Analysis of constraint systems using the Clairaut equation

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Abstract

We extend the Legendre transform to non-convex functions with vanishing Hessian using a mix of envelope and general solutions of the Clairaut equation. Applying this to constraint systems, the procedure of finding a Hamiltonian for a singular Lagrangian is just that of solving a corresponding Clairaut equation with a subsequent application of the proposed Legendre-Clairaut transformation. We obtain the unconstrained version of Hamilton's equations. We demonstrate the origin of the Dirac primary constraints, along with their explicit form, and this is done without using the Lagrange multiplier method.

Modern field theories are in fact degenerate dynamical systems whose key feature is the presence of constraints [1, 2]. The most common way to deal with such systems is to use the Dirac approach [3] based on extending a phase space and constructing the so-called total Hamiltonian. In spite of its general success, e.g. in describing systems with gauge symmetries and gravity [4, 5], the Dirac approach has limited applicability and some inner problems [6]. Therefore, it is worthwhile to reconsider several basic ideas of the Hamiltonian formalism per se starting from the Legendre transformation treated as a solution of the Clairaut equation also in the singular case. The Dirac approach is based on the following idea: in finding a Hamiltonian, to use the standard definition of momenta, then perform the Legendre transformation and add Lagrange multipliers, subsequently to be removed by imposing some relations between constraints [3].

Here we revisit the procedure of finding a Hamiltonian for both regular and singular cases. This procedure is reduced to that of solving the Clairaut partial differential equation. In the case of regular systems a Hamiltonian corresponds to its envelope solution [7], while adding the general solution of Clairaut equation leads to the total Hamiltonian of singular systems, and

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arbitrary constants correspond to the Lagrange multipliers within the Dirac approach [3]. Such solutions exist for smooth Lagrangian functions, while the standard Legendre transformation is applicable in the regular case only. To solve the Clairaut equation in the singular case, we introduce a half-envelope solution, which is an envelope solution in regular variables and a general solution in nonregular variables. We use the coordinate language which is convenient for making the basic idea transparent as well as for further applications.

First recall the standard Legendre transform definition [7]. The Legendre transform of a convex function[1] $F:\mathbb{R}^n\to\mathbb{R}$ is a map $\mathfrak{Leg}: F\longmapsto G$, where $G(\mathbf{p}):\mathbb{R}^{n*}\to\mathbb{R}$ is another convex function (in the dual space) such that $G(\mathbf{p})=\max_{\mathbf{x}}\tilde{G}(\mathbf{p},\mathbf{x})$, and $\tilde{G}(\mathbf{p},\mathbf{x})\overset{def}{=}\mathbf{p}\cdot\mathbf{x}-F(\mathbf{x})$. The maximum is attained, when

$$\frac{\partial\tilde{G}(\mathbf{p},\mathbf{x})}{\partial\mathbf{x}}=\mathbf{p}-\frac{\partial F(\mathbf{x})}{\partial\mathbf{x}}=0, \tag{1}$$

which, for a given $\mathbf{p}$, can determine $\mathbf{x}$ unambiguously, but not vice versa. The convexity implies that the Hessian $H_F(\mathbf{x})\overset{def}{=}\det\left\|\frac{\partial^2F(\mathbf{x})}{\partial\mathbf{x}^2}\right\|$ is positive, which ensures that (globally) there is only one critical point, the maximum [7].

Now we reformulate the Legendre transform in terms of the differential equation for $G(\mathbf{p})$ as above. Let us suppose that (1) has a solution $\mathbf{x}=\mathbf{X}(\mathbf{p})$ with $\mathbf{X}:\mathbb{R}^n\to\mathbb{R}^n$, then

$$G(\mathbf{p})\overset{def}{=}\tilde{G}(\mathbf{p},\mathbf{X}(\mathbf{p}))=\mathbf{p}\cdot\mathbf{X}(\mathbf{p})-F(\mathbf{X}(\mathbf{p}))\,. \tag{2}$$

Since $\frac{\partial^2G(\mathbf{p})}{\partial\mathbf{p}^2}\overset{(2)}{=}\left(\frac{\partial\mathbf{X}(\mathbf{p})}{\partial\mathbf{p}}\right)^2\frac{\partial^2F(\mathbf{x})}{\partial\mathbf{x}^2}$, the functions F and G belong to the same convexity class and in the case of convex functions one has $rank\left\|\frac{\partial^2G(\mathbf{p})}{\partial\mathbf{p}^2}\right\|=rank\left\|\frac{\partial^2F(\mathbf{x})}{\partial\mathbf{x}^2}\right\|$. Next we differentiate (2) and get $\frac{\partial G(\mathbf{p})}{\partial\mathbf{p}}\overset{(1)}{=}\mathbf{X}(\mathbf{p})$, which allows us to exclude $\mathbf{X}(\mathbf{p})$ from (2) and obtain

$$G^{Cl}(\mathbf{p})=\mathbf{p}\cdot\frac{\partial G^{Cl}(\mathbf{p})}{\partial\mathbf{p}}-F\left(\frac{\partial G^{Cl}(\mathbf{p})}{\partial\mathbf{p}}\right), \tag{3}$$

which is just the Clairaut equation [9]. Obviously, (3) has solutions even in cases, where (1) cannot be resolved in $\mathbf{x}$, and therefore the standard

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[1] We use vector notation [7] in which $\mathbf{x}\in\overbrace{\mathbb{R}\times\mathbb{R}\times\ldots\mathbb{R}}^{n}=\mathbb{R}^n$, and $\mathbf{x}\cdot\mathbf{y}$ is the scalar product. Functions and variables are denoted by capitals and lowercase letters, respectively. A scalar differentiable function $F:\mathbb{R}^n\to\mathbb{R}$ is denoted by $F(\mathbf{x})$. A vector $\partial F/\partial\mathbf{x}$ denotes a gradient of F whose entries are just partial derivatives $\partial F/\partial x^i$, $i=1,\ldots[i]=n$, and for $\partial^2F/\partial x^i\partial x^j$ we sometimes use the notation $\partial^2F/\partial\mathbf{x}^2$. Also, $\partial\mathbf{V}/\partial\mathbf{x}$ denotes the divergence of a vector function $\mathbf{V}:\mathbb{R}^n\to\mathbb{R}^n$ in a similar manner.

Legendre transform (2) does not exist (thus we add the superscript Cl). We call a map $\mathfrak{Leg}^{Cl}: F \longmapsto G^{Cl}$ defined by (3) a (generalized) Legendre-Clairaut transform with $F(\mathbf{x})$ being any (smooth) function (not necessarily convex). For a convex function $F(\mathbf{x})$, if $G(\mathbf{p}) = Leg_F(\mathbf{p})$ is its standard Legendre transform, then $G(\mathbf{p})$ satisfies (3), hence $G(\mathbf{p})$ coincides with $G^{Cl}(\mathbf{p}) = Leg_F^{Cl}(\mathbf{p})$, the Legendre-Clairaut transform of $F(\mathbf{x})$. Now we demonstrate the converse statement. Let us write a general solution of (3) as

$$\tilde{G}_{gen}^{Cl}(\mathbf{p},\mathbf{c}) = \mathbf{p}\cdot\mathbf{c} - F(\mathbf{c}), \tag{4}$$

where $\mathbf{c} \in \mathbb{R}^N$. The envelope solution can be obtained from the extremum condition $\frac{\partial \tilde{G}_{gen}^{Cl}(\mathbf{p},\mathbf{c})}{\partial \mathbf{c}} = \mathbf{p} - \frac{\partial F(\mathbf{c})}{\partial \mathbf{c}} = 0$, which coincides with the condition (1) (this $\tilde{G}_{gen}^{Cl}$ actually coincides with $\tilde{G}$ above). If $H_F(\mathbf{c}) > 0$ (F is convex), the extremum condition can be solved by $\mathbf{c} = \mathbf{X}(\mathbf{p})$. Then the envelope solution of (3) is

$$G_{env}^{Cl}(\mathbf{p}) \overset{def}{=} \tilde{G}_{gen}^{Cl}(\mathbf{p},\mathbf{X}(\mathbf{p})) = \mathbf{p}\cdot\mathbf{X}(\mathbf{p}) - F(\mathbf{X}(\mathbf{p})) \overset{(2)}{=} G(\mathbf{p}). \tag{5}$$

This means that, given a convex function $F(\mathbf{x})$, if $G^{Cl}(\mathbf{p}) = Leg_F^{Cl}(\mathbf{p})$, then $G^{Cl}(\mathbf{p}) = Leg_F(\mathbf{p}) = G(\mathbf{p})$. Also, only in this (convex) case both mappings $\mathfrak{Leg}^{Cl}$ and $\mathfrak{Leg}$ are involutive $\mathfrak{Leg}^{Cl} \circ \mathfrak{Leg}^{Cl} = id$, $\mathfrak{Leg} \circ \mathfrak{Leg} = id$. Thus, *the standard Legendre transforms in the class of convex functions are in 1-1 correspondence with the envelope solutions of the Clairaut equation* (3). This provides an exposition of the ordinary theory [7, 9] in a special way, which is convenient for our subsequent purposes that involve more general classes of functions related to the Hamiltonian structure of constraint systems [3, 5].

Indeed, let us consider the case of a non-convex function $F(\mathbf{x})$, when its Hessian $H_F(\mathbf{x})$ vanishes. The standard Legendre trick does not work, because (1) cannot be solved for $\mathbf{x}$ in this case [7]. On the other hand, the Clairaut equation (3) assumes nothing but smoothness about $F(\mathbf{x})$. Therefore, we can forget the condition of its obtaining and start from the Clairaut equation itself, then try to find the solutions. In this way, we can extend the Legendre-Clairaut transform to the degenerate case $H_F(\mathbf{x}) = 0$. Let $rank \left\| \frac{\partial^2 F(\mathbf{x})}{\partial \mathbf{x}^2} \right\| = k < n$ and k is constant on the domain of $F(\mathbf{x})$. Without loss of generality, we can assume the indices are rearranged in such a way that a non-singular minor of rank k is in the upper left-hand corner. Then, we express the index i as a pair $i = (i_1, i_2)$, $i_1 = 1, \dots k$, $i_2 = k+1, \dots n$, and correspondingly any vector variable is presented as $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1 \in \mathbb{R}^k$, $\mathbf{x}_2 \in \mathbb{R}^{n-k}$. We call the first and the second entry of $\mathbf{x}$ as regular and nonregular, respectively. In this notation, the Clairaut equation (3) becomes

$$G^{Cl}(\mathbf{p}_1,\mathbf{p}_2)=\mathbf{p}_1\cdot\frac{\partial G^{Cl}(\mathbf{p}_1,\mathbf{p}_2)}{\partial\mathbf{p}_1}+\mathbf{p}_2\cdot\frac{\partial G^{Cl}(\mathbf{p}_1,\mathbf{p}_2)}{\partial\mathbf{p}_2} \\ -F\left(\frac{\partial G^{Cl}(\mathbf{p}_1,\mathbf{p}_2)}{\partial\mathbf{p}_1},\frac{\partial G^{Cl}(\mathbf{p}_1,\mathbf{p}_2)}{\partial\mathbf{p}_2}\right). \tag{6}$$

By analogy with (4), the general solution is

$$\tilde{G}^{Cl}_{gen}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{c}_1,\mathbf{c}_2)=\mathbf{p}_1\cdot\mathbf{c}_1+\mathbf{p}_2\cdot\mathbf{c}_2-F(\mathbf{c}_1,\mathbf{c}_2), \tag{7}$$

where $\mathbf{c}_1\in\mathbb{R}^k$, $\mathbf{c}_2\in\mathbb{R}^{n-k}$. Our intention now is to search for the envelope solution (5) in regular variables only. But it is still the general solution (4) with arbitrary $\mathbf{c}_2$ with respect to non-regular variables. We call such a solution *a half-envelope solution of the Clairaut equation*. Differentiate (7) in $\mathbf{c}_1$ to get

$$\frac{\partial\tilde{G}^{Cl}_{gen}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{c}_1,\mathbf{c}_2)}{\partial\mathbf{c}_1}=\mathbf{p}_1-\frac{\partial F(\mathbf{c}_1,\mathbf{c}_2)}{\partial\mathbf{c}_1}=0 \tag{8}$$

Since the sub-Hessian $H^{(1)}_F(\mathbf{x})\overset{def}{=}\det\left\|\frac{\partial^2F(\mathbf{x}_1,\mathbf{x}_2)}{\partial\mathbf{x}_1^2}\right\|$ does not vanish, we can resolve (8) with respect to $\mathbf{c}_1$ and obtain $\mathbf{c}_1=\mathbf{X}(\mathbf{p}_1,\mathbf{c}_2)$. The subsequent substitution to (7) yields

$$G^{Cl}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{c}_2)=\tilde{G}^{Cl}_{gen}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{X}(\mathbf{p}_1,\mathbf{c}_2),\mathbf{c}_2) \\ =\mathbf{p}_1\cdot\mathbf{X}(\mathbf{p}_1,\mathbf{c}_2)+\mathbf{p}_2\cdot\mathbf{c}_2-F(\mathbf{X}(\mathbf{p}_1,\mathbf{c}_2),\mathbf{c}_2), \tag{9}$$

which can be treated as an explicit form of the Legendre-Clairaut transform for a non-convex function $F(\mathbf{x})$ having degenerate Hessian matrix. Note that the relation between the Legendre transform and the parallel curves (which is connected with the above general solution) was considered a long time ago [10, 11].

Now we apply the Legendre-Clairaut transform to the Hamiltonian procedure for constraint systems with finite number of degrees of freedom (in the language of classical mechanics). This is sufficient for exploring the main idea which can be easily generalized to a field theory (e.g. using DeWitt's condensed notation [12].

Let Q be a n-dimensional configuration space being a smooth manifold with local coordinates $\mathbf{q}=\left(q^1,\dots q^n\right)$ (all statements can be translated into the coordinate free language [13, 8]). A Lagrangian on Q is a continuous function $\mathcal{L}:T\mathsf{Q}\to\mathbb{R}$ that is smooth on the tangle bundle $T\mathsf{Q}\diagdown\{0\}$ which in local coordinates is determined by[2] $(\mathbf{q},\mathbf{v})$, where $\mathbf{v}(t)=\frac{d\mathbf{q}(t)}{dt}$ are velocities.

[2]As we consider time independent Lagrangians for conciseness, the time-dependent case can be treated similarly.

In this notation the Euler-Lagrange equations of motion are

$$\frac{d}{dt}\frac{\partial\mathcal{L}_{(\mathbf{q})}(\mathbf{v})}{\partial\mathbf{v}}-\frac{\partial\mathcal{L}_{(\mathbf{q})}(\mathbf{v})}{\partial\mathbf{q}}=\sum_{i=1}^{n}\left(\mathsf{W}_{(\mathbf{q})ij}(\mathbf{v})\,\dot{v}^{i}-K_{(\mathbf{q})i}(\mathbf{v})\right)=0, \tag{10}$$

where $\mathsf{W}_{(\mathbf{q})ij}(\mathbf{v})\overset{def}{=}\partial^{2}\mathcal{L}_{(\mathbf{q})}(\mathbf{v})\diagup\partial v^{i}\partial v^{j}$ is the Hessian matrix and $K_{(\mathbf{q})i}(\mathbf{v})\overset{def}{=}\frac{\partial\mathcal{L}_{(\mathbf{q})}(\mathbf{v})}{\partial q^{i}}-\sum_{j=1}^{n}v^{j}\frac{\partial^{2}\mathcal{L}_{(\mathbf{q})}(\mathbf{v})}{\partial v^{i}\partial q^{j}}$.

The standard Legendre transformation [7] is a local mapping $T\mathsf{Q}\rightarrow T^{*}\mathsf{Q}$ (the latter is the phase space which is $(\mathbf{q},\mathbf{p})$ in local coordinates) or $\mathfrak{Leg}:\mathcal{L}\rightarrow\mathcal{H}$, where $\mathcal{H}:T^{*}\mathsf{Q}\rightarrow\mathbb{R}$ is a Hamiltonian.

First consider the regular case, when the Hessian $H_{\mathcal{L}_{(\mathbf{q})}}(\mathbf{v})\overset{def}{=}\det\left\|W_{(\mathbf{q})ij}(\mathbf{v})\right\|$ is nonvanishing. Indeed, the above observations imply that it is the Legendre transform in velocities (considering $\mathbf{q}$ as parameters[3]) such that $\mathcal{H}_{(\mathbf{q})}(\mathbf{p})=\max_{\mathbf{v}}\tilde{\mathcal{H}}_{(\mathbf{q})}(\mathbf{p},\mathbf{v})$, where $\tilde{\mathcal{H}}_{(\mathbf{q})}(\mathbf{p},\mathbf{v})\overset{def}{=}\mathbf{p}\cdot\mathbf{v}-\mathcal{L}_{(\mathbf{q})}(\mathbf{v})$. The extremum occurs, when

$$\frac{\partial\tilde{\mathcal{H}}_{(\mathbf{q})}(\mathbf{p},\mathbf{v})}{\partial\mathbf{v}}=\mathbf{p}-\frac{\partial\mathcal{L}_{(\mathbf{q})}(\mathbf{v})}{\partial\mathbf{v}}=0, \tag{11}$$

which can be resolved with respect to velocities $\mathbf{v}=\mathbf{V}_{(\mathbf{q})}(\mathbf{p})$, since the Hessian is nonvanishing. Then in the regular case the Hamiltonian is

$$\mathcal{H}_{(\mathbf{q})}(\mathbf{p})\overset{def}{=}\tilde{\mathcal{H}}_{(\mathbf{q})}\left(\mathbf{p},\mathbf{V}_{(\mathbf{q})}(\mathbf{p})\right)=\mathbf{p}\cdot\mathbf{V}_{(\mathbf{q})}(\mathbf{p})-\mathcal{L}_{(\mathbf{q})}\left(\mathbf{V}_{(\mathbf{q})}(\mathbf{p})\right). \tag{12}$$

Now we differentiate $\mathcal{H}_{(\mathbf{q})}(\mathbf{p})$ and obtain

$$\frac{\partial\mathcal{H}_{(\mathbf{q})}(\mathbf{p})}{\partial\mathbf{p}}\overset{(11)}{=}\mathbf{V}_{(\mathbf{q})}(\mathbf{p}). \tag{13}$$

Because (13) holds for all $\mathbf{q},\mathbf{p}$ identically for a solution of (11), we are able to substitute $\mathbf{V}_{(\mathbf{q})}(\mathbf{p})$ into (12) and obtain the Clairaut equation for the Hamiltonian as follows (cf. (3))

$$\mathcal{H}_{(\mathbf{q})}^{Cl}(\mathbf{p})=\mathbf{p}\cdot\frac{\partial\mathcal{H}_{(\mathbf{q})}^{Cl}(\mathbf{p})}{\partial\mathbf{p}}-\mathcal{L}_{(\mathbf{q})}\left(\frac{\partial\mathcal{H}_{(\mathbf{q})}^{Cl}(\mathbf{p})}{\partial\mathbf{p}}\right). \tag{14}$$

We call this map *a (generalized) Legendre-Clairaut transformation* $\mathfrak{Leg}^{Cl}:\mathcal{L}\rightarrow\mathcal{H}^{Cl}$, because (14) has a solution also in the case of singular Lagrangians. In the regular case we follow the steps of the previous section in considering a general solution of (14)

$$\tilde{\mathcal{H}}_{(\mathbf{q})gen}^{Cl}(\mathbf{p},\mathbf{v})=\mathbf{p}\cdot\mathbf{v}-\mathcal{L}_{(\mathbf{q})}(\mathbf{v}), \tag{15}$$

[3]We write $\mathbf{q}$-dependence as a subscript to single out coordinates as passive variables or parameters under the Legendre transformation for which $\mathbf{v}$, $\mathbf{p}$ are the active variables.

where initially $\mathbf{v} = \mathbf{C}_{(\mathbf{q})}$ are constants with respect to the active variables $\mathbf{v}$, $\mathbf{p}$, i.e. arbitrary functions of $\mathbf{q}$ as parameters. The envelope solution of (15) is subject to the extremum condition

$$\frac{\partial \tilde{\mathcal{H}}^{Cl}_{(\mathbf{q})gen}(\mathbf{p},\mathbf{v})}{\partial \mathbf{v}} = \mathbf{p} - \frac{\partial \mathcal{L}_{(\mathbf{q})}(\mathbf{v})}{\partial \mathbf{v}} = 0, \tag{16}$$

which coincides with (11) and determines additional dependence on the momenta when we resolve (16) (which is possible because $H_{\mathcal{L}} \neq 0$), and we denote this solution by $\mathbf{v} = \mathbf{V}_{(\mathbf{q})}(\mathbf{p})$. After substituting into (15) we obtain the envelope solution of the Clairaut equation (14) as

$$\mathcal{H}^{Cl}_{(\mathbf{q})env}(\mathbf{p}) \stackrel{def}{=} \tilde{\mathcal{H}}^{Cl}_{(\mathbf{q})gen}(\mathbf{p},\mathbf{v})\,|_{\mathbf{v}=\mathbf{V}_{(\mathbf{q})}(\mathbf{p})} = \mathbf{p}\cdot\mathbf{V}_{(\mathbf{q})}(\mathbf{p}) - \mathcal{L}_{(\mathbf{q})}\left(\mathbf{V}_{(\mathbf{q})}(\mathbf{p})\right)), \tag{17}$$

which coincides with the standard Legendre transformation (12), as it should be in the regular case [9].

Consider a singular dynamical system for which the Hessian $H_{\mathcal{L}}$ vanishes. Direct application of the standard Legendre transformation is not possible now, because (11) cannot be solved with respect to velocities $\mathbf{v}$. But in the Clairaut equation (14) there are no restrictions on $\mathcal{L}_{(\mathbf{q})}(\mathbf{v})$ except smoothness, and so we are able to consider the (generalized) Legendre-Clairaut transformation (14) in the singular case $H_{\mathcal{L}_{(\mathbf{q})}}(\mathbf{v}) = 0$ as well.

Let the rank of the Hessian matrix be less than half of the configuration space dimension $rank\left\|\mathsf{W}_{(\mathbf{q})ij}(\mathbf{v})\right\| = k < n$ and k is constant. We rearrange the indices i, j in such a way that a non-singular minor of rank k will be in the upper left-hand corner. Then, we express the index i as a pair $i = (i_1, i_2)$, $i_1 = 1, \dots k$, $i_2 = k+1, \dots n$, and decompose sets of coordinates and momenta as $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2)$, $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$ calling the first and the second set as regular and nonregular coordinates/momenta, respectively. In this notation the Hessian matrix is $\mathsf{W}_{ij} = \begin{pmatrix} \mathsf{W}_{i_1 i_1} & \mathsf{W}_{i_1 i_2} \\ \mathsf{W}_{i_2 i_1} & \mathsf{W}_{i_2 i_2} \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} \mathsf{W}^{(11)} & \mathsf{W}^{(12)} \\ \mathsf{W}^{(21)} & \mathsf{W}^{(22)} \end{pmatrix}$, where $\mathsf{W}^{(11)}$ is nonsingular $\det \mathsf{W}^{(11)} \neq 0$. It is worthwhile to note that $rank\ \mathsf{W}^{(11)} = k$ and $rank\ \mathsf{W}^{(22)} = 0$. Then the Clairaut equation (14) acquires the form

$$\mathcal{H}^{Cl}_{(\mathbf{q})}(\mathbf{p}) = \mathbf{p}_1 \cdot \frac{\partial \mathcal{H}^{Cl}_{(\mathbf{q})}(\mathbf{p})}{\partial \mathbf{p}_1} + \mathbf{p}_2 \cdot \frac{\partial \mathcal{H}^{Cl}_{(\mathbf{q})}(\mathbf{p})}{\partial \mathbf{p}_2} - \mathcal{L}_{(\mathbf{q})}\left(\frac{\partial \mathcal{H}^{Cl}_{(\mathbf{q})}(\mathbf{p})}{\partial \mathbf{p}_1}, \frac{\partial \mathcal{H}^{Cl}_{(\mathbf{q})}(\mathbf{p})}{\partial \mathbf{p}_2}\right), \tag{18}$$

which we treat as a definition of $\mathcal{H}^{Cl}_{(\mathbf{q})}(\mathbf{p})$ in the case of singular Lagrangians.

We cannot derive this, as in the regular case, because there is no relation (16) for nonregular variables. A general solution of this partial differential equation is

$$\tilde{\mathcal{H}}^{Cl}_{(\mathbf{q})gen}(\mathbf{p},\mathbf{v}_1,\mathbf{v}_2) = \mathbf{p}_1\cdot\mathbf{v}_1 + \mathbf{p}_2\cdot\mathbf{v}_2 - \mathcal{L}_{(\mathbf{q})}(\mathbf{v}_1,\mathbf{v}_2), \tag{19}$$

where $\mathbf{v}_1 = \mathbf{C}_{1(\mathbf{q})}, \mathbf{v}_2 = \mathbf{C}_{2(\mathbf{q})}$ are arbitrary functions of the passive variables $\mathbf{q}$. As compared to (16), we can find the envelope solution for the regular part only, i.e. we obtain the half-envelope solution. The extremum condition for regular variables now is

$$\frac{\partial \tilde{\mathcal{H}}^{Cl}_{(\mathbf{q})gen}(\mathbf{p}, \mathbf{v}_1, \mathbf{v}_2)}{\partial \mathbf{v}_1} = \mathbf{p}_1 - \frac{\partial \mathcal{L}_{(\mathbf{q})}(\mathbf{v}_1, \mathbf{v}_2)}{\partial \mathbf{v}_1} = 0, \tag{20}$$

which can be solved (due to $\det \mathsf{W}^{(11)} \neq 0$) as $\mathbf{v}_1 = \mathbf{V}_{(\mathbf{q})}(\mathbf{p}_1, \mathbf{v}_2)\,|_{\mathbf{v}_2=\mathbf{C}_2(\mathbf{q})} = \mathbf{V}_{(\mathbf{q})}\left(\mathbf{p}_1, \mathbf{C}_{2(\mathbf{q})}\right)$. Then we substitute this solution into (19) and obtain the "half-Hamiltonian" (or unconstrained Hamiltonian) in the form

$$\begin{aligned} \mathcal{H}^{Cl}_{(\mathbf{q})half}(\mathbf{p}, \mathbf{v}_2) &\overset{def}{=} \tilde{\mathcal{H}}^{Cl}_{(\mathbf{q})gen}(\mathbf{p}, \mathbf{v}_1, \mathbf{v}_2)\,|_{\mathbf{v}_1=\mathbf{V}_{(\mathbf{q})}(\mathbf{p}_1,\mathbf{v}_2)} \\ &= \mathbf{p}_1 \cdot \mathbf{V}_{(\mathbf{q})}(\mathbf{p}_1, \mathbf{v}_2) + \mathbf{p}_2 \cdot \mathbf{v}_2 - \mathcal{L}_{(\mathbf{q})}\left(\mathbf{V}_{(\mathbf{q})}(\mathbf{p}_1, \mathbf{v}_2), \mathbf{v}_2\right), \end{aligned} \tag{21}$$

where $\mathbf{v}_2 = \mathbf{C}_{2(\mathbf{q})}$ remain arbitrary functions of the passive variables $\mathbf{q}$. In this picture the relation (20) is not a definition of the momenta, but rather a condition for the existence of the envelope solution for the regular part. A similar condition for the nonregular part does not exist. Therefore at this initial stage the nonregular momenta $\mathbf{p}_2$ have no connection with the Lagrangian (analogous to (20)), thus now a "true" (in the standard definition) phase space is formed by $(\mathbf{q}_1, \mathbf{p}_1) \in (T^*\mathsf{Q})_1$ only.

Note that in [14] the passage from the Lagrangian $\mathcal{L}_{(\mathbf{q})}(\mathbf{v})$ to the general solution $\tilde{\mathcal{H}}^{Cl}_{(\mathbf{q})gen}(\mathbf{p}, \mathbf{v}_1, \mathbf{v}_2)$ is called "a slow and careful Legendre transformation", while the further passage to the half-Hamiltonian $\mathcal{H}^{Cl}_{(\mathbf{q})half}(\mathbf{p}, \mathbf{v}_2)$ is called "a reduction of the global Hamiltonian Morse family". Also, the given Legendre-Clairaut transformation becomes exactly a generalized Legendre transformation as of [15].

Now we consider the full differential of both sides of (21) and use the extremum condition (20), which gives

$$\left.\frac{\partial \mathcal{H}^{Cl}_{(\mathbf{q})half}(\mathbf{p}, \mathbf{v}_2)}{\partial \mathbf{q}_{1,2}}\right|_{\mathbf{v}_2=\mathbf{C}_{2(\mathbf{q})}} = -\left.\frac{\partial \mathcal{L}_{(\mathbf{q})}(\mathbf{v}_1, \mathbf{v}_2)}{\partial \mathbf{q}_{1,2}}\right|_{\substack{\mathbf{v}_1=\mathbf{V}_{(\mathbf{q})}\left(\mathbf{p}_1,\mathbf{C}_{2(\mathbf{q})}\right)\\ \mathbf{v}_2=\mathbf{C}_{2(\mathbf{q})}}} + \mathbf{R}^{(1,2)}_{(\mathbf{q})}\left(\mathbf{p}, \mathbf{C}_{2(\mathbf{q})}\right), \tag{22}$$

$$\left.\frac{\partial \mathcal{H}^{Cl}_{(\mathbf{q})half}(\mathbf{p}, \mathbf{v}_2)}{\partial \mathbf{p}_1}\right|_{\mathbf{v}_2=\mathbf{C}_{2(\mathbf{q})}} = \mathbf{V}_{(\mathbf{q})}\left(\mathbf{p}_1, \mathbf{C}_{2(\mathbf{q})}\right), \tag{23}$$

$$\left.\frac{\partial \mathcal{H}^{Cl}_{(\mathbf{q})half}(\mathbf{p}, \mathbf{v}_2)}{\partial \mathbf{p}_2}\right|_{\mathbf{v}_2=\mathbf{C}_{2(\mathbf{q})}} = \mathbf{C}_{2(\mathbf{q})}, \tag{24}$$

where

$$R_{i_2}^{(1,2)}\left(\mathbf{q},\mathbf{p},\mathbf{C}_{2(\mathbf{q})}\right)=\sum_{i_2'=k+1}^{n}\Phi_{(\mathbf{q})i_2'}\left(\mathbf{p}\right)\frac{\partial C_{2(\mathbf{q}),i_2'}}{\partial q_{1,2}^{i_2}}, \tag{25}$$

$$\mathbf{\Phi}_{(\mathbf{q})}\left(\mathbf{p}\right)=\mathbf{p}_2-\mathbf{\Psi}_{(\mathbf{q})}\left(\mathbf{p}_1\right),\quad \mathbf{\Psi}_{(\mathbf{q})}\left(\mathbf{p}_1\right)=\left.\frac{\partial\mathcal{L}_{(\mathbf{q})}\left(\mathbf{v}_1,\mathbf{v}_2\right)}{\partial\mathbf{v}_2}\right|_{\substack{\mathbf{v}_1=\mathbf{V}\left(\mathbf{q},\mathbf{p}_1,\mathbf{C}_{2(\mathbf{q})}\right)\\ \mathbf{v}_2=\mathbf{C}_{2(\mathbf{q})}}} \tag{26}$$

and $\mathbf{C}_{2(\mathbf{q})}$ are still arbitrary. Note that $\mathbf{\Psi}_{(\mathbf{q})}\left(\mathbf{p}_1\right)$ and therefore $\mathbf{\Phi}_{(\mathbf{q})}\left(\mathbf{p}\right)$ have no dependence on the unsolved velocities $\mathbf{v}_2$, because, if some of them appeared there, we could derive them from (26), which contradicts the fact that the rank of the Hessian is k and $rank\,\mathsf{W}^{(22)}=0$. Then, using the Lagrange equations (10), we obtain "half-Hamilton equations of motion" (or unconstrained Hamilton equations)

$$\frac{\partial\mathcal{H}_{(\mathbf{q})half}\left(\mathbf{p}\right)}{\partial\mathbf{q}_1}=-\dot{\mathbf{p}}_1+\mathbf{R}_{(\mathbf{q})}^{(1)}\left(\mathbf{p},\dot{\mathbf{q}}_2\right), \tag{27}$$

$$\frac{\partial\mathcal{H}_{(\mathbf{q})half}\left(\mathbf{p}\right)}{\partial\mathbf{p}_1}=\dot{\mathbf{q}}_1, \tag{28}$$

$$\frac{\partial\mathcal{H}_{(\mathbf{q})half}\left(\mathbf{p}\right)}{\partial\mathbf{q}_2}=-\dot{\mathbf{p}}_2+\dot{\mathbf{\Phi}}_{(\mathbf{q})}\left(\mathbf{p}\right)+\mathbf{R}_{(\mathbf{q})}^{(2)}\left(\mathbf{p},\dot{\mathbf{q}}_2\right), \tag{29}$$

$$\frac{\partial\mathcal{H}_{(\mathbf{q})half}\left(\mathbf{p}\right)}{\partial\mathbf{p}_2}=\dot{\mathbf{q}}_2, \tag{30}$$

It can be shown that the system (27)–(30) leads to the equations of motion which are equivalent to the Lagrangian ones (10). Observe that (27)–(30) become the standard system of Hamilton equations, iff the following system of equations (generalized constraints) is valid

$$\mathbf{R}_{(\mathbf{q})}^{(1)}\left(\mathbf{p},\dot{\mathbf{q}}_2\right)=0,\qquad \dot{\mathbf{\Phi}}_{(\mathbf{q})}\left(\mathbf{p}\right)+\mathbf{R}_{(\mathbf{q})}^{(2)}\left(\mathbf{p},\dot{\mathbf{q}}_2\right)=0 \tag{31}$$

The generalized constraints (31) are sufficient to introduce the standard Poisson brackets and the "correct" time evolution. If the Hamiltonian does not depend on time explicitly, then it is a constant equal to the preserved energy [7].

In the Dirac formalism [3] one imposes

$$\mathbf{\Phi}_{(\mathbf{q})}\left(\mathbf{p}\right)=0,\qquad \dot{\mathbf{\Phi}}_{(\mathbf{q})}\left(\mathbf{p}\right)=0, \tag{32}$$

which are the standard primary scleronomous constraints (which should be functionally independent, otherwise see [16]). Obviously then, the conditions in (32) are more restrictive than those in (31). But, they lead to the "correct" phase space in regular and nonregular variables, in which for both sets of momenta the derivatives of the Lagrangian with respect to corresponding velocities, are initially treated as definitions [13]. Indeed, this allows one to consider $(\mathbf{q},\mathbf{p})$ as points of the entire "true" phase space $T^{*}\mathsf{Q}$, while the Legendre transformation becomes then a degenerate mapping with a kernel [13]. The "half-Hamiltonian" (21) can now be presented as the sum of $\mathcal{H}_{(\mathbf{q})}^{(0)}(\mathbf{p}_1)$ and the linear combination of the primary constraints

$$\partial\mathcal{H}_{(\mathbf{q})half}^{Cl}\left(\mathbf{p},\mathbf{C}_{2(\mathbf{q})}\right)=\mathcal{H}_{(\mathbf{q})}^{(0)}(\mathbf{p}_1)+\mathbf{C}_{2(\mathbf{q})}\cdot\mathbf{\Phi}_{(\mathbf{q})}(\mathbf{p}), \tag{33}$$

$$\begin{aligned}\mathcal{H}_{(\mathbf{q})}^{(0)}(\mathbf{p}_1)=\mathbf{p}_1\cdot\mathbf{V}_{(\mathbf{q})}\left(\mathbf{p}_1,\mathbf{C}_{2(\mathbf{q})}\right)-\mathcal{L}_{(\mathbf{q})}\left(\mathbf{V}_{(\mathbf{q})}\left(\mathbf{p}_1,\mathbf{C}_{2(\mathbf{q})}\right),\mathbf{C}_{2(\mathbf{q})}\right)\\ +\mathbf{C}_{2(\mathbf{q})}\cdot\mathbf{\Psi}_{(\mathbf{q})}(\mathbf{p}_1),\end{aligned} \tag{34}$$

where $\mathcal{H}_{(\mathbf{q})}^{(0)}(\mathbf{p}_1)$ does not depend on $\mathbf{p}_2$ or $\mathbf{C}_{2(\mathbf{q})}$ due to (20) and (24). In this case (33) coincides (formally) with the Dirac's total Hamiltonian [3].

So the concise treatment of constraint systems using the Clairaut partial differential equation presented here gives a different explanation of the Dirac primary constraints from the Lagrange multiplier method, as well as a different understanding of the nature of primary constraints. In some cases this can lead to possible generalizations. Moreover, it can be applied in the cases where the standard constraint methods do not work or are too cumbersome.

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