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The Janis-Newman-Winicour naked singularity in higher dimensions

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The paper explores the derivation of the D-dimensional Janis-Newman-Winicour (JNW) solution directly by solving the Einstein-massless scalar field equations. An analytical form of D-dimensional JNW metric is has been derived which covers the D-dimensional Schwarzschild solution and the metric arises pure scalar field in the higher dimension. The curvature invariants such as Ricci scalar R, Ricci square $R_{\alpha\beta}R^{\alpha\beta}$ and the Kretschmann scalar $K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ has been determined for obtained spacetime metric in order to investigate the structure of the geometry. It is also determined the equipotential surface and proper length for JNW metric in four and higher-dimensional space.

I. INTRODUCTION

The structure of spacetime around the gravitational compact objects, such that black holes, wormholes, even neutron star is a quite common phenomenon in general relativity (GR) and the alternative theory of gravity (ATG). Since the last few years, there has been a growing interest in studying scalar field theory and/or the gravitational collapse of scalar fields and the nature of singularities in the Einstein-scalar theory. In that sense, it is a very interesting and important task in GR and the alternative theory of gravity is to find the exact analytical solutions of the Einstein field equations.

It is known that the exact non-vacuum solutions of the Einstein field equations in D-spacetime and pure Gauss-Bonnet equation in $D+2$-spacetime are equivalent to each other. The motivation of work is to check whether this fact is true for the non-vacuum solutions of field equations in these theories. Before that we wish to present the exact solution of the Einstein-scalar field equations in D-spacetime.

In the early paper [1], the first spherically-symmetric static solutions of the Einstein-scalar field equations have been presented. Later similar solutions have been studied in [2–11]. In Ref. [8], it has shown that the general spherically-symmetric static solution obtained in [5] is the same as Janis, Newman, and Winicour (JNW) solution [2]. The exact solution of the Einstein equations for the wormhole with the scalar field has been recently studied [12]. In Ref. [13] an axisymmetric and static solution of Einstein equations with the self-gravitating scalar field has been studied. Contribution of the scalar field in the spacetime of static [14] and rotating [15] black holes have been also studied. In Ref. [16] new method has been developed to finding exact scalar-tensor solutions. Some approximate static solutions of the Einstein-scalar equations has been presented in [17–20] in the framework of Conformal Field Theory (CFT).

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The role of scalar field in geodesic motion [21–25], in the strong gravitational lensing [26–37], in optical appearence of the thin accretion [38, 39], in absorption and scattering of scalar wave [40], in iron line spectroscopy [41] in JNW spacetime have been studied.

In scalar-tensor gravity, the exterior spacetime of a static, spherically symmetric star sourcing a nontrivial scalar field in the Einstein frame is given

In Ref. [42], the Schwarzschild-de Sitter metric and Reissner-Nordstrom-de Sitter metric in D-dimensional space has been derived directly by solving the Einstein and Einstein-Maxwell equations, while in [43, 44] solution for Einstein-massless scalar field equations in D-dimensional has been studied.

In this research paper, we are interested in finding JNW solution in higher-dimensional space. The paper is organized as In Sec. II we provide in the very detailed derivation of the Einstein-scalar field equation in arbitrary D dimension. Section III is devoted to analyst JNW solution in $4D$ space. In Sec. IV we show the exact analytical static spherically symmetric solutions of the Einstein-scalar field equations in D dimension. Finally, in Sec. V we summarize obtained results and give future outlook related to the present work. Throughout the paper Greek (Latin) indices are taken to run from 0 to D (1 to D), where $D \geq 4$. For convenience, we use a system of units in which $G = c = \hbar = 1$ and a space-like signature $(-, \underbrace{+, +, +, ..., +}_{D-1})$.

II. BASIC EQUATIONS

In this section, we will show the derivation of the main equations of the Einstein-scalar field system in an arbitrary D-dimensional space. The action for the Einstein and massless scalar field system is given by

$$S = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left(R - \frac{D-2}{D-3} \partial_\alpha \Phi \partial^\alpha \Phi \right) , \tag{1}$$

where R is the Ricci scalar of the curvature, g is the determinant of the metric tensor, Φ is the scalar field which is the function of coordinates i.e. $\Phi = \Phi(x^\alpha)$ and ∂_α stands for partial derivative with respect to coordinate x^α.

Varying the action (1) with respect to the metric tensor, $g_{\alpha\beta}$, and the scalar field, Φ, we get the equation of motion for the system, namely, the Einstein field equations and the Klein-Gordon equation in the following forms

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta} , \tag{2}$$

$$\nabla_\alpha \nabla^\alpha \Phi = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} \partial^\alpha \Phi \right) = 0 , \tag{3}$$

where $G_{\alpha\beta}$ and $R_{\alpha\beta}$ are the Einstein and Ricci tensors, respectively. $T_{\alpha\beta}$ is the energy-momentum tensor for the scalar field Φ which can be defined as

$$T_{\alpha\beta} = 2 \frac{D-2}{D-3} \left(\partial_\alpha \Phi \partial_\beta \Phi - \frac{1}{2} g_{\alpha\beta} \partial_\mu \Phi \partial^\mu \Phi \right) . \tag{4}$$

Note that $g^\alpha_\alpha = \delta^\alpha_\alpha = D$ for D-dimensional space. Hereafter taking trace from both side of equation (2) and performing simple algebra one can rewrite the equation (2) in the following separable form

$$R = \frac{D-2}{D-3} \partial_\alpha \Phi \partial^\alpha \Phi , \qquad R_{\alpha\beta} = \frac{D-2}{D-3} \partial_\alpha \Phi \partial_\beta \Phi , \qquad R_{\alpha\beta} R^{\alpha\beta} = R^2 . \tag{5}$$

The first expression in (5) is one of the curvature invariant of the spacetime geometry. There are, actually, two more curvature invariants such as Ricci square $R_{\alpha\beta} R^{\alpha\beta}$ and the Kretschman scalar $K = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$. In order to find Ricci square we first determine Einstein "scalar" or the product of Einstein tensor (i.e. $G_{\alpha\beta} G^{\alpha\beta}$) as:

$$\begin{aligned} G_{\alpha\beta} G^{\alpha\beta} = & \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) \\ = & R_{\alpha\beta} R^{\alpha\beta} - \left(1 - \frac{D}{4} \right) R^2 = \frac{D}{4} R_{\alpha\beta} R^{\alpha\beta} , \end{aligned} \tag{6}$$

on the other hand taking into account equation (2) one has

$$R_{\alpha\beta} R^{\alpha\beta} = \frac{4}{D} T_{\alpha\beta} T^{\alpha\beta} = \left(\frac{D-2}{D-3} \right)^2 \partial_\alpha \Phi \partial_\beta \Phi \partial^\alpha \Phi \partial^\beta \Phi = R^2 . \tag{7}$$

Obviously, it is very difficult to obtain the exact analytical solutions of Einstein field equations (2), in particular, when the source term exists (non-vacuum case). But it can be solved under a reasonable assumption. In order to get analytical solution of equations (2) and (3), assume that the scalar field depends on the radial coordinate r only $\Phi = \Phi(r)$, then the diagonal element of the energy-momentum tensor can be written in the following form

$$T_0^0 = -T_1^1 = T_2^2 = ... = T_{D-1}^{D-1} = -\frac{D-2}{D-3}\partial_r\Phi\partial^r\Phi \ , \tag{8}$$

which are equivalent, form Einstein equations (2), to the following equations

$$G_0^0 = -G_1^1 = G_2^2 = ... = G_{D-1}^{D-1} = -\frac{D-2}{D-3}\partial_r\Phi\partial^r\Phi \ . \tag{9}$$

Consequently, the scalar field Φ satisfies the following equation

$$\partial_r\left(\sqrt{-g}g^{rr}\partial_r\Phi\right) = 0 \quad \text{or} \quad \sqrt{-g}g^{rr}\partial_r\Phi = \text{const} \ . \tag{10}$$

In this paper, our main aim is to find an analytical solution of the equations (9) and (10) in an arbitrary D-dimensional space. Before that let us focus on JNW solution in four-dimensional space.

III. JNW SOLUTION IN $4D$ SPACE

In this section, we will briefly study JNW solution in $4D$ space. Up to now, number of authors have been forund JNW solution in different ways (See, for example, [2, 5, 7, 8, 10]). The general form of JNW solution is given by

$$ds^2 = -\left(1-\frac{2\mathcal{M}}{r}\right)^n dt^2 + \left(1-\frac{2\mathcal{M}}{r}\right)^{-n} dr^2 + \left(1-\frac{2\mathcal{M}}{r}\right)^{1-n} r^2 d\Omega_2 \ , \tag{11}$$

with $d\Omega_2 = d\theta^2 + \sin^2\theta d\phi^2$ and the associated scalar field has a form

$$\Phi(r) = \frac{\sqrt{1-n^2}}{2}\ln\left(1-\frac{2\mathcal{M}}{r}\right) = \frac{\sigma}{2\mathcal{M}}\ln\left(1-\frac{2\mathcal{M}}{r}\right) \ , \tag{12}$$

where $\mathcal{M}$ is the induced mass of the compact object that can be defined as

$$\mathcal{M} = \sqrt{M^2+\sigma^2} \ , \qquad n = \frac{M}{\mathcal{M}} \le 1 \ , \tag{13}$$

with two free parameters, the total mass M of the gravitational compact object and σ is "scalar charge". Simple analysis shows that

- the well-known Schwarzschild solution can be easily obtained by putting $q = 0$,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2 \ , \qquad \Phi(r) = 0 \ , \tag{14}$$

  with the following curvature scalar invariants

$$R = R_{\alpha\beta}R^{\alpha\beta} = 0 \ , \qquad K = \frac{48M^2}{r^6} \ . \tag{15}$$

- one the other hand when the mass of the compact object vanishes $M = 0$, then one can see the effect of the only pure scalar field in the spacetime in the form

$$ds^2 = -dt^2 + dr^2 + \left(1-\frac{2\sigma}{r}\right)r^2 d\Omega_2 \ , \qquad \Phi(r) = \frac{1}{2}\ln\left(1-\frac{2\sigma}{r}\right) \ . \tag{16}$$

  In this case the curvature scalar invariants take the form

$$R = \frac{2\sigma^2}{r^2(r-2\sigma)^2} \ , \qquad R_{\alpha\beta}R^{\alpha\beta} = \frac{4\sigma^4}{r^4(r-2\sigma)^4} \ , \qquad K = \frac{12\sigma^4}{r^4(r-2\sigma)^4} \ . \tag{17}$$

Let us analysis the scalar field in equation (12), leading order of scalar field reads as

$$\Phi(r) \simeq -\frac{\sigma}{r} + \mathcal{O}\left(\frac{\sigma^2}{r^2}\right) , \tag{18}$$

which reduces to the Coulomb-type potential.

The components of the energy-momentum tensor can be expressed as

$$T_0^0 = -T_1^1 = T_2^2 = T_3^3 = (n^2-1)\frac{\mathcal{M}^2}{r^4}\left(1-\frac{2\mathcal{M}}{r}\right)^{n-2} = -\frac{\sigma^2}{r^4}\left(1-\frac{2\mathcal{M}}{r}\right)^{n-2} , \tag{19}$$

In order to investigate the formation of curvature singularities of the spacetime (11), we consider curvature invariants such as Ricci scalar R, Ricci square $R_{\alpha\beta}R^{\alpha\beta}$, and Kretschmann scalar $K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ that are given by

$$R = -\frac{2\mathcal{M}^2\left(n^2-1\right)}{r^4}\left(1-\frac{2\mathcal{M}}{r}\right)^{n-2} , \quad R_{\alpha\beta}R^{\alpha\beta} = \frac{4\mathcal{M}^4\left(n^2-1\right)^2}{r^8}\left(1-\frac{2\mathcal{M}}{r}\right)^{2(n-2)} , \tag{20}$$

and

$$K = \frac{48\mathcal{M}^2}{r^6}\left(1-\frac{2\mathcal{M}}{r}\right)^{2n-4}\left(n^2-\frac{2\mathcal{M}}{3r}A+\frac{\mathcal{M}^2}{12r^2}B\right) , \tag{21}$$

where $A = n(n+1)(2n+1)$ and $B = (n+1)^2(7n^2+2n+3)$.
From equation (21) we can see that the Kretschmann scalar has two singular points, the first is at $r = 0$ like in Schwarzschild spacetime and the second singular point appears at $r = 2\mathcal{M}(= 2\sqrt{M^2+\sigma^2})$ for non-zero value of both M and σ (i.e. $M \neq 0$ and $q \neq 0$). In the case when $\sigma = 0$ which is correspondence to Schwarzschild space, then there is an only single singular point ($r = 0$), and the radius of horizon takes $r_h = 2M$, while when mass vanishes $M = 0$ then Kretschmann scalar has still two singular points at $r = 0$ and $r = \sigma$ as shown in the equation (17) which means in this can horizon doesn't exit. That is why sometimes JNW metric calls as a solution of naked-singularity.
We now focus on the investigation of the formation of this singularity by examining the properties of the equipotential surfaces and of the closed curves on these surfaces as r approaches the value. The area of the equipotential surfaces is [6]

$$A = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\sqrt{g_{\theta\theta}g_{\phi\phi}} = 4\pi r^2\left(1-\frac{2\mathcal{M}}{r}\right)^{1-n} , \tag{22}$$

and the proper lengths are, respectively, to closed azimuthal curve $\theta = \pi/2$ and for a polar curve $\phi = \text{const}$, given by

$$L_\phi = \int_0^{2\pi} d\phi\sqrt{g_{\phi\phi}} = 2\pi r\left(1-\frac{2\mathcal{M}}{r}\right)^{(1-n)/2} , \tag{23}$$

$$L_\theta = 2\int_0^{\pi} d\theta\sqrt{g_{\theta\theta}} = 2\pi r\left(1-\frac{2\mathcal{M}}{r}\right)^{(1-n)/2} . \tag{24}$$

Unlike in Schwarzschild spacetime the quantities in equations (22)-(24) have singularities at $r = 0$ for $n < 1$ and at $r = 2\mathcal{M}$ for $n > 1$. It is now evident that the singularity at $r = 2\mathcal{M}$ has the topology of a point and the event horizon has therefore shrunk to a point. Thus we cannot speak of a black hole in the usual sense, even if the red-shift still approaches infinity as the radius of the body tends to zero.

IV. SOLUTION IN ARBITRARY D-DIMENSIONAL SPACE

In this section, we show the derivation of the scalar hairy black hole solution in an arbitrary D-dimensional space. Again we assume that the scalar field is a function radial coordinate as we did in the previous section. We can seek the solution in the following form

$$ds^2 = -f^a(r)dt^2 + f^b(r)dr^2 + f^c(r)r^2 d\Omega_{D-2} , \tag{25}$$

where $d\Omega_{D-2}$ is the solid angle in D-dimensional space, a, b, and c are unknown constants which can be found by solving the equations (9). Here one has to emphasize that in arbitrary dimensional space we assume JNW solution will be the same as in $4D$ space that is why we have chosen spacetime metric same as in equation (11). To be convenience the lapse function in equation (25) can be written as

$$f(r) = 1 - 2\left(\frac{\mathcal{M}}{r}\right)^{D-3} , \tag{26}$$

where $\mathcal{M}$ is again induced mass of the object in D-dimensional space which has the same dimension of length.

Once the explicit form of the metric is given then we can immediately calculate Einstein tensor $G_{\alpha\beta}$, and using the equations (9) one can easily find the following relations for the metric function for arbitrary dimension

$$G^0_0 + G^1_1 \equiv 0 = \frac{c[a-b+(D-2)c-2]f^{-b-2}}{2r^2} - \frac{((a-b)(c-1)+2(c-2)c)f^{-b-1}}{r^2} + \frac{(c-2)[a-b+(D-2)c-2]f^{-b}-4f^{-c}}{2r^2} , \tag{27}$$

$$G^0_0 - G^2_2 \equiv 0 = \frac{(c-a)[a-b+(D-2)c-2]f^{-b-2}}{4r^2} + \frac{\left[a(a-b)+(a+b)(c-1)-2(c-2)c\right] f^{-b-1}}{2r^2} + \frac{(a-c+2)[-a+b-(D-2)c+2]f^{-b}-4f^{-c}}{4r^2} . \tag{28}$$

Notice that here we have used the following useful notations

$$f'(r) = \frac{D-3}{r}\left(1-f(r)\right) , \qquad f''(r) = -\frac{(D-2)(D-3)}{r^2}\left(1-f(r)\right) ,$$

where prime and double primes denote the first and the second derivatives with respect to radial coordinate r. From the equations (27) and (28), we can get the following relations

$$a - b + (D-2)c - 2 = 0 , \qquad c = b+1 , \tag{29}$$

for unknown a, b and c constants. Note that the relations in equation (29) also satisfies the following $G^1_1 + G^2_2 \equiv 0$ condition. As we already mention before that JNW solution is non-vacuum solution and characterise with two free parameters, mass M and parameter n as shown in the metric (11). In that sense we now simply express unknown constants a, b and c in terms of constant n as in the form

$$a = n , \qquad b = \frac{1-n}{D-3} - 1 , \qquad c = \frac{1-n}{D-3} . \tag{30}$$

Finally, we obtain the following solution

$$ds^2 = -\left[1-2\left(\frac{\mathcal{M}}{r}\right)^{D-3}\right]^n dt^2 + \left[1-2\left(\frac{\mathcal{M}}{r}\right)^{D-3}\right]^{-\frac{n+D-4}{D-3}} dr^2 + \left[1-2\left(\frac{\mathcal{M}}{r}\right)^{D-3}\right]^{\frac{1-n}{D-3}} r^2 d\Omega_{D-2} , \tag{31}$$

Now we focus on finding an exact form of the associated scalar field to corresponding metric (25). The determinant of the metric tensor is

$$g = \det g_{\alpha\beta} = -r^{2(D-2)}\left[1-2\left(\frac{\mathcal{M}}{r}\right)^{D-3}\right]^{\frac{2(1-n)}{D-3}} \sin^{D-2}\theta_1 \sin^{D-3}\theta_2 ... \sin\theta_{D-2} , \tag{32}$$

then taking into account equations (31) and (32) Klein-Gordon equation (10) takes a form

$$\Phi'(r)\left[1-2\left(\frac{\mathcal{M}}{r}\right)^{D-3}\right] r^{D-2} = \text{const} . \tag{33}$$

By introducing new radial coordinate, $x = (r/\mathcal{M})^{D-3}$, equation (33) can be rewritten as

$$\frac{d\Phi(x)}{dx}\left(1-\frac{2}{x}\right)x^2 = \text{const} , \quad \to \quad \Phi(x) \simeq \ln\left(1-\frac{2}{x}\right) , \tag{34}$$

here the constant in equation (34) can found from equation (9) and finally, the associated scalar field corresponding to metric (31) is given by

$$\Phi(r) = \frac{\sqrt{1-n^2}}{2} \ln \left[1 - 2 \left(\frac{\mathcal{M}}{r} \right)^{D-3} \right] . \tag{35}$$

In order to explore spacetime geometric we determine the curvature invariants such as Ricci scalar, Ricci square and the Kretschmann scalar. However we could not find the general form of the curvature invariants in arbitrary space. That is why here we wish to present them at least for some spacial form. According to equation (7) we have $R_{\alpha\beta}R^{\alpha\beta} = R^2$, which means it is enough to present Ricci scalar along the Kretschmann scalar. In $D = 5$ we have

$$R = \frac{6\left(1-n^2\right)\mathcal{M}^4}{r^6} \left(1 - \frac{2\mathcal{M}^2}{r^2} \right)^{\frac{n-3}{2}} , \tag{36}$$

$$K = \frac{12\mathcal{M}^4}{r^{12}} \left(1 - \frac{2\mathcal{M}^2}{r^2} \right)^{n-3} \left[\mathcal{M}^4(n+1)^2(n(17n+2)+5) - 12\mathcal{M}^2 n(n+1)(3n+1)r^2 + 24n^2r^4 \right] , \tag{37}$$

in $D = 6$ they take the form

$$R = \frac{12\mathcal{M}^6\left(1-n^2\right)}{r^8} \left(1 - \frac{2\mathcal{M}^3}{r^3} \right)^{\frac{n-2}{3}} , \tag{38}$$

$$K = \frac{24\mathcal{M}^6}{r^{16}} \left(1 - \frac{2\mathcal{M}^3}{r^3} \right)^{\frac{2(n-4)}{3}} \left[\mathcal{M}^6(n+1)^2(n(31n+2)+7) - 16\mathcal{M}^3 n(n+1)(4n+1)r^3 + 40n^2r^6 \right] , \tag{39}$$

and in $D = 7$ one gets

$$R = \frac{20M^8\left(1-n^2\right)}{r^{10}} \left(1 - \frac{2\mathcal{M}^4}{r^4} \right)^{\frac{n-5}{4}} , \tag{40}$$

$$K = \frac{40M^8}{r^{20}} \left(1 - \frac{2M^4}{r^4} \right)^{\frac{n-5}{2}} \left[M^8(n+1)^2(n(49n+2)+9) - 20M^4 n(n+1)(5n+1)r^4 + 60n^2r^8 \right] , \tag{41}$$

Similarly, one can continue calculations for any higher dimensions. However, from the expressions (36), (36) and (40) one see that the curvature invariants have two singularities as shown in four-dimensional space. The first singularity at $r = 0$, while second one appears at $r = 2^{1/(D-3)}\mathcal{M}$ which corresponds to naked singularity.

We now determine the equipotential surfaces and proper lengths in D-dimensional spacetime. The area of the equipotential surfaces is

$$A_D = \frac{2\pi^{D/2}}{(D/2-1)!} r^{D-2} \left[1 - 2 \left(\frac{\mathcal{M}}{r} \right)^{D-3} \right]^{(D-2)(1-n)} , \tag{42}$$

and the proper lengths are, respectively, for each angular coordinates given by

$$L_{\theta_1} = 2 \int_0^{2\pi} d\theta_1 \sqrt{g_{\theta_1\theta_1}} = 2\pi r \left[1 - 2 \left(\frac{\mathcal{M}}{r} \right)^{D-3} \right]^{(D-2)(1-n)/2} , \tag{43}$$

$$L_{\theta_2} = 2 \int_0^{2\pi} d\theta_2 \sqrt{g_{\theta_2\theta_2}} = 2\pi r \left[1 - 2 \left(\frac{\mathcal{M}}{r} \right)^{D-3} \right]^{(D-2)(1-n)/2} , \tag{44}$$

$$\ldots$$

$$L_{\theta_{D-2}} = \int_0^{2\pi} d\theta_{D-2} \sqrt{g_{\theta_{D-2}\theta_{D-2}}} = 2\pi r \left[1 - 2 \left(\frac{\mathcal{M}}{r} \right)^{D-3} \right]^{(D-2)(1-n)/2} , \tag{45}$$

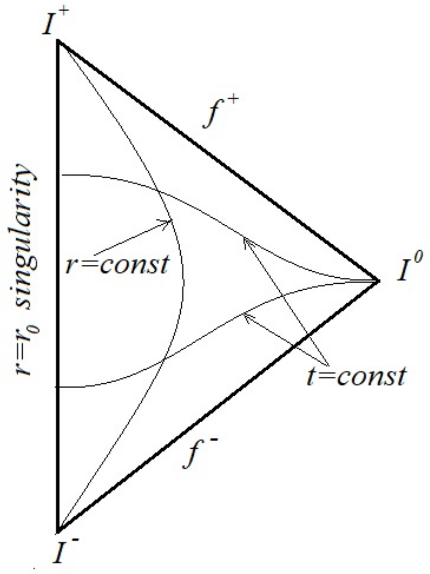

FIG. 1: Penrose diagram for JNW naked singularity in D-dimensional space, where $r_0 = 2^{1/(D-3)}\mathcal{M}$.

In the case when $n = 1$ the solution (31) reduces Schwarzschild solution in D-dimensional space. When the mass of the object vanishes then we have

$$ds^2 = -dt^2 + \left[1 - 2\left(\frac{\sigma}{r}\right)^{D-3}\right]^{-\frac{D-4}{D-3}} dr^2 + \left[1 - 2\left(\frac{\sigma}{r}\right)^{D-3}\right]^{\frac{1}{D-3}} r^2 d\Omega_{D-2} \ , \tag{46}$$

and

$$\Phi(r) = \frac{1}{2}\ln\left[1 - 2\left(\frac{\sigma}{r}\right)^{D-3}\right] \ . \tag{47}$$

From equation (46) one can see that unlike in four-dimensional space the spatial components of the metric is radially dependent.

The causal structure of the D-dimensional JNW solution can be summarized in the corresponding Penrose diagrams in Fig. 1. The topology of the spacelike singularity located at $r = 0$, is $R^1_t \times S^D$. Figure 1 draws the region conformal to the JNW spacetime (31), $r \in (2\mathcal{M}, \infty)$. It is asymptotically flat and has timelike curvature singularity is located at $r \in (2\mathcal{M}, \infty)$.

V. CONCLUSION

The explicit form of the JNW solution in higher dimensions has been investigated. The solution covers the well-known Schwarzschild metric in arbitrary dimensional space and D-dimensional metric arises from the pure scalar field. In order to verify the solution, we substitute it into the Einstein-scalar equations.

In order to investigate the structure of curvature singularities of the spacetime, we determine the curvature invariants such as Ricci scalar, Ricci square, and Kretschmann scalar. It has been shown that introducing a scalar field with minimal coupling to gravity prevents the formation of the event horizons which are present in all the well-known exact solutions of the exterior problem.

The physical meaning of the “scalar charge” is still clearly an open question. In particular, it might be not a good idea to think in a classical way, however on quantum approaches might explain this problem. For example, the capture of particles or antiparticles which are created in pairs in the curved vacuum surrounding the compact object during the gravitational collapse. There might also be a connection between scalar charge and Hawking radiation in order to describe black-hole evaporation.

We also calculate the area of the equipotential surface in the equatorial plane and the proper lengths in both four and higher-dimensional space. It has been shown that in a higher dimension the spatial components of the metric tensor survive unlike in a four-dimensional JNW solution. In the future, similar work can be done for the Einstein-Maxwell-scalar field system in higher-dimensional space.

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References

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[1] H. A. Buchdahl, Physical Review **116**, 1027 (1959).
[2] A. I. Janis, E. T. Newman, and J. Winicour, Physical Review Letters **20**, 878 (1968).
[3] R. Penney, Physical Review **174**, 1578 (1968).
[4] R. Penney, Physical Review **182**, 1383 (1969).
[5] M. Wyman, Phys. Rev. D **24**, 839 (1981).
[6] A. G. Agnese and M. La Camera, Phys. Rev. D **31**, 1280 (1985).
[7] T. Damour, Classical and Quantum Gravity **10**, S59 (1993).
[8] K. S. Virbhadra, International Journal of Modern Physics A **12**, 4831 (1997), gr-qc/9701021.
[9] A. Bhadra and K. K. Nandi, International Journal of Modern Physics A **16**, 4543 (2001).
[10] N. Dadhich and N. Banerjee, Modern Physics Letters A **16**, 1193 (2001), hep-th/0012015.
[11] B. Turimov, B. Ahmedov, and Z. Stuchlík, Physics of the Dark Universe **33**, 100868 (2021).
[12] G. W. Gibbons and M. S. Volkov, Journal of Cosmology and Astroparticle Physics **5**, 039 (2017), 1701.05533.
[13] B. Turimov, B. Ahmedov, M. Kološ, and Z. Stuchlík, Phys. Rev. D **98**, 084039 (2018), 1810.01460.
[14] C. A. R. Herdeiro and E. Radu, International Journal of Modern Physics D **24**, 1542014-219 (2015), 1504.08209.
[15] A. Sen, Physical Review Letters **69**, 1006 (1992), hep-th/9204046.
[16] B. Chauvineau, Phys. Rev. D **100**, 024051 (2019).
[17] Z.-Y. Fan and H. Lü, Phys. Rev. D **92**, 064008 (2015), 1505.03557.
[18] Z.-Y. Fan and H. Lu, ArXiv e-prints (2015), 1507.04369.
[19] Z.-Y. Fan and H. Lü, Physics Letters B **743**, 290 (2015), 1501.01727.
[20] Z.-Y. Fan and H. Lü, Journal of High Energy Physics **4**, 139 (2015), 1501.05318.
[21] A. N. Chowdhury, M. Patil, D. Malafarina, and P. S. Joshi, Phys.Rev. D **85**, 104031 (2012), 1112.2522.
[22] M. Patil and P. S. Joshi, Phys. Rev. D **85**, 104014 (2012), 1112.2525.
[23] S. Zhou, R. Zhang, J. Chen, and Y. Wang, International Journal of Theoretical Physics **54**, 2905 (2015), 1408.6041.
[24] G. Z. Babar, A. Z. Babar, and Y.-K. Lim, Phys. Rev. D **96**, 084052 (2017).
[25] F. Willenborg, S. Grunau, B. Kleihaus, and J. Kunz, Phys. Rev. D **97**, 124002 (2018), 1801.09769.
[26] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, Astron. Astrophys **337**, 1 (1998), astro-ph/9801174.
[27] T. Matos and R. Becerril, Classical and Quantum Gravity **18**, 2015 (2001).
[28] K. S. Virbhadra and G. F. Ellis, Phys. Rev. D **65**, 103004 (2002).
[29] V. Bozza, Phys. Rev. D **66**, 103001 (2002), gr-qc/0208075.
[30] K. K. Nandi, Y.-Z. Zhang, and A. V. Zakharov, Phys. Rev. D **74**, 024020 (2006), gr-qc/0602062.
[31] K. Sarkar and A. Bhadra, Classical and Quantum Gravity **23**, 6101 (2006), gr-qc/0602087.
[32] P. Amore and S. Arceo, Phys. Rev. D **73**, 083004 (2006), gr-qc/0602106.
[33] P. Amore, M. Cervantes, A. de Pace, and F. M. Fernández, Phys. Rev. D **75**, 083005 (2007), gr-qc/0610153.
[34] K. S. Virbhadra and C. R. Keeton, Phys. Rev. D **77**, 124014 (2008), 0710.2333.
[35] T. K. Dey and S. Sen, Modern Physics Letters A **23**, 953 (2008), 0806.4059.
[36] J. P. DeAndrea and K. M. Alexander, Phys. Rev. D **89**, 123012 (2014), 1402.5630.
[37] S. Roy and A. K. Sen, Zeitschrift Naturforschung Teil A **72**, 1113 (2017).
[38] G. Gyulchev, P. Nedkova, T. Vetsov, and S. Yazadjiev, Phys. Rev. D **100**, 024055 (2019), 1905.05273.
[39] R. Shaikh and P. S. Joshi, JCAP **2019**, 064 (2019), 1909.10322.
[40] P. Liao, J. Chen, H. Huang, and Y. Wang, General Relativity and Gravitation **46**, 1752 (2014).
[41] H. Liu, M. Zhou, and C. Bambi, JCAP **2018**, 044 (2018), 1801.00867.
[42] X. Dianyan, Classical and Quantum Gravity **5**, 871 (1988).

[43] T. Dereli, Physics Letters B **161**, 307 (1985).
[44] S. Abdolrahimi and A. A. Shoom, Phys. Rev. D **81**, 024035 (2010), 0911.5380.