x^e = p_2 = 0$ avoiding any differential equations. The solution for $\pi_\mu^{(2)}$ was then simplified with the help of three canonical transformations and an integration by parts. Using the symbolic computation features of Mathematica we determined the corresponding boundary term that was dropped in the main text:

$$
\begin{aligned}
&\frac{-3g^2\xi^2+4r^2+12rr_s}{8r^3}(q_2')^2-\frac{\left(l^2+l+2\right)r-r_s}{r^3}q_2'q_2+\left(-\frac{g^2\xi^2}{2r^2}+\frac{2r_s}{r}+2\right)q_1q_2'+-\frac{3\sqrt{4rr_s-g^2\xi^2}}{2r^2}p_1q_2'\\
&-\frac{\left(\left(l^2+l+2\right)r-r_s\right)\left(g^2\xi^2+12r^2-4rr_s\right)}{2r^2\left(-g^2\xi^2+4r^2+4rr_s\right)}q_1q_2-\frac{g^2\sqrt{l(l+1)}\xi}{2r^3}q_1A\\
&+\frac{2\left(\left(l^2+l+2\right)r-r_s\right)\sqrt{4rr_s-g^2\xi^2}\left(-g^2\xi^2+12r^2+4rr_s\right)}{r^2\left(-g^2\xi^2+4r^2+4rr_s\right)^2}p_1q_2\\
&+\frac{\sqrt{4rr_s-g^2\xi^2}\left(g^2\xi^2-4r\left(r_s+3r\right)\right)}{r\left(4r\left(r_s+r\right)-g^2\xi^2\right)}p_1q_1+\frac{g^2\xi\left(g^2\xi^2+\left(l^2+l+2\right)r^2-3rr_s\right)}{\sqrt{l}\sqrt{l+1}r\left(-g^2\xi^2+\left(l^2+l-2\right)r^2+3rr_s\right)}A^eq_1\\
&+\left(\frac{3}{2r}-\frac{32r^3}{\left(g^2\xi^2-4r\left(r_s+r\right)\right)^2}\right)(p_1)^2\\
&-\frac{1}{8r^5\left(-g^2\xi^2+4r^2+4rr_s\right)^2}\Big[-g^6l(l+1)\xi^6+g^4\left(11l^2+11l+2\right)\xi^4rr_s+8l(l+1)r^4\left(2\left(l^2+l+12\right)r_s^2-g^2\left(l^2+l+8\right)\xi^2\right)\\
&\qquad+8r^3\left(\left(6l^2+6l+4\right)r_s^3-g^2\left(l^4+2l^3+14l^2+13l+2\right)\xi^2r_s\right)+r^2\left(g^4l\left(l^3+2l^2+15l+14\right)\xi^4-8g^2\left(5l^2+5l+2\right)\xi^2r_s^2\right)\\
&\qquad-16\left(3l^4+6l^3+13l^2+10l+16\right)r^6+16\left(2l^4+4l^3+25l^2+23l+18\right)r^5r_s\Big](q_2)^2\\
&+\frac{1}{2l(l+1)r\left(-g^2\xi^2+\left(l^2+l-2\right)r^2+3rr_s\right)^2}\Big[-g^6\xi^6+2g^4\xi^4r\left(2l(l+1)r+5r_s+r\right)\\
&\qquad-g^2\xi^2r^2\left(2(l(l+1)(4l(l+1)-9)-2)r^2+(31l(l+1)+14)rr_s+30r_s^2\right)\\
&\qquad+r^3\left((l-1)(l+2)(l(l+1)(3l(l+1)-4)+4)r^3+(l-1)(l+2)(29l(l+1)+10)r^2r_s+3(19l(l+1)+10)rr_s^2+27r_s^3\right)\Big](q_1)^2\\
&+\frac{g^6\xi^4+4g^4\xi^2r\left(r-r_s\right)}{8r^3\left(-g^2\xi^2+\left(l^2+l-2\right)r^2+3rr_s\right)}(A^e)^2+\frac{g^6\xi^5-g^4\xi^3r\left(2\left(l^2+l-4\right)r+3r_s\right)+4g^2\left(l^2+l-5\right)\xi r^3r_s}{\sqrt{l^2+l-2}r^2\left(4r\left(r_s+r\right)-g^2\xi^2\right)\left(-g^2\xi^2+\left(l^2+l-2\right)r^2+3rr_s\right)}Q^eA^e\\
&+\frac{1}{8(l-1)(l+2)r^3\left(4rr_s-g^2\xi^2\right)\left(g^2\xi^2-4r\left(r_s+r\right)\right)^2\left(-g^2\xi^2+\left(l^2+l-2\right)r^2+3rr_s\right)}\Big[g^{10}(6-5l(l+1))\xi^{10}\\
&\qquad+g^8(85l(l+1)-174)\xi^8rr_s+2g^6\xi^6r^2\left(2g^2(l(l+1)(3l(l+1)+7)+6)\xi^2-3(95l(l+1)-278)r_s^2\right)\\
&\qquad+64(l-1)(l+2)r^8\left(g^2\left(l^2+l-2\right)\xi^2+\left(l(l+1)\left(l(l+1)\left(l^2+l+10\right)-25\right)-22\right)r_s^2\right)\\
&\qquad+32(l-1)(l+2)r^7r_s\left(4l(l+1)(5l(l+1)+23)r_s^2-g^2\left(l(l+1)\left(l(l+1)\left(l^2+l+10\right)-23\right)-38\right)\xi^2\right)\\
&\qquad+4g^4(l-1)(l+2)\left(l(l+1)\left(l(l+1)\left(l^2+l+10\right)-20\right)-56\right)\xi^4r^6\\
&\qquad-32g^2(l(l+1)(l(l+1)(16l(l+1)+55)-171)-30)\xi^2r^6r_s^2+192(l(l+1)(9l(l+1)-8)+43)r_s^4r^6\\
&\qquad+4g^2\xi^2r^4\left(g^4\left(28-3l(l+1)\left(l(l+1)\left(l^2+l+5\right)-12\right)\right)\xi^4+g^2(l(l+1)(223l(l+1)+156)+668)\xi^2r_s^2-8(95l(l+1)-469)r_s^4\right)\\
&\qquad+8r^5r_s\left(g^4(l(l+1)(l(l+1)(17l(l+1)+72)-198)-84)\xi^4-8g^2(l(l+1)(32l(l+1)-3)+121)\xi^2r_s^2+48(5l(l+1)-31)r_s^4\right)
\end{aligned}
\tag{A.0.1}
$$

$$+ 2g^4\xi^4 r^3 r_s \left(20(47l(l+1)-182)r_s^2 - g^2(l(l+1)(85l(l+1)+128)+204)\xi^2\right)$$
$$- 256\left(l^2+l-2\right)^2 r^9 r_s\Big](Q^e)^2$$

In order to make contact with the solution for $\pi_\mu^{(2)}$ in GP gauge, we needed to proof that the solution is weakly gauge invariant. $\pi_\mu^{(2)}$ has two contributions, on the one hand we have a gauge invariant integral over the bulk and on the other hand there is a gauge variant boundary term. With the help of Mathematica, we explicitly computed the boundary term which includes three contributions A_1, A_2, A_3. For the first one, we integrated by parts before inserting q_2 to simplify the expression. The corresponding boundary term is

$$\begin{aligned} A_1 =& \left(-\frac{3g^2\xi^2}{8r^3} + \frac{3r_s}{2r^2} + \frac{1}{2r}\right)(q_2')^2 + \frac{r_s - (l^2+l+2)\,r}{r^3} q_2' q_2 + \left(-\frac{g^2\xi^2}{2r^2} + \frac{2r_s}{r} + 2\right) q_2' q_1 - \frac{3\pi_\mu^{(0)}}{4r^2} q_2' p_1 \\ &- \frac{\sqrt{(l-1)l(l+1)(l+2)}}{2\sqrt{2}r^3} q_2 X^e + - \frac{\left((l^2+l+2)\,r - r_s\right)\left(g^2\xi^2 + 12r^2 - 4rr_s\right)}{8r^3\,(r_s+r) - 2g^2\xi^2 r^2} q_2 q_1 + \frac{\pi_\mu^{(0)}}{4r} q_2 p_2 \\ &+ \frac{2\left((l^2+l+2)\,r - r_s\right)\sqrt{4rr_s - g^2\xi^2}\,\left(4r\,(r_s+3r) - g^2\xi^2\right)}{r^2\left(g^2\xi^2 - 4r\,(r_s+r)\right)^2} q_2 p_1 - \frac{g^2\sqrt{l(l+1)}\xi}{2r^3} q_2 A \\ &+ \left(\frac{64r^4\left(r_s - (l^2+l+2)\,r\right)^2}{\left(4r^2+4rr_s-g^2\xi^2\right)^2} + \frac{l(l+1)g^2\xi^2 - l(l+1)r\left(\left(l^2+l+6\right)r + 3r_s\right) - 2rr_s}{4r^5}\right)(q_2)^2 \end{aligned} \tag{A.0.2}$$

After inserting the solution of $q_2^{(1)}$, we performed two more integrations by part. The first one, A_2, contains all the boundary terms involving X^e and reads

$$\begin{aligned} A_2 =& -\frac{1}{r^2}(X^e X^{e\prime}) - \frac{g^2 l(l+1)\xi}{2\sqrt{2(l+2)(l-1)}r^3} X^e O_A + \frac{1}{4\sqrt{2}\sqrt{(l-1)l(l+1)(l+2)}r^3\,(4rr_s-g^2\xi^2)\,(g^2\xi^2-4r\,(r_s+r))^2}\Big[g^8\left(l^2+l-10\right)\xi^8 \\ &- 2g^6\xi^6 r\left(l(l+1)\left(l^2+l+2\right)r + 8l(l+1)r_s - 92r_s - 72r\right) \\ &+ 8g^4\xi^4 r^2\left(2\left(l(l+1)\left(l^2+l+19\right)-2\right)r^2 + \left(3l(l+1)\left(l^2+l+2\right)-272\right)rr_s + (12l(l+1)-151)r_s^2\right) \\ &+ 32g^2\xi^2 r^3\left(-4\left(l(l+1)\left(l^2+l+19\right)-9\right)r^2 r_s - \left(3l(l+1)\left(l^2+l+2\right)-314\right)rr_s^2 + 3(l-1)l(l+1)(l+2)r^3 - 2(4l(l+1)-53)r_s^3\right) \\ &+ 128r^4 r_s\left(\left(2l(l+1)\left(l^2+l+19\right)-27\right)r^2 r_s + \left(l(l+1)\left(l^2+l+2\right)-114\right)rr_s^2 - 3(l-1)l(l+1)(l+2)r^3 + (2l(l+1)-27)r_s^3\right)\Big] X^e O_{q_1} \\ & \frac{1}{2\sqrt{2}\sqrt{l(l+1)\,(l^2+l-2)}r^3\,(4rr_s-g^2\xi^2)^{3/2}\,(4r\,(r_s+r)-g^2\xi^2)^3}\Big[-g^{10}\left(l^2+l+18\right)\xi^{10} + 8g^8(2l(l+1)+51)\xi^8 rr_s \\ &+ 256r^7 r_s\left(g^2(2-l(l+1)(7l(l+1)-2))\xi^2 + 4l(l+1)\left(l^2+l-28\right)r_s^2\right) \\ &+ 32r^3\left(g^6\left(l(l+1)\left(l^2+l+16\right)-87\right)\xi^6 r_s + g^4(8l(l+1)+486)\xi^4 r_s^3\right) \\ &+ 2g^6\xi^6 r^2\left(-g^2\left(l^2+l-4\right)\left(l^2+l+18\right)\xi^2 - 24(2l(l+1)+75)r_s^2\right) \\ &- 32r^6\left(-g^4 l(l+1)(7l(l+1)-2)\xi^4 + 8g^2\left(3l(l+1)\left(l^2+l-27\right)-4\right)\xi^2 r_s^2 + 16\left(l(l+1)\left(l^2+l+22\right)-117\right)r_s^4\right) \\ &+ 64r^5\left(g^4\left(3l(l+1)\left(l^2+l-26\right)-10\right)\xi^4 r_s + 4g^2\left(2l(l+1)\left(l^2+l+20\right)-219\right)\xi^2 r_s^3 + 432r_s^5\right) \\ &- 16r^4\left(g^6\left(l(l+1)\left(l^2+l-25\right)-6\right)\xi^6 + 6g^4\left(2l(l+1)\left(l^2+l+18\right)-199\right)\xi^4 r_s^2 + 16g^2\left(l^2+l+129\right)\xi^2 r_s^4\right) \\ &+ 512(l(l+1)(7l(l+1)-2)-3)r^8 r_s^2\Big] X^e O_{p_1} \\ &+ \frac{1}{2(l-1)l(l+1)(l+2)r^5(\pi_\mu^{(0)})^3\,(g^2\xi^2-4r\,(r+r_s))^4}\Big\{16r^6\left(4rr_s-g^2\xi^2\right)^3\left(-5g^4\xi^4 + 8g^2 r\,(3r+5r_s)\,\xi^2 + 16r^2\left(3r^2-6r_s r-5r_s^2\right)\right)l^8 \\ &+ 64r^6\left(4rr_s-g^2\xi^2\right)^3\left(-5g^4\xi^4 + 8g^2 r\,(3r+5r_s)\,\xi^2 + 16r^2\left(3r^2-6r_s r-5r_s^2\right)\right)l^7 \\ &+ 4r^2\left(4rr_s-g^2\xi^2\right)^3\Big[-2g^8\xi^8 + g^6 r\,(2r+29r_s)\,\xi^6 - 4g^4 r^2\left(56r^2-5r_s r+39r_s^2\right)\xi^4 + 16g^2 r^3\,(r+r_s)\left(146r^2-39r_s r+23r_s^2\right)\xi^2 \\ &\quad + 64r^4\left(4r^4-149r_s r^3-51r_s^2 r^2+9r_s^3 r-5r_s^4\right)\Big]l^6 \\ &- 4r^2\left(4rr_s-g^2\xi^2\right)^3\Big[6g^8\xi^8 - 3g^6 r\,(2r+29r_s)\,\xi^6 + 4g^4 r^2\left(98r^2-15r_s r+117r_s^2\right)\xi^4 \\ &\quad - 16g^2 r^3\left(354r^3+181r_s r^2-48r_s^2 r+69r_s^3\right)\xi^2 + 64r^4\left(30r^4+363r_s r^3+83r_s^2 r^2-27r_s^3 r+15r_s^4\right)\Big]l^5 \end{aligned} \tag{A.0.3}$$

$$+\left(g^2\xi^2-4rr_s\right)^2\Big[-g^{12}\xi^{12}+10g^{10}r\,(3r_s-2r)\,\xi^{10}+4g^8r^2\left(350r^2+81r_sr-90r_s^2\right)\xi^8$$

$$-16g^6r^3\left(793r^3+1513r_sr^2+127r_s^2r-140r_s^3\right)\xi^6-64g^4r^4\left(64r^4-2582r_sr^3-2432r_s^2r^2-95r_s^3r+120r_s^4\right)\xi^4$$

$$-256g^2r^5\left(17r^5-83r_sr^4+2754r_s^2r^3+1725r_s^3r^2+33r_s^4r-54r_s^5\right)\xi^2$$

$$+1024r^6r_s\left(17r^5-32r_sr^4+965r_s^2r^3+456r_s^3r^2+4r_s^4r-10r_s^5\right)\Big]l^4$$

$$+2\left(g^2\xi^2-4rr_s\right)^2\Big[-g^{12}\xi^{12}+10g^{10}r\,(3r_s-4r)\,\xi^{10}+2g^8r^2\left(710r^2+347r_sr-180r_s^2\right)\xi^8$$

$$-8g^6r^3\left(1726r^3+3011r_sr^2+594r_s^2r-280r_s^3\right)\xi^6+32g^4r^4\left(434r^4+5559r_sr^3+4759r_s^2r^2+500r_s^3r-240r_s^4\right)\xi^4$$

$$-128g^2r^5\left(98r^5+973r_sr^4+5878r_s^2r^3+3325r_s^3r^2+206r_s^4r-108r_s^5\right)\xi^2$$

$$+512r^6r_s\left(98r^5+513r_sr^4+2045r_s^2r^3+867r_s^3r^2+33r_s^4r-20r_s^5\right)\Big]l^3$$

$$-\left(g^2\xi^2-4rr_s\right)^2\Big[-7g^{12}\xi^{12}+2g^{10}r\,(71r_s-190r)\,\xi^{10}+40g^8r^2\left(114r^2+230r_sr-25r_s^2\right)\xi^8$$

$$+16g^6r^3\left(356r^3-5468r_sr^2-5487r_s^2r+149r_s^3\right)\xi^6+64g^4r^4\left(56r^4-860r_sr^3+9501r_s^2r^2+6424r_s^3r+53r_s^4\right)\xi^4$$

$$-256g^2r^5\left(24r^5+74r_sr^4-679r_s^2r^3+7119r_s^3r^2+3687r_s^4r+95r_s^5\right)\xi^2$$

$$+1024r^6r_s\left(24r^5+29r_sr^4-186r_s^2r^3+1946r_s^3r^2+830r_s^4r+29r_s^5\right)\Big]l^2$$

$$-4\left(g^2\xi^2-4rr_s\right)^2\Big[-2g^{12}\xi^{12}+g^{10}r\,(43r_s-106r)\,\xi^{10}+4g^8r^2\left(374r^2+623r_sr-85r_s^2\right)\xi^8$$

$$-4g^6r^3\left(520r^3+6972r_sr^2+5818r_s^2r-289r_s^3\right)\xi^6+16g^4r^4\left(328r^4+1956r_sr^3+11870r_s^2r^2+6705r_s^3r-67r_s^4\right)\xi^4$$

$$-64g^2r^5\left(80r^5+672r_sr^4+2294r_s^2r^3+8769r_s^3r^2+3804r_s^4r+41r_s^5\right)\xi^2$$

$$+256r^6r_s\left(80r^5+342r_sr^4+847r_s^2r^3+2375r_s^3r^2+849r_s^4r+19r_s^5\right)\Big]l$$

$$-4\left(-3g^6\xi^6+44g^4rr_s\xi^4-8g^2r^2r_s\,(2r+25r_s)\,\xi^2+48r^3r_s^2\,(r+6r_s)\right)^2\left(-3g^4\xi^4+24g^2r\,(r+r_s)\,\xi^2+16r^2\left(r^2-6r_sr-3r_s^2\right)\right)\Big\}(X^e)^2$$

$$-\frac{1}{r^3}(X^e)^2$$

$$+\frac{1}{2(l(l+1))^{3/2}\,(4rr_s-g^2\xi^2)\,(g^2\xi^2-4r\,(r_s+r))^2\,(-g^2\xi^2+(l^2+l-2)\,r^2+3rr_s)\sqrt{(l^2+l-2)\,(8rr_s-2g^2\xi^2)}}\Big[-g^{10}\left(l^2+l+18\right)\xi^{10}$$

$$-2g^8\xi^8r\left(\left(l^2+l-4\right)\left(l^2+l+18\right)r-4(2l(l+1)+51)r_s\right)$$

$$-16g^6\xi^6r^2\left(\left(l(l+1)\left(l^2+l-25\right)-6\right)r^2-2\left(l(l+1)\left(l^2+l+16\right)-87\right)rr_s+3(2l(l+1)+75)r_s^2\right)$$

$$+32g^4\xi^4r^3\left(2\left(3l(l+1)\left(l^2+l-26\right)-10\right)r^2r_s-3\left(2l(l+1)\left(l^2+l+18\right)-199\right)rr_s^2+l(l+1)(7l(l+1)-2)r^3+2(4l(l+1)+243)r_s^3\right)$$

$$-256g^2\xi^2r^4r_s\left(\left(3l(l+1)\left(l^2+l-27\right)-4\right)r^2r_s-\left(2l(l+1)\left(l^2+l+20\right)-219\right)rr_s^2+\left(l^2+l+129\right)r_s^3+(l(l+1)(7l(l+1)-2)-2)r^3\right)$$

$$+512r^5r_s^2\left(2l(l+1)\left(l^2+l-28\right)r^2r_s-\left(l(l+1)\left(l^2+l+22\right)-117\right)rr_s^2+(l(l+1)(7l(l+1)-2)-3)r^3+54r_s^3\right)\Big]X^ep_2'$$

$$+\frac{\sqrt{(l-1)(l+1)(l+2)}}{2\sqrt{2}(l+1)^2\,(l^2+l-2)\,r\,((l^2+l-2)\,r^2+3r_sr-g^2\xi^2)^2\,(l\,(4rr_s-g^2\xi^2))^{3/2}\,(4r\,(r+r_s)-g^2\xi^2)^3}\Big[-3g^{14}\left(l^2+l+30\right)\xi^{14}$$

$$+2g^{12}(29l(l+1)+1326)rr_s\xi^{12}-g^{10}r^2\left(g^2(l(l+1)(5l(l+1)+112)-900)\xi^2+4(103l(l+1)+8211)r_s^2\right)\xi^{10}$$

$$+4g^8r^3r_s\left(8g^2(l(l+1)(4l(l+1)+87)-720)\xi^2+9(29l(l+1)+6172)r_s^2\right)\xi^8$$

$$+2g^6r^4\Big(-g^4(l-3)(l+4)(l(l+1)(5l(l+1)+76)-28)\xi^4$$

$$-6g^2(l(l+1)(113l(l+1)+2340)-19768)r_s^2\xi^2+16(58l(l+1)-27759)r_s^4\Big)\xi^6$$

$$-1024(l-1)(l+2)(l(l+1)(l(l+1)(5l(l+1)-29)+11)+12)r^{12}r_s^2$$

$$+256r^{11}r_s\Big(g^2(l-1)(l+2)(l(l+1)(l(l+1)(10l(l+1)-57)+22)+16)\xi^2$$

$$+4(l(l+1)(l(l+1)(l(l+1)(2l(l+1)+95)-334)+404)-12)r_s^2\Big)$$

$$+64r^{10}\Big(-g^4(l-1)l(l+1)(l+2)(l(l+1)(5l(l+1)-28)+12)\xi^4$$

$$-8g^2(l(l+1)(l(l+1)(l(l+1)(3l(l+1)+134)-423)+547)-6)r_s^2\xi^2$$

$$+16\left(l(l+1)\left(l(l+1)\left(l^2+l-5\right)(3l(l+1)+52)+579\right)-1017\right)r_s^4\Big)$$

$$+64r^9\Big(16(l(l+1)(l(l+1)(5l(l+1)-86)-757)+36)r_s^5$$

$$
\begin{aligned}
&- 4g^2(l(l+1)(l(l+1)(l(l+1)(12l(l+1)+145)-704)+1153)-3848)\xi^2 r_s^3 \\
&+ g^4(l(l+1)(l(l+1)(l(l+1)(6l(l+1)+251)-688)+956)+80)\xi^4 r_s\Big) \\
+ 4r^5\Big(&g^8(l(l+1)(3l(l+1)(15l(l+1)+16)-8548)+4000)r_s\xi^8 \\
&+ 4g^6(7l(l+1)(68l(l+1)+1315)-79032)r_s^3\xi^6 - 32g^4(133l(l+1)-16413)r_s^5\xi^4\Big) \\
- 32r^8\Big(&g^6\left(l(l+1)\left(l(l+1)\left(l(l+1)\left(l^2+l+39\right)-88\right)+132\right)+48\right)\xi^6 \\
&- 2g^4(l(l+1)(l(l+1)(3l(l+1)(6l(l+1)+71)-556)-6)-5408)r_s^2\xi^4 \\
&+ 16g^2(l(l+1)(3l(l+1)(5l(l+1)-51)-2578)+906)r_s^4\xi^2 + 288(l(l+1)(3l(l+1)+41)-375)r_s^6\Big) \\
+ 4r^6\Big(&g^8(l(l+1)(l(l+1)(l(l+1)(3l(l+1)+34)+72)-568)-816)\xi^8 \\
&- 8g^6(l(l+1)(l(l+1)(40l(l+1)-67)-7549)+3864)r_s^2\xi^6 - 16g^4(l(l+1)(374l(l+1)+6615)-57687)r_s^4\xi^4 \\
&+ 1152g^2(8l(l+1)-591)r_s^6\xi^2\Big) \\
- 16r^7\Big(&1728\left(l^2+l-54\right)r_s^7 - 48g^2(l(l+1)(52l(l+1)+821)-7298)\xi^2 r_s^5 \\
&+ 4g^4(-l(l+1)(10l(l+1)(7l(l+1)-37)-12829)-6176)\xi^4 r_s^3 \\
&+ g^6(l(l+1)(l(l+1)(l(l+1)(12l(l+1)+139)-40)-1148)-3392)\xi^6 r_s\Big)\Big] X^e p_2
\end{aligned}
$$

The remaining boundary term is due to p_2 and it is given by

$$
\begin{aligned}
A_3 =& \frac{r^5\left(3g^4\xi^4 - 24g^2\xi^2 r\,(r_s+r) - 16r^2\left(r^2-6rr_s-3r_s^2\right)\right)}{2l^2(l+1)^2\left(-g^2\xi^2+(l^2+l-2)\,r^2+3rr_s\right)^2}(p_2')^2 \\
&+ \frac{1}{8l^2(l+1)^2r^2\left(4r\,(r_s+r)-g^2\xi^2\right)\Lambda^3}\Big[\left(-3g^4\xi^4+24g^2\xi^2 r\,(r_s+r)+16r^2\left(r^2-6rr_s-3r_s^2\right)\right)\times \\
&\quad\times\left(-5g^4\xi^4+g^2\xi^2 r\left(\left(l^2+l+10\right)r+34r_s\right)+2r^2\left((l-1)(l+2)\left(l^2+l-4\right)r^2-4\left(l^2+l+1\right)rr_s-27r_s^2\right)\right)\Big]p_2p_2' \\
&+ \frac{r^2\sqrt{4rr_s-g^2\xi^2}\left(g^2\xi^2-4r\,(r_s+3r)\right)}{l(l+1)\left(-g^2\xi^2+(l^2+l-2)\,r^2+3rr_s\right)}p_2'O_{q_1} + \frac{r^2\left(3g^4\xi^4-24g^2\xi^2 r\,(r_s+r)-16r^2\left(r^2-6rr_s-3r_s^2\right)\right)}{l(l+1)\left(4r\,(r_s+r)-g^2\xi^2\right)\left(-g^2\xi^2+(l^2+l-2)\,r^2+3rr_s\right)}p_2'O_{p_1} \\
&+ \frac{1}{4l(l+1)r^3\pi_\mu^{(0)}\left(4r\,(r_s+r)-g^2\xi^2\right)\Lambda^2}\Big[-9g^8\xi^8+g^6\xi^6 r\left((5l(l+1)+142)r+131r_s\right) \\
&\quad- 2g^4\xi^4 r^2\left(28\left(l^2+l+4\right)r^2+(31l(l+1)+750)rr_s+353r_s^2\right) \\
&\quad- 16g^2\xi^2 r^3\left((l-1)(l+2)(2l(l+1)-13)r^3-(33l(l+1)+65)r^2r_s-(16l(l+1)+321)rr_s^2-104r_s^3\right) \\
&\quad+ 32r^4 r_s\left(-11\left(l^2+l+16\right)rr_s^2+(l-1)(l+2)(4l(l+1)-27)r^3-(38l(l+1)+17)r^2r_s-45r_s^3\right)\Big]O_{q_1}p_2 \\
&+ \frac{1}{4l(l+1)r^3\left(g^2\xi^2-4r\,(r_s+r)\right)^2\Lambda^2}\Big[\left(-3g^4\xi^4+24g^2\xi^2 r\,(r_s+r)+16r^2\left(r^2-6rr_s-3r_s^2\right)\right)\times \\
&\quad\times\left(-5g^4\xi^4+g^2\xi^2 r\left(\left(l^2+l+10\right)r+34r_s\right)+2r^2\left((l-1)(l+2)\left(l^2+l-4\right)r^2-4\left(l^2+l+1\right)rr_s-27r_s^2\right)\right)\Big]O_{p_1}p_2 \\
&+ \frac{1}{32l^2(l+1)^2r^5\left(g^2\xi^2-4r\,(r+r_s)\right)^2\Lambda^4}\Big[128\left(l^2+l-2\right)^2(3l(l+1)-8)r^{12} \\
&\quad+ 64(l-1)(l+2)(l(l+1)(l(l+1)(7l(l+1)-47)+154)-224)r_sr^{11} \\
&\quad+ 16\Big(4(2l(l+1)-7)\left(l(l+1)\left(l(l+1)\left(l^2+l-31\right)+98\right)-32\right)r_s^2 \\
&\qquad- g^2(l-1)(l+2)(l(l+1)(l(l+1)(7l(l+1)-48)+196)-352)\xi^2\Big)r^{10} \\
&\quad+ 16r_s\Big(g^2(992-l(l+1)(l(l+1)(l(l+1)(4l(l+1)-111)+540)-460))\xi^2 \\
&\qquad- 4\left(l(l+1)\left(l(l+1)\left(l(l+1)\left(l^2+l+27\right)+57\right)-1817\right)+2904\right)r_s^2\Big)r^9 \\
&\quad+ 8\Big(g^4\left(l\left(l\left((l(l(l+4)-15)-59)l^3+103l+242\right)+180\right)-744\right)\xi^4 \\
&\qquad+ 2g^2\left(l(l+1)\left(3l(l+1)\left(l(l+1)\left(l^2+l+24\right)+191\right)-7778\right)+11032\right)r_s^2\xi^2 \\
&\qquad- 24(l(l+1)(l(l+1)(3l(l+1)+35)-803)+227)r_s^4\Big)r^8
\end{aligned} \qquad (A.0.4)
$$

$$+4r_s\Big(-g^4\left(l(l+1)\left(3l(l+1)\left(l(l+1)\left(l^2+l+21\right)+312\right)-9716\right)+12160\right)\xi^4$$
$$+12g^2(l(l+1)(l(l+1)(13l(l+1)+186)-3517)-696)r_s^2\xi^2-432\left(l^2+l-31\right)\left(l^2+l+6\right)r_s^4\Big)r^7$$
$$+\Big(g^6\left(l(l+1)\left(l(l+1)\left(l(l+1)\left(l^2+l+18\right)+420\right)-3752\right)+4000\right)\xi^6$$
$$-4g^4(3l(l+1)(7l(l+1)(3l(l+1)+49)-5534)-13664)r_s^2\xi^4$$
$$+48g^2(l(l+1)(51l(l+1)-1132)-10858)r_s^4\xi^2-1728\left(l^2+l-81\right)r_s^6\Big)r^6$$
$$+g^2\xi^2r_s\Big(g^4(l(l+1)(3l(l+1)(15l(l+1)+268)-11212)-18176)\xi^4$$
$$+12g^2(27416-l(l+1)(115l(l+1)-2213))r_s^2\xi^2+432(7l(l+1)-570)r_s^4\Big)r^5$$
$$+g^4\xi^4\Big(-g^4\left(l^2+l-10\right)\left(3l(l+1)\left(l^2+l+29\right)+182\right)\xi^4$$
$$+g^2(l(l+1)(387l(l+1)-6302)-101256)r_s^2\xi^2-12(183l(l+1)-14857)r_s^4\Big)r^4$$
$$+g^6\xi^6r_s\left(2g^2(7620-l(l+1)(27l(l+1)-362))\xi^2+7(121l(l+1)-9720)r_s^2\right)r^3$$
$$+g^8\xi^8\left(g^2(l(l+1)(3l(l+1)-32)-900)\xi^2-3(61l(l+1)-4816)r_s^2\right)r^2+3g^{10}(7l(l+1)-540)\xi^{10}r_sr-g^{12}\left(l^2+l-75\right)\xi^{12}\Big](p_2)^2$$

B

Some Formulas for the Even Parity Analysis in Generalised Gullstrand-Painlevé Gauge

In this section we list some equations used in the section on the generalised GP gauge. In the computation for the even parity contributions with $l \geq 2$ to the solution of $\pi_\mu^{(2)}$, we performed three canonical transformations. In the last one, we shifted the momentum P^e by the term $A_{\text{grav}} Q^e$. The explicit expression for A_{grav} reads

$$
\begin{aligned}
A_{\text{grav}} = \frac{2}{(l-1)(l+2)e^{6\mu+10\lambda}(\lambda')^3 M_1^2 (\pi_\mu^{(0)})^3 \Lambda} \Big[& -1024 e^{14\lambda} \Big(4g^4\xi^4 - 30e^\lambda g^2 r_s \xi^2 + 4e^{4\lambda}\left(l^2+l-2\right)^2 + 27e^{3\lambda}\left(l^2+l-2\right) r_s \\
& + e^{2\lambda}\left(54r_s^2 - 7g^2\left(l^2+l-2\right)\xi^2\right)\Big)\left(\lambda'\right)^9 \\
& + 1024 e^{14\lambda}\left(4g^4\xi^4 - 27e^\lambda g^2 r_s\xi^2 + 4e^{4\lambda}\left(l^2+l-2\right)^2 + 27e^{3\lambda}\left(l^2+l-2\right)r_s + e^{2\lambda}\left(45r_s^2 - 8g^2\left(l^2+l-2\right)\xi^2\right)\right)\mu'\left(\lambda'\right)^8 \\
& + 256e^{10\lambda}\Big(6e^{6\lambda+2\mu}l^6 + 18e^{6\lambda+2\mu}l^5 + 38e^{6\lambda+2\mu}l^4 - 12e^{5\lambda+2\mu}r_s l^4 + 4e^{8\lambda}\mu'' l^4 + 46e^{6\lambda+2\mu}l^3 - 24e^{5\lambda+2\mu}r_s l^3 + 8e^{8\lambda}\mu'' l^3 \\
& - 132e^{6\lambda+2\mu}l^2 - 15e^{2(\lambda+\mu)}g^4\xi^4 l^2 - 100e^{4\lambda+2\mu}g^2\xi^2 l^2 - 240e^{4\lambda+2\mu}r_s^2 l^2 + 384e^{5\lambda+2\mu}r_s l^2 + 120e^{3\lambda+2\mu}g^2\xi^2 r_s l^2 \\
& - 12e^{8\lambda}\mu'' l^2 - 8e^{6\lambda}g^2\xi^2\mu'' l^2 + 24e^{7\lambda}r_s\mu'' l^2 - 152e^{6\lambda+2\mu}l - 15e^{2(\lambda+\mu)}g^4\xi^4 l - 100e^{4\lambda+2\mu}g^2\xi^2 l - 240e^{4\lambda+2\mu}r_s^2 l \\
& + 396e^{5\lambda+2\mu}r_s l + 120e^{3\lambda+2\mu}g^2\xi^2 r_s l - 16e^{8\lambda}\mu'' l - 8e^{6\lambda}g^2\xi^2\mu'' l + 24e^{7\lambda}r_s\mu'' l + 176e^{6\lambda+2\mu} + 12e^{2\mu}g^6\xi^6 + 86e^{2(\lambda+\mu)}g^4\xi^4 \\
& - 558e^{3\lambda+2\mu}r_s^3 + 200e^{4\lambda+2\mu}g^2\xi^2 + 1128e^{4\lambda+2\mu}r_s^2 + 465e^{2(\lambda+\mu)}g^2\xi^2 r_s^2 - 4e^{4\lambda}\left(-g^2\xi^2 + e^{2\lambda}\left(l^2+l-2\right) + 3e^\lambda r_s\right)^2\left(\mu'\right)^2 \\
& - 744e^{5\lambda+2\mu}r_s - 129e^{\lambda+2\mu}g^4\xi^4 r_s - 624e^{3\lambda+2\mu}g^2\xi^2 r_s \\
& - 4e^{4\lambda}\left(4g^4\xi^4 - 27e^\lambda g^2 r_s\xi^2 + 4e^{4\lambda}\left(l^2+l-2\right)^2 + 27e^{3\lambda}\left(l^2+l-2\right)r_s + e^{2\lambda}\left(45r_s^2 - 8g^2\left(l^2+l-2\right)\xi^2\right)\right)\lambda'' \\
& + 16e^{8\lambda}\mu'' + 4e^{4\lambda}g^4\xi^4\mu'' + 16e^{6\lambda}g^2\xi^2\mu'' + 36e^{6\lambda}r_s^2\mu'' - 48e^{7\lambda}r_s\mu'' - 24e^{5\lambda}g^2\xi^2 r_s\mu''\Big)\left(\lambda'\right)^7 \\
& - 512e^{10\lambda}\left(-g^2\xi^2 + e^{2\lambda}\left(l^2+l-2\right) + 3e^\lambda r_s\right)\Big(\mu'\Big(e^{2\mu}\Big(- 6g^4\xi^4 + 44e^\lambda g^2 r_s\xi^2 + e^{4\lambda}\left(l^4 + 2l^3 + 21l^2 + 20l - 44\right) \\
& - 4e^{3\lambda}\left(5l^2+5l-31\right)r_s + e^{2\lambda}\left(g^2\left(5l^2+5l-34\right)\xi^2 - 81r_s^2\right)\Big) - 2e^{4\lambda}\left(-g^2\xi^2 + e^{2\lambda}\left(l^2+l-2\right) + 3e^\lambda r_s\right)\lambda''\Big) \\
& + 2e^{4\lambda}\left(-g^2\xi^2 + e^{2\lambda}\left(l^2+l-2\right) + 3e^\lambda r_s\right)\lambda^{(3)}\Big)\left(\lambda'\right)^6 \\
& + 16e^{6\lambda}\Big(48e^{4\lambda+2\mu}\left(g^2\xi^2 + 4e^{2\lambda} - 4e^\lambda r_s\right)\left(\mu'\right)^2\left(-g^2\xi^2 + e^{2\lambda}\left(l^2+l-2\right) + 3e^\lambda r_s\right)^2 \\
& + 64e^{8\lambda}\left(\lambda''\right)^2\left(-g^2\xi^2 + e^{2\lambda}\left(l^2+l-2\right) + 3e^\lambda r_s\right)^2 - 32e^{4\lambda+2\mu}\left(g^2\xi^2 + 4e^{2\lambda} - 4e^\lambda r_s\right)\mu''\left(-g^2\xi^2 + e^{2\lambda}\left(l^2+l-2\right) + 3e^\lambda r_s\right)^2 \\
& + e^{4\mu}\Big[- 36g^8\xi^8 + 540e^\lambda g^6 r_s\xi^6 + e^{2\lambda}g^4\left(g^2\left(7l^2+7l-430\right)\xi^2 - 3024r_s^2\right)\xi^4 - e^{3\lambda}g^2 r_s\left(g^2\left(201l^2+201l-4754\right)\xi^2 - 7500r_s^2\right)\xi^2 \\
& + 4e^{8\lambda}\left(l^2+l-2\right)^2\left(l^4+2l^3-51l^2-52l-156\right) + 8e^{7\lambda}\left(29l^6+87l^5-11l^4-167l^3-792l^2-694l+1548\right)r_s \\
& - 4e^{5\lambda}r_s\left(g^2\left(107l^4+214l^3+737l^2+630l-3320\right)\xi^2 + 18\left(27l^2+27l-298\right)r_s^2\right) \\
& + 4e^{6\lambda}\left(g^2\left(-15l^6-45l^5+37l^4+149l^3+394l^2+312l-832\right)\xi^2 + 24\left(6l^4+12l^3+67l^2+61l-254\right)r_s^2\right) \\
& - 4e^{4\lambda}\left(-2g^4\left(9l^4+18l^3+40l^2+31l-226\right)\xi^4 + g^2\left(-293l^2-293l+4378\right)r_s^2\xi^2 + 1737r_s^4\right)\Big] \\
& + 32e^{4\lambda+2\mu}\Big[6g^6\xi^6 - 62e^\lambda g^4 r_s\xi^4 + e^{6\lambda}\left(l^2+l-2\right)^2\left(l^2+l+22\right) - e^{5\lambda}\left(17l^4+34l^3-207l^2-224l+380\right)r_s \\
& - e^{3\lambda}r_s\left(g^2\left(-79l^2-79l+314\right)\xi^2 + 243r_s^2\right) + e^{4\lambda}\left(4g^2\left(l^4+2l^3-15l^2-16l+28\right)\xi^2 - 3\left(47l^2+47l-178\right)r_s^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + e^{2\lambda}\left(g^4\left(-11l^2-11l+46\right)\xi^4+213g^2r_s^2\xi^2\right)\Big]\lambda''\Big)\left(\lambda'\right)^5 \qquad \text{(B.0.1)}\\
&+16e^{6\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\Big[\mu'\Big(e^{2\mu}\Big(-36g^4\xi^4+267e^{\lambda}g^2r_s\xi^2\\
&\quad +16e^{4\lambda}\left(l^4+2l^3+8l^2+7l-18\right)-12e^{3\lambda}\left(8l^2+8l-65\right)r_s-4e^{2\lambda}\left(g^2\left(-5l^2-5l+54\right)\xi^2+123r_s^2\right)\Big)\\
&\quad -64e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda''\Big)+32e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda^{(3)}\Big]\left(\lambda'\right)^4\\
&-8e^{2(\lambda+\mu)}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\Big[18e^{4\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(\mu'\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2\\
&\quad +32e^{8\lambda}\left(\lambda''\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2-10e^{4\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\mu''\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2\\
&\quad +e^{4\mu}\Big[-4g^8\xi^8+64e^{\lambda}g^6r_s\xi^6-e^{2\lambda}g^4\left(g^2\left(11l^2+11l+54\right)\xi^2+374r_s^2\right)\xi^4+2e^{3\lambda}g^2r_s\left(5g^2\left(8l^2+8l+61\right)\xi^2+477r_s^2\right)\xi^2\\
&\quad +4e^{8\lambda}\left(l^2+l-2\right)^2\left(l^4+2l^3-15l^2-16l-24\right)+4e^{7\lambda}\left(22l^6+66l^5-49l^4-208l^3-189l^2-74l+432\right)r_s\\
&\quad +e^{5\lambda}r_s\left(g^2\left(-199l^4-398l^3-73l^2+126l+1768\right)\xi^2+36\left(l^2+l+78\right)r_s^2\right)\\
&\quad +4e^{6\lambda}\left(g^2\left(-6l^6-18l^5+23l^4+76l^3+41l^2-116\right)\xi^2+\left(73l^4+146l^3+108l^2+35l-812\right)r_s^2\right)\\
&\quad -e^{4\lambda}\left(16g^4\left(-2l^4-4l^3+l^2+3l+15\right)\xi^4+g^2\left(155l^2+155l+2276\right)r_s^2\xi^2+900r_s^4\right)\Big]\\
&\quad +2e^{4\lambda+2\mu}\Big[36g^6\xi^6-375e^{\lambda}g^4r_s\xi^4+16e^{6\lambda}\left(l^2+l-2\right)^2\left(l^2+l+9\right)-12e^{5\lambda}\left(4l^4+8l^3-105l^2-109l+202\right)r_s\\
&\quad -9e^{3\lambda}r_s\left(g^2\left(-47l^2-47l+218\right)\xi^2+164r_s^2\right)-4e^{4\lambda}\left(3\left(65l^2+65l-277\right)r_s^2-g^2\left(l^4+2l^3-91l^2-92l+180\right)\xi^2\right)\\
&\quad +e^{2\lambda}\left(1293g^2\xi^2r_s^2-8g^4\left(7l^2+7l-36\right)\xi^4\right)\Big]\lambda''\Big]\left(\lambda'\right)^3\\
&-4e^{2\lambda+4\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\Big[\mu'\Big(e^{2\mu}\Big(-8g^4\xi^4+63e^{\lambda}g^2r_s\xi^2\\
&\quad +8e^{4\lambda}\left(l^4+2l^3+4l^2+3l-10\right)-12e^{3\lambda}\left(l^2+l-17\right)r_s-8e^{2\lambda}\left(7g^2\xi^2+15r_s^2\right)\Big)\\
&\quad -52e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda''\Big)+20e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda^{(3)}\Big]\left(\lambda'\right)^2\\
&+e^{4\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)^2\Big[8e^{2(\lambda+\mu)}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(\mu'\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2\\
&\quad +16e^{6\lambda}\left(\lambda''\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2\\
&\quad -4e^{2(\lambda+\mu)}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\mu''\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2\\
&\quad +e^{4\mu}\Big[g^6\left(-5l^2-5l+2\right)\xi^6+e^{\lambda}g^4\left(45l^2+45l-22\right)r_s\xi^4+2e^{2\lambda}g^2\Big(2g^2\left(3l^4+6l^3-7l^2-10l+4\right)\xi^2-5\left(13l^2+13l-8\right)r_s^2\Big)\xi^2\\
&\quad +4e^{6\lambda}l\left(l^2+l-2\right)^2\left(l^3+2l^2-3l-4\right)+8e^{5\lambda}\left(l^2+l-2\right)^2\left(5l^2+5l-3\right)r_s\\
&\quad +2e^{3\lambda}r_s\left(g^2\left(-37l^4-74l^3+77l^2+114l-56\right)\xi^2+12\left(5l^2+5l-4\right)r_s^2\right)\\
&\quad +4e^{4\lambda}\left(l^2+l-2\right)\left(g^2\left(-3l^4-6l^3+7l^2+10l-4\right)\xi^2+3\left(9l^2+9l-8\right)r_s^2\right)\Big]\\
&\quad +4e^{2(\lambda+\mu)}\Big[8g^6\xi^6-87e^{\lambda}g^4r_s\xi^4+8e^{6\lambda}\left(l^2+l-2\right)^2\left(l^2+l+5\right)+12e^{5\lambda}\left(l^4+2l^3+26l^2+25l-54\right)r_s\\
&\quad -3e^{3\lambda}r_s\left(g^2\left(-25l^2-25l+166\right)\xi^2+120r_s^2\right)-4e^{4\lambda}\left(2g^2\left(l^4+2l^3+11l^2+10l-24\right)\xi^2+3\left(13l^2+13l-71\right)r_s^2\right)\\
&\quad +e^{2\lambda}\left(309g^2\xi^2r_s^2-8g^4\left(l^2+l-9\right)\xi^4\right)\Big]\lambda''\Big]\lambda'\\
&-4e^{2\lambda+6\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)^3\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2\left(3\mu'\lambda''-\lambda^{(3)}\right)\Big]
\end{aligned}
$$

Similar to the strict GP gauge, we simplified the solution for $\pi_\mu^{(2)}$ with an integration by parts. In the main text, we argue that we can drop the boundary term. Explicitly, it is given by

$$
\begin{aligned}
&\left(2e^{-\lambda-2\mu}-\frac{3e^{-3\lambda}}{\lambda'^2}\Delta\right)(q_2')^2+\frac{e^{-4\lambda}\left(r_s-e^{\lambda}\left(l^2+l+2\right)\right)}{\lambda'}q_2'q_2+\left(4e^{\lambda-4\mu}\lambda'-\frac{2e^{-2\mu-\lambda}}{\lambda'}\Delta\right)q_2'q_1-\frac{3}{4\lambda'}e^{-3\lambda}\pi_\mu^{(0)}q_2'p_1-\frac{1}{2}e^{-3\lambda}\sqrt{l(l+1)}g^2\xi Aq_2\\
&+\frac{4e^{2\mu-2\lambda}\left(e^{\lambda}\left(l^2+l+2\right)-r_s\right)\left(-20e^{4\lambda+2\mu}\left(\lambda'\right)^2\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)+e^{4\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)^2+64e^{8\lambda}\left(\lambda'\right)^4\right)}{\pi_0(r)\left(e^{2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)-8e^{4\lambda}\left(\lambda'\right)^2\right)^2}q_2p_1\\
&+\left[\frac{4e^{-3\mu-3\lambda}\left(e^{\lambda}\left(l^2+l+2\right)-r_s\right)\Delta}{\lambda'M_1}+\frac{1}{4}e^{-3\lambda-2\mu}\left(g^2\xi^2-2e^{2\lambda}l(l+1)-2e^{\lambda}r_s\right)-\frac{4e^{-\lambda-2\mu}\left(\lambda'\mu'-\lambda''\right)\Delta}{4\left(\lambda'\right)^2}-2e^{\lambda-4\mu}\left(\lambda''-\lambda'\mu'+\left(\lambda'\right)^2\right)\right]q_2q_1
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[-\frac{4l(l+1)e^{-7\lambda}\Delta^2\Lambda}{(\lambda')^2M_1^2}+\frac{1}{2}e^{-4\lambda}\left(4r_s-3e^{\lambda}\left(l^2+l+2\right)\right)+\frac{e^{-10\lambda-6\mu}}{4(\lambda')^2M_1^2}\Bigg(16e^{4\lambda+2\mu}\left(\lambda'\right)^2\Big(-2e^{\lambda}\left(6g^2\xi^2+17r_s^2\right)+9g^2\xi^2r_s\\
&\quad+4e^{3\lambda}\left(l^2+l-10\right)-2e^{2\lambda}\left(l^2+l-38\right)r_s\Big)+e^{4\mu}\left(g^2\xi^2+4e^{\lambda}\left(e^{\lambda}-r_s\right)\right)\Big(6e^{\lambda}\left(6r_s^2-g^2\left(l^2+l-2\right)\xi^2\right)-9g^2\xi^2r_s\\
&\quad+8e^{3\lambda}\left(l^4+2l^3-l+6\right)+4e^{2\lambda}(4l(l+1)-21)r_s\Big)+64e^{8\lambda}\left(\lambda'\right)^4\left(2e^{\lambda}\left(l^2+l+6\right)-9r_s\right)\Bigg]\Bigg](q_2)^2\\
&+\Bigg[\frac{g^2\xi e^{-3\lambda-6\mu}}{8\sqrt{l(l+1)}\Lambda^2}\Big[e^{2\mu}\lambda''\left(g^2\xi^2+4e^{\lambda}\left(e^{\lambda}-r_s\right)\right)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)+8e^{4\lambda}\left(\lambda'\right)^3\mu'\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\\
&\quad-e^{2\mu}\lambda'\mu'\left(g^2\xi^2+4e^{\lambda}\left(e^{\lambda}-r_s\right)\right)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)-8e^{4\lambda}\left(\lambda'\right)^4\left(-3g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+6e^{\lambda}r_s\right)\\
&\quad+\left(\lambda'\right)^2\Big(e^{2\mu}\left(-3g^4\xi^4-e^{2\lambda}\left(g^2\left(l^2+l+18\right)\xi^2+30r_s^2\right)+20g^2e^{\lambda}\xi^2r_s+2e^{4\lambda}(l-1)(l+2)\left(l^2+l+4\right)+48e^{3\lambda}r_s\right)\\
&\quad+8e^{4\lambda}\lambda''\left(g^2\xi^2-e^{2\lambda}\left(l^2+l-2\right)-3e^{\lambda}r_s\right)\Big)\Big]Aq_1 \qquad\qquad \text{(B.0.2)}\\
&+\frac{e^{-4\mu+\lambda}\left(-192e^{2\mu-2\lambda}\left(\lambda'\right)^2\Delta+48e^{4\mu-4\lambda}\Delta^2+128\left(\lambda'\right)^4\right)}{2(\lambda')^2M_1^2}(p_1)^2+\frac{2e^{-5\mu}\pi_\mu^{(0)}\left(4\left(\lambda'\right)^2-e^{2\mu-2\lambda}\Delta\right)}{\lambda'M_1}q_1p_1\\
&+\frac{g^2e^{-2\mu}\xi}{4\sqrt{l^2+l-2}}\Bigg[-\frac{2g^2\xi^2e^{-2\lambda}\left(\lambda'\right)^2}{\Lambda}+\frac{e^{-4\lambda}}{2\Lambda\lambda'e^{3\mu+3\lambda}M_1}\Big(-32e^{6\lambda}\left(\lambda'\right)^4\left(-2g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+6e^{\lambda}r_s\right)\\
&\quad+e^{4\mu}\left(g^2\xi^2+4e^{\lambda}\left(e^{\lambda}-r_s\right)\right)\left(-g^2\left(l^2+l+2\right)\xi^2+2e^{2\lambda}(l-1)(l+2)\left(l^2+l+2\right)+2e^{\lambda}\left(l^2+l+4\right)r_s\right)\\
&\quad-4e^{2(\lambda+\mu)}\left(\lambda'\right)^2\left(2g^4\xi^4+e^{2\lambda}\left(30r_s^2-g^2\left(l^2+l-10\right)\xi^2\right)-16g^2e^{\lambda}\xi^2r_s+4e^{3\lambda}\left(l^2+l-8\right)r_s+2e^{4\lambda}(l-1)l(l+1)(l+2)\right)\Big)\\
&\quad+4\lambda''-4\lambda'\mu'\Bigg]Q^eA^e+\frac{e^{-2\lambda}g^4\xi^2\Delta}{4\Lambda}(A^e)^2+\left(\frac{2(l^2+l+2)r}{l(l+1)(l+2)(l-1)}+O(r^0)\right)(q_1)^2+\left(-\frac{1}{r}+O(r^{-2})\right)(Q^e)^2
\end{aligned}
$$

In the formula we did not explicitly display the terms in front of $(q_1)^2$ and $(Q^e)^2$. We list them separately because they are quite lengthy and require a lot of space. The terms proportional to $(q_1)^2$ are

$$
\begin{aligned}
&\frac{1}{8l(l+1)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2\left(\lambda'\right)^2}\Big[e^{\lambda-10\mu}\Big(64e^{8\lambda}\left(-5g^2\xi^2+7e^{2\lambda}\left(l^2+l-2\right)+18e^{\lambda}r_s\right)\left(\lambda'\right)^8\\
&\quad-64e^{8\lambda}\left(-2g^2\xi^2+8e^{2\lambda}\left(l^2+l-2\right)+15e^{\lambda}r_s\right)\mu'\left(\lambda'\right)^7\\
&\quad+16e^{4\lambda}\Big(6e^{4\lambda+2\mu}l^4+12e^{4\lambda+2\mu}l^3-26e^{4\lambda+2\mu}l^2-9e^{2(\lambda+\mu)}g^2\xi^2l^2+34e^{3\lambda+2\mu}r_sl^2-4e^{6\lambda}\mu''l^2-32e^{4\lambda+2\mu}l-9e^{2(\lambda+\mu)}g^2\xi^2l\\
&\quad+34e^{3\lambda+2\mu}r_sl-4e^{6\lambda}\mu''l+40e^{4\lambda+2\mu}+7e^{2\mu}g^4\xi^4+34e^{2(\lambda+\mu)}g^2\xi^2+66e^{2(\lambda+\mu)}r_s^2+4e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\left(\mu'\right)^2\\
&\quad-104e^{3\lambda+2\mu}r_s-43e^{\lambda+2\mu}g^2\xi^2r_s+4e^{4\lambda}\left(-4g^2\xi^2+6e^{2\lambda}\left(l^2+l-2\right)+15e^{\lambda}r_s\right)\lambda''+8e^{6\lambda}\mu''+4e^{4\lambda}g^2\xi^2\mu''-12e^{5\lambda}r_s\mu''\Big)\left(\lambda'\right)^6\\
&\quad-16\Big(e^{4\lambda}\mu'\Big(e^{2\mu}\Big(3g^4\xi^4-25e^{\lambda}g^2r_s\xi^2+2e^{4\lambda}\left(l^4+2l^3-13l^2-14l+24\right)+4e^{3\lambda}\left(8l^2+8l-25\right)r_s+e^{2\lambda}\left(g^2\left(-9l^2-9l+22\right)\xi^2+54r_s^2\right)\Big)\\
&\quad+4e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda''\Big)-4e^{8\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda^{(3)}\Big)\left(\lambda'\right)^5\\
&\quad+\Big(-16e^{4\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\left(\mu'\right)^2-64e^{8\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\left(\lambda''\right)^2\\
&\quad+e^{4\mu}\Big(-9g^6\xi^6+92e^{\lambda}g^4r_s\xi^4+e^{2\lambda}g^2\left(g^2\left(19l^2+19l-86\right)\xi^2-296r_s^2\right)\xi^2+4e^{6\lambda}\left(3l^6+9l^5-17l^4-49l^3+30l^2+56l-32\right)\\
&\quad+4e^{5\lambda}\left(29l^4+58l^3-99l^2-128l+140\right)r_s+4e^{3\lambda}r_s\left(g^2\left(-37l^2-37l+130\right)\xi^2+75r_s^2\right)\\
&\quad+4e^{4\lambda}\left(3\left(23l^2+23l-62\right)r_s^2-4g^2\left(2l^4+4l^3-7l^2-9l+13\right)\xi^2\right)\Big)\\
&\quad+16e^{4\lambda+2\mu}\left(5g^4\xi^4-33e^{\lambda}g^2r_s\xi^2+2e^{4\lambda}\left(l^4+2l^3-9l^2-10l+16\right)+12e^{3\lambda}\left(2l^2+2l-7\right)r_s+e^{2\lambda}\left(g^2\left(-7l^2-7l+26\right)\xi^2+54r_s^2\right)\right)\lambda''\\
&\quad+16e^{4\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\mu''\Big)\left(\lambda'\right)^4\\
&\quad-e^{2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\Big(\mu'\Big(e^{2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(-4g^2\xi^2+6e^{2\lambda}\left(l^2+l-2\right)+15e^{\lambda}r_s\right)\\
&\quad-16e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda''\Big)+16e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda^{(3)}\Big)\left(\lambda'\right)^3\\
&\quad+e^{2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\Big(e^{2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\left(\mu'\right)^2\\
&\quad+16e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\left(\lambda''\right)^2+e^{2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(-6g^2\xi^2+4e^{2\lambda}\left(l^2+l-2\right)+15e^{\lambda}r_s\right)\lambda''\\
&\quad-e^{2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\mu''\Big)\left(\lambda'\right)^2\\
&\quad-e^{4\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\left(\mu'\lambda''-\lambda^{(3)}\right)\lambda'
\end{aligned}
$$

$$-e^{4\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\left(\lambda''\right)^2\Big)\Big]$$

The contributions proportional to $(Q^e)^2$ are

$$\frac{1}{8(l-1)(l+2)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda'\left(e^{2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)-8e^{4\lambda}\left(\lambda'\right)^2\right)^2\left(e^{2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)-4e^{4\lambda}\left(\lambda'\right)^2\right)}\times$$

$$\times\Big[e^{-3\lambda-2\mu}\Big(1024e^{14\lambda}\Big(-g^4\xi^4-21e^{\lambda}g^2r_s\xi^2+7e^{4\lambda}\left(l^2+l-2\right)^2+39e^{3\lambda}\left(l^2+l-2\right)r_s+2e^{2\lambda}\left(27r_s^2-5g^2\left(l^2+l-2\right)\xi^2\right)\Big)\left(\lambda'\right)^9$$

$$-1024e^{15\lambda}\left(8e^{\lambda}\left(l^2+l-2\right)+15r_s\right)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\mu'\left(\lambda'\right)^8$$

$$-256e^{10\lambda}\Big(6e^{6\lambda+2\mu}l^6+18e^{6\lambda+2\mu}l^5+58e^{6\lambda+2\mu}l^4+4e^{4\lambda+2\mu}g^2\xi^2l^4-30e^{5\lambda+2\mu}r_sl^4+4e^{8\lambda}\mu''l^4+86e^{6\lambda+2\mu}l^3+8e^{4\lambda+2\mu}g^2\xi^2l^3$$

$$-60e^{5\lambda+2\mu}r_sl^3+8e^{8\lambda}\mu''l^3-192e^{6\lambda+2\mu}l^2-21e^{2(\lambda+\mu)}g^4\xi^4l^2-136e^{4\lambda+2\mu}g^2\xi^2l^2-318e^{4\lambda+2\mu}r_s^2l^2+522e^{5\lambda+2\mu}r_sl^2$$

$$+162e^{3\lambda+2\mu}g^2\xi^2r_sl^2-12e^{8\lambda}\mu''l^2-8e^{6\lambda}g^2\xi^2\mu''l^2+24e^{7\lambda}r_s\mu''l^2-232e^{6\lambda+2\mu}l-21e^{2(\lambda+\mu)}g^4\xi^4l-140e^{4\lambda+2\mu}g^2\xi^2l$$

$$-318e^{4\lambda+2\mu}r_s^2l+552e^{5\lambda+2\mu}r_sl+162e^{3\lambda+2\mu}g^2\xi^2r_sl-16e^{8\lambda}\mu''l-8e^{6\lambda}g^2\xi^2\mu''l+24e^{7\lambda}r_s\mu''l+256e^{6\lambda+2\mu}+4e^{2\mu}g^6\xi^6$$

$$+62e^{2(\lambda+\mu)}g^4\xi^4-558e^{3\lambda+2\mu}r_s^3+264e^{4\lambda+2\mu}g^2\xi^2+1284e^{4\lambda+2\mu}r_s^2+411e^{2(\lambda+\mu)}g^2\xi^2r_s^2$$

$$-4e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2\left(\mu'\right)^2-984e^{5\lambda+2\mu}r_s-83e^{\lambda+2\mu}g^4\xi^4r_s-648e^{3\lambda+2\mu}g^2\xi^2r_s$$

$$-4e^{4\lambda}\Big(2g^4\xi^4-21e^{\lambda}g^2r_s\xi^2+6e^{4\lambda}\left(l^2+l-2\right)^2+33e^{3\lambda}\left(l^2+l-2\right)r_s+e^{2\lambda}\left(45r_s^2-8g^2\left(l^2+l-2\right)\xi^2\right)\Big)\lambda''$$

$$+16e^{8\lambda}\mu''+4e^{4\lambda}g^4\xi^4\mu''+16e^{6\lambda}g^2\xi^2\mu''+36e^{6\lambda}r_s^2\mu''-48e^{7\lambda}r_s\mu''-24e^{5\lambda}g^2\xi^2r_s\mu''\Big)\left(\lambda'\right)^7$$

$$+256e^{10\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\Big(\mu'\Big(e^{2\mu}\Big(-5g^4\xi^4+60e^{\lambda}g^2r_s\xi^2+2e^{4\lambda}\left(l^4+2l^3+35l^2+34l-72\right)$$

$$-4e^{3\lambda}\left(17l^2+17l-76\right)r_s+e^{2\lambda}\left(g^2\left(17l^2+17l-54\right)\xi^2-162r_s^2\right)\Big)-4e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda''\Big)$$

$$+4e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda^{(3)}\Big)\left(\lambda'\right)^6$$

$$-16e^{6\lambda}\Big(48e^{4\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(\mu'\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2$$

$$+64e^{8\lambda}\left(\lambda''\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2-32e^{4\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\mu''\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2$$

$$+e^{4\mu}\Big(-23g^8\xi^8+415e^{\lambda}g^6r_s\xi^6-3e^{3\lambda}g^2r_s\left(g^2\left(115l^2+115l-1446\right)\xi^2-2404r_s^2\right)\xi^2$$

$$+4e^{8\lambda}\left(l^2+l-2\right)^2\left(l^4+2l^3-51l^2-52l-192\right)+8e^{7\lambda}\left(29l^6+87l^5+19l^4-107l^3-966l^2-898l+1836\right)r_s$$

$$-8e^{5\lambda}r_s\left(g^2\left(49l^4+98l^3+475l^2+426l-1810\right)\xi^2+3\left(103l^2+103l-938\right)r_s^2\right)$$

$$+4e^{6\lambda}\left(12\left(10l^4+20l^3+165l^2+155l-566\right)r_s^2-5g^2\left(3l^6+9l^5-5l^4-25l^3-98l^2-84l+200\right)\xi^2\right)$$

$$+e^{2\lambda}\left(2g^6\left(11l^2+11l-166\right)\xi^6-2660g^4\xi^4r_s^2\right)$$

$$+e^{4\lambda}\left(g^4\left(69l^4+138l^3+449l^2+380l-1756\right)\xi^4+8g^2\left(205l^2+205l-2216\right)r_s^2\xi^2-6948r_s^4\right)\Big)$$

$$+16e^{4\lambda+2\mu}\Big(9g^6\xi^6-2e^{2\lambda}g^4\left(11l^2+11l-40\right)\xi^4+e^{4\lambda}g^2\left(11l^4+22l^3-129l^2-140l+236\right)\xi^2-486e^{3\lambda}r_s^3$$

$$-6e^{2\lambda}\left(e^{2\lambda}\left(53l^2+53l-190\right)-65g^2\xi^2\right)r_s^2+2e^{6\lambda}\left(l^2+l-2\right)^2\left(l^2+l+28\right)$$

$$+\left(-103e^{\lambda}g^4\xi^4+e^{3\lambda}g^2\left(167l^2+167l-610\right)\xi^2+e^{5\lambda}\left(-46l^4-92l^3+486l^2+532l-880\right)\right)r_s\Big)\lambda''\Big)\left(\lambda'\right)^5$$

$$-16e^{6\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\Big(\mu'\Big(e^{2\mu}\Big(-22g^4\xi^4+211e^{\lambda}g^2r_s\xi^2$$

$$+8e^{4\lambda}\left(2l^4+4l^3+23l^2+21l-50\right)-4e^{3\lambda}\left(38l^2+38l-223\right)r_s+e^{2\lambda}\left(2g^2\left(17l^2+17l-94\right)\xi^2-492r_s^2\right)\Big)$$

$$-64e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda''\Big)+32e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda^{(3)}\Big)\left(\lambda'\right)^4$$

$$+4e^{2(\lambda+\mu)}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\Big(36e^{4\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(\mu'\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2$$

$$+64e^{8\lambda}\left(\lambda''\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2-20e^{4\lambda+2\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\mu''\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2$$

$$+e^{4\mu}\Big(-7g^8\xi^8+119e^{\lambda}g^6r_s\xi^6-e^{2\lambda}g^4\left(g^2\left(19l^2+19l+98\right)\xi^2+728r_s^2\right)\xi^4+4e^{3\lambda}g^2r_s\left(g^2\left(34l^2+34l+297\right)\xi^2+477r_s^2\right)\xi^2$$

$$+8e^{8\lambda}\left(l^2+l-2\right)^2\left(l^4+2l^3-15l^2-16l-26\right)+8e^{7\lambda}\left(22l^6+66l^5-47l^4-204l^3-207l^2-94l+464\right)r_s$$

$$+2e^{5\lambda}r_s\left(g^2\left(-201l^4-402l^3-127l^2+74l+1856\right)\xi^2+12\left(l^2+l+238\right)r_s^2\right)$$

$$+8e^{6\lambda}\left(g^2\left(-6l^6-18l^5+23l^4+76l^3+47l^2+6l-128\right)\xi^2+\left(73l^4+146l^3+126l^2+53l-848\right)r_s^2\right)$$

$$-e^{4\lambda}\left(g^4\left(-65l^4-130l^3+11l^2+76l+476\right)\xi^4+2g^2\left(125l^2+125l+2312\right)r_s^2\xi^2+1800r_s^4\right)\Big)$$

$$+4e^{4\lambda+2\mu}\Big(32g^6\xi^6-347e^{\lambda}g^4r_s\xi^4+16e^{6\lambda}\left(l^2+l-2\right)^2\left(l^2+l+10\right)-4e^{5\lambda}\left(16l^4+32l^3-339l^2-355l+646\right)r_s$$

$$-3e^{3\lambda}r_s\left(g^2\left(-145l^2-145l+646\right)\xi^2+492r_s^2\right)-4e^{4\lambda}\left(9\left(23l^2+23l-95\right)r_s^2-2g^2\left(l^4+2l^3-47l^2-48l+92\right)\xi^2\right)$$

$$+e^{2\lambda}\left(1245g^2\xi^2r_s^2-8g^4\left(7l^2+7l-34\right)\xi^4\right)\right)\lambda''\Big)\left(\lambda'\right)^3$$
$$+\,4e^{2\lambda+4\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\Big(\mu'\Big(e^{2\mu}\Big(-6g^4\xi^4+55e^{\lambda}g^2r_s\xi^2$$
$$+8e^{4\lambda}\left(l^4+2l^3+5l^2+4l-12\right)-20e^{3\lambda}\left(l^2+l-11\right)r_s+2e^{2\lambda}\left(g^2\left(l^2+l-26\right)\xi^2-60r_s^2\right)\Big)$$
$$-52e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda''\Big)+20e^{4\lambda}\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)\lambda^{(3)}\Big)\left(\lambda'\right)^2$$
$$-\,e^{4\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)^2\Big(8e^{2(\lambda+\mu)}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\left(\mu'\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2$$
$$+\,16e^{6\lambda}\left(\lambda''\right)^2\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2-4e^{2(\lambda+\mu)}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)\mu''\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2$$
$$+\,e^{4\mu}\Big(g^6\left(-5l^2-5l+2\right)\xi^6+e^{\lambda}g^4\left(45l^2+45l-22\right)r_s\xi^4+2e^{2\lambda}g^2\Big(2g^2\left(3l^4+6l^3-7l^2-10l+4\right)\xi^2-5\left(13l^2+13l-8\right)r_s^2\Big)\xi^2$$
$$+\,4e^{6\lambda}l\left(l^2+l-2\right)^2\left(l^3+2l^2-3l-4\right)+8e^{5\lambda}\left(l^2+l-2\right)^2\left(5l^2+5l-3\right)r_s$$
$$+\,2e^{3\lambda}r_s\left(g^2\left(-37l^4-74l^3+77l^2+114l-56\right)\xi^2+12\left(5l^2+5l-4\right)r_s^2\right)+4e^{4\lambda}\left(l^2+l-2\right)\Big(g^2\left(-3l^4-6l^3+7l^2+10l-4\right)\xi^2$$
$$+3\left(9l^2+9l-8\right)r_s^2\Big)\Big)$$
$$+\,4e^{2(\lambda+\mu)}\Big(-360e^{3\lambda}r_s^3-3e^{2\lambda}\left(4e^{2\lambda}\left(13l^2+13l-71\right)-103g^2\xi^2\right)r_s^2$$
$$+\,3e^{\lambda}\left(-29g^4\xi^4+e^{2\lambda}g^2\left(25l^2+25l-166\right)\xi^2+4e^{4\lambda}\left(l^4+2l^3+26l^2+25l-54\right)\right)r_s$$
$$+8\left(e^{2\lambda}\left(l^2+l-2\right)-g^2\xi^2\right)^2\left(g^2\xi^2+e^{2\lambda}\left(l^2+l+5\right)\right)\Big)\lambda''\Big)\lambda'$$
$$+4e^{2\lambda+6\mu}\left(g^2\xi^2+4e^{2\lambda}-4e^{\lambda}r_s\right)^3\left(-g^2\xi^2+e^{2\lambda}\left(l^2+l-2\right)+3e^{\lambda}r_s\right)^2\left(3\mu'\lambda''-\lambda^{(3)}\right)\Big)\Big]$$

List of Symbols

Differential Geometry:

$\mathcal{M}$	General differential manifold, most of the time four-dimensional
$C^\infty(\mathcal{M})$	Space of smooth functions on $\mathcal{M}$
$T^a_b(\mathcal{M})$	Space of smooth tensor fields on $\mathcal{M}$
$\Lambda_n(\mathcal{M})$	Space of smooth n-forms on $\mathcal{M}$
$\mathcal{L}_v t$	Lie derivative of the tensor field t with respect to the vector field v
g_{ij}	Pseudo-Riemannian metric on $\mathcal{M}$ with signature $(-,+,+,+)$
∇	Connection on $\mathcal{M}$
Γ^i_{jk}	Connection coefficients of the connection ∇
$T^i{}_{jk}$	Torsion tensor of the connection ∇
$R^i{}_{jkl}, R_{ij}, R$	Riemann curvature tensor, Ricci tensor and Ricci scalar of the connection ∇
T_{ij}	Energy momentum tensor
M	Two-dimensional manifold with metric g_{ab} and connection ∇
S^2	Two-sphere with metric Ω_{AB} and connection D_A
$\mathcal{F} = \cup_t \Sigma_t$	Foliation of a spacetime $\mathcal{M}$ into Cauchy surfaces Σ_t
Σ	Hypersurface in a foliation of spacetime with metric $m_{\mu\nu}$, connection $\overline{\nabla}$, and extrinsic curvature $K_{\mu\nu}$

Indices:

$i, j, \ldots$	Index on the full manifold $\mathcal{M}$
$a, b, \ldots$	Index on the two-dimensional manifold M
$A, B, \ldots$	Index on the sphere S^2
$\mu, \nu, \ldots$	Index on the hypersurface of the foliation Σ

Hamiltonian Formulation:

V_0, V_μ	Full Hamiltonian and diffeomorphism constraints of general relativity coupled to matter
B	Boundary term of the constraints
C	Symmetric constraints with respect to some symmetry group
Z	Non-symmetric constraints with respect to some symmetry group
(x, y)	Non-symmetric gauge degrees of freedom
(X, Y)	Non-symmetric true degrees of freedom

Quantum theory:

$\mathcal{H}$	Hilbert space
$\mathcal{F}_S(\mathcal{H})$	Symmetric Fock space constructed from the Hilbert space $\mathcal{H}$
$a, a^\dagger$	Creation and annihilation operators
A_{IJ}, B_{ij}	Bogoliubov coefficients
$\langle\cdot,\cdot\rangle$	Klein-Gordon inner product

Constants: In the thesis we work in units $c = G = \hbar = 1$.

Λ	Cosmological constant
$\kappa = 16\pi$	Gravitational constant
g	Coupling constant of the electromagnetic field

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