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Localization of three-dimensional $\mathcal{N} = 2$ supersymmetric theories on $S^1 \times D^2$

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We study three-dimensional $\mathcal{N} = 2$ supersymmetric Chern–Simons matter theories on the direct product of a circle and a two-dimensional hemisphere ($S^1 \times D^2$) with specified boundary conditions by the method of localization. We construct boundary interactions to cancel the supersymmetric variation of the three-dimensional superpotential term and the Chern–Simons term and show inflows of the bulk–boundary anomalies. We find that the boundary conditions induce two-dimensional $\mathcal{N} = (0, 2)$-type supersymmetry on the boundary torus. We also study the relation between the three-dimensional–two-dimensional coupled partition function of our model and three-dimensional holomorphic blocks.

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1. Introduction

After the prominent work by Pestun [1], various supersymmetric gauge theories on curved spacetimes have been investigated and rigid supersymmetries on these backgrounds have been constructed. By evaluating fixed point sets of supersymmetric transformations combined with the method of localization, one can calculate exact partition functions of these supersymmetric theories. For example, a round sphere S^3 [2–4] and an ellipsoid S^3_b [5] were investigated as suitable backgrounds of three-dimensional (3d) supersymmetric theories. The superconformal indices on $S^1 \times S^2$ were studied in [6,7], which count the number of Bogomol'nyi–Prasad–Sommerfield (BPS) operators. These functions play important roles in studying M5-branes, infrared (IR) dualities among supersymmetric theories, and the anti-de Sitter / conformal field theory correspondence.

It is widely believed that these supersymmetric theories have interesting properties of the factorizations with fundamental building blocks. For example, in 3d cases, it was conjectured in Ref. [8] that $\mathcal{N} = 2$ supersymmetric partition functions on S^3_b and the superconformal indices on $S^1 \times S^2$ can be expressed as bilinear forms of two identical building blocks $\mathcal{B}^\alpha(x, q)$:

$$Z_{S^3_b \text{ or } S^1\times S^2} = \sum_\alpha \mathcal{B}^\alpha(x, q)\mathcal{B}^\alpha(\tilde{x}, \tilde{q}). \quad (1)$$

The authors of Ref. [8] proposed some general rules to write down these universal blocks $\mathcal{B}^\alpha(x, q)$, called *holomorphic blocks*. By analogy with pure Chern–Simon theory [9] and topological/anti-topological fusion in two dimensions [10], it is expected that holomorphic blocks should be partition functions of 3d Chern–Simons matter theories on a solid torus (Melvin cigar). In order to clarify the relation between their partition functions (indices) and holomorphic blocks, it is interesting to construct explicitly $\mathcal{N} = 2$ supersymmetric Chern–Simons matter theories on a solid torus. When the spacetime has boundaries, one may introduce physical degrees of freedom on the boundaries and classify what types of BPS boundaries are allowed in supersymmetric Chern–Simons matter theories.

For example, half-BPS boundary conditions have been studied in $\mathcal{N} = 1$ supersymmetric pure Chern–Simons theory [11] and supersymmetric Chern–Simons matter theories [12,13]. Theories on the boundary studied so far are realized as super Wess–Zumino–Witten (WZW) models, and it is natural to ask whether or not other types of boundary interactions exist. For example, in two-dimensional (2d) $\mathcal{N} = (2, 2)$ theories, the boundary interaction is described in terms of the matrix factorization so that the effect of the supersymmetric variation of the superpotential [14,15] is cancelled. Higher-dimensional analogues of the matrix factorization have not yet been studied much.

In this article we investigate 3d $\mathcal{N} = 2$ supersymmetric Chern–Simons matter theories on the direct product of a circle and a 2d hemisphere ($S^1 \times D^2$) toward understanding the properties of the building blocks and the factorization. We impose suitable boundary conditions for $\mathcal{N} = 2$ multiplets consistent with the supersymmetric transformations. Under the boundary conditions, the super Yang–Mills action and the kinetic action of the chiral multiplet are written as Q-exact forms without surface terms, and these actions are invariant under the supersymmetric transformations. On the other hand, in the presence of the boundary, the Chern–Simons term is invariant under neither the supersymmetric transformation nor the gauge transformation. Thus, we have to introduce some boundary term to compensate for these two variations. Here we propose two possible ways to make the theory gauge invariant. One is to lift gauge parameters to physical fields on the boundary and treat them in a chiral gauged WZW model. The other is to introduce $\mathcal{N} = (0, 2)$ theories on the boundary with chiral fermions. They induce gauge anomalies which compensate non-invariant terms coming from the 3d bulk theory under the gauge transformation.

Next, we evaluate the 3d–2d coupled indices on $S^1 \times D^2$ in terms of the supersymmetric localization and try to relate the 3d–2d indices to holomorphic blocks. In the cases of Abelian gauge theories, we find that the 3d–2d indices reproduce holomorphic blocks with a choice of the fugacity.

This paper is organized as follows. In Sect. 2 we study the super Yang–Mills action and the kinetic action of the chiral multiplet, and introduce consistent boundary conditions for the $\mathcal{N} = 2$ multiplets. In Sect. 3 we discuss BPS boundary interactions for the superpotential term (the 3d analogue of matrix factorization) and the Chern–Simons term. We also show that the restriction of the $\mathcal{N} = 2$ supersymmetric transformation to the boundary torus leads to an $\mathcal{N} = (0, 2)$ supersymmetric theory in two dimensions. In Sects. 4 and 5, we evaluate the one-loop determinants of 3d $\mathcal{N} = 2$ (vector and chiral) multiplets and 2d boundary $\mathcal{N} = (0, 2)$ (vector, chiral, and Fermi) multiplets to study the relation between the 3d–2d indices on $S^1 \times D^2$ and holomorphic blocks in three dimensions. In Sect. 6 we study properties of indices from several viewpoints. First, we consider a 3d analogue of the $\mathcal{N} = (2, 2)$ hemisphere partition function of the $\mathbb{C}\mathbf{P}^N$-model on D^2 and point out that the index of this model is related to the K-theoretic J-function of $\mathbb{C}\mathbf{P}^N$ and the q-deformed Whittaker function. Second, we study the structure of the index of $U(N)$ super QCD (SQCD) and its connection to K-theoretic vortex partition functions and surface operators. Third, we study the gauge/Bethe correspondence. We also analyze a domain wall on $S^1 \times S^2$ and evaluate the 3d index with a domain wall. The last topic is the action of Wilson loops and vortex loops on the 3d index. The last section is devoted to summary and discussion.

2. $\mathcal{N} = 2$ supersymmetric theory on $S^1 \times D^2$

In this section we will construct supersymmetric gauge theories on $S^1 \times D^2$ and introduce supersymmetric boundary conditions for the hemisphere. The construction of the supersymmetry and the Lagrangian is parallel to the $S^1 \times S^2$ case [7].

The hemisphere D^2 with radius r is specified by the coordinates (ϑ, φ), with $0 \leq \vartheta \leq \frac{\pi}{2}, 0 \leq \varphi \leq 2\pi$; the boundary of the hemisphere is defined by $\vartheta = \frac{\pi}{2}$. The circle S^1 is also parameterized by the coordinate τ with $0 \leq \tau \leq \beta r$, and βr is the circumference of S^1. We can write the metric of $S^1 \times D^2$ as

$$ds^2 = r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 + d\tau^2. \tag{2}$$

In the following, we use μ ($\mu = 1, 2, 3$) for superscripts and subscripts in the curved space, with $1 = \vartheta$, $2 = \varphi$, and $3 = \tau$. On the other hand, we take the symbol $\hat{a}$ ($a = 1, 2, 3$) for variables in the local Lorentz frame.

Now we construct supersymmetry in the curved space that is realized by conformal Killing spinors $\epsilon, \bar{\epsilon}$. These spinors should satisfy

$$\nabla_\mu \epsilon = \frac{1}{2r}\gamma_\mu \gamma_3 \epsilon, \qquad \nabla_\mu \bar{\epsilon} = -\frac{1}{2r}\gamma_\mu \gamma_3 \bar{\epsilon}. \tag{3}$$

The solutions of these equations are given by

$$\epsilon = e^{\frac{\tau}{2r}} e^{-\frac{i}{2}\gamma^2 \vartheta} e^{\frac{i}{2}\gamma^3 \varphi} \epsilon_o^{(2)}, \qquad \bar{\epsilon} = e^{-\frac{\tau}{2r}} e^{\frac{i}{2}\gamma^2 \vartheta} e^{\frac{i}{2}\gamma^3 \varphi} \bar{\epsilon}_o^{(2)}, \tag{4}$$

where we choose the constant spinors $\epsilon_o^{(2)} = \gamma_3 \epsilon_o^{(2)} = (\epsilon_o, 0)^{\mathrm{T}}$ and $\bar{\epsilon}_o^{(2)} = -\gamma_3 \bar{\epsilon}_o^{(2)} = (0, \bar{\epsilon}_o)^{\mathrm{T}}$ so that the component of the Killing vector $\bar{\epsilon}\gamma^\mu \epsilon$ along the ϑ-direction vanishes.

With the set of Killing spinors in Eq. (4), the supersymmetric transformation of the vector multiplet in Eq. (A.4) is expressed as

$$\delta A_\mu = \frac{i}{2}(\bar{\epsilon}\gamma_\mu \lambda - \bar{\lambda}\gamma_\mu \epsilon), \tag{5}$$

$$\delta\sigma = \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon), \tag{6}$$

$$\delta\lambda = -\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\epsilon - D\epsilon + i\gamma^\mu D_\mu \sigma \epsilon + \frac{i}{r}\sigma\gamma_3 \epsilon, \tag{7}$$

$$\delta\bar{\lambda} = -\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\bar{\epsilon} + D\bar{\epsilon} - i\gamma^\mu D_\mu \sigma \bar{\epsilon} + \frac{i}{r}\sigma\gamma_3 \bar{\epsilon}, \tag{8}$$

$$\delta D = -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu \lambda - \frac{i}{2}D_\mu \bar{\lambda}\gamma^\mu \epsilon + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] + \frac{i}{2}[\bar{\lambda}\epsilon, \sigma] + \frac{i}{4r}(\bar{\epsilon}\gamma_3 \lambda - \bar{\lambda}\gamma_3 \epsilon). \tag{9}$$

The Lagrangian density $\mathcal{L}_{\mathrm{vec}}$ of super Yang–Mills theory is written in the Q-exact form:

$$\begin{aligned} 2\mathcal{L}_{\mathrm{vec}} &= \frac{1}{\epsilon_2 \epsilon_1}\delta_{\epsilon_2}\delta_{\epsilon_1} \mathrm{Tr}\left[\frac{1}{2}\lambda\lambda\right] \\ &= \mathrm{Tr}\left[\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + D^\mu \sigma D_\mu \sigma + D^2 + \frac{1}{r^2}\sigma^2 - \epsilon^{\mu\nu\rho}F_{\mu\nu}D_\rho \sigma - \frac{1}{r}\epsilon^{\mu\nu 3}F_{\mu\nu}\sigma + \frac{2}{r}\sigma D_3 \sigma\right] \\ &\quad + \mathrm{Tr}\left[i\lambda\gamma^\mu D_\mu \bar{\lambda} + i\lambda[\bar{\lambda}, \sigma] - \frac{i}{2r}\bar{\lambda}\gamma_3 \lambda\right]. \end{aligned} \tag{10}$$

Here, δ_{ϵ_i} with $i = 1, 2$ denotes the supersymmetric transformation with $\bar{\epsilon} = 0$ and $\epsilon = \epsilon_i$. We impose the following boundary condition for this vector multiplet at $\vartheta = \frac{\pi}{2}$:

$$\sigma = 0, \quad A_1 = 0, \quad \partial_1 A_2 = 0, \quad \partial_1 A_3 = 0, \quad D_1(D - iD_{\hat{1}}\sigma) = 0,$$
$$\lambda_1 - \lambda_2 = 0, \quad \bar{\lambda}_1 - \bar{\lambda}_2 = 0, \quad \partial_1(\lambda_1 + \lambda_2) = 0, \quad \partial_1(\bar{\lambda}_1 + \bar{\lambda}_2) = 0, \tag{11}$$

with $\lambda = (\lambda_1, \lambda_2)^{\mathrm{T}}$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^{\mathrm{T}}$. This boundary condition is compatible with the supersymmetric transformation in Eqs. (5)–(9). When we impose the above boundary condition, induced surface terms vanish and the supercurrents have no components in the normal directions to the boundary.

Next, we consider the chiral multiplet (ϕ, ψ, F). The supersymmetric transformation of the chiral multiplet is given by

$$\delta\phi = \bar{\epsilon}\psi, \tag{12}$$
$$\delta\bar{\phi} = \epsilon\bar{\psi}, \tag{13}$$
$$\delta\psi = i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi + \frac{i\Delta}{r}\gamma_3\epsilon\phi + \bar{\epsilon}F, \tag{14}$$
$$\delta\bar{\psi} = i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} + i\bar{\phi}\sigma\bar{\epsilon} - \frac{i\Delta}{r}\bar{\phi}\gamma_3\bar{\epsilon} + \bar{F}\epsilon, \tag{15}$$
$$\delta F = \epsilon(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi) + \frac{i}{2r}(2\Delta - 1)\epsilon\gamma_3\psi, \tag{16}$$
$$\delta\bar{F} = \bar{\epsilon}(i\gamma^\mu D_\mu\bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\bar{\lambda}) - \frac{i}{2r}(2\Delta - 1)\bar{\epsilon}\gamma_3\bar{\psi}. \tag{17}$$

Here, Δ is the R-charge of ϕ. The kinetic term of the chiral multiplet is also given in the Q-exact form:

$$\begin{aligned}\mathcal{L}_{\text{chi}} &= \frac{1}{\epsilon_2\epsilon_1}\delta_{\epsilon_2}\delta_{\epsilon_1}\left(\bar{\phi}F\right)\\ &= -\bar{\phi}D^\mu D_\mu\phi + \bar{\phi}\sigma^2\phi + i\bar{\phi}D\phi + \bar{F}F + \frac{1-2\Delta}{r}\bar{\phi}D_3\phi + \frac{\Delta - \Delta^2}{r^2}\bar{\phi}\phi\\ &\quad - i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\psi}\sigma\psi + \frac{i(1-2\Delta)}{2r}\bar{\psi}\gamma_3\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi.\end{aligned} \tag{18}$$

We can introduce the Neumann boundary condition for the chiral multiplet at $\vartheta = \frac{\pi}{2}$:

$$\partial_1\phi = 0, \quad \partial_1\bar{\phi} = 0, \quad F = 0, \quad \bar{F} = 0,$$
$$\psi_1 + \psi_2 = 0, \quad \bar{\psi}_1 + \bar{\psi}_2 = 0, \quad \partial_1(\psi_1 - \psi_2) = 0, \quad \partial_1(\bar{\psi}_1 - \bar{\psi}_2) = 0, \tag{19}$$

with $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ and $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)^{\mathrm{T}}$.

The Dirichlet boundary condition is given by

$$\phi = 0, \quad \bar{\phi} = 0, \quad \partial_1(ie^{\frac{\tau}{r}}e^{i\varphi}\partial_{\hat{1}}\phi + F) = 0, \quad \partial_1(ie^{-\frac{\tau}{r}}e^{-i\varphi}\partial_{\hat{1}}\bar{\phi} + \bar{F}) = 0,$$
$$\psi_1 - \psi_2 = 0, \quad \bar{\psi}_1 - \bar{\psi}_2 = 0, \quad \partial_1(\psi_1 + \psi_2) = 0, \quad \partial_1(\psi_1 + \psi_2) = 0. \tag{20}$$

The Neumann (Dirichlet) boundary condition in Eq. (19) (Eq. (20)) is the 3d analogue of the Neumann (Dirichlet) boundary condition in 2d $\mathcal{N} = (2, 2)$ theories on D^2 [16,17], and different from the boundary condition imposed in Ref. [18].

3. BPS boundary interactions on the torus

Under the boundary conditions in Eqs. (11) and (19), the Lagrangians in Eqs. (10) and (18) are invariant under the supersymmetric transformations generated by $\delta_\epsilon, \delta_{\bar{\epsilon}}$. On the other hand, the supersymmetric transformations of the supersymmetric Chern–Simons term and the superpotential term do not vanish and we have to introduce supersymmetric boundary interactions to cancel surface terms coming from these terms.

3.1. Three-dimensional analogue of matrix factorization

In the presence of the boundary, the supersymmetric variation of the superpotential does not vanish. In 2d $\mathcal{N} = (2,2)$ theories, the boundary term for the superpotential is cancelled by the boundary interaction, which satisfies the matrix factorization [14,15]. The 3d analogue of the matrix factorization was first pointed out in Ref. [19]. In this subsection we study the matrix factorization for the 3d $\mathcal{N} = 2$ theories on $S^1 \times D^2$.

The supersymmetric transformations of the superpotentials W and $\bar{W}$ induce the following surface terms on the boundary:

$$\delta S_W = \int_{T^2} \sqrt{g} d^2x \sum_I \left(\epsilon \gamma_{\hat{1}} \psi_I \frac{\partial W}{\partial \phi_I} + \bar{\epsilon} \gamma_{\hat{1}} \bar{\psi}_I \frac{\bar{W}}{\partial \bar{\phi}_I} \right). \tag{21}$$

Here, T^2 is the boundary of $S^1 \times D^2$, and I in the sum labels the chiral multiplets with the Neumann boundary condition in the theory. We have to introduce boundary interactions which compensate for the above boundary terms. Now we restrict the supersymmetric transformations of the 3d $\mathcal{N} = 2$ multiplets to the boundary ($\vartheta = \frac{\pi}{2}$) in order to construct the BPS and gauge-invariant boundary interactions. First of all, we will consider the vector multiplet in Eqs. (5)–(9) restricted on the boundary. Under the condition in Eq. (11), associated supersymmetric transformations are given by

$$\begin{aligned}
\delta A_{\hat{2}} &= \frac{i}{2}(\bar{\epsilon}\gamma_{\hat{2}}\lambda - \bar{\lambda}\gamma_{\hat{2}}\epsilon) = \bar{\epsilon}'\lambda_1 - \bar{\lambda}_1\epsilon', \\
\delta A_{\hat{3}} &= \frac{i}{2}(\bar{\epsilon}\gamma_{\hat{3}}\lambda - \bar{\lambda}\gamma_{\hat{3}}\epsilon) = i(\bar{\epsilon}'\lambda_1 - \bar{\lambda}_1\epsilon'), \\
\delta \lambda &= -iF_{\hat{2}\hat{3}}\epsilon - \hat{D}\epsilon, \\
\delta \bar{\lambda} &= -iF_{\hat{2}\hat{3}}\bar{\epsilon} + \hat{D}\bar{\epsilon}, \\
\delta(-\hat{D} - iF_{\hat{2}\hat{3}}) &= 2\bar{\epsilon}'(D_{\hat{2}} + iD_{\hat{3}})\lambda_1 - \frac{2i}{r}\bar{\epsilon}'\lambda_1, \\
\delta(-\hat{D} + iF_{\hat{2}\hat{3}}) &= -2\epsilon'(D_{\hat{2}} + iD_{\hat{3}})\bar{\lambda}_1 - \frac{2i}{r}\epsilon'\bar{\lambda}_1.
\end{aligned} \tag{22}$$

Here we defined $\hat{D} := D - iD_{\hat{1}}\sigma$ and $\epsilon' := \frac{e^{\frac{\tau}{2r}} e^{i\frac{\varphi}{2}}}{\sqrt{2}}\epsilon_o$, $\bar{\epsilon}' := \frac{e^{-\frac{\tau}{2r}} e^{-i\frac{\varphi}{2}}}{\sqrt{2}}\bar{\epsilon}_o$. The commutation relations of these transformations are summarized in Appendix B. In the flat space limit $r \to \infty$, the above transformation set becomes that of the 2d $\mathcal{N} = (0,2)$ vector multiplet. We shall call this multiplet a boundary $\mathcal{N} = (0,2)$ vector multiplet. A Lagrangian can be expressed as a $\delta_{\epsilon'}$-exact form and invariant under the supersymmetric transformations in Eq. (22):

$$\begin{aligned}
\epsilon' \mathcal{L}_{\text{vec}}^{\mathcal{N}=(0,2)} &= \delta_{\epsilon'} \text{Tr}(-\hat{D} + iF_{\hat{2}\hat{3}})\lambda_1 \\
&= \epsilon' \left(F_{\hat{2}\hat{3}}^2 + \hat{D}^2 + 2\bar{\lambda}_1(D_{\hat{2}} + iD_{\hat{3}})\lambda_1 - \frac{2i}{r}\bar{\lambda}_1\lambda_1 \right).
\end{aligned} \tag{23}$$

When we take the Neumann boundary condition, the supersymmetric transformation of the chiral multiplet on the boundary is given by

$$\begin{aligned}\delta\phi &= \bar{\epsilon}\psi = \bar{\epsilon}'\psi',\\ \delta\bar{\phi} &= \epsilon\bar{\psi} = \epsilon'\bar{\psi}',\\ \delta\psi &= \gamma_{\hat{3}}\epsilon(D_{\hat{2}} + iD_{\hat{3}})\phi + \frac{i\Delta}{r}\gamma_3\epsilon\phi,\\ \delta\bar{\psi} &= \gamma_{\hat{3}}\bar{\epsilon}(D_{\hat{2}} + iD_{\hat{3}})\bar{\phi} - \frac{i\Delta}{r}\bar{\phi}\gamma_3\bar{\epsilon},\end{aligned} \tag{24}$$

where we defined $\psi' := \psi_1 - \psi_2$, $\bar{\psi}' := \bar{\psi}_1 - \bar{\psi}_2$. When we take the limit $r \to \infty$, the above transformation becomes the $\mathcal{N} = (0, 2)$ supersymmetric transformation of the boundary chiral multiplet (ϕ, ψ'). Then we introduce the Lagrangian as a $\delta_{\epsilon'}$-exact form:

$$\begin{aligned}\epsilon'\mathcal{L}_{\text{chi}}^{\mathcal{N}=(0,2)} &= \delta_{\epsilon'}\left(\frac{1}{2}\bar{\phi}(D_{\hat{2}} - iD_{\hat{3}})\psi' + i\bar{\phi}\lambda_1\phi\right)\\ &= \epsilon'\Big(\bar{\phi}(D_{\hat{2}} - iD_{\hat{3}})(D_{\hat{2}} + iD_{\hat{3}})\phi + \frac{1}{2}\bar{\psi}'(D_{\hat{2}} - iD_{\hat{3}})\psi'\\ &\qquad + \frac{i\Delta}{r}\bar{\phi}(D_{\hat{2}} - iD_{\hat{3}})\phi + i\bar{\phi}\bar{\lambda}_1\psi' + i\bar{\psi}_1\lambda_1\phi + \bar{\phi}(F_{\hat{2}\hat{3}} - i\hat{D})\phi\Big).\end{aligned} \tag{25}$$

If we take $r \to \infty$, $\mathcal{L}_{\text{chi}}^{\mathcal{N}=(0,2)}$ is reduced to the Lagrangian for the $\mathcal{N} = (0, 2)$ chiral multiplet in the flat space. These $\mathcal{N} = (0, 2)$ boundary (vector and chiral) multiplets are constructed through restriction of the bulk supersymmetry on the boundary. In addition, there are new multiplets in the $\mathcal{N} = (0, 2)$ theory characterized by holomorphic functions $E(\phi)$, namely Fermi multiplets, and we can construct the boundary interactions which cancel the variation of the superpotential term. We shall introduce the Fermi multiplet (Ψ, G) on T^2 coupled to the boundary $\mathcal{N} = (0, 2)$ (vector and chiral) multiplets in the supersymmetric way. The supersymmetric transformation of the boundary Fermi multiplet is given by

$$\delta\Psi = 2E\epsilon' + 2\bar{\epsilon}'G, \tag{26}$$

$$\delta\bar{\Psi} = 2\bar{E}\bar{\epsilon}' + 2\epsilon'\bar{G}, \tag{27}$$

$$\delta G = -\epsilon'\psi_E + \epsilon'(D_{\hat{2}} + iD_{\hat{3}})\Psi + \frac{i}{r}(\tilde{\Delta} - 1)\epsilon'\Psi, \tag{28}$$

$$\delta\bar{G} = -\bar{\epsilon}'\bar{\psi}_E + \bar{\epsilon}'(D_{\hat{2}} + iD_{\hat{3}})\bar{\Psi} + \frac{i}{r}(1 - \tilde{\Delta})\bar{\epsilon}'\bar{\Psi}. \tag{29}$$

Here, $\psi_E := \sum_I \frac{\partial E(\phi)}{\partial\phi_I}\psi'_I$, $\bar{\psi}_E := \sum_I \frac{\partial E(\bar{\phi})}{\partial\bar{\phi}_I}\bar{\psi}'_I$, and the (ϕ_I, ψ'_I) are $\mathcal{N} = (0, 2)$ boundary chiral multiplets. $\sum_I$ runs over the $\mathcal{N} = (0, 2)$ chiral multiplets in $E(\phi)$, and we also require the relation $\sum_I \Delta_I = \tilde{\Delta}$. One can show that commutators of these supersymmetries generate a translation, R-symmetry, and gauge transformations.

Next, we study the Dirichlet boundary condition. We remark that the supersymmetric transformations in Eqs. (12)–(17) with the Dirichlet boundary condition on the boundary torus lead to Eqs. (26)–(29) with $E(\phi) = 0$ by the following redefinition:

$$G := ie^{\frac{\tau}{r}}e^{i\varphi}D_{\hat{1}}\phi + F, \quad \bar{G} := ie^{\frac{-\tau}{r}}e^{-i\varphi}D_{\hat{1}}\bar{\phi} + \bar{F},$$

$$\Psi := \psi_1, \quad \bar{\Psi} := \bar{\psi}_1, \quad \tilde{\Delta} = 2\Delta. \tag{30}$$

This is the special case of the $\mathcal{N} = (0, 2)$ boundary Fermi multiplet.

Now we return to the boundary Fermi multiplet, the Lagrangian of which can be constructed as

$$\begin{aligned} \epsilon' \mathcal{L}^{\mathcal{N}=(0,2)}_{\text{Fermi}} &= \delta_{\epsilon'}(\bar{\Psi} G + \bar{E}\Psi) \\ &= \epsilon' \left(-\bar{\Psi}(D_{\hat{2}} + iD_{\hat{3}})\Psi + 2\bar{G}G + 2\bar{E}E - \bar{\psi}_E \Psi - \bar{\Psi}\psi_E + \frac{i}{r}(1 - \tilde{\Delta})\bar{\Psi}\Psi \right). \end{aligned} \tag{31}$$

We can also introduce the potential term for this multiplet,

$$\mathcal{L}_J = \sum_a \left(G_a J^a - \frac{1}{2}\Psi_a \psi_{J^a} \right) + \text{(c.c.)}, \tag{32}$$

which induces the following terms on T^2 through the variation:

$$\int_{T^2} \sqrt{g} d^2x \delta\mathcal{L}_J = \int_{T^2} \sqrt{g} d^2x \sum_{I,a} \left(-\epsilon' \psi'_I \frac{\partial(E_a J^a)}{\partial\phi_I} + \text{(c.c.)} \right). \tag{33}$$

Here, (c.c.) denotes the complex conjugate. By comparing this with the term in Eq. (21), we find that cancellation between Eqs. (21) and (33) occurs when the relation $\sum_a E_a J^a = W$ is satisfied. We further require that each monomial in $E_a J^a$ has R-charge 2 so that the relation $\sum_a E_a J^a = W$ can be regarded as *the 3d matrix factorization*.

3.2. *Supersymmetric Chern–Simons term*

In this subsection we consider $\mathcal{N} = 2$ supersymmetric Chern–Simons theory and investigate boundary terms induced by variations of the bulk action. The boundary effect was first studied for $\mathcal{N} = 1$ Chern–Simons theory [11], and the $\mathcal{N} = 2$ Abelian case was studied in Ref. [12].

We first treat the Chern–Simons term with the gauge group G and construct consistent boundary interactions. Later we treat gauge–flavor mixed Chern-Simons terms. The $\mathcal{N} = 2$ Chern–Simons theory has a gauge field A_μ, bosonic fields D and σ, and fermions λ and $\bar{\lambda}$. These fields take values in the Lie algebra of the gauge group G and transform in the adjoint representation. This model has a coupling constant κ called a Chern–Simons level, and the Chern–Simons action is written by

$$\begin{aligned} S_{\text{CS}} &= \frac{i\kappa}{4\pi} \int d^3x \, \mathcal{L}_{\text{CS}}, \\ \mathcal{L}_{\text{CS}} &= \varepsilon^{\mu\nu\rho}\text{Tr}\left(\partial_\mu A_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) + \sqrt{g}\, \text{Tr}(-\bar{\lambda}\lambda + 2\sigma D), \end{aligned} \tag{34}$$

where $\varepsilon^{\mu\nu\rho}$ is an antisymmetric tensor density. The supersymmetric variation of this term is evaluated as

$$\delta\mathcal{L}_{\text{CS}} = \frac{i}{2}\partial_\mu \text{Tr}\varepsilon^{\mu\nu\rho}\left(\bar{\epsilon}\gamma_\nu\lambda A_\rho - \bar{\lambda}\gamma_\nu\epsilon A_\rho \right) - i\partial_\mu \text{Tr}\sqrt{g}\left(\bar{\epsilon}\gamma^\mu\lambda\sigma + \bar{\lambda}\gamma^\mu\epsilon\sigma \right). \tag{35}$$

On $S^1 \times D^2$ with the boundary condition in Eq. (11), the second term in the above equation can be dropped and the first term leads to the following boundary term:

$$S_{\text{CS}} = -\frac{i\kappa}{8\pi} \int_{T^2} \sqrt{g^{(2)}} d^2x \sum_{\mu=2,3} \text{Tr}\left[A_\mu (\bar{\epsilon}\gamma^\mu\lambda - \bar{\lambda}\gamma^\mu\epsilon) \right], \tag{36}$$

where $\sqrt{g^{(2)}}$ is the measure of the 2d boundary and is related to the bulk 3d one by $\sqrt{g} = r\sqrt{g^{(2)}}$. By introducing a boundary Chern–Simons term

$$S_{\text{b.CS}} = \frac{-\kappa}{8\pi} \sum_{\mu=2,3} \int_{T^2} \sqrt{g^{(2)}} d^2x \, \text{Tr}(A_\mu A^\mu), \tag{37}$$

we find that $S_{\text{CS}} + S_{\text{b.CS}}$ preserves the supersymmetry generated by $\delta_{\epsilon'}, \delta_{\bar{\epsilon}'}$. But $S_{\text{CS}} + S_{\text{b.CS}}$ still breaks gauge invariance in the presence of the boundary, and we have to resolve this problem. There are two choices to recover the gauge symmetry:

(i) Introduce some boundary $\mathcal{N} = (0,2)$ multiplets to induce anomaly inflows.
(ii) Treat the gauge degrees of freedom as physical fields on the boundary and couple them with a chiral gauged WZW model.

We first explain choice (i). The gauge non-invariant term of $S_{\text{CS}} + S_{\text{b.CS}}$ is compensated by gauge anomalies from the boundary $\mathcal{N} = (0,2)$ chiral and Fermi multiplets when the following condition is satisfied:

$$\kappa \text{Tr}(T^a T^b) = \sum_{m:\text{chiral}} \text{Tr}_{\mathcal{R}_m}(T^a T^b) - \sum_{n:\text{Fermi}} \text{Tr}_{\mathcal{R}_n}(T^a T^b). \tag{38}$$

The left-hand side comes from an infinitesimal gauge transformation of $S_{\text{CS}} + S_{\text{b.CS}}$, and the symbol "Tr" denotes the trace over the Lie algebra of G. On the right-hand side, $\text{Tr}_{\mathcal{R}_m}(T^a T^b)$, (resp. $\text{Tr}_{\mathcal{R}_n}(T^a T^b)$) comes from the gauge anomaly coefficient of the chiral (resp. Fermi) multiplet in the representation $\mathcal{R}_m$ (resp. $\mathcal{R}_n$). The sums $\sum_{m:\text{chiral}}$ and $\sum_{n:\text{Fermi}}$ run over the number of boundary chiral and Fermi multiplets, respectively. This means the classical gauge anomaly (the left-hand side) and the one-loop gauge anomalies (the right-hand side) cancel each other. This type of cancellation has also been considered in the context of an $\mathcal{N} = (0,2)$ gauged WZW model [20].

But there is a subtle point. In the case of $\mathcal{N} = 2$ theories in three dimensions, the bare Chern–Simons level is shifted by one-loop effects of 3d dynamical fermions. This implies that the coefficient of the boundary term, i.e. the mass term of the 2d gauge field, is also shifted. In two dimensions, it is known that the gauge field acquires a mass term by the quantum effect [21], and the coefficient of the boundary Chern–Simons term is also shifted by the one-loop effects of 2d fermions. Then, the left-hand side of Eq. (38) is expected to be replaced by some effective Chern–Simons level. We will observe how this level shift appears in the context of the localization computation in Sect. 5.

Next, we shall consider quiver Chern–Simons theories, where gauge fields are not necessarily dynamical. In fact, there are possible mixing terms between the dynamical gauge fields and the background flavor gauge fields, namely, mixed Chern–Simons terms. For simplicity, we assume the group is Abelian, but it is straightforward to generalize to non-Abelian cases. We take the gauge and flavor groups as $U(1)_G^N \times U(1)_F^{N_\text{f}}$, with the dynamical gauge group $U(1)_G^N$ and the flavor symmetry group $U(1)_F^{N_\text{f}}$. Further, there are other possible mixings, for example the dynamical gauge symmetry and the $U(1)_R$ R-symmetry, or the flavor symmetry and the $U(1)_R$ R-symmetry. We will discuss such mixed Chern–Simons terms in Sect. 5.

$\mathcal{N} = 2$ Abelian quiver theory is described by the action

$$S_{\text{CS}} = \sum_{s,t=1}^{N+N_\text{f}} \frac{i\kappa_{st}}{4\pi} \int_{S^1 \times D^2} d^3x \left[\varepsilon^{\mu\nu\rho} \partial_\mu A_\nu^{(s)} A_\rho^{(t)} + \sqrt{g}(-\bar{\lambda}^{(s)}\lambda^{(t)} + 2\sigma^{(s)} D^{(t)}) \right]. \tag{39}$$

The κ_{st} with $s,t = 1,\dots,N+N_{\rm f}$ are the gauge–flavor mixed Chern–Simons levels. We will collect these levels into one symmetric $(N+N_{\rm f})\times(N+N_{\rm f})$ matrix κ_{st}. $A_\mu^{(s)}$ represents the collection of $U(1)_G^N$ dynamical gauge fields and $U(1)_F^{N_{\rm f}}$ background gauge fields. As in the case of the non-Abelian theory in Eq. (37), the supersymmetric boundary term is given by

$$S_{\rm b.CS} = \sum_{s,t=1}^{N+N_{\rm f}} \frac{-\kappa_{st}}{2\pi}\int_{T^2}\sqrt{g}d^2x A_z^{(s)}A_{\bar z}^{(t)}. \tag{40}$$

Here, we defined $A_z := (A_2 - iA_3)/2$ and $A_{\bar z} := (A_2 + iA_3)/2$. The infinitesimal $U(1)_G^N \times U(1)_F^{N_{\rm f}}$ transformation of the mixed Chern–Simons term leads to non-invariant terms on the boundary:

$$\delta_\alpha(S_{\rm CS}+S_{\rm b.CS}) = \sum_{s,t=1}^{N+N_{\rm f}}\frac{\kappa_{st}}{2\pi}\int_{T^2}F_{z\bar z}^{(s)}\alpha^{(t)}. \tag{41}$$

The gauge–gauge and gauge–flavor mixed parts in Eq. (41) can be cancelled from corresponding mixed anomalies in the boundary $\mathcal{N}=(0,2)$ (chiral and Fermi) multiplets when the following conditions are satisfied:

$$\kappa_{st} = \sum_{m:\text{2d.chiral}} Q^m_{\text{2d},s}Q^m_{\text{2d},t} - \sum_{n:\text{2d.Fermi}}\tilde{Q}^n_{\text{2d},s}\tilde{Q}^n_{\text{2d},t}. \tag{42}$$

Here, $Q^m_{\text{2d},s}$ denotes the charge of the sth $U(1)$ in $U(1)_G^N\times U(1)_F^{N_{\rm f}}$ for the mth $\mathcal{N}=(0,2)$ chiral multiplet. $\tilde{Q}^n_{\text{2d},s}$ denotes the charge of the sth $U(1)$ in $U(1)_G^N\times U(1)_F^{N_{\rm f}}$ for the nth $\mathcal{N}=(0,2)$ Fermi multiplet. The sum of each term runs over the $\mathcal{N}=(0,2)$ chiral and Fermi multiplets respectively. If the bare Chern–Simons levels κ are replaced by the effective Chern–Simons levels, Eq. (42) matches with the condition of Chern–Simons levels in the context of the holomorphic blocks [8] (see also Ref. [19]).

Next, we explain the second choice (ii). Under the finite gauge transformation $A_\mu^g = i\partial_\mu g g^{-1} + gA_\mu g^{-1}$ $(g\in G)$, the Chern–Simons term and the supersymmetric boundary term transform as

$$\begin{aligned}\varepsilon^{\mu\nu\rho}\mathrm{Tr}\left(\partial_\mu A_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right)^g &= \varepsilon^{\mu\nu\rho}\mathrm{Tr}\Big(-\frac{1}{3}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g g^{-1} + i\partial_\mu(A_\nu g^{-1}\partial_\rho g)\Big)\\ &\quad + \varepsilon^{\mu\nu\rho}\mathrm{Tr}\Big(\partial_\mu A_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\Big),\end{aligned} \tag{43}$$

$$\mathrm{Tr}(A_z A_{\bar z})^g = \mathrm{Tr}\Big(-\partial_z g g^{-1}\partial_{\bar z} g g^{-1} + ig^{-1}\partial_z g A_{\bar z} + iA_z g^{-1}\partial_{\bar z} g + A_z A_{\bar z}\Big). \tag{44}$$

Then, the transformation of the combined action is given by

$$(S_{\rm CS}+S_{\rm b.CS})^g = S_{\rm CS}+S_{\rm b.CS} - S_{\rm c.GWZW}, \tag{45}$$

with

$$\begin{aligned}S_{\rm c.GWZW}[g,A_{\bar z}] &= -\frac{\kappa}{2\pi}\int_{T^2}\mathrm{Tr}\big(\partial_z g g^{-1}\partial_{\bar z} g g^{-1}\big) + \frac{i\kappa}{12\pi}\int_{S^1\times D^2}\varepsilon^{\mu\nu\rho}\mathrm{Tr}\big(\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g g^{-1}\big)\\ &\quad + \frac{i\kappa}{\pi}\int_{T^2}\mathrm{Tr}\big(g^{-1}\partial_z g A_{\bar z}\big).\end{aligned} \tag{46}$$

The first line in Eq. (46) is the action of the G WZW model, and the second line is the chiral G/G gauged term with a right G-action. The supersymmetric transformation trivially acts on the G elements, and Eq. (46) is invariant by the supersymmetric transformation, since the fermionic superpartner of the chiral G/G' gauged WZW model takes values in the subspace orthogonal to $\mathrm{Lie}(G')$ in $\mathrm{Lie}(G)$. It is this combination of gauge fields $A_{\bar{z}}$ that is invariant under supersymmetric transformations.

The combination $S_{\mathrm{CS}} + S_{\mathrm{b.CS}} + S_{\mathrm{c.GWZW}}$ is also gauge invariant because we have identities under the right g'-action with $g \to gg'$ $(g, g' \in G)$:

$$(S_{\mathrm{CS}} + S_{\mathrm{b.CS}})^{g'} = S_{\mathrm{CS}} + S_{\mathrm{b.CS}} - S_{\mathrm{c.GWZW}}[g', A_{\bar{z}}], \tag{47}$$

$$S_{\mathrm{c.GWZW}}[gg', A_{\bar{z}}^{g'}] = S_{\mathrm{c.GWZW}}[g, A_{\bar{z}}] + S_{\mathrm{c.GWZW}}[g', A_{\bar{z}}]. \tag{48}$$

This is called the chiral gauged WZW model. When we treat the gauge degrees of freedom as physical fields, we find that $S_{\mathrm{CS}} + S_{\mathrm{b.CS}} + S_{\mathrm{c.GWZW}}$ is fully invariant under the G gauge transformation. The partition function of this WZW model with the background gauge field is known as the holomorphic wave functional $\Psi_{\mathrm{CS}}[A_{\bar{z}}]$ in the pure Chern–Simons theory [22].

It is possible to consider more general boundary interactions by gauging subgroup H of G and considering a left H-action: $g \to h^{-1}gg'$ $(g, g' \in G, h \in H)$. Although the chiral gauged WZW model is anomalous under the left H- and right G-actions, we can cancel the gauge anomaly for H by introducing appropriate $\mathcal{N} = (0, 2)$ (chiral and Fermi) multiplets coupled to the $\mathrm{Lie}(H)$-valued gauge field. Namely, when these $\mathcal{N} = (0, 2)$ multiplets satisfy some suitable relations to cancel the anomaly, the theory becomes consistent. It is interesting to study these boundary interactions in detail and evaluate such 3d–2d coupled partition functions. However, in the rest of this article we mainly consider case (i).

4. Index on $S^1 \times D^2$ and localization

In this section we evaluate the partition function (index) on $S^1 \times D^2$ via localization. On $S^1 \times S^2$, the supersymmetric variation parameters $\epsilon, \bar{\epsilon}$ cannot be periodic along the S^1 direction [7], so we impose the twisted boundary condition along this S^1 direction:

$$\Phi(\tau + \beta r) = e^{(-\mathsf{J}_3 - \mathsf{R})\beta_1 + \mathsf{J}_3\beta_2 + \sum_l F_l M_l}\,\Phi(\tau) \qquad (\beta = \beta_1 + \beta_2). \tag{49}$$

Here, Φ is a field in the supermultiplets, J_3 is the generator of a rotation along the φ-direction, R is the $U(1)$ R-charge, and the F_l are charges of global symmetry groups; β_1, β_2, and the M_l are fugacities for these charges. Under the twisted boundary condition, the partition function on $S^1 \times S^2$ defines a superconformal index:

$$\mathcal{I}_{S^1\times S^2} = \mathrm{tr}_{\mathcal{H}(S^2)}\left[(-1)^F e^{-\beta_1(\mathsf{D}-\mathsf{R}-\mathsf{J}_3)} e^{-\beta_2(\mathsf{D}+\mathsf{J}_3)} e^{-\sum_l F_l M_l}\right]. \tag{50}$$

Here, D is the generator of the translation along the S^1-direction. The superconformal index counts the number of BPS operators which saturate the bound $\mathsf{D} - \mathsf{R} - \mathsf{J}_3 \geq 0$ and does not depend on the fugacity β_1.

Similarly, the partition function on $S^1 \times D^2$ with the boundary conditions in Eqs. (11), (19), or (20) defines the following index:

$$\mathcal{I}_{S^1\times D^2} = \mathrm{tr}_{\mathcal{H}(D^2)}\left[(-1)^F e^{-\beta_1(\mathsf{D}-\mathsf{R}-\mathsf{J}_3)} e^{-\beta_2(\mathsf{D}+\mathsf{J}_3)} e^{-\sum_l F_l M_l}\right]$$
$$= \mathrm{tr}_{\mathcal{H}(D^2)}\left[(-1)^F e^{-\beta_2(\mathsf{R}+2\mathsf{J}_3)} e^{-\sum_l F_l M_l}\right]. \tag{51}$$

We will show that all the one-loop determinants on $S^1 \times D^2$ do not depend on β_1, which means that Eq. (51) counts the number of operators saturating the bound $\mathsf{D}-\mathsf{R}-\mathsf{J}_3 \geq 0$ on $S^1 \times D^2$.

Recently, the indices on $S^1 \times S_b^2$ and $S^1 \times \mathbb{R}\mathbf{P}^2$ were studied in Ref. [23]. These are trigonometric deformations of the partition functions on the 2d squashed sphere S_b^2 [24] and on the real projective space $\mathbb{R}\mathbf{P}^2$ [25] by the Kaluza–Klein modes of S^1. The index on $S^1 \times S_b^2$ does not depend on the squashing parameter b and agrees with the ordinary superconformal index. At least, the (3d) bulk part of the index on $S^1 \times D^2$ is also expected to be independent of the squashing deformation of the 2d hemisphere. Moreover, the index on $S^1 \times D^2$ does not explicitly depend on the radius r of the 2d hemisphere, since the radius of the hemisphere enters in the index on $S^1 \times D^2$ only through the matrix integral variables when the localization method is applied. When the boundary interactions are absent, the index on $S^1 \times D^2$ is expected to be independent of both the squashing parameter and the radius of the hemisphere. Then, the index on $S^1 \times D^2$ can be identified with an index on the circle times the 2d flat space, $\mathcal{I}_{S^1\times D^2} \simeq \mathcal{I}_{S^1\times\mathbb{R}^2}$:

$$\mathcal{I}_{S^1\times D^2} = \mathrm{tr}_{\mathcal{H}(\mathbb{R}^2)}\left[(-1)^F e^{-\beta_2(\mathsf{R}+2\mathsf{J}_3)} e^{-\sum_l F_l M_l}\right]. \tag{52}$$

This is a BPS index of 3d $\mathcal{N}=2$ supersymmetric theory. Our 3d–2d index should be thought of as a 3d BPS index with boundary interactions.

In the calculation of the localization of the super Yang–Mills Lagrangian in Eq. (10), we find that the zero locus (saddle point) of the Q-exact action is given by

$$A_3 = \text{constant} = a = \sum_{c=1}^{N} a^c H^c, \tag{53}$$

and the other fields are trivial. $\{H^c\}_{c=1}^N$ denotes the set of generators of the Cartan subalgebra of $\mathrm{Lie}(G)$, and N is the rank of $\mathrm{Lie}(G)$. The boundary Chern–Simons term at the saddle point in Eq. (53) is given by

$$\exp\left(S_{\mathrm{b.CS}}\right) = \exp\left(-\frac{\kappa}{4\beta}\mathrm{Tr}(i\beta r a)^2\right). \tag{54}$$

Note that Eq. (54) is not invariant under the large gauge transformation $i\beta r a \to i\beta r a + 2\pi i \sum_{c=1}^N d^c H^c$ with $d_c \in \mathbb{Z}$.

As another possibility for a Q-closed form, we have the Fayet–Iliopoulos term (FI term)

$$\mathcal{L}_{\mathrm{FI}} = i\zeta\,\mathrm{Tr}\left(\frac{A_3}{r} - D\right). \tag{55}$$

In fact, the supersymmetric transformation of this FI term does not have surface terms and becomes a Q-closed form under the boundary condition in Eq. (11). This FI term has a contribution at the saddle point:

$$-S_{\mathrm{FI}} = \int_{S^1\times D^2} \mathcal{L}_{\mathrm{FI}} = 2\pi r\zeta\,\mathrm{Tr}\, i\beta r a. \tag{56}$$

In the calculation of the localization, we deform the action by adding a Q-exact form $tQ\cdot V$ and take the limit $t\to\infty$. In general, it is possible to add boundary $\mathcal{N}=(0,2)$ vector multiplets which are independent of the bulk $\mathcal{N}=2$ vector multiplet, but we treat boundary $\mathcal{N}=(0,2)$ vector multiplets which have originated from the $\mathcal{N}=2$ bulk vector multiplets. Then, the 3d–2d coupled index on $S^1\times D^2$ can be evaluated by the one-loop calculation around the saddle point:

$$\mathcal{I}_{S^1\times D^2}=\lim_{t_1,t_2\to\infty}\int\mathcal{D}\Phi_{3\mathrm{d}}\mathcal{D}\Phi_{2\mathrm{d}}e^{-S[\Phi]-t_1Q_{3\mathrm{d}}\cdot V_{3d}[\Phi]-t_2Q_{2\mathrm{d}}\cdot V_{2d}[\Phi]}$$
$$=\frac{1}{|W_G|}\int\frac{d^N(\beta ra)}{(2\pi)^N}\left(\prod_{\alpha\neq0}\sinh\frac{i\beta r\alpha(a)}{2}\right)e^{-S_{\mathrm{cl}}}Z^{3\mathrm{d}}_{1\text{-loop}}Z^{2\mathrm{d}}_{1\text{-loop}}. \tag{57}$$

Here, the right-hand side of Eq. (57) is as follows. $\Phi_{3\mathrm{d}}$ (resp. $\Phi_{2\mathrm{d}}$) is the collection of 3d (resp. 2d) fields in the models, and $S[\Phi]:=S_{\mathrm{FI}}-S_{\mathrm{CS}}-S_{\mathrm{b.CS}}$. N is the rank of the gauge group G, and $|W_G|$ is the order of the Weyl group W_G of G. Since the values of βra related by the large gauge transformation are identified, the integration region for βra in Eq. (57) is taken as $[0,2\pi]^N$. $\prod_{\alpha\neq0}$ runs over the roots of Lie(G), and $\alpha(a)$ is the pairing of a root α and a Cartan element a in Eq. (53). S_{cl} is the saddle point value of $S[\Phi]$. We perform the path integrals for the 2d boundary fields, except for vector, chiral, and Fermi multiplets which are obtained by the restriction of the 3d bulk vector and chiral multiplets. The factor "sinh" comes from the additional gauge-fixing condition of A_τ.

For the closed manifolds $S^1\times S^2$ and S^3_b, the Chern–Simons terms are Q-closed but not Q-exact and have contributions in the classical level. On the other hand, in our case of $S^1\times D^2$, we will see that the Chern–Simons term contributes to one-loop determinants of the boundary $\mathcal{N}=(0,2)$ (chiral and Fermi) multiplets through anomaly inflows. We summarize the results of the bulk and boundary one-loop determinants. The derivation is given in Appendix D.

- The one-loop determinant of the 3d $\mathcal{N}=2$ vector multiplet:

$$\left(\prod_{\alpha\neq0}\sinh\frac{i\beta r\alpha(a)}{2}\right)Z^{3\mathrm{d.vec}}_{1\text{-loop}}=\prod_{\alpha\neq0}\exp\left\{\frac{-(i\beta r\alpha(a))^2}{8\beta_2}\right\}(e^{i\beta r\alpha(a)};q^2)_\infty. \tag{58}$$

  Here, we defined $q:=e^{-\beta_2}$, $(a;q)_\infty:=\prod_{n=0}^\infty(1-aq^n)$, and included the "sinh" factor coming from the additional gauge fixing.
- The one-loop determinant of the 3d $\mathcal{N}=2$ chiral multiplet with the Neumann boundary condition:

$$Z^{3\mathrm{d.chi.N}}_{1\text{-loop}}=\prod_\rho e^{\mathcal{E}(i\beta r\rho(a)+\Delta\beta_2+F_lM_l)}(e^{-i\beta r\rho(a)-F_lM_l}q^\Delta;q^2)^{-1}_\infty, \tag{59}$$

  with

$$\mathcal{E}(x):=\frac{\beta_2}{12}-\frac{1}{4}x+\frac{1}{8\beta_2}x^2. \tag{60}$$

  Here, ρ runs over the weight of the representation of the gauge group for the chiral multiplet. The factor $\mathcal{E}(x)$ comes from a zeta function regularization of zero-point energies.
- The one-loop determinant of the 3d $\mathcal{N}=2$ chiral multiplet with the Dirichlet boundary condition:

$$Z^{3\mathrm{d.chi.D}}_{1\text{-loop}}=\prod_\rho e^{-\mathcal{E}(-i\beta r\rho(a)+(2-\Delta)\beta_2-F_lM_l)}(e^{i\beta r\rho(a)+F_lM_l}q^{2-\Delta};q^2)_\infty. \tag{61}$$

- The one-loop determinant of the 2d $\mathcal{N} = (0, 2)$ vector multiplet:

$$Z^{\text{2d.vec}}_{\text{1-loop}} = \prod_{\alpha \neq 0} \exp\left\{-\frac{(i\beta r\alpha(a))^2}{4\beta_2}\right\} \theta(e^{i\beta r\alpha(a)}; q^2). \tag{62}$$

- The one-loop determinant of the 2d $\mathcal{N} = (0, 2)$ chiral multiplet:

$$Z^{\text{2d.chi}}_{\text{1-loop}} = \prod_{\rho} e^{2\mathcal{E}(i\beta r\rho(a)+\Delta\beta_2+F_lM_l)} \theta(e^{-i\beta r\rho(a)-F_lM_l} q^{\Delta}; q^2)^{-1}_{\infty}. \tag{63}$$

- The one-loop determinant of the 2d $\mathcal{N} = (0, 2)$ Fermi multiplet:

$$Z^{\text{2d.Fermi}}_{\text{1-loop}} = \prod_{\rho} e^{-2\mathcal{E}(i\beta r\rho(a)+\tilde{\Delta}\beta_2+F_aM_a)} \theta(e^{-i\beta r\rho(a)-F_aM_a} q^{\tilde{\Delta}}; q^2)_{\infty}. \tag{64}$$

We find that all the one-loop determinants do not depend on the fugacity β_1 explicitly. Unless specifically mentioned, we call the one-loop determinants without the anomalous terms simply the one-loop determinants.

Here, it is worth making several remarks on the properties of these one-loop determinants. First, when 2d $\mathcal{N} = (2, 2)$ gauge theories flow to nonlinear sigma models where the target spaces are Calabi–Yau three-folds, partition functions on the hemisphere compute central charges of a D-brane. In this case, we recall that there are two equivalent descriptions to express fluctuations along the normal directions to D-branes [16, 17]. One is to impose the Dirichlet boundary condition for 2d chiral multiplets whose lowest components label coordinates along the normal directions of D-branes. The other is to impose the Neumann boundary condition for the 2d chiral multiplets and introduce one-dimensional boundary interactions, which effectively transmutes the one-loop determinant with the Neumann boundary condition into that with the Dirichlet boundary condition. For the indices on $S^1 \times D^2$, although the direct connection to D-branes is not known, we have a similar relation between the boundary conditions for 3d multiplets and 2d boundary interactions. That is, if we combine a 3d Neumann-type chiral multiplet and a boundary $\mathcal{N} = (0, 2)$ Fermi multiplet with the common weight ρ and R-charge $\tilde{\Delta} = \Delta$, the boundary Fermi multiplet transmutes the Neumann boundary condition to a Dirichlet boundary condition as

$$Z^{\text{2d.Fermi}}_{\text{1-loop}} Z^{\text{3d.chi.N}}_{\text{1-loop}} = Z^{\text{3d.chi.D}}_{\text{1-loop}}. \tag{65}$$

When we consider the product of the Neumann-type and the Dirichlet-type chiral multiplets in three dimensions, we can construct a one-loop determinant of a chiral multiplet without magnetic charge $\mathbf{m} = 0$ on $S^1 \times S^2$:

$$Z^{\text{3d.chi.N}}_{\text{1-loop}} Z^{\text{3d.chi.D}}_{\text{1-loop}} = \frac{(e^{i\beta r\rho(a)+F_lM_l} q^{2-\Delta}; q^2)_{\infty}}{(e^{-i\beta r\rho(a)-F_lM_l} q^{\Delta}; q^2)_{\infty}}. \tag{66}$$

Next, we comment on the 2d limit of these determinants. When the size of S^1 goes to zero ($\beta \sim \beta_2 \sim 0$), the one-loop determinants of the 3d (vector and chiral) multiplets tend to the asymptotic forms, up to exponential factors,

$$Z^{\text{3d.vec}}_{\text{1-loop}} \sim \prod_{\alpha \neq 0} \prod_{j=0}^{\infty} (ir\alpha(a_{\text{2d}}) + j + 1),$$

$$Z^{\text{3d.chi.N}}_{\text{1-loop}} \sim \prod_{\rho}\prod_{j=0}^{\infty}\left(-ir\rho(a_{\text{2d}}) - F_l M_{\text{2d}.l} + \frac{\Delta}{2} + j\right)^{-1},$$

$$Z^{\text{3d.chi.D}}_{\text{1-loop}} \sim \prod_{\rho}\prod_{j=0}^{\infty}\left(ir\rho(a_{\text{2d}}) + F_l M_{\text{2d}.l} - \frac{\Delta}{2} + 1 + j\right). \tag{67}$$

Here, we defined $a_{\text{2d}} := a\beta/2\beta_2$ and $M_{\text{2d}.l} := M_l/2\beta_2$. These reproduce the one-loop determinants of the $\mathcal{N} = (2, 2)$ vector and chiral multiplets on the 2d hemisphere D^2. In this limit, fugacities of the flavor symmetries become the twisted masses in two dimensions.

5. Relation to holomorphic blocks

In this section we study the relation between the 3d–2d coupled partition function $\mathcal{I}_{S^1\times D^2}$ and the holomorphic block $\mathcal{B}_\alpha$ in three dimensions [8]. We assume that the boundary multiplets couple to the boundary gauge fields which have originated in the bulk vector multiplets.

First, we consider a single chiral multiplet. Up to the anomalous contribution $\mathcal{E}$, the one-loop determinant with the Dirichlet boundary condition is expressed as

$$Z^{\text{3d.chi.D}}_{\text{1-loop}} = (q^{2-\Delta}s^{-\rho}z_l^{-F_l}; q^2)_\infty, \tag{68}$$

where we defined $s^\rho := e^{-i\beta r\rho(a)}$. Equation (68) is precisely the same formula as the contribution of the chiral multiplet in the holomorphic block.

Next, we discuss the condition to cancel the anomalous terms. As we have seen in Eq. (54), there are anomalous contributions which break the single-valuedness under the large gauge transformation. First, let us study anomalous terms for dynamical gauge fields, i.e. the quadratic terms of the gauge fields around the saddle point. The condition for cancellation of the anomalous term is given by

$$\begin{aligned} &-\kappa + \frac{\beta}{2\beta_2}\left(-I_2(\text{Ad}) + \sum_{i:\text{3d.chiral}}(-1)^{|i|}I_2(R_i)\right) \\ &= \frac{\beta}{\beta_2}\left(\sum_{n:\text{2d.Fermi}} I_2(R_n) - \sum_{m:\text{2d.chiral}} I_2(R_m)\right), \end{aligned} \tag{69}$$

with

$$(-1)^{|i|} = \begin{cases} -1 & (i\text{th chiral = Dirichlet}) \\ +1 & (i\text{th chiral = Neumann}). \end{cases} \tag{70}$$

Here, $I_2(R)$ is the quadratic index of a representation R of the Lie algebra of the dynamical gauge group and is defined by the relation $\text{Tr}_R(T^aT^b) = I_2(R)\text{Tr}_\square(T^aT^b)$. The symbol $\square$ (resp. Ad in Eq. (69)) denotes the fundamental (resp. adjoint) representation.

The left-hand side of Eq. (69) is the contribution of the boundary Chern–Simons term in Eq. (54), the one-loop anomalous term of the vector multiplet in Eq. (58), and the one-loop anomalous terms of the chiral multiplets in Eqs. (59) and (61), respectively. The right-hand side in Eq. (69) is the contribution from the anomalous terms $\mathcal{E}$ of the boundary multiplets in Eqs. (63) and (64). The sums run over the 3d chiral multiplets, the 2d Fermi multiplets, and the 2d chiral multiplets, respectively. If we set $\beta_2 = \beta$, the left-hand side of Eq. (69) reproduces the effective Chern–Simons level for the gauge group G, and the right-hand side reproduces the gauge anomaly coefficient of the boundary

$\mathcal{N} = (0, 2)$ theory. Then, Eq. (69) is the anomaly inflow condition that describes the one-loop shift of the classical Chern–Simons terms in Eq. (38). A similar phenomenon is known in four-dimensional (4d) supersymmetric theory. In the localization computation of the 4d $\mathcal{N} = 1$ superconformal index, there exist anomalous terms which are not invariant under large gauge transformations. It was observed in Ref. [26] that the anomalous terms in the superconformal index are proportional to anomalies in four dimensions.

If there is a pair of a fundamental and an anti-fundamental chiral multiplet, we have an accidental cancellation. That is, we take a Neumann (Dirichlet) boundary condition for the fundamental (anti-fundamental) chiral multiplet respectively, then anomalous terms cancel out. Moreover, if we include an adjoint chiral multiplet, the contribution from the adjoint chiral multiplet cancels the anomalous factor coming from the vector multiplet. This means that the Chern–Simons level on $S^1 \times D^2$ cannot be shifted in $\mathcal{N}(\geq 3)$ supersymmetric cases, and the condition in Eq. (69) for $\mathcal{N} \geq 3$ cases matches with Eq. (38). This observation is similar to the level shifts on the flat space [27].

Next, we consider the cancellation of anomalous terms for the mixed Chern–Simons terms between the ath central $U(1)$ gauge symmetry and the lth $U(1)$ flavor symmetry:

$$-\kappa_{a\,l} + \frac{\beta}{2\beta_2} \sum_{i:\text{3d.chiral}} (-1)^{|i|} Q^i_a F^i_l \frac{\beta}{\beta_2} \left(\sum_{n:\text{2d.Fermi}} \tilde{Q}^n_{\text{2d},a} \tilde{F}^n_{\text{2d},l} - \sum_{m:\text{2d.chiral}} Q^m_{\text{2d},a} F^m_{\text{2d},l} \right). \tag{71}$$

Here, we used the fact that the background gauge field for the lth flavor symmetry is $A^{(l)}_\mu = (0, 0, -iM_l/\beta r)$. Q^i_a is the ath central $U(1)$ gauge charge for the ith 3d chiral multiplet. $\tilde{Q}^n_{\text{2d},a}$ (resp. $Q^m_{\text{2d},a}$) denotes the ath central $U(1)$ gauge charge for the 2d Fermi (resp. chiral) multiplet. $\tilde{F}^n_{\text{2d},l}$ (resp. $F^m_{\text{2d},l}$) denotes the lth $U(1)$ flavor charge for the 2d nth Fermi (resp. mth chiral) multiplet.

In addition to the gauge and flavor mixed CS terms, we have the R-symmetry mixed Chern–Simons levels $\kappa_{s\,R}$. Here, the subscript "R" means the R-symmetry and "s" runs over the center of the gauge symmetry and the Cartan of flavor symmetries. Then, the mixed Chern–Simons terms are given by

$$S^{sR}_{\text{CS}} = \frac{i\kappa_{s\,R}}{4\pi} \int_{S^1 \times D^2} A^{(s)} dA^{(R)}. \tag{72}$$

Here, $A^{(R)}_\mu$ is the background gauge field coupled to the R-symmetry current. As in the cases of the gauge–gauge or gauge–flavor mixed boundary Chern–Simons terms of Eq. (40), these boundary terms are expected to contain quadratic terms of the gauge field:

$$\int_{T^2} A^{(s)}_z A^{(R)}_{\bar{z}}. \tag{73}$$

Since the twisted boundary condition in Eq. (51) on $S^1 \times D^2$ is the same as the condition in Eq. (50) on $S^1 \times S^2$, the background gauge field $A^{(R)}_\mu$ on $S^1 \times D^2$ is given by the one on $S^1 \times S^2$. When we put $\beta_2 = \beta$, the background gauge field [28] is written as

$$A^{(R)}_\mu = \left(0, 0, -\frac{i}{r}\right). \tag{74}$$

Then, the condition for the cancellation of anomalous terms including the gauge and R-symmetries is given by

$$-\kappa_{aR} + \sum_{i:\text{3d.chiral}} \frac{(-1)^{|i|}}{2} Q^i_a (\Delta_i - 1) = \sum_{n:\text{2d.Fermi}} \tilde{Q}^n_{\text{2d},a} (\tilde{\Delta}_n - 1) - \sum_{m:\text{2d.chiral}} Q^m_{\text{2d},a} (\Delta_m - 1), \tag{75}$$

Table 1. Left: The charge assignments of the scalar in the 3d chiral multiplet. $U(1)_G$ is the dynamical gauge group and $U(1)_J$ is the topological flavor group. Right: The set of mixed Chern–Simons levels.

	$U(1)_G$	$U(1)_J$	$U(1)_R$
ϕ	1	0	0

κ_{st}^{eff}	$U(1)_G$	$U(1)_J$	$U(1)_R$
$U(1)_G$	1	1	−1
$U(1)_J$	1	0	0
$U(1)_R$	−1	0	0

Table 2. The charge assignments of the scalar in the 2d boundary chiral multiplet and the fermion in the Fermi multiplet.

	$U(1)_G$	$U(1)_J$	$U(1)_R$
ϕ'	1	1	0
Ψ	0	1	−1

and that for the flavor and R-symmetries is given by

$$-\kappa_{lR} + \sum_{i:\text{3d.chiral}} \frac{(-1)^{|i|}}{2} F_l^i(\Delta_i - 1) = \sum_{n:\text{2d.Fermi}} \tilde{F}^n_{\text{2d},l}(\tilde{\Delta}_n - 1) - \sum_{m:\text{2d.chiral}} F^m_{\text{2d},l}(\Delta_m - 1). \tag{76}$$

When we take Abelian gauge groups and put $\beta_2 = \beta$, the conditions in Eqs. (69), (71), (75), and (76) agree with the decomposition rule for the effective mixed Chern–Simons levels proposed in the holomorphic blocks [8]. Let us study several examples of 3d–2d indices and compare holomorphic blocks with our 3d–2d indices in the following subsections. We take mixed bare Chern–Simons levels and charges for gauge, flavor, and R-symmetries as in Ref. [8].[1] Since 2d multiplets are chosen to cancel the anomalous terms of 3d bulk theory, we call the one-loop determinants without anomalous terms simply the one-loop determinants.

5.1. *Mirror of T_Δ*

As a first example, we take $G = U(1)_G$ with a bare dynamical Chern–Simons level $k_{GG} = +\frac{1}{2}$ and consider a single chiral multiplet with charge $+1$. This model is the 3d mirror dual of the tetrahedron T_Δ. The charge assignments and the effective mixed Chern–Simons levels[2] are listed in Table 1. Then, the one-loop determinant of the 3d bulk chiral multiplet is given by

$$Z^{\text{3d.chi.D}}_{\text{1-loop}} = (q^2 s^{-1}; q^2)_\infty, \tag{77}$$

where $s := e^{-i\beta r a}$. In order to satisfy the conditions in Eqs. (69), (71), (75), and (76), we introduce a boundary chiral multiplet and a Fermi multiplet, whose lowest components are denoted by a scalar ϕ' and a fermion Ψ, respectively. The charge assignments of these boundary multiplets are listed in Table 2, and the one-loop contribution of the boundary multiplets is given by

[1] As mentioned in Sect. 4.2 of [8], the holomorphic blocks are not uniquely determined. In our case, this ambiguity is equivalent to adding the anomaly-free boundary multiplets. When two holomorphic blocks are glued into a partition function (index) on a closed three-manifold, it is not clear whether the ambiguity is cancelled out.

[2] The effective Chern–Simons level is related to the bare Chern–Simons level through the equation $\kappa_{st}^{\text{eff}} = \kappa_{st}^{\text{bare}} + \sum_l \frac{1}{2}\text{sign}(m_l) Q_s^l Q_t^l$. In Ref. [8], signatures of fermion masses are taken as $\text{sign}(m_l) = +1$.

Table 3. The set of mixed effective Chern–Simons levels for the Neumann boundary condition.

κ_{st}^{eff}	$U(1)_G$	$U(1)_J$	$U(1)_R$
$U(1)_G$	0	1	0
$U(1)_J$	1	0	0
$U(1)_R$	0	0	0

$$Z_{\text{1-loop}}^{\text{2d}} = \frac{\theta(x;q^2)}{\theta(sx;q^2)}. \tag{78}$$

From Eqs. (77) and (78), we obtain a 3d–2d index of this model T'_Δ mirror dual to the tetrahedron theory T_Δ:

$$\mathcal{I}_{S^1\times D^2}^{T'_\Delta} = \int \frac{ds}{2\pi is}\frac{\theta(x;q^2)}{\theta(sx;q^2)}(q^2s^{-1};q^2)_\infty. \tag{79}$$

Equation (79) is exactly the same as the holomorphic block of T'_Δ.[3] That is to say, in our language, the contribution of the chiral multiplet in the holomorphic blocks [8] corresponds to the one-loop determinant of 3d chiral multiplets with the *Dirichlet boundary condition*.

Next, we consider the Neumann boundary condition and study the relation between our 3d–2d index and the holomorphic blocks. At least when the gauge group is Abelian and the superpotential is absent, we can reproduce the homolorphic blocks from the 3d–2d index. To see this, we take the Neumann boundary condition for the chiral multiplet in T'_Δ. The one-loop determinant of the 3d chiral multiplet is given by

$$Z_{\text{1-loop}}^{\text{3d.chi.N}} = (s;q^2)_\infty^{-1}. \tag{80}$$

In this case, the level shift from the 3d anomalous factor has the opposite sign to the Dirichlet boundary condition. Then, the dynamical Chern–Simons level is shifted by $-\frac{1}{2}$. The effective Chern–Simons levels for the Neumann boundary condition are listed in Table 3. In addition, we have to satisfy the conditions in Eqs. (69), (71), (75), and (76) for the anomaly cancellation and introduce appropriate boundary multiplets. The charge assignments of the boundary multiplets are listed in Table 4. Then, the one-loop determinant of the boundary multiplets is given by

$$Z_{\text{1-loop}}^{\text{2d}} = \frac{\theta(s;q^2)\theta(x;q^2)}{\theta(sx;q^2)}. \tag{81}$$

The 3d–2d index with the Neumann boundary condition,

$$\mathcal{I}_{S^1\times D^2}^{T'_\Delta.\text{Neu}} = \int \frac{ds}{2\pi is}(s;q^2)_\infty^{-1}\frac{\theta(s;q^2)\theta(x;q^2)}{\theta(sx;q^2)}, \tag{82}$$

gives the same result as the 3d–2d index with the Dirichlet boundary condition in Eq. (79).

5.2. *XYZ model*

As a second example, we consider the XYZ model that consists of three chiral multiplets in bulk three dimensions (Table 5). When we impose the Dirichlet boundary conditions on the bulk chiral

[3] The convention of the theta function is $\theta(x;q)(\text{here}) = \theta(q^{\frac{1}{2}}x;q)$([8]). The normalization of the fugacity q is different from that of Ref. [8]. In addition, an extra sign difference comes from $(-1)^R$ which is used in the holomorphic blocks instead of $(-1)^F$. Then, the identification becomes $q(\text{here}) = -q^{\frac{1}{2}}$([8]).

Table 4. The charge assignments of the scalar in the 2d boundary chiral multiplet and fermions in the Fermi multiplets.

	$U(1)_G$	$U(1)_J$	$U(1)_R$
ϕ''	1	1	0
Ψ_1	1	0	-1
Ψ_2	0	1	-1

Table 5. Left: The charge assignments of scalars in the 3d chiral multiplets. Right: The set of mixed Chern–Simons levels.

	$U(1)_x$	$U(1)_y$	$U(1)_R$
ϕ_1	1	0	0
ϕ_2	0	1	0
ϕ_3	-1	-1	2

κ_{st}^{eff}	$U(1)_x$	$U(1)_y$	$U(1)_R$
$U(1)_x$	1	1	-1
$U(1)_y$	1	1	-1
$U(1)_R$	-1	-1	1

Table 6. The charge assignment of the scalar in the 2d boundary chiral multiplet.

	$U(1)_x$	$U(1)_y$	$U(1)_R$
ϕ	1	1	0

multiplets, the one-loop determinant is given by

$$Z^{\text{3d.chi}}_{\text{1-loop}} = (q^2x^{-1};q^2)_\infty(q^2y^{-1};q^2)_\infty(xy;q^2)_\infty. \tag{83}$$

Next we introduce a boundary chiral multiplet ϕ to cancel the bulk-boundary anomalies (Table 6). Then this multiplet has the contribution at the one-loop level:

$$Z^{\text{2d.chi}}_{\text{1-loop}} = \theta(xy;q^2)^{-1}. \tag{84}$$

From Eqs. (83) and (84), the 3d–2d index on $S^1 \times D^2$ becomes

$$\mathcal{I}^{\text{XYZ}}_{S^1\times D^2} = \frac{(q^2x^{-1};q^2)_\infty(q^2y^{-1};q^2)_\infty(xy;q^2)_\infty}{\theta(xy;q^2)}. \tag{85}$$

This matches with the holomorphic block of the XYZ model.

5.3. SQED

In this subsection we consider the SQED model. From Table 7, the one-loop determinant of 3d chiral multiplets ϕ_1, ϕ_2 is given by

$$Z^{\text{3d.chi.D}}_{\text{1-loop}} = (s^{-1}q^2;q^2)_\infty(sq^2x^{-1};q^2)_\infty. \tag{86}$$

A pair of boundary multiplets should be introduced to cancel anomalous terms. It is a pair of a boundary $\mathcal{N} = (0,2)$ chiral multiplets and a Fermi multiplet whose lowest components are respectively a scalar ϕ' and a fermion Ψ. Their charge assignments are listed in Table 8. Then, the one-loop contributions of the boundary multiplets are given by

$$Z^{\text{2d.chi}}_{\text{1-loop}}Z^{\text{2d.Fermi}}_{\text{1-loop}} = \frac{\theta(y;q^2)}{\theta(sy;q^2)}. \tag{87}$$

Table 7. Left: The charge assignments of scalars in the 3d chiral multiplets. Right: The set of effective mixed Chern–Simons levels.

	$U(1)_G$	$U(1)_x$	$U(1)_y$	$U(1)_R$
ϕ_1	1	0	0	0
ϕ_2	−1	1	0	0

κ_{st}^{eff}	$U(1)_G$	$U(1)_x$	$U(1)_y$	$U(1)_R$
$U(1)_G$	1	0	1	−1
$U(1)_x$	0	0	0	0
$U(1)_y$	1	0	0	0
$U(1)_R$	−1	0	0	0

Table 8. The charge assignment of the scalar (resp. fermion) in the boundary chiral (resp. Fermi) multiplet.

	$U(1)_G$	$U(1)_x$	$U(1)_y$	$U(1)_R$
ϕ'	1	0	1	0
Ψ	0	0	1	0

Thus, the 3d–2d index of this model becomes

$$\mathcal{I}^{\text{SQED}}_{S^1\times D^2} = \int \frac{ds}{2\pi i s}(s^{-1}q^2;q^2)_\infty(sq^2x^{-1};q^2)_\infty\frac{\theta(y;q^2)}{\theta(sy;q^2)}. \tag{88}$$

This has the same expression as the result in the holomorphic block for the SQED. The holomorphic block for the SQED also matches that in the XYZ model. Thus, 3d–2d indices for these two models produce identical results. We make a comment here: the SQED and XYZ models flow to the same IR fixed point, and a pair of these models is the simplest example of the $\mathcal{N}=2$ mirror symmetry in three dimensions [29]. The half-BPS boundary conditions for the SQED and XYZ models were studied in Ref. [30], where it is shown that the $\mathcal{N}=(0,2)$-type BPS boundary condition in the SQED is mapped to the $\mathcal{N}=(0,2)$-type supersymmetry in the XYZ model. Our result is consistent with their analysis of the boundary conditions because the 3d–2d index on $S^1\times D^2$ preserves the boundary $\mathcal{N}=(0,2)$ supersymmetry. This situation is different from two dimensions. For a mirror pair in two dimensions, the A-type boundary supersymmetry is mapped to the B-type boundary supersymmetry [31–33].

5.4. A remark on non-Abelian gauge theories

For non-Abelian gauge theories, there is a difference between the one-loop determinant of the vector multiplet in Eq. (58) and the contribution of the vector multiplet in the holomorphic blocks. In our calculation, up to the anomalous term, the contribution of the vector multiplet is given by the one-loop determinant:

$$\prod_{\alpha>0}(s^\alpha;q^2)_\infty(s^{-\alpha};q^2)_\infty. \tag{89}$$

On the other hand, the holomorphic block in Ref. [8] leads to the contribution of the vector multiplet with a non-Abelian gauge group,

$$\prod_{\alpha>0}\frac{(qs^\alpha;q^2)_\infty(q^{-1}s^{-\alpha};q^2)_\infty}{(q^2s^\alpha;q^2)_\infty(q^{-2}s^{-\alpha};q^2)_\infty}. \tag{90}$$

In general, we expect that our indices for non-Abelian gauge theories are different from the holomorphic blocks for non-Abelian gauge theories. We will see later that our index for 3d $\mathcal{N}=4$ $U(N)$

gauge theory without boundary degrees of freedom agrees with an index on $S^1 \times \mathbb{C}$ proposed in Ref. [34].

6. Several models

In this section we study properties of indices in several examples. Unless specifically mentioned, we choose the bare Chern–Simons levels to cancel the anomalous terms coming from the one-loop determinants.

6.1. *q-deformed Whittaker function and K-theoretic J-function for* $\mathbb{C}\mathbf{P}^N$

In this subsection we explain the mathematical aspects of the index of $G = U(1)$ with $N + 1$ chiral multiplets with gauge charges $+1$:

$$\mathcal{I}^{\mathbb{C}\mathbf{P}^N}_{S^1\times D^2} = \oint \frac{ds}{2\pi i s} \frac{s^{-2\pi r\zeta}}{\prod_{l=1}^{N+1}\prod_{j=0}^{\infty}(1-sq^{2j}z_l)}. \tag{91}$$

Here, $s = e^{-i\beta r a}$, $q = e^{-\beta_2}$, $z_l = e^{-M_l}$, and the M_l $(l = 1, \ldots, N + 1)$ represent the set of fugacities of $SU(N + 1)$ flavor symmetry. Since $\beta r a \in [0, 2\pi]$, the integration contour for s is the unit circle.

First, let us recall the mathematical aspects of the hemisphere partition function. In two dimensions, the partition function on the hemisphere D^2 is related to the (equivariant) J-function of the $\mathbb{C}\mathbf{P}^N$ model in the large-volume regime. In order to clarify the geometric data in our model, we evaluate the above integral explicitly. When we assume $|z_l| = 1$ and $|q| < 1$, the pole of s inside the unit circle is the origin. The residue at the origin is rewritten as the sum of the residues outside the unit circle: $s = q^{-2k}z_{l'}^{-1}$ $(k = 0, 1, 2\ldots, l' = 1, \ldots, N + 1)$. Up to the overall sign, Eq. (91) is given by

$$\begin{aligned}\mathcal{I}^{\mathbb{C}\mathbf{P}^N}_{S^1\times D^2} &= \sum_{l'=1}^{N+1}\sum_{k=0}^{\infty}\oint_{s=z_{l'}^{-1}} \frac{ds}{2\pi i s}\frac{s^{-2\pi r\zeta}q^{4\pi r\zeta k}}{\prod_{l=1}^{N+1}\prod_{j=0}^{\infty}(1-sz_lq^{2j-2k})}\\ &= \sum_{l'=1}^{N+1}\oint_{s=z_{l'}^{-1}} \frac{ds}{2\pi i s}\frac{s^{-2\pi r\zeta}}{\prod_{l=1}^{N+1}\prod_{j=0}^{\infty}(1-sz_lq^{2j})}\\ &\qquad\times\left(\sum_{k=0}^{\infty}\frac{Q^k}{\prod_{l=1}^{N+1}\prod_{j=1}^{k}(1-sz_lq^{-2j})}\right).\end{aligned} \tag{92}$$

Here, we defined $Q := q^{4\pi r\zeta}$. The region for $\zeta > 0$ corresponds to the Higgs branch in the 2d limit. The third line in Eq. (92),

$$J^{\mathbb{C}\mathbf{P}^N}(Q, s, z, q) := \sum_{k=0}^{\infty}\frac{Q^k}{\prod_{l=1}^{N+1}\prod_{j=1}^{k}(1-sz_lq^{-2j})}, \tag{93}$$

agrees with the equivariant K-theoretic J-function of $\mathbb{C}\mathbf{P}^N$ [35] by rescaling parameters appropriately. This function $J^{\mathbb{C}\mathbf{P}^N}$ is reduced to the ordinary K-theoretic J-function in the limit $z_l \to 1$. In order to compare our model to 2d cases, we take the 2d limit in Eq. (67). Then, the index in Eq. (91) is reduced to the hemisphere partition function of the 2d model where the moduli space of the Higgs branch is $\mathbb{C}\mathbf{P}^N$:

$$\lim_{\beta\to 0}\mathcal{I}_{S^1\times D^2} \sim \int \frac{dy}{2\pi i}e^{2\pi i\zeta_{2d}y}\prod_{l=1}^{N+1}\Gamma(y - M_l). \tag{94}$$

In other words, this has the same formula as the 2d hemisphere partition function for the $\mathcal{N} = (2, 2)$ $U(1)$ theory with $N + 1$ chiral multiplets with gauge charges $+1$ and twisted masses M_l.

In Refs. [36,37], an eigenfunction of the Hamiltonian of the q-deformed $\mathfrak{gl}_{N+1}$-Toda chain is constructed, that is, the q-deformed Whittaker function. This q-deformed Whittaker function $\Psi_{z_i}^{\mathfrak{gl}_{N+1}}(n, k)$ has the contour integral representation

$$\Psi_{\mathbf{z}}^{\mathfrak{gl}_{N+1}}(n, k) = \left(\prod_{l=1}^{N} z_l^k\right) \oint \frac{ds}{2\pi i s} s^{-n} \prod_{l=1}^{N+1} (z_l s; q)_\infty^{-1}. \tag{95}$$

When we set $n = 2\pi r\zeta$ and replace $q \to q^2$ in the above equation, the index in Eq. (91) on $S^1 \times D^2$ agrees with this q-deformed Whittaker function $\Psi_{\mathbf{z}}^{\mathfrak{gl}_{N+1}}(n, k)$ up to an overall constant $\left(\prod_{l=1}^{N} z_l^k\right)$. Here, the contour is chosen to enclose all the poles except for the pole at the origin. We can also include the factor $\left(\prod_{l=1}^{N} z_l^k\right)$ in the index on $S^1 \times D^2$ by turning on the FI term in Eq. (55) for the flavor gauge field with the FI parameter $\frac{k}{2\pi i}$. The geometrical interpretation of $\Psi_{\mathbf{z}}^{\mathfrak{gl}_M}(n, k)$ was conjectured in Ref. [37] as

$$\left(\prod_{l=1}^{N} z_l^k\right) \mathcal{I}_{S^1\times D^2}^{\mathbb{C}\mathbf{P}^N} = \int_{\mathcal{QM}_\infty(\mathbb{C}\mathbf{P}^N)} \mathrm{Ch}_G(\mathcal{L}_k \otimes \mathcal{O}(n)) \mathrm{Td}_G(T\mathcal{QM}_\infty(\mathbb{C}\mathbf{P}^N)). \tag{96}$$

Here, $\mathcal{QM}_\infty(\mathbb{C}\mathbf{P}^N)$ is the space of the degree-∞ quasimaps $\mathbb{C}\mathbf{P}^1 \to \mathbb{C}\mathbf{P}^N$, and Ch_G and Td_G are the $G = U(1) \times GL(N+1)$-equivariant Chern character and Todd class, respectively.

Remarkably, it was also conjectured in Ref. [38] that the q-deformed Whittaker function is related to the partition function of an equivariant A-type twisted model on $S^1 \times D^2$. A 3d version of the A-type twisted Chern–Simons matter theory was also constructed in Ref. [39], but the supersymmetry considered in this paper is not topologically twisted supersymmetry. It would be interesting to reveal the precise relation among the 3d–2d index studied in this paper, quasimaps, and the A-type twisted theories in three dimensions.

6.2. *Vortex partition function and surface operator*

The vortex partition functions [40–42] are vortex counterparts of the Nekrasov instanton partition functions [43]. It was shown in Refs. [44–46] that the partition function on S_b^3 ($S^1 \times S^2$) is respectively factorized into a pair of vortex and anti-vortex partition functions. From the viewpoint of Higgs branch localization in three dimensions [47,48], we can also construct some Q-exact term whose saddle points admit point-like vortices at the north pole of the hemisphere. Thus, we expect that the index on $S^1 \times D^2$ contains contributions from vortex partition functions.

Here we consider a $U(N)$ SQCD; the gauge group is $G = U(N)$ and the flavor symmetry is $SU(N_\mathrm{f}) \times SU(\tilde{N}_\mathrm{f})$ with $N_\mathrm{f} \geq \tilde{N}_\mathrm{f}$. We have N_f fundamental chiral multiplets with the Neumann boundary conditions and $\tilde{N}_\mathrm{f}$ anti-fundamental chiral multiplets with the Dirichlet boundary conditions. The index of this model is given by

$$\mathcal{I}_{S^1\times D^2} = \frac{1}{N!} \int \prod_{a=1}^{N} \frac{ds_a}{2\pi i s_a} s_a^{-2\pi r\zeta} \prod_{1\leq a\neq b\leq N} \prod_{j=0}^{\infty} (1 - s_a s_b^{-1} q^{2j}) \prod_{a=1}^{N} \frac{\prod_{m=1}^{\tilde{N}_f} (1 - s_a q^{2j+2} \tilde{z}_m^{-1})}{\prod_{l=1}^{N_\mathrm{f}} (1 - s_a q^{2j} z_l)}. \tag{97}$$

Here, $s_b = e^{-i\beta r a_b}$, $q = e^{-\beta_2}$, $z_l = e^{-M_l}$, and the M_l $(l = 1, \ldots, N_\mathrm{f})$ represent the set of fugacities of the $SU(N_\mathrm{f})$ flavor symmetry. Also, we put $\tilde{z}_m := e^{-\tilde{M}_m}$, and the $\tilde{M}_m$ $(m = 1, \ldots, \tilde{N}_\mathrm{f})$ denote the set of fugacities of the flavor symmetry $SU(\tilde{N}_\mathrm{f})$.

When we take residues at poles $s_a = q^{-2j'_a} z_{l'_a}^{-1}$ ($l'_a = 1, \ldots, N_{\rm f}$; $a = 1, \ldots, N$; $j'_a = 0, 1, 2, \ldots$), then the partition function is written as a combination of classical terms, one-loop terms, and vortex partition functions:

$$\mathcal{I}_{S^1 \times D^2} = \sum_{\{l'\} \subset \{N_{\rm f}\}} Z_{\rm cl}^{\{l'\}} Z_{\text{1-loop}}^{\{l'\}} Z_{\text{K-theory}}^{\text{vortex},\{l'\}}, \tag{98}$$

with

$$Z_{\rm cl}^{\{l'\}} = \prod_{a=1}^{N} (z_{l'_a}^{-1})^{2\pi r \zeta}, \tag{99}$$

$$Z_{\text{1-loop}}^{\{l'\}} = \prod_{j=0}^{\infty} \frac{\prod_{m=1}^{\tilde{N}_{\rm f}} (1 - q^{2j+2} z_{l'_a}^{-1} \tilde{z}_m^{-1})}{(1 - q^{2j+2})^N \prod_{a=1}^{N} \prod_{l \notin \{l'\}} (1 - z_l z_{l'_a}^{-1} q^{2j})^{-1}}, \tag{100}$$

$$Z_{\text{K-theory}}^{\text{vortex},\{l'\}} = \sum_{\{k\}} Q^{\sum_{a=1}^{N} k_a}$$
$$\times \frac{\prod_{j=1}^{k_a} \prod_{s=1}^{\tilde{N}_{\rm f}} (1 - q^{-2j+2} z_{l'_a}^{-1} \tilde{z}_s^{-1})}{\left(\prod_{1 \le a,b \le N} \prod_{j=0}^{k_b - 1} (1 - z_{l'_b} z_{l'_a}^{-1} q^{2j - 2k_a}) \right) \left(\prod_{a=1}^{N} \prod_{l \notin \{l'\}} \prod_{j=1}^{k_a} (1 - z_l z_{l'_a}^{-1} q^{-2j}) \right)}. \tag{101}$$

Here, we defined $\{l'\} := \{l_1, l_2, \ldots, l_N\}$, with $1 \le l_1 < l_2 < \cdots < l_N \le N_{\rm f}$ and $\{N_{\rm f}\} := \{1, 2 \ldots, N_{\rm f}\}$. Also, the sum is defined by $\sum_{\{j'\}} := \sum_{i=1}^{N} \sum_{j_i=0}^{\infty}$. In three dimensions, BPS vortices are particle-like objects and the K-theoretic vortex partition functions contribute to the BPS index. The appearance of the K-theoretic vortex partition functions in Eq. (98) is also consistent with the observation that the index on $S^1 \times D^2$ is related to the 3d $\mathcal{N} = 2$ BPS index on $S^1 \times \mathbb{R}^2$. This is analogous to the fact that instantons on $S^1 \times \mathbb{R}^4$ are particle-like objects and the K-theoretic instanton partition functions [49] contribute to the BPS index in five dimensions.

Next, we study the vortex partition function in Eq. (101) from the viewpoint of instanton counting with surface operators. In the geometric engineering, instanton counting with surface operators in five dimensions is expected to be encoded in partition functions of open-closed (refined) topological strings. Vortex partition functions of $U(1)$ gauge theories arise in a certain limit of topological string partition functions [41]. The correspondence between vortex partition functions and topological strings has been extended to non-Abelian gauge theories [46]. On the other hand, it has not been studied how vortex partition functions of non-Abelian gauge theories appear in the instanton partition functions with surface operators. We will show that the vortex counting for non-Abelian $U(N)$ gauge theory with $N_{\rm f}$ fundamental chiral multiplets arises in the sector of vanishing instanton number.

We consider a five-dimensional $\mathcal{N} = 1$ pure $SU(N_{\rm f})$ gauge theory on $S^1 \times \mathbb{C}^2$. We take the surface operator specified by a Levi subgroup $\mathbb{L} = S(U(N) \times U(N_{\rm f} - N)) \subset SU(N_{\rm f})$. Then, the instanton counting with the surface operator is replaced by instanton counting on the orbifold $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_2$ [50].[4] By the orbifold action $(z, \omega) \to (z, -\omega)$, the set of Atiyah–Drinfield–Hitchin–Manin (ADHM) data $(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{I}, \boldsymbol{J})$,

$$\boldsymbol{B}_1, \boldsymbol{B}_2 \in \mathrm{End}(V), \qquad \boldsymbol{I} \in \mathrm{Hom}(W, V), \qquad \boldsymbol{J} \in \mathrm{Hom}(V, W), \tag{102}$$

[4] Also, H. Kanno, Lecture at Saint Paul's University (In Japanese) (2012).

is divided into two groups with $\mathbb{Z}_2$-grading indices "i" ($i = 1, 2$),

$$V = V_1 \oplus V_2, \qquad W = W_1 \oplus W_2, \tag{103}$$

and

$$\begin{aligned} A_i &= \boldsymbol{B}_1\big|_{V_i} \in \mathrm{End}(V_i), \qquad B_i = \boldsymbol{B}_2\big|_{V_i} \in \mathrm{Hom}(V_i, V_{i+1}), \\ I_i &= \boldsymbol{I}\big|_{W_i} \in \mathrm{Hom}(W_i, V_i), \qquad J_i = \boldsymbol{J}\big|_{V_i} \in \mathrm{Hom}(V_i, W_{i+1}). \end{aligned} \tag{104}$$

The dimensions of V_i, W_i are $\dim V_i = k_i$, $\dim W_1 = N$, and $\dim W_2 = N_{\mathrm{f}} - N$. Here, the instanton number is k_2 and the vortex number is $k_1 - k_2$. Then, the moduli space is defined by a quotient:

$$\mathcal{M}_{\mathbb{L},k_1,k_2} = \left\{ (A_i, B_i, I_i, J_i) \middle| A_{i+1}B_i - B_iA_i + I_{i+1}J_i = 0 \right\} \Big/ \prod_i GL(k_i, \mathbb{C}), \tag{105}$$

with some stability condition. The equivariant character for the tangent space of $\mathcal{M}_{\mathbb{L},k_1,k_2}$ is evaluated as

$$\begin{aligned} \chi(T_p\mathcal{M}_{\mathbb{L},k_1,k_2}) = \sum_{i,j=1}^{2} \Big[& e^{\varepsilon_1 + \frac{\varepsilon_2}{2}} \chi(W_i)\chi(V_j^*) + \chi(V_i)\chi(W_j^*) \\ & - (1 - e^{\varepsilon_1})(1 - e^{\frac{\varepsilon_2}{2}})\chi(V_i)\chi(V_j^*) \Big] \Big|_{\mathbb{Z}_2\text{-even}}, \end{aligned} \tag{106}$$

with

$$\begin{aligned} \chi(W_1) &= e^{\frac{\varepsilon_2}{2}} \sum_{a=1}^{N} e^{M_a}, \quad \chi(W_2) = \sum_{b=1}^{N_{\mathrm{f}}-N} e^{M'_b}, \\ \chi(V_1) &= e^{\frac{\varepsilon_2}{2}} \sum_{a=1}^{N} e^{M_a} \sum_{(i,2j-1)\in Y_a} e^{-(i-1)\varepsilon_1 - (j-1)\varepsilon_2} + \sum_{b=1}^{N_{\mathrm{f}}-N} e^{M'_b} \sum_{(i,2j)\in X_b} e^{-(i-1)\varepsilon_1 - (j-\frac{1}{2})\varepsilon_2}, \\ \chi(V_2) &= e^{\frac{\varepsilon_2}{2}} \sum_{a=1}^{N} e^{M_a} \sum_{(i,2j)\in Y_a} e^{-(i-1)\varepsilon_1 - (j-1)\varepsilon_2} + \sum_{b=1}^{N_{\mathrm{f}}-N} e^{M'_b} \sum_{(i,2j-1)\in X_b} e^{-(i-1)\varepsilon_1 - (j-\frac{1}{2})\varepsilon_2}. \end{aligned} \tag{107}$$

Here, the Y_a and X_b are Young diagrams. The collection $(M_1, \ldots, M_N, M'_1, \ldots, M'_{N_{\mathrm{f}}-N})$ is the set of Coulomb branch parameters and the ε_i are Ω-background parameters. In the above equation for the equivariant character, the symbol "$\mathbb{Z}_2$-even" means that we remove terms expressed as $e^{r\varepsilon_2}$ ($r = \frac{1}{2}, \frac{3}{2}, \ldots$). The non-negative integers k_i are related to the number of boxes by the following equations:

$$k_1 = \sum_{a=1}^{N} \#\{(i, 2j-1) | (i, 2j-1) \in Y_a\} + \sum_{b=1}^{N_{\mathrm{f}}-N} \#\{(i, 2j) | (i, 2j) \in X_b\}, \tag{108}$$

$$k_2 = \sum_{a=1}^{N} \#\{(i, 2j) | (i, 2j) \in Y_a\} + \sum_{b=1}^{N_{\mathrm{f}}-N} \#\{(i, 2j-1) | (i, 2j-1) \in X_b\}, \tag{109}$$

with $i, j \in \mathbb{N}$. Here, $\#\{\cdots\}$ expresses the cardinality of the set.

Now we put the instanton number $k_2 = 0$. Then the ADHM data on $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_2$ is reduced to A_1, I_1, and J_1:

$$A_1 \in \mathrm{End}(V_1), \qquad I_1 \in \mathrm{Hom}(W_1, V_1), \qquad J_1 \in \mathrm{Hom}(V_1, W_2). \tag{110}$$

This (A_1, I_1, J_1) is precisely the same as the data of the moduli space of the k_1-vortex of the $G = U(N)$ gauge theory with the $SU(N_{\mathrm{f}})$ flavor symmetry [51]. Moreover, from the relation in Eq. (109), we find that $Y_a = \{(i,1) \mid i = 1, \ldots, k_1^a\}$ with $\sum_{a=1}^N k_1^a = k_1$, but the X_b are absent. Thus, the equivariant character $\chi(T_p\mathcal{M}_{\mathbb{L},k_1,k_2})$ is reduced to

$$\begin{aligned}\chi(T_p\mathcal{M}_{\mathbb{L},k_1,k_2=0}) &= e^{\varepsilon_1+\frac{\varepsilon_2}{2}}\chi(W_2)\chi(V_1^*) + \chi(V_1)\chi(W_1^*) - (1-e^{\varepsilon_1})\chi(V_1)\chi(V_1^*)\\ &= \sum_{a,a'=1}^{N} e^{M_a - M_{a'}} \sum_{i=1}^{k_1^a} e^{(k_1^{a'}+1-i)\varepsilon_1} + \sum_{b=1}^{N_{\mathrm{f}}-N}\sum_{a=1}^{N} e^{M'_b - M_a}\sum_{i=1}^{k_1^a} e^{i\varepsilon_1},\end{aligned} \tag{111}$$

with

$$\chi(W_1) = e^{\frac{\varepsilon_2}{2}}\sum_{a=1}^{N} e^{M_a}, \quad \chi(W_2) = \sum_{b=1}^{N_{\mathrm{f}}-N} e^{M'_b}, \quad \chi(V_1) = e^{\frac{\varepsilon_2}{2}}\sum_{a=1}^{N} e^{M_a}\sum_{i=1}^{k_1^{(a)}} e^{-(i-1)\varepsilon_1}. \tag{112}$$

$\chi(T_p\mathcal{M}_{\mathbb{L},k_1,k_2=0})$ does not depend on the equivariant parameter ε_2 and is precisely the same as the equivariant character of the tangent space of the k_1-vortex moduli space of the $G = U(N)$ gauge theory with N_{f}-fundamental chiral multiplets. The latter has the fixed point p labeled by $(k_1^1, \ldots, k_1^N)$ under the equivariant action. In the context of the 3d theory on $S^1 \times \mathbb{C}$, the Coulomb branch parameters correspond to real masses or fugacities associated to the $SU(N_{\mathrm{f}})$-flavor symmetry.

The K-theoretic vortex partition function can be obtained from the equivariant character $\sum_i \pm e^{\omega_{i,p}}$ by the replacement $\sum_i \pm e^{\omega_{i,p}} \to \sum_p \prod_i (1 - e^{\omega_{i,p}})^{\mp 1}$ as in the case of the K-theoretic instanton partition function. Here, $p = (k_1^1, \ldots, k_1^N)$ denotes a fixed point under the $U(1)_M^{N_{\mathrm{f}}-1} \times U(1)_{\varepsilon_1}$-equivariant action and the $\omega_{i,p}$ denote the equivariant weights at the point p. Then, from the equivariant character in Eq. (111), the vortex partition function with vortex number k_1 is written as

$$Z_{k_1\text{-vortex}} = \sum_{\sum_a k_1^a = k_1}\prod_{a,a'=1}^{N}(1 - e^{M_a - M_{a'} + (k_1^{a'}+1-i)\varepsilon_1})^{-1}\prod_{b=1}^{N_{\mathrm{f}}-N}\prod_{a=1}^{N}\prod_{i=1}^{k_1^a}(1 - e^{M'_b - M_a + i\varepsilon_1})^{-1}. \tag{113}$$

When we identify the parameters as $e^{\varepsilon_1} = q^{-2}$, $e^{M'_b} = z_l$ $(b + N = l)$, and $e^{M_a} = z_{l'}$ $(a = l')$, Eq. (113) agrees with the vortex number k_1 sector of Eq. (101) with $\tilde{N}_{\mathrm{f}} = 0$.

6.3. *Calabi–Yau model and 3d matrix factorization*

So far, all the models we have considered do not have surface terms of superpotentials. In this section we will consider a simple model which has a nontrivial 3d analogue of the matrix factorization and study its 2d limit.

In this section we consider an analogue of an $\mathcal{N} = (2,2)$ gauged linear sigma model which flows in the IR limit to some nonlinear sigma model whose target space is a Calabi–Yau $(N-2)$-fold CY_{N-2} defined by a degree-N homogeneous polynomial $f(x_1, \ldots, x_N) = 0$ in $\mathbb{C}\mathbf{P}^{N-1}$ in the large-volume regime. As a setup, we consider an Abelian model with $G = U(1)$ and take a set of chiral multiplets P, ϕ_I $(I = 1, \ldots, N)$ with a superpotential

$$W(P, \phi_I) = P \cdot f(\phi_I). \tag{114}$$

Here, $f(\phi_I)$ is the homogeneous polynomial of degree N. The charge assignments of the chiral multiplets are listed in Table 9. We impose the Neumann boundary conditions for these chiral

Table 9. The charge assignments of scalars in the 3d chiral multiplets.

	$U(1)_G$	$U(1)_R$
P	$-N$	$+2$
ϕ_I	$+1$	0

multiplets. Then, the one-loop contribution of the bulk 3d chiral multiplets is given by

$$Z^{\text{3d.chi.N}}_{\text{1-loop}} = (e^{-i\beta ra}; q^2)^{-N}_\infty (e^{iN\beta ra} q^2; q^2)^{-1}_\infty . \tag{115}$$

In addition, we have another contribution from the 2d boundary, and the corresponding boundary theory is characterized by functions E_a and J^a. This boundary effect is correlated to the bulk 3d theory through some kind of 3d matrix factorization. We choose E_a and J^a with $E_a J^a = W$ in order to realize this factorization:

$$E(P, \phi_I) = P, \qquad J(P, \phi_I) = f(\phi_I). \tag{116}$$

Then, the boundary contribution in the partition function comes from the one-loop determinant of the Fermi multiplet coupled to $E(P, \phi_I)$ and is given by

$$Z^{\text{2d.Fermi}}_{\text{1-loop}} = \theta(e^{Ni\beta ra} q^2; q^2). \tag{117}$$

We can also include the FI term and write down the expression of the partition function of the model:

$$\begin{aligned}
\mathcal{I}^{\text{CY}_{N-2}}_{S^1\times D^2} &= \int \frac{d(\beta ra)}{2\pi} e^{-S_{FI}} Z^{\text{3d.chi.N}}_{\text{1-loop}} Z^{\text{2d.Fermi}}_{\text{1-loop}} \\
&= \int \frac{d(\beta ra)}{2\pi} e^{2\pi r\zeta(i\beta ra)} \frac{\theta(e^{Ni\beta ra} q^2; q^2)}{(e^{-i\beta ra}; q^2)^N_\infty (e^{iN\beta ra} q^2; q^2)_\infty} \\
&= \int \frac{d(\beta ra)}{2\pi} e^{2\pi r\zeta(i\beta ra)} \frac{(e^{-Ni\beta ra}; q^2)_\infty}{(e^{-i\beta ra}; q^2)^N_\infty}.
\end{aligned} \tag{118}$$

In the 2d limit $\beta \to 0$, keeping $t_{\text{2d}} := 2\pi r\beta\zeta$ finite, Eq. (118) is reduced to

$$\begin{aligned}
\lim_{\beta\to 0} \mathcal{I}^{\text{CY}_{N-2}}_{S^1\times D^2} &\sim \int \frac{dy}{2\pi i} e^{t_{2d}\, y} \Gamma(1-Ny)\Gamma(y)^N (e^{iN\pi y} - e^{-iN\pi y}) \\
&= \int dy\, e^{t_{2d}\, y} \frac{\Gamma(y)^N}{\Gamma(Ny)}.
\end{aligned} \tag{119}$$

Here, we also defined $y := ira$. The first line in Eq. (119) agrees with the partition function on D^2 with a loop operator wrapping on $S^1 = \partial D^2$ associated with a matrix factorization, where the Neumann boundary condition is imposed for the chiral multiplets. The boundary condition of a one-loop determinant changes from the Neumann type to the Dirichlet type by the effect of the loop operator. Then, we obtain the second line in Eq. (119), which is related to the Γ-class of the Calabi–Yau $(N-2)$-fold CY_{N-2}. Thus, Eq. (118) is thought of as a 3d extension of the hemisphere partition function for CY_{N-2}, and the ratio of the q-Pochhammer symbols in Eq. (118) can be regarded as a q-deformation of the Γ-class of CY_{N-2}.

Table 10. $U(1)_R \times U(1)_F$ R-charge assignments for the $\mathcal{N}=4$ vector multiplet.

	A_μ	λ	σ	D	σ'	λ'	D'
$U(1)_R$	0	1	0	0	1	0	-1
$U(1)_F$	0	0	0	0	1	1	1

Table 11. $U(1)_R \times U(1)_F$ R-charge assignments for the $\mathcal{N}=4$ hypermultiplet.

	ϕ	ψ	$\tilde{\phi}$	$\tilde{\psi}$
$U(1)_R$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$U(1)_F$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

6.4. *Three-dimensional $\mathcal{N} \geq 3$ models and the gauge/Bethe correspondence*

In this subsection we study 3d $\mathcal{N} \geq 3$ supersymmetric theories and the Bethe ansatz for quantum integrable models. We consider the $G = U(N)$ theory with N_f fundamental hypermultiplets. In the language of the $\mathcal{N}=2$ multiplets, an $\mathcal{N}=4$ vector multiplet consists of an $\mathcal{N}=2$ vector multiplet $(A_\mu, \sigma, D, \lambda, \bar{\lambda})$ and an $\mathcal{N}=2$ adjoint chiral multiplet (σ', λ', D'). On the other hand, an $\mathcal{N}=4$ fundamental hypermultiplet consists of an $\mathcal{N}=2$ fundamental chiral multiplet (ϕ, ψ, F) and an anti-fundamental chiral multiplet $(\tilde{\phi}, \tilde{\psi}, \tilde{F})$.

The R-symmetry group of the $\mathcal{N}=4$ supersymmetric theory is $SU(2)_L \times SU(2)_R$, and the Cartan generator of $SU(2)_L$ (resp. $SU(2)_R$) is J_L (resp. J_R). The $U(1)_R$ R-symmetry in the $\mathcal{N}=2$ supersymmetry is generated by a generator R, which is related to the $SU(2)_L \times SU(2)_R$ generators by $R = J_L - J_R$. On the other hand, the combination $F = J_L + J_R$ defines another global charge which commutes with $U(1)_R$, and we introduce the fugacity t for $U(1)_F$. The R-charge assignments for the $\mathcal{N}=4$ multiplets are given in Tables 10 and 11. Now we impose the Neumann boundary conditions on the adjoint chiral multiplet (σ', λ', D') and on the fundamental chiral multiplet (ϕ, ψ, F). We impose the Dirichlet boundary conditions on the anti-fundamental chiral multiplet $(\tilde{\phi}, \tilde{\psi}, \tilde{F})$. In this choice, all the anomalous terms from the 3d multiplets cancel each other, and thus the Chern–Simons levels are not shifted from the bare ones. If we introduce the Chern–Simons term, the supersymmetry is broken to $\mathcal{N}=3$. When we take the bare Chern–Simons levels as zero, the 3d index is well defined without introducing any boundary multiplets. Then, the $\mathcal{N}=4$ index on $S^1 \times D^2$ is given by

$$\mathcal{I}^{\mathcal{N}=4}_{S^1\times D^2} = \frac{(t;q^2)_\infty^{-N}}{N!} \int \prod_{a=1}^{N} \frac{ds_a}{2\pi i s_a} \prod_{1\leq a\neq b\leq N} \frac{(s_a s_b^{-1};q^2)_\infty}{(s_a s_b^{-1} t q;q^2)_\infty} \prod_{l=1}^{N_\text{f}} \prod_{a=1}^{N} \frac{(s_a q^{\frac{3}{2}} t^{\frac{1}{2}} z_l;q^2)_\infty}{(s_a t^{-\frac{1}{2}} q^{\frac{1}{2}} z_l;q^2)_\infty}. \tag{120}$$

This equation has the important property that the one-loop determinant of the 3d $\mathcal{N}=4$ vector multiplet is expressed by the Macdonald measure and agrees with the index for the $\mathcal{N}=4$ theory on $S^1 \times \mathbb{C}$ computed in Ref. [34] by some redefinitions of the fugacities.

Let us study the relation to integrable spin models. First, we examine the relation between $\mathcal{N}=4$ SQCD and the spin-$\frac{1}{2}$ XXZ quantum spin chain. It is shown in Ref. [52] (see also Ref. [53]) that the saddle point equation of the twisted effective superpotential for mass-deformed $\mathcal{N}=4$ SQCD corresponds to the Bethe ansatz equation for the spin-$\frac{1}{2}$ XXZ model. We recall that the partition function on the 3d ellipsoid S_b^3 depends on the squashing parameter b. In the limit $b \to 0$, an effective twisted superpotential appears. Similarly, we will see that the effective twisted superpotential appears in the limit $q \to 1$, keeping the other parameters finite. This is different from the 2d limit. In the

Abelian case (the rank of the gauge group $N = 1$), semiclassical behaviors of the holomorphic blocks correctly reproduce the Bethe ansatz equation for the XXZ spin chain [54] (see also Ref. [55]). Then, we expect the Bethe ansatz equation to also appear in the limit $q \to 1$ of the 3d index in Eq. (120) for the generic rank N. To see this, we take the limit $q \to 1$ and look at the behavior of the index:

$$\mathcal{I}^{\mathcal{N}=4}_{S^1\times D^2} \sim \frac{1}{N!}\int \prod_{a=1}^{N} \frac{ds_a}{2\pi i s_a}\ \exp\left(\frac{1}{2\beta}\mathcal{W}^{\mathcal{N}=4}_{\text{eff}}\right) \qquad (q \to 1), \tag{121}$$

with

$$\mathcal{W}^{\mathcal{N}=4}_{\text{eff}} = \sum_{a\neq b}\left(\text{Li}_2(s_a s_b^{-1}) - \text{Li}_2(t s_a s_b^{-1})\right) + \sum_{l=1}^{N_{\text{f}}}\sum_{a=1}^{N}\left(\text{Li}_2(s_a t^{\frac{1}{2}} z_l) - \text{Li}_2(s_a t^{-\frac{1}{2}} z_l)\right). \tag{122}$$

The saddle point equation $\exp\left(s_a\partial_{s_a}\mathcal{W}^{\mathcal{N}=4}_{\text{eff}}\right) = 1$ gives

$$\prod_{\substack{b=1\\ b\neq a}}^{N} \frac{\sinh(y_b - y_a - c)}{\sinh(y_a - y_b - c)} = \prod_{l=1}^{N_{\text{f}}} \frac{\sinh(y_a + \frac{c}{2} + M_l/2)}{\sinh(y_a - \frac{c}{2} + M_l/2)}. \tag{123}$$

Here, we defined $2y_a = \log s_a$, $2c := \log t$. This is equivalent to the Bethe ansatz equation for the $\mathfrak{sl}(2)$ spin-$\frac{1}{2}$ inhomogeneous XXZ quantum spin chain by some redefinitions of the parameters.

The parameters in the gauge theory are related to those of the spin chain as follows. The rank of the gauge group N corresponds to the number of excitations, and the number of hypermultiplets N_{f} corresponds to the number of sites of the spin chain. So far, the periodic boundary condition is imposed in the spin chain system. The twisted boundary condition in the spin chain is reproduced by introducing the FI term. Also, the generalization to the $\mathfrak{sl}(K)$ case is straightforward by considering a quiver gauge theory.

The correspondence between mass deformations of $\mathcal{N} = 4$ SQCD and the XXZ model is already known in the context of the gauge/Bethe correspondence. But we will propose a new example for the gauge/Bethe correspondence: the "gauge" side is the pure $\mathcal{N} = 3$ Chern–Simons matter theory; namely, we take $N_{\text{f}} = 0$ in Eq. (120) and introduce a dynamical Chern–Simons term with level κ. The "Bethe" side (quantum integrable model) is the q-boson hopping model [56]. The existence of the gauge/Bethe correspondence for Chern–Simons (matter) theories was first pointed out in Refs. [57,58]. The boundary Chern–Simons terms are evaluated at the saddle point as

$$e^{S_{b.CS}} = \exp\left(-\sum_{a=1}^{N}\frac{\kappa}{4\beta}(\log s_a)^2\right). \tag{124}$$

Thus, we can read off the effective twisted superpotential,

$$\mathcal{W}^{\mathcal{N}=3}_{\text{eff}} = -\frac{\kappa}{2}\sum_{a=1}^{N}\log^2(s_a) + \sum_{a\neq b}\left(\text{Li}_2(s_a s_b^{-1}) - \text{Li}_2(t s_a s_b^{-1})\right), \tag{125}$$

and obtain the set of saddle point equations,

$$s_a^{\kappa} = \prod_{\substack{b=1\\ b\neq a}}^{N} \frac{s_a t - s_b}{s_a - t s_b}. \tag{126}$$

This reproduces the Bethe ansatz equation for the q-boson hopping model in the N-particle sector with the periodic boundary condition. The number of sites corresponds to the Chern–Simons level κ.

6.5. *Domain wall index on $S^1 \times S^2$*

In four dimensions, 4d–3d coupled partition functions or superconformal indices were introduced in Refs. [59,60]. In this section we briefly mention a 3d–2d domain wall index on $S^1 \times S^2$.

The $\mathcal{N} = (0,2)$ domain wall theory lives on $T^2 = S^1 \times S^1_{\vartheta=\frac{\pi}{2}} \subset S^1 \times S^2$ and couples to two different theories, each of which lives on a northern part $S^1 \times D^2_{\mathrm{N}}(\vartheta \leq \frac{\pi}{2})$ or a southern part $S^1 \times D^2_{\mathrm{S}}(\vartheta \geq \frac{\pi}{2})$. Let G_{N} and G_{S} be gauge groups of the $\mathcal{N} = 2$ theories on $S^1 \times D^2_{\mathrm{N}}$ and $S^1 \times D^2_{\mathrm{S}}$, respectively. Let $Z^{\text{1-loop}}_{S^1\times D^2_{\mathrm{N}}}(s)$ and $Z^{\text{1-loop}}_{S^1\times D^2_{\mathrm{S}}}(\tilde{s})$ be the one-loop determinants of the $\mathcal{N} = 2$ theories on $S^1 \times D^2_{\mathrm{N}}$ and $S^1 \times D^2_{\mathrm{S}}$. Here, the s_a and $\tilde{s}_b$ label Cartan parts of exponentiated holonomies of G_{N} and G_{S} along the S^1-direction. On the boundary torus, the multiplets of the $\mathcal{N} = (0,2)$ theory have charges associated with the gauge groups $G_{\mathrm{N}} \times G_{\mathrm{S}}$, and its partition function $Z^{\text{1-loop}}_{T^2}(s,\tilde{s})$ is given by the one-loop determinant of the $\mathcal{N} = (0,2)$ theory. By collecting these functions, we can write down the domain wall index:

$$\mathcal{I}^{\mathrm{DW}}_{S^1\times S^2} = \int\int \prod_{a=1}^{\mathrm{rk}(G_{\mathrm{N}})} \frac{ds_a}{2\pi s_a} \prod_{b=1}^{\mathrm{rk}(G_{\mathrm{S}})} \frac{d\tilde{s}_b}{2\pi \tilde{s}_b} Z^{\text{1-loop}}_{S^1\times D^2_{\mathrm{N}}}(s) Z^{\text{1-loop}}_{T^2}(s,\tilde{s}) Z^{\text{1-loop}}_{S^1\times D^2_{\mathrm{S}}}(\tilde{s}). \tag{127}$$

The matter contents and two bare Chern–Simons levels have to be chosen to cancel the total bulk–boundary anomalies.

6.6. *Wilson–vortex loops and the q-shift operator*

In this subsection we study properties of flavor Wilson loops and vortex loops on the 3d index $\mathcal{I}_{S^1\times D^2}(q^2, z)$. When we turn on the lth flavor background gauge field $A^{(l)}_{\mu} = (0, 0, -iM_l/\beta r)$, an associated flavor Wilson loop with a charge Q is defined by

$$W^{(l)}_{\mathrm{F}} = \exp\left(iQ \int A^{(l)}_{\tau} d\tau\right). \tag{128}$$

Then, the expectation value of the flavor Wilson loop is given by using the index $\langle W^{(l)}_{\mathrm{F}} \rangle = z_l^{-Q} \mathcal{I}_{S^1\times D^2}$ with $z_l = e^{M_l}$. The localization computation of vortex loops was studied in Refs. [61,62] for S^3 and $S^1 \times S^2$. The vortex loops are defined as some defect operators specified by line singularities similar to 't Hooft loops. This means that the boundary conditions for the component fields in 3d theories are modified near the vortex loops. Since the appropriate equivariant index theorem for the manifold with the boundary is not yet known, it is difficult to directly evaluate the effects of the vortex loops on the manifold with the boundary, for example $S^1 \times D^2$. Instead of direct computation, we apply the method studied in Ref. [61] to our model. That is, the vortex loop is obtained by acting an S-transformation on the flavor Wilson loop:

$$V^{(l)}_{\mathrm{F}} = S^{-1} W^{(l)}_{\mathrm{F}} S, \qquad S \in SL(2,\mathbb{Z}). \tag{129}$$

Here, the S-transformation is defined by adding an FI term for the lth flavor background gauge field and by gauging this background field,

$$(S \cdot \mathcal{I}_{S^1\times D^2})(q^2, \zeta_l) = \int dM_l\, \mathcal{I}_{S^1\times D^2}\, e^{2\pi i \zeta_l M_l}. \tag{130}$$

The transformation S^{-1} is also given as the inverse transformation of S, and the vortex loop acts on the 3d index as

$$(S^{-1}W_{\mathrm{F}}^{(l)}S)\cdot\mathcal{I}_{S^1\times D^2}(q,z_l') = \int d\zeta_l\, e^{-2\pi i M_l'\zeta_l}e^{2\pi i Q\zeta_l}\int dM_l\,\mathcal{I}_{S^1\times D^2}\,e^{2\pi i\zeta_l M_l}$$
$$= \mathcal{I}_{S^1\times D^2}(q^2, z_l' q^Q). \tag{131}$$

Thus, we obtain the expectation value of the vortex loop for the lth flavor gauge field by shifting the lth flavor fugacity $\langle V_{\mathrm{F}}^{(l)}\rangle = \mathcal{I}_{S^1\times D^2}(q^2, z_l q^Q)$. As a result, the flavor vortex loops are regarded as q-shift operators. The successive actions of Wilson vortex loops on the 3d index do not commute, but they satisfy the commutation relations

$$\langle V_{\mathrm{F}}^{(l)}W_{\mathrm{F}}^{(k)}\rangle = q^{-Q}\delta_{lk}\langle W_{\mathrm{F}}^{(k)}V_{\mathrm{F}}^{(l)}\rangle. \tag{132}$$

7. Summary and discussion

We have evaluated the partition functions of the $\mathcal{N}=2$ supersymmetric Chern–Simons matter theories on $S^1\times D^2$ in terms of localization techniques. In the particular choice of the fugacity $\beta_2=\beta$, we find that the conditions to cancel anomalous terms are reduced to the decomposition rule for effective mixed Chern–Simons levels in the holomorphic blocks for Abelian gauge theories. In these cases, our 3d–2d indices reproduce the holomorphic blocks. On the other hand, in non-Abelian gauge theories, there might be a mismatch in the sector of the vector multiplet and effective Chern–Simons levels. One possibility for mismatch comes from the difference in the metrics of $S^1\times D^2$ and the Melvin cigar. Both spaces have the same topology as the solid torus, but have different metrics. It is desirable to study this point further to reveal the origin of this discrepancy. We postpone this problem to future work.

We have also studied the connection between our indices on $S^1\times D^2$ and several topics: the K-theoretic J-function for the $\mathbb{C}\mathbf{P}^N$ model, vortex partition functions and surface operators, the 3d analogue of matrix factorization, the gauge/Bethe correspondence, and loop operators.

We have constructed boundary interactions which can be regarded as a 3d analogue of matrix factorization. Although the boundary interactions in three dimensions (the $\mathcal{N}=(0,2)$ superpotential term in EQ. (33)) have quite different expressions from 2d ones (roughly speaking, Wilson loops for superconnections), after the localization computation is performed, the partition functions can reproduce the 2d partition functions on the hemisphere by dimensional reduction.

We have not studied the boundary interaction described by the G/G chiral gauged WZW model in detail. It would be interesting to study the 3d–2d index with this boundary interaction.

In Ref. [19], the 2d–4d correspondence was proposed. The 2d side describes the $\mathcal{N}=(0,2)$ flavored elliptic genus [63,64] and the 4d side is related to the Vafa–Witten partition functions [65] on four-manifolds. In this paper we have realized $\mathcal{N}=(0,2)$ theories as the boundary interactions of 3d $\mathcal{N}=2$ supersymmetric theories. The 3d–2d coupled index is expected to be related to some Vafa–Witten partition function with degrees of freedom on the 3d boundary, which is realized as the asymptotic boundary of the four-manifold. It will be interesting to explore the connection between the 3d–2d indices and the partition functions of the 3d–4d coupled systems.

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Appendix A. Conventions of 3d $\mathcal{N} = 2$ theory on $S^1 \times D^2$

We use gamma matrices $\gamma_{\hat{a}}$ in the local Lorentz frame:

$$\gamma_{\hat{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_{\hat{2}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma_{\hat{3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

In curved spaces, one can define $\gamma_\mu = e^{\hat{a}}_\mu \gamma_{\hat{a}}$ by using a dreibein $e^{\hat{a}}_\mu$. The charge conjugation matrix is expressed by

$$C_{\alpha\beta} = -i\gamma_{\hat{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.2}$$

which satisfies $C\gamma^\mu C^{-1} = -(\gamma^\mu)^{\mathrm{T}}$. With this C, the spinor product is defined by

$$\epsilon\psi := \epsilon^\alpha \psi_\alpha = \epsilon^\alpha C_{\alpha\beta}\psi^\beta = \epsilon^{\mathrm{T}} C\psi. \tag{A.3}$$

The supersymmetric transformation of the $\mathcal{N} = 2$ vector multiplet is given by

$$\begin{aligned}
\delta A_\mu &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon), \\
\delta\sigma &= \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon), \\
\delta\lambda &= -\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\epsilon - D\epsilon + i\gamma^\mu D_\mu\sigma\epsilon + \frac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon, \\
\delta\bar{\lambda} &= -\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\bar{\epsilon} + D\bar{\epsilon} - i\gamma^\mu D_\mu\sigma\bar{\epsilon} - \frac{2i}{3}\sigma\gamma^\mu D_\mu\bar{\epsilon}, \\
\delta D &= -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] + \frac{i}{2}[\bar{\lambda}\epsilon, \sigma] - \frac{i}{6}(D_\mu\bar{\epsilon}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu D_\mu\epsilon).
\end{aligned} \tag{A.4}$$

For the $\mathcal{N} = 2$ chiral multiplet, the supersymmetric transformation is expressed as

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi, \\
\delta\bar{\phi} &= \epsilon\bar{\psi}, \\
\delta\psi &= i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi + \frac{2i\Delta}{3}\gamma^\mu D_\mu\epsilon\phi + \bar{\epsilon}F, \\
\delta\bar{\psi} &= i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} + i\bar{\phi}\sigma\bar{\epsilon} + \frac{2i\Delta}{3}\bar{\phi}\gamma^\mu D_\mu\bar{\epsilon} + \bar{F}\epsilon, \\
\delta F &= \epsilon(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi) + \frac{i}{3}(2\Delta - 1)D_\mu\epsilon\gamma^\mu\psi, \\
\delta\bar{F} &= \bar{\epsilon}(i\gamma^\mu D_\mu\bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\bar{\lambda}) + \frac{i}{3}(2\Delta - 1)D_\mu\bar{\epsilon}\gamma^\mu\bar{\psi}.
\end{aligned} \tag{A.5}$$

Here, the covariant derivative is defined by $D_\mu = \nabla_\mu + iA_\mu^a T_{\mathcal{R}}^a$ with the vector boson $A_\mu = A_\mu^a T_{\mathcal{R}}^a$ in the representation $\mathcal{R}$.

Appendix B. 2d $\mathcal{N} = (0, 2)$ supersymmetry on the boundary torus

The generators of the supersymmetric transformations are defined by the restriction of the 3d Killing spinors ϵ' and $\bar{\epsilon}'$ on the boundary torus:

$$\begin{aligned}
&(D_{\hat{2}} + iD_{\hat{3}})\epsilon' = \frac{i}{r}\epsilon', \qquad (D_{\hat{2}} - iD_{\hat{3}})\epsilon' = 0, \\
&(D_{\hat{2}} + iD_{\hat{3}})\bar{\epsilon}' = \frac{-i}{r}\bar{\epsilon}', \qquad (D_{\hat{2}} - iD_{\hat{3}})\bar{\epsilon}' = 0.
\end{aligned} \tag{B.1}$$

The set of commutators of the supersymmetric transformations of the vector multiplet is given by

$$\begin{aligned}
&[\delta_1, \delta_2](A_{\hat{2}} - iA_{\hat{3}}) = \alpha(-2i)F_{\hat{2}\hat{3}}, \\
&[\delta_1, \delta_2]\lambda_1 = \alpha\left[-2(D_{\hat{2}} + iD_{\hat{3}})\lambda_1 + \frac{2i}{r}\lambda_1\right], \\
&[\delta_1, \delta_2]\bar{\lambda}_1 = \alpha\left[-2(D_{\hat{2}} + iD_{\hat{3}})\bar{\lambda}_1 - \frac{2i}{r}\bar{\lambda}_1\right], \\
&[\delta_1, \delta_2]\hat{D} = \alpha\left[2(D_{\hat{2}} + iD_{\hat{3}})\hat{D}\right].
\end{aligned} \tag{B.2}$$

Here, we defined $\alpha = \bar{\epsilon}_2'\epsilon'_1 - \bar{\epsilon}_1'\epsilon'_2$. Next, the set of commutators of the supersymmetric transformations of the chiral multiplet is expressed as

$$\begin{aligned}
&[\delta_1, \delta_2]\phi = \alpha\left[2(D_{\hat{2}} + iD_{\hat{3}})\phi + 2\frac{i\Delta}{r}\phi\right], \\
&[\delta_1, \delta_2]\psi' = \alpha\left[-2(D_{\hat{2}} + iD_{\hat{3}})\psi' - \frac{2i}{r}(\Delta - 1)\psi'\right], \\
&[\delta_1, \delta_2]\bar{\phi} = \alpha\left[2(D_{\hat{2}} + iD_{\hat{3}})\bar{\phi} - 2\frac{i\Delta}{r}\bar{\phi}\right], \\
&[\delta_1, \delta_2]\bar{\psi}' = \alpha\left[-2(D_{\hat{2}} + iD_{\hat{3}})\bar{\psi}' + \frac{2i}{r}(\Delta - 1)\bar{\psi}'\right].
\end{aligned} \tag{B.3}$$

The set of commutators of the supersymmetric transformations of the Fermi multiplet is given by

$$\begin{aligned}
&[\delta_1, \delta_2]\Psi = \alpha\left[2(D_{\hat{2}} + iD_{\hat{3}})\Psi + \frac{2i}{r}(\tilde{\Delta} - 1)\Psi\right], \\
&[\delta_1, \delta_2]G = \alpha\left[2(D_{\hat{2}} + iD_{\hat{3}})G + \frac{2i}{r}(\tilde{\Delta} - 2)G\right], \\
&[\delta_1, \delta_2]\bar{\Psi} = \alpha\left[2(D_{\hat{2}} + iD_{\hat{3}})\bar{\Psi} + \frac{2i}{r}(1 - \tilde{\Delta})\bar{\Psi}\right], \\
&[\delta_1, \delta_2]\bar{G} = \alpha\left[2(D_{\hat{2}} + iD_{\hat{3}})\bar{G} + \frac{2i}{r}(2 - \tilde{\Delta})\bar{G}\right].
\end{aligned} \tag{B.4}$$

Appendix C. Definitions of functions

The dilogarithm function is defined by

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \tag{C.1}$$

and an integral representation of this dilogarithm function is given by

$$\mathrm{Li}_2(x) = -\int_0^x dt\, \frac{\log(1-t)}{t}. \tag{C.2}$$

Next, the q-Pochhammer symbol is defined by

$$(a;q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \tag{C.3}$$

and the q-theta function is defined, for $|q| < 1$, as

$$\theta(y;q) = \prod_{n=0}^{\infty} (1 - yq^n)(1 - y^{-1}q^{n+1}), \qquad y \in \mathbb{C}^*,\ |q| < 1. \tag{C.4}$$

The quantum dilogarithm function is defined by

$$\mathrm{Li}_2(x;q) = \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^n)}, \qquad |x|, |q| < 1. \tag{C.5}$$

The q-Pochhammer symbol is expressed by the quantum dilogarithm as

$$(x;q)_\infty = \exp(-\mathrm{Li}_2(x;q)), \tag{C.6}$$

and the semiclassical limit is given by

$$\mathrm{Li}_2(x;e^{2\hbar}) \sim -\frac{1}{2\hbar}\mathrm{Li}_2(x), \qquad \hbar \to 0. \tag{C.7}$$

Appendix D. Derivation of one-loop determinants

D.1. Three-dimensional vector multiplet

In this subsection we evaluate one-loop determinants of super Yang–Mills theory, Eq. (10). The evaluation of the one-loop determinant on $S^1 \times D^2$ can be performed in the similar manner to Refs. [16,17]. Because we treat bosonic fields on $S^1 \times D^2$, we introduce scalar harmonics Y_{jm} and vector harmonics $(C^\lambda_{jm})_i$ on S^2 labeled by sets of (j, m) with $j \leq |m|$ $(j = 1, 2, 3, \ldots)$. Each field in the multiplet is expanded by these harmonics and generators E_α associated with roots α in the Cartan Weyl basis (Cartan parts for the fluctuations are omitted):

$$\sigma = \sum_j {\sum_m}' \sum_{\alpha>0} \sigma^\alpha Y_{jm} E_\alpha + (\mathrm{h.c.}), \tag{D.1}$$

$$A_i = \sum_{\lambda=1}^{2} \sum_{j=1}^{\infty} {\sum_{m=-j}^{j}}' \sum_{\alpha>0} A^{\alpha\lambda}_{jm} (C^\lambda_{jm})_i E_\alpha + (\mathrm{h.c.}) \qquad (i = 1, 2), \tag{D.2}$$

$$A_3 = \sum_{\alpha>0}\sum_{j=1}^{\infty}\sum_{\substack{m=-j\\ j-m=\text{even}}}^{j} A^{\alpha 3}_{jm} Y_{jm} E_\alpha + (\text{h.c.}). \tag{D.3}$$

Here, the symbol "(h.c.)" denotes the Hermitian conjugate, and the sum $\sum'$ runs over the following modes under the boundary condition in Eq. (11):

$$Y_{jm} : j - m = \text{even} \qquad (\text{Neumann}), \tag{D.4}$$

$$Y_{jm} : j - m = \text{odd} \qquad (\text{Dirichlet}), \tag{D.5}$$

$$C^1_{jm} : j - m = \text{even}, \qquad C^2_{jm} : j - m = \text{odd}. \tag{D.6}$$

Next, we turn to the ghost fields $(c, \bar{c})$. The theory has gauge symmetry and we need to introduce ghost terms for fixing the symmetry:

$$\mathcal{L}_{\text{ghost+g.f.}} = -\bar{c}\nabla^i D_i c + \frac{1}{2\xi}(\nabla^i A_i)^2. \tag{D.7}$$

We take the Neumann boundary condition for the ghost fields $(c, \bar{c})$ and expand them by the harmonics:

$$c = \sum_j {\sum_m}' \sum_{\alpha\neq 0} c^\alpha_{jm} Y_{jm} E_\alpha. \tag{D.8}$$

One can express the ghost part $\mathcal{L}_{\text{ghost}}$ and the bosonic part $\mathcal{L}_{\text{vec.b}}$ of super Yang–Mills at the quadratic order as

$$\int_{D^2} \mathcal{L}_{\text{ghost}} = \frac{1}{2r^2}\sum_\alpha {\sum_{j,m}}' \text{Tr}(E_\alpha E_{-\alpha}) \cdot (-1)^m j(j+1)\, \bar{c}^\alpha_{j,m} c^{-\alpha}_{j,-m},$$
$$\int_{D^2} \mathcal{L}_{\text{vec.b}} = \frac{1}{2r^2}\sum_{\alpha>0} {\sum_{j,m}}' \text{Tr}(E_{-\alpha} E_\alpha) \cdot \mathcal{V}^{\alpha\dagger}_{jm} \cdot \mathcal{M} \cdot \mathcal{V}^\alpha_{jm}\,, \tag{D.9}$$

with

$$\mathcal{V}^\alpha_{jm} = \begin{pmatrix} A^{\alpha 1}_{jm} & A^{\alpha 2}_{jm} & A^{\alpha 3}_{jm} & \sigma^\alpha_{jm} \end{pmatrix}^{\text{T}},$$

$$\mathcal{M} = \begin{pmatrix} \xi^{-1}\cdot j(j+1) - r^2 D^3 D_3 & 0 & \sqrt{j(j+1)}\cdot rD_3 & 0 \\ 0 & j(j+1) - r^2 D^3 D_3 & 0 & -\sqrt{j(j+1)} \\ -\sqrt{j(j+1)}\cdot rD_3 & 0 & j(j+1) & 0 \\ 0 & -\sqrt{j(j+1)} & 0 & j(j+1) + 1 - r^2 D^3 D_3 \end{pmatrix}. \tag{D.10}$$

Then, one-loop determinants of the bosonic and the ghost parts respectively become products of the modes (j, m):

$$Z_{\text{vec.b}} = \prod_{\alpha\neq 0}\prod_{j-m=\text{even}} [j(j+1)]^{-1}$$
$$\times \prod_{\alpha\neq 0}\prod_{j-m=\text{odd}} [(j+1+rD_3)(j-rD_3)(j+1-rD_3)(j+rD_3)]^{-\frac{1}{2}}, \tag{D.11}$$

$$Z_{\text{ghost}} = \prod_{\alpha\neq 0}\prod_{j-m=\text{even}} [j(j+1)]. \tag{D.12}$$

Here, Z_{ghost} is cancelled by the first line on the right-hand side of Eq. (D.11).

Next, we shall evaluate eigenvalues of the operator D_3. Let $\mathcal{O}_{n,m}$ be an operator (field) with an R-charge R and a flavor charge F_l, which satisfies

$$\partial_\tau \mathcal{O}_{n,m} = \frac{2\pi i n}{\beta r}\mathcal{O}_{n,m}, \qquad \mathsf{J}_3\mathcal{O}_{n,m} = m\mathcal{O}_{n,m}. \tag{D.13}$$

Then D_3 acts on $\mathcal{O}_{n,m}$ as

$$\beta r D_3\mathcal{O}_{n,m} = [2\pi i n + i\beta r\rho(a) - (R+m)\beta_1 + m\beta_2 + F_l M_l]\,\mathcal{O}_{n,m}. \tag{D.14}$$

For example, the R-charge for the gauge field is 0 and that for the fermion λ (resp. $\bar{\lambda}$) is -1 (resp. $+1$). Then, the one-loop determinant of the bosonic part is given by

$$\prod_{\alpha\neq 0}\prod_{n\in\mathbb{Z}}\prod_{j=1}^{\infty}\prod_{\substack{m=-j+1\\ j-m=\text{odd}}}^{j-1} (2\pi i n + i\beta r\alpha(a) + (j-m+1)\beta_1 + (j+m+1)\beta_2)^{-1}$$
$$\times (2\pi i n + i\beta r\alpha(a) + (j+m)\beta_1 + (j-m)\beta_2)^{-1}.$$

We make a remark here. We have fixed the gauge symmetry with $\nabla^i A_i = 0$, but there is a residual gauge symmetry $\delta A_\tau = D_\tau\kappa$ with parameter $\kappa(\tau)$. In order to fix this symmetry, we impose a condition $\partial_\tau\omega = 0$ with $\omega := \frac{1}{\text{vol}(D^2)}\int_{D^2} A_\tau$ and introduce a set of a ghost and an anti-ghost. When we integrate these ghost fields, they induce a contribution $\det D_\tau$ to the partition function. It is evaluated up to an overall constant:

$$\det D_\tau = \prod_{n\neq 0}\prod_{\alpha}\left(2\pi i\frac{n}{\beta r} + i\alpha(\lambda)\right) \approx \prod_{\alpha>0}\frac{1}{\alpha(\lambda)^2}\sin^2\frac{\beta r\alpha(\lambda)}{2}, \tag{D.15}$$

where λ_i $(i = 1, 2, \ldots, N)$ is the set of eigenvalues of the matrix ω and α is the root. The measure of the matrix integral is expressed by $d\omega = \prod_i d\lambda_i \prod_{\alpha>0}\alpha(\lambda)^2$ and $d\omega\cdot\det D_\tau = \prod_i d\lambda_i\prod_{\alpha>0}\sin^2\frac{\beta r\alpha(\lambda)}{2}$.

Next, we shall consider the contribution of the fermions to the one-loop determinant. We consider the fermionic part of the Yang–Mills Lagrangian and evaluate the fluctuations around the saddle point at the quadratic order. We expand the gaugino in terms of the spinor harmonics $\chi^{\pm}_{j,m}(\vartheta,\varphi)$:

$$\lambda = \sum_{\alpha\neq 0}\sum_{s=\pm}\sum_{j}\sum_{m}{}' \lambda^{\alpha,s}_{jm}\chi^{s}_{jm}E_\alpha. \tag{D.16}$$

Here, the sum $\sum'$ runs over the following modes under the boundary condition in Eq. (11):

$$r\gamma_3\gamma^i D_i\chi^{\pm}_{jm} = \pm\left(j+\frac{1}{2}\right)\chi^{\pm}_{jm},$$
$$\chi^{+}_{jm} : j-m = \text{even}, \qquad \chi^{-}_{jm} : j-m = \text{odd}. \tag{D.17}$$

By using this spinor harmonics, we can write down the fermion part of the Yang–Mills Lagrangian,

$$S^{(2)}_{\text{vec}}\Big|_{\text{fer}} = \int_{S^1} \sum_{\alpha\neq 0} \sum_{j=\frac{1}{2}}^{\infty} \frac{i}{4r} \Bigg[\sum_{m:j-m=\text{even}} (-1)^{-m+\frac{1}{2}} \bar{\lambda}^{\alpha,-}_{j,-m} (j + rD_3) \lambda^{-\alpha,+}_{j,m} + \sum_{m:j-m=\text{odd}} (-1)^{-m+\frac{1}{2}} \bar{\lambda}^{\alpha,+}_{j,-m} (j - rD_3 + 1) \lambda^{-\alpha,-}_{j,m} \Bigg] \mathrm{Tr}(E_\alpha E_{-\alpha}),$$

and calculate the one-loop determinant as

$$\prod_{j=\frac{1}{2}}^{\infty} \prod_{m:j-m=\text{even}} \mathrm{Det}(j + rD_3) \prod_{m:j-m=\text{odd}} \mathrm{Det}(j - rD_3 + 1). \tag{D.18}$$

First, we can evaluate the factor $\mathrm{Det}(j - rD_3 + 1)$ with $j = j' + \frac{1}{2}$, $m = m' + \frac{1}{2}$:

$$\begin{aligned} &\mathrm{Det}(\beta(j+1) - \beta r D_3) \\ &= \prod_{n\in\mathbb{Z}} \prod_{\alpha\neq 0} \prod_{j'=0}^{\infty} \prod_{\substack{m'=-j'-1 \\ j'-m'=\text{odd}}}^{j'-1} (2\pi i n + i\beta r \alpha(a) + (j' + m' + 1)\beta_1 + (j' + 1 - m')\beta_2). \end{aligned}$$

Similarly, the other factor $\mathrm{Det}(j + rD_3)$ is evaluated with $j = j' - \frac{1}{2}$, $m = m' + \frac{1}{2}$:

$$\mathrm{Det}(\beta j + \beta r D_3) = \prod_{n\in\mathbb{Z}} \prod_{\alpha\neq 0} \prod_{j'=1}^{\infty} \prod_{\substack{m'=-j'+1 \\ j'-m'=\text{odd}}}^{j'-1} (2\pi i n + i\beta r \alpha(a) + (j' - m')\beta_1 + (j' + m')\beta_2).$$

Then, the one-loop determinant of the vector multiplet results in the product formula

$$Z^{\text{3d.vec}}_{\text{1-loop}} = \prod_{\alpha\neq 0} e^{\frac{-(i\beta r \alpha(a))^2}{8\beta_2}} (q^2 e^{-i\beta r \alpha(a)}; q^2). \tag{D.19}$$

Here, we adopted the zeta function regularization used in Ref. [23]. In the evaluation of the one-loop determinant in the following subsections we use the common regularization scheme. As expected, the one-loop determinant of the vector multiplet does not depend on the fugacity β_1.

D.2. Three-dimensional chiral multiplet

D.2.1. Neumann boundary condition

We first evaluate the one-loop bosonic determinant for the chiral multiplet. When the Neumann boundary condition in Eq. (19) is imposed, ϕ can be expanded as follows:

$$\phi = \sum_{\rho} \sum_{j=0}^{\infty} \sum_{\substack{m=-j \\ j-m=\text{even}}}^{j} \phi^{\rho}_{jm} Y_{jm}(\vartheta, \varphi) E_\rho. \tag{D.20}$$

Here, ρ runs over the weight of the representation $\mathcal{R}$ of the Lie algebra Lie(G). At the quadratic order of fluctuations, the action of the chiral multiplet is expanded in terms of the scalar harmonics:

$$S^{(2)}_{\text{chi}}\Big|_{\text{bos}} = \frac{1}{2r^2} \int_{S^1} \sum_{j=0}^{\infty} \sum_{\substack{m=-j \\ j-m=\text{even}}}^{j} \sum_{\rho} \bar{\phi}^{\rho}_{j,m} \,(j + \Delta + rD_3)\,(j + 1 - \Delta - rD_3)\, \phi^{\rho}_{jm}. \tag{D.21}$$

Under the twisted boundary condition of Eq. (49), the factor $(j+\Delta+rD_3)$ in Eq. (D.21) contributes to the one-loop determinant of the bosonic fields as

$$\prod_{j=0}^{\infty}\prod_{\substack{m=-j\\ j-m=\text{even}}}^{j}\text{Det}\left((j+\Delta)\beta+\beta rD_3\right)^{-1}$$

$$=\prod_{n\in\mathbb{Z}}\prod_{j=0}^{\infty}\prod_{\substack{m=-j\\ j-m=\text{even}}}^{j}\prod_{l}(2\pi in+i\beta r\rho(a)+(j-m)\beta_1+(j+\Delta+m)\beta_2+F_lM_l)^{-1}. \quad \text{(D.22)}$$

Similarly, the other factor $(j+1-\Delta-rD_3)$ contributes to the one-loop determinant of the bosonic fields as

$$\prod_{j=0}^{\infty}\prod_{\substack{m=-j\\ j-m=\text{even}}}^{j}\left((j+1-\Delta)\beta-\beta rD_3\right)^{-1}$$

$$=\prod_{n\in\mathbb{Z}}\prod_{j=0}^{\infty}\prod_{\substack{m=-j\\ j-m=\text{even}}}^{j}(-2\pi in-i\beta r\rho(a)+(j+m+1)\beta_1+(j+1-\Delta-m)\beta_2-F_lM_l)^{-1}. \quad \text{(D.23)}$$

Next, we evaluate the one-loop determinant of the fermions. We expand ψ by the spinor harmonics χ^s_{jm} as

$$\psi=\sum_{\rho}\sum_{s=\pm}\sum_{j=\frac{1}{2}}^{\infty}\sum_{m=-j}^{j}{}' \psi^{\rho,s}_{jm}(\tau)\chi^s_{jm}(\vartheta,\varphi)E_\rho. \quad \text{(D.24)}$$

Here, the sum $\sum'$ runs over the following modes under the boundary condition in Eq. (19):

$$\chi^+_{jm}: j-m=\text{odd},\qquad \chi^-_{jm}: j-m=\text{even}. \quad \text{(D.25)}$$

Then, the action of the chiral multiplet in Eq. (18) can be expanded at the quadratic order as

$$S^{(2)}_{\text{chi}}\Big|_{\text{fer}}=\frac{i}{2r}\int_{S^1}\sum_{\rho}\sum_{j}\sum_{m;j-m=\text{even}}\bar{\psi}^{\rho,+}_{j,-m}\left(j-rD_3+1-\Delta\right)\psi^{\rho,-}_{j,m}\cdot(-1)^{m-1/2}$$

$$+\frac{i}{2r}\int_{S^1}\sum_{\rho}\sum_{j}\sum_{m;j-m=\text{odd}}\bar{\psi}^{\rho,-}_{j,-m}\left(j+rD_3+\Delta\right)\psi^{\rho,+}_{j,m}\cdot(-1)^{m-1/2}. \quad \text{(D.26)}$$

The factor $(j-rD_3+1-\Delta)$ contributes to the one-loop determinant of the chiral multiplet with $j=j'+\frac{1}{2}$, $m=m'+\frac{1}{2}$:

$$\prod_{j=\frac{1}{2}}^{\infty}\text{Det}\left((j-\Delta+1)\beta-\beta rD_3\right)$$

$$=\prod_{n\in\mathbb{Z}}\prod_{j'=0}^{\infty}\prod_{\substack{m'=-j'\\ j'-m'=\text{even}}}^{j'}\prod_{l}(-2\pi in-i\beta r\rho(a)+(j'+m'+1)\beta_1+(j'+1-\Delta-m')\beta_2-F_lM_l). \quad \text{(D.27)}$$

The other factor $(j + rD_3 + \Delta)$ contributes to the one-loop determinant of the chiral multiplet with $j = j' - \frac{1}{2}$, $m = m' + \frac{1}{2}$:

$$\prod_{j=\frac{1}{2}}^{\infty} \mathrm{Det}\left((j+\Delta)\beta + \beta r D_3\right)$$

$$= \prod_{n\in\mathbb{Z}} \prod_{j'=1}^{\infty} \prod_{\substack{m'=-j' \\ j'-m'=\mathrm{odd}}}^{j'-2} \prod_{l} (2\pi i n + i\beta r\rho(a) + (j'-m')\beta_1 + (j'+\Delta+m')\beta_2 + F_l M_l). \quad \text{(D.28)}$$

Then, the one-loop determinant of the chiral multiplet with the Neumann boundary condition is given by

$$Z^{\text{3d.chi.N}}_{\text{1-loop}} = \prod_{\rho} \prod_{n\in\mathbb{Z}} \prod_{j=0}^{\infty} \prod_{l} (2\pi i n + i\beta r\rho(a) + (2j+\Delta)\beta_2 + F_l M_l)^{-1}$$
$$= \prod_{\rho} \prod_{l} e^{\mathcal{E}(i\beta r\rho(a)+\Delta\beta_2+F_l M_l)} (e^{-i\beta r\rho(a)-F_l M_l} q^{\Delta}; q^2)^{-1}. \quad \text{(D.29)}$$

The one-loop determinant of the chiral multiplet also does not depend on the parameter β_1.

D.2.2. Dirichlet boundary condition

Next, we evaluate the bosonic one-loop determinant for the chiral multiplet with the Dirichlet boundary condition of Eq. (20). At the quadratic order, the bosonic part of the action is expanded as

$$S^{(2)}_{\text{chi}}\Big|_{\text{bos}} = \frac{1}{2r^2} \int_{S^1} \sum_{j=0}^{\infty} \sum_{\substack{m=-j \\ j-m=\mathrm{odd}}}^{j} \sum_{\rho} \bar{\phi}^{\rho}_{j,m} \,(j+\Delta+rD_3)\,(j+1-\Delta-rD_3)\,\phi^{\rho}_{jm}. \quad \text{(D.30)}$$

The factor $(j + \Delta + rD_3)$ in Eq. (D.30) contributes to the one-loop determinant of the bosonic fields as

$$\prod_{j=1}^{\infty} \prod_{\substack{m=-j+1 \\ j-m=\mathrm{odd}}}^{j-1} \mathrm{Det}\left((j+\Delta)\beta + \beta r D_3\right)^{-1}$$

$$= \prod_{n\in\mathbb{Z}} \prod_{j=1}^{\infty} \prod_{\substack{m=-j+1 \\ j-m=\mathrm{odd}}}^{j-1} \prod_{l} (2\pi i n + i\beta r\rho(a) + (j-m)\beta_1 + (j+\Delta+m)\beta_2 + F_l M_l)^{-1}.$$

The other factor $(j + 1 - \Delta - rD_3)$ contributes to the one-loop determinant of the bosonic fields as

$$\prod_{j=1}^{\infty} \prod_{\substack{m=-j+1 \\ j-m=\mathrm{odd}}}^{j-1} \prod_{l} \left((j+1-\Delta)\beta - \beta r D_3\right)^{-1}$$

$$= \prod_{n\in\mathbb{Z}} \prod_{j=1}^{\infty} \prod_{\substack{m=-j+1 \\ j-m=\mathrm{odd}}}^{j-1} (-2\pi i n - i\beta r\rho(a) + (j+m+1)\beta_1 + (j+1-\Delta-m)\beta_2 - F_l M_l)^{-1}.$$

Next, we evaluate the one-loop determinant of the fermions. In this case, the sum $\sum'$ in Eq. (D.24) runs over the following modes under the boundary condition in Eq. (20):

$$\chi^+_{jm} : j - m = \text{even}, \qquad \chi^-_{jm} : j - m = \text{odd}. \tag{D.31}$$

The action of the chiral multiplet in Eq. (18) can be expanded at the quadratic order:

$$\begin{aligned} S^{(2)}_{\text{chi}}\Big|_{\text{fer}} &= \frac{i}{2r}\int_{S^1}\sum_\rho\sum_j\sum_{m;j-m=\text{odd}} \bar{\psi}^{\rho,+}_{j,-m}\,(j - rD_3 + 1 - \Delta)\,\psi^{\rho,-}_{j,m}\cdot(-1)^{m-1/2} \\ &\quad + \frac{i}{2r}\int_{S^1}\sum_\rho\sum_j\sum_{m;j-m=\text{even}} \bar{\psi}^{\rho,-}_{j,-m}\,(j + rD_3 + \Delta)\,\psi^{\rho,+}_{j,m}\cdot(-1)^{m-1/2}. \end{aligned}$$

Then, the factor $(j - rD_3 + 1 - \Delta)$ contributes to the one-loop determinant of the chiral multiplet as

$$\begin{aligned} &\prod_{j=\frac{1}{2}}^{\infty}\text{Det}\,((j - \Delta + 1)\beta - \beta r D_3) \\ &= \prod_{n\in\mathbb{Z}}\prod_{j'=0}^{\infty}\prod_{\substack{m'=-j'-1 \\ j'-m'=\text{odd}}}^{j'-1}\prod_l(-2\pi i n - i\beta r\rho(a) + (j' + m' + 1)\beta_1 + (j' + 1 - \Delta - m')\beta_2 - F_l M_l), \end{aligned}$$

where we defined $j' = j - \frac{1}{2}, m' = m - \frac{1}{2}$. The other factor $(j + rD_3 + \Delta)$ contributes to the one-loop determinant of the chiral multiplet as

$$\begin{aligned} &\prod_{j=\frac{1}{2}}^{\infty}\text{Det}\,((j + \Delta)\beta + \beta r D_3) \\ &= \prod_{n\in\mathbb{Z}}\prod_{j'=1}^{\infty}\prod_{\substack{m'=-j'+1 \\ j'-m'=\text{odd}}}^{j'-1}\prod_l(2\pi i n + i\beta r\rho(a) + (j' - m')\beta_1 + (j' + \Delta + m')\beta_2 + F_l M_l). \end{aligned}$$

Here, we defined $j' = j + \frac{1}{2}, m' := m - \frac{1}{2}$. Thus, the one-loop determinant of the chiral multiplet with the Dirichlet boundary condition is given by

$$\begin{aligned} Z^{\text{3d.chi.D}}_{\text{1-loop}} &= \prod_\rho\prod_{n\in\mathbb{Z}}\prod_{j=0}^{\infty}\prod_l(-2\pi i n - i\beta r\rho(a) + (2j + 2 - \Delta)\beta_2 - F_l M_l) \\ &= \prod_\rho\prod_l e^{-\mathcal{E}(-i\beta r\rho(a) + (2-\Delta)\beta_2 - F_l M_l)}(e^{i\beta r\rho(a) - F_l M_l}q^{2-\Delta}; q^2)_\infty. \end{aligned} \tag{D.32}$$

D.3. *Two-dimensional $\mathcal{N} = (0, 2)$ chiral multiplet*

We have boundary theories with the $\mathcal{N} = (0, 2)$ supersymmetry. The Lagrangian for the $\mathcal{N} = (0, 2)$ chiral multiplet is given by

$$\begin{aligned} \mathcal{L}^{\mathcal{N}=(0,2)}_{\text{chi}} &= \bar{\phi}(D_{\hat{2}} - iD_{\hat{3}})(D_{\hat{2}} + iD_{\hat{3}})\phi + \frac{1}{2}\bar{\psi}'(D_{\hat{2}} - iD_{\hat{3}})\psi' \\ &\quad + \frac{i\Delta}{r}\bar{\phi}(D_{\hat{2}} - iD_{\hat{3}})\phi + i\bar{\phi}\bar{\lambda}_1\psi' + i\bar{\psi}_1\lambda_1\phi + \bar{\phi}(F_{\hat{2}\hat{3}} - i\hat{D})\phi. \end{aligned} \tag{D.33}$$

The one-loop determinant of the bosonic part is given by

$$\begin{aligned}
&\mathrm{Det}\left(D_{\hat{2}} + iD_{\hat{3}} + \frac{i\Delta}{r}\right)(D_{\hat{2}} - iD_{\hat{3}}) \\
&\cong \prod_{\rho}\prod_{l}\prod_{m\in\mathbb{Z}}\prod_{n\in\mathbb{Z}}(2\pi in + i\beta r\rho(a) + (2m+\Delta)\beta_2 + F_l M_l) \\
&\quad\times \prod_{\rho}\prod_{l}\prod_{m\in\mathbb{Z}}\prod_{n\in\mathbb{Z}}(2\pi in + i\beta r\rho(a) - (\Delta+2m)\beta_1 + F_l M_l).
\end{aligned} \tag{D.34}$$

The one-loop determinant of the fermionic part is given by

$$\mathrm{Det}(D_{\hat{2}} - iD_{\hat{3}}) \cong \prod_{\rho}\prod_{m'\in\mathbb{Z}}\prod_{n\in\mathbb{Z}}(2\pi in + i\beta\rho(a) - (\Delta+2m')\beta_1 + F_l M_l). \tag{D.35}$$

Then, the one-loop determinant of the chiral multiplet can be written by

$$Z^{\text{2d.chi}}_{\text{1-loop}} = \prod_{\rho}\prod_{l} e^{2\mathcal{E}(i\beta r\rho(a)+\Delta\beta_2+F_l M_l)}\theta(e^{-i\beta r\rho(a)-F_l M_l}q^{\Delta};q^2)^{-1}. \tag{D.36}$$

D.4. Two-dimensional $\mathcal{N}=(0,2)$ Fermi multiplet

Next, we evaluate the one-loop determinant of the Fermi multiplet on the torus T^2:

$$\mathcal{L}^{\mathcal{N}=(0,2)}_{\text{Fermi}} = -\bar{\Psi}(D_{\hat{2}} + iD_{\hat{3}})\Psi + 2\bar{G}G + 2\bar{E}E - \bar{\psi}_E\Psi - \bar{\Psi}\psi_E + \frac{i}{r}(1-\tilde{\Delta})\bar{\Psi}\Psi.$$

At the quadratic order, the one-loop contribution comes from the determinant of the fermion:

$$\begin{aligned}
Z^{\text{2d.Fermi}}_{\text{1-loop}} &= \mathrm{Det}\left[-(D_{\hat{2}} + iD_{\hat{3}}) + \frac{i}{r}(1-\tilde{\Delta})\right] \\
&\cong \prod_{a}\prod_{\rho}\prod_{m'\in\mathbb{Z}}\prod_{n\in\mathbb{Z}}(2\pi in + (2m'+\tilde{\Delta})\beta_2 + i\beta r\rho(a) + F_a M_a) \\
&= \prod_{a}\prod_{\rho} e^{-2\mathcal{E}(i\beta r\rho(a)+\tilde{\Delta}\beta_2+F_a M_a)}\theta(e^{-i\beta r\rho(a)-F_a M_a}q^{\tilde{\Delta}};q^2).
\end{aligned} \tag{D.37}$$

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