Higher gauge theory, BV formalism and self-dual theories from twistor space

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Thesis submitted to the University of Surrey
for the degree of Doctor of Philosophy

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Scientific abstract

In this Thesis we investigate higher homotopy structures arising in ordinary classical field theory, as well as in string and M-theory. First, we review L_∞-algebras and we discuss their homotopy Maurer–Cartan theory. Our perspective is adapted to an application towards higher gauge theory from the outset. We observe that homotopy Maurer–Cartan theory always allows for a supersymmetric extension by auxiliary fields, just as ordinary Chern–Simons theory does. Then, we review in detail the Batalin–Vilkovisky formalism for Lagrangian field theories and its mathematical foundations, with an emphasis on higher algebraic structures and classical field theories. We explain that, with the help of this formalism, any classical field theory admitting an action can be fully described by an L_∞-algebra, encoding its symmetry structure, the field contents, the equations of motion, as well as the Noether identities. Moreover, the classical action is given by the homotopy Maurer–Cartan action of its L_∞-algebra. We employ the L_∞-perspective of the Batalin–Vilkovisky formalism, with an eye to gauge theory and twistor theory. In particular, we show how quasi-isomorphisms between L_∞-algebras correspond to classical equivalences of field theories. As examples, we explore Yang–Mills theory and we discuss in great detail higher (categorified) Chern–Simons theory, providing some useful shortcuts in usually rather involved computations. Moreover, we employ the fact that the ideas of higher gauge theory can be combined with those of twistor geometry to formulate self-dual higher gauge theory. We propose a twistor space action for non-Abelian self-dual tensor field theory in six-dimensions in terms of holomorphic higher Chern–Simons theory for a Lie 2-algebra. We explicitly show how both the Abelian and non-Abelian twistor space actions descend to six-dimensional Euclidean space-time and we comment about possible advantages of the L_∞-perspective in this setting.

Keywords: L_∞-algebra, Higher gauge theory, Batalin–Vilkovisky formalism, Yang–Mills theory, Chern–Simons theory, Twistor geometry, Self-duality

Lay summary

Gauge theory is a very profound concept at the core of modern theoretical physics. Interestingly, the idea of gauge symmetry does not reflect a true property of Nature, but rather a redundancy in our description of Nature itself. Yet, such a redundancy turns out to be very useful, endowing the physical theories that enjoy it with incredible richness and complexity. In geometrical terms, gauge theory is about the parallel transport of point particles along curves. The pivotal role of gauge theory emerges when we consider it in the context of quantum physics. Quantum gauge fields are the building blocks governing the Standard Model of Elementary Particles, a physical theory describing three of the four fundamental forces in the universe. However, a quantum theory of gravity is still missing. In the long sought attempt of unifying all fundamental interactions of Nature in the same framework, String Theory is our most promising candidate. However, such theory comes with some tricky mathematics. For instance, it is not formulated in terms of point-like objects, but in terms of higher dimensional ones, known as strings and branes. Hence, a mathematical description of the dynamics of these objects requires a generalisation of ordinary gauge theory. This generalisation is known as Higher Gauge Theory and describes the parallel transport of extended objects along surfaces. The underlying language is that of category theory, which brings into the game new mathematical structures, that are more complex compared to their field theoretic counterparts. Investigating these kind of structures might contribute to a deeper understanding of String Theory. Also from a purely mathematical point of view, Higher Gauge Theory is a very interesting topic which is worth further investigation.

Acknowledgements

I would like to express my most sincere gratitude to numerous people I have met during my time at the University of Surrey.

First of all, my supervisor Martin Wolf. For your patience and your constant support. For guiding me through this journey. For your many advices and insightful discussions. I appreciate this more than you will ever know.

My collaborators, Christian Saemann and Branislav Jurco. It was a real pleasure and a privilege to collaborate and share ideas with you. Thank you Brano for hosting me in Prague and for the wonderful opportunity.

The staff and the students in the Department of Mathematics at Surrey. For creating a very pleasant environment and for making the working hours more enjoyable. In particular, thanks to Michele Bartuccelli, Cesare Tronci and Paul Skerritt.

A special thanks goes to the members of the FSG group for the many discussions and the time spent together. In particular Jan Gutowski, my co-supervisor, Alessandro Torrielli, Jock McOrist, James Grant, Andrea Prinsloo, Andrea Fontanella, and Joakim Stromvall. It was a real pleasure to be part of this group.

Finally, to the friends I have met during these years. Roberto, Emanuele and Marcel, thank you guys for the many memories.

Ci sono altre persone nella mia vita che vorrei ringraziare.

La mia famiglia. Per avermi supportato in questi anni. Per esserci sempre. Grazie di cuore.

I miei amici. Anche se ci vediamo meno ora che ognuno ha preso la sua strada per me siete molto importanti.

Laura. La cosa più bella di tutta questa esperienza è stata averti conosciuta.

Declaration

This Thesis is a result of my own efforts. The work to which it refers is based on my PhD research projects, done in collaboration with Dr. Martin Wolf, Prof. Christian Saemann, Prof. Branislav Jurčo and Tommaso Macrelli, which are the published works

1. B. Jurčo, **L. Raspollini**, C. Sämann, and M. Wolf, *L_∞-Algebras of Classical Field Theories and the Batalin-Vilkovisky Formalism*, Fortsch. Phys. 67 (2019) 1900025 [arXiv:1809.09899 [hep-th]].

2. B. Jurčo, T. Macrelli, **L. Raspollini**, C. Sämann, and M. Wolf, *L_∞-Algebras, the BV Formalism, and Classical Fields*, Fortsch. Phys. 67 (2019) 1910025 [arXiv:1903.02887 [hep-th]].

and the ongoing work

3. **L. Raspollini** and M. Wolf, *Self-dual actions from twistor space*, Work in progress.

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Sincerely yours,

Lorenzo Raspollini

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Introduction

1.1. Motivation

Nowadays, string theory arguably provides the most promising approach to quantum gravity. At the heart of string theory lies the problem of its full non-perturbative completion, that is, the proper formulation of M-theory [1] (see [2, Section 12] for exposition). Working within string theory and M-theory one faces the problem of dealing with higher homotopy structures, or *higher structures* for short. These emerge naturally as a consequence of the gauge principle and appear as higher degree objects, which are the categorification of their field theoretic counterparts. In this context, ordinary geometric and algebraic notions such as smooth manifolds, Lie algebras and principal fibre bundles need to be generalised. Making use of higher category theory, the formalism of *higher gauge theory* [3–8] allows to incorporate higher degree gauge potentials in a systematic and geometric fashion. For instance, the bosonic string couples to the Kalb–Ramond B-field, which is part of the connective structure on a gerbe [9]. This is the higher degree-2 version of the coupling of the charged particle to a Maxwell gauge potential, the latter being a connection on a principal circle bundle. In particular, higher algebraic structures become fundamental in string field theory [10]. Explicitly, homotopy algebras encode the structure of Hilbert spaces of both closed and open classical string field theory. Hence, if one believes in the fundamental nature of string theory in physics, it is hardly surprising to find out that these categorified structures make their appearance in several other physical contexts. In particular, they also play a relevant role in ordinary quantum field theory. This is made manifest by employing the classical part of the *Batalin–Vilkovisky formalism* [11–15]. Such formalism, that represents a general procedure for quantising a classical field theory, manifestly relies on higher algebraic structures. Much of the current research activity in string theory focuses on the so-called $(2,0)$-theory [16]. This is a superconformal theory in six dimensions whose field content comprises a two-form potential with self-dual field strength. A deeper comprehension of such theory would largely improve our understanding of M-theory, as many features of string theory can be understood from the perspective of the $(2,0)$-theory. A natural setting for describing self-dual theories is provided by the geometry of *twistor theory* [17]. In particular, the combination of the ideas of higher gauge theory and twistor theory in this framework appears very promising.

The aim of this Thesis is twofold. First, we investigate how homotopy algebraic structures arise in ordinary field theory, with applications towards Yang–Mills theory and (higher) Chern–Simons theory. Secondly, we study an application of the language of higher gauge theory in the context of string and M-theory, by using the ideas of twistor geometry. In the remainder of the Introduction, we shall provide a brief review of the main frameworks underlying this Thesis, namely higher gauge theory and twistor theory. The Chapter ends with a description of the various chapters and appendices.

1.1.1. Higher Gauge Theory

Higher gauge theory (HGT) provides the natural framework for describing the parallel transport for extended objects along higher dimensional surfaces, generalising what ordinary gauge theory does for point-like objects along one-dimensional paths. The transition from ordinary gauge theory to higher gauge theory is realised by a 'categorification' of the kinematical data of gauge theory, such as gauge groups and connections on principal fibre bundles. Under this operation such notions are, roughly speaking, replaced by their category-theoretic counterparts with the result that the original structure equations hold only up to isomorphisms. This procedure results in higher extensions of the concept of Lie groups, known as *Lie 2-groups* [18, 19], as well as categorified notions of principal bundles known as *principal 2-bundles* [20, 5, 6].[1] Hence, ordinary connections on principal bundles generalise to connective structures on principal 2-bundles, containing differential forms of degree greater than one. Higher categorifications lead to the notion of principal n-bundles [23].

Besides being an appealing mathematical subject, HGT finds immediate applications in string theory and M-theory, where point-like particles are replaced by higher dimensional objects such as strings, M2-branes and M5-branes. As already mentioned, the Kalb–Ramond B-field

[1] The notion of principal 2-bundle also includes the non-Abelian version of gerbes [21, 22] as special cases.

that generalises the ordinary electromagnetic potential to strings, belongs to the connective structure of some topologically non-trivial gerbe. This is a particular higher bundle with a categorified Lie group as its structure group. Exploiting this point of view, the Freed–Witten anomaly cancellation [24] in superstring theory, governed by the B-field, is understood as the principal 2-bundles version of the Dirac charge quantisation. Another physically relevant class of higher structures in this context is provided by the so-called *string structures* [25, 26]. These are higher generalisations of spin structures for spinning particles and arise from lifting the spinor group Spin(n) to the string group String(n), in the same way as the former is a lift of the special orthogonal group SO(n), as required by Dirac's theory of the electron. For instance, the Green–Schwarz anomaly cancellation mechanism [27], which plays a key role in heterotic string theory, requires to consider string structures [28, 29]. Moreover, these higher structures appeared [30] in the formulation of the non-Abelian version of the self-dual string soliton [31]. Similarly, going one step higher in the homotopy Whitehead tower of the orthogonal group, *fivebrane structures* [32, 29] have been used in the interpretation of the magnetic dual Green–Schwarz mechanism [33, 34]. While M-theory is well-known, its underlying principles are not well-understood yet. The path to a better understanding of M-theory goes through a deeper insight into its fundamental ingredients, namely M2-branes and M5-branes. The description of single M2-branes or single M5-branes have been already known for quite a long time. M2-branes are now relatively well-understood and the world-volume theory of multiple coincident M2-branes is described via three-dimensional Chern–Simons superconformal field theories [35–38]. However, M5-branes remain still mysterious. The low energy dynamics of multiple M5-branes is described by the aforementioned six-dimensional superconformal field theory with $\mathcal{N} = (2, 0)$ supersymmetry, known as the $(2, 0)$-theory. This theory plays a central role in M-theory, similar to the one that $\mathcal{N} = 4$ SYM serves in string theory, rendering its understanding of utmost importance. For instance, it provides a unifying picture in the intricate web of string theory dualities. There are rather clear indications suggesting that the $(2, 0)$-theory is a higher gauge theory, see e.g. [39] for a review.

Higher algebraic structures also appear in several other contexts within string theory. In particular, when considering the formalism of string field theory, they become omnipresent. Closed string field theory [10] is fundamentally based on *homotopy Maurer–Cartan theory*, the vastly generalised analogue of Chern–Simons theory to strong homotopy Lie algebras, also known as L_∞*-algebras*. These are L_∞-categorifications of the notion of a Lie algebra, in which the Jacobi identity holds only up to homotopies. Similarly, the construction of open string field theory is based on other higher homotopy algebras, known as A_∞*-algebras* [40, 41], that are generalisation of associative algebras. Because of the basic role that string field theory is believed to play in physics, it is quite natural to expect that the relevance of homotopy algebras also extends to ordinary field theory. As already mentioned, the connection between Lagrangian field theories and homotopy algebras is realised by the *Batalin–Vilkovisky (BV) formalism*. Mathematically, this connection is immediate. The natural framework in which HGT is formulated is that of *higher differential geometry*, where differential geometry and homotopy theory are combined. As we shall see, this is indeed the very ambient theory where the BV formalism is rooted. Essentially, any BV quantisable theory can be described by an L_∞-algebra encoding both its kinematics and its dynamics, and the original classical action is reproduced by the corresponding homotopy Maurer–Cartan action, see e.g [42, 43]. This result invites to adopt a purely algebraic point of view on field theory. This is the very perspective examined along this Thesis. For instance, minimal models of L_∞-algebras encode tree-level scattering amplitudes of the corresponding field theory. This can be also extended to loop scattering amplitudes considering quantum versions of L_∞-algebras, namely L_∞-algebras satisfying the quantum master equation in the BV formalism. Recently, many results have appeared discussing recursion relations and S-matrix in the L_∞-language. See e.g. [44] for a condensed review of the subject and pointers to references. In the second part of the Thesis, we shall also see how these ideas can be combined with those of twistor theory.

This brief overview of the role of higher dimensional structures in the realm of string and M-theory already suggests that the investigation of HGT is of great interest and worthy of further explorations. As one can easily imagine, its relevance goes far beyond this context. For instance, higher degree differential form fields arising naturally in supergravity theory are part of the connective structure of higher principal bundles, having categorified Lie groups as their structure groups. Similarly, in the context of generalised geometry, the Courant algebroid underlying T-duality is better understood within higher symplectic geometry [45]. This is the higher analog of symplectic geometry, where symplectic forms of degree greater than two are introduced. We shall refer the interested reader to the recent review [46] (and references therein) for a concise overview of the applications of higher structures in theoretical physics. For completeness, we just mention that the language of higher gauge theory has been applied to several other branches of theoretical physics, such as loop quantum gravity and condensed matter physics. The interested reader may consult the review [8] for an exhaustive introduction to the beautiful world of higher gauge theory and its applications, as well as pointers to references, which is well beyond the scope of this work.

1.1.2. Twistor Theory

Twistor theory was first introduced by Roger Penrose in the late 1960s [17] as a new mathematical framework for unifying the description of quantum field theory and gravity. In the following years, twistors played a fundamental role in the exploration of gauge and gravity theories, particularly integrable ones. At the heart of twistor geometry there is the idea of replacing space-time as a background for physical processes by an auxiliary complex space, called *twistor space*. Its principal role consists in setting up a general correspondence which translates certain important data on space-time into data on twistor space. The motivation for finding such a correspondence, in general, is the attempt to bypass the equations of physics by deriving them from the rigidity of complex geometry. Explicitly, via this correspondence, differentially constrained data on space-time correspond to differentially unconstrained complex analytic data on twistor space. Concretely, solutions to field equations translate into elements of cohomology groups. This allows for an elegant picture representing physics by complex geometry. The prime examples capturing this picture are provided by the following classic results. Via the *Penrose transform* [17, 47–49], zero-rest-mass fields on four-dimensional space-time are encoded into cohomology groups on a complex three-dimensional auxiliary space, usually called *Penrose's twistor space* [17]. This integral-geometric method represents a cornerstone of twistor theory. As an extension of this construction, Ward [50] proved that all solutions to the non-linear self-dual Yang–Mills equation [51] on flat space-time have a natural interpretation in terms of holomorphic principal bundles, subject to certain triviality conditions, over Penrose's twistor space. One often refers to this approach as the *Penrose–Ward transform*. This construction is both valid in Minkowski and Euclidean space, where it has yielded striking results [52]. Moreover, self-dual Riemannian oriented four-dimensional manifolds, that is, manifolds with self-dual Weyl tensor, were mapped to integrable complex structures on twistor space (the non-linear graviton) [53]. For a generalisation to the curved setting see [54–56]. In addition, solutions to the full ordinary Yang–Mills and supersymmetric Yang–Mills equations have been also described in terms of holomorphic data using twistor methods [57–59], by replacing Penrose's twistor space by the so-called *ambitwistor space* [57, 58]. The same setting allowed also to capture the full non-linear second-order Einstein equations and their supersymmetric extensions [60–63]. We invite the reader interested in detailed expositions on twistor theory and its applications to consult, for example, the textbooks [64–68] or the recent reviews [69–72].

However, despite many interesting initial achievements, twistor theory has fallen well short of its original aspirations. Indeed, its next contributions have been mainly limited to the context of pure mathematics, where twistors have been used as tools for the study of integrable systems and geometry, see e.g. [67, 73–75]. One of the most significant setbacks[1] was represented by the difficulty to use twistor theory to compute physical observables in quantum field theory, like scattering amplitudes or cross-sections. Such scenario dramatically changed in the early 2000s. In 2003 Witten realised [77] that the tree-level scattering amplitudes of planar maximally supersymmetric Yang–Mills theory in four dimensions ($\mathcal{N} = 4$ SYM) could be computed by combining string perturbation theory with twistor geometry. This result established a meaningful connection between twistors and quantum field theory, stimulating a renewed and broader interest in twistor theory. Twistor string theory led to new powerful approaches to Yang–Mills theory such as worldsheet formulae for all tree-level amplitudes in $\mathcal{N} = 4$ SYM [78–80], the MHV formalism [81] and the Britto–Cachazo–Feng–Witten recursion relations [82]. See e.g. [83] for a review. In the meanwhile, a twistor space action functional for Yang–Mills was introduced in [84, 85]. This led to a proper derivation of twistor string theory and constructive proof of related formulas, such as the MHV diagram formalism [86, 87] and scattering amplitude/Wilson loop duality [88, 89, 70]. Moreover, it yields perturbative Yang–Mills theory without the unwanted contributions from conformal supergravity [90], allowing to compute physical observables to all orders in perturbation theory. Twistor actions have been also introduced for many field theories, including for instance conformal gravity [84, 91], self-dual (super-)gravity [92, 93] and (super-)Yang-Mills-Higgs theory in three dimensions [94]. See e.g. [95] for a review and for more references.

Considering the great success of M2-brane models [35, 37, 38], it appears natural to wonder about the role of twistor theory in the description of solutions of more general gauge theories. In the context of M-theory, a crucial role is played by the $(2, 0)$-theory arising on the world-volume of parallel M5-branes. Besides its relevance in the web of string dualities, such a theory reduces to $\mathcal{N} = 4$ SYM after compactifying down to four dimensions. Moreover, the

[1] Another long-standing issue is represented by the so-called 'googly problem'. See e.g. [76].

self-dual string equation in four-dimensions [31], a BPS equation describing configurations of M2-branes ending on M5-branes, can be obtained reducing the six-dimensional theory of self-dual three-forms. We have already seen that an appropriate language for describing M5-branes is provided by higher gauge theory. It turns out that a promising approach towards finding a classical description of the $(2,0)$-theory is to combine the ideas of higher gauge theory and twistor geometry. In particular, a twistorial description of self-dual tensor field theories can be obtained along this line. The relevant twistor space in this context is given by the complex six-dimensional twistor space considered in [96–99]. Such a space encodes the description of six-dimensional self-dual 3-forms in terms of holomorphic Abelian gerbes [97,98]. Moreover, solutions to the field equations for the non-Abelian $\mathcal{N}=(2,0)$ tensor multiplet in six dimensions, taking value in some categorified Lie group, have been obtained via a Penrose–Ward transform of certain higher holomorphic principal bundles over this twistor space [99–103]. This identification is lifted to the level of an L_∞-quasi-isomorphism. Interestingly, this description also yields a twistor space actions, as shown in [97–99] for the Abelian case. These might represent the twistor space version of the space-time action of Pasti, Sorokin & Tonin [104–107] or Sen [108, 109]. Moreover, a non-Abelian extension in terms of higher holomorphic Chern–Simons theory on twistor space has been constructed in [43]. Higher holomorphic Chern–Simons theories can be of use even in the case of ordinary field theories. In [110], an on-shell equivalence between $\mathcal{N}=4$ SYM in four-dimensions and holomorphic higher CS theory for a Lie 3-algebra on the ambitwistor superspace of [77] was obtained. This would allow, at least in principle, for computing $\mathcal{N}=4$ SYM scattering amplitudes within twistor string theory. We believe that the application of the homotopic algebraic perspective at the level of twistor space might help to shed some light on particular aspects of certain physical theories.

1.2. Outline and main results

Here, we shall briefly expose the contents of the Thesis, trying to facilitate the reader for a better comprehension of the manuscript. It is our intention to be highly self-contained in the exposition. In this perspective, each chapter shall start with an introductory section, explaining the motivation behind it and its structure. However, this Section may provide some further help. Chapters 2-5 are based on [42, 43], while Chapter 7 is based on [43] and [111]. This Thesis is organised as follows:

Chapter 2. This Chapter consists of a detailed review of L_∞-algebras. These objects arise as a particular categorification of the notion of a Lie algebra that is described in terms of graded vector spaces and higher brackets. Much more directly, L_∞-algebras are introduced by means of certain geometric structures called Q-*manifolds*. In this approach, one defines L_∞-algebras by generalising the equivalent definition of a Lie algebra via its differential graded algebra (dga), known as the *Chevalley-Eilenberg algebra*. In addition, the Batalin–Vilkovisky formalism can be formulated in the same language. In order to formulate physical theories, we shall focus on L_∞-algebras endowed with an inner product, which are called *cyclic* L_∞-*algebras*. After that we define L_∞-morphisms and, in particular, the very important notion of *quasi-isomorphisms* of L_∞-algebras, arguing that quasi-isomorphisms induce equivalences on the space of L_∞-algebras. Then, we recall some relevant structural theorems of L_∞-algebras. These results come in handy in the homological algebraic approach to field theory. In particular, the *minimal model theorem* basically tells us how to restrict from the kinematical data of a certain gauge theory to its physical states. We shall make extensive use of it in Chapter 5. We conclude by defining a representation of an L_∞-algebra, a concept needed in order to define higher supersymmetric field theories with matter content.

Chapter 3. Here, we discuss homotopy Maurer–Cartan theory by using the language of L_∞-algebras. Ordinary gauge transformations and their higher versions are derived. Then, for cyclic L_∞-algebras, the homotopy *Maurer–Cartan action* is constructed. We show that quasi-isomorphisms of L_∞-algebras induce isomorphisms of moduli spaces of Maurer–Cartan elements. We further observe that homotopy Maurer–Cartan theory, just as ordinary Chern–Simons theory, always allows for a supersymmetric extension by auxiliary fields. This is important if one wishes to compute path integrals via supersymmetric localisation techniques. Most importantly, our perspective is adapted to an application towards higher gauge theory. We see how, given an L_∞-algebra, the kinematical data of the corresponding higher gauge theory can be immediately derived. Interestingly, by considering the cyclic L_∞-algebra given by the tensor product of a gauge algebra and the de Rham (Doulbeault) complex, we can directly make contact with ordinary (holomorphic) Chern–Simons theory in $d = 3$. In this setting, the higher dimensional generalisation is then immediate and makes use of Lie n-algebras.

Chapter 4. The generalities of the Batalin–Vilkovisky formalism together with its application to homotopy Maurer–Cartan theory are discussed in this Chapter. The first part is dedicated to a detailed analysis of the classical BV formalism for ordinary field theory along with its formulation in the language of symplectic Q-manifolds. Such a formalism allows for the quantisation of classical field theories with open gauge symmetries. Readers familiar with the formalism may just skip this part. Then, we explain how this framework provides the natural connection between Lagrangian field theories and L_∞-algebras, even for theories without gauge symmetries. Mathematically, the BV formalism consists of a two-step resolution of the algebra of classical observables. This procedure yields a differential graded commutative algebra, called the *BV complex*, that is one of the equivalent descriptions of an L_∞-algebra. Moreover, the action of such a theory can be recast as a Maurer–Cartan–Batalin–Vilkovisky action. Hence, any variational field theory is a homotopy Maurer–Cartan theory. We also see how the BV formalism is applied to homotopy MC theory and observe that, at least formally, the homotopy MC action satisfies the quantum master equation.

Chapter 5. This Chapter is devoted to some concrete examples, analysing the L_∞-structure underlying classical field theory, with a particular focus to Yang–Mills and (higher) Chern–Simons theory. In the L_∞-framework, an equivalence of classical field theories is given by a quasi-isomorphism of L_∞-algebras. Firstly, we discuss scalar field theory, providing an example where the L_∞-structure underlies a field theory without gauge symmetry. Secondly, we consider Yang–Mills theory in its first and second order formulation. We construct their minimal models, yielding a minimal representation of the classical dynamical data and we show that the corresponding L_∞-algebras are quasi-isomorphic, providing an explicit quasi-isomorphism. In particular, we construct higher Chern–Simons theory for a Lie n-algebra and we perform detailed computations for the $d = 3, 4$ cases. We develop the BV formalism and we construct the minimal model of the corresponding gauge L_∞-algebra. Finally, after a brief summary of the AKSZ construction, we discuss supersymmetric extensions of higher CS theory and YM theory in the L_∞-framework and, because of its relevance to M-theory, we also briefly review the Bagger–Lambert–Gustavsson model in this setting.

Chapter 6. Here we present an example where the ideas of higher gauge theory have been proficiently combined with the language of twistor theory. In particular, we review how certain conformal gauge theories in six dimensions can be reformulated in terms of holomorphic categorified principal bundles over an appropriate twistor space. First, we briefly discuss zero-rest-mass fields by using the six-dimensional spinor-helicity formalism. Then, we review the geometry of the relevant twistor space, P^6, and we discuss Euclidean reality conditions. We proceed by reviewing the Penrose transform, mapping cohomology groups on twistor space to certain chiral zero-rest-mass fields on space-time. Finally, the Penrose–Ward transform is also discussed, both in the Abelian and non-Abelian setting. The former associates Abelian self-dual 3-forms in six dimensions to certain holomorphic Abelian gerbes over P^6. The latter establishes a correspondence between certain higher groupoid bundles over P^6 and six-dimensional non-Abelian self-dual tensor field theories.

Chapter 7. This Chapter is based on the ongoing work [111]. Here, we discuss twistor space actions on P^6 for chiral two-form fields in six dimensions, both in the Abelian and non-Abelian case. In particular, the non-Abelian twistor space action comes from higher gauge theory. We are interested in establishing an off-shell correspondence between these actions and space-time actions such as the PST action or the Sen action. In particular, we provide a pedagogical review of Sen's mechanism. Indeed, such a formalism emerges naturally from homotopic algebraic considerations and it looks particularly promising for our purposes. Explicitly, we show how the twistorial actions descend to Euclidean six-dimensional space-time. This is done in three steps. We first impose Euclidean reality conditions on twistor space, we expand the fields in an adapted basis by using non-holomorphic coordinates and finally we integrate along the fibres of the twistor fibration. This lays the groundwork for a further analysis by using the language of L_∞-algebras that we plan to conduct in the near future.

Appendices. The Thesis ends with several appendices. We use them to collect many computational proofs, in order to avoid them cluttering our discussion, together with definitions and conventions which are not introduced in the body of the text.

Appendix A describes L_∞-algebras in the codifferential graded coalgebras perspective,[1] where the higher products are packaged in a single codifferential and we discuss morphisms between L_∞-algebras as coalgebra morphisms. In Appendix B we briefly discuss cochain complexes underlying L_∞-algebras and *abstract Hodge–Kodaira decomposition*. Appendix C contains some rather technical computational proofs related to homotopy Maurer–Cartan theory. Frequently, these results are provided in more than one of the possible descriptions of L_∞-algebras, as sometimes one picture is more illuminating than the others. In Appendix D

[1] This is the proper picture for performing perturbation theory.

we introduce some basic notions of category theory and sheaves. We also summarise recent developments in quasi-groups with regard to higher gauge theory with Lie quasi-groupoids as gauge structure. In Appendix E we collect a few basic notions of complex geometry and we review important concepts behind twistor theory, such as Čech–Dolbeault correspondence and the Penrose transform. Finally, Appendix F collects calculations on twistor space. We introduce local coordinates on P^6, rewriting useful quantities and we discuss some aspects of fibre integration.

2

L_∞-Algebras

2.1. Motivation

Mathematically, the local description of gauge theories is based on Lie algebra-valued differential forms. Let M be a manifold and $\mathfrak{g}$ a Lie algebra, then a corresponding gauge theory has a gauge potential $A \in \Omega^1(M, \mathfrak{g}) := \Omega^1(M) \otimes \mathfrak{g}$ and a curvature $F := \mathrm{d}A + \frac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g})$. Moreover, gauge parameters, Bianchi identities, Noether currents, and equations of motion also involve elements of $\Omega^\bullet(M, \mathfrak{g})$.

To obtain a natural (i.e. category theoretical) description of gauge theory, we should therefore bring together differential forms and Lie algebras in a common framework. This framework was provided long ago by Henri Cartan [112, 113], by moving from a Lie algebra to its dual differential graded algebra, known as the *Chevalley–Eilenberg algebra*. In particular, the differential graded algebras in which we are interested arise as functions on particular graded manifolds. On these manifolds we have a vector field Q that induces a differential on the algebra of functions. Henceforth, they are referred to as *Q-manifolds*, cf. [114].

Formulating gauge theories using differential graded algebras arising from Q-manifolds has several advantages. Firstly, one can define vast generalisations of ordinary gauge theory [7] which appear naturally in string and M-theory in various contexts. Secondly, because of its mathematical naturality, it is not surprising that a powerful framework such as the Batalin–Vilkovisky (BV) formalism is best formulated in this language [115]. Thirdly, this is how they appear directly in the Alexandrov–Kontsevich–Schwarz–Zaboronsky (AKSZ) construction [114].

In this Chapter, we shall provide a few mathematical tools. We shall introduce graded manifolds, Q-manifolds and finally L_∞-algebras and morphisms between them. Our perspective will be oriented towards discussing applications in higher gauge theory and follows faithfully [42, 43].

2.2. Differential graded algebras

To set up the stage, we shall first review some definitions about graded structures. For more details on this topic, see e.g. [116, 117].

$\mathbb{Z}$-graded vector spaces and shifts. By a *$\mathbb{Z}$-graded vector space* we mean a direct sum $\mathsf{V} = \bigoplus_{k\in\mathbb{Z}} \mathsf{V}_k$ of vector spaces over a field of characteristic zero.[1] The V_k are called the components of V of degree k and the degree of a homogeneous element $v \in \mathsf{V}$ is denoted by $|v| \in \mathbb{Z}$. In the following we shall make intensive use of the degree-shift operation. For a $\mathbb{Z}$-graded vector space V, we define the *degree shift* by $l \in \mathbb{Z}$ according to

$$\mathsf{V}[l] \;=\; \bigoplus_{k\in\mathbb{Z}} (\mathsf{V}[l])_k \quad \text{with} \quad (\mathsf{V}[l])_k \;:=\; \mathsf{V}_{k+l} \quad \text{for} \quad l \;\in\; \mathbb{Z}\,. \tag{2.1}$$

This convention (which is one of two commonly used ones) indicates the shift of the *coordinate functions* and the *opposite direction* of the shift of the vectors themselves. For example, given an ordinary vector space V, the degree-shifted vector space $V[1]$ consists of vectors v of degree -1, since only $(V[1])_{-1} = V$ is non-trivial. Moreover, we define the dualisation by flipping the signs of the shifts. Explicitly, we shall denote the dual of a $\mathbb{Z}$-graded vector space V by $\mathsf{V}^* = \oplus_{k\in\mathbb{Z}}(\mathsf{V}^*)_k$, with homogeneous subspaces $(\mathsf{V}^*)_k := (\mathsf{V}_{-k})^*$, for all $k \in \mathbb{Z}$. Let V, W be two $\mathbb{Z}$-graded vector spaces. The direct sum $\mathsf{V}\oplus\mathsf{W}$ is a graded vector space defined by $\mathsf{V}\oplus\mathsf{W} = \bigoplus_{k\in\mathbb{Z}}(\mathsf{V}\oplus\mathsf{W})_k$, where $(\mathsf{V}\oplus\mathsf{W})_k := \mathsf{V}_k\oplus\mathsf{W}_k$. Similarly, the tensor product $\mathsf{V}\otimes\mathsf{W}$ is again a $\mathbb{Z}$-graded vector space whose degree k component is given by $(\mathsf{V}\otimes\mathsf{W})_k := \bigoplus_{i+j=k}\mathsf{V}_i\otimes\mathsf{W}_j$. Both constructions are associative. Note that $(\mathsf{V}\otimes\mathsf{W})[l] = \mathsf{V}[l]\otimes\mathsf{W} = \mathsf{V}\otimes\mathsf{W}[l]$ and $(\mathsf{V}[l])^* = \mathsf{V}^*[-l]$ for all $l\in\mathbb{Z}$. Finally, a *morphism* of $\mathbb{Z}$-graded vector spaces $f : \mathsf{V}\to\mathsf{W}$ is a linear map which preserves the degree of homogeneous elements: $f(\mathsf{V}_k)\subseteq\mathsf{W}_k$, for all $k\in\mathbb{Z}$. The category of $\mathbb{Z}$-graded vector spaces is a symmetric monoidal category. The symmetric monodical structure is inherited from the category of ordinary vector

[1] Here and in the rest of the Thesis all the algebraic structures have to be considered over a field $\mathbb{K}$ of characteristic zero. We always have in mind $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

spaces. In particular, the symmetric braiding, for any $\mathbb{Z}$-graded vector spaces V and W, is given by the the graded linear isomorphism $B_{\mathsf{V},\mathsf{W}} : \mathsf{V} \otimes \mathsf{W} \to \mathsf{W} \otimes \mathsf{V}$, defined by

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \ , \tag{2.2}$$

for v, w homogeneous elements. The choice of the sign pre-factor in the definition of the braiding takes the name of *Koszul convention*.

A note about coordinates. Let us already now stress an important point for our whole discussion. With respect to a basis τ_α of degree -1 of the above example $V[1]$ of a grade-shifted ordinary vector space V, the coordinate functions $\xi^\alpha : \mathsf{V}[1] \to \mathbb{R}$ are of degree 1,

$$\xi^\alpha \ : \ V[1] \ \to \ \mathbb{R} \quad \text{with} \quad \xi^\alpha(v) \ = \ \xi^\alpha(v^\beta \tau_\beta) \ := \ v^\beta \underbrace{\xi^\alpha(\tau_\beta)}_{=:\,\delta^\alpha_\beta} \ = \ v^\alpha \ . \tag{2.3}$$

There is now much room for confusion between the coordinates v^α and the coordinate functions ξ^α; Nick Woodhouse [118] coined the term *first fundamental confusion of calculus* for this phenomenon. While in the ungraded case, this confusion is usually reasonably controlled, it can get out of hands in the graded case, since the degree of the object and the coordinate functions acting on it will be inverse to each other.

In the context of both the Becchi–Rouet–Stora–Tyutin (BRST) and the BV formalisms, this problem with degrees is exacerbated by the fact that there is an additional, implicit shift in degree by -1. For example, consider a gauge parameter $c \in \Omega^0(M, \mathfrak{g})$ and a gauge potential $A \in \Omega^1(M, \mathfrak{g})$, where, as before, M is a manifold and $\mathfrak{g}$ a Lie algebra. These fields belong to the graded vector space

$$\mathsf{V} \ = \ \mathsf{V}_0 \oplus \mathsf{V}_1 \ := \ \Omega^0(M, \mathfrak{g}) \oplus \Omega^1(M, \mathfrak{g}) \ . \tag{2.4}$$

The corresponding BRST complex, however, is that of $\mathsf{V}[1] = \mathsf{V}[1]_{-1} \oplus \mathsf{V}[1]_0$. Consequently, we obtain coordinate functions of degrees 1 and 0,

$$c^\alpha(x) \ : \ \Omega^0(M, \mathfrak{g})[1] \ \to \ \mathbb{R} \quad \text{and} \quad A^\alpha_\mu(x) \ : \ \Omega^1(M, \mathfrak{g})[1] \ \to \ \mathbb{R} \ , \tag{2.5}$$

with τ_α and $\mathrm{d}x^\mu$ are bases[1] of $\mathfrak{g}$ and $\Omega^1(M)$, respectively. For convenience, we shall often contract the coordinate functions with the basis of V to arrive at the *contracted coordinate*

[1]Note that $\mathrm{d}x^\mu$ is a basis of $\Omega^1(M)$ regarded as a module over $\mathscr{C}^\infty(M)$. More appropriately, we should be using the infinite-dimensional basis of $\Omega^1(M)$ regarded as a vector space over $\mathbb{R}$. To avoid the related, potentially distracting technicalities, we are slightly sloppy here.

functions, which we denote by the same letters c and A as customary in the discussion of BRST/BV quantisation. These contracted coordinate functions are always of total degree 1. Whether we mean vectors or their coordinate functions should always be clear from the context, and we hope that no confusion will arise. The degree of the vectors c and A in V will be called the L_∞-*degree*, while the degree of the (uncontracted) coordinate functions $c^a(x)$ and $A^\alpha_\mu(x)$ will be called the *ghost degree*. The latter agrees with the general nomenclature.

Commutative dg-algebras. A *differential graded commutative*[1] *algebra* (or a dg(c)-algebra for short) is an associative unital commutative algebra A which is simultaneously a $\mathbb{Z}$-graded algebra and a differential algebra in a way that all structures are compatible. Specifically, the $\mathbb{Z}$-grading means that we have the decomposition $\mathsf{A} = \bigoplus_{k\in\mathbb{Z}} \mathsf{A}_k$ and non-zero elements of A_k will be called *homogeneous* and *of degree* $k \in \mathbb{Z}$. In addition, the product $\mathsf{A} \times \mathsf{A} \to \mathsf{A}$ is graded commutative,

$$a_1 a_2 \;=\; (-1)^{|a_1||a_2|} a_2 a_1 \;, \tag{2.6}$$

for $a_{1,2} \in \mathsf{A}$ of homogeneous degrees $|a_{1,2}| \in \mathbb{Z}$. Being differential means that A is equipped with differential derivations $\mathrm{d}_k : \mathsf{A}_k \to \mathsf{A}_{k+1}$ of homogeneous degree 1, which we collectively denote by d. Specifically, d satisfies $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0 \Leftrightarrow \mathrm{d}_{k+1} \circ \mathrm{d}_k = 0$ and obeys the graded Leibniz rule

$$\mathrm{d}(a_1 a_2) \;=\; (\mathrm{d}a_1)a_2 + (-1)^{|a_1|} a_1(\mathrm{d}a_2) \tag{2.7}$$

for $a_{1,2} \in \mathsf{A}$ and a_1 of homogeneous degree $|a_1| \in \mathbb{Z}$. We shall write (A, d) in the following to indicate a dg-algebra.

Cochain complexes. Notice that d gives A the structure of a cochain complex,

$$\cdots \xrightarrow{\mathrm{d}} \mathsf{A}_{-1} \xrightarrow{\mathrm{d}} \mathsf{A}_0 \xrightarrow{\mathrm{d}} \mathsf{A}_1 \xrightarrow{\mathrm{d}} \cdots \;, \tag{2.8}$$

and its cohomology $H^\bullet(\mathsf{A}, \mathrm{d})$ is a graded algebra. For convenience, we shall use the cochain convention and use the terms *cochain* and *cohomology* versus the slightly more common *chain* and *homology*, see the remark at the beginning of Appendix B. More abstractly, we can define a dg-algebra also as a monoid object in the monoidal category of (co)chain complexes.

[1]We shall only be concerned with commutative algebras and hence drop the adjective 'commutative' in the following.

de Rham complex. The prime example in view of applications to gauge theory is the differential graded algebra given by the de Rham complex $(\Omega^\bullet(M), \mathrm{d})$ on a d-dimensional manifold M,

$$\Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^d(M) , \tag{2.9}$$

with the wedge product being the multiplication and d being the exterior derivative.

Dolbeault complex. Let now X be a complex manifold of dimension d.[1] The Dolbeault complex of X is the cochain complex

$$\Omega^{0,0}(\mathsf{X}) \xrightarrow{\bar\partial} \Omega^{0,1}(\mathsf{X}) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{0,d}(\mathsf{X}) , \tag{2.10}$$

where $\Omega^{0,\bullet}(\mathsf{X})$ is the graded vector space of smooth antiholomorphic forms over X and $\bar\partial$ is the antiholomorphic part of the de Rham differential given by the complex structure. Hence, $(\Omega^{0,\bullet}(\mathsf{X}), \bar\partial)$ is naturally a (commutative) dg-algebra. See Appendix E for details.

Another fundamental example is that of the Chevalley–Eilenberg algebra described in Section 2.4..

Morphisms of dg-algebras. A *morphism* $f : (\mathsf{A}, \mathrm{d}) \to (\mathsf{A}', \mathrm{d}')$ between two dg-algebras (A, d) and $(\mathsf{A}', \mathrm{d}')$ is a collection f of degree 0 maps $f_k : \mathsf{A}_k \to \mathsf{A}'_k$ for all $k \in \mathbb{Z}$, which respects the differential in the sense that $f \circ \mathrm{d} = \mathrm{d}' \circ f \Leftrightarrow f_{k+1} \circ \mathrm{d}_k = \mathrm{d}'_k \circ f_k$. An *isomorphism of dg-algebras* is an invertible morphism. This notion of isomorphism will turn out to be too strict for our purposes, mainly due to our interpretation of dg-algebras as categorified Lie algebras. More appropriately, we should use *quasi-isomorphisms of dg-algebras*. We shall return to this point and explain it in detail in Section 2.5.. We shall exhaust the subject by providing a useful example.

Tensor algebras and shift isomorphism. The (real) tensor algebra of a graded vector space V is defined by[2]

$$\bigotimes{}^\bullet \mathsf{V} := \mathbb{R} \oplus \mathsf{V} \oplus (\mathsf{V} \otimes \mathsf{V}) \oplus \cdots = \bigoplus_{k \geq 0} \bigotimes{}^k \mathsf{V} . \tag{2.11a}$$

[1] We shall always use sans serif capital letters, such as $\mathsf{X}, \mathsf{Y}, \mathsf{M}$ to denote complex manifolds.

[2] Note that $\bigotimes^\bullet \mathsf{V}$ is a bigraded vector space on which the tensor product naturally describes the unital algebra structure that is compatible with both gradings.

It has two totally graded symmetric and graded antisymmetric subalgebras,

$$\begin{aligned} \bigodot^\bullet \mathsf{V} &:= \mathbb{R} \oplus \mathsf{V} \oplus (\mathsf{V} \odot \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 0} \bigodot^k \mathsf{V} \,, \\ \bigwedge^\bullet \mathsf{V} &:= \mathbb{R} \oplus \mathsf{V} \oplus (\mathsf{V} \wedge \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 0} \bigwedge^k \mathsf{V} \,, \end{aligned} \tag{2.11b}$$

while their reduced counterparts are[1]

$$\begin{aligned} \bigodot^\bullet_0 \mathsf{V} &:= \mathsf{V} \oplus (\mathsf{V} \odot \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 1} \bigodot^k \mathsf{V} \,, \\ \bigwedge^\bullet_0 \mathsf{V} &:= \mathsf{V} \oplus (\mathsf{V} \wedge \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 1} \bigwedge^k \mathsf{V} \,. \end{aligned} \tag{2.11c}$$

We introduce the notion of a *shift isomorphism*

$$s : \mathsf{V} \to \mathsf{V}[1] \,, \tag{2.12}$$

which lowers the degree of every element of V, that is $s : \mathsf{V}_k \to (\mathsf{V}[1])_{k-1}$. It induces an isomorphism of graded algebras,

$$\begin{aligned} s^\bullet &: \bigwedge^\bullet \mathsf{V} \to \bigodot^\bullet \mathsf{V}[1] \,, \\ s^{\otimes i} &: v_1 \wedge \ldots \wedge v_i \mapsto (-1)^{\sum_{j=1}^{i-1}(i-j)|v_j|} s v_1 \odot \cdots \odot s v_i \end{aligned} \tag{2.13}$$

for $v_1, \ldots, v_i \in \mathsf{V}$. The sign arises from the usual *Koszul sign rule*,

$$(s \otimes s)(v_1 \otimes v_2) := (-1)^{|v_1|} s v_1 \otimes s v_2 \tag{2.14}$$

for $v_1, v_2 \in \mathsf{V}$. The inverse map is given by

$$(s^{\otimes i})^{-1} = (-1)^{\frac{1}{2}i(i-1)} (s^{-1})^{\otimes i} \,. \tag{2.15}$$

This shift isomorphism will be crucial in treating Lie algebras and higher Lie algebras as differential graded algebras.

Differential graded Lie algebras. We define a *graded Lie algebra* to be a graded vector space $\mathfrak{g}$[2] endowed with a bilinear degree 0 bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying graded skew-symmetry and graded Jacobi identity, that is

$$[v,w] = -(-1)^{|v||w|}[w,v] \,, \quad (-1)^{|v||z|}[v,[w,z]] + (-1)^{|v||w|}[w,[z,v]] + (-1)^{|w||z|}[z,[v,w]] = 0 \,, \tag{2.16a}$$

[1] Other common notations are $\bar{\bigodot}^\bullet \mathsf{V}$ and $\bar{\bigwedge}^\bullet \mathsf{V}$.

[2] In the following, with $\mathfrak{g}$ we shall denote a Lie algebra as well as the graded vector space underlying a graded Lie algebra or a dg-Lie algebra. It will always be clear by the context which one we are referring to.

for homogeneous elements $v, w, z \in \mathfrak{g}$. A *differential graded Lie algebra (dg-Lie algebra)* is the data of a graded Lie algebra $\mathfrak{g}$ together with a differential d, such that

$$\mathrm{d}[v,w] = [\mathrm{d}v, w] + (-1)^{|v|}[v, \mathrm{d}w] \ , \tag{2.16b}$$

for homogeneous elements $v, w \in \mathfrak{g}$. The graded Leibniz rule (2.16b) implies that the bracket induces the structure of a graded Lie algebra on the cohomology $H^\bullet(\mathfrak{g})$ of $\mathfrak{g}$. A morphism of dg-Lie algebras is a morphism of the underlying graded vector spaces that is compatible with brackets and differentials. Dg-Lie algebras assemble into a category called DGLA. As we shall see in Section 2.4., L_∞-algebras can be seen as generalisation of differential graded Lie algebras in which the Jacobi identity is satisfied only up to homotopies. The following result holds. Let $\mathfrak{g}$ be a dg-Lie algebra and A a dg-algebra. The tensor product $\mathfrak{g} \otimes \mathsf{A}$ is a dg-Lie algebra and it is functorial in both arguments. In particular $\mathsf{V} \otimes \mathsf{A}$ is equipped with the bracket

$$[v \otimes a, w \otimes b] = (-1)^{|a||w|}[v,w] \otimes ab \ , \tag{2.17}$$

for $v, w \in \mathfrak{g}$ and $a, b \in \mathsf{A}$. We shall come back to this examples later in this Chapter. We just mention here that dg-Lie algebras are fundamental objects in deformation theory. According to a general paradigm in mathematics, adopted by Quillen, Deligne, Drinfeld, Kontsevich and others, every deformation problem in characteristic 0 is naturally governed by a dg-Lie algebra via the Maurer-Cartan equation modulo gauge action. See e.g. [119, 120] for details on the subject.

2.3. Q-Manifolds

L_∞-algebras are most straightforwardly introduced by means of Q-manifolds. Here, we shall review the language of Q-manifolds and their relation to differential graded algebras and L_∞-algebras.

Motivation. To motivate the notion of Q-manifolds, let us recall the fact that differential forms $\Omega^\bullet(M)$ on a d-dimensional smooth manifold M can be regarded as the smooth functions $\mathscr{C}^\infty(T[1]M)$ on the degree-shifted tangent bundle $T[1]M$. Indeed, working locally with coordinates x^μ, $\mu = 1, \ldots, \dim(M)$, on M and coordinates ξ^μ on the fibres of $T[1]M$, functions on $T[1]M$ are simply polynomials in ξ^μ, that is, they are of the form $f(x,\xi) = f^\circ(x) + \xi^\mu f_\mu(x) + \frac{1}{2}\xi^\mu\xi^\nu f_{\mu\nu}(x) + \cdots \in \mathscr{C}^\infty(T[1]M)$. Identifying ξ^μ with $\mathrm{d}x^\mu$

amounts to the identification $\mathscr{C}^\infty(T[1]M) \cong \Omega^\bullet(M)$. Under this identification, the de Rham differential d becomes the vector field $Q : \mathscr{C}^\infty(T[1]M) \to \mathscr{C}^\infty(T[1]M)$, defined by $Q = \xi^\mu \frac{\partial}{\partial x^\mu}$ and satisfying $Q^2 = 0$. The manifold $T[1]M$, together with Q, forms an important example of a *Q-manifold* [115, 121, 114]. These Q-manifolds provide a very efficient way of encoding a categorified Lie algebra in the form of an L_∞-algebra or a categorified Lie algebroid in the form of an L_∞-algebroid, as we shall see later. Moreover, such a geometrical point of view brings significant simplification when considering morphisms between L_∞-algebroids or L_∞-algebras. Let us first recall some preliminary notions.

$\mathbb{Z}$-graded manifolds with body $\mathbb{R}^d$. Consider $\mathbb{R}^d$ as a manifold. Furthermore, let V be a $\mathbb{Z}$-graded vector space. We may enlarge the ring of smooth functions $\mathscr{C}^\infty(\mathbb{R}^d)$ on $\mathbb{R}^d$ by considering the tensor product $\bigodot^\bullet \mathsf{V}^* \otimes \mathscr{C}^\infty(\mathbb{R}^d)$, where $\bigodot^\bullet$ denotes the graded symmetric tensor algebra (2.11b). We call the result the algebra of functions

$$\mathscr{C}^\infty(M) \ := \ \bigodot{}^\bullet \mathsf{V}^* \otimes \mathscr{C}^\infty(\mathbb{R}^d) \tag{2.18}$$

on the *$\mathbb{Z}$-graded manifold* M and the underlying ordinary manifold $\mathbb{R}^d$ is called the *body* M° of M. By *coordinates* on the $\mathbb{Z}$-graded manifold M, we mean a set of ordinary coordinates on M° together with a set of generators of $\bigodot^\bullet V^*$, say ξ^α with $\alpha \in I$ for some index set I.

Elements of $\mathscr{C}^\infty(M)$ are clearly polynomials in the generators ξ^α whose coefficients are functions on M°,

$$f(x,\xi) \ = \ f^\circ(x) + \xi^\alpha f_\alpha(x) + \tfrac{1}{2!}\xi^\alpha\xi^\beta f_{\alpha\beta}(x) + \cdots \tag{2.19}$$

with $f^\circ, f_\alpha, f_{\alpha\beta}, \ldots \in \mathscr{C}^\infty(M^\circ)$. We shall make extensive use of the natural decomposition

$$\mathscr{C}^\infty(M) \ \cong \ \bigoplus_{k\in\mathbb{Z}} \mathscr{C}^\infty_k(M) \ , \tag{2.20}$$

where $\mathscr{C}^\infty_k(M)$ are the *homogeneous functions of degree* k. These are spanned by the monomials of $\mathbb{Z}$-degree k, i.e. monomials $\xi^{\alpha_1}\cdots\xi^{\alpha_n} f_{\alpha_1\cdots\alpha_n}(x)$, where $f_{\alpha_1\cdots\alpha_n}(x) \in \mathscr{C}^\infty(M^\circ)$ and $|\xi^{\alpha_1}| + \cdots + |\xi^{\alpha_n}| = k$.

For most of our purposes, the above local picture (2.18) of the ring of functions is sufficient. For some aspects, as e.g. the correct definition of morphisms between $\mathbb{Z}$-graded manifolds, however, the full mathematical definition[1] can be helpful. We therefore recall it in the following.

[1]This definition also resolves naive paradoxes concerning super and graded manifolds.

General $\mathbb{Z}$-graded manifolds. The proper definition of Q-manifolds requires the machinery of locally ringed spaces. A *ringed space* M is a pair $(|M|, \mathscr{S}_M)$, where $|M|$ is a topological space and $\mathscr{S}_M$ a sheaf of rings on $|M|$, called the *structure sheaf* of M.[1] A *locally ringed space* is a ringed space $(|M|, \mathscr{S}_M)$ such that all stalks of $\mathscr{S}_M$ are local rings, that is, they have unique maximal ideals.

A *morphism* of $(|M|, \mathscr{S}_M) \to (|M'|, \mathscr{S}_{M'})$ of locally ringed spaces is defined to be a pair $(f, f^\sharp)$, where $f : |M| \to |M'|$ is a morphism of topological spaces and $f^\sharp : \mathscr{S}_{M'} \to f_*\mathscr{S}_M$ a comorphism of local rings, that is, a map that respects the maximal ideals. Here, $f_*\mathscr{S}_M$ is the zeroth direct image of $\mathscr{S}_M$ under f, that is, for any open subset U' of $|M'|$ there is a comorphism $f^\sharp_{U'} : \mathscr{S}_{M'}|_{U'} \to \mathscr{S}_M|_{f^{-1}(U')}$. If the structure sheaves carry extra structure such as a $\mathbb{Z}$-grading, then we require the morphism to respect this structure.

For instance, an ordinary smooth manifold M can be defined as a locally ringed space $(|M|, \mathscr{S}_M)$ for a topological manifold $|M|$ such that for each $x \in |M|$ there is an open neighbourhood $U \ni x$ and an isomorphism of locally ringed spaces $(U, \mathscr{S}_M|_U) \cong (U', \mathscr{C}^\infty_{U'})$ where $\mathscr{C}^\infty_{U'}$ is the sheaf of smooth functions on the open set $U' \subseteq \mathbb{R}^d$. The stalk of $\mathscr{S}_M$ at a point $x \in |M|$ is the set of all germs of smooth functions at $x \in |M|$, and the maximal ideal of the stalk are the functions vanishing at $x \in |M|$. Furthermore, if $f : |M| \to |M'|$ is a continuous function between two topological manifolds $|M|$ and $|M'|$ for two smooth manifolds $(|M|, \mathscr{S}_M)$ and $(|M'|, \mathscr{S}_{M'})$ and if there is a comorphism $F : \mathscr{S}_{M'} \to f_*\mathscr{S}_M$ of local rings, then f must also be smooth and $F = f^\sharp$. See Appendix D for details.

A *smooth $\mathbb{Z}$-graded manifold* is then defined to be a locally ringed space $M = (|M|, \mathscr{S}_M)$, for $|M|$ a topological manifold, such that for each $x \in |M|$ there is an open neighbourhood $U \ni x$ and an isomorphism of locally ringed spaces,

$$(U, \mathscr{S}_M|_U) \;\cong\; (U', \odot^\bullet \mathscr{V}^*_{U'} \otimes \mathscr{C}^\infty_{U'}) \;, \tag{2.21}$$

where $U' \subseteq \mathbb{R}^d$ open, $\mathscr{C}^\infty_{U'}$ the sheaf of smooth functions on U', and $\mathscr{V}_{U'}$ a locally free $\mathbb{Z}$-graded sheaf of $\mathscr{C}^\infty_{U'}$-modules on U'. Hence, we require the sheaf of functions to look locally like (2.18). We shall denote by $\mathscr{C}^\infty_M$ the sheaf of smooth functions on $|M|$ (which is a subsheaf of the structure sheaf $\mathscr{S}_M$), the *body* of $M = (|M|, \mathscr{S}_M)$ and denote it by M°. We shall also write $\mathscr{C}^\infty(M) := \Gamma(|M|, \mathscr{S}_M)$ for the global functions on M.

[1]See Appendix D for details about sheaves.

Examples. A convenient way of obtaining $\mathbb{Z}$-graded manifolds is by degree shifting the fibres of a vector bundle. We already mentioned the simplest example of the degree-shifted tangent bundle $T[1]M$. In general, it can be shown [122, 123] that any smooth real $\mathbb{Z}$-graded manifold must take the form of a vector bundle $E \to M^\circ$ over an ordinary smooth manifold M° with the typical fibre being a $\mathbb{Z}$-graded vector space.[1] The resulting manifolds are usually called *globally split*. Note, however, that complex $\mathbb{Z}$-graded manifolds are not globally split in general, which is basically due to the non-existence of a holomorphic partition of unity.

Vector fields and differential forms on $\mathbb{Z}$-graded manifolds. A *vector field* V on a $\mathbb{Z}$-graded manifold M is a graded derivation of the sheaf of functions, $V : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$. That is, for homogeneous V of degree $|V| \in \mathbb{Z}$ and homogeneous $f, g \in \mathscr{C}^\infty(M)$, we have

$$V(fg) \;=\; V(f)g + (-1)^{|V|\,|f|} f\, V(g)\ . \tag{2.22}$$

As in the ordinary case, we define the tangent bundle TM of a $\mathbb{Z}$-graded manifold M to be the disjoint union of the tangent spaces which in turn are the vector space of derivations at particular points of M.

A particular example of a vector field is the *Euler vector field* Υ which is defined by its action $\Upsilon f := |f| f$ on any homogeneous $f \in \mathscr{C}^\infty(M)$ of degree $|f| \in \mathbb{Z}$. The Euler vector field itself is homogeneous and of degree 0.

Note that the definition of differential forms on an ordinary manifold M as functions on the degree-shifted tangent bundle, $\Omega^\bullet(M) \cong \mathscr{C}^\infty(T[1]M)$, as discussed above, generalises straightforwardly to $\mathbb{Z}$-graded manifolds. That is, we may define

$$\Omega^\bullet(M) \;:=\; \mathscr{C}^\infty(T[1]M) \tag{2.23}$$

also for M a $\mathbb{Z}$-graded manifold. The shift in the degree-shifted tangent bundle may now be regarded as an additional grading, extending that of the $\mathbb{Z}$-graded manifold to a bi-grading. Since we are not interested in any further generalisation of this grading, we simply use the ordinary notation for differential forms on manifolds to $\mathbb{Z}$-graded manifolds and write again d and $V \lrcorner$ for de Rham differential and interior product on a $\mathbb{Z}$-graded manifold,

$$\mathrm{d} \;\Longleftrightarrow\; \xi^\mu \frac{\partial}{\partial x^\mu} \quad \text{and} \quad V^\mu \frac{\partial}{\partial x^\mu} \lrcorner \;\Longleftrightarrow\; V^\mu \frac{\partial}{\partial \xi^\mu}\ , \tag{2.24}$$

[1] This is essentially due to the existence of a partition of unity and the fact that any smooth $\mathbb{Z}$-graded manifold can be smoothly deformed into said vector bundle.

where the x^μ are now local $\mathbb{Z}$-graded coordinates on M and ξ^μ local $\mathbb{Z}$-graded fibre coordinates on $T[1]M$.

In the following, we shall also make use of the *Lie derivative* which is now defined by the graded version of Cartan's formula

$$\mathcal{L}_V\omega \;:=\; V \lrcorner\, \mathrm{d}\omega + (-1)^{|V|}\,\mathrm{d}(V \lrcorner\, \omega) \tag{2.25}$$

for $\omega \in \Omega^\bullet(M)$ and V a homogeneous vector field of degree $|V| \in \mathbb{Z}$. Note that $[\mathcal{L}_V, \mathrm{d}] = 0$, which implies $\mathcal{L}_\Upsilon \mathrm{d}f = \mathrm{d}\mathcal{L}_\Upsilon f = |f|\,\mathrm{d}f$. Consequently, $\mathcal{L}_\Upsilon$ extracts the $\mathbb{Z}$-degree of a differential form while ignoring its form degree. We say that $\omega \in \Omega^\bullet(M)$ is of *degree* $k \in \mathbb{Z}$ if and only if $\mathcal{L}_\Upsilon\omega = k\omega$.

$\mathbb{Z}$-graded vector bundles. The tangent bundle of a $\mathbb{Z}$-graded manifold as introduced above is an example of a $\mathbb{Z}$-graded vector bundle. Generally, a *$\mathbb{Z}$-graded vector bundle* over a $\mathbb{Z}$-graded manifold $M = (|M|, \mathscr{S}_M)$ is defined to be a locally free sheaf $\mathscr{E}_M$ of $\mathbb{Z}$-graded $\mathscr{S}_M$-modules over M. In addition, for a morphism $(f, f^\sharp) : (|M|, \mathscr{S}_M) \to (|M'|, \mathscr{S}_{M'})$ of locally ringed spaces, the *pull-back* of a $\mathbb{Z}$-graded vector bundle $\mathscr{E}_{M'}$ over M' to M is the locally free sheaf $f^*\mathscr{E}_{M'} := \mathscr{S}_M \otimes_{f^{-1}\mathscr{S}_{M'}} f^{-1}\mathscr{E}_{M'}$ over M.[1]

Q-manifolds. We now have introduced all the ingredients necessary to define a Q-manifold. A *Q-manifold* is a pair (M, Q), where M is a $\mathbb{Z}$-graded manifold and Q is a *homological vector field*, that is, a homogeneous vector field Q of degree 1 which satisfies $Q^2 = 0$. Note that the homological vector field induces a differential on the algebra of functions and the pair $(\mathscr{C}^\infty(M), Q)$ is a dg-algebra[2]. Moreover, if the $\mathbb{Z}$-grading reduces to a non-negative or $\mathbb{N}$-grading, then we also speak of an *NQ-manifold*. A morphism of Q-manifolds (M, Q) and (M', Q') is a smooth map $f : M \to M'$, inducing the homomorphism $f^\# : C^\infty(M') \to C^\infty(M)$ of differential algebras, that is the usual chain property holds: $Q \circ f^\# = f^\# \circ Q'$. As a composition of two morphisms is apparently a Q-morphism, we have a well-defined category of Q-manifolds.

Historically, the term Q-manifolds has its origin in physics, in particular in the study of supersymmetry, where the letter Q was a standard notation for a super-charge. These objects played a key role in the seminal work of Alexandrov–Kontsevich–Schwarz–Zaboronsky

[1]Here, $f^{-1}\mathscr{S}_{M'}$ denotes the inverse image of $\mathscr{S}_{M'}$ (and likewise for $\mathscr{E}_{M'}$). See c.f. Appendix D for a definition.

[2]The action of Q is naturally extended from $C^\infty(M)$ to the whole tensor algebra of M.

(AKSZ) [114], while Kontsevich has shown the link to deformation quantisation [119]. Homological vector fields were studied by Vaintrob [124, 125], but even before they were formalised as a mathematical notion, they had existed in physics parlance as BRST symmetries.

Examples. The simplest example of a Q-manifold is any ordinary manifold M together with $Q = 0$. Another simple but more interesting example is the degree-shifted tangent bundle $T[1]M$, with the canonical vector field turning into the de Rham differential on the algebra of functions given in the motivational paragraph. We shall encounter many more examples in Section 2.4..

2.3.1. Symplectic Q-manifolds

A *graded symplectic structure of degree* k on a $\mathbb{Z}$-graded manifold M is a closed non-degenerate differential two-form $\omega \in \Omega^2(M)$ of degree k. The non-degeneracy means that $V \lrcorner \omega = 0$ is equivalent to $V = 0$ for vector fields V.

A *symplectic Q-manifold of degree* k is a Q-manifold (M, Q) equipped with a graded symplectic structure ω of degree k for which Q is symplectic, that is, $\mathcal{L}_Q\omega = 0$. It is rather straightforward to see that ω must be exact for $k \neq 0$ and Q Hamiltonian for $k \neq -1$, respectively [45]. Indeed,

$$k\omega \;=\; \mathcal{L}_\Upsilon\omega \;=\; \Upsilon \lrcorner \mathrm{d}\omega + \mathrm{d}(\Upsilon \lrcorner \omega) \;=\; \mathrm{d}(\Upsilon \lrcorner \omega) \quad \Longrightarrow \quad \omega \;=\; \mathrm{d}\Big(\tfrac{1}{k}\Upsilon \lrcorner \omega\Big)\ . \tag{2.26}$$

Likewise, to verify that Q is Hamiltonian, we first note that $\mathrm{d}(Q \lrcorner \omega) = 0$ since Q is symplectic. Then,[1]

$$\begin{aligned} -Q \lrcorner \omega \;&=\; [Q, \Upsilon] \lrcorner \omega \;=\; Q \lrcorner \mathrm{d}(\Upsilon \lrcorner \omega) - \mathrm{d}(Q \lrcorner \Upsilon \lrcorner \omega) \\ &=\; kQ \lrcorner \omega + \mathrm{d}(\Upsilon \lrcorner Q \lrcorner \omega)\ , \end{aligned} \tag{2.27a}$$

where in the second step we have used (2.26). Consequently,

$$Q \lrcorner \omega \;=\; \mathrm{d}S \quad \text{with} \quad S \;:=\; \tfrac{1}{k+1} Q \lrcorner \Upsilon \lrcorner \omega\ . \tag{2.27b}$$

[1] Cartan's formula (2.25) together with the fact that the Lie derivative is a graded derivation that commutes with the contraction, $\mathcal{L}_V(W \lrcorner \omega) = [V, W] \lrcorner \omega + (-1)^{|V||W|} W \lrcorner \mathcal{L}_V\omega$, imply that $[V, W] \lrcorner \omega = V \lrcorner W \lrcorner \mathrm{d}\omega + V \lrcorner \mathrm{d}(W \lrcorner \omega) - (-1)^{|V|(|W|+1)} W \lrcorner \mathrm{d}(V \lrcorner \omega) + (-1)^{|V|}\, \mathrm{d}(V \lrcorner W \lrcorner \omega)$.

Poisson structure. As on ordinary manifolds, a symplectic structure on a Q-manifold induces a Poisson structure. Concretely, let (M, Q, ω) be a smooth symplectic degree k manifold. For any $f \in \mathscr{C}^\infty(M)$ let V_f be the corresponding Hamilton vector field given by

$$V_f \lrcorner \omega \;=\; \mathrm{d}f \ . \tag{2.28}$$

For homogeneous $f \in \mathscr{C}^\infty(M)$ of degree $|f| \in \mathbb{Z}$, this equation implies $|V_f| = |f| - k$ because $|V_f \lrcorner \omega| = |V_f| + |\omega|$ and $|\mathrm{d}f| = |f|$. We then define the *graded Poisson structure*

$$\{f, g\} \;:=\; V_f \lrcorner V_g \lrcorner \omega \;=\; V_f g \ . \tag{2.29}$$

For homogeneous elements $f, g \in \mathscr{C}^\infty(M)$ of degrees $|f|, |g| \in \mathbb{Z}$, we have that $|\{f, g\}| = |f| + |g| - k$. Using Cartan's formula (2.25), we immediately find the standard result $V_{\{f,g\}} = (-1)^{|f|-k}[V_f, V_g]$. Furthermore, the Poisson structure is graded antisymmetric,

$$\{f, g\} \;=\; -(-1)^{(|f|-k)(|g|-k)}\{g, f\} \ , \tag{2.30a}$$

satisfies a graded Jacobi identity,

$$\begin{aligned} \{f, \{g, h\}\} + (-1)^{(|f|-k)(|g|+|h|)+|f|-k}\{g, \{h, f\}\} +& \\ + (-1)^{(|h|-k)(|f|+|g|)+|h|-k}\{h, \{f, g\}\} \;=&\; 0 \ , \end{aligned} \tag{2.30b}$$

as well as a graded Leibniz rule,

$$\{f, gh\} \;=\; \{f, g\}h + (-1)^{(|f|-k)|g|} g\{f, h\} \ . \tag{2.30c}$$

Using (2.27b) and the fact that $|S| = k + 1$, we find

$$Q \;=\; \{S, -\} \ . \tag{2.31}$$

The Jacobi identity then implies that $Q^2 = \frac{1}{2}\{\{S, S\}, -\}$. Consequently, $\{S, S\} = QS$ must be locally constant for $Q^2 = 0$. Since $|\{S, S\}| = 2 + k$, we may conclude that for $k \neq -2$ the condition $Q^2 = 0$ is equivalent to saying that

$$\{S, S\} \;=\; 0 \ . \tag{2.32}$$

For S the (classical) Batalin–Vilkovisky action, this is called the *classical master equation*. In this special case, the Poisson bracket is of degree 1. General Poisson algebras of degree 1 are known as *Gerstenhaber algebras*. We shall return to the Batalin–Vilkovisky action in Section 4.3..

Examples. The following examples were first given in [126], see also [45] and [116] for further details. A symplectic NQ-manifold of degree 0 is simply a symplectic manifold. A symplectic NQ-manifold of degree 1 is in one-to-one correspondence with a Poisson manifold. Such a manifold can be shown to be symplectomorphic to $T^*[1]M$ with canonical symplectic structure. A compatible homological vector field Q corresponds to a bi-vector field on M and the condition $Q^2 = 0$ amounts to this bivector being a Poisson tensor. Similarly, symplectic NQ-manifolds of degree 2 are Courant algebroids. These objects arise in string theory when studying the geometry underlying T-duality and features prominently in generalised geometry and double field theory. As a last, more involved, example we mention $T^*[n]T[1]M$, a shifted cotangent bundle of the shifted tangent bundle of an ordinary manifold M. This is a class of NQ-manifolds containing the Vinogradov algebroid $TM \oplus \Lambda^{n-1}T^*M$, see [127, 128] for details. Here, the cotangent functor T^* gives extra coordinates with opposite degree to the fibers in $T[1]$. For $n = 2$ the Vinogradov algebroid is an exact Courant algebroid[1] and one has local coordinates $(x^\mu, \xi^\mu, \xi_\mu, p_\mu)$, with $\mu = 1, \ldots, \dim(M)$ of degree $(0, 1, 1, 2)$, respectively. A canonical choice of the homological vector field reads $Q := \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \xi_\mu}$.

2.4. L_∞-algebras and L_∞-algebroids

2.4.1. Motivation

The previous example of a Courant algebroid as well as the Q-manifolds $T[1]M$ and $T^*[1]M$ connect Q-manifolds to Lie algebroids. This connection can be vastly generalised as we shall see in the following.

Firstly, let us see what happens for a Q-manifold M with body M° a point. In that case, we simply have a $\mathbb{Z}$-graded vector space V. Let us further simplify V such that it is non-trivial only in degree -1, i.e. $\mathsf{V} = \mathfrak{g}[1]$ for an ordinary vector space $\mathfrak{g}$. Let ξ^α be local coordinates on V of degree 1. Then, the most general degree 1 vector field Q is necessarily of the form

$$Q \;=\; -\tfrac{1}{2} f_{\alpha\beta}{}^{\gamma} \xi^\alpha \xi^\beta \frac{\partial}{\partial \xi^\gamma} \;, \tag{2.33}$$

for some constants $f_{\alpha\beta}{}^{\gamma} = -f_{\beta\alpha}{}^{\gamma}$. It is straightforward to check that the identity $Q^2 = 0$ is

[1] The data specifying a symplectic structure on $T^*[2]T[1]M$ are equivalent to the data specifying a Courant algebroid structure on the bundle $TM \oplus T^*M$, [45].

equivalent to requiring the $f_{\alpha\beta}{}^\gamma$ to satisfy the Jacobi identity

$$f_{\alpha\beta}{}^\delta f_{\gamma\delta}{}^\epsilon + f_{\beta\gamma}{}^\delta f_{\alpha\delta}{}^\epsilon + f_{\gamma\alpha}{}^\delta f_{\beta\delta}{}^\epsilon \;=\; 0\;. \tag{2.34}$$

Therefore, the constants $f_{\alpha\beta}{}^\gamma$ are, in fact, the structure constants of a Lie algebra structure on $\mathfrak{g}$. We conclude that a Q-manifold with body a point and concentrated in degree -1 is therefore a Lie algebra $\mathfrak{g}$.

The latter generalises to the following statements, which we shall explain in more detail in the remainder of this Section. A Q-manifold with body a point is an *L_∞-algebra*. If the only non-trivial coordinates are of degrees $1,\ldots,n$ we shall speak of a *Lie n-algebra*.[1] These algebras are expected to be the correct infinitesimal symmetry structure for gauge theories of extended objects. Similarly, a Q-manifold with non-trivial body and coordinates of degrees $0,\ldots,n$ is a *Lie n-algebroid* and a general Q-manifold is an *L_∞-algebroid*.

Chevalley–Eilenberg complex and Q-manifolds. Let us now link the above discussion to standard mathematical nomenclature, introducing the language that is natural for describing BV quantisation. The differential graded algebra[2] $(\bigwedge^\bullet \mathfrak{g}^*, \mathrm{d}_{\mathrm{CE}}) \cong (\mathscr{C}^\infty(\mathfrak{g}[1]), Q)$, for some finite dimensional[3] Lie algebra $\mathfrak{g}$, is called the *Chevalley–Eilenberg algebra* $\mathrm{CE}(\mathfrak{g})$ of $\mathfrak{g}$ and the differential d_{CE}, induced by the homological vector field Q, is identified with the *Chevalley–Eilenberg differential*. Similarly, we have the identification $\mathrm{CE}(T[1]M) := (\Omega^\bullet(M), \mathrm{d})$, defining the the Chevalley–Eilenberg algebra of the grade-shifted tangent bundle $T[1]M$ as the de Rham complex of some manifold M. This language is well-known from Lie algebra cohomology and it allows a natural extension to the case of L_∞-algebras.

The Chevalley–Eilenberg algebra is a special case of the Chevalley–Eilenberg complex for a $\mathfrak{g}$-module $\mathscr{E}$,

$$0 \longrightarrow \mathsf{Hom}(\textstyle\bigwedge^0\mathfrak{g},\mathscr{E}) \;\cong\; \mathscr{E} \xrightarrow{\mathrm{d}_{\mathrm{CE}}} \mathsf{Hom}(\textstyle\bigwedge^1\mathfrak{g},\mathscr{E}) \xrightarrow{\mathrm{d}_{\mathrm{CE}}} \mathsf{Hom}(\textstyle\bigwedge^2\mathfrak{g},\mathscr{E}) \xrightarrow{\mathrm{d}_{\mathrm{CE}}} \cdots \tag{2.35}$$

[1] Strictly speaking, they are n-term L_∞-algebras, but for all intents and purposes, they can be regarded as (categorically) equivalent to Lie n-algebras. The categorical equivalence has been proven for Lie 2-algebras and 2-term L_∞-algebras [4]; the extension to Lie n-algebras and n-term L_∞-algebras should be very involved, but ultimately a mere technicality.

[2] The isomorphism is the shift isomorphism $s^\bullet$ defined in (2.13). See Appendix A for details. Moreover, homomorphisms of a Lie algebra $\mathfrak{g}$ are in one-to-one correspondence to chain maps of the complex $\mathrm{CE}(\mathfrak{g})$.

[3] Special care has to be taken in the infinite-dimensional case. See e.g. [129] for details on this point.

with Chevalley–Eilenberg differential

$$
\begin{aligned}
\mathrm{d}_{\mathrm{CE}} \,:\, \mathsf{Hom}(\textstyle\bigwedge^p \mathfrak{g}, \mathscr{E}) \;&\rightarrow\; \mathsf{Hom}(\textstyle\bigwedge^{p+1}\mathfrak{g}, \mathscr{E})~,\\
(\mathrm{d}_{\mathrm{CE}}F)(X_1,\dots,X_{p+1}) \;&:=\\
:=\; &\sum_{i=1}^{p+1}(-1)^{i+1}X_i \rhd F(X_1,\dots,\hat{X}_i,\dots,X_{p+1})+\\
&+\sum_{1\leq i<j\leq p+1}(-1)^{i+j}F([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1})~.
\end{aligned}
\tag{2.36}
$$

We now explain that in the case of the trivial $\mathfrak{g}$-module $\mathscr{E}=\mathbb{R}$, we recover the Chevalley–Eilenberg algebra and the action of d_{CE} is essentially that of Q. First, note that a function $F\in\mathsf{Hom}(\bigwedge^k\mathfrak{g},\mathbb{R})$ corresponds to an element in $\mathscr{C}^\infty_k(\mathfrak{g}[1])$ according to

$$
F(\tau_{\alpha_1}\wedge\dots\wedge\tau_{\alpha_k}) \;=\; F_{\alpha_1\cdots\alpha_k} \quad\Longleftrightarrow\quad \frac{1}{k!}\xi^{\alpha_1}\cdots\xi^{\alpha_k}F_{\alpha_1\cdots\alpha_k}~. \tag{2.37}
$$

Under this isomorphism, the Chevalley–Eilenberg differential is indeed mapped to Q. Concretely, we have that

$$
\begin{aligned}
(\mathrm{d}_{\mathrm{CE}}F)(\tau_{\alpha_1}\wedge\dots\wedge\tau_{\alpha_{k+1}}) \;&=\; \sum_{1\leq i<j\leq k+1}(-1)^{i+j}F([\tau_{\alpha_i},\tau_{\alpha_j}],\tau_{\alpha_1},\dots,\hat{\tau}_{\alpha_i},\dots,\hat{\tau}_{\alpha_j},\dots,\tau_{k+1})\\
&=\; -\frac{(k+1)!}{2(k-1)!}f_{[\alpha_1\alpha_2}{}^{\beta}F_{\beta\alpha_3\dots\alpha_{k+1}]}
\end{aligned}
\tag{2.38}
$$

corresponds to

$$
Q\frac{1}{k!}\xi^{\alpha_1}\cdots\xi^{\alpha_k}F_{\alpha_1\cdots\alpha_k} \;=\; -\frac{1}{2(k-1)!}\xi^{\alpha_1}\cdots\xi^{\alpha_{k+1}}f_{\alpha_1\alpha_2}{}^{\beta}F_{\beta\alpha_3\cdots\alpha_{k+1}}~. \tag{2.39}
$$

Remark. This is exactly the language used by Cartan to describe local connection forms on principal fibre bundles, as mentioned in the introduction to this Chapter. Let M be a manifold and $\mathfrak{g}$ a Lie algebra with basis τ_α and structure constants $f^\alpha_{\beta\gamma}$. Morphisms of differential graded algebras

$$
\mathcal{A}:\mathrm{CE}(\mathfrak{g})\rightarrow\mathrm{CE}(T[1]M)~, \tag{2.40}
$$

preserve the graded algebra structure and therefore are fixed by the image of ξ^α, $\mathcal{A}(\xi^\alpha)=:A^\alpha\in\Omega^1(M)$, that defines a Lie algebra-valued differential form or local connection 1-form $A:=A^\alpha\tau_\alpha$ on M. Compatibility with the differentials of $\mathrm{CE}(\mathfrak{g})$ and $\mathrm{CE}(T[1]M)$, that is $(\mathrm{d}\circ\mathcal{A})(\xi^\alpha)=(\mathcal{A}\circ Q)(\xi^\alpha)$, enforces flatness of this connection

$$
\mathrm{d}A^\alpha=\mathcal{A}(-\tfrac{1}{2}f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma)=-\tfrac{1}{2}f^\alpha_{\beta\gamma}A^\beta\wedge A^\gamma \quad\Longrightarrow\quad F:=\mathrm{d}A+\tfrac{1}{2}[A,A]=0~. \tag{2.41}
$$

Gauge transformations are encoded in partially flat homotopies between two morphisms of type (2.40). Moreover, to describe non-flat connections one needs to enlarge the Chevalley-Eilenberg algebra and construct the so called *Weil algebra*, [130, 7]. We shall not discuss this approach further, which in its full extent is due to [7] . The interested reader may also consult [131] for a review of the subject. Along this Thesis, we shall use a different approach to describe kinematical data of higher gauge, as explained in Chapter 3.

Contracted coordinate functions I. We now come to an important technicality which, however, will greatly simplify our notation. Relying on basis dependent equations as $Q\xi^\alpha = -\frac{1}{2}f_{\beta\gamma}{}^\alpha\xi^\beta\xi^\gamma$ is both inconvenient and inelegant. We can, however, contract both sides of the equation by the basis vector τ_α, obtaining the basis independent version

$$Q\xi \;=\; -\tfrac{1}{2}[\xi,\xi] \quad \text{with} \quad \xi \;:=\; \xi^\alpha\otimes\tau_\alpha\;. \tag{2.42}$$

Note that ξ is *not* an element of $\mathfrak{g}[1]$ but rather an element of $(\mathfrak{g}[1])^*\otimes\mathfrak{g}$ which, in turn, is a subset of $\mathfrak{g}_{\mathscr{C}} := \mathscr{C}^\infty(\mathfrak{g}[1])\otimes\mathfrak{g}$. We thus extended the Lie bracket from $\mathfrak{g}$ to $\mathfrak{g}_{\mathscr{C}}$. Since $[-,-]$ does not carry any degree, this extension by linearity is unique. This is similar to the extension of the Lie bracket from $\mathfrak{g}$ to $\Omega^\bullet(M,\mathfrak{g})$, the set of Lie algebra valued differential forms, which is often used in gauge theories. In the general case of L_∞-algebras, however, we will have to be more careful.

2.4.2. L_∞-algebras

As stated above, the differential graded algebra $(C^\infty(\mathfrak{g}[1]),Q)$ provides an alternative description of the Lie algebra $(\mathfrak{g},[-,-])$, with $[-,-]$ the Lie bracket via the identification with the Chevalley–Eilenberg algebra $\mathsf{CE}(\mathfrak{g}) := (\bigwedge^\bullet\mathfrak{g}^*,\mathrm{d}_{\mathrm{CE}})$ of $\mathfrak{g}$. This statement readily generalises to categorised structures. Considering a Q-manifold[1] M concentrated in degrees $1,\cdots,n$ we declare the pair $(\mathscr{C}^\infty(M),Q)$ to be the Chevalley–Eilenberg algebra $\mathsf{CE}(\mathsf{L})$ of an n-*term* L_∞-*algebra* (L,μ_i) over $\mathbb{R}$, with μ_i being the higher brackets generalising the ordinary Lie bracket. Indeed, such a Q-manifold is necessarily of the form $M=\mathsf{L}[1]$, for a $\mathbb{Z}$-graded vector space $\mathsf{L}=\oplus_{k=-n}^{0}\mathsf{L}_k$.

Defining ξ^α to be local coordinates of degree $|\xi^\alpha|\in\{1,\ldots,n\}$, then the vector field (2.33)

[1] Recall, that we here are considering Q-manifolds with body a point.

generalises to

$$Q := \sum_{i=1}^{n} \frac{(-1)^{\frac{1}{2}i(i+1)}}{i!}\xi^{\alpha_1}\cdots\xi^{\alpha_i} f_{\alpha_1\cdots\alpha_i}{}^{\beta}\frac{\partial}{\partial\xi^{\beta}} . \tag{2.43}$$

Note that the $f_{\alpha_1\cdots\alpha_i}{}^{\beta}$ are constants, but not all them are non-zero because of the requirement of the vector field Q being of degree 1. These constants encode totally graded anti-symmetric multilinear maps[1]

$$\mu_i : \underbrace{\mathsf{L}\times\cdots\times\mathsf{L}}_{i\text{ copies}} \rightarrow \mathsf{L} , \tag{2.44a}$$

of degree $2-i$. Indeed, letting τ_α be a basis of L with $|\tau_\alpha| = -|\xi_\alpha| + 1 \in \{-n,\dots,0\}$ we may write

$$\mu_i(\tau_{\alpha_1},\dots,\tau_{\alpha_i}) := f_{\alpha_1\cdots\alpha_i}{}^{\beta}\tau_\beta . \tag{2.44b}$$

The nilpotency condition $Q^2 = 0$ amounts to requiring that the higher maps satisfy the *higher* or *homotopy Jacobi identities*,

$$\sum_{j+k=i}\sum_{\sigma\in\mathrm{Sh}(j;i)} \chi(\sigma;\ell_1,\dots,\ell_i)(-1)^k\mu_{k+1}(\mu_j(\ell_{\sigma(1)},\dots,\ell_{\sigma(j)}),\ell_{\sigma(j+1)},\dots,\ell_{\sigma(i)}) = 0 , \tag{2.44c}$$

for $\ell_1,\dots,\ell_i \in \mathsf{L}$ and $i \in \mathbb{N}_0$, as a straightforward but lengthy calculation shows. Here, the sum is taken over all $(j;i)$ *shuffles* σ which consist of permutations σ of $\{1,\dots,i\}$ such that the first j and the last $i-j$ images of σ are ordered: $\sigma(1) < \cdots < \sigma(j)$ and $\sigma(j+1) < \cdots < \sigma(i)$. Moreover, $\chi(\sigma;\ell_1,\dots,\ell_i)$ is the *graded Koszul sign* defined via the equation

$$\ell_1\wedge\ldots\wedge\ell_i = \chi(\sigma;\ell_1,\dots,\ell_i)\,\ell_{\sigma(1)}\wedge\ldots\wedge\ell_{\sigma(i)} \tag{2.44d}$$

in the graded exterior algebra for homogeneous elements. More generally, a $\mathbb{Z}$-graded vector space L together with the i-ary multilinear totally antisymmetric products μ_i defined above satisfying (2.44c) is called *L_∞-algebra* or *strong homotopy Lie algebra* [10, 132–134]. The homotopy Jacobi identities (2.44c) for $i = 1$ and $i = 2$ can be written as

$$\begin{aligned} \mu_1(\mu_1(\ell_1)) &= 0 , \\ \mu_1(\mu_2(\ell_1,\ell_2)) &= \mu_2(\mu_1(\ell_1),\ell_2) + (-1)^{|\ell_1|_{\mathsf{L}}}\mu_2(\ell_1,\mu_1(\ell_2)) , \end{aligned} \tag{2.45}$$

where $|\ell|_{\mathsf{L}}$ denotes the L_∞-degree of $\ell \in \mathsf{L}$. The first identity tells us that the unary product μ_1 is a differential, making L into a cochain complex, while the second identity shows that μ_1

[1] Please note that our notation differs from another commonly used one, where higher products are denoted by l_i or ℓ_i. For us, the latter are often elements of the L_∞-algebras.

is a graded derivation which is compatible with the binary product μ_2. In addition, the corresponding relation for $i = 3$ captures the failure of the binary product μ_2 to satisfy the standard Jacobi identity. We just mentions here that L_∞-algebras and their homotopy Jacobi relations can be also elegantly described in the codifferential graded coalgebras perspective, which is due to [133, 134]. The interested reader may find the translation between the differential graded algebra, coalgebra and higher brackets pictures in Appendix A.

Note that in the same sense as L_∞-algebras generalise Lie algebras, the higher analogue of an associative algebra is an A_∞*-algebra or strong homotopy associative algebra*, as introduced in [40, 41]. Roughly speaking, while an L_∞-algebra generalises the commutator in a matrix algebra to a structure violating the Jacobi identity, an A_∞-algebra generalises the matrix product in a matrix algebra to a non-associative structure. These A_∞-algebra come with higher homotopy associative products and antisymmetrising these products yields the L_∞-algebra products μ_i on the underlying graded vector space. A unifying description of all the homotopy algebras[1] and their cyclic structures is provided by the formalism of operads, where A_∞-algebras and L_∞-algebra are defined as homotopy algebras over the associative operad and the Lie operad, respectively. For our purposes, we are not interested in reaching this level of abstraction. See e.g. [136, 137].

Special cases. It is useful to introduce some nomenclature. We call the L_∞-algebra

$$L = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots , \tag{2.46}$$

where 0 denotes the zero-dimensional vector space, the *trivial* L_∞*-algebra*. An L_∞-algebra is called *minimal* whenever $\mu_1 = 0$. For example, ordinary Lie algebras form minimal L_∞-algebras, while differential graded Lie algebras are not minimal in general. Furthermore, a *linearly contractible* L_∞-algebra is one in which the only non-trivial higher product is the differential μ_1 and the corresponding cohomology vanishes. For example, the L_∞-algebra $V[1] \xrightarrow{\mathrm{id}} V$ with $\mu_1 = \mathrm{id}$ and all higher products trivial is linearly contractible. Also the trivial pairs introduced later for gauge fixing in the BV formalism are linearly contractible. Similarly, L_∞-algebras for which $\mu_i = 0$ for $i \geq 3$ are called *strict* L_∞*-algebras*. Such L_∞-algebras are the same as differential graded Lie algebras. Finally, we recall that when an L_∞-algebra is

[1]The term homotopy algebra generally refers to either a L_∞-, A_∞- or C_∞-algebra, that is, an A_∞-algebra that also satisfies homotopy commutativity relations, see [135] for details.

non-trivial only in particular degrees, we say that it is concentrated in these degrees. L_∞-algebras concentrated in degrees $-n+1,\dots,0$ corresponds to an n-fold categorification of a Lie algebra (with morphisms up to degree $n-1$).

2.4.3. Cyclic L_∞-algebras

The appropriate notion of an *inner product* on an L_∞-algebra L is a graded symmetric non-degenerate bilinear pairing

$$\langle -,-\rangle_{\mathsf{L}} \,:\, \mathsf{L}\times\mathsf{L} \;\to\; \mathbb{R} \tag{2.47a}$$

which is cyclic in the sense of

$$\langle \ell_1,\mu_i(\ell_2,\dots,\ell_{i+1})\rangle_{\mathsf{L}} \;=\; (-1)^{i+i(|\ell_1|_{\mathsf{L}}+|\ell_{i+1}|_{\mathsf{L}})+|\ell_{i+1}|_{\mathsf{L}}\sum_{j=1}^{i}|\ell_j|_{\mathsf{L}}}\langle \ell_{i+1},\mu_i(\ell_1,\dots,\ell_i)\rangle_{\mathsf{L}} \tag{2.47b}$$

for all $i\in\mathbb{N}$ for homogeneous $\ell_1,\dots,\ell_{i+1}\in\mathsf{L}$ with $|\ell_i|_{\mathsf{L}}$ the L_∞-degree of $\ell_i\in\mathsf{L}$, cf. [138,139] for the original introduction of cyclic structures.

Whenever an L_∞-algebra is equipped with such an inner product, we shall call it a *cyclic L_∞-algebra*. When a cyclic L_∞-algebra is Hilbert (i.e. complete), the non-degeneracy of the inner product induces the isomorphisms $\mathsf{L}\cong(\mathsf{L}[k])^*\cong\mathsf{L}^*[-k]$ where $k:=|\langle -,-\rangle_{\mathsf{L}}|_{\mathsf{L}}$.

In the Q-manifold picture, a cyclic inner product corresponds to a symplectic form and the cyclicality is encoded in the requirement of the vector field to be symplectic with respect to the symplectic form. Let us again illustrate this starting form the simple example of a Lie algebra $\mathfrak{g}$. Let ξ^α be coordinates on $\mathfrak{g}[1]$ with respect to a basis τ_α of $\mathfrak{g}$. As we have seen above, the homological vector field is then given by $Q=-\frac{1}{2}f_{\alpha\beta}{}^\gamma\xi^\alpha\xi^\beta\frac{\partial}{\partial\xi^\gamma}$. A symplectic structure on $\mathsf{L}[1]$ is necessarily of degree 2 and thus takes the form $\omega=\frac{1}{2}\omega_{\alpha\beta}\mathrm{d}\xi^\alpha\wedge\mathrm{d}\xi^\beta$ with $\omega_{\alpha\beta}=\omega_{\beta\alpha}\in\mathbb{R}$. The fact that Q is symplectic implies $\mathcal{L}_Q\omega=-\mathrm{d}(Q\lrcorner\,\omega)=0$ which, together with $Q\lrcorner\,\mathrm{d}\xi^\alpha=Q\xi^\alpha$ leads to

$$\tfrac{1}{2}\omega_{\alpha\beta}\left(\mathrm{d}(Q\xi^\alpha)\wedge\mathrm{d}\xi^\beta-\mathrm{d}\xi^\alpha\wedge\mathrm{d}\big(Q\xi^\beta\big)\right)\;=\;0\quad\Longleftrightarrow\quad\omega_{\delta(\alpha}f_{\beta)\gamma}{}^\delta\;=\;0\,. \tag{2.48}$$

This is precisely the cyclicity condition for the symmetric inner product $\langle\tau_\alpha,\tau_\beta\rangle_{\mathfrak{g}}=\omega_{\alpha\beta}$ on the Lie algebra $\mathfrak{g}$,

$$\langle\tau_\alpha,[\tau_\beta,\tau_\gamma]\rangle_{\mathfrak{g}}\;=\;\langle\tau_\gamma,[\tau_\alpha,\tau_\beta]\rangle_{\mathfrak{g}}\,. \tag{2.49}$$

Finally, note that the Hamiltonian (2.27b) for Q is given by $S=\frac{1}{3!}\xi^\alpha\xi^\beta\xi^\gamma\omega_{\delta[\alpha}f_{\beta\gamma]}{}^\delta$. It is not difficult to see that this treatment generalises to $\mathsf{L}[1]$ with $\mathsf{L}=\bigoplus_{k\in\mathbb{Z}}\mathsf{L}_k$. That is, a smooth symplectic Q-manifold of the form $(\mathsf{L}[1],Q,\omega)$ encodes a cyclic L_∞-algebra L.

2.4.4. L_∞-algebras from tensor products

An important observation which we shall heavily rely upon is that the graded vector space obtained from the tensor product of an L_∞-algebra and a differential graded commutative algebra carries a natural L_∞-algebra structure, see e.g. [102]. As we shall see in the rest of the Thesis, this concept is crucial in describing the kinematical data of a higher gauge theory having some L_∞-algebra as its structure group.

Let (A, d) be a differential graded commutative algebra and (L, μ_i) be an L_∞-algebra. Then we have a new L_∞-algebra L_A with underlying graded vector space

$$\mathsf{L}_\mathsf{A} \ := \ \bigoplus_{k\in\mathbb{Z}} (\mathsf{A}\otimes\mathsf{L})_k \quad \text{with} \quad (\mathsf{A}\otimes\mathsf{L})_k \ := \ \bigoplus_{i+j=k} \mathsf{A}_i\otimes\mathsf{L}_j \tag{2.50a}$$

so that the homogeneous degree in L_A is given by $|a\otimes\ell|_{\mathsf{L}_\mathsf{A}} = |a|_\mathsf{A} + |\ell|_\mathsf{L}$ for homogeneous $a\in\mathsf{A}$ and $\ell\in\mathsf{L}$. The higher products $\hat\mu_i$ on L_A read as

$$\hat\mu_i \ := \ \begin{cases} \mathrm{d}\otimes\mathrm{id} + \mathrm{id}\otimes\mu_1 & i=1\ , \\ m_i\otimes\mu_i & \text{else}\ , \end{cases} \tag{2.50b}$$

where $m_i(a_1,\ldots,a_i) := a_1\cdots a_i$ is the commutative, associative product on A.

Applied to $(a_1\otimes\ell_1,\ldots,a_i\otimes\ell_i)$ with homogeneous $a_1,\ldots,a_i\in\mathsf{A}$ and $\ell_1,\ldots,\ell_i\in\mathsf{L}$, we obtain

$$\begin{aligned} \hat\mu_1(a_1\otimes\ell_1) \ &:= \ \mathrm{d}a_1\otimes\ell_1 + (-1)^{|a_1|_\mathsf{A}} a_1\otimes\mu_1(\ell_1)\ , \\ \hat\mu_i(a_1\otimes\ell_1,\ldots,a_i\otimes\ell_i) \ &:= \ (-1)^{i\sum_{j=1}^i |a_j|_\mathsf{A} + \sum_{j=2}^i |a_j|_\mathsf{A} \sum_{k=1}^{j-1} |\ell_k|_\mathsf{L}} \times \\ &\qquad \times (a_1\cdots a_i)\otimes\mu_i(\ell_1,\ldots,\ell_i) \end{aligned} \tag{2.51a}$$

for $i\geq 2$, and they extend to general elements by linearity. It is shown in Appendix C that these products satisfy the homotopy Jacobi identities (2.44c).

In addition, if both A and L come with inner products $\langle -,-\rangle_\mathsf{A}$ and $\langle -,-\rangle_\mathsf{L}$, then L_A admits a natural inner product defined by

$$\langle a_1\otimes\ell_1, a_2\otimes\ell_2\rangle_{\mathsf{L}_\mathsf{A}} \ := \ (-1)^{|a_2|_\mathsf{A}|\ell_1|_\mathsf{L}} \langle a_1,a_2\rangle_\mathsf{A}\langle\ell_1,\ell_2\rangle_\mathsf{L} \tag{2.51b}$$

for homogeneous $a_1,a_2\in\mathsf{A}$ and $\ell_1,\ell_2\in\mathsf{L}$. Clearly, this inner product is graded symmetric and its cyclicity is shown in Appendix C.

Remark: L_∞-algebra structures on graded modules. Let us remark that our previous discussion of L_∞-algebra structures on graded vector spaces translates to the case of graded modules. This is particularly important for the application to field theory, as here the field of real numbers is essentially always replaced by a ring of functions or, more generally, by a ring of sections of some vector bundle. In the following, a (cyclic) L_∞-algebra will have an underlying graded module.

Example: Contracted coordinate functions II. Another fundamental example is the following. Let (L,μ_i) be an L_∞-algebra. Recall that to use the simplifying notation in Equation (2.42), we introduced the contracted coordinate functions $\xi := \xi^\alpha \otimes \tau_\alpha$ of total degree $|\xi| = 1$, extending the Lie algebra $\mathfrak{g}$ to the tensor product $\mathfrak{g}_{\mathscr{C}} := \mathscr{C}^\infty(\mathfrak{g}[1]) \otimes \mathfrak{g}$. In the case of L the analogue extension is

$$\mathsf{L}_{\mathscr{C}} := \mathscr{C}^\infty(\mathsf{L}[1]) \otimes \mathsf{L} , \tag{2.52a}$$

where we regard $\mathscr{C}^\infty(\mathsf{L}[1])$ as a differential graded algebra with trivial differential. This leads to higher products obtained by linearly extending

$$\begin{aligned} \hat\mu_1(\zeta_1 \otimes \ell_1) &:= (-1)^{|\zeta_1|_{\rm gh}} \zeta_1 \otimes \mu_1(\ell_1) , \\ \hat\mu_i(\zeta_1 \otimes \ell_1, \ldots, \zeta_i \otimes \ell_i) &:= (-1)^{i\sum_{j=1}^i |\zeta_i|_{\rm gh} + \sum_{j=2}^i |\zeta_j|_{\rm gh} \sum_{k=1}^{j-1} |\ell_k|_{\mathsf{L}}} (\zeta_1 \cdots \zeta_i) \otimes \mu_i(\ell_1, \ldots, \ell_i) \end{aligned} \tag{2.52b}$$

for homogeneous $\zeta_j \in \mathscr{C}^\infty(\mathsf{L}[1])$, of degree $|\zeta_j|_{\rm gh} \in \mathbb{Z}$, and $\ell_j \in \mathsf{L}$. These give the $\mathbb{Z}$-graded vector space (2.52a) an L_∞-structure.

For L a cyclic L_∞-algebra with cyclic inner product of degree k, we also define a non-degenerate graded symmetric pairing $\mathsf{L}_{\mathscr{C}} \times \mathsf{L}_{\mathscr{C}} \to \mathscr{C}^\infty(\mathsf{L}[1])$ given by

$$\langle \zeta_1 \otimes \ell_1, \zeta_2 \otimes \ell_2 \rangle_{\mathsf{L}_{\mathscr{C}}} := (-1)^{k(|\zeta_1|_{\rm gh} + |\zeta_2|_{\rm gh}) + |\ell_1|_{\mathsf{L}} |\zeta_2|_{\rm gh}} (\zeta_1 \zeta_2) \langle \ell_1, \ell_2 \rangle_{\mathsf{L}} \tag{2.52c}$$

for homogeneous $\zeta_{1,2} \in \mathscr{C}^\infty(\mathsf{L}[1])$ and $\ell_{1,2} \in \mathsf{L}$.

This tensor product L_∞-algebra now allows us to write the action of Q in a very compact form, extending formula (2.42). We introduce again the contracted coordinate functions $\xi = \xi^\alpha \otimes \tau_\alpha \in \mathsf{L}_{\mathscr{C}}$ with $|\xi| = 1$, where τ_α is a basis for L. As shown in (A.25), we then have

$$Q\xi = -\sum_{i\geq 1} \frac{1}{i!} \hat\mu_i(\xi, \ldots, \xi) . \tag{2.53}$$

This formula should be interpreted as acting on elements of $\bigodot^\bullet \mathsf{L}[1]$. The $\mu_i(\xi,\ldots,\xi)$ act non-trivially only on elements of $\bigodot^i \mathsf{L}[1]$ and when moving ξ past elements in $\mathsf{L}[1]$, one should insert Koszul signs accordingly.

Such a construction, together with the one discussed in the next two paragraphs, will be central in the BRST/BV formalism analysed in Chapter 4.

2.4.5. L_∞-algebroids

It remains to discuss the case of a Q-manifold (M,Q) with non-trivial body M°. As stated above, these correspond to L_∞-algebroids. We call the dg-algebra $(\mathscr{C}^\infty(M),Q)$ the Chevalley–Eilenberg algebra of an *L_∞-algebroid*.

The Q-manifolds underlying ordinary Lie algebroids are simply degree-shifted vector bundles. Let $E\to M^\circ$ be a vector bundle over an ordinary manifold M°. On the degree-shifted vector bundle $M=E[1]$, we introduce local coordinates x^μ,ξ^α with $|x^\mu|=0$ and $|\xi^\alpha|=1$, the homological vector field Q must be of the form

$$Q \;=\; \rho^\mu_\alpha(x)\xi^\alpha\frac{\partial}{\partial x^\mu}-\tfrac{1}{2}f_{\alpha\beta}{}^\gamma(x)\xi^\alpha\xi^\beta\frac{\partial}{\partial\xi^\gamma}\;. \tag{2.54}$$

The condition $Q^2=0$ then amounts to requiring that the $f_{\alpha\beta}{}^\gamma$ encode a Lie bracket on the space of sections of E satisfying a Leibniz rule and the ρ^μ_α encode a Lie algebra morphism $E\to TM^\circ$.

Example: action Lie algebroid. An important example of a Lie algebroid is the *action Lie algebroid*. To define it, let $M^\circ=(|M|,\mathscr{C}^\infty_M)$ be an ordinary manifold together with an action $\rho:\mathsf{G}\times M^\circ\to M^\circ$ of a Lie group G. We are then often interested in the orbit space M°/G. This space is can be badly behaved and hard to get under control. For example, the action of G may not be free (i.e. it contains fixed points) which leads to singularities when trying to regard M°/G as a smooth manifold. But even if this is not the case, it may be hard to find an explicit and useful description of the quotient space M°/G.

One way to circumvent this problem is to use the *derived quotient* instead. This is just modern terminology for considering the *action Lie groupoid*.[1] The morphisms of the latter are the maps transforming $x\in M^\circ$ by a group element $g\in\mathsf{G}$ to $g\rhd x\in M^\circ$,

$$x\;\xrightarrow{(g,x)}\;g\rhd x\;. \tag{2.55}$$

[1]See Appendix D for a definition of the Lie groupoid. Recall that a Lie group is a Lie groupoid with the base manifold being a point and similarly a Lie algebra is a Lie algebroid with the base manifold being a point

The action Lie groupoid is thus the category $\mathsf{G} \ltimes M^\circ \rightrightarrows M^\circ$. The structure maps are rather obvious. The identity map $\mathrm{id}_x : M^\circ \to \mathsf{G} \ltimes M^\circ$ is simply $x \mapsto (\mathbb{1}_\mathsf{G}, x)$ and composition of morphisms $(g_2, g_1 \rhd x)$ and (g_1, x) is given by $(g_2, g_1 \rhd x) \circ (g_1, x) = ((g_2 g_1) \rhd x, x)$. The inverse of a morphism (g, x) is $(g, x)^{-1} = (g^{-1}, g \rhd x)$.

Just as a Lie group differentiates to a Lie algebra, a Lie groupoid differentiates to a Lie algebroid and a very general prescription for the Lie differentiation of L_∞-groupoids is found in [140], see also [103] for all details. The action Lie algebroid is now the trivial vector bundle $\mathfrak{g} \times M^\circ \to M^\circ$, where $\mathfrak{g}$ is the Lie algebra of G. The corresponding Q-manifold is $M = \mathfrak{g}[1] \times M^\circ$ with local coordinates x^μ and ξ^α of degrees 0 and 1, respectively, and the homological vector field reads as (2.54). Here, ρ^μ_α is given by linearising the Lie group action $\rho : \mathsf{G} \times M^\circ \to M^\circ$ and $f_{\alpha\beta}{}^\gamma$ are the structure constants of $\mathfrak{g}$.

As we shall see in Section 4, when dealing with gauge field theories, the groupoid perspective turns out to be a much more refined approach than the naive gauge orbit space perspective. In particular, the action Lie algebroid is the mathematical structure underlying the *Becchi–Rouet–Stora–Tyutin (BRST) complex*. Finally, also note that this construction generalises to actions of L_∞-algebras on manifolds.

Comments on generalisations. Our treatment of L_∞-algebras and L_∞-algebroids as particular Q-manifolds extends to cases where the $\mathbb{Z}$-graded vector bundles become infinite-dimensional. Here, however, care needs to be taken in the dualisation from the dg-algebra picture involving the Chevalley–Eilenberg differential Q to the coalgebra picture with codifferential D.[1]

Also, the cyclicity condition for L_∞-algebras (2.47b) can certainly be extended to bilinear maps on a module A over a ring R to that ring R, just as in the case of Lie algebras. Recall that such maps are used e.g. when defining Lagrangians, where A are representation space-valued differential forms and R is the ring of functions (or even densities) on a manifold.

2.5. Morphisms of L_∞-algebras and quasi-isomorphisms

Morphisms between L_∞-algebras, also known as L_∞-morphisms, generalise the notion of Lie algebra morphisms, which are maps preserving the Lie bracket. Because of the higher categorical nature of L_∞-algebras, it shouldn't be surprising that the former are much richer than

[1] See e.g. [141] and [142] for discussions on the subject.

the latter. The notion of morphism of L_∞-algebras is naturally induced by their description in terms of differential graded algebras. In the following, we shall translate this notion to the multilinear maps μ_i and clarify the appropriate notion of isomorphism, called *quasi-isomorphism*.[1] A key references for this Section is [143], where corresponding definitions and results are found in the more general case of A_∞-algebras.

2.5.1. L_∞-morphisms

L_∞-morphisms are most straightforwardly understood via the description of L_∞-algebras in terms of Q-manifolds or via their interpretation as codifferential coalgebras, where there is an obvious notion of morphism.The technical details of the derivation are found in Appendix A, which we summarise here as follows. In the Q-manifold picture an L_∞-morphism is described by a degree-0 morphism $(f, f^\#) : (M, Q) \to (M', Q')$ of $\mathbb{Z}$-graded manifolds, preserving the homological vector fields in the sense that $Q \circ f^\# = f^\# \circ Q'$. In the L_∞-picture, this can be rephrased as follows. A *morphism* between two L_∞-algebras (L, μ_i) and (L', μ_i') is a collection of mutlilinear totally graded anti-symmetric maps

$$\phi_i : \mathsf{L} \times \cdots \times \mathsf{L} \to \mathsf{L}' \ , \tag{2.56}$$

of degree $1 - i$ for $i \in \mathbb{N}$, which obey

$$\begin{aligned}
&\sum_{j+k=i} \sum_{\sigma \in \mathrm{Sh}(j;i)} (-1)^k \chi(\sigma; \ell_1, \ldots, \ell_i) \phi_{k+1}(\mu_j(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \ldots, \ell_{\sigma(i)}) \\
&\quad = \sum_{j=1}^{i} \frac{1}{j!} \sum_{k_1+\cdots+k_j=i} \sum_{\sigma \in \mathrm{Sh}(k_1,\ldots,k_{j-1};i)} \chi(\sigma; \ell_1, \ldots, \ell_i) \zeta(\sigma; \ell_1, \ldots, \ell_i) \times \qquad \text{(2.57a)} \\
&\qquad \times \mu_j'\Big(\phi_{k_1}\Big(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(k_1)}\Big), \ldots, \phi_{k_j}\Big(\ell_{\sigma(k_1+\cdots+k_{j-1}+1)}, \ldots, \ell_{\sigma(i)}\Big)\Big) \ .
\end{aligned}$$

Here, $\chi(\sigma; \ell_1, \ldots, \ell_i)$ is the Koszul sign and $\zeta(\sigma; \ell_1, \ldots, \ell_i)$ for a $(k_1, \ldots, k_{j-1}; i)$-shuffle σ is defined as

$$\zeta(\sigma; \ell_1, \ldots, \ell_i) \ := \ (-1)^{\sum_{1 \le m < n \le j} k_m k_n + \sum_{m=1}^{j-1} k_m (j-m) + \sum_{m=2}^{j} (1-k_m) \sum_{k=1}^{k_1+\cdots+k_{m-1}} |\ell_{\sigma(k)}|_{\mathsf{L}}} \ . \tag{2.57b}$$

An L_∞-morphism for which $\phi_i = 0$ for $i \geq 2$ is called *strict*. Clearly, L_∞-morphisms concentrated in degree 0 are of this type and for those the relation (2.57) reduces to

$$\phi_1(\mu_2(\ell_1, \ell_2))) \ = \ \mu_2'(\phi_1(\ell_1), \phi_1(\ell_2)) \ , \tag{2.58}$$

[1] They are most appropriate from a ∞-categorical point of view. For example, all definitions of the gerbes governing the higher form fields work up to quasi-isomorphisms.

that is, the expected relation for a morphism of Lie algebras. The notion of a weak morphism between 2-term L_∞-algebras was derived in [4], where also many more details on 2-term L_∞-algebras can be found. Morphisms of L_∞-algebras are composable, and the formulas for the composition map can be derived using the coalgebra picture in Appendix A in which composition is evident. As we shall see, there are two notions of isomorphisms between L_∞-algebras.

L_∞-isomorphisms. An L_∞-morphism is (strictly) invertible if and only if the map ϕ_1 is invertible. This is already suggested by (2.57), which shows that the higher products on either (L, μ_i) or (L', μ_i') can be reconstructed from the respective others if ϕ_1^{-1} is known. In this case, the map $\phi_1 : \mathsf{L} \to \mathsf{L}'$ is an isomorphism, and, correspondingly, we call such L_∞-morphisms *isomorphisms of L_∞-algebras* or *L_∞-isomorphisms*. They allow us to formulate the next theorem.

Decomposition theorem. Any L_∞-algebra L is L_∞-isomorphic to the direct sum of a minimal L_∞-algebra (that is, an L_∞-algebra with $\mu_1 = 0$) and a linearly contractible L_∞-algebra (that is, an L_∞-algebra with $\mu_i = 0$ for $i > 1$ and trivial cohomology), see [143] for the more general case of A_∞-algebras.

Because an L_∞-isomorphism is in particular a cochain map, it follows that the minimal L_∞-algebra in the decomposition should have the cohomology complex $(H^\bullet_{\mu_1}(\mathsf{L}), 0)$ of the complex (L, μ_1) as its differential graded vector space. Thus, we have in general

$$(\mathsf{L}, \mu_i) \;\cong\; (\mathsf{L}', \mu_i') \;:=\; H^\bullet_{\mu_1}(\mathsf{L}) \oplus \mathsf{L}/H^\bullet_{\mu_1}(\mathsf{L}) \tag{2.59}$$

and $\mathsf{L}/H^\bullet_{\mu_1}(\mathsf{L})$ is the linearly contractible part.

2.5.2. L_∞-quasi-isomorphisms

Often, L_∞-isomorphisms do not yield the physically relevant equivalence classes; see e.g. Section 3.1.. Instead, one should consider the following weaker notion of an isomorphism: a *quasi-isomorphism* is a morphism that induces an isomorphism on the level of cohomology. Explicitly, a quasi-isomorphism of L_∞-algebras is a L_∞-morphism $(\mathsf{L}, \mu_i) \to (\mathsf{L}', \mu_i')$ for which ϕ_1 induces an isomorphism $H^\bullet_{\mu_1}(\mathsf{L}) \cong H^\bullet_{\mu_1'}(\mathsf{L}')$ of graded vector spaces.[1] We will argue below

[1]We observe that the condition (2.57) for $i = 1$ reads $\phi_1(\mu_1(\ell)) = \mu_1'(\phi_1(\ell))$, for $\ell \in \mathsf{L}$. Hence, ϕ_1 is a cochain map between (L, μ_1) and (L', μ_1') and thus descends to a homomorphism on cohomology.

that L_∞-quasi-isomorphy indeed induces an equivalence on the space of L_∞-algebras. Since a Lie algebra is isomorphic to its cohomology, the difference between quasi-isomorphisms and L_∞-isomorphisms is a new feature of L_∞-algebras, which is not present in the case of Lie algebras. We also call two differential graded algebras $(\mathscr{C}^\infty(\mathsf{L}[1]), Q)$ and $(\mathscr{C}^\infty(\mathsf{L}'[1]), Q')$ *quasi-isomorphic*, if they form the Chevalley–Eilenberg algebras of two quasi-isomorphic L_∞-algebras L and L'.

Besides the decomposition theorem, there are two additional structure theorems, which generally hold for homotopy algebras and are useful for the homological algebraic perspective on field theories.

Strictification theorem. It can be shown [144, 145], that any L_∞-algebra is L_∞-quasi-isomorphic to a strict L_∞-algebra. Recall that a strict L_∞-algebra is a differential graded Lie algebra because only the differential and the binary product are non-vanishing. This result is known as *strictification*[1] of an L_∞-algebra. Whilst this is a crucial result for making general statements about L_∞-algebras, in practical applications, however, it often turns out that the transition to the strict model of an L_∞-algebra is either hard to begin with or not very convenient and too restrictive for further computations. We just mention here the relevance of this result by saying that the strictification theorem implies that any field theory is equivalent to a field theory with only cubic interaction terms. See [146] or [42] and 5.2.2. for applications to physical theories.

Minimal model theorem. Here, we present another crucial result, companion to the above one. It states that any L_∞-algebra (L, μ_i) is quasi-isomorphic to an L_∞-algebra (L', μ_i') with $\mu_1' = 0$, that is a minimal one. This result follows from the decomposition theorem [143] that guarantees that there are L_∞-quasi-isomorphisms

$$\begin{aligned} p \;:\; \mathsf{L} \;&\xrightarrow{\cong}\; H^\bullet_{\mu_1}(\mathsf{L}) \oplus \mathsf{L}/H^\bullet_{\mu_1}(\mathsf{L}) \;\xrightarrow{\pi}\!\!\!\!\rightarrow\; H^\bullet_{\mu_1}(\mathsf{L}) \;, \\ e \;:\; H^\bullet_{\mu_1}(\mathsf{L}) \;&\xhookrightarrow{\iota}\; H^\bullet_{\mu_1}(\mathsf{L}) \oplus \mathsf{L}/H^\bullet_{\mu_1}(\mathsf{L}) \;\xrightarrow{\cong}\; \mathsf{L} \;. \end{aligned} \tag{2.60}$$

Here, $\pi : H^\bullet_{\mu_1}(\mathsf{L}) \oplus \mathsf{L}/H^\bullet_{\mu_1}(\mathsf{L}) \twoheadrightarrow H^\bullet_{\mu_1}(\mathsf{L})$ is the projection and $\iota : H^\bullet_{\mu_1}(\mathsf{L}) \hookrightarrow H^\bullet_{\mu_1}(\mathsf{L}) \oplus \mathsf{L}/H^\bullet_{\mu_1}(\mathsf{L})$ the inclusion. Both π and ι are strict, but neither p nor e are, in general. The existence of the second L_∞-quasi-isomorphism in (2.60) is indeed known as the *minimal*

[1] In the mathematical literature it is often used the term *rectification*.

model theorem [147, 143], which historically predates the decomposition theorem. The L_∞-structure on $H^\bullet_{\mu_1}(\mathsf{L})$ is also called a *minimal model*, i.e. a minimal representative of the quasi-isomorphism class of L. We note that minimal models are unique up to L_∞-isomorphisms.

Roughly speaking, the restriction to cohomology $\mathsf{L} \to H^\bullet_{\mu_1}(\mathsf{L})$ is the restriction from the kinematical data of a gauge theory to its physical states, and we shall come back to this in Section 5. Hence, as we shall see, quasi-isomorphisms constitute the correct notion of equivalence of physics theories [42, 43]. In particular, the minimal model of a classical field theory describes precisely the tree-level scattering amplitudes, as it corresponds to a field theory that is equivalent to the original field theory, but without any propagating degrees of freedom [148, 143, 42, 43]. In [42] we constructed these minimal models for Yang–Mills and higher Chern–Simons theory. We shall see this in Chapter 5.

The construction of a minimal model for (L, μ_i) amounts to computing the L_∞-structure given by the brackets on the cohomology ring $H^\bullet_{\mu_1}(\mathsf{L}) =: \mathsf{L}'$. Explicitly, let (L, μ_i) be an L_∞-algebra and write $\mathrm{d}_k : \mathsf{L}_k \to \mathsf{L}_{k+1}$ for μ_1. Consider the complex (L, d) and denote its cohomology complex by $(H^\bullet_{\mathrm{d}(\mathsf{L})}, 0)$. The minimal model theorem then tells us that we have cochain maps p and e

$$h \circlearrowright \mathsf{L} \underset{e}{\overset{p}{\rightleftarrows}} H^\bullet_{\mathrm{d}(\mathsf{L})} \,, \tag{2.61a}$$

with $p \circ e = 1$ and h is a *contracting homotopy*. Specifically, h is a collection of maps $h_k : \mathsf{L}_k \to \mathsf{L}_{k-1}$ such that

$$\mathrm{d}_k = \mathrm{d}_k \circ h_{k+1} \circ \mathrm{d}_k \,. \tag{2.61b}$$

It follows that we can construct the three projectors

$$P_k := e_k \circ p_k \,, \quad h_{k+1} \circ \mathrm{d}_k \,, \quad \text{and} \quad \mathrm{d}_{k-1} \circ h_k \tag{2.62a}$$

with

$$1 = P_k + h_{k+1} \circ \mathrm{d}_k + \mathrm{d}_{k-1} \circ h_k \,, \tag{2.62b}$$

that is, they allow for the decomposition

$$\mathsf{L} \cong \mathrm{im}(P) \oplus \mathrm{im}(h \circ \mathrm{d}) \oplus \mathrm{im}(\mathrm{d} \circ h) \quad \text{with} \quad \mathrm{im}(P) \cong H^\bullet_{\mathrm{d}(\mathsf{L})} \,. \tag{2.63}$$

This decomposition is also known as the *abstract Hodge–Kodaira decomposition*, see e.g. [143] as well as Appendix B for more details.

Explicit minimal model. Let us see how the homological perturbation lemma translates to the dual picture. To write down the L_∞-structure on $H^\bullet_{\mathrm{d}(\mathsf{L})}$, let us set $\mathsf{L}' := H^\bullet_{\mathrm{d}(\mathsf{L})}$ and $\mu_1' := 0$. Following [143], we define totally graded anti-symmetric multilinear maps $\phi_i : \mathsf{L}' \times \cdots \times \mathsf{L}' \to \mathsf{L}$ of homogeneous degree $1-i$ recursively by setting

$$
\begin{aligned}
\phi_1(\ell_1') &:= e(\ell') \,, \\
\phi_2(\ell_1',\ell_2') &:= -h(\mu_2(e(\ell_1'),e(\ell_2'))) \,, \\
&\;\;\vdots \\
\phi_i(\ell_1',\ldots,\ell_i') &:= -\sum_{j=2}^{i}\frac{1}{j!}\sum_{k_1+\cdots+k_j=i}\ \sum_{\sigma\in\mathrm{Sh}(k_1,\ldots,k_{j-1};i)}\chi(\sigma;\ell_1',\ldots,\ell_i')\zeta(\sigma;\ell_1',\ldots,\ell_i')\times \\
&\quad\times h\Big\{\mu_j\Big(\phi_{k_1}\Big(\ell'_{\sigma(1)},\ldots,\ell'_{\sigma(k_1)}\Big),\ldots,\phi_{k_j}\Big(\ell'_{\sigma(k_1+\cdots+k_{j-1}+1)},\ldots,\ell'_{\sigma(i)}\Big)\Big)\Big\}\,,
\end{aligned}
\tag{2.64}
$$

where $\ell_1',\ldots,\ell_i' \in \mathsf{L}'$. Here, h and e are again the maps from (2.61), $\chi(\sigma;\ell_1',\ldots,\ell_i')$ is the Koszul sign defined in (2.44d), and $\zeta(\sigma;\ell_1',\ldots,\ell_i')$ the sign factor introduced in (2.57). Recall that e is a cochain map and thus so is ϕ_1. The maps ϕ_i provide an L_∞-quasi-isomorphism from L' to L provided the higher products μ_i' on L' are constructed recursively as

$$
\begin{aligned}
\mu_1'(\ell_1') &:= 0 \,, \\
\mu_2'(\ell_1',\ell_2') &:= p(\mu_2(e(\ell_1'),e(\ell_2'))) \,, \\
&\;\;\vdots \\
\mu_i'(\ell_1',\ldots,\ell_i') &:= \sum_{j=2}^{i}\frac{1}{j!}\sum_{k_1+\cdots+k_j=i}\ \sum_{\sigma\in\mathrm{Sh}(k_1,\ldots,k_{j-1};i)}\chi(\sigma;\ell_1',\ldots,\ell_i')\zeta(\sigma;\ell_1',\ldots,\ell_i')\times \\
&\quad\times p\Big\{\mu_j\Big(\phi_{k_1}\Big(\ell'_{\sigma(1)},\ldots,\ell'_{\sigma(k_1)}\Big),\ldots,\phi_{k_j}\Big(\ell'_{\sigma(k_1+\cdots+k_{j-1}+1)},\ldots,\ell'_{\sigma(i)}\Big)\Big)\Big\}\,.
\end{aligned}
\tag{2.65}
$$

Using the identities $p\circ\phi_1 = p\circ e = 1$, $p\circ\mu_1 = p\circ\mathrm{d} = 0$, and $\mu_1(e(\ell')) = \mathrm{d}(e(\ell')) = 0$ for all $\ell' \in \mathsf{L}' = H^\bullet_{\mathrm{d}(\mathsf{L})}$ together with the decomposition (2.62b) and the higher homotopy Jacobi identities (2.44c) for the products μ_i on L, it is rather straightforward to see that (2.64) together with (2.65) satisfy the definition (2.57) of an L_∞-morphism.[1]

Weak inverses of L_∞-quasi-isomorphisms. Using the decomposition theorem, any L_∞-quasi-isomorphism $\phi : (\mathsf{L},\mu_i) \to (\mathsf{L}',\mu_i')$ can be *weakly inverted* by going through the corresponding minimal models. Specifically, using the L_∞-quasi-isomorphisms (2.60), we obtain an

[1] The calculations are much simplified if one instead works with the contracting homotopy $\tilde h := h - h\circ h\circ\mu_1$ in (2.64) since then $\tilde h\circ\tilde h\circ\mu_1 = \tilde h\circ\tilde h\circ\mathrm{d} = 0$. See [146] for details.

L_∞-isomorphism

$$\phi_{\rm rd}\ :\ H^\bullet_{\mu_1}(\mathsf{L})\ \longrightarrow\ \mathsf{L}\ \xrightarrow{\ \phi\ }\ \mathsf{L}'\ \longrightarrow\ H^\bullet_{\mu'_1}(\mathsf{L}') \tag{2.66}$$

which can be inverted and composed to give the inverse L_∞-quasi-isomorphism,

$$\phi^{-1}\ :\ \mathsf{L}'\ \longrightarrow\ H^\bullet_{\mu'_1}(\mathsf{L}')\ \xrightarrow{\ \phi^{-1}_{\rm rd}\ }\ H^\bullet_{\mu_1}(\mathsf{L})\ \longrightarrow\ \mathsf{L}\ . \tag{2.67}$$

For instance, the quasi-isomorphisms (2.60) are trivially weakly inverses of each other. Since L_∞-quasi-isomorphism can be weakly inverted, L_∞-quasi-isomorphy induces an equivalence relation on the space of all L_∞-algebras.

Examples. Note that linearly contractible L_∞-algebras, such as $V[1]\xrightarrow{\rm id}V$, have trivial cohomology and therefore they are quasi-isomorphic to the trivial L_∞-algebra. The decomposition theorem therefore implies the minimal model theorem.

It is always possible to extend an L_∞-algebra with underlying graded vector space

$$\mathsf{L}\ =\ \cdots\ \to\ 0\ \to\ 0\ \to\ \mathsf{L}_i\ \xrightarrow{\ \mu_1\ }\ \mathsf{L}_{i+1}\ \xrightarrow{\ \mu_1\ }\ \mathsf{L}_{i+2}\ \to\ \cdots\ , \tag{2.68}$$

to an L_∞-algebra structure on

$$\mathsf{L}'\ =\ \cdots\ \longrightarrow\ 0\ \longrightarrow\ \ker(\mu_1)\hookrightarrow\mathsf{L}_i\ \xrightarrow{\ \mu_1\ }\ \mathsf{L}_{i+1}\ \xrightarrow{\ \mu_1\ }\ \mathsf{L}_{i+2}\ \longrightarrow\ \cdots\ . \tag{2.69}$$

Note, however, that L' is L_∞-quasi-isomorphic to an L_∞-algebra structure on

$$\mathsf{L}''\ =\ \cdots\ \longrightarrow\ 0\ \longrightarrow\ 0\ \longrightarrow\ {\rm coker}(\mu_1)\ \xrightarrow{\ \mu_1\ }\ \mathsf{L}_{i+1}\ \xrightarrow{\ \mu_1\ }\ \mathsf{L}_{i+2}\ \longrightarrow\ \cdots\ , \tag{2.70}$$

effectively reducing the L_∞-algebra L to L'' by extending it to L'. It is therefore important to distinguish between the L_∞-algebras L and L'.

Morphisms of cyclic L_∞-algebras. The definition of a morphism of cyclic L_∞-algebras is induced from the description in terms of a differential graded algebra: it is simply a morphism of symplectic differential graded manifolds. However, such a morphism $\Phi:(M,Q,\omega)\to(M',Q',\omega')$ would imply that $\omega=\Phi^*\omega'$ which, due to the non-degeneracy of ω implies that Φ is injective. This is often too restrictive, and one usually switches to *Lagrangian correspondences*, see e.g. [149]. For our purposes, however, this is not necessary.

Moreover, we shall restrict our morphisms a bit further with an eye to homotopy Maurer–Cartan theory, which we shall discuss in Section 3. In this context, we are dealing with constant symplectic forms and we wish that morphisms of cyclic L_∞-algebras preserve the homotopy Maurer–Cartan action. Following [143], we thus define a *morphism of cyclic L_∞-algebras* $\phi : (\mathsf{L}, \mu_i, \langle -,-\rangle_{\mathsf{L}}) \to (\mathsf{L}', \mu_i', \langle -,-\rangle_{\mathsf{L}'})$ as an L_∞-morphism $\phi : (\mathsf{L}, \mu_i) \to (\mathsf{L}', \mu_i')$ such that in addition

$$\langle \phi_1(\ell_1), \phi_1(\ell_2)\rangle_{\mathsf{L}'} \;=\; \langle \ell_1, \ell_2\rangle_{\mathsf{L}} \tag{2.71a}$$

and for all $i \geq 3$ and $\ell_1, \ldots, \ell_i \in \mathsf{L}$,

$$\sum_{\substack{j+k=i\\ j,k\geq 1}} \langle \phi_j(\ell_1, \ldots, \ell_j), \phi_k(\ell_{j+1}, \ldots, \ell_{j+k}))\rangle_{\mathsf{L}'} \;=\; 0\ . \tag{2.71b}$$

As before, for the sake of brevity, we shall also refer to such morphisms as *cyclic L_∞-morphisms*.

Decomposition theorem for cyclic L_∞-algebras. As shown in [143] for cyclic A_∞-algebras, the decomposition theorem extends to cyclic L_∞-algebras. That is, any cyclic L_∞-algebra is isomorphic to the direct sum of a minimal cyclic L_∞-algebra and a linearly contractible cyclic L_∞-algebra.

Quasi-isomorphisms of cyclic L_∞-algebras. We indicated above that quasi-isomorphisms allow us to describe an equivalence between data that is the same up to some gauge symmetry. To extend this notion to action principles, we need a preservation of the cyclic inner product on the relevant parts, which are the cohomology. We therefore define that a *quasi-isomorphism of cyclic L_∞-algebras* is a morphism of cyclic L_∞-algebras $\phi : \mathsf{L} \to \mathsf{L}'$, which descends to an isomorphism of cyclic L_∞-algebras between $H^\bullet_{\mu_1}(\mathsf{L})$ and $H^\bullet_{\mu_1}(\mathsf{L}')$.

2.5.3. Representations of L_∞-algebras

To define (higher) supersymmetric field theories with matter content, we need to specify what we mean by a representation of an L_∞-algebra. The first ingredient is a higher analogue of a vector space carrying the representation. There is a variety of definitions in the literature already for the simplest case of a 2-vector space. Fortunately, supersymmetry requires us to use the same type of categorified vector space that underlies our L_∞-algebras. We can thus restrict ourselves to dg-vector spaces. Note that these can be regarded as Abelian L_∞-algebras with all higher brackets μ_i trivial for $i \geq 2$.

L_∞-representations. There are now (at least) three evident ways of defining a representation of an L_∞-algebra (L,μ_i) on a dg-vector space (V,d):

(i) Via an action of elements of L on V with compatibility relations as done in [134, Definition 5.1], see also [150];

(ii) As an L_∞-morphism of L_∞-algebras from L to End (V), cf. e.g. [151, Definition 4.3];

(iii) As a semidirect product of L_∞-algebras $\mathsf{L}\ltimes\mathsf{V}$, which can be regarded as a short exact sequence of L_∞-algebras $\mathsf{V}\hookrightarrow\mathsf{L}\ltimes\mathsf{V}\rightarrow\mathsf{L}$, cf. e.g. [152, Definition 11.1.1.1].

Theorem 5.4 of [134] shows that (i) and (ii) are equivalent, and we choose to work with the latter. Recall that any dg-vector space (V,d) comes with a dg-algebra $(\mathsf{End}\,(\mathsf{V}),\mathrm{d}_{\mathsf{End}\,(\mathsf{V})})$, which is defined by

$$\mathsf{End}\,(\mathsf{V})\ :=\ \bigoplus_{i\in\mathbb{Z}}\mathsf{End}_{\,i}(\mathsf{V})\quad\text{with}\quad\mathsf{End}_{\,i}(\mathsf{V})\ :=\ \prod_{j\in\mathbb{Z}}\mathsf{Hom}(\mathsf{V}_j,\mathsf{V}_{j+i})\tag{2.72a}$$

together with

$$\mathrm{d}_{\mathsf{End}\,(\mathsf{V})}T\ :=\ \mathrm{d}\circ T-(-1)^{|T|}T\circ\mathrm{d}\tag{2.72b}$$

for $T\in\mathsf{End}\,(\mathsf{V})$. Together with the commutator $[S,T]:=S\circ T-(-1)^{|T||S|}T\circ S$ for $S,T\in\mathsf{End}\,(\mathsf{V})$, End (V) becomes a dg-Lie algebra.

An *L_∞-representation* of an L_∞-algebra (L,μ_i) on a differential graded vector space (V,d) is an L_∞-morphism, as defined in (2.57), from (L,μ_i) to $(\mathsf{End}\,(\mathsf{V}),\mathrm{d}_{\mathsf{End}\,(\mathsf{V})})$.

Example. As an example, let us consider the case of a representation of a Lie 2-algebra, that is, an L_∞-algebra $(\mathsf{L},\mu_i)=(\mathsf{L}_{-1}\oplus\mathsf{L}_0,\mu_i)$ concentrated in degrees -1 and 0 on the differential graded vector space $(\mathsf{V},\mathrm{d})=(\mathsf{V}_{-1}\oplus\mathsf{V}_0,\mathrm{d})$. We note that

$$\mathsf{End}\,(\mathsf{V})\ =\ \Big(\mathsf{Hom}(\mathsf{V}_0,\mathsf{V}_{-1})\xrightarrow{\mathrm{d}_{\mathsf{End}\,(\mathsf{V})}}\mathsf{End}\,(\mathsf{V}_0)\oplus\mathsf{End}\,(\mathsf{V}_{-1})\xrightarrow{\mathrm{d}_{\mathsf{End}\,(\mathsf{V})}}\Big(\mathsf{Hom}(\mathsf{V}_{-1},\mathsf{V}_0)\Big)\tag{2.73}$$

and therefore a representation of (L,μ_i) on (V,d) consists of a cochain map ϕ_1

$$\begin{array}{ccccc}\mathsf{L}_{-1} & \xrightarrow{\mu_1} & \mathsf{L}_0 & \xrightarrow{0} & 0\\ \big\downarrow{\scriptstyle\phi_1} & & \big\downarrow{\scriptstyle\phi_1} & & \big\downarrow{\scriptstyle\phi_1}\\ \mathsf{Hom}(\mathsf{V}_0,\mathsf{V}_{-1}) & \xrightarrow{\mathrm{d}_{\mathsf{End}\,(\mathsf{V})}} & \mathsf{End}\,(\mathsf{V}_0)\oplus\mathsf{End}\,(\mathsf{V}_{-1}) & \xrightarrow{\mathrm{d}_{\mathsf{End}\,(\mathsf{V})}} & \mathsf{Hom}(\mathsf{V}_{-1},\mathsf{V}_0)\end{array}\tag{2.74a}$$

together with a map

$$\phi_2 \;:\; \mathsf{L}_0 \times \mathsf{L}_0 \;\to\; \mathsf{Hom}(\mathsf{V}_0, \mathsf{V}_{-1}) \tag{2.74b}$$

such that

$$\begin{aligned}
\phi_1(\mu_1(y)) &= \mathrm{d}_{\mathsf{End}\,(\mathsf{V})}\phi_1(y) \;=\; \mathrm{d} \circ \phi_1(y) + \phi_1(y) \circ \mathrm{d}\ ,\\
\phi_1(\mu_2(x_1,x_2)) &= [\phi_1(x_1),\phi_1(x_2)] + \mathrm{d}_{\mathsf{End}\,(\mathsf{V})}\phi_2(x_1,x_2)\\
&= [\phi_1(x_1),\phi_1(x_2)] + \mathrm{d} \circ \phi_2(x_1,x_2) + \phi_2(x_1,x_2) \circ \mathrm{d}\ ,\\
\phi_1(\mu_2(x,y)) &= [\phi_1(x),\phi_1(y)] + \phi_2(x,\mu_1(y))\ ,\\
\phi_1(\mu_3(x_1,x_2,x_3)) &= \phi_2(\mu_2(x_1,x_2),x_3) - [\phi_1(x_1),\phi_2(x_2,x_3)] + \text{cyclic}
\end{aligned} \tag{2.74c}$$

for al $x_{1,2,3} \in \mathsf{L}_0$ and $y \in \mathsf{L}_{-1}$, see (2.57).

As examples of concrete applications, let us specialise to the case $\phi_2 = 0$. For $(\mathsf{V}, \mathrm{d}) = (\mathsf{L}_{-1} \oplus \mathsf{L}_0, \mu_1)$ and $\phi_1 : \mathsf{L}_{-1} \to \mathsf{Hom}(\mathsf{L}_0, \mathsf{L}_{-1})$ trivial, we recover the representations underlying the models of [153]. For a suitable choice of L, $\mathsf{V}_{-1} = 0$ and V_0 the tensor fields on a manifold, we obtain the representations relevant in generalised geometry and double field theory, see the discussion in [154].

3 Homotopy Maurer–Cartan theory

Homotopy Maurer–Cartan theory was first developed in [10] in the context of the second quantisation of string theory, known as string field theory. In the same paper also L_∞-algebras were defined for the first time[1], taking inspiration from the older definition of A_∞-algebras [40, 41]. Explicitly, the structures of the Hilbert spaces of string field theories are encoded in higher algebraic structures: L_∞-algebras in the case of closed string field theory and A_∞-algebras in the case of open string field theory[2]. At the classical level, the relevant action is simply the canonical action associated with an L_∞-algebra (or A_∞-algebra), which is known as the homotopy Maurer-Cartan action. As mentioned in the Introduction, this structure is somehow reflected in ordinary field theory. This is anything but unexpected if one believes in the fundamental role that string theory, and therefore string field theory, is supposed to play in physics. We shall explore this connection in detail in Chapters 4 and 5.

Given an L_∞-algebra L, we shall see how to construct the kinematical data of a corresponding higher gauge theory[3] from the homotopy Maurer–Cartan equation of a certain L_∞-algebra. As mentioned in the Introduction, such a higher gauge theory should indeed describe higher connections taking values in categorified Lie algebras and living on a higher principal bundle, see Appendix D. That is, following [102] we shall see how to specify notions of gauge potentials, curvatures, (higher) gauge transformations and Bianchi identities. In particular, homotopy Maurer–Cartan theory turns out to be a vast generalisation of Chern–Simons theory, which contains higher Chern–Simons theories as special cases. Here, the discussion

[1] already together with their quantum variants.

[2] The construction of open-closed string field theory is based on a combination of both.

[3] There are several different ways of doing this, generalising the usual definition of connections on principal bundles.

follows the detailed review [42].

3.1. Homotopy Maurer–Cartan equation and action

Along this Section, we show how any L_∞-algebra comes naturally with an associated homotopy Maurer–Cartan theory. In the following, some results require quite technical and tedious computational proofs, which we collect in Appendix C, in order to avoid them cluttering our discussion.

Differential graded Lie algebras. We start by recalling the Maurer–Cartan theory for a differential graded Lie algebra $(\mathfrak{g}, \mathrm{d}, [-,-])$. A Maurer-Cartan element a is an element of degree 1 satisfying

$$\mathrm{d}a + \frac{1}{2}[a,a] = 0 \ , \tag{3.1}$$

which is called Maurer–Cartan equation. This reduces to $\mathrm{d}\omega + \omega \wedge \omega = 0$, for a matrix Lie algebra valued differential one-form ω. Let us assume that $\mathfrak{g}$ is endowed with an invariant inner product compatible with the differential, that is a bilinear, graded symmetric, and non-degenerate map $\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ such that

$$\langle \mathrm{d}a_1, a_2 \rangle + (-1)^{|a_1|} \langle a_1, \mathrm{d}a_2 \rangle = 0 \quad \text{and} \quad \langle [a_1, a_2], a_3 \rangle + (-1)^{|a_1||a_2|} \langle a_2, [a_1, a_3] \rangle = 0 \ , \tag{3.2}$$

for $a_1, a_2, a_3 \in \mathfrak{g}$. Then, the equation (3.1) is the equation of motion of the Maurer–Cartan action

$$S_{\mathsf{MC}} := \frac{1}{2}\langle a, \mathrm{d}a \rangle + \frac{1}{3!}\langle a, [a,a] \rangle \ . \tag{3.3}$$

As L_∞-algebras are generalisations of differential graded Lie algebras this construction naturally generalises.

Gauge potentials and curvatures. In the following, let L be an L_∞-algebra with higher products μ_i and define $|\ell|_{\mathsf{L}} \in \mathbb{Z}$ as the L-*degree* of a homogeneous element $\ell \in \mathsf{L}$. We call an element $a \in \mathsf{L}_1$ a *gauge potential*, and we define its *curvature* $f \in \mathsf{L}_2$ as

$$f \ := \ \sum_{i \geq 1} \frac{1}{i!} \mu_i(a, \ldots, a) \ . \tag{3.4}$$

This terminology originates from particular choices of L in which a and f indeed reduce to the gauge potential and the curvature of (higher) gauge theory.

A gauge potential $a \in \mathsf{L}_1$ is called a *Maurer–Cartan (MC) element* provided it satisfies the *homotopy Maurer–Cartan equation*

$$f \;=\; \mu_1(a) + \tfrac{1}{2}\mu_2(a,a) + \cdots \;=\; 0\ . \tag{3.5}$$

This equation describes an abstract form of flatness of the gauge potential and generalises (3.1).

As can be easily seen, due to the higher homotopy Jacobi identities (2.44c), the curvature satisfies the *Bianchi identity*

$$\sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+1}(f,a,\ldots,a) \;=\; 0\ . \tag{3.6}$$

We invite the interested reader to consult Appendix C for detailed proofs, one using direct but tedious computation and another, shorter one using the evaluation map ξ and formula (2.53).

3.1.1. Gauge transformations

Elements of L_{-k} for $k \geq 0$ are the gauge parameters of infinitesimal gauge transformations (also called level 0 gauge transformations for $k = 0$) and infinitesimal higher gauge transformations (also called level k gauge transformations for $k \geq 1$). Mathematically, level 0 gauge transformations between two gauge potentials, are encoded in partially flat homotopies between them. These homotopies are captured by gauge potentials for the tensor product L_∞-algebra $\Omega^\bullet(I,\mathsf{L}) := \Omega^\bullet(I)\otimes\mathsf{L}$ with $I := [0,1] \subseteq \mathbb{R}$, where the tensor product is as defined in Section 2.4.. We can decompose $\Omega^\bullet_1(I,\mathsf{L}) \cong \mathscr{C}^\infty(I,\mathsf{L}_1)\oplus\Omega^1(I,\mathsf{L}_0)$ and an element $\mathsf{a} \in \Omega^\bullet_1(I,\mathsf{L})$ takes the form

$$\mathsf{a}(t) \;=\; a(t) + \mathrm{d}t\otimes c(t)\ , \tag{3.7}$$

with $t \in I$, $a(t) \in \mathscr{C}^\infty(I,\mathsf{L}_1)$ and $c(t) \in \mathscr{C}^\infty(I,\mathsf{L}_0)$. Moreover, since $\Omega^\bullet_2(I,\mathsf{L}) \cong \mathscr{C}^\infty(I,\mathsf{L}_2)\oplus\Omega^1(I,\mathsf{L}_1)$, the curvature $\mathsf{f} \in \Omega^\bullet_2(I,\mathsf{L})$ reads as

$$\begin{aligned}\mathsf{f}(t) \;&=\; \sum_{i\geq 1}\frac{1}{i!}\hat\mu_i(\mathsf{a}(t),\ldots,\mathsf{a}(t),\mathsf{a}(t))\\ &=\; f(t) + \mathrm{d}t\otimes\left\{\frac{\partial}{\partial t}a(t) - \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a(t),\ldots,a(t),c(t))\right\},\end{aligned} \tag{3.8}$$

where the higher products μ_i are agnostic about the form degree of their arguments. Partial flatness $\frac{\partial}{\partial t}\lrcorner f = 0$ or, equivalently, $\mathsf{f} \in \mathscr{C}^\infty(I,\mathsf{L}_2)$, leads to the differential equation

$$\frac{\partial}{\partial t}a(t) - \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a(t),\ldots,a(t),c(t)) \;=\; 0\ , \tag{3.9}$$

which describes the changes of $a(t)$. At $t = 0$, we can read off the gauge transformation $a \mapsto a + \delta_{c_0} a$ of the gauge potential $a := a(0)$ parametrised by the gauge parameter $c_0 := c(0) \in \mathsf{L}_0$,

$$\delta_{c_0} a \;:=\; \frac{\partial}{\partial t}\bigg|_{t=0} a(t) \;=\; \sum_{i\geq 0} \frac{1}{i!}\mu_{i+1}(a,\ldots,a,c_0)\ . \tag{3.10}$$

Furthermore, the curvature transforms under gauge transformations as

$$\delta_{c_0} f \;:=\; \frac{\partial}{\partial t}\bigg|_{t=0} f(t) \;=\; \sum_{i\geq 0} \frac{(-1)^i}{i!}\mu_{i+2}(f,a,\ldots,a,c_0)\ , \tag{3.11}$$

the proof of this equation is found in Appendix C .

Always in Appendix C, it is also demonstrated that the commutator of two successive gauge transformations with gauge parameters $c_0, c_0' \in \mathsf{L}_0$ is given by

$$[\delta_{c_0}, \delta_{c_0'}]a \;=\; \delta_{c_0''} a + \sum_{i\geq 0} \frac{1}{i!}(-1)^i \mu_{i+3}(f,a,\ldots,a,c_0,c_0') \tag{3.12a}$$

with

$$c_0'' \;:=\; \sum_{i\geq 0} \frac{1}{i!}\mu_{i+2}(a,\ldots,a,c_0,c_0')\ , \tag{3.12b}$$

where we have made repeated use of the higher Jacobi identities (2.44c). This shows that for general L_∞-algebras, gauge transformations only close up to terms proportional to the curvature f. For instance, the standard gauge transformations always close for strict L_∞-algebras, for which only the differential and the 2-product are non-trivial.

3.1.2. Higher gauge transformations

It is important to stress that, in general, the gauge parameters $c_0 \in \mathsf{L}_0$ may enjoy a gauge freedom themselves which is mediated by next-to-lowest gauge parameters $c_{-1} \in \mathsf{L}_{-1}$. Likewise, the next-to-lowest gauge parameters $c_{-1} \in \mathsf{L}_{-1}$ may enjoy gauge freedom that is mediated by next-to-next-to-lowest gauge parameters $c_{-2} \in \mathsf{L}_{-2}$, and so on. These are known as the *higher gauge transformations*. Mathematically, they are described by homotopies between homotopies. In particular, consider a level 1 gauge transformation between two level 0 gauge transformations. These are captured by gauge potentials on the tensor product $\Omega^\bullet(I^2, \mathsf{L})$ with $I^2 := I \times I$ and $I := [0,1] \subseteq \mathbb{R}$. This time, we have the decomposition $\Omega^\bullet_1(I^2,\mathsf{L}) \cong \mathscr{C}^\infty(I^2,\mathsf{L}_1) \oplus \Omega^1(I^2,\mathsf{L}_0) \oplus \Omega^2(I^2,\mathsf{L}_{-1})$ and hence, $\mathsf{a} \in \Omega^\bullet_1(I^2,\mathsf{L})$ takes the form

$$\mathsf{a}(t,s) \;=\; a(t,s) + \mathrm{d}t \otimes c^{(1)}(t,s) + \mathrm{d}s \otimes c^{(2)}(t,s)\mathrm{d}s + (\mathrm{d}t \wedge \mathrm{d}s) \otimes \sigma(t,s)\ , \tag{3.13a}$$

where $(t,s)\in I^2$, $a(t,s)\in\mathscr{C}^\infty(I^2,\mathsf{L}_1)$, $c^{(1,2)}(t,s)\in\mathscr{C}^\infty(I^2,\mathsf{L}_0)$, and $\sigma(t,s)\in\mathscr{C}^\infty(I^2,\mathsf{L}_{-1})$. The fact that a is a homotopy between homotopies is reflected in the boundary conditions

$$a(0,s) \;=\; a \quad\text{and}\quad c^{(1)}(t,0) \;=\; c(t)\ . \tag{3.13b}$$

The geometric shape underlying this homotopy between homotopies is not a square but a bigon with a coordinate degeneracy in s at $t=0$ and $t=1$. Therefore, we have to supplement the above boundary conditions by

$$c^{(2)}(0,s) \;=\; c^{(2)}(1,s) \;=\; 0\ . \tag{3.13c}$$

Moreover, $\Omega^\bullet_2(I^2,\mathsf{L})$ decomposes as $\Omega^\bullet_2(I^2,\mathsf{L})\cong\mathscr{C}^\infty(I^2,\mathsf{L}_2)\oplus\Omega^1(I^2,\mathsf{L}_1)\oplus\Omega^2(I^2,\mathsf{L}_0)$, and upon imposing the partial flatness condition $\mathsf{f}\in\mathscr{C}^\infty(I^2,\mathsf{L}_2)$, we obtain the level 0 gauge transformation (3.10) with the gauge parameter $c_0:=c^{(1)}(0,0)\in\mathsf{L}_0$ together with the level 1 gauge transformation

$$\delta_{c_{-1}}c_0 \;:=\; \frac{\partial}{\partial s}\bigg|_{t=s=0} c^{(1)}(t,s) \;=\; \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a,\ldots,a,c_{-1})\ , \tag{3.14}$$

where $c_{-1}:=\sigma(0,0)\in\mathsf{L}_{-1}$.

The derivation of level k gauge transformations from certain components of a partially flat curvature on I^{k+1} makes it clear that this can be iterated further by considering $\Omega^\bullet(I^{k+1},\mathsf{L})$ for $k\geq 0$. Ultimately, we obtain the level $k+1$ gauge transformation,

$$\delta_{c_{-k-1}}c_{-k} \;=\; \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a,\ldots,a,c_{-k-1})\ , \tag{3.15}$$

for $c_{-k}\in\mathsf{L}_{-k}$.

It is important to point out that, as for gauge transformations, the condition $f=0$ is also a sufficient condition for the higher gauge transformations to close. Indeed, computing the commutator of two successive gauge-of-gauge transformation we have

$$\begin{aligned}
\delta_{c_{-1}}(\delta_{c_0}a) \;&:=\; \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a,\ldots,a,\delta_{c_{-1}}c_0)\\
&=\; \sum_{i\geq 0}\frac{1}{i!}\mu_{i+2}(f,a,\ldots,a,c_{-1})\ ,\\
\delta_{c_{-k-2}}(\delta_{c_{-k-1}}c_{-k}) \;&:=\; \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a,\ldots,a,\delta_{c_{-k-2}}c_{-k-1})\\
&=\; \sum_{i\geq 0}\frac{1}{i!}\mu_{i+2}(f,a,\ldots,a,c_{-k-2})\ ,
\end{aligned} \tag{3.16}$$

for all $k \geq 0$, as is demonstrated in Appendix C. Hence, for MC elements (3.5), this vanishes identically and gauge transformations of level k gauge parameters leave the outcome of level k gauge transformations unchanged (as expected).[1]

Covariant derivative. Given an L_∞-algebra (L, μ_i), consider now $\varphi \in \mathsf{L}_k$, for some $k \in \mathbb{Z}$ and require that, under infinitesimal gauge transformations, φ transforms *adjointly*, that is,

$$\delta_{c_0}\varphi \ := \ -\sum_{i\geq 0}\frac{1}{i!}\mu_{i+2}(a,\ldots,a,c_0,\varphi) \tag{3.17}$$

for $c_0 \in \mathsf{L}_0$. We define the *covariant derivative* $\nabla : \mathsf{L}_k \to \mathsf{L}_{k+1}$ of φ by

$$\nabla\varphi \ := \ \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a,\ldots,a,\varphi)\ , \tag{3.18}$$

for $a \in \mathsf{L}_1$. Under the gauge transformations (3.10) and (3.17), $\nabla\varphi$ behaves as

$$\delta_{c_0}(\nabla\varphi) \ = \ -\sum_{i\geq 0}\frac{1}{i!}\mu_{i+2}(a,\ldots,a,c_0,\nabla\varphi) + \sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+3}(f,a,\ldots,a,c_0,\varphi) \tag{3.19}$$

as is demonstrated in Appendix C. Hence, $\nabla\varphi$ transforms adjointly[2] up to terms proportional to the curvature f. Furthermore, in Appendix C is also shown that we have the standard result

$$\nabla^2\varphi \ = \ \sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+2}(f,a,\ldots,a,\varphi)\ . \tag{3.20}$$

3.1.3. Homotopy Maurer–Cartan action

Let L now be a cyclic L_∞-algebra with an inner product $\langle -,-\rangle_{\mathsf{L}}$ of degree -3. Then the Maurer–Cartan equation is variational.

Action. The MC equation (3.5) describes the stationary locus of the action functional

$$S_{\rm MC}[a] \ := \ \sum_{i\geq 1}\frac{1}{(i+1)!}\langle a,\mu_i(a,\ldots,a)\rangle_{\mathsf{L}}\ . \tag{3.21}$$

Using the cyclicity (2.47b), it is a straightforward exercise to show that the extrema of $S_{\rm MC}$ are given by solutions to the MC equation (3.5). We shall refer to the action (3.21) as the *homotopy Maurer–Cartan action*.

[1] Strictly speaking, it is only in this case that the geometric shape underlying the homotopy between homotopies becomes a bigon.

[2] It will always transform adjointly in case of strict L_∞-algebras.

The homotopy MC action (3.21) is invariant under the gauge transformations (3.10). Indeed, we have

$$\delta_{c_0} S_{\rm MC}[a] \;=\; \langle f, \delta_{c_0} a\rangle_{\mathsf{L}} \;=\; -\sum_{i\geq 0}\frac{(-1)^i}{i!}\langle c_0, \mu_{i+1}(f,a,\ldots,a)\rangle_{\mathsf{L}} \;=\; 0\ , \tag{3.22}$$

where we have used graded symmetry and the cyclicity (2.47b) of the inner product as well as the Bianchi identity (3.6).

Further bosonic symmetries. Whilst gauge transformations themselves do not close off-shell in general, their action on the action functional (3.21) does. This is due to an additional invariance of $S_{\rm MC}[a]$ under transformations of the form

$$\delta_{\ell_1,\ldots,\ell_i} a \;:=\; \sum_{j\geq 0}\frac{\gamma_j}{j!}\mu_{i+j+1}(f,a,\ldots,a,\ell_1,\ldots,\ell_i)\ , \tag{3.23}$$

for $\ell_1,\ldots,\ell_i \in \mathsf{L}$ with $\sum_{j=1}^i |\ell_j|_{\mathsf{L}} \;=\; i-2$ and $\gamma_j \in \mathbb{R}$. The invariance follows directly from cyclicity of the inner product (2.47b) and the fact that $\mu_{i+2}(f,f,\ldots) \;=\; 0$ for $i \;\geq\; 0$. Since these symmetries vanish on-shell and therefore do not affect classical observables, they are referred to as *trivial symmetries*. They are of no physical significance. In particular, they neither lead to conserved quantities nor do they prevent perturbation theory.

3.1.4. Maurer–Cartan elements and L_∞-morphisms

Let us now study the behaviour of Maurer–Cartan elements and gauge transformations under L_∞-morphisms $\phi : \mathsf{L} \to \mathsf{L}'$ as introduced in (2.57). Here, we just present the results, leaving the technical details of the computations in the Appendix C.[1] We just point out that, even one may be led to assume that ϕ should act on a gauge potential a as $a' = \phi_1(a)$, this however turns out not to be the case, as it does not give the desired compatibility with the L_∞-algebra structures. Instead, one gets the following.

Let (L,μ_i) and (L',μ_i') be two L_∞-algebras. For any L_∞-morphism ϕ between (L,μ_i) and (L',μ_i') there is a natural morphism of gauge potentials,

$$a \mapsto a' \;:=\; \sum_{i\geq 1}\frac{1}{i!}\phi_i(a,\ldots,a)\ , \tag{3.24a}$$

[1] See also [148] for a similar discussion in the case of A_∞-algebras.

for $a \in \mathsf{L}_1$ and $a' \in \mathsf{L'}_1$. Correspondingly,

$$f \;\mapsto\; f' \;=\; \sum_{i\geq 0} \frac{(-1)^i}{i!} \phi_{i+1}(f, a, \ldots, a) \;, \tag{3.24b}$$

for the curvatures

$$f \;=\; \sum_{i\geq 1} \frac{1}{i!} \mu_i(a, \ldots, a) \in \mathsf{L}_2 \quad \text{and} \quad f' \;=\; \sum_{i\geq 1} \frac{1}{i!} \mu'_i(a', \ldots, a') \in \mathsf{L'}_2 \;. \tag{3.24c}$$

Therefore, we may conclude that, under L_∞-morphisms (2.57), MC elements are mapped to MC elements.

In addition, let us see what happens at the level of gauge transformations. A gauge transformation $a \mapsto a + \delta_{c_0} a$ of a MC element $a \in \mathsf{L}_1$, mediated by a gauge parameter $c_0 \in \mathsf{L}_0$, is transformed under an L_∞-morphism to $a' \mapsto a' + \delta_{c'_0} a'$, where $a' \in \mathsf{L'}_1$ is given by (3.24a) and

$$c_0 \;\mapsto\; c'_0 \;:=\; \sum_{i\geq 0} \frac{1}{i!} \phi_{i+1}(a, \ldots, a, c_0) \in \mathsf{L}'_0 \;. \tag{3.25}$$

Consequently, the gauge orbits of MC elements a are mapped to the gauge orbits of MC elements a' under L_∞-morphisms. In other words, gauge equivalence classes of Maurer-Cartan elements are mapped to gauge equivalence classes of Maurer-Cartan elements.

The above can be extended so that whenever ϕ is a quasi-isomorphism of L_∞-algebras L and L', we have an isomorphism

$$\mathcal{M}_{\mathsf{L}} \;\cong\; \mathcal{M}_{\mathsf{L}'} \tag{3.26}$$

between the moduli spaces of MC elements in L and L'. Explicitly, the spaces of solutions to the homotopy Maurer-Cartan equation modulo gauge transformations in L and L' are isomorphic.

3.1.5. Supersymmetric extension

To close the Chapter, we comment about supersymmetric homotopy Maurer–Cartan theory. A useful property of Chern–Simons theory in three dimensions is that it is trivially $\mathcal{N} = 2$ supersymmetric, cf. [155]. That is, it can be extended to an $\mathcal{N} = 2$ supersymmetric action such that all superpartners of the gauge potential are auxiliary fields. The latter come with algebraic equations that can be integrated out and one is left with the usual Chern–Simons action. The supersymmetries can be linearly combined into the odd symmetry required for computing path integrals of Chern–Simons theory via supersymmetric localisation techniques, see e.g. [156].

It was shown [42] that a similar supersymmetric extension exists in general homotopy MC theory. Hopefully, this feature shall allow to exactly compute path integrals and other physical observables by making use of (higher) supersymmetric localisation techniques. To avoid introducing the spinors involved in a true supersymmetric extension, we consider here an example of, roughly, an analogue of a topological twist with only one real supercharge (which would be sufficient for localisation).

We introduce superpartners $(\varphi_k, \vartheta_k) \in \mathsf{L}_k \oplus \Pi\mathsf{L}_k$ for $k = 0, \ldots, 3$, where Π is the Graßmann-parity changing functor, which transform under gauge transformations (3.17). These fields are thus a generalisation of matter fields transforming in the adjoint representation of some gauge Lie algebra. A gauge invariant action functional is then

$$S_{\rm SMC}[a, \varphi, \vartheta] \;:=\; S_{\rm MC}[a] + \langle\vartheta_0, \vartheta_3\rangle_{\mathsf{L}} + \langle\vartheta_1, \vartheta_2\rangle_{\mathsf{L}} + \langle\varphi_0, \varphi_3\rangle_{\mathsf{L}} + \langle\varphi_1, \varphi_2\rangle_{\mathsf{L}} \tag{3.27}$$

with $S_{\rm MC}$ the homotopy MC action (3.21). Note that gauge invariance follows directly from the cyclicity (2.47b) and the graded symmetry of the inner product $\langle -, - \rangle_{\mathsf{L}}$. Since the extra fields φ_k and ϑ_k appear only algebraically and do not mix with the gauge potential, the two actions $S_{\rm SMC}$ and $S_{\rm MC}$ are clearly equivalent. We stress that the new fields are auxiliary, but physical. They are *not* to be regarded as ghosts or antifields for any value of k.

The action $S_{\rm SMC}$ is also invariant under the following fermionic transformations:

$$\begin{aligned}
Qa \;&:=\; \vartheta_1\;,\\
Q\varphi_1 \;&:=\; \vartheta_1\;,\\
Q\varphi_2 \;&:=\; -\tfrac12\nabla\vartheta_1 + \tfrac12\sum_{i\geq 0}\frac{1-2\gamma_{i0}}{i!}\mu_{i+2}(\underbrace{a,\ldots,a}_{i\text{ copies}},\vartheta_1,\varphi_1) +\\
&\qquad + \sum_{i,j\geq 0}\frac{\gamma_{i+1j}+\gamma_{ij+1}}{i!(j+2)!}\mu_{i+j+3}(\underbrace{a,\ldots,a}_{i\text{ copies}},\vartheta_1,\underbrace{\varphi_1,\ldots,\varphi_1}_{j+2\text{ copies}})\;,\\
Q\vartheta_2 \;&:=\; f + \varphi_2 - \tfrac12\nabla\varphi_1 + \tfrac12\sum_{i\geq 0}\frac{\gamma_{i0}}{i!}\mu_{i+2}(\underbrace{a,\ldots,a}_{i\text{ copies}},\varphi_1,\varphi_1) +\\
&\qquad + \sum_{i,j\geq 0}\frac{\gamma_{ij+1}}{i!(j+3)!}\mu_{i+j+3}(\underbrace{a,\ldots,a}_{i\text{ copies}},\underbrace{\varphi_1,\ldots,\varphi_1}_{j+3\text{ copies}})
\end{aligned} \tag{3.28a}$$

and

$$Q\varphi_0 \;:=\; Q\varphi_3 \;:=\; Q\vartheta_0 \;:=\; Q\vartheta_1 \;:=\; Q\vartheta_3 \;:=\; 0\;, \tag{3.28b}$$

for

$$\gamma_{i0} \;=\; \frac13 \quad\text{and}\quad \gamma_{i+1j} + \gamma_{ij+1} \;=\; -\frac{1}{j+3}\;. \tag{3.28c}$$

Here, ∇ is the covariant derivative defined in (3.18), f is the curvature (3.4) and it is easy to see that $Q^2 = 0$. We shall see an explicit example in Section 5.3..

3.2. Kinematical data of higher gauge theory

Via the above construction, given an L_∞-algebra L, the kinematical data of a corresponding (very abstract) higher gauge theory can be naturally constructed. Here, we shall introduce some concreteness on the subject by providing some examples. Let us consider the L_∞-algebra $\mathsf{L} = \Omega^\bullet(M)$, given by the de Rham complex of some d-dimensional manifold M. Homotopy Maurer–Cartan theory for $\Omega^\bullet(M)$ provides us with the kinematical data for the topologically trivial sector of an ordinary Abelian gauge theory. Moreover, considering the degree-shift of the de Rham complex $\mathsf{L}_\bullet = \Omega^{\bullet+k}(M)$, for $k > 1$, one obtains the kinematical data of higher Abelian gauge theories on topologically trivial higher principal bundles. The restriction to the topologically trivial higher principal bundles[1] is due to our restriction to global differential forms and therefore to the topologically trivial sector.

3.2.1. Non-Abelian higher gauge theory

The generalisation to non-Abelian higher gauge theory is quite straightforward. Recall from 2.4.4. that the tensor product of an L_∞-algebra and a differential graded commutative algebra carries a natural L_∞-algebra structure. The key example regarding higher gauge theory is discussed in the next paragraph.

$\Omega^\bullet(M, \mathsf{L})$. Let L be some L_∞-algebra and M a smooth manifold of dimension d. The tensor product of L and the de Rham complex $(\Omega^\bullet(M), \mathrm{d})$ yields the graded vector space

$$\Omega^\bullet(M, \mathsf{L}) \ := \ \bigoplus_{k\in\mathbb{Z}} \Omega^\bullet_k(M, \mathsf{L}) \tag{3.29a}$$

with

$$\begin{aligned} \Omega^\bullet_k(M, \mathsf{L}) \ &:= \ \Omega^0(M) \otimes \mathsf{L}_k \ \oplus \ \Omega^1(M) \otimes \mathsf{L}_{k-1} \ \oplus \ \cdots \ \oplus \ \Omega^d(M) \otimes \mathsf{L}_{k-d} \\ &= \ \bigoplus_{\substack{i+j=k \\ 0\leq i\leq d \\ j\in\mathbb{Z}}} \Omega^i(M) \otimes \mathsf{L}_j \ . \end{aligned} \tag{3.29b}$$

[1] For topologically non-trivial higher principal bundles, one would have to consider local gauge potentials on patches of the manifold and provide gluing prescriptions on overlaps of the patches. The general discussion can be found e.g. in [7].

The total degree is the sum of the individual degrees, $|\alpha\otimes\ell|_{\Omega^\bullet(M,\mathsf{L})} = |\alpha|_{\Omega^\bullet(M)} + |\ell|_{\mathsf{L}}$ for homogeneous $\alpha\in\Omega^\bullet(M)$ and $\ell\in\mathsf{L}$. This graded vector space carries an L_∞-algebra structure which is the linear extension of the higher products

$$\begin{aligned}\hat\mu_1(\alpha_1\otimes\ell_1) &:= \mathrm{d}\alpha_1\otimes\ell_1 + (-1)^{|\alpha_1|_{\Omega^\bullet(M)}}\alpha_1\otimes\mu_1(\ell_1)\ ,\\ \hat\mu_i(\alpha_1\otimes\ell_1,\ldots,\alpha_i\otimes\ell_i) &:= (-1)^{i\sum_{j=1}^i|\alpha_j|_{\Omega^\bullet(M)}+\sum_{j=0}^{i-2}|\alpha_{i-j}|_{\Omega^\bullet(M)}\sum_{k=1}^{i-j-1}|\ell_k|_{\mathsf{L}}}\times\\ &\qquad\times(\alpha_1\wedge\ldots\wedge\alpha_i)\otimes\mu_i(\ell_1,\ldots,\ell_i)\end{aligned}\tag{3.29c}$$

for $i\geq 2$, $\alpha_1,\ldots,\alpha_i\in\Omega^\bullet(M)$ and $\ell_1,\ldots,\ell_i\in\mathsf{L}$ (with obvious linear continuation to inhomogeneous elements).

Let now M be a manifold of dimension d and L a Lie n-algebra. For the L_∞-algebra $(\Omega^\bullet(M,\mathsf{L}),\hat\mu_i)$ as defined in (3.29), one obtains the potentials and curvatures of non-Abelian higher gauge theory on topologically trivial higher principal bundles, cf. also [102, 42]. We shall use this case to illustrate our constructions throughout the remainder of this Section.

Case $d=3$. As a concrete example, let L be an ordinary Lie algebra[1] $\mathfrak{g}$. Here, a gauge potential is a $\mathfrak{g}$-valued one-form $a=A\in\Omega^1(M,\mathfrak{g})$ and its curvature is simply

$$f = F\quad\text{with}\quad F := \mathrm{d}A+\tfrac{1}{2}\hat\mu_2(A,A) = \mathrm{d}A+\tfrac12[A,A]\ .\tag{3.30}$$

The Bianchi identity reads as $\nabla F=0$ and the homotopy MC equation reduces to the ordinary MC equation $\mathrm{d}A+\frac12[A,A]=0$. The formulas for gauge transformations (3.10) and (3.11) reproduce the familiar transformations,

$$\begin{aligned}A &\mapsto A+\delta_{c_0}A = A+\mathrm{d}c_0+[A,c_0]\ ,\\ F &\mapsto F+\delta_{c_0}F = F+[F,c_0]\ .\end{aligned}\tag{3.31}$$

Since $\mu_3=0$, the gauge algebra closes also for $F\neq 0$. The same is true in the case of higher gauge theories corresponding to $\Omega^\bullet(M,\mathsf{L})$ for L an L_∞-algebra with $\mu_i=0$ for $i\geq 3$.

Case $d=4$. As a second concrete example, let L be a Lie 2-algebra[2], i.e. an L_∞-algebra concentrated in degrees 0 and -1: $\mathsf{L}=\mathsf{L}_{-1}\oplus\mathsf{L}_0$. The gauge potential a is a degree 1 element in $\Omega^1(M,\mathsf{L})$ and decomposes as

$$a = A+B\quad\text{with}\quad A\in\Omega^1(M,\mathsf{L}_0)\quad\text{and}\quad B\in\Omega^2(M,\mathsf{L}_{-1})\ .\tag{3.32a}$$

[1] Recall that every Lie algebra is a differential graded Lie algebra concentrated in degree 0.

[2] Note that any Lie 2-algebra is categorically equivalent to one with $\mu_1=0$ [4].

It represents a higher connection on a topologically trivial principal 2-bundle with structure algebra L, see Appendix D for details. The curvature reads as

$$f \;=\; \hat{\mu}_1(a) + \tfrac{1}{2}\hat{\mu}_2(a,a) + \tfrac{1}{3!}\hat{\mu}_3(a,a,a) := \mathcal{F} + H \;, \tag{3.32b}$$

with

$$\begin{aligned} \mathcal{F} &:= \; \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) + \mu_1(B) \;\in\; \Omega^2(M,\mathsf{L}_0) \;, \\ H &:= \mathrm{d}B + \mu_2(A,B) - \tfrac{1}{3!}\mu_3(A,A,A) \;\in\; \Omega^3(M,\mathsf{L}_{-1}) \;. \end{aligned} \tag{3.32c}$$

Here, the higher products μ_i only see the gauge algebra, not the degree of the differential forms in the arguments, yielding sign factors e.g. $\hat{\mu}_3(a,a,a) = \hat{\mu}_3(A,A,A) = -\mu_3(A,A,A)$. All components of f except for the form of highest degree are usually called *fake curvatures*, in the context of non-Abelian gerbes [21, 22]. In the case of a Lie 2-algebra, there is only one fake curvature, $\mathcal{F}$. We shall further discuss the meaning of the fake curvature $\mathcal{F}$ later in this Section and in 3.1.. The MC equation is simply total flatness, $\mathcal{F} = 0$ and $H = 0$. Moreover, the components of the curvature satisfy each a Bianchi identity, which can be readily computed, together with gauge transformations of the components of the gauge potential and the curvature. We will need these explicit expressions in the second part of the Thesis.

Since any homogeneous degree 0 element of $\Omega^\bullet(M,\mathsf{L})$ decomposes into $c \in \Omega^0(M,\mathsf{L}_0)$ and $\Lambda \in \Omega^1(M,\mathsf{L}_{-1})$, the gauge transformations (3.10) and (3.11) reduce to

$$\begin{aligned} \delta_{c,\Lambda}A &= \mathrm{d}c + \mu_2(A,c) + \mu_1(\Lambda) \;, \\ \delta_{c,\Lambda}B &= -\mu_2(c,B) + \mathrm{d}\Lambda + \mu_2(A,\Lambda) + \tfrac{1}{2}\mu_3(c,A,A) \end{aligned} \tag{3.33a}$$

and

$$\begin{aligned} \delta_{c,\Lambda}\mathcal{F} &= -\mu_2(c,\mathcal{F}) \;, \\ \delta_{c,\Lambda}H &= -\mu_2(c,H) + \mu_2(\mathcal{F},\Lambda) - \mu_3(\mathcal{F},A,c) \;. \end{aligned} \tag{3.33b}$$

To conclude the example, we look at the covariant derivative. For $\varphi \in \Omega^\bullet(M,\mathsf{L})$ the equation (3.18) takes the form

$$\nabla\varphi \;=\; \mathrm{d}\varphi + \mu_1(\varphi) + \mu_2(A,\varphi) + \mu_2(B,\varphi) + \tfrac{1}{2}\mu_3(A,A,\varphi) \;. \tag{3.34}$$

It is clear now how to this generalises to any dimension $d > 4$. In this manner, we can in principle compute the kinematical data for the topologically trivial sector of any higher gauge theory. In Chapter 5 we shall also see the homotopy MC description behind certain supersymmetric field theories.

(Higher) Chern–Simons actions. Note that the homotopy MC action is a vast generalisation of the Chern–Simons action functional. We shall come back to this point in Section 5.3. and we limit ourselves now to the following observation. Let L be a cyclic L_∞-algebra and M a compact, oriented manifold without boundary. Then, the L_∞-algebra $\Omega^\bullet(M,\mathsf{L})$ admits a natural cyclic inner product,

$$\langle \alpha_1\otimes\ell_1,\alpha_2\otimes\ell_2\rangle_{\Omega^\bullet(M,\mathsf{L})} \;:=\; (-1)^{|\alpha_2|_{\Omega^\bullet(M)}|\ell_1|_{\mathsf{L}}}\int_M \alpha_1\wedge\alpha_2\;\langle\ell_1,\ell_2\rangle_{\mathsf{L}}\;, \tag{3.35}$$

where $\alpha_{1,2}\in\Omega^\bullet(M)$ and $\ell_{1,2}\in\mathsf{L}$.

Indeed, if M is a compact oriented three-dimensional manifold and $\hat{\mathsf{L}}=\Omega^\bullet(M,\mathfrak{g})$ with $\mathfrak{g}$ a metric Lie algebra endowed with the obvious cyclic structure given in 2.4.2., then one has the gauge potential $a=A\in\Omega^1(M,\mathfrak{g})$ and the homotopy Maurer–Cartan action for $\hat{\mathsf{L}}$ reads

$$S_{\rm MC}[a] \;=\; \int_M\left\{\tfrac12\langle A,\mathrm{d}A\rangle_{\mathfrak{g}}+\tfrac{1}{3!}\langle A,[A,A]\rangle_{\mathfrak{g}}\right\}\,. \tag{3.36}$$

We thus recover the ordinary Chern–Simons action. We shall return to higher Chern–Simons theory in 5.3., when $\mathfrak{g}$ will be replaced by a general Lie n-algebra L.

Holomorphic (higher) gauge theory. This construction can be generalised to Calabi–Yau manifolds to define *higher holomorphic Chern–Simons theory* [110]. Let X be a complex manifold, with $\dim_{\mathbb{C}}\mathsf{X}=d$. In this case, we consider a gauge L_∞-algebra (L,μ_i) and tensor it with the Dolbeault complex $(\Omega^{0,\bullet}(X),\bar\partial)$, resulting in homotopy Maurer–Cartan theory for the L_∞-algebra $\Omega^{0,\bullet}(\mathsf{X},\mathsf{L})$. The higher non trivial products on $\Omega^{0,\bullet}(\mathsf{X},\mathsf{L})$ trivially follows as in (3.29c), that is

$$\hat\mu_1(\alpha\otimes\ell):=\bar\partial\alpha\otimes\ell+(-1)^{|\alpha|_{\Omega^{0,\bullet}(X,\mathsf{L})}}\alpha\otimes\mu_1(\ell)\;, \tag{3.37}$$

for $\alpha\in\Omega^{0,\bullet}(\mathsf{X})$, $\ell\in\mathsf{L}$ and similarly for higher products. We shall see concrete examples along the rest of the Thesis. Here, we just see how to make contact with ordinary holomorphic Chern–Simons theory. Given $\alpha_{1,2}\in\Omega^{0,\bullet}(\mathsf{X})$ and $\ell_{1,2}\in\mathsf{L}$ the cyclic structure (2.51b) specialised to $\Omega^{0,\bullet}(\mathsf{X},\mathsf{L})$ reads

$$\langle\alpha_1\otimes\ell_1,\alpha_2\otimes\ell_2\rangle_{\Omega^{0,\bullet}(X,\mathsf{L})}=\int_{\mathsf{X}}\Omega^{d,0}\wedge\alpha_1\wedge\alpha_2\langle\ell_1,\ell_2\rangle_{\mathsf{L}}\;, \tag{3.38}$$

where $\Omega^{d,0}$ is the holomorphic volume form on X. Hence, specialising to $d=3$ and $\mathsf{L}=\mathfrak{g}$ being a metric Lie algebra, one has $a=A^{0,1}\in\Omega^{0,1}(\mathsf{X},\mathfrak{g})$ and the homotopy MC action (3.21) gives the holomorphic CS action, as desired.

We shall discuss these theories more in detail in Chapters 6,7. Explicitly, we shall seen how to formulate holomorphic higher gauge theory by making use of the ideas of higher geometry and twistor geometry.

Vanishing curvature and kinematical data. It is now worth to take some time to comment about the observation regarding the formula (3.12). Explicitly, the fact that general gauge transformations only close if the condition $f = 0$ is satisfied. In particular, this can now be interpreted in two possible ways. Firstly, we could regard the condition $f = 0$ as a *dynamical* equation, i.e. an equation of motion and postulate that the condition (3.12) indicates that the gauge algebra is an *open algebra*, namely one that only closes on-shell. As we shall discuss in the next Chapter, these gauge symmetries require using the BV formalism in the quantisation of the theory. Secondly, we could regard $f = 0$ as a constraint on the *kinematical* data of the higher gauge theory and therefore as an equation that is also imposed off-shell.

We should note that the approach to the kinematical data of higher gauge theory presented above fits the interpretation of higher connections as a way of capturing a higher-dimensional parallel transport[1]. From this perspective, the second interpretation is favoured and the fake curvature should indeed be part of the kinematical data. Beyond equations (3.12) and (3.16), there are a number of crucial points observed in the literature. Firstly, for the kinematical data (3.32) of a Lie 2-algebra, it was shown that a consistent, reparametrisation-invariant parallel transport along surfaces requires the fake curvature $\mathcal{F}$ defined in (3.32b) to vanish[2] [5]. Secondly, it was observed [102] that for semistrict Lie 2-algebras with non-trivial μ_3, infinitesimal gauge transformations can only be concatenated if the fake curvature vanishes. This is simply a special case of equation (3.12). Thirdly, equation (3.11) shows that the curvature appears itself in gauge transformations of the curvature, which makes it essentially impossible to write down covariant equations of motions beyond $f = 0$ in the general setting.

We stress that, however, there is an alternative approach to defining higher potentials, curvatures and their gauge transformations that has been worked out for the special case of (twisted) string structures in [7, 29]. In this approach, things become significantly simpler, and

[1]For a consistent higher dimensional parallel transport in the context of higher non-abelian principal bundles see [21, 22] and e.g. [157].

[2]This fake flatness condition also arises from a higher Stokes' theorem, guaranteeing invariance of the induced higher parallel transport under reparametrisations. This fact locally renders the connection gauge equivalent to the connection on an abelian gerbe, as pointed out in [158, 159]. See [159] for an alternative definition of curvature by using special gauge algebras.

the resulting structures have been applied in the context of self-dual strings and six-dimensional superconformal field theories e.g. in [160, 30]. See also [131]. The precise relation between both approaches has not been fully worked out yet.

4

Batalin–Vilkovisky formalism

The Batalin-Vilkovisky (BV) formalism was developed in the late 70ies, early 80ies as a very general approach to the quantisation of classical field theories with complicating symmetries, such as gauge theories. It generalises the Becchi–Rouet–Stora–Tyutin (BRST) quantisation method which, in turn, is the cohomological and manifestly gauge invariant version of the Faddeev–Popov approach. This is also known as the BV/BRST formalism, the antifield formalism or it is included in the term BRST formalism. The main references are represented by the series of papers [11–15, 161] as well as [115], where deeper explanations of the geometry and the meaning of the classical part of the BV formalism were given and the famous [114] in the context of topological field theories. Here, the discussion follows the papers [42, 43]. For more detailed reviews, we refer the reader to Section 8 of [162] as well as the papers [163–167] and [168–170].

This Chapter starts with an outline of the Batalin–Vilkovisky formalism in the context of ordinary gauge theory. We try to highlight its conceptual origins and its formulation in terms of the language of symplectic Q-manifolds introduced in Chapter 2. Then, we proceed explaining how this formalism applies to the much more general example of homotopy Maurer–Cartan theory. The fundamental papers [10, 115] established the link between L_∞-algebras and BV formalism, which, in turn, provides the bridge connecting Lagrangian field theories and homotopy algebras, via homotopy Maurer–Cartan theory. As we shall see in Chapter 5, at the classical level any BV quantisable field theory can be rewritten as a homotopy Maurer–Cartan theory [42, 43].

4.1. Motivation and outline

The BV formalism is commonly used in the quantisation of classical field theories with open gauge symmetries, i.e. gauge symmetries that only close on-shell. This was also the historical motivation for its development. The corresponding setup of a classical field theory for quantisation, however, exposes much of the theory's internal structure in a way that is mathematically very precise and useful at the same time. The underlying language is mostly that of homological algebra, i.e. that of cochain complexes and differential graded algebras. The result is a cohomological description of the algebra of gauge-invariant functions on the critical locus of a classical action functional.

Classical field theories. We start by introducing the setup for the BV formalism. Classical field theory is a formalism for describing a physical system in terms of objects called *fields*. Given a space-time manifold M, the space of fields $\mathfrak{F}$ is generally considered to be a space of sections of some sheaf over M. For instance, it is given by a space of connections on a principal G-bundle over M, in the case of gauge theories with structure group G.[1] In the Lagrangian approach, a classical field theory is specified by defining a (usually real or complex valued) functional on the space of fields

$$S \ : \ \mathfrak{F} \to \mathbb{R} \ \ (\text{or} \ \ \mathbb{C}) \ , \tag{4.1}$$

called an *action functional*. This is usually required to satisfy a locality criterion, meaning that it is of the form

$$S[\Phi] = \int_M \mathcal{L}(\Phi, \partial\Phi, \partial^2\Phi, \cdots) \ . \tag{4.2}$$

Here, $\mathcal{L}$ is the Lagrangian density[2], i.e. a density on M depending on the value of the field $\Phi \in \mathfrak{F}$ and only finitely many of its derivatives at the point of integration on M. For definiteness, mathematicians often drop the action and work with the Lagrangian instead. The classical physics of such a system is completely captured by the critical locus of S, that is

$$\text{Crit}(S) := \{\Phi \in \mathfrak{F} \mid \mathrm{d}S[\Phi] = 0\} \ . \tag{4.3}$$

[1] In mathematical parlance, a classical gauge theory is a classical field theory whose space of fields is not modeled by a set, but rather by a stack.

[2] Mathematically, given some vector bundle E over M, one constructs the bundle of ∞-jets $J(E)$. The space of Lagrangians on E then is given by the algebra of functions $\mathcal{O}(J(E)) \setminus \mathbb{R}$. See e.g. [171] for an exhaustive explanation.

Using the calculus of variations, this defines the space of *solutions to the classical equations of motion* or Euler–Lagrange equations. However, if the functional S is not sufficiently well-behaved, this space can be highly singular, even in the finite-dimensional case. We shall see how the classical part of Batalin–Vilkovisky formalism addresses this problem by considering the *derived critical locus* [172, 173].

Classical BV structure. The classical part of the BV formalism applied to a gauge field theory consists essentially of a two-step resolution of the space of classical observables. The first one is the usual starting point of BRST quantisation[1] [174, 175], which expresses the symmetries[2] of a field theory as the action Lie algebroid introduced in Section 2.4.. Such interpretation may seem at first unnecessarily abstract, but it clarifies the mathematical origin of the fermionic ghosts as well as the generalisation to arbitrarily complicated gauge theories. Concretely, starting from classical field theory data, the BRST formalism yields a complex $(\mathfrak{F}_{\rm BRST}, Q_{\rm BRST})$. The $\mathbb{Z}$-graded vector space $\mathfrak{F}_{\rm BRST}$ is known as *minimal set of fields*, while differential operator $Q_{\rm BRST}$, the *BRST operator*, is nothing but the Chevalley–Eilenberg differential encoding all the information about the Lie algebra of gauge symmetry (and higher gauge symmetry) and its action on $\mathfrak{F}$. Such a complex can be extended to the set of fields $\mathfrak{F}_{\rm eBRST}$ needed for gauge fixing in the BRST formalism.

In many cases, the latter complex exists only on-shell, and therefore needs to be lifted before quantisation. This is true in particular in higher gauge theories if one regards the vanishing of fake curvatures as a dynamical instead of a kinematical condition, recall the discussion in 3.1.2.. The off-shell lift is provided by a Koszul–Tate resolution, cf. [168, 169], which amounts to introducing antifields and, if necessary, anti-ghosts and higher anti-ghosts. Moreover, these two steps are combined by homological perturbation theory to obtain the relevant algebra of observables as the cohomology some differential. Explicitly, we extend the set of fields $\mathfrak{F}_{\rm BRST}$ further to $\mathfrak{F}_{\rm BV} := T^*[-1]\mathfrak{F}_{\rm BRST}$, which is endowed with the differential $Q_{\rm BV}$ given by a derived bracket $Q_{\rm BV} := \{S_{\rm BV}, -\}$. Here, $S_{\rm BV}$ is the classical BV action satisfying the classical master equation $\{S_{\rm BV}, S_{\rm BV}\} = 0$. The fact that, under some reasonable conditions, $S_{\rm BV}$ exists and is unique for a general k-th stage reducible theory (that is, k-th

[1]We shall always distinguish between the BRST formalism, which involves ghosts and where antifields only enter when gauge fixing, and the BV formalism, which involves antifields from the outset.

[2]Usually, one is only concerned with the local symmetries, as these are the ones complicating the quantisation.

level gauge invariance) was proved in [176] using homological perturbation theory.

The result is then an L_∞-algebra[1] that encodes the fields, the gauge structure, gauge invariant observables, field equations, Noether identities and consistent deformations of the theory; in short, everything one needs to know about a classical theory. The minimal model of this L_∞-algebra yields a minimal representation of the classical dynamical data and an equivalence of classical theories is a quasi-isomorphism of L_∞-algebras.

Quantisation. To quantise a classical field theory $(\mathfrak{F}, S)$ means to make sense of the path integral

$$Z(S) \;=\; \int_{\mathfrak{F}} \mu_{\mathfrak{F}}(\Phi)\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\Phi]}\ , \tag{4.4}$$

where $\hbar$ is a parameter of the quantisation, $\mu_{\mathfrak{F}}(\Phi)$ is a measure on the space of fields $\mathfrak{F}$ and S is the action functional defined above. Unfortunately, such a quantity is in general very difficult to compute, in particular we have to make sense of an integration over an infinite-dimensional space of functions. Path integrals of quantum field theories on spaces M with Minkowski signature are oscillatory functional integrals. Such functional integrals can be computed perturbatively by the stationary phase formula (see [167, Section 1.2.4] for details). This, however, requires the stationary points of S to be isolated or, equivalently, the Hessian to be non-degenerate at the stationary points.

This is not the case in gauge theories due to the large degeneracy arising from gauge orbits. Let G be a Lie group inducing a group $\mathfrak{G}$ of gauge transformations[2] acting on the space $\mathfrak{F}$. Gauge symmetry of S implies that the Hessian in any stationary point is degenerate and the perturbative expansion is not well-defined. One could in principle remedy the problem of degenerate critical points by restricting to gauge orbits,

$$\int_{\mathfrak{F}} \mu_{\mathfrak{F}}(\Phi)\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\Phi]} \quad\rightarrow\quad \int_{\mathfrak{F}/\mathfrak{G}} \mu_{\mathfrak{F}/\mathfrak{G}}(\Phi)\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\Phi]|_{\mathfrak{F}/\mathfrak{G}}}\ , \tag{4.5}$$

where $\mu_{\mathfrak{F}/\mathfrak{G}}(\Phi)$ is the measure induced on $\mathfrak{F}/\mathfrak{G}$ by $\mu_{\mathfrak{F}}(\Phi)$. However, there are various reasons why a restriction to the orbit space is not feasible in practise, chief of all the fact that the orbit space is not well-behaved in general.

[1]or L_∞-algebroid in the most general setting.

[2]In our notation we will always identify the gauge group G and the gauge Lie algebra with the structure group and structure Lie algebra of the (higher) principal bundle underlying the (higher) gauge theory. The gauge group is thus different from the resulting group of gauge transformations $\mathfrak{G}$.

The classical BV formalism provides also the starting point for a very convenient gauge fixing procedure[1]. As observed above, introducing antifields in the Koszul–Tate resolution corresponds to extending the space of fields and ghosts $\mathfrak{F}_{\rm BRST}$ to its cotangent bundle $\mathfrak{F}_{\rm BV} := T^*[-1]\mathfrak{F}_{\rm BRST}$. The original action functional corresponds to evaluating the BV action for the zero section, but we can choose a different Lagrangian submanifold. This can be done such that the resulting restricted BV action functional has isolated stationary points and that corresponding functional integral equals[2] the original functional integral. The choice of section is encoded in a functional known as the gauge fermion. Very roughly speaking, this procedure is analogous to the computation of a real integral by going to the complex plane: one doubles the number of variables and extends the original integrand to the new variables. The final integral is still taken along a half-dimensional contour.

Clearly, expectation values should be independent of the choice of gauge fermion, which is tantamount to the *quantum master equation*, a deformation of the classical master equation in $\mathcal{O}(\hbar)$. This requires, in general, to deform the classical BV action to a formal power series in $\hbar$, the quantum BV action. The latter action is then the starting point for all further, e.g. perturbative computations.

4.2. Becchi–Rouet–Stora–Tyutin quantisation

4.2.1. Gauge Lie algebroid

As stated above, we shall focus on the example of ordinary gauge theories, but we shall present the steps in a way that allows for a straightforward extension to higher gauge theory. Such a discussion will show how $\mathbb{Z}$-graded vector spaces and homological vector fields discussed in the first Chapter enter naturally into the description of a gauge field theory. For a detailed discussion along traditional lines, see also [177] and in particular [163] for the case of *open algebras*, i.e. gauge algebroids where the gauge symmetries close only on-shell.

Action Lie algebroid. As already mentioned, quotient spaces as e.g. $\mathfrak{F}/\mathfrak{G}$ appearing in (4.5) are often badly behaved and a useful way to circumvent this issue is to consider the *derived quotient*[3] $[\mathfrak{F}/\mathfrak{G}]$, which amounts to considering the corresponding *action Lie groupoid* as

[1] Meaning the problem of constructing the perturbatively well-defined functional integral for a gauge theory.

[2] after applying heuristics generalised from ordinary integration to functional integration

[3] or model for the homotopy quotient

discussed in Section 2.4..

Let $\mathfrak{F}$ and $\mathfrak{G}$ be again the space of fields and the group of gauge transformations, respectively. Then the action Lie groupoid has objects $\mathfrak{F}$ and morphisms $\mathfrak{G} \ltimes \mathfrak{F}$. A morphism (g, Φ) is of the form

$$\Phi \xrightarrow{(g,\Phi)} g \rhd \Phi \tag{4.6}$$

with the obvious concatenation and identity morphisms, cf. Section 2.4..

For many purposes, and in particular for the BV formalism over contractible manifolds or with trivial principal (gauge) bundle, the infinitesimal picture in terms of Lie algebra actions is sufficient.[1] The corresponding action Lie algebroid is most readily described in terms of Q-manifolds, as seen in Section 2.4., and looks like

$$\mathfrak{F}_{\rm BRST} := \mathsf{Lie}(\mathfrak{G})[1] \ltimes \mathfrak{F} , \tag{4.7}$$

where $\mathsf{Lie}(\mathfrak{G})[1]$ is the Lie algebra $\mathsf{Lie}(\mathfrak{G})$ of the group of gauge transformations $\mathfrak{G}$, whose underlying vector space is degree-shifted by -1, cf. Equation (2.1). The group product and the action of the gauge transformation are now encoded in the homological vector field $Q_{\rm BRST}$, whose form is determined by actions on the contracted coordinate functions on $\mathfrak{F}_{\rm BRST}$,[2]

$$Q_{\rm BRST}\Phi := \delta_c\Phi \quad \text{and} \quad Q_{\rm BRST}c := -\tfrac{1}{2}[c, c] . \tag{4.8}$$

Here, $\delta_c\Phi$ denotes an infinitesimal gauge transformation of Φ parametrised by $c \in \mathsf{Lie}(\mathfrak{G})[1]^* \otimes \mathsf{Lie}(\mathfrak{G})$ and $[-,-]$ is the Lie bracket on $\mathsf{Lie}(\mathfrak{G})$. Hence, together with $Q_{\rm BRST}$, the algebra of functions on $\mathfrak{F}_{\rm BRST}$ forms a dg-algebra.

Let us stress here the important distinction between elements c of $\mathsf{Lie}(\mathfrak{G})$, which parametrise gauge transformations via

$$A \mapsto A' := A + \delta A \quad \text{with} \quad \delta A := \mathrm{d}c + [A, c] \tag{4.9}$$

and the corresponding coordinate functions $c \in \mathscr{C}^\infty(\mathfrak{F}_{\rm BRST})$ appearing in (4.8), which is of degree 1. Nevertheless, we follow the common convention of using the same letter for a vector and its (contracted) coordinate functions. The degree shift is due to the Q-manifold description of the gauge algebroid $\mathfrak{F}_{\rm BRST}$ in which the degree of $\mathsf{Lie}(\mathfrak{G})$ is shifted by -1 and

[1] A framework to discuss global aspects of gauge theory in the BV formalism is provided by derived algebraic geometry [178–180].

[2] Recall our remark about the first fundamental confusion of calculus in Section 2.2.. For instance, here $c = c^\alpha \otimes \tau_\alpha$, for $c^\alpha : \mathsf{Lie}(\mathfrak{G})[1] \to \mathbb{R}$ and τ_α a basis for $\mathsf{Lie}(\mathfrak{G})$.

therefore the coordinate function on Lie$(\mathfrak{G})[1]$ has degree $+1$. The coordinate functions c are known as *ghosts* and the degree shift is the origin of their fermionic character. In general, we call the natural degree of functions on $\mathfrak{F}_{\rm BRST}$ the *ghost number*.

If we are dealing with a higher action Lie algebroid encoding gauge symmetries between gauge symmetries, as will be the case for higher gauge theory, then we also have *ghosts for ghosts*, which are functions of homogeneous degree greater than one in $\mathfrak{F}_{\rm BRST}$. In this case, $Q^2_{\rm BRST}=0$ only on the proper kinematical data which we shall discuss later.

$Q_{\rm BRST}$-cohomology. Note that the functions on $\mathfrak{F}_{\rm BRST}$ form a cochain complex with differential $Q_{\rm BRST}$:

$$0 \longrightarrow \mathscr{C}^\infty_0(\mathfrak{F}_{\rm BRST}) \xrightarrow{Q_{\rm BRST}} \mathscr{C}^\infty_1(\mathfrak{F}_{\rm BRST}) \xrightarrow{Q_{\rm BRST}} \cdots , \tag{4.10}$$

where $\mathscr{C}^\infty_i(\mathfrak{F}_{\rm BRST})$ are functions of ghost degree i. Gauge invariant functionals $F[\Phi]\in\mathscr{C}^\infty(\mathfrak{F})$, such as the action $S[\Phi]$, satisfy

$$Q_{\rm BRST}F[\Phi] \;=\; 0 \tag{4.11}$$

because $Q_{\rm BRST}$ encodes the action of gauge transformations on $\mathfrak{F}$. The analogue of restricting to the isomorphism classes in the derived quotient $[\mathfrak{F}/\mathfrak{G}]$ therefore corresponds to restricting to the $Q_{\rm BRST}$-cohomology $H^0(\mathfrak{F}_{\rm BRST})$,

$$\mathscr{C}^\infty(\mathfrak{F}/\mathfrak{G}) \;\cong\; H^0(\mathfrak{F}_{\rm BRST}) . \tag{4.12}$$

We shall return to this point below.

Note that the idea that cohomological considerations should play a key role in functional integration is motivated by the following heuristics: let M be a smooth compact manifold with volume form μ. We thus obtain a map $\mathscr{C}(M)\to\Omega^n(M)$ by $F\mapsto F\mu$, for an observable $F\in\mathscr{C}^\infty(M)$. Then, the expectation value of F is computed as

$$\langle F\rangle_\mu \;:=\; \frac{\int_M \mu F}{\int_M \mu} \;=\; \frac{[F\mu]}{[\mu]} , \tag{4.13}$$

where $[-]$ denotes the cohomology class of a differential form and we used Stokes theorem and the fact that $\mathrm{d}\mu=0$. Thus, the computation of the expectation value of an observable can be reduced from an integral to a comparison of cohomology classes.[1] The isomorphism (4.12) is also a first step into this direction for the case of path integrals.

[1]To our knowledge, this point of view was first emphasised in [181].

Q_{BRST}-complex. Let us briefly consider the above from a mathematical perspective. Recall that we replaced the naive quotient $\mathfrak{F}/\mathfrak{G}$ by the derived quotient $[\mathfrak{F}/\mathfrak{G}]$, where $\mathfrak{F}/\mathfrak{G}$ equals the isomorphism classes of objects in the action Lie groupoid $[\mathfrak{F}/\mathfrak{G}]$. At the infinitesimal level, this corresponds to considering the cohomology of the *Chevalley–Eilenberg (cochain) complex* for the Lie$(\mathfrak{G})$-module $\mathscr{C}^\infty(\mathfrak{F})$, which is given in (4.10).

Recall from Section 2.4. that the p-cochains of the Chevalley–Eilenberg complex for a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module $\mathscr{E}$ are given by Hom$(\bigwedge^p \mathfrak{g}, \mathscr{E})$, and the differential arises from the action of $\mathfrak{g}$ on $\mathscr{E}$ as well as the Lie algebra structure on $\mathfrak{g}$. In our case $\mathscr{E} = \mathscr{C}^\infty(\mathfrak{F})$, and we have

$$\mathsf{Hom}(\textstyle\bigwedge^p \mathsf{Lie}(\mathfrak{G}), \mathscr{C}^\infty(\mathfrak{F})) \;\cong\; \mathscr{C}^\infty_p(\mathfrak{F}_{\mathrm{BRST}})\;, \tag{4.14}$$

which indeed reproduces the complex (4.10) in the case of ordinary gauge theory. Note also that the Chevalley–Eilenberg complex of a Lie algebra has a straightforward generalisation for modules of L_∞-algebras.

This complex allows us to characterise the gauge invariant functionals as a certain cohomology, since we can extend the Chevalley–Eilenberg complex on the left to

$$0 \longrightarrow \mathscr{C}^\infty(\mathfrak{F}/\mathfrak{G}) \;\cong\; H^0(\mathfrak{F}/\mathfrak{G}) \hookrightarrow \mathscr{C}^\infty_0(\mathfrak{F}_{\mathrm{BRST}}) \xrightarrow{Q_{\mathrm{BRST}}} \cdots\;. \tag{4.15}$$

Note that two other low cohomology groups have an interesting interpretation. Firstly, $H^1(\mathfrak{F}_{\mathrm{BRST}})$ is the set of derivations modulo inner derivations. Here, a *derivation* is a map

$$\delta\;:\;\mathsf{Lie}(\mathfrak{G}) \;\to\; \mathscr{C}^\infty(\mathfrak{F}_{\mathrm{BRST}}) \tag{4.16a}$$

so that

$$\delta([c,c']) \;=\; c \rhd \delta(c') - c' \rhd \delta(c) \quad \text{for} \quad c,c' \in \mathsf{Lie}(\mathfrak{G}) \tag{4.16b}$$

and *inner derivations* are derivations of the form $\delta_f(c) = c \rhd f$ for some $f \in \mathscr{C}^\infty(\mathfrak{F})$.

Secondly, $H^2(\mathfrak{F}_{\mathrm{BRST}})$ is isomorphic to the equivalence classes of Lie algebra extensions $\widehat{\mathsf{Lie}(\mathfrak{G})}$ by $\mathscr{C}^\infty(\mathfrak{F})$, i.e. short exact sequences

$$0 \longrightarrow \mathscr{C}^\infty(\mathfrak{F}) \longrightarrow \widehat{\mathsf{Lie}(\mathfrak{G})} \longrightarrow \mathsf{Lie}(\mathfrak{G}) \longrightarrow 0\;. \tag{4.17}$$

Further details are found, e.g., in [182].

To summarise, gauge-trivial observables are functions on field space which are Q_{BRST}-exact and gauge invariant observables are functions which are Q_{BRST}-closed. In the next Section we shall reduce $H^0(\mathfrak{F}_{\mathrm{BRST}})$, via a (projective) resolution, to the classical observables, which are obtained after taking the quotient by the ideal of functionals vanishing on classical solutions. First, we discuss gauge fixing in the BRST formalism.

4.2.2. Gauge fixing

If the symmetries of a classical theory close off-shell, which amounts to $Q^2_{\rm BRST} = 0$ without any further restriction on the fields $\mathfrak{F}_{\rm BRST}$, then the BRST formalism is sufficient for quantisation. We briefly outline this approach in the following.

Gauge fixing and Faddeev–Popov determinant. The gauge fixing itself is encoded in the *gauge fixing function*, which is a map $F : \mathfrak{F} \to \mathsf{Lie}(\mathfrak{G})$ such that each point in $F^{-1}(0)$ represents a different orbit of $\mathfrak{G}$. We can restrict to $F^{-1}(0)$ by inserting a factor of $\delta(F(\Phi))$ into the functional integral (4.18) with δ the functional analogue of the δ-distribution. This also requires the insertion of the *Faddeev–Popov (FP) determinant* $\det(M_{\rm FP}(\Phi))$ to render the construction invariant under deformations of F. Schematically, we obtain

$$\int_{\mathfrak{F}} \mu_{\mathfrak{F}}(\Phi)\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\Phi]} \quad\to\quad \int_{\mathfrak{F}} \mu_{\mathfrak{F}}(\Phi)\ \det(M_{\rm FP}(\Phi))\ \delta(F(\Phi))\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\Phi]}\ . \tag{4.18}$$

BRST quantisation. Instead of dealing with these two insertions as they are, we can encode them in an extended action functional on an enlarged field space. Since the action must be of total homogeneous degree 0, this requires an extension of $\mathfrak{F}_{\rm BRST}$ by fields of negative degrees.[1] It turns out that an appropriate choice is $\mathfrak{F}_{\rm eBRST}$ which contains the fields Φ and ghost c as well as the Lagrange multipliers $b \in \mathsf{Lie}(\mathfrak{G})[0]$ and the antighosts $\bar{c} \in \mathsf{Lie}(\mathfrak{G})[-1]$, that is,

$$\mathfrak{F}_{\rm eBRST} \ :=\ (\mathsf{Lie}(\mathfrak{G})[1] \oplus \mathsf{Lie}(\mathfrak{G})[0] \oplus \mathsf{Lie}(\mathfrak{G})[-1]) \ltimes \mathfrak{F}\ . \tag{4.19}$$

The homological vector field $Q_{\rm BRST}$ is extended to the homological vector field

$$Q_{\rm eBRST}\Phi \ :=\ \delta_c\Phi\ ,\quad Q_{\rm eBRST}c \ :=\ -\tfrac{1}{2}[c,c]\ ,\quad Q_{\rm eBRST}\bar{c} \ :=\ b\ ,\quad Q_{\rm eBRST}b \ :=\ 0\ . \tag{4.20}$$

Note that the antighosts and Lagrange multipliers form the dg-subalgebra $\mathsf{Lie}(\mathfrak{G})[-1] \xrightarrow{\rm id} \mathsf{Lie}(\mathfrak{G})[0]$ of $\mathfrak{F}_{\rm eBRST}$. The corresponding L_∞-algebra is linearly contractible, and therefore the action algebroid $\mathfrak{F}_{\rm eBRST}$ is quasi-isomorphic to $\mathfrak{F}_{\rm BRST}$. In this sense, we have not extended the data of the theory.

[1] More appropriately, one should speak of the graded ring of functions on the action Lie algebroid and extending it by generators of negative degree.

Using elements of $\mathfrak{F}_{\mathrm{eBRST}}$, we now rewrite (4.18) as

$$\int_{\mathfrak{F}} \mu_{\mathfrak{F}}(\Phi)\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\Phi]} \quad \to \quad \int_{\mathfrak{F}_{\mathrm{eBRST}}} \mu_{\mathrm{eBRST}}(\Phi,b,c,\bar{c})\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\Phi]+\langle b,F(\Phi)\rangle+\langle \bar{c},M_{\mathrm{FP}}(\Phi)c\rangle}\ . \tag{4.21}$$

Here $\mu_{\mathrm{eBRST}}(\Phi,b,c,\bar{c})$ is a natural extension of $\mu_{\mathfrak{F}}(\Phi)$ to $\mathfrak{F}_{\mathrm{eBRST}}$ and $\langle -,-\rangle$ is an appropriate pairing between Lie algebra valued fields, including the integral over space-time. We thus achieved our goal of replacing a functional integral with degenerate Hessian at stationary points by a technically equivalent, non-degenerate functional integral over a larger function space.

Moreover, there is a function[1] $\Psi \in \mathscr{C}^\infty(\mathfrak{F}_{\mathrm{eBRST}})$ of homogeneous degree -1, called the *gauge fixing fermion*, such that

$$\begin{aligned}\int_{\mathfrak{F}_{\mathrm{eBRST}}} \mu_{\mathrm{eBRST}}(\Phi,b,c,\bar{c})\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\Phi]+\langle b,F(\Phi)\rangle+\langle \bar{c},M_{\mathrm{FP}}(\Phi)c\rangle} &= \\ = \int_{\mathfrak{F}_{\mathrm{BRST}}} \mu_{\mathrm{BRST}}(\Phi,b,c,\bar{c})\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S[\Phi]+Q_{\mathrm{eBRST}}\Psi)}\ ,\end{aligned} \tag{4.22}$$

and the measure $\mu_{\mathrm{eBRST}}(\Phi,b,c,\bar{c})$ is compatible with Q_{eBRST} in the sense that

$$\int_{\mathfrak{F}_{\mathrm{eBRST}}} \mu_{\mathrm{eBRST}}(\Phi,b,c,\bar{c})\ Q_{\mathrm{eBRST}}f \ = \ 0 \tag{4.23}$$

for all reasonable test functions $f \in \mathscr{C}^\infty(\mathfrak{F}_{\mathrm{eBRST}})$.

Just as before gauge fixing, we have again a cochain complex $(\mathscr{C}^\infty(\mathfrak{F}_{\mathrm{eBRST}}), Q_{\mathrm{eBRST}})$. Its cohomology encodes potential observables: because of (4.23) and $Q_{\mathrm{eBRST}}S = 0$, Q_{eBRST}-exact terms vanish under the functional integral. Also, only the expectation values of Q_{eBRST}-closed functions are independent of the gauge fixing. Thus, equation (4.22) shows that the BRST approach to quantisation renders gauge invariance manifest.

4.3. Batalin–Vilkovisky complex and classical master equation

Above, we have seen how gauge-invariant observables were contained in the cohomology of the BRST operator. However, the BRST formalism is not suitable for the general treatment of gauge theories. In the case of *open symmetries*, which are symmetries that are only satisfied on-shell, the BRST complex is only a complex up to equations of motion. For many purposes including quantisation, however, we require an off-shell description. This can be obtained by

[1] We shall describe this function in more detail in Section 4.3.1..

a further extension of the BRST complex, and this extension is known as the BV formalism. The idea is to double the field content and to construct the Q-manifold

$$\mathfrak{F}_{\rm BV} \;:=\; T^*[-1]\mathfrak{F}_{\rm BRST}\;, \tag{4.24}$$

which allows for a homological vector field $Q_{\rm BV}$ for which $Q_{\rm BV}^2 = 0$ off-shell. The functional integral is then performed over a Lagrangian submanifold of $\mathfrak{F}_{\rm BV}$ which extends $\mathfrak{F}_{\rm BRST}$.

Symplectic structure. Since $\mathfrak{F}_{\rm BV}$ is a cotangent bundle, we have a natural symplectic structure $\omega_{\rm BV}$ of degree -1. Let Φ^A be local coordinates on $\mathfrak{F}_{\rm BRST}$ (i.e. the fields) and let Φ^+_A (i.e. the antifields) be fibre coordinates on $\mathfrak{F}_{\rm BV} \to \mathfrak{F}_{\rm BRST}$, where $A, B, \ldots$ are multi-indices running over fields, ghosts, and their antighosts as well as all their labels such as momenta, tensor and gauge labels. In terms of these Darboux coordinates, the canonical symplectic form reads as[1]

$$\omega_{\rm BV} \;:=\; (-1)^{|\Phi_A|}\delta\Phi^A \wedge \delta\Phi^+_A\;, \tag{4.25}$$

where δ is the exterior differential on $\mathfrak{F}_{\rm BV}$. This symplectic form $\omega_{\rm BV}$, in turn, induces a Poisson bracket $\{-,-\}_{\rm BV}$ and we have $|\{F,G\}_{\rm BV}| = |F|+|G|+1$ for homogeneous $F, G \in \mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$. As seen in Section 2.3., $\{-,-\}_{\rm BV}$ is graded antisymmetric and obeys a graded Leibniz rule and a graded Jacobi identity. This Poisson bracket is also known as the *antibracket*. It is of degree 1 and therefore $\mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$ forms a *Gerstenhaber algebra*.

Batalin–Vilkovisky complex. We now wish to extend the homological vector field $Q_{\rm BRST}$ to a homological vector field $Q_{\rm BV}$ such that

$$Q_{\rm BV}|_{\mathfrak{F}_{\rm BRST}} \;=\; Q_{\rm BRST} \tag{4.26a}$$

and $Q_{\rm BV}$ is Hamiltonian with respect to the symplectic structure $\omega_{\rm BV}$, that is,

$$Q_{\rm BV} \lrcorner\, \omega_{\rm BV} \;=\; \delta S_{\rm BV} \quad \text{with} \quad S_{\rm BV} \;\in\; \mathscr{C}^\infty(\mathfrak{F}_{\rm BV}) \tag{4.26b}$$

or, equivalently,[2]

$$Q_{\rm BV} \;=\; \{S_{\rm BV}, -\}_{\rm BV}\;. \tag{4.26c}$$

[1]Note, that here we are implicitly summing over the multi-index A, i.e. summing when the indices are discrete and integrating when they are continuous.

[2]Recall that $Q_{\rm BV}$ is acting on $\bigodot^\bullet \mathfrak{F}^*_{\rm BV}$, which is the symmetric tensor algebra (over $\mathbb{R}$) of the space Hom$(\mathfrak{F}_{\rm BV}, \mathbb{R})$ of continuous $\mathbb{R}$-linear functionals on $\mathfrak{F}_{\rm BV}$. See e.g. [141, 171] for further details.

This makes $(\mathfrak{F}_{\rm BV}, Q_{\rm BV}, \omega_{\rm BV})$ a symplectic Q-manifold of degree -1. Recall from Section 2.3. that the Hamiltonian condition is equivalent to requiring that $Q_{\rm BV}$ is a symplectomorphism on symplectic Q-manifolds of degree k except for $k=-1$, which is the case at hand.

Equation (4.26c), together with the Jacobi identity of the Poisson bracket, provides the equivalence between $Q_{\rm BV}^2 = 0$ and the *classical master equation*,

$$\big\{S_{\rm BV}, S_{\rm BV}\big\}_{\rm BV} \;=\; 0\;. \tag{4.27}$$

Solutions to the classical master equation. We have some freedom in choosing a solution $S_{\rm BV}$ to equation (4.27), and we use this to impose a boundary condition beyond (4.26a). We require that

$$S_{\rm BV}|_{\mathfrak{F}_{\rm BRST}} \;=\; S\;, \tag{4.28}$$

where $S \in \mathscr{C}^\infty(\mathfrak{F})$ is the original action of our field theory. Thus, $S_{\rm BV}$ encodes simultaneously our action and the gauge structure of the fields. One important consequence of the choice (4.28) is that the classical equations of motion are now encoded in $Q_{\rm BV}$ via

$$\{S_{\rm BV}, \Phi^+_A\}\Big|_{\mathfrak{F}_{\rm BRST}} \;=\; \delta_{\Phi^A} S\;. \tag{4.29}$$

A solution $S_{\rm BV}$ also defines a Lagrangian subspace $\mathfrak{L}_{S_{\rm BV}}$ of $\mathfrak{F}_{\rm BV}$ through its stationary points. It is called *proper* provided the rank of the Hessian of $S_{\rm BV}$ on $\mathfrak{L}_{S_{\rm BV}}$ equals the number of fields Φ^A. For such a proper solution, one finds that $S_{\rm BV}$ has precisely the gauge invariance required to eliminate all auxiliary fields. It can be shown that a proper solution always exists, see [164] and references therein for details.

A proper solution can be written as a power series in the antifields,

$$S_{\rm BV} \;=\; S + \Phi^+_A\, R^A{}_B\, \Phi^B + \mathcal{O}((\Phi^+_A)^2)\;, \tag{4.30}$$

where the coefficients $R^A{}_B$ vanish unless the ghost number of Φ^A is one less than that of Φ^B so that the total ghost number of $S_{\rm BV}$ indeed vanishes. From the power series expansion, we can iteratively determine the relevant proper solution for a given action S and gauge symmetries $Q_{\rm BRST}$.

Koszul–Tate resolution. Let us briefly look at the BV complex induced by $Q_{\rm BV}$ on $\mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$ from a more mathematical perspective. Given a field theory with a set of classical fields $\mathfrak{F}$, the classical states are given by the subset (4.3) that solves the equations of motion of the theory. The functionals which vanish on solutions to the equations of motion form an ideal $\mathfrak{I}$ of $\mathscr{C}^\infty(\mathfrak{F})$, and the classical observables are given by $\mathscr{C}^\infty(\mathfrak{F})/\mathfrak{I}$. In the first step of the classical BV formalism, the space of gauge invariant functionals on $\mathfrak{F}$ have been characterised by using the Chevalley–Eilenberg complex[1], the quotient $\mathscr{C}^\infty(\mathfrak{F})/\mathfrak{I}$ shall be replaced by a resolution encoded in a suitable differential graded algebra. In the language of algebraic geometry, this corresponds to finding the derived critical locus [172, 173] of the action functional S.[2] This is precisely what the BV formalism does.

Consider first the case of a general field theory with action S, ignoring potential gauge symmetries. Let $\mathscr{C}^\infty(\mathfrak{F})$ be the functionals on the fields $\mathfrak{F}$ and $\mathfrak{I}$ the ideal induced by the critical locus of S. We then have the cochain complex of functions on $T^*[-1]\mathfrak{F}$,

$$\cdots \xrightarrow{Q_{\rm BV}} \mathscr{C}^\infty_{-1}(T^*[-1]\mathfrak{F}) \xrightarrow{Q_{\rm BV}} \mathscr{C}^\infty_0(T^*[-1]\mathfrak{F}) \;\cong\; \mathscr{C}^\infty(\mathfrak{F}) \longrightarrow 0\,, \tag{4.31}$$

where $Q_{\rm BV} = \{S_{\rm BV}, -\}$ with a BV action $S_{\rm BV}$ satisfying (4.28). Because $\mathscr{C}^\infty_{-1}(T^*[-1]\mathfrak{F})$ consists of functionals linear in the antifields, equation (4.29) implies that the ideal $\mathfrak{I}$ is simply the image of $Q_{\rm BV}$

$$Q_{\rm BV}(\mathscr{C}^\infty_{-1}(T^*[-1]\mathfrak{F})) \;=\; \mathfrak{I}\,. \tag{4.32}$$

Thus, the cohomology group $H^0(T^*[-1]\mathfrak{F})$ consists of the desired quotient $\mathscr{C}^\infty(\mathfrak{F})/\mathfrak{I}$. We can extend the above cochain complex by the projection onto the latter, which leads to the resolution

$$\cdots \xrightarrow{Q_{\rm BV}} \mathscr{C}^\infty_{-1}(T^*[-1]\mathfrak{F}) \xrightarrow{Q_{\rm BV}} \mathscr{C}^\infty_0(T^*[-1]\mathfrak{F}) \xrightarrow{\varepsilon} H^0(T^*[-1]\mathfrak{F}) \longrightarrow 0\,, \tag{4.33}$$

and this is the *Koszul–Tate resolution* of $\mathscr{C}^\infty(\mathfrak{F})/\mathfrak{I}$. In the absence of ghosts, functions on field space differing by $Q_{\rm BV}$-exact terms are on-shell equivalent.

To incorporate gauge symmetry, we replace $\mathfrak{F}$ by $\mathfrak{F}_{\rm BRST}$ and construct a new homological vector field $Q_{\rm BV}$ satisfying the boundary condition (4.26a). The result is the complex

$$\cdots \xrightarrow{Q_{\rm BV}} \mathscr{C}^\infty_{-1}(\mathfrak{F}_{\rm BV}) \xrightarrow{Q_{\rm BV}} \mathscr{C}^\infty_0(\mathfrak{F}_{\rm BV}) \xrightarrow{Q_{\rm BV}} \mathscr{C}^\infty_1(\mathfrak{F}_{\rm BV}) \xrightarrow{Q_{\rm BV}} \cdots\,. \tag{4.34}$$

[1] In the language of homological algebra the space of functionals on $\mathfrak{F}/\mathfrak{G}$ is replaced by the space of derived Lie($\mathfrak{G}$)-invariants of functionals on $\mathfrak{F}$, which is given by $(\mathsf{Hom}(\bigodot^\bullet \mathsf{Lie}(\mathfrak{G})[1]^*, \mathcal{O}(\mathfrak{F})), Q_{\rm BRST})$. See e.g [141] for further details.

[2] Recall that if S is a function on M, the derived critical locus is the derived intersection of graph$(\mathrm{d}S)$ with the 0-section of the cotangent bundle of M, $\mathsf{Crit}(S) = \mathsf{graph}(\mathrm{d}S) \times_{T^*M} M$. Derived algebraic geometry suggests to replacing the tensor product in $\mathcal{O}(\mathsf{Crit}(S)) = \mathcal{O}(\mathsf{graph}(\mathrm{d}S)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M)$ with the derived tensor product. For a complete discussion of the BV formalism in this setting see e.g. [178, 179].

The image of $Q_{\rm BV}$ in $\mathscr{C}^\infty_0(\mathfrak{F}_{\rm BV})$ are now the functionals vanishing on the equations of motions (for fields, ghosts, etc.) and the kernel of $Q_{\rm BV}$ in $\mathscr{C}^\infty_0(\mathfrak{F}_{\rm BV})$ are the gauge invariant functionals. The cohomology of the BV differential therefore contains in particular the classical observables: classical states which are gauge invariant.

Classical L_∞-algebra structure. We note already here that the BV formalism assigns to any Lagrangian field theory an L_∞-algebra structure on the graded vector space $\mathfrak{F}_{\rm BV}[-1]$.[1] The question now is clearly to determine the structure of this L_∞-algebra dual to the Batalin-Vilkovisky complex. As we shall see in detail in Chapter 5, such an L_∞-algebra captures the essence of a classical (gauge) field theory.

4.3.1. Quantum master equation

Gauge fixing. Having constructed the BV action, we now need to implement gauge fixing in the BV formalism before we can quantise the theory. To this end, we return to the gauge fixing fermion $\Psi\in\mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$, a field of ghost number -1 which we have already encountered in Section 4.2.. The idea is then to eliminate all antifields by imposing the equation [14]

$$\Phi^+ ~=~ \frac{\delta}{\delta\Phi}\Psi~, \tag{4.35}$$

which defines a Lagrangian submanifold $\mathfrak{L}_\Psi$ in $\mathfrak{F}_{\rm BV}$. The restriction $S_{\rm BV}|_{\mathfrak{L}_\Psi}$ is then the gauge-fixed BV action, i.e. its Hessian it is not degenerate.

In a functional integral, gauge fixing would be implemented by a delta functional, and we would define expectation values of an observable $F\in\mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$ as

$$\langle F\rangle_\Psi ~:=~ \int_{\mathfrak{F}_{\rm BV}}\mu_{\rm BV}(\Phi,\Phi^+)~\delta\left(\Phi^+-\frac{\delta}{\delta\Phi}\Psi\right)F[\Phi,\Phi^+]~\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S^\hbar_{\rm BV}[\Phi,\Phi^+]}~. \tag{4.36}$$

Here, $S^\hbar_{\rm BV}$ is the quantum generalisation of $S_{\rm BV}$ with the boundary condition

$$S^\hbar_{\rm BV}|_{\hbar=0} ~=~ S_{\rm BV}~. \tag{4.37}$$

Moreover, $\mu_{\rm BV}(\Phi,\Phi^+)$ is the functional measure on $\mathfrak{F}_{\rm BV}$ compatible with the symplectic structure $\omega_{\rm BV}$, i.e. coordinate transformations between Darboux charts are measure preserving.

[1] This conversion reflects the duality between differential graded commutative algebras and L_∞-algebras discussed in Chapter 2.

Quantum master equation. Clearly, for physically meaningful statements, we would want $\langle F\rangle_\Psi$ to be independent of the gauge fixing fermion Ψ. We observe that besides the canonical symplectic form (4.25), one may also define a canonical second-order differential operator on $C^\infty(\mathfrak{F}_{\rm BV})$, called the *Batalin–Vilkovisky Laplacian*; see e.g. [164] for details. This is defined by

$$\Delta_{\rm BV} \;:=\; (-1)^{|\Phi^A|+1}\frac{\overleftarrow{\delta}}{\delta\Phi^A}\frac{\overleftarrow{\delta}}{\delta\Phi^+_A} \tag{4.38}$$

and satisfies[1]

$$\begin{aligned}\Delta^2_{\rm BV} \;=\; 0\;,\quad \Delta_{\rm BV}(FG) \;&=\; F\Delta_{\rm BV}G+(-1)^{|G|}(\Delta_{\rm BV}F)G+(-1)^{|G|}\big\{F,G\big\}_{\rm BV}\;,\\ \Delta_{\rm BV}\big\{F,G\big\}_{\rm BV} \;&=\; \big\{F,\Delta_{\rm BV}G\big\}_{\rm BV}-(-1)^{|G|}\big\{\Delta_{\rm BV}F,G\big\}_{\rm BV}\end{aligned} \tag{4.39}$$

for $F,G\in\mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$. Such an object, plays a key role in the path integral quantisation. It turns out that that the expectation value $\langle F\rangle_\Psi$ is invariant under deformations of Ψ if and only if

$$\Delta_{\rm BV}\Big(F[\Phi,\Phi^+]\;{\rm e}^{\frac{\rm i}{\hbar}S^\hbar_{\rm BV}[\Phi,\Phi^+]}\Big) \;=\; 0\;. \tag{4.40}$$

This can be easily seen by imposing[2] $F_{\Psi+\delta\Psi}=F_\Psi$ in (4.36).

For $F=1$, the condition (4.40) reduces to

$$\Delta_{\rm BV}{\rm e}^{\frac{\rm i}{\hbar}S^\hbar_{\rm BV}[\Phi,\Phi^+]} \;=\; 0 \quad\Longleftrightarrow\quad \{S^\hbar_{\rm BV},S^\hbar_{\rm BV}\}_{\rm BV}-2{\rm i}\hbar\Delta_{\rm BV}S^\hbar_{\rm BV} \;=\; 0\;, \tag{4.41}$$

which is known as the *quantum master equation*. Using this equation and the boundary condition (4.37), one can compute constraints on the coefficients of the power series expansion of $S^\hbar_{\rm BV}$ in $\hbar$. A solution to the quantum master equation can then be found iteratively.

In many cases, and in particular in Chern–Simons theory, it turns out that a solution $S_{\rm BV}$ of the classical master equation satisfies $\Delta_{\rm BV}S_{\rm BV}=0$ (at least formally, before regularisation) and therefore also solves the quantum master equation. We shall comment this fact in the next Section for Maurer–Cartan theories.

[1]Note that the BV Laplacian is ill-defined on local functionals. However, the property in the second line of (4.39) implies that it is well-defined on the subalgebra generated by those local functionals which are killed by $\Delta_{\rm BV}$. We shall come back to this point later.

[2]The space of gauge-fixing fermions Ψ (whose Hessian may not be invertible) is contactable, so F_Ψ is globally independent of Ψ.

Quantum L_∞-algebra structure. It is now an interesting question to ask what survives of the classical L_∞-algebra structure noticed in Section 4.3. and discussed further in Section 5. We may define the nilquadratic differential operator[1]

$$\mathfrak{d}_{\rm BV} \;=\; -\mathrm{i}\hbar\Delta_{\rm BV} + \{S_{\rm BV}, -\}_{\rm BV} \quad \text{with} \quad \mathfrak{d}_{\rm BV}^2 \;=\; 0\;. \tag{4.42}$$

Just as the homological vector field $Q_{\rm BV} = \{S_{\rm BV}, -\}_{\rm BV}$ induces a classical L_∞-algebra structure on $\mathfrak{F}_{\rm BV}$, so $\mathfrak{d}_{\rm BV}$ induces a *quantum L_∞-algebra* or *loop homotopy Lie algebra* on the same graded vector space. For more details, see [10] and [183–189]. In particular, the decomposition theorem can be proved and the minimal model constructed explicitly, e.g. by a direct application of the homological perturbation lemma. This approach also leads directly to a homotopy between a quantum L_∞-algebra and its minimal model.

Finally, let us mention some further highly relevant papers addressing closely related issues: the fibre BV integral and its relation to the homotopy transfer of quantum L_∞-structures is subject of [184–186, 188], see also [190] and references therein. BV canonical transformations as isomorphisms of homotopy algebraic structures are discussed in [184–186, 188]. The fact that the propagator is the chain homotopy is explained in [184, 186, 188], see also [190] and references therein.

Gauge fixing and trivial pairs. Since the gauge fixing fermion Ψ is a function of degree -1 and because we wish to use it to eliminate the antifields via (4.35), we have to introduce additional fields of negative degree to construct such a Ψ, and these fields must be trivial in a certain sense. For this to be consistent, these fields have to have an interpretation as coordinate functions on a symplectic Q-manifold of degree -1.

We note that one can always add *trivial pairs* $\bar{c} \in V[l]$ and $b \in V[l+1]$ for V some vector space and $l \in \mathbb{Z}$ and consider the Q-manifold $V[l] \oplus V[l+1]$ with $Q\bar{c} = b$. For example, $V = \mathrm{Lie}(\mathfrak{G})$ and $l = -1$ are used in BRST quantisation, cf. Section 4.2.. The corresponding L_∞-algebra has trivial cohomology and it is therefore quasi-isomorphic to the trivial L_∞-algebra, cf. Section 2.5.. Adding trivial pairs therefore does not affect the data of the classical theory.

To use a trivial pair in the BV formalism, we have to minimally extend it to a symplectic Q-manifold of degree -1. This is simply done by adding corresponding antifields $\bar{c}^+$ and b^+,

[1] Note that contrary to the classical version, the quantum version (4.42) is no longer a derivation, but it's still a differential.

which yields

$$\mathfrak{F}_{\rm tp} \;:=\; T^*[-1]\Big(\mathsf{Lie}(\mathfrak{G})[l]\oplus\mathsf{Lie}(\mathfrak{G})[l+1]\Big) \tag{4.43}$$

with symplectic structure

$$\omega_{\rm tp} \;:=\; (-1)^l\langle\delta\bar c,\delta\bar c^+\rangle+(-1)^{l+1}\langle\delta b,\delta b^+\rangle \tag{4.44}$$

of degree -1 and Hamiltonian

$$S_{\rm tp} \;:=\; -\langle b,\bar c^+\rangle \tag{4.45}$$

of homogeneous degree 0. The resulting homological vector field $Q_{\rm tp}=\{S_{\rm tp},-\}_{\rm tp}$ is still a shift isomorphism,

$$Q_{\rm tp}\bar c \;=\; b\ ,\quad Q_{\rm tp}\bar c^+ \;=\; 0\ ,\quad Q_{\rm tp}b \;=\; 0\ ,\quad Q_{\rm tp}b^+ \;=\; (-1)^l\bar c^+\ . \tag{4.46}$$

Consequently, the resulting cyclic L_∞-algebra is still trivial by quasi-isomorphism.

It is now rather straightforward to see that all structures add up properly when adding trivial pairs to the outcome of the BV formalism. We define $\omega_{\rm eBV}:=\omega_{\rm BV}+\omega_{\rm tb}$, $\{-,-\}_{\rm eBV}:=\{-,-\}_{\rm BV}+\{-,-\}_{\rm tp}$ and $\Delta_{\rm eBV}:=\Delta_{\rm BV}+(-1)^{l+1}\big\langle\frac{\overleftarrow{\delta}}{\delta\bar c},\frac{\overleftarrow{\delta}}{\delta\bar c^+}\big\rangle+(-1)^l\big\langle\frac{\overleftarrow{\delta}}{\delta b},\frac{\overleftarrow{\delta}}{\delta b^+}\big\rangle$. If an action functional $S^\hbar_{\rm BV}$ satisfies the quantum master equation for $\{-,-\}_{\rm BV}$ and then $S^\hbar_{\rm eBV}:=S^\hbar_{\rm BV}+S_{\rm tp}$ satisfies the quantum master equation for $\{-,-\}_{\rm eBV}$ and $\Delta_{\rm eBV}$.

4.4. Batalin–Vilkovisky complex of homotopy Maurer–Cartan theory

Let us now discuss how the BV formalism is adapted to the context of L_∞-algebras and the Maurer–Cartan action (3.21).

4.4.1. Batalin–Vilkovisky formalism and L_∞-algebras

To this end, let us consider a cyclic L_∞-algebra (L,μ_i) with inner product $\langle-,-\rangle_\mathsf{L}$ of L_∞-degree -3 and ghost degree 0. As before, we start with the BRST complex. The action of infinitesimal gauge symmetries on the gauge potential a and the gauge parameters c_{-k} and their symmetry structure are captured by the elements of the L_∞-algebra L of degree $i\leq 1$ as displayed in Figure 4.1.

Becchi–Rouet–Stora–Tyutin complex. The BRST complex therefore corresponds to a truncation of L,

$$\mathsf{L}_{\rm trunc} \;=\; \bigoplus_{i\leq 1}\mathsf{L}_i\;, \tag{4.47}$$

which we endow with the higher products μ_i of L, but putting all μ_i with images outside of $\mathsf{L}_{\rm trunc}$ to zero.

	a	c_0	c_{-1}	$\cdots$	c_{-k}	$\cdots$
L_∞-degree	1	0	-1	$\cdots$	$-k$	$\cdots$

Figure 4.1: Becchi–Rouet–Stora–Tyutin fields

In the case where L is concentrated in degrees 0 and 1 (e.g. in the case corresponding to ordinary Chern–Simons theory), as well as in many other special cases of L_∞-algebras, the truncated higher products still satisfy the homotopy Jacobi identity (2.44c). The truncated L_∞-algebra $\mathsf{L}_{\rm trunc}$ forms then an action L_∞-algebroid and the BRST complex is the Chevalley–Eilenberg algebra of this L_∞-algebroid as discussed in sections 4.2.1. and 4.2.. In general, however $\mathsf{L}_{\rm trunc}$ is not an L_∞-algebra, as we shall explain now.

We still can switch to a graded manifold description underlying $\mathsf{L}_{\rm trunc}$, performing the usual shift by 1 to obtain

$$\mathfrak{F}_{\rm BRST} \;:=\; \mathsf{L}_{\rm trunc}[1] \;=\; \bigoplus_{i\leq 1}\mathsf{L}_i[1]\;, \tag{4.48}$$

where the higher brackets μ_i yield a degree 1 vector field $Q_{\rm BRST}$. The fact that $\mathsf{L}_{\rm trunc}$ is no longer an L_∞-algebra amounts then to $Q_{\rm BRST}$ being no longer homological, that is, $Q_{\rm BRST}^2=0$ is no longer guaranteed.

To simplify our notation, we again combine the coordinate functions on $\mathfrak{F}_{\rm BRST}$ with the basis on $\mathsf{L}_{\rm trunc}$ to form the contracted coordinate functions

$$\mathsf{L}_{{\rm trunc},\mathscr{C}} \;:=\; \mathscr{C}^\infty(\mathfrak{F}_{\rm BRST})\otimes\mathsf{L}_{\rm trunc} \tag{4.49}$$

with higher products $\hat\mu_i$ as defined in (2.52b). Objects in $\mathsf{L}_{{\rm trunc},\mathscr{C}}$ now have a bi-degree, and we refer to the one from $\mathsf{L}_{\rm trunc}$ as the L_∞-degree and the one from $\mathscr{C}^\infty(\mathfrak{F}_{\rm BRST})$ as the *ghost degree*. The latter indeed matches the usual nomenclature. We use here the same letter for both the elements of $\mathsf{L}_{\rm trunc}$ and their contracted coordinate functions, hoping to avoid the first fundamental confusion of calculus.

We see that the field type is determined by the parity of the ghost degree, i.e. by the field regarded as a contracted coordinate function, as expected.

	a	c_0	c_{-1}	$\cdots$	c_{-k}	$\cdots$
L_∞-degree	1	0	-1	$\ldots$	$-k$	$\cdots$
ghost degree	0	1	2	$\cdots$	$k+1$	$\cdots$
field type	b	f	b	$\cdots$	f/b	$\cdots$

Figure 4.2: Becchi–Rouet–Stora–Tyutin fields; a 'b' stands for boson and an 'f' for fermion.

The action of the BRST operator Q_{BRST} on elements of $\mathfrak{F}_{\mathrm{BRST}}$ is then described using formula (2.53), combining all contracted coordinate functions into a single object,

$$\mathsf{a} \;:=\; a + \sum_{k\geq 0} c_{-k}\ . \tag{4.50}$$

We then have

$$Q_{\mathrm{BRST}}\mathsf{a} \;=\; -\sum_{i\geq 1}\frac{1}{i!}\hat{\mu}_i(\mathsf{a},\ldots,\mathsf{a})\ . \tag{4.51}$$

Let us briefly look at the individual components of a. First of all, we have

$$Q_{\mathrm{BRST}}a \;:=\; -\sum_{i\geq 0}\frac{1}{i!}\hat{\mu}_{i+1}(a,\ldots,a,c_0) \;=\; \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a,\ldots,a,c_0)\ , \tag{4.52a}$$

where the μ_i only respect the L_∞-degree of the arguments and they are agnostic about the ghost degree. This explains the additional sign in going from $\hat{\mu}_{i+1}$ to μ_{i+1}: we need to move the coordinate function in c_0, which is of degree 1, past the degree 1 basis vectors of the i arguments a and the bracket $\hat{\mu}_{i+1}$. Similarly, we have

$$\begin{aligned}
Q_{\mathrm{BRST}}c_{-k} \;&:= \\
:=\;& \sum_{\substack{i,n\geq 0\\ i+n\geq 1}}\frac{1}{i!}\sum_{\pi_n}\frac{1}{n_0!\cdots n_{k+1}!}\hat{\mu}_{i+n}\big(\underbrace{a,\ldots,a}_{i},\underbrace{c_0,\ldots,c_0}_{n_0},\underbrace{c_{-1},\ldots,c_{-1}}_{n_1},\ldots,\underbrace{c_{-k-1},\ldots,c_{-k-1}}_{n_{k+1}}\big)\\
=\;& \sum_{\substack{i,n\geq 0\\ i+n\geq 1}}\frac{1}{i!}\sum_{\pi_n}\frac{\chi(\pi_n)}{n_0!\cdots n_{k+1}!}\mu_{i+n}\big(\underbrace{a,\ldots,a}_{i},\underbrace{c_0,\ldots,c_0}_{n_0},\underbrace{c_{-1},\ldots,c_{-1}}_{n_1},\ldots,\underbrace{c_{-k-1},\ldots,c_{-k-1}}_{n_{k+1}}\big)\ ,
\end{aligned} \tag{4.52b}$$

with the sum over π_n running over all weighted partition π_n of n with $n = n_0 + \cdots + n_{k+1}$. The sign

$$\chi(\pi_n) \;=\; (-1)^{\sum_{j=0}^{k+1}\sum_{m=1}^{n_k+1}(j+1)\left(n_j - m + \sum_{l=j+1}^{k+1} n_l\right)} \tag{4.52c}$$

arises again by moving all coordinate functions past the basis vectors and the higher product $\hat{\mu}_{i+n}$.

We note that $Q_{\rm BRST}$ governs the gauge transformations of fields and ghosts,

$$Q_{\rm BRST} a \;:=\; \delta_{c_0} a \quad \text{and} \quad Q_{\rm BRST} c_{-k} \;:=\; \delta_{c_{-k-1}} c_{-k} + \cdots \;, \tag{4.53}$$

and it also incorporates the symmetry structure of the ghosts themselves.

As shown in Appendix C, we have

$$Q_{\rm BRST}^2 a \;=\; \sum_{i\geq 0} \frac{(-1)^i}{i!}\Big[-\mu_{i+2}(f,a,\ldots,a,c_{-1}) + \frac{1}{2!}\mu_{i+3}(f,a,\ldots,a,c_0,c_0)\Big] \tag{4.54}$$

and similar equations for c_{-k}. This reflects the fact that the truncation from L to $\mathsf{L}_{\rm trunc}$ breaks the homotopy Jacobi relation on the truncated higher products. We see that for ordinary Lie algebras and, consequently, ordinary gauge theory, we have neither μ_i for $i>2$ nor the higher ghosts c_{-k} with $k>0$ and so, $Q^2_{\rm BRST}=0$. Therefore, the BRST formalism is sufficient for ordinary gauge theory. In the general case, however, we would have to impose $f=0$ to close the gauge algebra, which is usually phrased as the gauge algebra only closes on-shell.

To obtain an off-shell formulation, e.g. for a quantisation of the field theory, we need to extend the BRST formalism. In the case of homotopy MC theory it is very obvious what this extension should be. Instead of truncating the original L_∞-algebra L to $\mathsf{L}_{\rm trunc}$ yielding the BRST complex, we should have simply kept all of L and put $\mathfrak{F}_{\rm BV}=\mathsf{L}[1]$. This is indeed what the BV formalism does.

Batalin–Vilkovisky fields. As discussed in Section 4.4., to cure the problem we need to double the space of fields[1], by adding an antifield for every field and ghost, so that

$$\mathfrak{F}_{\rm BV} \;:=\; T^*[-1]\mathfrak{F}_{\rm BRST}\;. \tag{4.55}$$

Note that in the case of homotopy MC theory, an inner product $\langle\tau_\alpha,\tau_\beta\rangle_{\mathsf{L}}=\omega_{\alpha\beta}$ of degree -3 with respect to some basis τ_α of L induces a symplectic form $\omega=\frac12\omega_{\alpha\beta}\mathrm{d}\xi^\alpha\wedge\mathrm{d}\xi^\beta$ on $\mathsf{L}[1]$ of degree $-|\tau_\alpha|+1-|\tau_\beta|+1=-3+2=-1$. Non-degeneracy of $\langle-,-\rangle_{\mathsf{L}}$ therefore implies that $T^*[-1]\mathfrak{F}_{\rm BRST}\cong\mathsf{L}[1]$ as claimed above.

For clarity, let us summarise the L_∞-degrees and ghost degrees again in Figure 4.3.

Note that the above does not yet include the additional trivial pairs needed for gauge fixing; we shall come to these later.

[1]As shown, ghosts appear as Chevalley–Eilenberg generators and antifields arise as Koszul generators.

	$\cdots$	c^+_{-k}	$\cdots$	c^+_{-1}	c^+_0	a^+	a	c_0	c_{-1}	$\cdots$	c_{-k}	$\cdots$
L_∞-degree	$\cdots$	$3+k$	$\cdots$	4	3	2	1	0	-1	$\cdots$	$-k$	$\ldots$
ghost degree	$\cdots$	$-k-2$	$\cdots$	-3	-2	-1	0	1	2	$\cdots$	$k+1$	$\ldots$
field type	$\cdots$	f/b	$\cdots$	f	b	f	b	f	b	$\cdots$	f/b	$\cdots$

Figure 4.3: Batalin–Vilkovisky fields; a 'b' stands for boson and an 'f' for fermion.

Since $\mathfrak{F}_{\rm BV} = T^*[-1]\mathfrak{F}_{\rm BRST}$, it comes with the canonical symplectic structure (4.25)

$$\omega_{\rm BV} \;=\; \langle {\rm d}a, {\rm d}a^+\rangle_{\mathsf{L}} + \sum_{k\geq 0}(-1)^{k+1}\langle {\rm d}c_{-k}, {\rm d}c^+_{-k}\rangle_{\mathsf{L}} \;. \tag{4.56}$$

Note that $\omega_{\rm BV}$ is of degree -1 precisely when $\langle -,-\rangle_{\mathsf{L}}$ is of degree -3 after exchanging coordinate functions for the actual fields. We can conveniently combine all fields, ghosts and all their antifields into the superfield (contracted coordinate function)[1]

$$\mathsf{a} \;:=\; a + a^+ + \sum_{k\geq 0}(c_{-k} + c^+_{-k}) \;. \tag{4.57}$$

In terms of the superfield a the symplectic form simply reads as

$$\omega_{\rm BV} \;:=\; -\tfrac{1}{2}\langle {\rm d}\mathsf{a}, {\rm d}\mathsf{a}\rangle_{\mathsf{L}_{\mathscr{C}}} \;. \tag{4.58}$$

Recall that $\mathsf{L}_{\mathscr{C}} = \mathscr{C}^\infty(\mathsf{L}[1]) \otimes \mathsf{L}$ and $\langle -,-\rangle_{\mathsf{L}_{\mathscr{C}}}$ denotes the naturally induced inner product on $\mathsf{L}_{\mathscr{C}}$. In terms of a, this symplectic form induces the Poisson structure

$$\{F,G\}_{\rm BV} \;=\; F\left\langle \frac{\overleftarrow{\delta}}{\delta\mathsf{a}}, \frac{\overrightarrow{\delta}}{\delta\mathsf{a}}\right\rangle^*_{\mathsf{L}_{\mathscr{C}}} G \;, \tag{4.59a}$$

where $F,G \in \mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$ and $\langle -,-\rangle^*_{\mathsf{L}_{\mathscr{C}}}$ denotes the dual BV pairing.[2] Here, the left and right functional derivatives satisfy

$$F\frac{\overleftarrow{\delta}}{\delta\mathsf{a}} \;=\; (-1)^{|F|_{\mathsf{L}_{\mathscr{C}}}+1}\frac{\overrightarrow{\delta}}{\delta\mathsf{a}}F \;. \tag{4.59b}$$

It remains to construct the BV action $S_{\rm BV}$ satisfying the classical master equation $\{S_{\rm BV}, S_{\rm BV}\}_{\rm BV} \;=\; 0$ and which induces the homological vector field $Q_{\rm BV} \;:=\; \{S_{\rm BV}, -\}$ on $\mathfrak{F}_{\rm BV}$.

[1] Explicitly, $\mathsf{a} \;= \mathsf{a}^{\mathsf{A}} \otimes \mathsf{e}_{\mathsf{A}} = \Phi^A \otimes \mathsf{e}_A + \Phi^+_A \otimes \mathsf{e}^A$, where the index A runs over all fields, ghosts, higher ghosts and the corresponding anti-fields, as well as Lie algebra and space-time indices.

[2] Explicitly, $\frac{\delta}{\delta\mathsf{a}} := \frac{\delta}{\delta\mathsf{a}^{\mathsf{A}}} \otimes \mathsf{e}^{\mathsf{A}}$. Here, we have used $\mathsf{e}^{\mathsf{A}} := \mathsf{e}_{\mathsf{B}}\omega^{\mathsf{BA}}$ with ω^{AB} being understood as the inverse of ω_{AB} of the symplectic form (4.25) written as $\omega_{\rm BV} = \frac{1}{2}{\rm d}\mathsf{a}^{\mathsf{A}} \wedge \omega_{\mathsf{AB}}{\rm d}\mathsf{a}^{\mathsf{B}}$.

Batalin–Vilkovisky action. We could follow the construction of $S_{\rm BV}$ discussed in Section 4.4., but for homotopy MC theory, there exists a significant shortcut. Recall that we require $S_{\rm BV}$ to agree with $S_{\rm MC}$ after all ghosts and antifields are put to zero. Also, we require

$$\{S_{\rm BV}, -\}_{\rm BV}|_{\mathfrak{F}_{\rm BRST}} \;=\; Q_{\rm BRST} \;, \tag{4.60}$$

where $Q_{\rm BRST}$ has the action (4.51). An obvious ansatz is therefore given by the superfield version of the homotopy Maurer–Cartan action

$$S_{\rm BV}[\mathsf{a}] \;:=\; \sum_{i\geq 1} \frac{1}{(i+1)!}\langle \mathsf{a}, \hat{\mu}_i(\mathsf{a},\dots,\mathsf{a})\rangle_{\mathsf{L}_{\mathscr{C}}} \tag{4.61}$$

with a defined in (4.57). Note that $S_{\rm BV}[\mathsf{a}]$ is still a function on $\mathfrak{F}_{\rm BV}$ and we compute[1]

$$\big\{S_{\rm BV}, S_{\rm BV}\big\}_{\rm BV} \;=\; -\langle \mathsf{f},\mathsf{f}\rangle_{\mathsf{L}_{\mathscr{C}}} \;, \tag{4.62a}$$

where f is the curvature of a,

$$\mathsf{f} \;=\; \sum_{i\geq 0} \frac{1}{i!}\hat{\mu}_i(\mathsf{a},\dots,\mathsf{a}) \;. \tag{4.62b}$$

By virtue of the identity (C.62) proved in Appendix C, the expression $\langle \mathsf{f},\mathsf{f}\rangle_{\mathsf{L}_{\mathscr{C}}}$ vanishes identically. Consequently, $S_{\rm BV}$ satisfies the classical master equation

$$\big\{S_{\rm BV}, S_{\rm BV}\big\}_{\rm BV} \;=\; 0 \;. \tag{4.63}$$

Defining the homological vector field

$$Q_{\rm BV} \;:=\; \big\{S_{\rm BV}, -\big\}_{\rm BV} \;, \tag{4.64}$$

the triple $(\mathfrak{F}_{\rm BV}, Q_{\rm BV}, \omega_{\rm BV})$ is the data of a symplectic Q-manifold of degree -1. We note that[2]

$$Q_{\rm BV}\mathsf{a} \;=\; -\mathsf{f} \quad \text{and} \quad Q_{\rm BV}\mathsf{f} \;=\; 0 \;. \tag{4.65}$$

From (4.65), we can derive the action of $Q_{\rm BV}$ on the individual contracted coordinate functions on $\mathfrak{F}_{\rm BV}$.

[1] Recall that $\frac{\overrightarrow{\delta}}{\delta \mathsf{a}} S_{\rm BV}[\mathsf{a}] = \mathsf{f}$. The minus sign in (4.62a) comes from (4.59b).

[2] Relations (4.65) are shown in C.2.. Note that by using such relations and the definition of $Q_{\rm BV}$, the result (4.62a) again easily follows.

Example. As an explicit example, consider the L_∞-algebra $\mathsf{L} = \mathsf{L}_{-1} \oplus \mathsf{L}_0 \oplus \mathsf{L}_1 \oplus \mathsf{L}_2$ with the only non-trivial higher brackets being μ_1, μ_2 and μ_3. The superfield reads

$$\mathsf{a} \;=\; a + a^+ + c_0 + c_0^+ + c_{-1} + c_{-1}^+ \;, \tag{4.66}$$

with

$$\begin{aligned} S_{\rm BV}[\mathsf{a}] \;&=\; \sum_{i\geq 1} \frac{1}{(i+1)!} \langle \mathsf{a}, \hat\mu_i(\mathsf{a},\ldots,\mathsf{a})\rangle_{\mathsf{L}_{\mathscr{C}}} \\ &=\; S_{\rm MC}[a] - \langle c_0, \mu_1(a^+)\rangle_{\mathsf{L}} + \langle c_{-1}, \mu_1(c_0^+)\rangle_{\mathsf{L}} + \\ &\qquad + \langle a, \mu_2(c_{-1}, c_0^+)\rangle_{\mathsf{L}} + \langle a, \mu_2(a^+, c_0)\rangle_{\mathsf{L}} + \tfrac{1}{2}\langle a^+, \mu_2(a^+, c_{-1})\rangle_{\mathsf{L}} + \\ &\qquad + \tfrac{1}{2}\langle c_0^+, \mu_2(c_0, c_0)\rangle_{\mathsf{L}} - \langle c_{-1}^+, \mu_2(c_{-1}, c_0)\rangle_{\mathsf{L}} + \\ &\qquad + \tfrac{1}{2}\langle a, \mu_3(a, a^+, c_0)\rangle_{\mathsf{L}} + \tfrac{1}{2}\langle a, \mu_3(a, c_0^+, c_{-1})\rangle_{\mathsf{L}} + \tfrac{1}{2}\langle a, \mu_3(a^+, a^+, c_{-1})\rangle_{\mathsf{L}} + \\ &\qquad + \tfrac{1}{2}\langle a, \mu_3(c_0, c_0, c_0^+)\rangle_{\mathsf{L}} - \langle a, \mu_3(c_0, c_{-1}, c_{-1}^+)\rangle_{\mathsf{L}} - \tfrac{1}{2\cdot 2}\langle a^+, \mu_3(a^+, c_0, c_0)\rangle_{\mathsf{L}} + \\ &\qquad - \langle a^+, \mu_3(c_0, c_0^+, c_{-1})\rangle_{\mathsf{L}} - \tfrac{1}{2}\langle a^+, \mu_3(c_{-1}, c_{-1}, c_{-1}^+)\rangle_{\mathsf{L}} + \\ &\qquad - \tfrac{1}{3!}\langle c_0, \mu_3(c_0, c_0, c_{-1}^+)\rangle_{\mathsf{L}} + \tfrac{1}{2\cdot 2}\langle c_0^+, \mu_3(c_0^+, c_{-1}, c_{-1})\rangle_{\mathsf{L}} \;. \end{aligned} \tag{4.67a}$$

Here, the higher products μ_i are agnostic about the ghost degree of the enclosed fields, and the signs arise again from moving coordinate functions past graded basis vectors and the $\hat\mu_i$. Moreover, $S_{\rm MC}$ is the homotopy MC action for L,

$$S_{\rm MC}[a] \;=\; \tfrac{1}{2}\langle a, \mu_1(a)\rangle_{\mathsf{L}} + \tfrac{1}{3!}\langle a, \mu_2(a,a)\rangle_{\mathsf{L}} + \tfrac{1}{4!}\langle a, \mu_3(a,a,a)\rangle_{\mathsf{L}} \;. \tag{4.67b}$$

The homological vector field induced by $S_{\rm BV}$ acts as follows on the individual contracted coordinate functions on L:

$$\begin{aligned} Q_{\rm BV} a \;&=\; \mu_1(c_0) + \mu_2(a, c_0) + \tfrac{1}{2}\mu_3(a,a,c_0) + \mu_2(c_{-1}, a^+) - \\ &\qquad - \mu_3(a, a^+, c_{-1}) - \mu_3(c_{-1}, c_0, c_0^+) + \tfrac{1}{2}\mu_3(c_0, c_0, a^+) + \tfrac{1}{2}\mu_3(c_{-1}, c_{-1}, c_{-1}^+) \;, \\ Q_{\rm BV} c_0 \;&=\; -\mu_1(c_{-1}) - \mu_2(a, c_{-1}) - \tfrac{1}{2}\mu_3(a,a,c_{-1}) - \tfrac{1}{2}\mu_2(c_0, c_0) - \\ &\qquad - \tfrac{1}{2}\mu_3(a, c_0, c_0) + \mu_3(c_0, c_{-1}, a^+) - \tfrac{1}{2}\mu_3(c_{-1}, c_{-1}, c_0^+) \;, \\ Q_{\rm BV} c_{-1} \;&=\; \mu_2(c_{-1}, c_0) + \mu_3(a, c_{-1}, c_0) + \tfrac{1}{3!}\mu_3(c_0, c_0, c_0) + \tfrac{1}{2}\mu_3(c_{-1}, c_{-1}, a^+) \;, \end{aligned} \tag{4.68}$$

$$\begin{aligned} Q_{\rm BV}a^+ &= -\mu_1(a) - \tfrac{1}{2}\mu_2(a,a) - \tfrac{1}{3!}\mu_3(a,a,a) - \mu_2(c_0,a^+) - \mu_2(c_{-1},c_0^+) + \\ &\quad + \mu_3(a,c_0,a^+) - \mu_3(a,c_{-1},c_0^+) - \tfrac{1}{2}\mu_3(c_{-1},a^+,a^+) - \\ &\quad - \tfrac{1}{2}\mu_3(c_0,c_0,c_0^+) - \mu_3(c_{-1},c_0,c_{-1}^+)\ , \\ Q_{\rm BV}c_0^+ &= \mu_1(a^+) + \mu_2(a,a^+) + \tfrac{1}{2}\mu_3(a,a,a^+) - \mu_2(c_0,c_0^+) + \mu_2(c_{-1},c_{-1}^+) - \\ &\quad - \mu_3(a,c_0,c_0^+) + \mu_3(a,c_{-1},c_{-1}^+) + \mu_3(a,c_{-1},c_{-1}^+) + \\ &\quad + \tfrac{1}{2}\mu_3(a^+,a^+,c_0) - \mu_3(c_{-1},a^+,c_0^+) + \tfrac{1}{2}(c_0,c_0,c_{-1}^+)\ , \\ Q_{\rm BV}c_{-1}^+ &= -\mu_1(c_0^+) - \tfrac{1}{2}\mu_2(a,c_0^+) - \tfrac{1}{2}\mu_3(a,a,c_0^+) - \tfrac{1}{2}\mu_2(a^+,a^+) - \mu_2(c_0,c_{-1}^+) - \\ &\quad - \tfrac{1}{2}\mu_3(a,a^+,a^+) - \mu_3(a^+,c_0,c_0^+) + \mu_3(a^+,c_{-1},c_{-1}^+) - \\ &\quad - \tfrac{1}{2}\mu_3(c_{-1},c_0^+,c_0^+) - \mu_3(a,c_0,c_{-1}^+)\ . \end{aligned} \tag{4.69}$$

Quantum master equation. Following (4.38), we introduce the BV Laplacian by its action on an $F \in \mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$,

$$\Delta_{\rm BV} F \ := \ -\frac{1}{2} F \left\langle \frac{\overleftarrow{\delta}}{\delta \mathsf{a}}, \frac{\overleftarrow{\delta}}{\delta \mathsf{a}} \right\rangle^*_{\mathsf{L}_\mathscr{C}} \ . \tag{4.70}$$

Since the inner product $\langle -,-\rangle_{\mathsf{L}_\mathscr{C}}$ is graded symmetric and since the higher products $\hat\mu_i$ for $i \geq 2$ are graded anti-symmetric, it follows immediately from the cyclicity of the inner product that

$$\Delta_{\rm BV} \langle \mathsf{a}, \hat\mu_i(\mathsf{a},\ldots,\mathsf{a})\rangle_{\mathsf{L}_\mathscr{C}} \ = \ 0\ , \tag{4.71a}$$

for $i \ \geq \ 2$. We also have

$$\Delta_{\rm BV} \langle \mathsf{a}, \hat\mu_1(\mathsf{a})\rangle_{\mathsf{L}_\mathscr{C}} \ = \ 0\ , \tag{4.71b}$$

since both the L-degree and the ghost degree of $\langle \mathsf{a}, \hat\mu_1(\mathsf{a})\rangle_{\mathsf{L}_\mathscr{C}}$ are zero so that a field and its antifield cannot pair up in $\langle \mathsf{a}, \hat\mu_1(\mathsf{a})\rangle_{\mathsf{L}_\mathscr{C}}$. Hence, the BV action (4.61) obeys

$$\Delta_{\rm BV} S_{\rm BV} \ = \ 0\ . \tag{4.72}$$

Altogether, we conclude that $S_{\rm BV}$ satisfies the quantum master equation

$$\big\{S_{\rm BV}, S_{\rm BV}\big\}_{\rm BV} - 2\mathrm{i}\hbar\, \Delta_{\rm BV} S_{\rm BV} \ = \ 0 \quad \Longleftrightarrow \quad \Delta_{\rm BV} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\rm BV}} \ = \ 0\ . \tag{4.73}$$

It is important to stress, however, that this is only true formally since the BV Laplacian is singular and regularisation needs to be taken into account in general.[1] Furthermore, the above

[1] Regularisation and renormalisation in Euclidean signature were addressed in [171] via a heat kernel approach.

observation, namely that the two terms in the quantum master equation vanish separately, was made earlier in the context of BF-theory [191].

4.4.2. Gauge fixing

We shall conclude the Chapter by briefly summarising the gauge fixing procedure, although quantum computations are beyond our purposes, and adding an outlook on the quantum master equation for homotopy MC theory.

Additional fields. To gauge-fix the BV action (4.61), we will have to expand the field space further by introducing trivial pairs, that is, *antighosts* $\bar{c}_{i,j}$ and *Lagrange multipliers* $b_{i,j}$, together with their antifields, the *antifield antighost* $\bar{c}^+_{i,j}$ and the *antifield Lagrange multiplier* $b^+_{i,j}$, as encountered in sections 4.2. and 4.3.1..

In the L_∞-algebra picture, the necessary extension is given in Figure 4.5. In general, we have additional quadruples of fields for all $i \leq 0$ and $i-1 \leq j \leq -i-1$ as displayed in Figure 4.4.

	$\bar{c}_{i,j}$	$b_{i,j}$	$\bar{c}^+_{i,j}$	$b^+_{i,j}$
takes values in a copy of	L_i	L_i	L_{3-i}	L_{3-i}
which is added to L in ghost degree	j	$j+1$	$-j-1$	$-j-2$
or, equivalently, in L_∞-degree	$1-j$	$-j$	$2+j$	$j+3$

Figure 4.4: L_∞-degrees and ghost degrees of the trivial pairs and their antifields.

Put differently, we extend L to

$$\mathsf{L}_{\rm e} \;:=\; \mathsf{L} \oplus \bigoplus_{\substack{i\leq 0\\ i-1\leq j\leq -i-1}} \Big(\mathsf{L}_i[j-1] \oplus \mathsf{L}_i[j] \oplus \mathsf{L}_{3-i}[-j-2] \oplus \mathsf{L}_{3-i}[-j-3]\Big) \tag{4.74}$$

and the BV complex $\mathfrak{F}_{\rm BV}$ correspondingly reads as

$$\begin{aligned} \mathfrak{F}_{\rm eBV} \;&:=\; \mathsf{L}_{\rm e}[1] \\ &\cong\; T^*[-1]\left(\mathfrak{F}_{\rm BRST} \oplus \bigoplus_{\substack{i\leq 0\\ i-1\leq j\leq -i-1}} \Big(\mathsf{L}_i[j] \oplus \mathsf{L}_i[j+1]\Big)\right) \\ &\cong\; \mathfrak{F}_{\rm BV} \oplus \bigoplus_{\substack{i\leq 0\\ i-1\leq j\leq -i-1}} \Big(\mathsf{L}_i[j] \oplus \mathsf{L}_i[j+1] \oplus \mathsf{L}_{3-i}[-j-1] \oplus \mathsf{L}_{3-i}[-j-2]\Big)\,. \end{aligned} \tag{4.75}$$

L_∞-degree	-2	-1	0	1	2	3	4
ghost degree	3	2	1	0	-1	-2	-3
BV fields	$\cdots \xrightarrow{\mu_1} (\mathsf{L}_{-2}, c_{-2}) \xrightarrow{\mu_1}$	$(\mathsf{L}_{-1}, c_{-1}) \xrightarrow{\mu_1}$	$(\mathsf{L}_0, c_0) \xrightarrow{\mu_1}$	$(\mathsf{L}_1, a) \xrightarrow{\mu_1}$	$(\mathsf{L}_2, a^+) \xrightarrow{\mu_1}$	$(\mathsf{L}_3, c_0^+) \xrightarrow{\mu_1}$	$(\mathsf{L}_4, c_{-1}^+) \xrightarrow{\mu_1} \cdots$
				$\oplus$	$\oplus$		
trivial pair				$(\mathsf{L}_0, b_{0,-1}) \xrightarrow{\mathrm{id}}$	$(\mathsf{L}_0, \bar{c}_{0,-1})$		
				$\oplus$	$\oplus$		
antifields				$(\mathsf{L}_3, \bar{c}^+_{0,-1}) \xrightarrow{-\mathrm{id}}$	$(\mathsf{L}_3, b^+_{0,-1})$		
			$\oplus$	$\oplus$	$\oplus$	$\oplus$	
trivial pairs			$(\mathsf{L}_{-1}, b_{-1,0}) \xrightarrow{\mathrm{id}}$	$(\mathsf{L}_{-1}, \bar{c}_{-1,0})$	$(\mathsf{L}_{-1}, b_{-1,-2}) \xrightarrow{\mathrm{id}}$	$(\mathsf{L}_{-1}, \bar{c}_{-1,-2})$	
			$\oplus$	$\oplus$	$\oplus$	$\oplus$	
antifields			$(\mathsf{L}_4, \bar{c}^+_{-1,-2}) \xrightarrow{\mathrm{id}}$	$(\mathsf{L}_4, b^+_{-1,-2})$	$(\mathsf{L}_4, \bar{c}^+_{-1,0}) \xrightarrow{\mathrm{id}}$	$(\mathsf{L}_4, b^+_{-1,0})$	
		$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
trivial pairs		$(\mathsf{L}_{-2}, b_{-2,1}) \xrightarrow{\mathrm{id}}$	$(\mathsf{L}_{-2}, \bar{c}_{-2,1})$	$(\mathsf{L}_{-2}, b_{-2,-1}) \xrightarrow{\mathrm{id}}$	$(\mathsf{L}_{-2}, \bar{c}_{-2,-1})$	$(\mathsf{L}_{-2}, b_{-2,-3}) \xrightarrow{\mathrm{id}}$	$(\mathsf{L}_{-2}, \bar{c}_{-2,-3})$
		$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
antifields		$(\mathsf{L}_5, \bar{c}^+_{-2,-3}) \xrightarrow{-\mathrm{id}}$	$(\mathsf{L}_5, b^+_{-2,-3})$	$(\mathsf{L}_5, \bar{c}^+_{-2,-1}) \xrightarrow{-\mathrm{id}}$	$(\mathsf{L}_5, b^+_{-2,-1})$	$(\mathsf{L}_5, \bar{c}^+_{-2,1}) \xrightarrow{-\mathrm{id}}$	$(\mathsf{L}_5, b^+_{-2,1})$
$\vdots$				$\vdots$			

Figure 4.5: L_∞-algebra picture of the Batalin–Vilkovisky fields including trivial pairs

A diagram of the additional fields is found in Figure 4.5.

The reason for introducing families of antighosts and Lagrange multipliers for each level k can be understood as follows: the lowest level antighosts and Lagrange multipliers are needed to fix the gauge symmetries of the fields and ghosts, the next-to-lowest level antighosts and Lagrange multipliers are needed to fix the gauge symmetries of the lowest level antighosts, and so on [11]. Reducing to merely the antighost of each quadruple of new fields, one obtains the so-called *Batalin–Vilkovisky triangle* [11, 42] displayed in Figure 4.6.

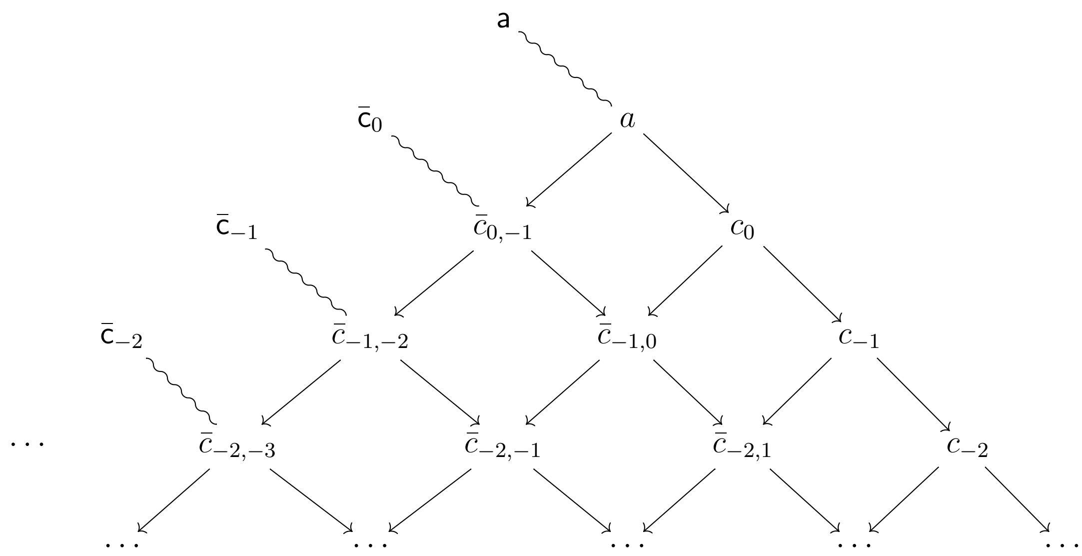

Figure 4.6: Batalin–Vilkovisky triangle.

Symplectic Q-manifold structure. The graded vector space $\mathfrak{F}_{\rm eBV}$ comes with the canonical symplectic structure

$$\begin{aligned}\omega_{\rm eBV} &= \langle {\rm d}a, {\rm d}a^+\rangle_{\mathsf{L}} + \sum_{i\leq 0}(-1)^{i+1}\langle {\rm d}c_i, {\rm d}c_i^+\rangle_{\mathsf{L}} + \\ &\quad + \sum_{i\leq 0}\sum_{j=i-1}^{-i-1}\Big[(-1)^j\langle {\rm d}\bar{c}_{i,j}, {\rm d}\bar{c}^+_{i,j}\rangle_{\mathsf{L}} + (-1)^{j+1}\langle {\rm d}b_{i,j}, {\rm d}b^+_{i,j}\rangle_{\mathsf{L}}\Big]~. \end{aligned} \tag{4.76}$$

Also, the extension of the BV action to a solution of the classical master equation is given by

$$S_{\rm eBV}[a, c_i, \ldots, \bar{c}_{i,j}, \ldots] = S_{\rm BV}[a, c_i, \ldots] - \sum_{i\leq 0}\sum_{j=i-1}^{-i-1}\langle b_{i,j}, \bar{c}^+_{i,j}\rangle_{\mathsf{L}}~, \tag{4.77}$$

as discussed in Section 4.3.1.. We set again $Q_{\rm eBV} := \{S_{\rm eBV}, -\}_{\rm eBV}$ and its action on all the fields is (4.65) together with

$$Q_{\rm eBV}\bar{c}_{i,j} = b_{i,j} \ , \quad Q_{\rm eBV}b_{i,j} = 0 \ , \quad Q_{\rm eBV}\bar{c}^+_{i,j} = 0 \ , \quad Q_{\rm eBV}b^+_{i,j} = (-1)^j\bar{c}^+_{i,j} \ . \tag{4.78}$$

Note that the new fields $\bar{c}_{i,j}$, etc., denote contracted coordinate functions here.

As before, it is convenient to combine the additional fields arising from trivial pairs into superfields $\bar{\mathsf{c}}_i$ and b_i such that

$$Q_{\rm eBV}\bar{\mathsf{c}}_i = \mathsf{b}_i \quad \text{and} \quad Q_{\rm eBV}\mathsf{b}_i = 0 \ . \tag{4.79}$$

We can put

$$\bar{\mathsf{c}}_i := \begin{cases} \sum_{j\geq 0}(-1)^j\bar{c}_{i-j,i+j-1} & \text{for} \quad i \leq 0 \\ \sum_{j\geq 0}(j)b^+_{1-i-j,j-i} & \text{for} \quad i > 1 \end{cases} \tag{4.80a}$$

and

$$\mathsf{b}_i := \begin{cases} \sum_{j\geq 0}(-1)^j b_{i-j,i+j-1} & \text{for} \quad i \leq 0 \\ \sum_{j\geq 0}(-1)^{j-i}\bar{c}^+_{1-i-j,j-i} & \text{for} \quad i > 1 \end{cases} \tag{4.80b}$$

To obtain the component fields, as before, one simply projects onto the corresponding ghost degree.

The symplectic form (4.76) and the extended BV action (4.77) then read as

$$\begin{aligned} \omega_{\rm eBV} &:= -\tfrac{1}{2}\langle \mathrm{d}\mathsf{a}, \mathrm{d}\mathsf{a}\rangle_{\mathsf{L}_{\mathscr{C}}} + \sum_{i+j=1}\langle \mathrm{d}\mathsf{b}_i, \mathrm{d}\bar{\mathsf{c}}_j\rangle_{\mathsf{L}_{\mathrm{e}\mathscr{C}}} \ , \\ S_{\rm eBV}[\mathsf{a},\mathsf{b}] &:= S_{\rm BV}[\mathsf{a}] - \sum_{\substack{i+j=1 \\ i\leq j}}(-1)^{i+1}\langle \mathsf{b}_i, \mathsf{b}_j\rangle_{\mathsf{L}_{\mathrm{e}\mathscr{C}}} \ . \end{aligned} \tag{4.81}$$

Quantum master equation. The fact that the BV action satisfies the quantum master equation is preserved after the extension by trivial pairs. In particular, the BV Laplacian $\Delta_{\rm eBV}$ on $\mathsf{L}_{\rm e}$ annihilates $S_{\rm eBV}$ and we have altogether

$$\{S_{\rm eBV}, S_{\rm eBV}\}_{\rm eBV} = 0 \quad \text{and} \quad \Delta_{\rm eBV}S_{\rm eBV} = 0 \tag{4.82}$$

so that

$$\big\{S_{\rm eBV}, S_{\rm eBV}\big\}_{\rm eBV} - 2\mathrm{i}\hbar\,\Delta_{\rm eBV}S_{\rm eBV} = 0 \quad \Longleftrightarrow \quad \Delta_{\rm eBV}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\rm eBV}} = 0 \ . \tag{4.83}$$

As before, this is only true formally as regularisation needs to be taken into account.

Gauge fixing. To gauge-fix the extended BV action (4.77), we introduce a gauge fixing fermion

$$\Psi \;:=\; \sum_{i\leq 0}\langle \bar{\mathsf{c}}_i, \mathsf{G}_i(\mathsf{a}, \bar{\mathsf{c}}_0, \bar{\mathsf{c}}_{-1}, \ldots)\rangle_{\mathsf{L}_\mathrm{e}} \tag{4.84}$$

such that $\frac{\partial}{\partial\phi}\Psi$ for any field $\phi \in \mathsf{L}_e$ takes values in the same homogeneously graded vector subspace of L_e as its antifield, ϕ^+. The gauge fixed quantum BV action is then given by

$$S_{\mathrm{qBV}}[a, c, \bar{c}, b] \;:=\; S_{\mathrm{eBV}}[\mathsf{a}, \mathsf{b}]\Big|_{\phi^+=\frac{\vec{\partial}}{\partial\phi}\Psi}\;. \tag{4.85}$$

5 Classical L_∞-structure of field theories

5.1. Interpretation of the BV L_∞-algebra

Outline. As we have seen in the previous Chapter, the classical part of the BV formalism maps the data of a classical field theory $(\mathfrak{F}, S)$ to a set of BV fields $\mathfrak{F}_{\mathrm{BV}}$ together with a BV action S_{BV}. As explained in Section 2.4., via the BV bracket, the BV action defines a homological vector field Q_{BV}, which in turn encodes an L_∞-algebra structure on the graded vector space $\mathfrak{F}_{\mathrm{BV}}$. This L_∞-algebra encodes all relevant classical information about the field theory in question. It captures the field content and its gauge symmetry structure, the equations of motion, as well as the Noether identities. Moreover, the original action is recovered by the homotopy Maurer–Cartan theory of that L_∞-algebra. At the classical level, Lagrangian field theories are thus equivalently described by cyclic L_∞-algebras [42, 43]. This is not a new fact and it is certainly well-known by experts on the BV formalism, see [176, 192–205] for earlier and partial accounts. However, the structural advantages of this description, have not been yet fully exploited, as pointed out in e.g. [42, 43, 206, 146].

At the classical level, two physical theories are equivalent if they have an isomorphic space of observables. In the L_∞-framework this should translate to L_∞-algebras which are isomorphic in some sense. In particular, this appropriate notion of equivalence is given by quasi-isomorphisms and physically equivalent classical field theories have quasi-isomorphic L_∞-algebras, see e.g. [42].[1] Hence, two equivalent field theories have a common minimal model. Recall that this corresponds to a field theory equivalent to the original, but without any propagating degrees of freedom and where the non-triviality of the action is fully absorbed

[1] at the classical level; however equivalence of quantum field theories is more involved, cf. e.g. [44].

in the higher products. The minimal model and its higher products encode precisely the tree-level scattering amplitudes of the original field theory [148, 143, 42, 43]. The explicit application of this idea is rather recent. Quasi-isomorphisms induce useful recursion relations for the computations of S-matrices at tree level [146, 207–209]. This has also been extended to obtain the full scattering amplitudes of quantum field theories by computing minimal models of the corresponding quantum algebras[1] via the homological perturbation lemma [189, 206, 210].

This Chapter is devoted to analyse the L_∞-structure underlying classical field theory, with a particular focus to Yang–Mills and (higher) Chern–Simons theory and follows [42]. Before discussing explicit examples, let us make a few more observations about the L_∞-algebra structure which we expect.

***L_∞*-algebra structure.** The vector space $\mathfrak{F}_{\rm BV}$ is graded in particular with respect to the ghost degree, $\mathfrak{F}_{\rm BV} := \bigoplus_{i\in\mathbb{Z}} \mathfrak{F}^i_{\rm BV}$. The usual correspondence between Q-manifolds and L_∞-algebras suggests that we need to shift the degree by one and invert it, for an L_∞-algebra with higher products μ_i of degree $2-i$. We thus arrive at the L_∞-algebra

$$\cdots \xrightarrow{\mu_1} \underbrace{\mathfrak{F}^{-1}_{\rm BV}}_{=:\mathsf{L}_0} \xrightarrow{\mu_1} \underbrace{\mathfrak{F}^{0}_{\rm BV}}_{=:\mathsf{L}_1} \xrightarrow{\mu_1} \underbrace{\mathfrak{F}^{1}_{\rm BV}}_{=:\mathsf{L}_2} \xrightarrow{\mu_1} \underbrace{\mathfrak{F}^{2}_{\rm BV}}_{=:\mathsf{L}_3} \xrightarrow{\mu_1} \cdots \tag{5.1}$$

That is, L_0 is given by the ghosts, L_1 by the fields, L_2 by the antifields and L_3 by the antighosts. This extends in an obvious manner to cases with ghosts-for-ghosts and trivial pairs.

The map μ_1 is encoded in the linear part of the action $Q_{\rm BV} := \{S_{\rm BV}, -\}_{\rm BV}$ on the field corresponding to the *image* of μ_1. Explicitly, $\mu_1 : \mathsf{L}_0 \to \mathsf{L}_1$ is encoded in the linear part of the explicit expression for $Q_{\rm BV}A$ and therefore encodes the linearised gauge transformations. The map $\mu_1 : \mathsf{L}_1 \to \mathsf{L}_2$ is obtained by linearising $Q_{\rm BV}A^+$, which yields the linearised variation of the classical action with respect to the field and therefore the linearised classical equations of motion. The map $\mu_1 : \mathsf{L}_2 \to \mathsf{L}_3$ is the linearised part of the action of $Q_{\rm BV}$ on c^+ and these encode precisely the Noether identities as we shall explain below. This is all the structure necessary to describe a classical (gauge) field theory; for higher gauge theories, one obtains an extension beyond the homogeneously graded vector subspaces L_j with $0 \leq j \leq 3$.

The higher brackets then fulfil the task of making the linearised expressions covariant and to allow for higher interaction terms. In general, an interaction term of nth order in the fields will be encoded in a higher product μ_i with $i = n-1$.

[1] Recall that L_∞-algebras can be generalised to quantum L_∞-algebras, corresponding to solutions to the quantum master equation in the BV formalism.

From our discussion in Chapter 3, it is also clear that the homotopy Maurer–Cartan action for L reproduces the original action S and the homotopy Maurer–Cartan action for $\mathsf{L}_{\mathscr{C}}$ reproduces the BV action $S_{\rm BV}$.

$\cdots$	L_{-1}	L_0	L_1	L_2	L_3	L_4	$\cdots$
$\cdots$	gauge-of-gauge transf.	gauge transf.	physical fields	equations of motion	Noether identities	higher Noether	$\cdots$

Figure 5.1: Summary of the structure of the L_∞-algebra of a classical field theory. While the labels under L_i for $i \leq 1$ describe the spaces, the meaning of the labels changes for $i \geq 2$: L_2, for instance, is *not* the space of the equations of motion, but the element ℓ of L that is forced to zero by the equations of motion $\ell = 0$ takes values in L_2.

Noether identities. Here, we discuss Noether identities as done in [42]. In particular, we are concerned with Noether's second theorem, generalising the more familiar first one. In this picture, also gauge symmetries give rise to Noether identities.

The motivation for considering Noether identities is twofold. First of all, they are at the heart of the BV formalism: they correspond precisely to the degeneracies of the Hessian which make the application of the stationary phase formula in the interpretation of the path integral impossible. Secondly, they are an important part of the classical structure of a field theory and also contained in its L_∞-algebra.

Let M be a manifold with local coordinates x^μ. Consider an infinitesimal group action on a set of fields Φ^A on M parametrised by infinitesimal parameters $\varepsilon = (\varepsilon^I)$ as

$$\delta\Phi^A(x) \;:=\; R^A_I(\Phi)\varepsilon^I(x) \;, \tag{5.2}$$

where $R^A_I(\Phi)$ are field-dependent differential operators, possibly containing terms of order 0. Alternatively, we can write

$$\delta\Phi^A(x) \;=\; \int_M \mu(y) R^A_I(x,y,\Phi)\varepsilon^I(y) \;, \tag{5.3}$$

where μ is a suitable measure on M. If this action is a symmetry of an action $S[\Phi]$, then we have the Noether identity

$$\int_M \mu(x) \frac{\delta S[\Phi]}{\delta\Phi^A(x)} \frac{\delta\Phi^A(x)}{\delta\varepsilon^I(y)} \;=\; 0 \tag{5.4}$$

or

$$\int_M \mu(x) \frac{\delta S[\Phi]}{\delta \Phi^A(x)} R_I^A(x, y, \Phi) \;=\; 0 \;. \tag{5.5}$$

If we vary this equation with respect to $\delta\Phi^B(z)$ and restrict Φ to the stationary surface, we have

$$\frac{\delta^2 S[\Phi]}{\delta\Phi^B(z)\delta\Phi^A(x)} R_I^A(x, z, \Phi) \;=\; 0 \;, \tag{5.6}$$

which implies that the R_I^A encode degeneracies of the Hessian, that is, they are the eigenvectors of the Hessian with eigenvalue zero.

The Noether identities also imply that the vector fields $Q_{\rm BV}$ decompose as

$$Q_{\rm BV} \;=\; Q_{\rm KT} + \cdots \;, \tag{5.7}$$

where $Q_{\rm KT}$ is the part purely responsible for the Koszul–Tate resolution acting non-trivially only on the antifields of fields and ghosts, with $Q_{\rm KT}^2 = 0$. In the BV picture, the symmetry transformation (5.2) is encoded in $Q_{\rm BV}\Phi^A$, which contains the operators R_I^A. Since $Q_{\rm BV}\Phi^A$ is related to the variation of $S_{\rm BV}$ with respect to Φ_A^+, we have a term in the BV action of the form $\langle \Phi_A^+, R_I^A c^I\rangle$, where the inner product is usually given by an integral over some space-time. This implies that the adjoint of R_I^A appears in the $Q_{\rm BV} c_I^+ = \pm (R^\dagger)_I^A \Phi_A^+$, which is the variation of $S_{\rm BV}$ with respect to c^I. Here, $R^\dagger$ denotes the adjoint of R with respect to $\langle -, -\rangle$. The Noether identity then implies that $Q_{\rm KT}^2 = 0$, at least when acting on the antifields of ghosts:

$$Q_{\rm KT}^2 c_I^+ \;=\; \pm Q_{\rm KT}\Phi_A^+ (R^\dagger)_I^A \;=\; \pm \frac{\delta S[\Phi]}{\delta\Phi^A} (R^\dagger)_I^A \;=\; 0 \;. \tag{5.8}$$

For more details on Noether identities, see e.g. [164, 169, 211, 162].

5.1.1. Scalar field theory

As an introductory example illustrating the construction of an L_∞-algebra for a classical field theory, let us consider scalar field theory on Minkowski space $\mathbb{R}^{1,d}$ as a simple example. This is mainly to stress the point that the BV formalism also provides the L_∞-algebra structure underlying classical field theory without gauge symmetries. Further examples will follow below. As another unusual point, let us include global symmetries into the BRST formalism. This is clearly not necessary for the quantisation of the path integral, and it is usually not even desirable, as it reduces the space of solutions to globally symmetric ones. It will, however, allow us to obtain the usual Noether identities in the L_∞-algebra picture.

Let $\varphi \in \mathscr{C}^\infty(\mathbb{R}^{1,d})$ be a real scalar field with action functional

$$S \;:=\; \int_{\mathbb{R}^{1,d}} \mathrm{d}^{d+1}x \left\{ \tfrac{1}{2}(\partial_\mu\varphi)^2 - \tfrac{1}{2}m^2\varphi^2 - \tfrac{\lambda}{4!}\varphi^4 \right\} . \tag{5.9}$$

We extend the field space $\mathfrak{F} = \mathscr{C}^\infty(\mathbb{R}^{1,d})$ to the action groupoid for the Poincaré group,

$$(\mathsf{SO}(1,d) \ltimes \mathbb{R}^{1,d}) \ltimes \mathscr{C}^\infty(\mathbb{R}^{1,d}) \;\rightrightarrows\; \mathscr{C}^\infty(\mathbb{R}^{1,d}) \;, \tag{5.10}$$

which differentiates to the action algebroid

$$\mathfrak{F}_{\rm BRST} \;=\; (\mathfrak{so}(1,d) \ltimes \mathbb{R}^{1,d}) \ltimes \mathscr{C}^\infty(\mathbb{R}^{1,d}) \;. \tag{5.11}$$

In addition to φ, we have also ghosts $c = c^I = c^\mu{}_\nu + c^\mu \in \mathfrak{so}(1,d) \ltimes \mathbb{R}^{1,d}$. Those are not fields but rather constants on Minkowski space $\mathbb{R}^{1,d}$. The actions of the BRST operator $Q_{\rm BRST}$ on c and φ capture the Poincaré Lie algebra as well as its action on the field φ and read as

$$\begin{aligned} Q_{\rm BRST}(c^\mu{}_\nu + c^\mu) \;&:=\; c^\mu{}_\kappa c^\kappa{}_\nu + c^\mu{}_\kappa c^\kappa \;, \\ Q_{\rm BRST}\varphi \;&:=\; c \rhd \varphi \;:=\; c^\mu \partial_\mu\varphi + c^\mu{}_\nu x^\nu \partial_\mu \varphi \;. \end{aligned} \tag{5.12}$$

We now perform the Koszul–Tate resolution by including antifields φ^+ and c^+. The BV bracket is induced by the canonical symplectic form,

$$\omega_{\rm BV} \;:=\; -\mathrm{d}c^I \wedge \mathrm{d}c^+_I + \int_{\mathbb{R}^{1,d}} \mathrm{d}^{d+1}x \left\{ \delta\varphi(x) \wedge \delta\varphi^+(x) \right\} \tag{5.13}$$

and the BV action functional reads as

$$S_{\rm BV} \;:=\; c^+_I [c,c]^I + \int_{\mathbb{R}^{1,d}} \mathrm{d}^{d+1}x \left\{ \tfrac{1}{2}(\partial_\mu\varphi)^2 - \tfrac{1}{2}m^2\varphi^2 - \tfrac{\lambda}{4!}\varphi^4 + \varphi^+(c \rhd \varphi) \right\} , \tag{5.14}$$

from which the action of $Q_{\rm BV}$ is read off as $Q_{\rm BV} = \{S_{\rm BV}, -\}_{\rm BV}$.

The differential graded vector space underlying the L_∞-algebra is

$$\underbrace{\mathfrak{so}(1,d) \ltimes \mathbb{R}^{1,d}}_{=:\,\mathsf{L}_0} \xrightarrow{\;0\;} \underbrace{\mathscr{C}^\infty(\mathbb{R}^{1,d})}_{=:\,\mathsf{L}_1} \xrightarrow{\;-\partial_\mu\partial^\mu - m^2\;} \underbrace{\mathscr{C}^\infty(\mathbb{R}^{1,d})}_{=:\,\mathsf{L}_2} \xrightarrow{\;0\;} \underbrace{(\mathfrak{so}(1,d) \ltimes \mathbb{R}^{1,d})^*}_{=:\,\mathsf{L}_3} \tag{5.15a}$$

and the non-trivial higher brackets take the form

$$\begin{gathered} \mu_2(c_1,c_2) \;:=\; [c_1,c_2] \;, \quad \mu_2(c_1,\varphi_1) \;:=\; c_1 \rhd \varphi_1 \;, \\ \mu_2(c_1,\varphi_1^+) \;:=\; c_1 \rhd \varphi^+ \;, \quad \mu_2(c_1,c_1^+) \;:=\; c_1 \rhd c_1^+ \;, \\ \mu_3(\varphi_1,\varphi_2,\varphi_3) \;:=\; -\lambda\varphi_1\varphi_2\varphi_3 \end{gathered} \tag{5.15b}$$

for $c_1 \in \mathsf{L}_0$, $\varphi_{1,2,3} \in \mathsf{L}_1$, $\varphi_1^+ \in \mathsf{L}_2$ and $c_1^+ \in \mathsf{L}_3$.

One sees that the homotopy MC action (4.61) of this L_∞-algebra is indeed the BV action (5.14). In addition, we note that the Noether identities follow. For example, we have

$$Q_{\rm KT}^2 c_\mu^+ \;=\; Q_{\rm KT}(\varphi^+\partial_\mu\varphi + c_\nu^+ c_\mu{}^\nu) \;=\; \frac{\delta S}{\delta\varphi}\partial_\mu\varphi + \cdots \;=\; \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi)}\right)\partial_\mu\varphi + \cdots\;, \tag{5.16}$$

where the ellipsis denote ghost terms. We thus see indeed the emergence of the usual Noether identities.

5.1.2. Equivalence of classical field theories

Finally, let us comment a bit more on the role of quasi-isomorphisms. In 2.5., we have seen the mathematical reasons for using them rather than ordinary isomorphisms, but there is a deeper, physical, reason. Classically, equivalent physical theories have the same observable and can be related by field redefinition, factoring out symmetries, integrating out fields, etc. Now, considering what we have learnt about the BV formalism, this implies that classically equivalent physical theories have isomorphic $Q_{\rm BV}$-cohomology. In the dual picture, we have the correspondence between classical field theories and L_∞-algebras. Such a correspondence implies that physical equivalence of classical field theories amounts to equivalence between (cyclic) L_∞-algebras, whose appropriate mathematical notion is indeed that of quasi-isomorphisms. We note that a related notion of equivalence was discussed in [212].

To clarify this point, let us see an example of field theories that are related by integrating out auxiliary fields. Consider two classical field theories with actions

$$\begin{aligned} S \;&:=\; \int_{\mathbb{R}^{1,d}} \mathrm{d}^{d+1}x\, \left\{\tfrac12\varphi(-\partial_\mu\partial^\mu - m^2)\varphi - \tfrac{\lambda}{4!}\varphi^4\right\}\,,\\ \tilde S \;&:=\; \int_{\mathbb{R}^{1,d}} \mathrm{d}^{d+1}x\, \left\{\tfrac12\varphi(-\partial_\mu\partial^\mu - m^2)\varphi + \tfrac12 X^2 + \tfrac12\sqrt{\tfrac{\lambda}{3}}X\varphi^2\right\}\,, \end{aligned} \tag{5.17}$$

where φ and X are real scalar fields on Minkowski space $\mathbb{R}^{1,d}$. The equations of motion read as

$$\begin{aligned} S &: (-\partial_\mu\partial^\mu - m^2)\varphi - \tfrac{\lambda}{3!}\varphi^3 \;=\; 0\,,\\ \tilde S &: (-\partial_\mu\partial^\mu - m^2)\varphi + \sqrt{\tfrac{\lambda}{3}}X\varphi \;=\; 0 \quad\text{and}\quad X + \tfrac12\sqrt{\tfrac{\lambda}{3}}\varphi^2 \;=\; 0 \end{aligned} \tag{5.18}$$

and S clearly arises from $\tilde S$ by integrating out the auxiliary field X.

Following the BV formalism and introducing antifields (which amounts to the Koszul–Tate resolution), leads to two L_∞-algebras L and $\tilde{\mathsf{L}}$ given by

$$\begin{gathered}\underbrace{0}_{=:\,\mathsf{L}_0}\ \longrightarrow\ \underbrace{\mathscr{C}^\infty(\mathbb{R}^{1,d})}_{=:\,\mathsf{L}_1}\ \longrightarrow\ \underbrace{\mathscr{C}^\infty(\mathbb{R}^{1,d})}_{=:\,\mathsf{L}_2}\ \longrightarrow\ \underbrace{0}_{=:\,\mathsf{L}_3}\ ,\\ \mu_1(\varphi_1)\ :=\ (-\partial_\mu\partial^\mu - m^2)\varphi_1\ ,\\ \mu_3(\varphi_1,\varphi_2,\varphi_3)\ :=\ -\lambda\varphi_1\varphi_2\varphi_3\end{gathered} \tag{5.19a}$$

and

$$\begin{gathered}\underbrace{0}_{=:\,\tilde{\mathsf{L}}_0}\ \longrightarrow\ \underbrace{\mathscr{C}^\infty(\mathbb{R}^{1,d})\oplus\mathscr{C}^\infty(\mathbb{R}^{1,d})}_{=:\,\tilde{\mathsf{L}}_1}\ \longrightarrow\ \underbrace{\mathscr{C}^\infty(\mathbb{R}^{1,d})\oplus\mathscr{C}^\infty(\mathbb{R}^{1,d})}_{=:\,\tilde{\mathsf{L}}_2}\ \longrightarrow\ \underbrace{0}_{=:\,\tilde{\mathsf{L}}_3}\ ,\\ \tilde\mu_1(\varphi_1+X_1)\ :=\ (-\partial_\mu\partial^\mu - m^2)\varphi_1 + X_1\ ,\\ \tilde\mu_2(\varphi_1+X_1,\varphi_2+X_2)\ :=\ \sqrt{\tfrac{\lambda}{3}}\Big((X_1\varphi_2+X_2\varphi_1)+\varphi_1\varphi_2\Big)\ .\end{gathered} \tag{5.19b}$$

The identity map contained in $\tilde\mu_1$ makes it obvious that the graded vector spaces underlying L and $\tilde{\mathsf{L}}$ have the same cohomology and we define a cochain map $\phi_1 : \tilde{\mathsf{L}} \to \mathsf{L}$ by setting

$$\begin{aligned}\phi_1\ &:\ \tilde{\mathsf{L}}_1\ \to\ \mathsf{L}_1\quad\text{with}\quad \phi_1(\varphi+X)\ :=\ \varphi\ ,\\ \phi_1\ &:\ \tilde{\mathsf{L}}_2\ \to\ \mathsf{L}_2\quad\text{with}\quad \phi_1(\zeta+Y)\ :=\ \zeta\ ,\end{aligned} \tag{5.20}$$

that is, we obtain a quasi-isomorphism of cochain complexes. To extend this to a quasi-isomorphism between $\tilde{\mathsf{L}}$ and L, we note that all higher products are of the form $\tilde\mu_i : \tilde{\mathsf{L}}_1 \times \cdots \times \tilde{\mathsf{L}}_1 \to \tilde{\mathsf{L}}_2$ and $\mu_i : \mathsf{L}_1 \times \cdots \times \mathsf{L}_1 \to \mathsf{L}_2$. Thus, we are only interested in the defining equation of L_∞-morphisms, (2.57), for all arguments in $\tilde{\mathsf{L}}_1$. Upon reducing to the non-trivial higher products and truncating the morphism at the level 2, we obtain

$$\begin{aligned}i\ =\ 1\ &:\ \mu_1(\phi_1(\varphi_1+X_1))\ =\ \phi_1(\tilde\mu_1(\varphi_1+X_1))\ ,\\ i\ =\ 2\ &:\ \mu_1(\phi_2(\varphi_1+X_1,\varphi_2+X_2))\ =\\ &\qquad =\ \phi_1(\tilde\mu_2(\varphi_1+X_1,\varphi_2+X_2))\,-\\ &\qquad\qquad -\,\phi_2(\tilde\mu_1(\varphi_1+X_1),\varphi_2+X_2)-\phi_2(\tilde\mu_1(\varphi_2+X_2),\varphi_1+X_1)\ ,\\ i\ =\ 3\ &:\ \mu_3(\phi_1(\varphi_1+X_1),\phi_1(\varphi_2+X_2),\phi_1(\varphi_3+X_3))\ =\\ &\qquad =\ -[\phi_2(\tilde\mu_2(\varphi_1+X_1,\varphi_2+X_2),\varphi_3+X_3)+\text{cyclic}]\ .\end{aligned} \tag{5.21}$$

These equations are fulfilled by setting

$$\begin{aligned}\phi_2\ &:\ \tilde{\mathsf{L}}_1\times\tilde{\mathsf{L}}_1\ \to\ \mathsf{L}_1\quad\text{with}\quad \phi_2(\varphi_1+X_1,\varphi_2+X_2)\ :=\ 0\ ,\\ \phi_2\ &:\ \tilde{\mathsf{L}}_1\times\tilde{\mathsf{L}}_2\ \to\ \mathsf{L}_2\quad\text{with}\quad \phi_2(\varphi_1+X_1,\zeta_1+Y_1)\ :=\ \sqrt{\tfrac{\lambda}{3}}\varphi_1 Y_1\ .\end{aligned} \tag{5.22}$$

Hence, we may conclude that the L_∞-algebras L and $\tilde{\mathsf{L}}$ corresponding to classically equivalent field theories are quasi-isomorphic, as mentioned. We shall see more examples in the next few sections.

5.2. Yang–Mills theory

We start our discussion about gauge theories by studying Yang–Mills theory. In the following, let M be a smooth compact Riemannian manifold without boundary and real dimension d. In addition, let G be a Lie group with metric Lie algebra $(\mathfrak{g}, [-,-], \langle -,- \rangle_{\mathfrak{g}})$. We define L to be $\Omega^\bullet(M,\mathfrak{g}) := \Omega^\bullet(M) \otimes \mathfrak{g}$, the differential graded Lie algebra of $\mathfrak{g}$-valued differential forms on M. Furthermore, let d be the exterior derivative and $\mathrm{d}^\dagger$ its adjoint with respect to the standard inner product $\langle \alpha_1, \alpha_2 \rangle = \int_M \alpha_1 \wedge \star\alpha_2$ for $\alpha_{1,2} \in \Omega^k(M)$, with $\star$ the Hodge operator for the given metric.

5.2.1. Yang–Mills theory: second-order formulation

The field content of plain Yang–Mills theory in second-order formulation (YM_2) consists of a gauge potential $A \in \Omega^1(M,\mathfrak{g})[0]$ with curvature $F = \mathrm{d}A + \frac{1}{2}[A,A]$. We add a ghost $c \in \Omega^0(M,\mathfrak{g})[1]$ and complete thus the BRST complex, the differential graded algebra description of the gauge algebroid, as explained in Chapter 4.[1] Recall that $[k]$ for $k \in \mathbb{Z}$ indicates the ghost degree of the object.

BV action and BV operator. To complete the BV complex, we also add the antifield $A^+ \in \Omega^{d-1}(M,\mathfrak{g})[-1]$ as well as the antifield of the ghost field, $c^+ \in \Omega^d(M,\mathfrak{g})[-2]$. On this space of fields $\mathfrak{F}_{\mathrm{YM_2BV}}$, we have the canonical symplectic form

$$\omega_{\mathrm{YM_2BV}} \ := \ \int_M \Big\{ \langle \delta A, \delta A^+ \rangle_{\mathfrak{g}} - \langle \delta c, \delta c^+ \rangle_{\mathfrak{g}} \Big\} \ , \tag{5.23}$$

as introduced in (4.25).

The BV action for Yang–Mills theory is derived to be

$$S_{\mathrm{YM_2BV}} \ := \ \int_M \Big\{ \tfrac{1}{2} \langle F, \star F \rangle_{\mathfrak{g}} - \langle A^+, \nabla c \rangle_{\mathfrak{g}} + \tfrac{1}{2} \langle c^+, [c,c] \rangle_{\mathfrak{g}} \Big\} \ , \tag{5.24}$$

[1] Recall that A and c denote contracted coordinate functions. For instance $A = \int_M A^\alpha_\mu(x) \otimes \tau_\alpha \otimes \mathrm{d}x^\mu \otimes s_x$, where τ_α, $\mathrm{d}x^\mu$ and s_x are basis vectors on $\mathfrak{g}$, $T^*_x M$ and $\mathscr{C}^\infty(M)$, respectively.

cf. e.g. [14]. Denoting by $\{-,-\}_{\mathrm{YM_2BV}}$ the Poisson bracket induced by (5.23), it is a straightforward exercise to show that $S_{\mathrm{YM_2BV}}$ satisfies the classical master equation

$$\{S_{\mathrm{YM_2BV}}, S_{\mathrm{YM_2BV}}\}_{\mathrm{YM_2BV}} = 0 \ . \tag{5.25}$$

Hence, we may define the homological vector field

$$Q_{\mathrm{YM_2BV}} \ := \ \{S_{\mathrm{YM_2BV}}, -\}_{\mathrm{YM_2BV}} \quad \text{with} \quad Q^2_{\mathrm{YM_2BV}} \ = \ 0 \ , \tag{5.26}$$

whose action on a functional $F \in \mathscr{C}^\infty(\mathfrak{F}_{\mathrm{YM_2BV}})$ reads as[1]

$$\begin{aligned} Q_{\mathrm{YM_2BV}}F \ = \ \int_M \Big\{ &-\left\langle \frac{\delta S_{\mathrm{BV}}}{\delta A^+}, \frac{\delta F}{\delta A}\right\rangle^*_{\mathfrak{g}} - \left\langle \frac{\delta S_{\mathrm{BV}}}{\delta A}, \frac{\delta F}{\delta A^+}\right\rangle^*_{\mathfrak{g}} - \\ &- \left\langle \frac{\delta S_{\mathrm{BV}}}{\delta c^+}, \frac{\delta F}{\delta c}\right\rangle^*_{\mathfrak{g}} - \left\langle \frac{\delta S_{\mathrm{BV}}}{\delta c}, \frac{\delta F}{\delta c^+}\right\rangle^*_{\mathfrak{g}} \Big\} \ . \end{aligned} \tag{5.27}$$

On the contracted coordinate functions on $\mathfrak{F}_{\mathrm{YM_2BV}}$, we have

$$\begin{aligned} Q_{\mathrm{YM_2BV}}c \ &= \ -\tfrac{1}{2}[c,c] \ , \\ Q_{\mathrm{YM_2BV}}A \ &= \ \nabla c \ = \ \mathrm{d}c + [A,c] \ , \\ Q_{\mathrm{YM_2BV}}A^+ \ &= \ -\nabla{\star}F - [c,A^+] \\ &= \ -\mathrm{d}\star\mathrm{d}\,A + \tfrac{1}{2}\mathrm{d}\star[A,A] + [A,{\star}\mathrm{d}A] + \tfrac{1}{2}[A,{\star}[A,A]] - [c,A^+] \ , \\ Q_{\mathrm{YM_2BV}}c^+ \ &= \ \nabla A^+ - [c,c^+] \\ &= \ \mathrm{d}A^+ + [A,A^+] - [c,c^+] \ . \end{aligned} \tag{5.28}$$

L_∞-algebra structure. The differential graded algebra $(\mathscr{C}^\infty(\mathfrak{F}_{\mathrm{YM_2BV}}), Q_{\mathrm{YM_2BV}})$ is dual to an L_∞-algebra structure on the graded vector space[2]

$$\underbrace{\Omega^0(M,\mathfrak{g})}_{=:\,\mathsf{L}_0} \xrightarrow{\mu_1:=\mathrm{d}} \underbrace{\Omega^1(M,\mathfrak{g})}_{=:\,\mathsf{L}_1} \xrightarrow{\mu_1:=\mathrm{d}{\star}\mathrm{d}} \underbrace{\Omega^{d-1}(M,\mathfrak{g})}_{=:\,\mathsf{L}_2} \xrightarrow{\mu_1:=\mathrm{d}} \underbrace{\Omega^d(M,\mathfrak{g})}_{=:\,\mathsf{L}_3} \ . \tag{5.29a}$$

[1]Recall that the inner product on $\Omega^\bullet(M,\mathfrak{g})$ is $\int_M\langle-,-\rangle_{\mathfrak{g}}$. Here and in the following, the notation $\int_M\langle-,-\rangle^*_{\mathfrak{g}}$ shall denote the dual BV pairing as discussed in Section 4.4.. In (5.27) for instance, $\frac{\delta}{\delta A} = \frac{\delta}{\delta A^\alpha_\mu}\otimes e^\alpha\otimes\partial_\mu$, for e^α and ∂_μ dual bases of τ_α and $\mathrm{d}x^\mu$ respectively and similarly for the other fields.

[2]In principle, one may apply the natural isomorphisms $\Omega^{d-i}(M,\mathfrak{g}) \cong \Omega^i(M,\mathfrak{g})$ to be able to identify the second μ_1 with the Hodge Laplacian. This, however, is somewhat unnatural from the BV perspective, as it will modify the canonical symplectic structure (5.23) on $\mathfrak{F}_{\mathrm{YM_2BV}}$. It would also make our computations below less straightforward.

We call this complex the *second-order Yang–Mills complex*. Note that the higher products μ_i are read off the action (5.28) of the homological vector field on the fields using formula (2.53): μ_1 is given by the linear part of the right-hand side of (5.28), μ_2 by the quadratic part, etc. The coordinate functions on L_0, L_1, L_2, and L_3 are, respectively, c, A, A^+, and c^+. The L_∞-algebra is then defined by the non-vanishing higher products

$$
\begin{gathered}
\mu_1(c_1) \;:=\; \mathrm{d}c_1 \;, \quad \mu_1(A_1) \;:=\; \mathrm{d}\star\mathrm{d}A_1 \;, \quad \mu_1(A_1^+) \;:=\; \mathrm{d}A_1^+ \;, \\
\mu_2(c_1,c_2) \;:=\; [c_1,c_2] \;, \quad \mu_2(c_1,A_1) \;:=\; [c_1,A_1] \;, \\
\mu_2(c_1,A_2^+) \;:=\; [c_1,A_2^+] \;, \quad \mu_2(c_1,c_2^+) \;:=\; [c_1,c_2^+] \;, \\
\mu_2(A_1,A_2^+) \;:=\; [A_1,A_2^+] \;, \\
\mu_2(A_1,A_2) \;:=\; \mathrm{d}\star[A_1,A_2] + [A_1,\star\mathrm{d}A_2] + [A_2,\star\mathrm{d}A_1] \;, \\
\mu_3(A_1,A_2,A_3) \;:=\; [A_1,\star[A_2,A_3]] + [A_2,\star[A_3,A_1]] + [A_3,\star[A_1,A_2]] \;,
\end{gathered}
\tag{5.29b}
$$

for elements $c_i \in \mathsf{L}_0$, $A_i \in \mathsf{L}_1$, $A_i^+ \in \mathsf{L}_2$, and $c_i^+ \in \mathsf{L}_3$. We shall denote this L_∞-algebra by $\mathsf{L}_{\mathrm{YM}_2}$. Note that as expected from the formalism, $\mathsf{L}_{\mathrm{YM}_2}$ is cyclic with cyclic structure induced by the components of the symplectic form (5.23). This L_∞-algebra has been rediscovered several times in the literature. It was first given in [195, 196] in its dual formulation as a differential graded algebra. The same L_∞-algebra was then rederived from string field theory considerations and further discussed in [197, 199].

We stress again that the L_∞-algebra $\mathsf{L}_{\mathrm{YM}_2}$ encodes all classical information about Yang–Mills theory: it contains the field content, the gauge symmetries, the equations of motions as well as the Noether identities.

We note that in the case of classical gauge theories with Abelian gauge group, for which L is concentrated in degrees $0,\ldots,3$, the underlying complex has been studied under the name of *detour complex* [213, 214]. The non-abelian detour complex differs from the complex underlying L by terms covariantising the μ_1. This relation was explained in [201], where also the homotopy algebra structures underlying the complex (5.29a) were studied.

Homotopy Maurer–Cartan action. Let us now plug $\mathsf{L}_{\mathrm{YM}_2}$ into the homotopy MC action (3.21). With $a = A$ we have

$$
\begin{aligned}
\tfrac{1}{2}\langle a, \mu_1(a)\rangle_{\mathsf{L}_{\mathrm{YM}_2}} &= \tfrac{1}{2}\int_M \langle \mathrm{d}A, \star \mathrm{d}A\rangle_{\mathfrak{g}}\ , \\
\tfrac{1}{3!}\langle A, \mu_2(a,a)\rangle_{\mathsf{L}_{\mathrm{YM}_2}} &= \tfrac{1}{2}\int_M \langle \mathrm{d}A, \star[A,A]\rangle_{\mathfrak{g}} \\
&= \tfrac{1}{4}\int_M \big\{\langle [A,A], \star \mathrm{d}A\rangle_{\mathfrak{g}} + \langle \mathrm{d}A, \star[A,A]\rangle_{\mathfrak{g}}\big\}\ , \\
\tfrac{1}{4!}\langle a, \mu_3(a,a,a)\rangle_{\mathsf{L}_{\mathrm{YM}_2}} &= \tfrac{1}{8}\int_M \langle [A,A], \star[A,A]\rangle_{\mathfrak{g}}\ .
\end{aligned}
\tag{5.30}
$$

Consequently, the homotopy Maurer–Cartan action becomes the Yang–Mills action,

$$
S_{\mathrm{MC}} = \sum_{i=1}^{3} \frac{1}{(i+1)!}\langle a, \mu_i(a,\ldots,a)\rangle_{\mathsf{L}_{\mathrm{YM}_2}} = \tfrac{1}{2}\int_M \langle F, \star F\rangle_{\mathfrak{g}}\ . \tag{5.31}
$$

In addition, the Yang–Mills equation translates into the flatness condition

$$
\star\nabla\star F = \mathrm{d}\star F + [A, \star F] = 0 \quad \rightarrow \quad \sum_{i=1}^{3}\frac{1}{i!}\mu_i(a,\ldots,a) = 0\ . \tag{5.32}
$$

We can also reproduce the BV action (5.24) from the BV action (4.61), see also (4.67a), using $\mathsf{a} = c_0 + a + a^+ + c_0^+ = c + A + A^+ + c^+$. Indeed, we find

$$
\begin{aligned}
S_{\mathrm{BV}} &= \sum_{i=1}^{3}\frac{1}{(i+1)!}\langle a, \mu_i(a,\ldots,a)\rangle_{\mathsf{L}_{\mathrm{YM}_2}} - \langle c_0, \mu_1(a^+)\rangle_{\mathsf{L}_{\mathrm{YM}_2}} + \langle a, \mu_2(a^+, c_0)\rangle_{\mathsf{L}_{\mathrm{YM}_2}} \\
&= S_{\mathrm{YM}_2\mathrm{BV}}\ .
\end{aligned}
\tag{5.33}
$$

In summary, we have obtained a reformulation of Yang–Mills theory as a homotopy MC theory, which is closely related to Chern–Simons theory.

Minimal model. Above we observed that the structures of classical Yang–Mills theory are fully captured by the L_∞-algebra $\mathsf{L}_{\mathrm{YM}_2}$. As explained earlier, the appropriate isomorphisms, namely the quasi-isomorphisms, are supposed to lead to theories which are physically equivalent at the classical level. A particularly interesting quasi-isomorphic L_∞-algebra is certainly a minimal model of $\mathsf{L}_{\mathrm{YM}_2}$ in which all possible equivalences have been divided out.

Recall that the minimal model can be constructed recursively by using the homological perturbation lemma [215–217]. In particular, to compute the minimal model $\mathsf{L}'_{\mathrm{YM}_2}$ of $\mathsf{L}_{\mathrm{YM}_2}$ one needs to find a contracting homotopy

$$
h \circlearrowright \mathsf{L}_{\mathrm{YM}_2} \underset{e}{\overset{p}{\rightleftarrows}} \mathsf{L}'_{\mathrm{YM}_2} := H^\bullet_{\mathrm{d}}(\mathsf{L}_{\mathrm{YM}_2})\ . \tag{5.34}
$$

Explicitly, we shall make use of the Hodge decomposition

$$\Omega^k(M,\mathfrak{g}) \;\cong\; \Omega^k_{\rm h}(M,\mathfrak{g}) \oplus {\rm d}\Omega^{k-1}(M,\mathfrak{g}) \oplus {\rm d}^\dagger\Omega^{k+1}(M,\mathfrak{g})\;, \tag{5.35a}$$

together with the projectors

$$\begin{gathered} P_{\rm h}\,:\,\Omega^k(M,\mathfrak{g}) \;\to\; \Omega^k_{\rm h}(M,\mathfrak{g})\;,\quad P_{\rm e}\,:\,\Omega^k(M,\mathfrak{g}) \;\to\; {\rm d}\Omega^{k-1}(M,\mathfrak{g})\;,\\ P_{\rm c}\,:\,\Omega^k(M,\mathfrak{g}) \;\to\; {\rm d}^\dagger\Omega^{k+1}(M,\mathfrak{g})\;, \end{gathered} \tag{5.35b}$$

which extract the harmonic, exact, and coexact parts, respectively. The cohomology complex

$$\mathsf{L}'_{\rm YM_2} \;:=\; H^\bullet_{\mu_1}(\mathsf{L}_{\rm YM_2}) \tag{5.36}$$

of the second-order Yang–Mills complex (5.29a) is then given by

$$\underbrace{\Omega^0_{\rm h}(M,\mathfrak{g})}_{=:\,\mathsf{L}'_0} \xrightarrow{\mu'_1:=0} \underbrace{\Omega^1_{\rm h}(M,\mathfrak{g})}_{=:\,\mathsf{L}'_1} \xrightarrow{\mu'_1:=0} \underbrace{\Omega^{d-1}_{\rm h}(M,\mathfrak{g})}_{=:\,\mathsf{L}'_2} \xrightarrow{\mu'_1:=0} \underbrace{\Omega^d_{\rm h}(M,\mathfrak{g})}_{=:\,\mathsf{L}'_3}\;, \tag{5.37}$$

as we shall now argue. First, L'_0 and L'_3 are obvious; note also that $\Omega^0_{\rm h}(M,\mathfrak{g}) \cong \mathfrak{g}$.[1] On $\Omega^d(M,\mathfrak{g})$, we have $\ker\big({\rm d}\star{\rm d}\big) \cong \ker({\rm d}^\dagger{\rm d}) \cong P_{\rm c}\Omega^d(M,\mathfrak{g})$ and so, $H^1_{\mu_1}(\mathsf{L}_{\rm YM_2}) \cong H^1_{\rm dR}(M,\mathfrak{g}) \cong \Omega^1_{\rm h}(M,\mathfrak{g})$ using the Hodge theorem. Moreover,

$$\begin{aligned} \ker({\rm d}) \;&\cong\; \Omega^{d-1}_{\rm h}(M,\mathfrak{g}) \oplus {\rm d}\Omega^{d-2}(M,\mathfrak{g})\\ &\cong\; \Omega^{d-1}_{\rm h}(M,\mathfrak{g}) \oplus {\rm d}{\rm d}^\dagger\Omega^{d-1}(M,\mathfrak{g})\\ &\cong\; \Omega^1_{\rm h}(M,\mathfrak{g}) \oplus {\rm im}(-\,{\rm d}\star{\rm d}) \end{aligned} \tag{5.38}$$

using the Hodge decomposition and Hodge duality, and therefore $H^2_{\mu_1}(\mathsf{L}_{\rm YM_2}) \cong \Omega^1_{\rm h}(M,\mathfrak{g})$.

As discussed in Section 2.5., the complex $\mathsf{L}'_{\rm YM_2}$ admits an L_∞-structure. To construct the higher products μ'_i for $i>1$ on $\mathsf{L}'_{\rm YM_2}$, we first note that the second-order Yang–Mills complex (5.29a) is split in the sense of (2.61). To see this, set

$${\rm d}_0 \;:=\; {\rm d}\;,\quad {\rm d}_1 \;:=\; {\rm d}\star{\rm d}\;,\quad \text{and}\quad {\rm d}_2 \;:=\; {\rm d}\;. \tag{5.39}$$

Next, we need to find a contracting homotopy $h_k : \mathsf{L}_k \to \mathsf{L}_{k-1}$, that is, ${\rm d}_k = {\rm d}_k \circ h_{k+1} \circ {\rm d}_k$ for $k = 0,1,2$. Using the Green operator G defined in (5.92), we can put

$$h_1 \;:=\; {\rm d}^\dagger G\;,\quad h_2 \;:=\; (-1)^{d+1}G\star\;,\quad \text{and}\quad h_3 \;:=\; {\rm d}^\dagger G\;. \tag{5.40}$$

[1]This condition holds if M is connected.

This is seen using the identities (5.94). Furthermore, setting $(\mathrm{d}_{-1}, h_0) := (0,0)$ and $(\mathrm{d}_3, h_4) := (0,0)$, we have the projectors P_k defined by

$$1 \;=\; P_k + h_{k+1} \circ \mathrm{d}_k + \mathrm{d}_{k-1} \circ h_k \;, \tag{5.41}$$

which are the compositions of the projections $p_k : \mathsf{L}_k \twoheadrightarrow H^k_{\mu_1}(\mathsf{L}_{\mathrm{YM}_2})$ and the injections $e_k : H^k_{\mu_1}(\mathsf{L}_{\mathrm{YM}_2}) \hookrightarrow \mathsf{L}_k$. That is, the P_k form a projector $P_\mathrm{h} : \Omega^\bullet(M,\mathfrak{g}) \to \Omega^\bullet_\mathrm{h}(M,\mathfrak{g})$ onto the harmonic forms using the Hodge decomposition. Using the contracting homotopy (5.40) it is now easy to adapt the formulas (2.64) for the quasi-isomorphism between $\mathsf{L}_{\mathrm{YM}_2}$ and $\mathsf{L}'_{\mathrm{YM}_2} = H^\bullet_{\mu_1}(\mathsf{L}_{\mathrm{YM}_2})$. For the sake of clarity, we shall again only display the formulas in homogeneous degree 1. We obtain

$$\begin{aligned} \phi_1(a') &= e(a')\;, \\ \phi_2(a',a') &= -G\mathrm{d}^\dagger[e(a'),e(a')]\;, \\ &\;\;\vdots \end{aligned} \tag{5.42}$$

Hence, the higher products on $\mathsf{L}'_{\mathrm{YM}_2} = H^\bullet_{\mu_1}(\mathsf{L}_{\mathrm{YM}_2})$ defined in (2.65) are then given for degree 1 elements by

$$\begin{aligned} \mu'_1(a') &= 0\;, \\ \mu'_2(a',a') &= 0\;, \\ \mu'_3(a',a',a') &= 3p\Big(\big[e(a'),\star P_\mathrm{h}[e(a'),e(a')]\big]\Big)\;, \\ &\;\;\vdots \end{aligned} \tag{5.43}$$

where we have used the projectors (5.35b).

Altogether, we note that the simplification of the configuration space in the quasi-isomorphism induced a much more complicated structure in the interaction terms. See [206, 146] for details.

5.2.2. Yang–Mills theory: first-order formulation

It is well-known [218] that Yang–Mills theory in four dimensions admits an alternative formulation which only makes use of first-order rather than second-order differential operators and has only cubic interactions. Again, it can be formulated in L_∞-language [204, 42].

Hence, let us consider $d = 4$. The metric on M induces the decomposition of differential 2-forms

$$\Omega^2(M,\mathfrak{g}) \;\cong\; \Omega^2_+(M,\mathfrak{g}) \oplus \Omega^2_-(M,\mathfrak{g}) \tag{5.44}$$

into self-dual and anti-self-dual parts. Letting $P_\pm := \frac{1}{2}(1 \pm \star)$ be the corresponding projectors where $\star$ is, as before, the Hodge operator associated with the given metric, we may write $\Omega^2_\pm(M, \mathfrak{g}) = P_\pm \Omega^2(M, \mathfrak{g})$.

BV action and BV operator. The field content of Yang–Mills theory in first-order formulation consists of a gauge potential $A \in \Omega^1(M, \mathfrak{g})[0]$, with usual curvature $F = \mathrm{d}A + \frac{1}{2}[A, A]$, together with an additional Lie-algebra valued, self-dual two-form $B_+ \in \Omega^2_+(M, \mathfrak{g})[0]$, subject to the gauge transformations

$$\delta_c A \ := \ \nabla c \ , \quad \delta_c B_+ \ := \ -[c, B_+] \ , \quad \text{and} \quad \delta_c F \ = \ -[c, F] \tag{5.45}$$

for $c \in \Omega^0(M, \mathfrak{g})$. The action reads as

$$S_{\mathrm{YM}_1} \ := \ \int_M \Big\{ \langle F, B_+ \rangle_{\mathfrak{g}} + \tfrac{\varepsilon}{2} \langle B_+, B_+ \rangle_{\mathfrak{g}} \Big\} \ , \tag{5.46}$$

where the parameter ε is a positive real number. Note that the Yang–Mills equation $\nabla \star F = 0$ is equivalent to $\nabla F_\pm = 0$ due to the Bianchi identity $\nabla F = 0$. Hence, the equations of motion following from (5.46), $B_+ = -\frac{1}{\varepsilon} F_+$ and $\nabla B_+ = 0$, imply the Yang–Mills equations so that (5.46) is classically equivalent to Yang–Mills theory.

The action (5.46) is extended to a BV action by adding ghosts $c \in \Omega^0(M, \mathfrak{g})[1]$ as well as the antifields $A^+ \in \Omega^3(M, \mathfrak{g})[-1]$, $B^+_+ \in \Omega^2_+(M, \mathfrak{g})[-1]$, and $c^+ \in \Omega^4(M, \mathfrak{g})[-2]$. The canonical symplectic form on the space of BV fields $\mathfrak{F}_{\mathrm{BV}}$ is

$$\omega_{\mathrm{YM}_1\mathrm{BV}} \ := \ \int_M \Big\{ \langle \delta A, \delta A^+ \rangle_{\mathfrak{g}} + \langle \delta B_+, \delta B^+_+ \rangle_{\mathfrak{g}} - \langle \delta c, \delta c^+ \rangle_{\mathfrak{g}} \Big\} \tag{5.47}$$

and the BV action reads as

$$\begin{aligned} S_{\mathrm{YM}_1\mathrm{BV}} \ := \ \int_M \Big\{ & \langle F, B_+ \rangle_{\mathfrak{g}} + \tfrac{\varepsilon}{2} \langle B_+, B_+ \rangle_{\mathfrak{g}} - \\ & - \langle A^+, \nabla c \rangle_{\mathfrak{g}} - \langle B^+_+, [B_+, c] \rangle_{\mathfrak{g}} + \tfrac{1}{2} \langle c^+, [c, c] \rangle_{\mathfrak{g}} \Big\} \ . \end{aligned} \tag{5.48}$$

Hence, we may define

$$Q_{\mathrm{YM}_1\mathrm{BV}} \ := \ \{ S_{\mathrm{YM}_1\mathrm{BV}}, - \}_{\mathrm{YM}_1\mathrm{BV}} \quad \text{with} \quad Q^2_{\mathrm{YM}_1\mathrm{BV}} \ = \ 0 \ , \tag{5.49}$$

where $\{-,-\}_{\mathrm{YM_1BV}}$ is the Poisson bracket induced by (5.47). We then obtain

$$
\begin{aligned}
Q_{\mathrm{YM_1BV}} c &= -\tfrac{1}{2}[c,c]\,,\\
Q_{\mathrm{YM_1BV}}(B_+ + A) &= -[c,B_+] + \nabla c \;=\; \mathrm{d}c + [B_+ + A, c]\,,\\
Q_{\mathrm{YM_1BV}}(B_+^+ + A^+) &= -(F_+ + \varepsilon B_+ + [c,B_+^+]) - (\nabla B_+ + [c,A^+])\,,\\
&= -\varepsilon B_+ - P_+\mathrm{d}A - \mathrm{d}B_+ -\\
&\qquad - \tfrac{1}{2}P_+[A,A] - [A,B_+] - [c, B_+^+ + A^+]\,,\\
Q_{\mathrm{YM_1BV}} c^+ &= \nabla A^+ + [B_+, B_+^+] - [c,c^+]\\
&= \mathrm{d}A^+ + [A,A^+] + [B_+,B_+^+] - [c,c^+]\,.
\end{aligned}
\tag{5.50}
$$

L_∞-algebra structure. Note that (5.46) is only cubic in the interactions and hence the corresponding equations of motion are at most quadratic. The L_∞-algebra $\mathsf{L}_{\mathrm{YM}_1}$ has now the underlying graded vector space [219]

$$
\underbrace{\Omega^0(M,\mathfrak{g})}_{=:\,\mathsf{L}_0} \xrightarrow{\mu_1 := \mathrm{d}} \underbrace{\Omega^2_+(M,\mathfrak{g}) \oplus \Omega^1(M,\mathfrak{g})}_{=:\,\mathsf{L}_1} \xrightarrow{\mu_1 := (\varepsilon + \mathrm{d}) + P_+\mathrm{d}}
\underbrace{\Omega^2_+(M,\mathfrak{g}) \oplus \Omega^3(M,\mathfrak{g})}_{=:\,\mathsf{L}_2} \xrightarrow{\mu_1 := 0 + \mathrm{d}} \underbrace{\Omega^4(M,\mathfrak{g})}_{=:\,\mathsf{L}_3}\,,
\tag{5.51a}
$$

that we call the *first-order Yang–Mills complex*, together with the non vanishing higher products

$$
\begin{gathered}
\mu_1(c_1) := \mathrm{d}c_1\,,\quad \mu_1(B_{+1} + A_1) := (\varepsilon B_{+1} + P_+\mathrm{d}A_1) + \mathrm{d}B_{+1}\,,\quad \mu_1(A_1^+) := \mathrm{d}A_1^+\,,\\
\mu_2(c_1,c_2) := [c_1,c_2]\,,\quad \mu_2(c_1, B_{+1} + A_1) := [c_1, B_{+1}] + [c, A_1]\,,\\
\mu_2(c_1, B_{+1}^+ + A_1^+) := [c_1, B_{+1}^+] + [c, A_1^+]\,,\quad \mu_2(c_1, c_2^+) := [c_1, c_2^+]\,,\\
\mu_2(B_{+1} + A_1, B_{+2} + A_2) := P_+[A_1,A_2] + [A_1, B_{+2}] + [A_2, B_{+1}]\,,\\
\mu_2(B_{+1} + A_1, B_{+2}^+ + A_2^+) := [A_1, A_2^+] + [B_1, B_{+2}^+]\,,
\end{gathered}
\tag{5.51b}
$$

which are read off the action of the homological vector field $Q_{\mathrm{YM_1BV}}$ on $\mathfrak{F}_{\mathrm{BV}} = \mathsf{L}_{\mathrm{YM}_1}[1]$ as given in (5.50). Here, $c_i \in \mathsf{L}_0$, $(B_{+i} + A_i) \in \mathsf{L}_1$, $(B_{+i}^+ + A_i^+) \in \mathsf{L}_2$, and $c_i^+ \in \mathsf{L}_3$ for $i = 1,2$. Moreover, an inner product on $\mathsf{L}_{\mathrm{YM}_1}$ is induced by the symplectic form (5.47) and reads as

$$
\langle \alpha_1 \otimes t_1, \alpha_2 \otimes t_2 \rangle_{\mathsf{L}_{\mathrm{YM}_1}} := \int_M \alpha_1 \wedge \alpha_2 \, \langle t_1, t_2 \rangle_{\mathfrak{g}}\,.
\tag{5.52}
$$

Homotopy Maurer–Cartan action. Again, let us briefly check that the homotopy MC action (3.21) for the L_∞-algebra $\mathsf{L}_{\mathrm{YM}_1}$ reproduces the classical action (5.46) and, after extension to shifted copies, the BV action (5.48). For degree 1 elements $a \in \mathsf{L}_1$ we have

$a = B_+ + A \in \Omega^2_+(M,\mathfrak{g})[0] \oplus \Omega^1(M,\mathfrak{g})[0]$ and so

$$\begin{aligned} \tfrac{1}{2}\langle a,\mu_1(a)\rangle_{\mathsf{L}_{\mathrm{YM}_1}} &= \int_M \Big\{\langle \mathrm{d}A, B_+\rangle_{\mathfrak{g}} + \tfrac{\varepsilon}{2}\langle B_+,B_+\rangle_{\mathfrak{g}}\Big\}~, \\ \tfrac{1}{3!}\langle a,\mu_2(a,a)\rangle_{\mathsf{L}_{\mathrm{YM}_1}} &= \tfrac{1}{2}\int_M \langle [A,A],B_+\rangle_{\mathfrak{g}}~. \end{aligned} \tag{5.53}$$

Consequently, the homotopy MC action (3.21) becomes (5.46).

Furthermore, the BV action (5.48) inducing the transformations (5.50) is obtained from the BV action (4.61), see also (4.67a), using $\mathsf{a} = c_0+a+a^+ + c_0^+ = c+B_++A+B_+^+ + A^+ + c^+$.

Minimal model. As in the second-order formalism, it is a rather straightforward exercise to compute a minimal model of the L_∞-algebra $\mathsf{L}_{\mathrm{YM}_1}$. We start from the cohomology complex

$$\mathsf{L}'_{\mathrm{YM}_1} ~:=~ H^\bullet_{\mu_1}(\mathsf{L}_{\mathrm{YM}_1}) \tag{5.54}$$

of the first-order Yang–Mills complex (5.51a) using the Hodge decomposition and the Hodge theorem. We obtain the complex

$$\underbrace{\Omega^0_{\mathrm{h}}(M,\mathfrak{g})}_{=:\mathsf{L}'_0} \xrightarrow{\mu'_1:=0} \underbrace{\Omega^1_{\mathrm{h}}(M,\mathfrak{g})}_{=:\mathsf{L}'_1} \xrightarrow{\mu'_1:=0} \underbrace{\Omega^3_{\mathrm{h}}(M,\mathfrak{g})}_{=:\mathsf{L}'_2} \xrightarrow{\mu'_1:=0} \underbrace{\Omega^4_{\mathrm{h}}(M,\mathfrak{g})}_{=:\mathsf{L}'_3}~, \tag{5.55}$$

which is the same as the complex (5.37). Indeed,

$$H^0_{\mu_1}(\mathsf{L}_{\mathrm{YM}_1}) ~\cong~ \Omega^0_{\mathrm{h}}(M,\mathfrak{g}) \quad\text{and}\quad H^3_{\mu_1}(\mathsf{L}_{\mathrm{YM}_1}) ~\cong~ \Omega^4_{\mathrm{h}}(M,\mathfrak{g}) \tag{5.56}$$

follow trivially. Furthermore,

$$\ker(P_+\mathrm{d}+(\varepsilon+\mathrm{d})) ~\cong~ \ker(\mathrm{d}^\dagger\mathrm{d}|_{\Omega^1(M,\mathfrak{g})}) \tag{5.57a}$$

so that

$$H^1_{\mu_1}(\mathsf{L}_{\mathrm{YM}_1}) ~\cong~ \ker(\mathrm{d}^\dagger\mathrm{d}|_{\Omega^1(M,\mathfrak{g})})/\mathrm{im}(\mathrm{d}) ~\cong~ \Omega^1_{\mathrm{h}}(M,\mathfrak{g}) \tag{5.57b}$$

as was already shown in the previous Section. Since

$$1|_{\Omega^2_\pm(M,\mathfrak{g})} ~=~ (P_{\mathrm{h}}+2P_\pm\circ P_{\mathrm{e}})|_{\Omega^2_\pm(M,\mathfrak{g})} ~=~ (P_{\mathrm{h}}+2P_\pm\circ P_{\mathrm{c}})|_{\Omega^2_\pm(M,\mathfrak{g})}~, \tag{5.58}$$

where P_{h}, P_{e}, and P_{c} were introduced in (5.35b), we obtain

$$\begin{aligned} \ker(\mathrm{d}) ~&\cong~ \Omega^3_{\mathrm{h}}(M,\mathfrak{g}) \oplus \mathrm{d}\Omega^2(M,\mathfrak{g}) \oplus \Omega^2_+(M,\mathfrak{g})~, \\ \mathrm{im}(P_+\mathrm{d}+(\varepsilon+\mathrm{d})) ~&\cong~ \Omega^2_+(M,\mathfrak{g}) \oplus \mathrm{d}\Omega^2(M,\mathfrak{g})~. \end{aligned} \tag{5.59a}$$

Hence,

$$H^2_{\mu_1}(\mathsf{L}_{\mathrm{YM}_1}) \;\cong\; \Omega^3_{\mathrm{h}}(M,\mathfrak{g}) \;. \tag{5.59b}$$

To complete the quasi-isomorphism, let us again construct a contracting homotopy $h_k : \mathsf{L}_k \to \mathsf{L}_{k-1}$. We set

$$\mathrm{d}_0 \;:=\; \begin{pmatrix} 0 \\ \mathrm{d} \end{pmatrix} , \quad \mathrm{d}_1 \;:=\; \begin{pmatrix} \varepsilon & P_+\mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} , \quad \text{and} \quad \mathrm{d}_2 \;:=\; \big(0,\mathrm{d}\big) \;. \tag{5.60}$$

Then, we wish to find h_k such that $\mathrm{d}_k = \mathrm{d}_k \circ h_{k+1} \circ \mathrm{d}_k$ for $k = 0,1,2$. Using the Green operator (5.92) and (5.58), we obtain

$$h_1 \;=\; \big(0,\mathrm{d}^\dagger G\big) \;, \quad h_2 \;=\; \begin{pmatrix} \frac{1}{\varepsilon}P_{\mathrm{h}} & 2P_+\mathrm{d}^\dagger G \\ 2\mathrm{d}^\dagger G & 2\varepsilon\mathrm{d}^\dagger G\mathrm{d}G\star \end{pmatrix} , \quad \text{and} \quad h_3 \;=\; \begin{pmatrix} 0 \\ \mathrm{d}^\dagger G \end{pmatrix} . \tag{5.61}$$

Furthermore, setting $(\mathrm{d}_{-1},h_0) := (0,0)$ and $(\mathrm{d}_3,h_4) := (0,0)$, we have the projectors P_k defined by

$$1 \;=\; P_k + h_{k+1}\circ \mathrm{d}_k + \mathrm{d}_{k-1}\circ h_k \tag{5.62}$$

projecting L_k onto $H^k_{\mu_1}(\mathsf{L}_{\mathrm{YM}_1})$. That is, the P_k yield the projector $P_{\mathrm{h}} : \Omega^\bullet(M,\mathfrak{g}) \to \Omega^\bullet_{\mathrm{h}}(M,\mathfrak{g})$. Using the contracting homotopy (5.61), we now adapt the formulas (2.64) for the quasi-isomorphism between $\mathsf{L}_{\mathrm{YM}_1}$ and $\mathsf{L}'_{\mathrm{YM}_1} = H^\bullet_{\mu_1}(\mathsf{L}_{\mathrm{YM}_1})$. As before, for the sake of clarity, we shall only display the formulas in homogeneous degree 1. We obtain

$$\begin{aligned} \phi_1(a') \;&=\; e(a') \;, \\ \phi_2(a',a') \;&=\; -\Big(\tfrac{1}{\varepsilon}P_{\mathrm{h}} + 2\mathrm{d}^\dagger G\Big)P_+[e(a'),e(a')] \;, \\ &\;\;\vdots \end{aligned} \tag{5.63}$$

Hence, the higher products on $\mathsf{L}'_{\mathrm{YM}_1} = H^\bullet_{\mu_1}(\mathsf{L}_{\mathrm{YM}_1})$ defined in (2.65) are then given for degree 1 elements by

$$\begin{aligned} \mu'_1(a') \;&=\; 0 \;, \\ \mu'_2(a',a') \;&=\; 0 \;, \\ \mu'_3(a',a',a') \;&=\; -\tfrac{3}{\varepsilon}p\Big(\Big[\big[e(a'),P_{\mathrm{h}}P_+[e(a'),e(a')]\big]\Big]\Big) \;, \\ &\;\;\vdots \end{aligned} \tag{5.64}$$

Integrating out fields. Before showing that both formulations of Yang–Mills theory are L_∞-quasi-isomorphic, we demonstrate, as a warm up, that both formulations are equivalent by 'integrating out fields' [171, 42]. Starting from the action

$$S_{\mathrm{YM}_1} = \int_M \left\{ \langle F, B_+\rangle_{\mathfrak{g}} + \tfrac{\varepsilon}{2}\langle B_+, B_+\rangle_{\mathfrak{g}} \right\} \tag{5.65}$$

of Yang–Mills theory in the first-order formulation one can easily integrate out B_+, as it only appears algebraically. We obtain

$$S_{\mathrm{YM}_1,\,\mathrm{eff}} = -\tfrac{1}{2\varepsilon}\int_M \langle F_+, F_+\rangle_{\mathfrak{g}} = -\tfrac{1}{4\varepsilon}\int_M \langle F, \star F\rangle_{\mathfrak{g}} - \tfrac{1}{4\varepsilon}\int_M \langle F, F\rangle_{\mathfrak{g}} , \tag{5.66}$$

that is, we find the Yang–Mills action in the second-order formulation plus a topological term, which is irrelevant for perturbation theory. Hence, the two formulations of Yang–Mills theory are equivalent at the level of their equations of motion.

Next, let us recall the BV action (5.48)

$$\begin{aligned} S_{\mathrm{YM}_1\mathrm{BV}} = \int_M \Big\{ &\langle F, B_+\rangle_{\mathfrak{g}} + \tfrac{\varepsilon}{2}\langle B_+, B_+\rangle_{\mathfrak{g}} - \\ &- \langle A^+, \nabla c\rangle_{\mathfrak{g}} - \langle B_+^+, [B_+, c]\rangle_{\mathfrak{g}} + \tfrac{1}{2}\langle c^+, [c,c]\rangle_{\mathfrak{g}} \Big\} , \end{aligned} \tag{5.67}$$

of Yang–Mills theory in the first-order formulation. Since the ghosts and all the anti-fields are present, integrating out B_+ and B_+^+ is not as straightforward as above even though they appear only algebraically. Following [42] we consider the symplectomorphism given by the Hamiltonian[1]

$$H := \tfrac{1}{2\varepsilon}\int_M \langle c, [B_+^+, B_+^+]\rangle_{\mathfrak{g}} \tag{5.68}$$

for the symplectic form (5.47). Concretely,

$$\begin{aligned} A &\mapsto A + \{H, A\}_{\mathrm{YM}_1\mathrm{BV}} = A , \\ B_+ &\mapsto B_+ + \{H, B_+\}_{\mathrm{YM}_1\mathrm{BV}} = B_+ - \tfrac{1}{\varepsilon}[c, B_+^+] , \\ c &\mapsto c + \{H, c\}_{\mathrm{YM}_1\mathrm{BV}} = c , \\ A^+ &\mapsto A^+ + \{H, A^+\}_{\mathrm{YM}_1\mathrm{BV}} = A^+ , \\ B_+^+ &\mapsto B_+^+ + \{H, B_+^+\}_{\mathrm{YM}_1\mathrm{BV}} = B_+^+ , \\ c^+ &\mapsto c^+ + \{H, c^+\}_{\mathrm{YM}_1\mathrm{BV}} = c^+ + \tfrac{1}{2\varepsilon}[B_+^+, B_+^+] , \end{aligned} \tag{5.69}$$

[1] Costello [171] proves an homotopy equivalence of classical field theories by using $H := \frac{1}{\varepsilon}\int_M \langle F, B_+^+\rangle_{\mathfrak{g}}$ instead.

where $\{-,-\}_{\mathrm{YM_1BV}}$ is the Poisson structure induced by (5.47). Furthermore, it is easy to see that this symplectomorphism preserves the Darboux path integral measure. Upon performing the transformation (5.69), the BV action (5.67) becomes

$$\begin{aligned}\tilde{S}_{\mathrm{YM_1BV}} &:= S_{\mathrm{YM_1BV}} + Q_{\mathrm{YM_1BV}} H \\ &= \int_M \Big\{ -\tfrac{1}{2\varepsilon}\langle F_+, B_+\rangle_{\mathfrak{g}} + \tfrac{\varepsilon}{2}\langle B_+, B_+\rangle_{\mathfrak{g}} - \\ &\qquad - \tfrac{1}{\varepsilon}\langle F_+, [c, B_+^+]\rangle_{\mathfrak{g}} - \langle A^+, \nabla c\rangle_{\mathfrak{g}} + \tfrac{1}{2}\langle c^+, [c,c]\rangle_{\mathfrak{g}} \Big\} .\end{aligned} \tag{5.70}$$

Now we can straightforwardly integrate out B_+ and B_+^+. Indeed, we obtain

$$\tilde{S}_{\mathrm{YM_1BV,eff}} = \int_M \Big\{ -\tfrac{1}{4\varepsilon}\langle F, \star F\rangle_{\mathfrak{g}} - \langle A^+, \nabla c\rangle_{\mathfrak{g}} + \tfrac{1}{2}\langle c^+, [c,c]\rangle_{\mathfrak{g}} \Big\} - \tfrac{1}{4\varepsilon}\int_M \langle F, F\rangle_{\mathfrak{g}} , \tag{5.71}$$

that is, we find (5.24) in the second-order formulation plus a topological term. Hence, the two formulations of Yang–Mills theory are also equivalent in the BV formalism.

L_∞-quasi-isomorphism between the formulations of Yang–Mills theory. L_∞-algebras for the first-order and second-order formulation are L_∞-quasi-isomorphic [204,42]. Since $\mathsf{L}_{\mathrm{YM_1}}$ and $\mathsf{L}_{\mathrm{YM_2}}$ describe equivalent classical field theories, this should not be surprising according to our general discussion in Section 5.1.. In the following we verify the classical equivalence by giving the explicit quasi-isomorphism in the Q-manifold language, as in [42].

Clearly, we expect a quasi-isomorphism that is based on an L_∞-morphism which is an injective L_∞-morphism $\mathsf{L}_{\mathrm{YM_2}} \hookrightarrow \mathsf{L}_{\mathrm{YM_1}}$.[1] In the dga-picture, this corresponds to a surjection $\Phi : \mathscr{C}^\infty(\mathfrak{F}_{\mathrm{YM_1BV}}) \to \mathscr{C}^\infty(\mathfrak{F}_{\mathrm{YM_2BV}})$. For this surjection to be a cochain map we have to verify that

$$Q_{\mathrm{YM_2BV}} \circ \Phi = \Phi \circ Q_{\mathrm{YM_1BV}} . \tag{5.72}$$

Trying to construct such a Φ, one is lead to the surjection defined on the coordinate functions as

$$\begin{aligned} \Phi(c) &:= c , \quad \Phi(B_+) := -\tfrac{1}{\varepsilon}F_+ , \quad \Phi(A) := A , \\ \Phi(B_+^+) &:= 0 , \quad \Phi(A^+) = A^+ , \quad \Phi(c^+) := c^+ . \end{aligned} \tag{5.73}$$

The left-hand side of (5.72) reads as

$$\begin{aligned} &Q_{\mathrm{YM_2BV}}\Phi(c + B_+ + A + B_+^+ + A^+ + c^+) = \\ &= Q_{\mathrm{YM_2BV}}(c + A - \tfrac{1}{\varepsilon}F_+ + A^+ + c^+) \\ &= -\tfrac{1}{2}[c,c] + \nabla c - \tfrac{1}{\varepsilon}[F_+, c] + \nabla\star F + [c, A^+] - \nabla A^+ - [c, c^+] , \end{aligned} \tag{5.74}$$

[1] Obviously, here we are considering $\mathsf{L}_{\mathrm{YM_2}}$ for $d = 4$.

while the right-hand side of (5.72) evaluates to

$$
\begin{aligned}
&\Phi(Q_{\mathrm{YM_1BV}}(c + B_+ + A + B_+^+ + A^+ + c^+)) \ = \\
&= \ \Phi\Big(-\tfrac{1}{2}[c,c] - [c,B_+] + \nabla c - (F_+ + \varepsilon B_+ + [c,B_+^+]) - \\
&\qquad\quad - \nabla B_+ - [c,A^+] + \nabla A^+ + [B_+,B_+^+] - [c,c^+]\Big) \\
&= \ -\tfrac{1}{2}[c,c] + \nabla c - \tfrac{1}{\varepsilon}[F_+,c] + \nabla(1+\star)F - [c,A^+] + \nabla A^+ - [c,c^+] \\
&= \ -\tfrac{1}{2}[c,c] + \nabla c - \tfrac{1}{\varepsilon}[F_+,c] + \nabla\star F - [c,A^+] + \nabla A^+ - [c,c^+] \ .
\end{aligned}
\tag{5.75}
$$

Since both results agree, Φ defines indeed a morphism of L_∞-algebras. Moreover, this isomorphism is surjective, and because we know that the cohomologies of $\mathsf{L}_{\mathrm{YM}_1}$ and $\mathsf{L}_{\mathrm{YM}_2}$ agree, the L_∞-morphism induces an isomorphism on cohomology. One trivially notes that the symplectic from on $\mathsf{L}_{\mathrm{YM}_2}$ is the pull-back of that on $\mathsf{L}_{\mathrm{YM}_1}$ along Φ. Altogether, Φ defines a cyclic quasi-isomorphism, as expected. Hence, the L_∞-algebra $(\mathsf{L}_{\mathrm{YM}_1}, \mu_i)$ provides a strictification of the four-dimensional Yang–Mills L_∞-algebra $(\mathsf{L}_{\mathrm{YM}_2}, \mu_i)$.

This short computation shows the power of going back and forth between the bracket formulation of L_∞-algebras and the dga-picture. The direct construction of a quasi-isomorphism in the bracket formulation would have been somewhat lengthier, as was showing the equivalence of the BV actions by integrating out fields in the previous paragraph.

5.3. Chern–Simons theory and its higher analogues

Homotopy Maurer–Cartan theory is clearly a vast generalisation of Chern–Simons theory for Lie algebras and one may derive higher Chern–Simons theories directly by constructing suitable L_∞-algebras, cf. [110]. Here, motivated by their prominent role in both quantum field theory and string theory, we further specialise our discussion in 3.2.1. to the case of (higher) Chern–Simons theory.

The setup is the following. Let M be a d-dimensional smooth compact oriented manifold with $d \geq 3$ and $\mathsf{L} := \bigoplus_{k=-d+3}^{0} \mathsf{L}_k$ be a cyclic L_∞-algebra to which we shall refer as the *gauge L_∞-algebra* in the following. The tensor product L_∞-algebra $\Omega^\bullet(M,\mathsf{L})$ as defined in (3.29) then has a cyclic structure of degree $-d-(-d+3) = -3$ and we can write down the corresponding homotopy MC action (3.21). This action defines higher Chern–Simons theory with trivial underlying principal ∞-bundles. For instance, in the case when $d = 3$, the gauge L_∞-algebra L is an ordinary Lie algebra and we recover ordinary Chern–Simons theory, see (3.30) and (3.36). In the case when $d = 4$, L is a 2-term L_∞-algebra or, equivalently, a Lie 2-algebra, see (3.32). To obtain the classical L_∞-structure, we simply enlarge the space of fields consisting of the gauge potentials by all ghosts and higher ghosts and then once more by all corresponding antifields.

5.3.1. BV formalism and L_∞-structure

Let us now review the BV formalism applied to (higher) Chern–Simons theory.

Case $d = 3$. In the case of ordinary Chern–Simons theory in three dimensions, we have additional ghosts $c \in \Omega^0(M, \mathfrak{g})[1]$ and antifields $A^+ \in \Omega^2(M, \mathfrak{g})[-1]$ and $c^+ \in \Omega^3(M, \mathfrak{g})[-2]$. Hence, with $\mathsf{a} = c + A + A^+ + c^+$, the symplectic form $\omega_{\rm BV}$ on $\mathfrak{F}_{\rm BV}$ defined in (4.58) becomes

$$\omega_{\rm BV} \; = \; \int_M \left\{ \langle \delta A, \delta A^+ \rangle_{\mathfrak{g}} - \langle \delta c, \delta c^+ \rangle_{\mathfrak{g}} \right\} \, . \tag{5.76}$$

Thus, the induced the Poisson bracket (4.59) reads explicitly as

$$\begin{aligned} \{F, G\}_{\rm BV} \; &= \; \int_M \left\{ F \left\langle \frac{\overleftarrow{\delta}}{\delta A^+}, \frac{\overrightarrow{\delta}}{\delta A} \right\rangle_{\mathfrak{g}}^* G - F \left\langle \frac{\overleftarrow{\delta}}{\delta A}, \frac{\overrightarrow{\delta}}{\delta A^+} \right\rangle_{\mathfrak{g}}^* G - \right. \\ &\qquad \left. - F \left\langle \frac{\overleftarrow{\delta}}{\delta c^+}, \frac{\overrightarrow{\delta}}{\delta c} \right\rangle_{\mathfrak{g}}^* G + F \left\langle \frac{\overleftarrow{\delta}}{\delta c}, \frac{\overrightarrow{\delta}}{\delta c^+} \right\rangle_{\mathfrak{g}}^* G \right\} \end{aligned} \tag{5.77}$$

for $F, G \in \mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$. The BV action (4.61) reads as

$$\begin{aligned} S_{\rm BV} \; &= \; \int_M \left\{ \tfrac{1}{2} \langle A, {\rm d}A \rangle_{\mathfrak{g}} + \tfrac{1}{3!} \langle A, [A, A] \rangle_{\mathfrak{g}} - \right. \\ &\qquad \left. - \langle c, {\rm d}A^+ \rangle_{\mathfrak{g}} - \langle c, [A, A^+] \rangle_{\mathfrak{g}} + \tfrac{1}{2} \langle c^+, [c, c] \rangle_{\mathfrak{g}} \right\} , \end{aligned} \tag{5.78}$$

which is the Hamiltonian for the vector field $Q_{\rm BV}$, which acts on an element $F \in \mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$ according to

$$\begin{aligned} Q_{\rm BV} F \; &:= \; \{S_{\rm BV}, F\}_{\rm BV} \\ &= \; \int_M \left\{ - \left\langle \frac{\delta S_{\rm BV}}{\delta A^+}, \frac{\delta F}{\delta A} \right\rangle_{\mathfrak{g}}^* - \left\langle \frac{\delta S_{\rm BV}}{\delta A}, \frac{\delta F}{\delta A^+} \right\rangle_{\mathfrak{g}}^* - \right. \\ &\qquad \left. - \left\langle \frac{\delta S_{\rm BV}}{\delta c^+}, \frac{\delta F}{\delta c} \right\rangle_{\mathfrak{g}}^* - \left\langle \frac{\delta S_{\rm BV}}{\delta c}, \frac{\delta F}{\delta c^+} \right\rangle_{\mathfrak{g}}^* \right\} . \end{aligned} \tag{5.79}$$

For the coordinate functions, we obtain explicitly

$$
\begin{aligned}
Q_{\rm BV} c \;&=\; -\tfrac{1}{2}[c,c]\ ,\quad Q_{\rm BV} A \;=\; {\rm d}c+[A,c]\ ,\\
Q_{\rm BV} A^+ \;&=\; -{\rm d}A-\tfrac{1}{2}[A,A]-[c,A^+]\ ,\quad Q_{\rm BV} c^+ \;=\; {\rm d}A^+ + [A,A^+]-[c,c^+]\ .
\end{aligned}
\tag{5.80}
$$

Note that we can also specialise the extended BV action (4.77), which is suitable for gauge fixing. Here, the above field content is further extended by the trivial pair $(\bar c,b)\in\Omega^1(M,\mathfrak{g})[-1]\oplus\Omega^1(M,\mathfrak{g})[0]$ together with a trivial pair of corresponding antifields $(\bar c^+,b^+)\in\Omega^2(M,\mathfrak{g})[0]\oplus\Omega^2(M,\mathfrak{g})[-1]$. The extended BV action (4.77) then reads as

$$
\begin{aligned}
S_{\rm eBV} \;=\; \int_M\Big\{&\tfrac{1}{2}\langle A,{\rm d}A\rangle_{\mathfrak{g}}+\tfrac{1}{3!}\langle A,[A,A]\rangle_{\mathfrak{g}}-\\
&-\langle c,{\rm d}A^+\rangle_{\mathfrak{g}}+\langle A,[A^+,c]\rangle_{\mathfrak{g}}+\tfrac{1}{2}\langle c^+,[c,c]\rangle_{\mathfrak{g}}-\langle b,\bar c^+\rangle_{\mathfrak{g}}\Big\}\ ,
\end{aligned}
\tag{5.81}
$$

resulting in

$$
\begin{aligned}
Q_{\rm eBV} c \;&=\; -\tfrac{1}{2}[c,c]\ ,\quad Q_{\rm eBV} A \;=\; {\rm d}c+[A,c]\ ,\\
Q_{\rm eBV} A^+ \;&=\; -{\rm d}A-\tfrac{1}{2}[A,A]-[c,A^+]\ ,\quad Q_{\rm eBV} c^+ \;=\; {\rm d}A^+ + [A,A^+]-[c,c^+]\ ,\\
Q_{\rm eBV}\bar c \;&=\; b\ ,\quad Q_{\rm eBV}\bar c^+ \;=\; 0\ ,\quad Q_{\rm eBV} b^+ \;=\; -\bar c^+\ ,\quad Q_{\rm eBV} b \;=\; 0\ .
\end{aligned}
\tag{5.82}
$$

Case $d = 4$. Next, let us discuss the simplest higher case $d=4$ in detail. Here, the gauge algebra is a Lie 2-algebra[1] whose underlying graded vector space is of the form $\mathsf{L}=\mathsf{L}_{-1}\oplus\mathsf{L}_0$. The inner product identifies $(\mathsf{L}_{-1})^*\cong\mathsf{L}_0$ so that L_{-1} and L_0 must be of the same dimension. We have the decomposition $\Omega^\bullet_1(M,\mathsf{L})\cong\Omega^1(M,\mathsf{L}_0)\oplus\Omega^2(M,\mathsf{L}_{-1})$ of homogeneous degree 1 elements in $\Omega^\bullet(M,\mathsf{L})$. Consequently, we obtain a 1-form gauge potential $A\in\Omega^1(M,\mathsf{L}_0)[0]$ and a 2-form gauge potential $B\in\Omega^2(M,\mathsf{L}_{-1})[0]$, respectively. The homotopy MC action (3.21), with $a=A+B$, is then higher Chern–Simons theory on a four-dimensional manifold M,

$$
S_{\rm MC} \;=\; \int_M\Big\{\langle B,{\rm d}A+\tfrac{1}{2}\mu_2(A,A)+\tfrac{1}{2}\mu_1(B)\rangle_{\mathsf{L}}+\tfrac{1}{4!}\langle\mu_3(A,A,A),A\rangle_{\mathsf{L}}\Big\}\ ,
\tag{5.83}
$$

and the curvature (3.4) reduces to $f\;:=\;\mathcal{F}+H$ where $\mathcal{F}$, H given in (3.32c). As previously mentioned, in these and the following formulas, the higher products μ_i will not see the form

[1] Non-trivial examples of Lie 2-algebras relevant for physical applications are not easy to find. We mention here the *string Lie 2-algebras* (corresponding to the string Lie 2-group $\mathsf{String}(n)$) introduced in [4] and we refer to the papers [7, 30, 160] (and references therein) together with the *T-duality Lie 2-algebra* [220] for applications in string theory. The interested reader may also consult c.f. [7, 221] for further pointers to the physics literature on Lie 2-algebras and the recent review [222] for more geometrical applications.

degree of the various fields. Since any homogeneous degree 0 element of $\Omega^\bullet(M,\mathsf{L})$ decomposes into $c\in\Omega^0(M,\mathsf{L}_0)[0]$ and $\Lambda\in\Omega^1(M,\mathsf{L}_{-1})[0]$, the gauge transformations of the components of the gauge potential and the curvature reduce to (3.33a) and (3.33b), respectively.

To write down the BV action (4.61), see also (4.67a), we recall that one needs to add antifields leading to the decompositions

$$\begin{aligned} a &= A+B \in \Omega^1(M,\mathsf{L}_0)[0]\oplus\Omega^2(M,\mathsf{L}_{-1})[0]~,\\ a^+ &= A^++B^+ \in \Omega^3(M,\mathsf{L}_{-1})[-1]\oplus\Omega^2(M,\mathsf{L}_0)[-1]~, \end{aligned} \tag{5.84a}$$

while for the ghosts we obtain

$$\begin{aligned} c_0 &= c_0^0+c_0^1 \in \Omega^0(M,\mathsf{L}_0)[1]\oplus\Omega^1(M,\mathsf{L}_{-1})[1]~,\\ c_0^+ &= c_0^{0+}+c_0^{1+} \in \Omega^4(M,\mathsf{L}_{-1})[-2]\oplus\Omega^3(M,\mathsf{L}_0)[-2]~,\\ c_{-1} &\in \Omega^0(M,\mathsf{L}_{-1})[2]~,\quad c_{-1}^+ \in \Omega^4(M,\mathsf{L}_0)[-3]~. \end{aligned} \tag{5.84b}$$

The full BV action (4.61) is then $S_{\rm BV} = S_{\rm MC}+S_{\rm gh}$ with $S_{\rm MC}$ given by (5.83) and

$$\begin{aligned} S_{\rm gh} := \int_M \Big\{ &\langle {\rm d}A^+,c_0^0\rangle_\mathsf{L}+\langle {\rm d}B^+,c_0^1\rangle_\mathsf{L}+\langle c_0^1,\mu_1(A^+)\rangle_\mathsf{L}+\langle c_{-1},{\rm d}c_0^{1+}+\mu_1(c_0^{0+})\rangle_\mathsf{L}-\\ &-\langle\mu_2(c_0^{1+},c_{-1})+\mu_2(B^+,c_0^1)+\mu_2(A^+,c_0^0),A\rangle_\mathsf{L}+\langle B,\mu_2(B^+,c_0^0)\rangle_\mathsf{L}-\\ &-\tfrac12\langle\mu_2(B^+,c_{-1}),B^+\rangle_\mathsf{L}+\tfrac12\langle c_0^{0+},\mu_2(c_0^0,c_0^0)\rangle_\mathsf{L}-\langle\mu_2(c_0^0,c_0^1),c_0^{1+}\rangle_\mathsf{L}-\\ &-\langle\mu_2(c_0^0,c_{-1}),c_{-1}^+\rangle_\mathsf{L}+\tfrac12\langle\mu_3(A,B^+,c_0^0),A\rangle_\mathsf{L}-\tfrac12\langle\mu_3(A,c_0^{1+},c_0^0),A\rangle_\mathsf{L}+\\ &+\tfrac12\langle\mu_3(c_0^0,c_0^0,c_0^{1+}),A\rangle_\mathsf{L}+\tfrac{1}{2\cdot2}\langle\mu_3(B^+,c_0^0,c_0^0),B^+\rangle_\mathsf{L}+\\ &+\tfrac{1}{3!}\langle\mu_3(c_0^0,c_0^0,c_{-1}^+),c_0^0\rangle_\mathsf{L}\Big\}~. \end{aligned} \tag{5.85}$$

It induces a homological vector field $Q_{\rm BV}$ on $\mathfrak{F}_{\rm BV}$ acting on $F\in\mathscr{C}^\infty(\mathfrak{F}_{\rm BV})$ as

$$\begin{aligned} Q_{\rm BV}F := \int_M\Big\{ &-\left\langle\frac{\delta S_{\rm BV}}{\delta A^+},\frac{\delta F}{\delta A}\right\rangle_\mathsf{L}^*-\left\langle\frac{\delta S_{\rm BV}}{\delta A},\frac{\delta F}{\delta A^+}\right\rangle_\mathsf{L}^*-\left\langle\frac{\delta S_{\rm BV}}{\delta B^+},\frac{\delta F}{\delta B}\right\rangle_\mathsf{L}^*-\\ &-\left\langle\frac{\delta S_{\rm BV}}{\delta B},\frac{\delta F}{\delta B^+}\right\rangle_\mathsf{L}^*-\left\langle\frac{\delta S_{\rm BV}}{\delta c_0^{0+}},\frac{\delta F}{\delta c_0^0}\right\rangle_\mathsf{L}^*-\left\langle\frac{\delta S_{\rm BV}}{\delta c_0^0},\frac{\delta F}{\delta c_0^{0+}}\right\rangle_\mathsf{L}^*-\\ &-\left\langle\frac{\delta S_{\rm BV}}{\delta c_0^{1+}},\frac{\delta F}{\delta c_0^1}\right\rangle_\mathsf{L}^*-\left\langle\frac{\delta S_{\rm BV}}{\delta c_0^1},\frac{\delta F}{\delta c_0^{1+}}\right\rangle_\mathsf{L}^*+\left\langle\frac{\delta S_{\rm BV}}{\delta c_{-1}^+},\frac{\delta F}{\delta c_{-1}}\right\rangle_\mathsf{L}^*+\\ &+\left\langle\frac{\delta S_{\rm BV}}{\delta c_{-1}},\frac{\delta F}{\delta c_{-1}^+}\right\rangle_\mathsf{L}^*\Big\}~. \end{aligned} \tag{5.86}$$

Explicitly, we have

$$
\begin{aligned}
Q_{\rm BV}A &= {\rm d}c_0^0+\mu_2(A,c_0^0)-\mu_1(c_0^1)~,\\
Q_{\rm BV}B &= -\mu_2(c_0^0,B)+{\rm d}c_0^1+\mu_2(A,c_0^1)+\tfrac12\mu_3(c_0^0,A,A)-\\
&\quad-\mu_2(B^+,c_{-1})+\tfrac12\mu_3(B^+,c_0^0,c_0^0)~,\\
Q_{\rm BV}A^+ &= -{\rm d}B-\mu_2(A,B)+\tfrac{1}{3!}\mu_3(A,A,A)-\mu_2(A^+,c_0^0)-\mu_2(B^+,c_0^1)+\\
&\quad+\mu_2(c_{-1},c_0^{1+})+\mu_3(A,B^+,c_0^0)+\tfrac12\mu_3(c_0^0,c_0^0,c_0^{1+})~,\\
Q_{\rm BV}B^+ &= -{\rm d}A-\tfrac12\mu_2(A,A)-\mu_1(B)-\mu_2(B^+,c_0^0)~,\\
Q_{\rm BV}c_0^{0+} &= -{\rm d}A^++\mu_2(A,A^+)+\mu_2(B,B^+)-\mu_2(c_0^0,c_0^{0+})+\mu_2(c_0^1,c_0^{1+})\\
&\quad+\mu_2(c_{-1},c_{-1}^+)+\tfrac12\mu_3(A,A,B^+)+\tfrac12\mu_3(B^+,B^+,c_0^0)\\
&\quad-\mu_3(A,c_0^0,c_0^{1+})+\tfrac12\mu_3(c_0^0,c_0^0,c_{-1}^+)~,\\
Q_{\rm BV}~c_0^1 &= -{\rm d}c_{-1}-\mu_2(A,c_{-1})-\mu_2(c_0^0,c_0^1)+\tfrac12\mu_3(A,c_0^0,c_0^0)~,\\
Q_{\rm BV}c_0^{1+} &= -{\rm d}B^+-\mu_1(A^+)+\mu_2(A,B^+)-\mu_2(c_0^0,c_0^{1+})~,\\
Q_{\rm BV}c_{-1} &= -\mu_2(c_0^0,c_{-1})+\tfrac{1}{3!}\mu_3(c_0^0,c_0^0,c_0^0)~,\\
Q_{\rm BV}c_{-1}^+ &= -{\rm d}c_0^{1+}-\mu_1(c_0^{0+})+\mu_2(A,c_0^{1+})-\tfrac12\mu_2(B^+,B^+)-\mu_2(c_{-1}^+,c_0^0)~,
\end{aligned}
\tag{5.87}
$$

cf. (4.68).

The extension (4.77) of $S_{\rm BV}$ by trivial pairs requires the introduction of the additional quadruples

$$
\begin{aligned}
\bar c_{0,-1} &= \bar c^0_{0,-1}+\bar c^1_{0,-1} \in \big(\Omega^0(M,\mathsf{L}_0)\oplus\Omega^1(M,\mathsf{L}_{-1})\big)[-1]~,\\
b_{0,-1} &= b^0_{0,-1}+b^1_{0,-1} \in \big(\Omega^0(M,\mathsf{L}_0)\oplus\Omega^1(M,\mathsf{L}_{-1})\big)[0]~,\\
\bar c^+_{0,-1} &= \bar c^{3+}_{0,-1}+\bar c^{2+}_{0,-1} \in \big(\Omega^3(M,\mathsf{L}_0)\oplus\Omega^2(M,\mathsf{L}_{-1})\big)[0]~,\\
b^+_{0,-1} &= b^{3+}_{0,-1}+b^{2+}_{0,-1} \in \big(\Omega^3(M,\mathsf{L}_0)\oplus\Omega^2(M,\mathsf{L}_{-1})\big)[-1]~,
\end{aligned}
\tag{5.88a}
$$

and

$$
\begin{aligned}
\bar c_{-1,-2} &\in \Omega^0(M,\mathsf{L}_{-1})[-2]~, & b_{-1,-2} &\in \Omega^0(M,\mathsf{L}_{-1})[-1]~,\\
\bar c^+_{-1,-2} &\in \Omega^3(M,\mathsf{L}_{-1})[1]~, & b^+_{-1,-2} &\in \Omega^3(M,\mathsf{L}_{-1})[0]~,\\
\bar c_{-1,0} &\in \Omega^0(M,\mathsf{L}_{-1})[0]~, & b_{-1,0} &\in \Omega^0(M,\mathsf{L}_{-1})[1]~,\\
\bar c^+_{-1,0} &\in \Omega^3(M,\mathsf{L}_{-1})[-1]~, & b^+_{-1,0} &\in \Omega^3(M,\mathsf{L}_{-1})[-2]~,
\end{aligned}
\tag{5.88b}
$$

where all $\Omega^i(M,\mathsf{L}_j)$ are regarded as ungraded vector spaces with elements of degree 0. The additional contribution to the extended BV action (4.77) is

$$S_{\rm tp} \;:=\; \int_M \Big\{ -\langle b^0_{0,-1}, \bar c^{3+}_{0,-1}\rangle_{\mathsf{L}} - \langle b^1_{0,-1}, \bar c^{2+}_{0,-1}\rangle_{\mathsf{L}} + \langle b_{-1,-2}, \bar c^{+}_{-1,-2}\rangle_{\mathsf{L}} + \langle b_{-1,0}, \bar c^{+}_{-1,0}\rangle_{\mathsf{L}} \Big\} \,. \tag{5.89}$$

Minimal model and L_∞-quasi-isomorphism. We now construct the minimal model of the L_∞-algebra $\Omega^\bullet(M,\mathsf{L})$ underlying higher Chern–Simons theory following our discussion in Section 2.5..

We start by noting that the cochain complex underlying the L_∞-algebra (3.29) is the tensor product of two cochain complexes: the de Rham complex and the complex arising from the gauge L_∞-algebra L. Since the cohomology $H^\bullet_{\mu_1}(\mathsf{L})$ of L is evidently free, Künneth's theorem, see e.g. [223], yields the isomorphisms

$$\begin{aligned} H^\bullet_{\hat\mu_1}(\Omega^\bullet(M,\mathsf{L})) \;&:=\; \bigoplus_{k\in\mathbb{Z}} H^k_{\hat\mu_1}(\Omega^\bullet(M,\mathsf{L}))\;,\\ H^k_{\hat\mu_1}(\Omega^\bullet(M,\mathsf{L})) \;&\cong\; \bigoplus_{\substack{i+j=k\\ 0\le i\le d\\ -n+1\le j\le 0}} H^i_{\rm dR}(M)\otimes H^j_{\mu_1}(\mathsf{L})\;, \end{aligned} \tag{5.90}$$

where $H^i_{\rm dR}(M)$ denotes the i-th de Rham cohomology group.

To construct the L_∞-structure on $H^\bullet_{\hat\mu_1}(\Omega^\bullet(M,\mathsf{L}))$, we note that another consequence of the cochain complex underlying the L_∞-algebra (3.29) being the tensor product of cochain complexes of vector spaces is that it splits in the sense of (2.61); see also Appendix B. Hence, we have

$$h_{\hat\mu_1} \circlearrowright \Omega^\bullet(M,\mathsf{L}) \underset{e_{\hat\mu_1}}{\overset{p_{\hat\mu_1}}{\rightleftarrows}} H^\bullet_{\hat\mu_1}(\Omega^\bullet(M,\mathsf{L}))\;, \tag{5.91}$$

where $p_{\hat\mu_1}\circ e_{\hat\mu_1}={\rm id}$ and $h_{\hat\mu_1}:\Omega^\bullet(M,\mathsf{L})\to\Omega^\bullet(M,\mathsf{L})$ with $\hat\mu_1=\hat\mu_1\circ h_{\hat\mu_1}\circ\hat\mu_1$ a contracting homotopy of $P_{\hat\mu_1}:=e_{\hat\mu_1}\circ p_{\hat\mu_1}$. To construct $h_{\hat\mu_1}$ explicitly, we assume that we have already found[1] a contracting homotopy $h_{\mu_1}:\mathsf{L}\to\mathsf{L}$ of $P_{\mu_1}:=e_{\mu_1}\circ p_{\mu_1}$ and construct a contracting homotopy $h_{\rm d}:\Omega^\bullet(M)\to\Omega^\bullet(M)$ of $P_{\rm d}:=e_{\rm d}\circ p_{\rm d}$. In order to write down the latter, we fix a Riemannian metric on M and let ${\rm d}^\dagger$ be the adjoint of ${\rm d}$ with respect to the standard inner product $\langle\alpha_1,\alpha_2\rangle=\int_M\alpha_1\wedge\star\alpha_2$ for $\alpha_{1,2}\in\Omega^k(M)$ with $\star$ the Hodge operator for the chosen metric. Using the Green operator

$$G|_{{\rm im(d)}\oplus{\rm im(d^\dagger)}} \;:=\; \Delta^{-1} \quad\text{and}\quad G|_{\ker(\Delta)} \;:=\; 0 \quad\text{with}\quad \Delta \;:=\; {\rm d}{\rm d}^\dagger+{\rm d}^\dagger{\rm d}\;, \tag{5.92}$$

[1] See Appendix B for an explicit example.

we define

$$h_{\mathrm{d}} \;:=\; \mathrm{d}^\dagger G~. \tag{5.93}$$

Together with the identities

$$G\mathrm{d} \;=\; \mathrm{d}G~,\quad G\mathrm{d}^\dagger \;=\; \mathrm{d}^\dagger G~,\quad\text{and}\quad \star G \;=\; G\star~, \tag{5.94}$$

it is then easily seen that $\mathrm{d} = \mathrm{d}\circ h_{\mathrm{d}}\circ\mathrm{d}$. We thus obtain

$$1 \;=\; P_{\mathrm{d}} + h_{\mathrm{d}}\circ\mathrm{d} + \mathrm{d}\circ h_{\mathrm{d}} \;=\; P_{\mathrm{d}} + \Delta G \quad\Longrightarrow\quad P_{\mathrm{d}} \;=\; 1-\Delta G \tag{5.95}$$

which is the projector onto the harmonic forms $\Omega^\bullet_{\mathrm{h}}(M)$ under the Hodge decomposition

$$\Omega^k(M) \;\cong\; \Omega^k_{\mathrm{h}}(M)\oplus\mathrm{d}\Omega^{k-1}(M)\oplus\mathrm{d}^\dagger\Omega^{k+1}(M)~, \tag{5.96}$$

cf. Appendix B. Postcomposing the projector with the Hodge isomorphism $\Omega^\bullet_{\mathrm{h}}(M)\cong H^\bullet_{\mathrm{dR}}(M)$, we obtain a projector on de Rham cohomology. We now combine the homotopies h_{d} and h_{μ_1} as

$$h_{\hat\mu_1} \;:=\; \tfrac12(h_{\mathrm{d}}\otimes 1 + 1\otimes h_{\mu_1} + P_{\mathrm{d}}\otimes h_{\mu_1} + h_{\mathrm{d}}\otimes P_{\mu_1})~. \tag{5.97}$$

Using $\hat\mu_1 = \mathrm{d}\otimes 1 + 1\otimes\mu_1$, it now follows that $\hat\mu_1 = \hat\mu_1\circ h_{\hat\mu_1}\circ\hat\mu_1$, as desired.

Using the contracting homotopy (5.97), it is now easy to adapt the formulas (2.64) for the quasi-isomorphism between $\Omega^\bullet(M,\mathsf{L})$ and $H^\bullet_{\hat\mu_1}(\Omega^\bullet(M,\mathsf{L}))$. For the sake of clarity, we shall only display the formulas in homogeneous degree 1. We obtain

$$\begin{aligned}
\phi_1(a') \;&=\; e_{\hat\mu_1}(a')~,\\
\phi_2(a',a') \;&=\; -h_{\hat\mu_1}\Big(\hat\mu_2\big(e_{\hat\mu_1}(a'),e_{\hat\mu_1}(a')\big)\Big)~,\\
\phi_3(a',a',a') \;&=\; -3h_{\hat\mu_1}\Big(\hat\mu_2\big(\phi_2(a',a'),e_{\hat\mu_1}(a')\big)\Big) - h_{\hat\mu_1}\Big(\hat\mu_3\big(e_{\hat\mu_1}(a'),e_{\hat\mu_1}(a'),e_{\hat\mu_1}(a')\big)\Big)~,\\
\phi_4'(a',a',a',a') \;&=\; -3h_{\hat\mu_1}\Big(\hat\mu_2\big(\phi_2(a',a'),\phi_2(a',a')\big)\Big) - 4h_{\hat\mu_1}\Big(\hat\mu_2\big(e_{\hat\mu_1}(a'),\phi_3(a',a',a')\big)\Big) -\\
&\qquad - 6h_{\hat\mu_1}\Big(\hat\mu_3\big(e_{\hat\mu_1}(a'),e_{\hat\mu_1}(a'),\phi_2\,(a',a')\,\big)\Big) -\\
&\qquad - h_{\hat\mu_1}\Big(\hat\mu_4\big(e_{\hat\mu_1}(a'),e_{\hat\mu_1}(a'),e_{\hat\mu_1}(a'),e_{\hat\mu_1}(a')\big)\Big)~,\\
&\;\;\vdots
\end{aligned} \tag{5.98}$$

Hence, the higher products $\hat{\mu}'_i$ on $H^\bullet_{\hat{\mu}_1}(\Omega^\bullet(M,\mathsf{L}))$ defined in (2.65) are then given for degree 1 elements by

$$\begin{aligned}
\hat{\mu}'_1(a') &= 0\,,\\
\hat{\mu}'_2(a',a') &= p_{\hat{\mu}_1}\Big(\hat{\mu}_2\big(e_{\hat{\mu}_1}(a'),e_{\hat{\mu}_1}(a')\big)\Big)\,,\\
\hat{\mu}'_3(a',a',a') &= 3p_{\hat{\mu}_1}\Big(\hat{\mu}_2\big(\phi_2(a',a'),e_{\hat{\mu}_1}(a')\big)\Big)+p_{\hat{\mu}_1}\Big(\hat{\mu}_3\big(e_{\hat{\mu}_1}(a'),e_{\hat{\mu}_1}(a'),e_{\hat{\mu}_1}(a')\big)\Big)\,,\\
\hat{\mu}'_4(a',a',a',a') &= 3p_{\hat{\mu}_1}\Big(\hat{\mu}_2\big(\phi_2(a',a'),\phi_2(a',a')\big)\Big)+4p_{\hat{\mu}_1}\Big(\hat{\mu}_2\big(e_{\hat{\mu}_1}(a'),\phi_3(a',a',a')\big)\Big)+\\
&\quad+6p_{\hat{\mu}_1}\Big(\hat{\mu}_3\big(e_{\hat{\mu}_1}(a'),e_{\hat{\mu}_1}(a'),\phi_2\,(a',a')\,\big)\Big)+\\
&\quad+p_{\hat{\mu}_1}\Big(\hat{\mu}_4\big(e_{\hat{\mu}_1}(a'),e_{\hat{\mu}_1}(a'),e_{\hat{\mu}_1}(a'),e_{\hat{\mu}_1}(a')\big)\Big)\,,\\
&\;\;\vdots
\end{aligned}\tag{5.99}$$

We conclude that the resulting classical field theory equivalent to higher Chern–Simons theory has a much simpler or reduced space of fields, however this is compensated by the interactions becoming much more involved.

Remark on fake curvatures. To conclude the Section, we briefly consider fake curvatures in the case of higher Chern–Simons theories, recalling our discussion on the general role of fake curvatures in Section 3.1..

For general $d\geq 3$, the MC potential $a\in\Omega^\bullet_1(M,\mathsf{L})$ decomposes as $a=A_1+A_2+\cdots+A_{d-2}$ with $A_k\in\Omega^k(M,\mathsf{L}_{-k+1})$ while the curvature f has the decomposition $f=F_2+F_3+\cdots+F_{d-1}$ with $F_k\in\Omega^k(M,\mathsf{L}_{-k+2})$. The curvatures F_k for $k=2,\ldots,d-2$ are known as the k*-form fake curvatures*.

Analogously, the gauge parameters of level k gauge transformations, c_{-k}, decompose into forms of varying degrees and $c_{-k}\in\Omega^0(M,\mathsf{L}_{-k})\oplus\cdots\oplus\Omega^{d-k-3}(M,\mathsf{L}_{-d+2})$. We see that the formula (3.11) for the gauge transformation of the curvature form F_{d-1} contains a covariant term of the form $\mu_2(F_{d-1},\alpha)$, where $\alpha\in\Omega^0(M,\mathsf{L}_0)$ is a component of c_0 and all other terms are proportional to lower curvatures, F_j with $j<d-1$.

We also note that the successive action of two gauge transformations (3.16) does not contain the highest form component of the curvature, F_{d-1}: we have $k\leq -1$, and matching the L_∞-degrees makes an appearance of F_{d-1} impossible.

Altogether, requiring $F_2=\cdots=F_{d-2}=0$ renders the gauge transformations of the highest curvature F_{d-1} covariant and allows to close general gauge transformations. The first point is particularly important for writing down interesting equations of motions, coupling the higher gauge potentials to matter fields.

5.3.2. Alexandrov–Kontsevich–Schwarz–Zaboronsky construction

As we have seen, in the classical approach to BV, one starts from a classical field theory and constructs a corresponding BV action. There are, however, more modern approaches to directly construct (possibly infinite dimensional) classical field theories directly in their BV form. The most important of these is the Alexandrov–Kontsevich–Schwarz–Zaboronsky (AKSZ) formalism [114]. This method is essentially a way to induce a differential graded symplectic structure on the space of maps between two graded manifolds, endowed with opportune geometrical structures. We briefly summarise the AKSZ construction in the following and derive Chern–Simons theory as an example. For more details and examples, see also [224–227] and in particular [228] for a modern perspective.[1] Examples of AKSZ descriptions of non-topological gauge field theories are found, e.g., in [231, 232].

Alexandrov–Kontsevich–Schwarz–Zaboronsky data. We start from an NQ-manifold (Σ, Q_Σ), which is endowed with a non-degenerate Q_Σ-invariant measure μ of degree $-n-1$, for some positive integer n. The canonical example here is $\Sigma = T[1]\Sigma_0$ for a compact oriented $(n+1)$-dimensional manifold Σ_0 without boundary and with Q_Σ the de Rham differential. The manifold Σ is called the *source* and Σ_0 often corresponds to the world volume of the described objects (point particles, strings, etc.).

We also define a *target* (thought of as an extended form of the target space), which is a symplectic NQ-manifold (M, Q_M, ω_M) of degree n. As explained in 2.3.1., Q_M is Hamiltonian with Hamiltonian function Θ satisfying $\{\Theta, \Theta\} = 0$, where $\{-,-\}$ is the Poisson bracket induced by ω_M.

The space of fields $\mathfrak{F}_{\rm BV}$ is now the space of maps from Σ to M. This is, in fact, a very general construction. For example, the mechanics of point particles can be described by maps from their worldline $\mathbb{R}$ into space-time. It also leads to vast generalisations of gauge theories after appropriate refinement, see [7]. In particular, in the case $\Sigma = T[1]\Sigma_0$ and

[1] For the AKSZ construction of higher Chern–Simons theories see also [229] and [230] for an earlier account.

$M = \mathfrak{g}[1]$ with Q_M the Chevalley–Eilenberg differential of the Lie algebra $\mathfrak{g}$, we obtain the kinematical data of Chern–Simons theory as morphisms of degree 0. Let ξ^α be again the coordinate functions on $\mathfrak{g}[1]$ and let $\mathrm{d}x^\mu$ be the generators of $\Omega^\bullet(\Sigma_0)$ over $\mathscr{C}^\infty(\Sigma_0)$. Then, a dga-morphism $a : \mathscr{C}^\infty(\mathfrak{g}[1]) \to \Omega^\bullet(\Sigma_0)$ maps

$$a : \xi^\alpha \;\mapsto\; \mathrm{d}x^\mu A^\alpha_\mu(x) \;=:\; A^\alpha \tag{5.100}$$

such that $a \circ Q_M = Q_\Sigma \circ a$ or, equivalently,

$$(a \circ Q_M)\xi^\alpha \;=\; a(-\tfrac{1}{2} f_{\beta\gamma}{}^\alpha \xi^\beta \xi^\gamma) \;=\; -\tfrac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma \;=\; \mathrm{d}A^\alpha \;=\; (Q_\Sigma \circ a)\xi^\alpha\ . \tag{5.101}$$

We thus obtain a gauge potential $A \in \Omega^1(\Sigma_0, \mathfrak{g})$ whose curvature $F := \mathrm{d}A + \frac{1}{2}[A, A]$ vanishes.

Batalin–Vilkovisky structure. Note that $\mathfrak{F}_{\rm BV}$ is naturally graded, and the degree will be the ghost number of the fields. In addition, the structures on Σ and M endow $\mathfrak{F}_{\rm BV}$ with a homological vector field and a symplectic form. To obtain the symplectic form, note that there is the evaluation map $\mathrm{ev} : \mathfrak{F}_{\rm BV} \times \Sigma \to M$, acting as $\mathrm{ev}(\phi, x) := \phi(x)$.

We can pull back any differential form $\alpha \in \Omega^\bullet(M)$ along ev to $\mathfrak{F}_{\rm BV} \times \Sigma$ and subsequently integrate over Σ, leading to the map

$$p(\alpha) \;:=\; \int_\Sigma \mu\ \mathrm{ev}^*\alpha\ . \tag{5.102}$$

We can use this map to define the symplectic form

$$\omega_{\rm BV} \;=\; p(\omega_M) \tag{5.103}$$

inducing the BV bracket. From the degrees of ω_M and μ it is clear that $\omega_{\rm BV}$ is of degree -1. Also, $\omega_{\rm BV}$ is non-degenerate if μ is non-degenerate.

To construct the homological vector field $Q_{\rm BV}$, note that diffeomorphisms on both Σ and M induce an action on $\mathfrak{F}_{\rm BV}$, by pre-composition or post-composition, respectively. Therefore, the two homological vector fields Q_Σ and Q_M induce vector fields $\hat{Q}_\Sigma$ and $\hat{Q}_M$ on $\mathfrak{F}_{\rm BV}$ and we can choose any linear combination of these to form $Q_{\rm BV}$. The compatibility between $Q_{\rm BV}$ and $\omega_{\rm BV}$ is readily checked, cf. [224, 225]. In particular, one can show that the map p defined in (5.102) is a symplectomorphism. Thus,

$$\{\Theta, \Theta\} \;=\; 0 \;\Longleftrightarrow\; \{p(\Theta), p(\Theta)\}_{\rm BV} \;=\; 0\ , \tag{5.104}$$

and therefore $p(\Theta)$ is the Hamiltonian of a homological vector field $\hat{Q}_M$. The contribution of $\hat{Q}_M$ to the Hamiltonian of $Q_{\rm BV}$ is thus a multiple of $p(\Theta)$.

On the other hand, μ is invariant under Q_Σ and so is ev under the simultaneous action on $\mathfrak{F}_{\rm BV}$ and Σ, which leads to

$$\mathcal{L}_{\hat{Q}_\Sigma} p \;=\; \hat{Q}_\Sigma \lrcorner \mathrm{d}p + \mathrm{d}\big(\hat{Q}_\Sigma \lrcorner p\big) \;=\; 0\ . \tag{5.105}$$

If the symplectic form ω_M is exact, $\omega_M = \mathrm{d}\vartheta$, then the Hamiltonian of $\hat{Q}_\Sigma$ is therefore $\hat{Q}_\Sigma \lrcorner p(\vartheta)$. Moreover, if $\vartheta = \vartheta_\alpha(\xi)\mathrm{d}\xi^\alpha$ in some coordinates ξ^α on M, then

$$\hat{Q}_\Sigma \lrcorner p(\vartheta) \;=\; \int_\Sigma \vartheta_\alpha(\phi)\delta\phi^\alpha\ , \tag{5.106}$$

where ϕ^α is the coordinate corresponding to ξ^α on $\mathfrak{F}_{\rm BV}$ under the map $\phi : \Sigma \to M$.

Altogether, the Hamiltonian of $Q_{\rm BV}$, which is a linear combination of $\hat{Q}_\Sigma \lrcorner p(\vartheta)$ and $p(\Theta)$, is the classical BV action. For a more precise argument regarding to which linear combinations are preferable, see [226,228]. The gauge invariant classical action, may be eventually recovered from a solution of the classical master equation setting to zero all fields with negative degree.

Example: Chern–Simons theory. As a simple example, consider the case $\Sigma = T[1]\Sigma_0$ with $Q_\Sigma = \mathrm{d}$ for a compact oriented three-dimensional manifold Σ_0. As target, choose $M = \mathfrak{g}[1]$ where $\mathfrak{g}$ is a metric Lie algebra with coordinates ξ^α and metric $\langle\tau_\alpha,\tau_\beta\rangle_\mathfrak{g} = \omega_{\alpha\beta}$ inducing a symplectic form $\omega_\mathfrak{g} = \frac{1}{2}\omega_{\alpha\beta}\mathrm{d}\xi^\alpha \wedge \mathrm{d}\xi^\beta$ of degree 2. The Hamiltonian Θ of $Q_{\mathfrak{g}[1]}$ is $\Theta = \frac{1}{3!}f_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma$ with $f_{\alpha\beta\gamma} := f_{\alpha\beta}{}^\delta\omega_{\delta\gamma}$ and corresponds to the 3-cocycle $\langle -,[-,-]\rangle_\mathfrak{g}$.

The maps from Σ to $\mathfrak{g}[1]$ form the space of $\mathfrak{g}$-valued forms on Σ_0 and we have $\mathfrak{F}_{\rm BV} = \Omega^\bullet(\Sigma_0,\mathfrak{g})$. The symplectic form reads as

$$\omega_{\rm BV} \;:=\; \int_{\Sigma_0} \tfrac{1}{2}\omega_{\alpha\beta}\delta\phi^\alpha \wedge \delta\phi^\beta \;=\; \int_{\Sigma_0} \langle\delta\phi,\delta\phi\rangle_\mathfrak{g} \tag{5.107}$$

for $\phi \in \Omega^\bullet(\Sigma_0,\mathfrak{g})$. Note that $\omega_\mathfrak{g}$ is exact (see Section 2.3.) with symplectic potential $\vartheta = \frac{1}{2}\xi^\alpha\omega_{\alpha\beta}\mathrm{d}\xi^\beta$, and the two contributions to $S_{\rm BV}$ from degree 0 maps are

$$\hat{Q}_\Sigma \lrcorner p(\vartheta) \;=\; \tfrac{1}{2}\int_{\Sigma_0}\langle\phi,\mathrm{d}\phi\rangle_\mathfrak{g} \quad \text{and} \quad p(\Theta) \;=\; \tfrac{1}{3!}\int_{\Sigma_0}\langle\phi,[\phi,\phi]\rangle_\mathfrak{g}\ , \tag{5.108}$$

where $\phi \in \Omega^\bullet(\Sigma_0,\mathfrak{g})$. To see this, note that the pull-back along the evaluation map yields

$$\mathrm{ev}^*\xi^\alpha \;=\; (x,\phi^\alpha(x)) \tag{5.109}$$

and

$$\hat{Q}_\Sigma \lrcorner\, p(\vartheta) \;=\; \hat{Q}_\Sigma \lrcorner \int_{\Sigma_0} \tfrac{1}{2}\omega_{\alpha\beta}\phi^\alpha \mathrm{D}\phi^\beta \;=\; \tfrac{1}{2}\int_{\Sigma_0} \phi^\alpha\omega_{\alpha\beta}\mathrm{d}\phi^\beta \;, \tag{5.110}$$

where D is the de Rham differential on $\mathfrak{F}_{\mathrm{BV}} \times \Sigma$.

If we decompose ϕ into forms of homogeneous degree, $\phi = c + A + A^+ + c^+$, and linearly combine both of the above contributions, we obtain the classical BV action of Chern–Simons theory,

$$\begin{aligned} S_{\mathrm{BV}} \;=\; \int_{\Sigma_0} \Big\{ &\tfrac{1}{2}\langle A, \mathrm{d}A\rangle_{\mathfrak{g}} + \tfrac{1}{3!}\langle A, [A,A]\rangle_{\mathfrak{g}} - \\ &- \langle c, \mathrm{d}A^+\rangle_{\mathfrak{g}} + \langle A, [A^+, c]\rangle_{\mathfrak{g}} + \tfrac{1}{2}\langle c^+, [c,c]\rangle_{\mathfrak{g}} \Big\} \;. \end{aligned} \tag{5.111}$$

5.4. Supersymmetric Extensions

In the following we shall comment on the supersymmetric extensions of the field theories examined above, starting from higher Chern–Simons theory. In addition, due to its relevance to the description of M-theory, which is ultimately the source of much motivation of higher structures, we shall also quickly review the Bagger–Lambert–Gustavsson (BLG) model [35–37].

Higher CS theory: topological setting. Let us briefly consider the supersymmetric extension of the above classical higher Chern–Simons theory along the lines discussed in Section 3.1.3.. That is, we introduce the additional fields

$$\begin{aligned} \phi_1 \;&=\; Y + \phi \;\in\; \Omega^\bullet_1(M,\mathsf{L}) \;\cong\; \Omega^1(M,\mathsf{L}_0) \oplus \Omega^2(M,\mathsf{L}_{-1}) \;, \\ \phi_2 \;&=\; \big[D - \tfrac{1}{2}\mu_1(\phi)\big] - Z \;\in\; \Omega^\bullet_2(M,\mathsf{L}) \;\cong\; \Omega^2(M,\mathsf{L}_0) \oplus \Omega^3(M,\mathsf{L}_{-1}) \;, \\ \psi_1 \;&=\; \lambda_1 + \chi_2 \;\in \Pi\Omega^\bullet_1(M,\mathsf{L}) \;\cong\; \Pi\Omega^1(M,\mathsf{L}_0) \oplus \Pi\Omega^2(M,\mathsf{L}_{-1}) \;, \\ \psi_2 \;&=\; \lambda_2 + \chi_3 \;\in\; \Pi\Omega^\bullet_2(M,\mathsf{L}) \;\cong\; \Pi\Omega^2(M,\mathsf{L}_0) \oplus \Pi\Omega^3(M,\mathsf{L}_{-1}) \;, \end{aligned} \tag{5.112}$$

where, as before, Π is the Graßmann-parity changing functor. The action (5.83) is then extended to the appropriate specialisation of the action (3.27),

$$\begin{aligned} S_{\mathrm{STMC}} \;:=\; \int_M \Big\{ &\langle B, F - \tfrac{1}{2}\mu_1(B)\rangle_{\mathsf{L}} + \tfrac{1}{4!}\langle \mu_3(A,A,A), A\rangle_{\mathsf{L}} + \\ &+ \langle \chi_2, \lambda_2\rangle_{\mathsf{L}} + \langle \chi_3, \lambda_1\rangle_{\mathsf{L}} + \langle \phi, D - \tfrac{1}{2}\mu_1(\phi)\rangle_{\mathsf{L}} - \langle Z, Y\rangle_{\mathsf{L}} \Big\} \;. \end{aligned} \tag{5.113}$$

The fermionic transformations (3.27) read as

$$\begin{aligned} Q\lambda_1 &= 0\ , \quad Q\lambda_2 = \mathcal{F} + D\ , \\ QA &= \lambda_1\ , \quad QY = \lambda_1\ , \\ QD &= -\nabla\lambda_1 + \mu_1(\chi_2) \end{aligned} \tag{5.114a}$$

and

$$\begin{aligned} Q\chi_2 &= 0\ , \\ Q\chi_3 &= H - \nabla\phi - \tfrac{1}{2}\mu_2(Y, B-\phi) + \tfrac{1}{4}\mu_3(Y,A,A) - \tfrac{1}{3!}\mu_3(Y,Y,A) + \tfrac{1}{4!}\mu_3(Y,Y,Y) + Z\ , \\ QB &= \chi_2\ , \quad Q\phi = \chi_2\ , \\ QZ &= -\tfrac{1}{2}\mu_2(\lambda_1, B-\phi) + \tfrac{1}{4}\mu_3(\lambda_1,A,A) - \tfrac{1}{3!}\mu_3(\lambda_1,Y,A) + \tfrac{1}{4!}\mu_3(\lambda_1,Y,Y)\ . \end{aligned} \tag{5.114b}$$

where the curvatures $\mathcal{F}$ and H were defined in (3.32c). As shown in 3.1.3., we have $Q^2 = 0$ off-shell.

Higher CS theory: physical setting. The above supersymmetric extension is similar to a topological twist of supersymmetric higher Chern–Simons theory for the $d = 4$, $\mathcal{N} = 2$ tensor multiplet coupled to the $\mathcal{N} = 2$ vector multiplet. The Abelian part of the action was already presented in [233, 234] and with our framework, we extend the action to the non-Abelian case on $M = \mathbb{R}^4$, as we shall now briefly discuss.

For convenience, we switch to spinor notation and replace the coordinates x^μ by the coordinates $x^{\alpha\dot\alpha}$ with $\alpha, \dot\alpha = 1, 2$. This is possible due to the factorisation of the tangent bundle $TM \otimes \mathbb{C} \cong S_+ \otimes S_-$ into the tensor product of the chiral and anti-chiral spin bundles S_+ and S_-. We also use R-symmetry indices $i, j = 1, 2$. The $\mathcal{N} = 2$ vector multiplet then consists of a one-form gauge potential, two pairs of Weyl spinors and 5 real auxiliary fields,

$$\{A_{\alpha\dot\alpha}, \lambda^i_\alpha, \hat\lambda_{i\dot\alpha}, D^{ij} = D^{ji}, Y, \hat Y\}\ , \tag{5.115}$$

which all take values in L_0. The $\mathcal{N} = 2$ tensor multiplet consists of a two-form gauge potential, two pairs of Weyl spinors and 5 real auxiliary fields,

$$\{B_{\alpha\beta} = B_{\beta\alpha}, B_{\dot\alpha\dot\beta} = B_{\dot\beta\dot\alpha}, \chi_{i\alpha}, \hat\chi^i_{\dot\alpha}, \phi^{ij} = \phi^{ji}, Z, \hat Z\}\ , \tag{5.116}$$

which all take values in L_{-1}. Here, $B_{\alpha\beta}$ and $B_{\dot\alpha\dot\beta}$ encode the self-dual and antiself-dual parts of the two-form B. The curvatures (3.32c) become

$$\begin{aligned}\mathcal{F}_{\alpha\beta} &= F_{\alpha\beta} - \mu_1(B_{\alpha\beta}) , \quad \mathcal{F}_{\dot\alpha\dot\beta} = F_{\dot\alpha\dot\beta} - \mu_1(B_{\dot\alpha\dot\beta}) ,\\ H_{\alpha\dot\beta} &= \varepsilon^{\dot\gamma\dot\delta}\nabla_{\alpha\dot\gamma}B_{\dot\beta\dot\delta} - \varepsilon^{\gamma\delta}\nabla_{\gamma\dot\beta}B_{\alpha\delta} - \tfrac{1}{3!}[\mu_3(A,A,A)]_{\alpha\dot\beta} .\end{aligned} \tag{5.117}$$

The action functional

$$\begin{aligned}S_{\rm SPMC} := \int \mathrm{d}^4x \Big\{ &\mathrm{i}\Big[\langle B^{\alpha\beta}, F_{\alpha\beta} - \tfrac12\mu_1(B_{\alpha\beta})\rangle_{\mathsf{L}} - \langle B^{\dot\alpha\dot\beta}, F_{\dot\alpha\dot\beta} - \tfrac12\mu_1(B_{\dot\alpha\dot\beta})\rangle_{\mathsf{L}} +\\ &+ \tfrac{1}{2\cdot 4!}\langle[\mu_3(A,A,A)]^{\alpha\dot\beta}, A_{\alpha\dot\beta}\rangle_{\mathsf{L}}\Big] + \langle\chi^{i\alpha},\lambda_{i\alpha}\rangle_{\mathsf{L}} + \langle\hat\chi^{i\dot\alpha},\hat\lambda_{i\dot\alpha}\rangle_{\mathsf{L}} -\\ &- \langle Z,Y\rangle_{\mathsf{L}} - \langle\hat Z,\hat Y\rangle_{\mathsf{L}} - \langle\phi^{ij}, D_{ij} - \tfrac12\mu_1(\phi_{ij})\rangle_{\mathsf{L}}\Big\}\end{aligned} \tag{5.118}$$

is then invariant under the supersymmetry transformations

$$\begin{aligned}\delta_{\rm SUSY}A_{\alpha\dot\beta} &:= \epsilon^i_\alpha\hat\lambda_{i\dot\beta} - \hat\epsilon_{i\dot\beta}\lambda^i_\alpha ,\\ \delta_{\rm SUSY}\lambda^i_\alpha &:= \mathrm{i}\epsilon^{i\beta}\mathcal{F}_{\alpha\beta} - \tfrac12\epsilon_{j\alpha}[D^{ij} - \mu_1(\phi^{ij})] + \hat\epsilon^{i\dot\beta}\nabla_{\alpha\dot\beta}Y + \epsilon^i_\alpha\mu_2(Y,\hat Y) ,\\ \delta_{\rm SUSY}\hat\lambda_{i\dot\alpha} &:= \mathrm{i}\hat\epsilon^{\dot\beta}_i\mathcal{F}_{\dot\alpha\dot\beta} - \tfrac12\hat\epsilon^j_{\dot\alpha}[D_{ij} - \mu_1(\phi_{ij})] - \epsilon^\beta_i\nabla_{\beta\dot\alpha}\hat Y + \hat\epsilon_{i\dot\alpha}\mu_2(Y,\hat Y) ,\\ \delta_{\rm SUSY}D^{ij} &:= \epsilon^{(i\alpha}\nabla_{\alpha\dot\beta}\hat\lambda^{j)\dot\beta} + \hat\epsilon^{(i\dot\alpha}\nabla_{\beta\dot\alpha}\lambda^{j)\beta} + \epsilon^{(i\alpha}\mu_2(Y,\lambda^{j)}_\alpha) + \hat\epsilon^{(i\dot\alpha}\mu_2(\hat Y,\hat\lambda^{j)}_{\dot\alpha}) ,\\ \delta_{\rm SUSY}Y &:= \epsilon^{i\alpha}\lambda_{i\alpha} , \quad \delta_{\rm SUSY}\hat Y := \hat\epsilon^{i\dot\alpha}\hat\lambda_{i\dot\alpha} ,\end{aligned} \tag{5.119a}$$

and

$$\begin{aligned}\delta_{\rm SUSY}B_{\alpha\beta} &:= -\epsilon^i_{(\alpha}\chi_{i\beta)} , \quad \delta_{\rm SUSY}B_{\dot\alpha\dot\beta} := -\hat\epsilon_{i(\dot\alpha}\hat\chi^i_{\dot\beta)} ,\\ \delta_{\rm SUSY}\chi_{i\alpha} &:= [\mathrm{i}H_{\alpha\dot\beta}\varepsilon_{ij} - \nabla_{\alpha\dot\beta}\phi_{ij}]\hat\epsilon^{j\dot\beta} + \epsilon_{i\alpha}Z + \epsilon^j_\alpha\mu_2(Y,\phi_{ij}) ,\\ \delta_{\rm SUSY}\hat\chi^i_{\dot\alpha} &:= [\mathrm{i}H_{\beta\dot\alpha}\varepsilon^{ij} + \nabla_{\beta\dot\alpha}\phi^{ij}]\epsilon^\beta_j + \hat\epsilon^i_{\dot\alpha}\hat Z - \hat\epsilon_{j\dot\alpha}\mu_2(\hat Y,\phi^{ij}) ,\\ \delta_{\rm SUSY}\phi^{ij} &:= \epsilon^{(i\alpha}\chi^{j)}_\alpha - \hat\epsilon^{(i\dot\alpha}\hat\chi^{j)}_{\dot\alpha} ,\\ \delta_{\rm SUSY}Z &:= -\hat\epsilon^{i\dot\beta}\nabla_{\alpha\dot\beta}\chi^\alpha_i + \hat\epsilon^{\dot\alpha}_i\mu_2(\hat Y,\hat\chi^i_{\dot\alpha}) , \quad \delta_{\rm SUSY}\hat Z := -\epsilon^\beta_i\nabla_{\beta\dot\alpha}\hat\chi^{i\dot\alpha} + \epsilon^{i\alpha}\mu_2(Y,\chi_{i\alpha}) .\end{aligned} \tag{5.119b}$$

As in ordinary Chern–Simons theory, all fields except for the gauge potentials appear merely algebraically and are therefore auxiliary and can be integrated out.

Yang–Mills theory. We consider the example of $\mathcal{N}=1$ supersymmetric Yang–Mills theory on a ten-dimensional compact Riemannian spin manifold M. The lower-dimensional cases simply follow by dimensional reduction, see e.g. [235]. The spin bundle decomposes into the bundles of chiral and anti-chiral spinors $S_\pm$ and we have $S_\pm \cong S^*_\mp$. Let $\sigma_\pm : TM\otimes\mathbb{C} \to \odot^2 S_\pm$

and consider the complex

$$\underbrace{\Omega^0(M,\mathfrak{g})}_{=:\mathsf{L}_0} \xrightarrow{\mu_1:=\mathrm{d}} \underbrace{\Omega^1(M,\mathfrak{g})\oplus\Gamma(M,\Pi S_+\otimes\mathfrak{g})}_{=:\mathsf{L}_1}$$
$$\xrightarrow{\mu_1:=\mathrm{d}\star\mathrm{d}+\not{D}} \underbrace{\Omega^{d-1}(M,\mathfrak{g})\oplus\Gamma(M,\Pi S_-\otimes\mathfrak{g})}_{=:\mathsf{L}_2} \xrightarrow{\mu_1:=\mathrm{d}} \underbrace{\Omega^d(M,\mathfrak{g})}_{=:\mathsf{L}_3}\ , \tag{5.120}$$

where $\not{D}$ is the Dirac operator on M and for any one-form $\omega\in\Omega^1(M)$ we set $\not\omega := \sigma_- \lrcorner\, \omega$ with the non-vanishing higher products

$$\begin{gathered}
\mu_1(c_1) \;:=\; \mathrm{d}c_1\ ,\quad \mu_1(A_1+\psi_1) \;:=\; \mathrm{d}\star\mathrm{d}A_1+\not{D}\psi_1\ ,\quad \mu_1(A_1^+) \;:=\; \mathrm{d}A_1^+\ ,\\
\mu_2(c_1,c_2) \;:=\; [c_1,c_2]\ ,\quad \mu_2(c_1,A_1+\psi_1) \;:=\; [c_1,A_1+\psi_1]\ ,\\
\mu_2(c_1,A_1^++\psi_1^+) \;:=\; [c_1,A_1^++\psi_1^+]\ ,\quad \mu_2(c_1,c_2^+) \;:=\; [c_1,c_2^+]\ ,\\
\mu_2(A_1+\psi_1,A_2^++\psi_2^+) \;:=\; [A_1,A_2^+]+[\psi_1,\psi_2^+]\ ,\\
\mu_2(A_1+\psi_1,A_2+\psi_2) \;:=\; \mathrm{d}\star[A_1,A_2]+[A_1,\star\mathrm{d}A_2]+[A_2,\star\mathrm{d}A_1]+\\
+\psi_1(\sigma_-\lrcorner(\star1))\psi_2+[\not{A}_1,\psi_2]-[\not{A}_2,\psi_1]\ ,\\
\mu_3(A_1,A_2,A_3) \;:=\; [A_1,\star[A_2,A_3]]+[A_2,\star[A_3,A_1]]+[A_3,\star[A_1,A_2]]\ .
\end{gathered}\tag{5.121}$$

Here, $c_i\in\mathsf{L}_0$, $\{A_i+\psi_i\}\in\mathsf{L}_1$, $\{A_i^++\psi_i^+\}\in\mathsf{L}_2$, and $c_i^+\in\mathsf{L}_3$ for $i=1,2,3$. Following the same discussion as in the previous sections, one can check that the MC action (3.21) with $a=A+\psi$ becomes

$$S_{\mathrm{MC}} \;=\; \tfrac{1}{2}\int_M \Big\{\langle F,\star F\rangle_{\mathfrak{g}}+\langle\psi,\star\not{\nabla}\psi\rangle_{\mathfrak{g}}\Big\}\ , \tag{5.122}$$

where $\not{\nabla}$ is the covariant Dirac operator, involving the gauge connection one-form A. We note that extensions of the Yang–Mills L_∞-algebra $\mathsf{L}_{\mathrm{YM}_2}$ by scalars and Dirac spinor fields coupling to the gauge field were already given in [236].

Finally, we observe that by using these results about $\mathcal{N}=1$ supersymmetric Yang–Mills theory in ten dimensions the quasi-isomorphism discussed at the end of 5.2.2. also extends to $\mathcal{N}=4$ supersymmetric Yang–Mills theory.

5.4.1. Bagger–Lambert–Gustavsson model

Finally, let us review the Bagger–Lambert–Gustavsson model in this setting.[1] It describes the effective dynamics of stacks of M2-branes and relies on Lie 3-algebras. A sub-L_∞-algebra of the L_∞-algebra structure of this model was identified previously in [202].

[1] Note that other Chern–Simons matter theories lead to analogous results.

Review of the model. Let $M = \mathbb{R}^{1,2}$. It is convenient to describe the gauge structure of the BLG model using the metric 3-Lie algebra A_4. This 3-Lie algebra is a vector space $A_4 \cong \mathbb{R}^4$ with basis σ_a and 3-algebra relation and metric structure

$$[\sigma_a, \sigma_b, \sigma_c] \;=\; \varepsilon_{abc}{}^d \sigma_d \quad \text{and} \quad \langle \sigma_a, \sigma_b \rangle_{A_4} \;=\; \delta_{ab} \; . \tag{5.123}$$

This 3-Lie algebra comes with an associated Lie algebra $\mathfrak{g}_{A_4} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ of inner derivations acting on A_4 and the metric $\langle -,- \rangle_{A_4}$ on A_4 induces a metric $\langle -,- \rangle_{\mathfrak{g}_{A_4}}$ of split signature on $\mathfrak{g}_{A_4}$.

The matter fields of the BLG model consist of eight scalars $X^I \in \Omega^0(M, A_4) \otimes \mathbb{R}^8$ on M with $I, J, \ldots = 1, \ldots, 8$ and a Graßmann-odd Majorana spinor $\Psi \in \Gamma(M, \Pi S \otimes A_4)$ on M in $\mathbb{R}^{1,10}$, reduced to 3 dimensions, both taking values in A_4. In addition, we have a gauge potential $A \in \Omega^1(M, \mathfrak{g}_{A_4})$ taking values in the Lie algebra $\mathfrak{g}_{A_4}$ associated with A_4. Let us decompose the gamma matrices Γ_M for SO$(1,10)$ as $\Gamma_M \to (\Gamma_\mu, \Gamma_I)$ with $\mu, \nu, \ldots = 0,1,2$ and we shall write $\Gamma_{IJK\cdots}$ for the corresponding normalised totally antisymmetric products. The action of the BLG model reads as [35, 37]

$$\begin{aligned} S_{\text{BLG}} \;:=\; \int_M \Big\{ & \tfrac{1}{2}\langle A, \mathrm{d}A \rangle_{\mathfrak{g}_{A_4}} + \tfrac{1}{3!}\langle A, [A,A] \rangle_{\mathfrak{g}_{A_4}} + \tfrac{1}{2}\langle X^I, \nabla{\star}\nabla X^I \rangle_{A_4} + \tfrac{\mathrm{i}}{2}\langle \bar\Psi, {\star}\not{\nabla}\Psi \rangle_{A_4} + \\ & + \tfrac{\mathrm{i}}{4}\langle \bar\Psi, {\star}\Gamma_{IJ}[X^I, X^J, \Psi] \rangle_{A_4} - \tfrac{1}{2\cdot 3!}\langle [X^I, X^J, X^K], {\star}[X^I, X^J, X^K] \rangle_{A_4} \Big\} \end{aligned} \tag{5.124}$$

with equations of motion

$$\begin{aligned} \nabla{\star}\nabla X^I + {\star}\tfrac{1}{2}[X^J, X^K, [X^I, X^J, X^K]] \;&=\; 0 \; , \\ \not{\nabla}_A \Psi + \tfrac{1}{2}\Gamma_{IJ}[X^I, X^J, \Psi] \;&=\; 0 \; , \\ \underbrace{\mathrm{d}A + \tfrac{1}{2}[A,A]}_{=:F} + {\star}(X^I \wedge \nabla X^I + \tfrac{\mathrm{i}}{2}\bar\Psi \wedge \Gamma\Psi) \;&=\; 0 \; , \end{aligned} \tag{5.125}$$

where, in local coordinates x^μ, we define $\Gamma := \mathrm{d}x^\mu \Gamma_\mu$.

Batalin–Vilkovisky action. The action (5.124) is extended to the corresponding BV action

$$S_{\text{BLGBV}} \;:=\; S_{\text{BLG}} + S_{\text{gh}} \; , \tag{5.126}$$

containing the same ghosts and antifields as Chern–Simons theory, $c \in \Omega^0(M, \mathfrak{g})[1]$, $A^+ \in \Omega^2(M, \mathfrak{g})[-1]$ and $c^+ \in \Omega^3(M, \mathfrak{g})[-2]$, as well as the additional two antifields

$X^{I+} \in \Omega^3(M, A_4) \otimes \mathbb{R}^8[-1]$ and $\Psi^+ \in \Gamma(M, \Pi S \otimes A_4)[-1]$. Explicitly,

$$
\begin{aligned}
S_{\rm gh} \;:=\; \int_M \Big\{ &- \langle c, {\rm d}A^+\rangle_{\mathfrak{g}} - \langle c, [A, A^+]\rangle_{\mathfrak{g}} + \tfrac{1}{2}\langle c^+, [c, c]\rangle_{\mathfrak{g}} + \\
&+ \langle X^{I+}, c \rhd X^I\rangle_{A_4} + \star\langle \Psi^+, c \rhd \Psi\rangle_{A_4} \Big\} \,.
\end{aligned}
\tag{5.127}
$$

As always, the BV action $S_{\rm BLGBV}$ is the Hamiltonian function for the homological vector field with respect to the canonical symplectic form. The latter encodes the L_∞-algebra structure of the BLG model, and we directly jump to its description.

L_∞-structure. The action (5.124) and the equations (5.125) can be re-written in L_∞-language. In particular, we consider the complex

$$
\begin{aligned}
&\underbrace{\Omega^0(M, \mathfrak{g}_{A_4})}_{=:\,\mathsf{L}_0} \xrightarrow{\mu_1 := {\rm d}} \underbrace{\Omega^1(M, \mathfrak{g}_{A_4}) \oplus \Omega^0(M, A_4) \otimes \mathbb{R}^8 \oplus \Gamma(M, \Pi S \otimes A_4)}_{=:\,\mathsf{L}_1} \\
&\xrightarrow{\mu_1 := {\rm d} + {\rm d}\star{\rm d} + \not{D}} \underbrace{\Omega^2(M, \mathfrak{g}_{A_4}) \oplus \Omega^3(M, A_4) \otimes \mathbb{R}^8 \oplus \Gamma(M, \Pi S \otimes A_4)}_{=:\,\mathsf{L}_2} \\
&\xrightarrow{\mu_1 := {\rm d}} \underbrace{\Omega^3(M, \mathfrak{g}_{A_4})}_{=:\,\mathsf{L}_3} \,,
\end{aligned}
\tag{5.128a}
$$

which we call the *Bagger–Lambert–Gustavsson complex*, together with the non-vanishing higher products

$$
\begin{aligned}
\mu_1(c_1) \;=\; {\rm d}c_1 \,, \quad \mu_1(A_1 + X_1 + \Psi_1) \;&=\; {\rm d}A_1 + \star{\rm d}\star{\rm d}X_1 + \not{D}\Psi_1 \,, \\
\mu_1(A_1^+ + X_1^+ + \Psi_1^+) \;&=\; {\rm d}A_1^+ \\
\mu_2(c_1, c_2) \;=\; [c_1, c_2] \,, \quad \mu_2(c_1, c_2^+) \;&=\; [c_1, c_2^+] \,, \\
\mu_2(c_1, A_1 + X_1 + \Psi_1) \;&=\; [c_1, A_1] + c_1 \rhd (X_1 + \Psi_1) \,, \\
\mu_2(c_1, A_1^+ + X_1^+ + \Psi_1^+) \;&=\; [c_1, A_1^+] + c_1 \rhd (X_1^+ + \Psi_1^+) \,,
\end{aligned}
\tag{5.128b}
$$

$$
\begin{aligned}
&\mu_2(A_1 + X_1 + \Psi_1, A_2 + X_2 + \Psi_2) \;= \\
&\quad=\; [A_1, A_2] + \Big\{ \not{A}_1 \rhd \Psi_2 + \star\,{\rm d}\star(A_1 \rhd X_2^I) + \\
&\qquad\qquad + \star(A_1 \rhd \star{\rm d}X_2^I) + \star(X_1^I \wedge {\rm d}X_2^I + \tfrac{\rm i}{2}\bar{\Psi}_1^I \wedge \Gamma\Psi_2^I) + (1 \leftrightarrow 2) \Big\} \,, \\
&\mu_2(A_1 + X_1 + \Psi_1, A_2^+ + X_2^+ + \Psi_2^+) \;=\; [A_1, A_2^+] + \mathfrak{d}(X_1^I, X_2^{+I}) + \mathfrak{d}(\Psi_1, \Psi_2^+) \,,
\end{aligned}
$$

$$
\begin{aligned}
&\mu_3(A_1 + X_1 + \Psi_1, \ldots, A_3 + X_3 + \Psi_3) \;= \\
&\qquad=\; \star A_1 \rhd (\star A_2 \rhd X_3) + \tfrac{1}{2}\Gamma_{IJ}[X_1^I, X_2^J, \Psi_3] + \star X_1^I \wedge A_2 \rhd X_3^I + \text{cyclic} \,,
\end{aligned}
\tag{5.128c}
$$

$$
\mu_5(A_1 + X_1 + \Psi_1, \ldots, A_5 + X_5 + \Psi_5) \;=\; \tfrac{1}{2}[X_1^J, X_2^K, [X_3, X_4^J, X_5^K]] + \text{cyclic} \,,
$$

where $c_i \in \mathsf{L}_0$, $A_i + X_i + \Psi_i \in \mathsf{L}_1$, $A_i^+ + X_i^+ + \Psi_i^+ \in \mathsf{L}_2$, and $c_i^+ \in \mathsf{L}_3$ for $i = 1, \ldots, 5$. In addition, the flavour indices on the X_i and X_i^+ are contracted with some basis λ_I, e.g. $X_i = X_i^I \lambda_I$ and $\mathfrak{d} : A_4 \times A_4 \to \mathfrak{g}$ maps two elements in A_4 to the corresponding inner derivation $\mathfrak{d}(\tau_a, \tau_b) := [\tau_a, \tau_b, -]$.

It is rather easy to see that the MC equation (3.5) translates into (5.125). We can endow the above L_∞-algebra with the cyclic inner product

$$\begin{aligned}\langle \ell_1, \ell_2 \rangle_{\mathsf{L}} \;:=\; \int_M \Big\{ &- \langle c_1, c_2^+ \rangle_{\mathfrak{g}} - \langle c_1^+, c_2 \rangle_{\mathfrak{g}} + \langle A_1, A_2^+ \rangle_{\mathfrak{g}} + \langle A_1^+, A_2 \rangle_{\mathfrak{g}} + \\ &+ \langle X_1, X_2^+ \rangle_{A_4} + \langle X_1^+, X_2 \rangle_{A_4} + \star \langle \Psi_1, \Psi_2^+ \rangle_{A_4} + \star \langle \Psi_1^+, \Psi_2 \rangle_{A_4} \Big\} ,\end{aligned} \tag{5.129}$$

where $\ell_i = c_i + A_i + X_i + \Psi_i + A_i^+ + X_i^+ + \Psi_i^+ + c_i^+$ in the notation used above and spinor indices are contracted with the $\mathsf{Spin}(1, 2)$-invariant metric on ΠS. With this inner product, the MC action (3.21) for the L_∞-algebra (5.128) becomes the BLG action (5.124), as expected.

6
Six-dimensional self-dual fields from twistors

6.1. Motivation and outline

We are interested now in discussing an example where the language of higher gauge theory can be fruitfully combined with that of twistor geometry. In particular, we shall consider higher holomorphic Chern–Simons theory. Via an extension of the Čech–Dolbeault correspondence [237–239], which is discussed for the Abelian case in [97], classical solutions to the higher holomorphic Chern–Simons equations of motion correspond to topologically trivial higher holomorphic principal bundles. Over twistor spaces, the latter can be mapped to solutions of various field equations on space-time via a Penrose–Ward transform. Particularly interesting in this context is the complex six-dimensional twistor space considered in [96–98], that is associated with flat complexified space-time in six dimensions. We recall that the advantage of twistor geometry is that the equations of motion and the gauge transformations of a certain gauge theory follow directly from complex algebraic data on twistor space. Higher holomorphic bundles over this space yield solutions to the self-duality equation $H = \star H$, for a three-form curvature $H = \mathrm{d}B$ in six dimensions. Our motivation to better understand these objects comes from the important role that they play in string theory and M-theory. In fact, two-forms with self-dual field strength are part of the field content of the $(2,0)$-theory [16], a six-dimensional superconformal field theory that plays a key role in the web of string theory dualities. In particular, Abelian self-dual three forms on six-dimensional space-time can be represented by certain holomorphic gerbes over the aforementioned twistor space [97,98]. The twistor description presents several advantages. First, both supersymmetric and non-Abelian extension of the self-duality equations are found by switching to supertwistor spaces and non-Abelian higher principal bundles, see the discussions in the papers [99–103]. Explicitly, it has been shown how twistor theory naturally identifies the moduli space of solutions to the constraint system of supercurvatures containing the non-Abelian tensor multiplet with the moduli space of holomorphic higher principal $\mathscr{G}$-bundles, for $\mathscr{G}$ a Lie quasi group, over this twistor space. In fact, this identification is lifted to the level of an L_∞-quasi-isomorphism.

The field equations for non-Abelian supersymmetric self-dual strings in four dimensions [31] can be also easily obtained [97, 100] by performing a dimensional reduction at the level of twistor geometry, in terms of the complex three-dimensional twistor space introduced in [97]. Moreover, in this language the problem of searching for a classical candidate $(2,0)$-theory gets essentially reduced to the search of the suitable higher gauge structure, see [160, 240].

In this Chapter, we shall review the geometry of the relevant twistor space, that we denote by P^6 and notational conventions that we will use. As usual in twistor theory, we will always work in the complex setting, unless otherwise mentioned. In particular, since we are interested in discussing field theories in six dimensions, we shall start by studying physics on flat six-dimensional complexified space-time $\mathsf{M}^6 := \mathbb{C}^6$. However, we recall that one can recover results on the desired space-time signature by imposing appropriate real structures at any stage of the construction. This particular aspect will be discussed in Section 6.3. and then exploited in the next Chapter. See the paper [97] for more details on this point. Moreover, we shall review how self-dual tensor field theories in six dimensions, for general gauge structure, can be described in the twistor approach. In particular, we shall review both Penrose and Penrose–Ward transforms for the construction of chiral zero-rest-mass spinor fields on six-dimensional space-time M^6 by using the twistor space P^6.

6.2. Spinors and zero-rest-mass fields in six dimensions

First, we provide a brief review of spinors and free fields in six dimensions. See e.g. [241, 97, 98] for an exhaustive and pedagogical treatment of the subject. This allows for an immediate introduction of twistor theory, which is naturally defined for complexified space-time and formulated in terms of spinor variables.

6.2.1. Spinors in six dimensions

The spin bundle of M^6 is of rank eight and decomposes into the direct sum $\widetilde{S} \oplus S$ of two complex rank-four bundles. The $\widetilde{S}$ and S are simply the bundles of chiral spinors and anti-chiral spinors, respectively. In dimension six there exist the natural isomorphisms $S \cong \widetilde{S}^*$ and $\widetilde{S} \cong S^*$, between the spaces of spinors and their duals.[1] Hence, we may identify the tangent bundle $T\mathsf{M}^6$ with the antisymmetric tensor product $S \wedge S$ ($\cong \widetilde{S} \wedge \widetilde{S}$) of the anti-chiral spin bundle over the spin manifold M^6. Note that a factorisation of this form is equivalent to choosing a (holomorphic) conformal structure on the spin manifold M^6, since the isomorphism $T\mathsf{M}^6 \cong S \wedge S$ yields canonically the complex line subbundle $\det S^* := \Lambda^4 S^*$ in $T\mathsf{M}^6 \odot T\mathsf{M}^6$, see [97, Remark 3.2] for details. Therefore, we shall use (upper) lower capital Latin indices $A, B = 1, \cdots, 4$ to label (anti-chiral) chiral spinors. This reflects the fact that the complexified Lorentz group in six dimensions is SO$(6, \mathbb{C})$, which is locally isomorphic to SL$(4, \mathbb{C})$. Here, S^A and S^*_A correspond to the two independent Weyl spinors representations that transform in the fundamental and anti-fundamental of SL$(4, \mathbb{C})$, respectively. Correspondingly, points in M^6 can be parametrised using spinor coordinates by $x^{AB} = -x^{BA}$. In particular, we coordinatise M^6 by x^M, for $M, N, \cdots = 1, \ldots, 6$ and we use the identification $x^{AB} = \widetilde{\sigma}_M^{AB} x^M$ and $x^M = \frac{1}{4}\sigma^M_{AB} x^{AB}$, where the 4×4 antisymmetric matrices σ^M_{AB}, $\widetilde{\sigma}^{AB}_M$ are the off-diagonal blocks in the six-dimensional γ-matrices satisfying the corresponding Clifford algebra, see [97]. Moreover, the flat metric in spinor coordinates becomes $g_{ABCD} = \frac{1}{2}\varepsilon_{ABCD}$, where ε_{ABCD} is the Levi-Civita symbol in six dimensions. Hence, anti-symmetric pairs of spinor indices can be raised and lowered by[2] $x_{AB} = \frac{1}{2}\varepsilon_{ABCD} x^{CD}$ and $x^{AB} = \frac{1}{2}\varepsilon^{ABCD} x_{CD}$. Accordingly, we introduce partial derivatives with respect to x^{AB} as $\partial_{AB} = \frac{1}{4}\sigma^M_{AB}\partial_M = \frac{1}{2}\varepsilon_{ABCD}\partial^{CD}$, with $\partial_{AB} x^{CD} = \delta^C_{[A}\delta^D_{B]}$. Here and in the following, brackets denote normalised anti-symmetrisation of the enclosed indices. Similarly, we shall use parentheses to denote normalised symmetrisation.

6.2.2. Free fields in six dimensions

Our main interest in the following will be discussing six-dimensional zero-rest-mass chiral fields, as done in [97, 98]. We shall start by briefly recalling the spinor-helicity formalism in six dimensions, as explained in [241]. Let us stress that, here, we are considering complex

[1] This was shown by Penrose and Rindler [242] and is due to an automorphism of the Clifford algebra under charge conjugation. Recall that in our notation V^* denotes the dual of some given linear space V.

[2] Note that the four index object ε_{ABCD} is the only non-trivial Spin$(6, \mathbb{C})$-invariant tensor. Hence, there is no tensor that can raise or lower a single spinor index. Moreover, we recall that it satisfies relations

$$\varepsilon^{ABCD}\varepsilon_{ABCD} = 4!\ , \quad \varepsilon^{ABCD}\varepsilon_{EBCD} = 3!\delta^A_E\ , \quad \varepsilon^{ABCD}\varepsilon_{EFCD} = 2!\delta^A_{[E}\delta^B_{F]}\ , \quad \varepsilon^{ABCD}\varepsilon_{EFGD} = 3!\delta^A_{[E}\delta^B_F\delta^C_{G]}\ .$$

momenta, for which the associated spinors need not satisfy any reality conditions.

Spinor-helicity formalism. Let us consider a momentum six-vector $p = p_M$. Similar to four dimensions, the six-dimensional spinor-helicity provides the spinor representation of null momentum vectors. In spinor notation, the null condition $p^2 = 0$ implies $\det p_{AB} = 0 = \det p^{AB}$. Hence, the most general solution can be represented either by chiral or anti-chiral spinors[1] [241]

$$p_{AB} = k_{Aa} k_{Bb} \varepsilon^{ab} \ , \qquad p^{AB} = \tilde{k}^{A\dot{a}} \tilde{k}^{B\dot{b}} \varepsilon_{\dot{a}\dot{b}} \ . \tag{6.1}$$

Here, the indices $a, b, \ldots, \dot{a}, \dot{b}, \cdots = 1, 2$ turn out to be precisely the indices for the little group $\mathsf{SL}(2,\mathbb{C}) \times \widetilde{\mathsf{SL}(2,\mathbb{C})}$ inside the stabilizer of the vector p^{AB} under the Lorentz group $\mathsf{SO}(6,\mathbb{C})$. Little group indices will be raised and lowered with the help of the the usual invariant tensors $\varepsilon^{ab} = -\varepsilon^{ba}$ and $\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{\dot{b}\dot{a}}$, with $\varepsilon^{12} = \varepsilon_{\dot{1}\dot{2}} = 1$. In fact, transformations of the form $k_{Aa} \mapsto M_a^b k_{Ab}$ and $\tilde{k}^{A\dot{a}} \mapsto \tilde{M}_{\dot{b}}^{\dot{a}} \tilde{k}^{A\dot{b}}$, with $\det M = 1 = \det \tilde{M}$, will leave p invariant. It should be noted that k_{Aa} has five independent components and that k_{Aa} and $\tilde{k}^{A\dot{b}}$ are not independent, as $k_{Aa} \tilde{k}^{A\dot{b}} = 0$, since $p_{AB} = \frac{1}{2} \varepsilon_{ABCD} p^{CD}$. Hence, (6.1) constitutes spinor representations of the null-momentum p_M that transform appropriately under the Lorentz and little groups.

Since we shall make extensive use of differential forms in spinor coordinates, it is useful to briefly establish a dictionary between vector and spinor notation. In spinor notation, a differential 1-form A on M^6 has components $A_{AB} = -A_{BA}$, while a differential 2-form B is given by trace-less matrix $B_A{}^B$. Moreover, a differential 3-form H on space-time in spinor coordinates is the pair of symmetric bi-spinors $H = (H_{AB}, H^{AB})$, where $H_{AB} = H_{BA}$ contains the self-dual part of H, while $H^{AB} = H^{BA}$ contains the anti-self-dual part. By imposing either self-duality or anti-self-duality onto H in spinor notation, one of the bi-spinors is put to zero. Moreover, $H = \mathrm{d}B$ takes the form

$$H = (H_{AB}, H^{AB}) = (\partial_{C(A} B_{B)}^{\ C}, \partial^{C(A} B_C^{\ B)}) \tag{6.2}$$

and transforms as the $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$ of the little group. We invite the interested reader to consult [97] where the explicit translation between vector and spinor notation for differential

[1] This is due to the fact that null momenta are matrices p^{AB} of rank two, due to the antisymmetry of the spinor indices. Hence,

$$p^2 = \frac{1}{8} \varepsilon_{ABCD} p^{AB} p^{CD} = \frac{1}{8} \varepsilon_{ABCD} \tilde{k}^{A\dot{a}} \tilde{k}^{B\dot{b}} \tilde{k}^{C\dot{c}} \tilde{k}^{D\dot{d}} \varepsilon_{\dot{a}\dot{b}} \varepsilon_{\dot{c}\dot{d}} = 0 \ .$$

forms can be found. We just mention here that $A_{AB} = \frac{1}{4}\sigma^M_{AB} A_M$, while the connection to the ordinary space-time expression of the self-dual three-form is through $H_{AB} = \sigma^{MNK}_{AB} H_{MNK}$, where the projector σ^{MNK}_{AB} is given by some totally antisymmetric product of sigma-matrices.[1]

Zero-rest-mass fields. Let us discuss now some linear field equations on space-time M^6. From the above considerations, the spin label of six-dimensional fields consists of a pair of integers, corresponding to irreducible representations of the little group. *Chiral zero-rest-mass fields* form representations $(\mathbf{2h+1}, \mathbf{1})$, for $h \in \frac{1}{2}\mathbb{N}_0$, of $\mathsf{SL}(2,\mathbb{C}) \times \widetilde{\mathsf{SL}(2,\mathbb{C})}$. These are conformally invariant fields, of helicity h, that carry $2h$ symmetrised lower spinor indices and satisfy the free equations of motion

$$\partial^{AA_1}\psi_{A_1\ldots A_{2h}} = 0 \ , \quad \text{for } h \geq \tfrac{1}{2} \quad \text{and} \quad \Box\psi = 0 \ , \quad \text{for } h = 0 \ . \tag{6.3}$$

Explicitly, we define the sheaf of germs of solutions to zero-rest-mass equations for helicity h, denoted by $\mathcal{Z}_h$, as [97, 43]

$$\mathcal{Z}_h := \begin{cases} \ker\left\{\partial^{AB} : \odot^{2h-1}S^* \otimes \det S^* \to (\odot^{2h-1}S^* \otimes S)_0 \otimes \otimes^2 \det S^*\right\} \ , & \text{for } \ h \geq \frac{1}{2}, \\ \ker\left\{\Box := \frac{1}{4}\partial^{AB}\partial_{AB} : \det S^* \to \otimes^2 \det S^*\right\} \ , & \text{for } \ h = 0 \ . \end{cases} \tag{6.4}$$

Here, the subscript zero refers to the totally trace-less part, while the powers of the determinant of S^* are included to render the zero-rest-mass field equations conformally invariant[2], in analogy to the four dimensional case [243, 242, 66].

Example: $\mathcal{N} = (2,0)$ tensor multiplet. As a motivational example let us consider the $(2,0)$-*theory*, arising on the world-volume of parallel M5-branes. From supersymmetry we know that the field content is given by the $\mathcal{N} = (2,0)$ tensor multiplet in six dimensions [244]. This is a chiral multiplet and consists of a self-dual three-form curvature $H = \mathrm{d}B$, which transforms as the $(\mathbf{3}, \mathbf{1})$ of the little group, as well as five scalars $\phi^{IJ} = -\phi^{JI}$[3] with $I, J = 1, \ldots 4$, in the trivial representation $(\mathbf{1}, \mathbf{1})$, parametrising transverse fluctuations of the M5-brane, together with four Weyl spinors ψ^I_A in the $(\mathbf{2}, \mathbf{1})$. The zero-rest-mass field equations for the fields in

[1] Explicitly, $\sigma^{MNK}_{AB} := \frac{1}{3!}(\sigma^M_{AC}\sigma^{N\ CD}\sigma^K_{DB} \pm \text{ permutations }) = \sigma^{MNK}_{BA}$.

[2] Here, we refer to the notion of *conformal weights*, as introduced in [66].

[3] The six scalars ϕ^{IJ} are constrained by the condition $\phi^{IJ}\Omega_{IJ} = 0$, where Ω_{IJ} is an invariant form of the underlying $\mathsf{Sp}(2)_R \cong \mathsf{SO}(5)_R/\mathbb{Z}_2$ R-symmetry group, see e.g. [245].

the tensor multiplet read as[1]

$$H^{AB} = 0\ , \quad \partial^{AB}\psi_B = 0 \quad \text{and} \quad \Box\phi = 0\ . \tag{6.5}$$

Recall that, in our conventions, the first equation is nothing but self-duality, i.e. the field H can be described as the chiral field H_{AB} with symmetric subscripts. Moreover, $H_{AB} = \partial_{C(A}B_{B)}^C$ implies[2] the Bianchi identity $\partial^{AC}H_{CB} = 0$, which is equivalent to the field equation for self-dual three-forms. The corresponding plane waves

$$H_{ABab} = k_{A(a}k_{Bb)}e^{ip\cdot x}\ , \quad \psi_{Aa} = k_{Aa}e^{ip\cdot x} \quad \text{and} \quad \phi = e^{ip\cdot x}\ , \tag{6.6}$$

show, for instance, the three polarisation states of a helicity 1 field H_{AB}. Moreover, gauge transformations are mediated by the gauge parameter $\Lambda_{AB} = -\Lambda_{BA}$ via

$$B_B{}^A \mapsto B_B{}^A + \partial^{AC}\Lambda_{CB} - \partial_{BC}\Lambda^{CA}\ . \tag{6.7}$$

Potential formulation. As in four dimensions, there is a formulation of six-dimensional conformally invariant massless fields in terms of potentials. Here, we closely follow [97], that generalises the four dimensional case [66]. Let us consider $h \in \frac{1}{2}\mathbb{N}_0$. The potential field of a chiral zero-rest-mass field of helicity h is the spinor field

$$B_A^{A_1\ldots A_{2h}} = B_A^{(A_1\ldots A_{2h})} \in H^0(\mathsf{M}^6, (\odot^{2h-1}S \otimes_{\mathcal{O}_{\mathsf{M}^6}} S^*)_0 \otimes \det S^*)\ , \tag{6.8}$$

Then, one defines the corresponding field strength according to

$$H_{A_1\ldots A_{2h}} := \partial_{(A_1B_1}\cdots\partial_{A_{2h-1}B_{2h-1}}B^{B_1\cdots B_{2h-1}}_{A_{2h})} \in H^0(\mathsf{M}^6, (\odot^{2h}S) \otimes \det S^*))\ . \tag{6.9}$$

Moreover, the equations

$$H^{A_1\ldots A_{2h}} := \partial^{A(A_1}B_A^{A_2\ldots A_{2h})} = 0\ . \tag{6.10}$$

immediately imply

$$\partial^{AA_1}H_{A_1\ldots A_{2h}} = 0 \tag{6.11}$$

There is a gauge freedom in the potential (6.8), namely the transformations

$$B_A^{A_1\ldots A_{2h}} \mapsto B_A^{A_1\ldots A_{2h}} + \left(\partial_{CB}\Lambda^{C(AA_1\ldots A_{2h-2})} - \partial^{C(A}\Lambda_{CB}^{A_1\ldots A_{2h-2})}\right)_0\ , \tag{6.12}$$

[1]Here, for simplicity, we have suppressed the R-symmetry indices.

[2]See Appendix F.

in terms of the totally trace-less gauge parameter $\Lambda_{AB}^{A_1\ldots A_{2h-2}} = \Lambda_{[AB]}^{(A_1\ldots A_{2h-2})}$, give the same pair of spinors $(H^{A_1,\ldots,A_{2h}}, H_{A_1,\ldots,A_{2h}})$. Hence, the spinor field $H_{A_1,\ldots,A_{2h}}$ can be regarded as a section of the sheaf $\mathcal{Z}_h$. In the following, we will refer to general theories containing a self-dual 3-form field strength as *self-dual tensor field theories*. Via twistor approach, it is possible to employ the representation of solutions to (6.11) in terms of sheaf cohomology on the corresponding twistor space. One immediately notice that for $h = 1$ this formalism reproduces the 3-form field (6.2). Moreover, (6.10) implies the equations of motion for H_{AB} and (6.12) gives the desired gauge transformations.

6.3. Twistor space of six-dimensional space-time

In the following, we give an overview of the geometry of twistor space P^6, underlying the description of self-dual 3-forms on flat six-dimensional space-time $\mathbb{C}^6$. Such a twistor space has a long history, see e.g. [246–253, 96, 254–256]. Here, we shall expose its construction following [96–98], see also [43] for a brief account of the subject. Then, we shall focus our attention on Euclidean signature by means of an appropriate anti-holomorphic involution on twistor space that picks the desired real slice on complexified space-time.

In the following, we will make use of the standard notation $\mathcal{O}_{\mathsf{X}}(k)$ for the sheaf of holomorphic functions on a complex manifold X which are homogeneous of degree $k \in \mathbb{Z}$ and Ω^p_{X} for the sheaf of holomorphic differential p-forms on X.[1] The interested reader may consult Appendices D and E for a review of the notions of algebraic and complex geometry necessary to understand the constructions described throughout this Chapter.

6.3.1. Geometry of twistor space

Following a standard approach such space can be constructed as follows. First, we define the so called *correspondence space*

$$\mathsf{F}^9 := \mathbb{P}(S^*) \cong \mathbb{C}^6 \times \mathbb{P}^3 \,, \tag{6.13}$$

as the projectivisation of the dual of S. This is a nine-dimensional complex manifold, that can be equipped with coordinates (x^{AB}, λ_A), with λ_A being homogenous coordinates on $\mathbb{P}^3$.[2]

[1] Notice that we shall freely switch between the notions of vector bundles and their corresponding sheaves of sections, as we are working with locally free sheaves [97].

[2] Here, $\mathbb{P}^3$ is understood as a copy of the projectivised fibres of the dual of the anti-chiral spin bundle, hence, the chiral (i.e. lower-script) spinor index.

Then, we consider the vector fields

$$V^A = \lambda_B \partial^{AB} \tag{6.14}$$

on F^9 and the induced distribution $D :=$ span$\{V^A\}$, called the *twistor distribution*. Because of the relation $\lambda_A V^A = 0$, this is a rank-3 distribution. Moreover, because of the condition $[V_A, V_B] = 0$, the twistor distribution is integrable, that is, $[D, D] \subseteq D$. Consequently, there is a foliation of F^9 by three-dimensional complex manifolds and the *twistor space* is defined to be the the associated six-dimensional leaf space, $\mathsf{P}^6 := \mathsf{F}^9/D$. Hence, we may now establish a twistor correspondence, which is captured by the following double fibration[1]

$$\begin{array}{ccc} & \mathsf{F}^9 & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ \mathsf{P}^6 & & \mathsf{M}^6 \end{array} \tag{6.15}$$

Let (z^A, λ_A) be homogeneous coordinates the complex projective space $\mathbb{P}^7$, identified up to overall re-scalings, i.e. $(z^A, \lambda_A) \sim (tz^A, t\lambda_A)$, for any $t \in \mathbb{C}_0$. Here, we assume $(\lambda_A) \neq (0,0,0,0)$, meaning that we are working on the open subset $\mathbb{P}^7_\circ := \mathbb{P}^7 \setminus \mathbb{P}^3$, where the removed projective space $\mathbb{P}^3$ is given by $\lambda_A = 0$ and $z^A \neq 0$. Clearly, the projection π_2 is the trivial projection $(x, \lambda) \mapsto x$, while π_1 is given by

$$\pi_1 : (x^{AB}, \lambda_A) \mapsto (z^A, \lambda_A) = (x^{AB}\lambda_B, \lambda_A) \ . \tag{6.16}$$

Since the matrix x^{AB} is skew-symmetric, the constraint $z^A \lambda_A = 0$ holds. It follows that the twistor space P^6 is the complex quadric hypersurface given by the zero locus

$$\mathsf{P}^6 := \Big\{ (z^A, \lambda_A) \in \mathbb{P}^7_\circ \mid z^A \lambda_A = 0 \Big\} \ , \tag{6.17}$$

embedded into the total space of the holomorphic fibration[2] $\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \to \mathbb{P}^3$, with fibre coordinates z^A as well as base coordinates λ_A. Here and in the following $\mathcal{O}_{\mathbb{P}^3}(1)$ denotes the dual *tautological bundle*[3] (or *hyperplane bundle*) over $\mathbb{P}^3$. See Appendix E for details.

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[1]See Appendix E for geometrical treatment of the double fibrations appearing in twistor theory.

[2]Twistor space P^6 may be seen as a rank-3 subbundle of $\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \to \mathbb{P}^3$ or, alternatively, as the normal bundle of $\mathbb{P}^3$ inside the hypersurface $\mathbb{Q} \hookrightarrow \mathbb{P}^7$ given by $z^A \lambda_A = 0$. See Appendix E and [97] for details.

[3]We shall also use the standard notation $\mathcal{O}_{\mathbb{P}^3}(k) := \otimes^k \mathcal{O}_{\mathbb{P}^3}(1)$ and $\mathcal{O}_{\mathbb{P}^3}(-k) := \otimes^k \mathcal{O}^*_{\mathbb{P}^3}(1)$, $k > 0$ as well as $\mathcal{O}_{\mathbb{P}^3}(0) := \mathcal{O}_{\mathbb{P}^3}$.

We notice that the relation

$$z^A = x^{AB}\lambda_B \tag{6.18}$$

is a six-dimensional generalisation of the *Penrose's incidence relation*, relating twistor space to space-time. By virtue of the relation (6.18), a geometric correspondence between M^6 and P^6 is realised. Explicitly, a point $x \in \mathsf{M}^6$ corresponds to a three dimensional submanifold $\pi_1(\pi_2^{-1}(x)) \hookrightarrow \mathsf{P}^6$ that is bi-holomorphic to $\mathbb{P}^3$, while a point (z,λ) in twistor space corresponds to a submanifold $\pi_2(\pi_1^{-1}(z,\lambda))$ in space-time given by

$$x^{AB} = x_0^{AB} + \varepsilon^{ABCD}\mu_C\lambda_D \ . \tag{6.19}$$

This defines a totally null self-dual three-plane[1] (as three is the rank of the twistor distribution), for some arbitrary μ_C defined modulo terms proportional[2] to λ_C and x_0^{AB} being a particular solution to the incidence relation corresponding to a reference point in space-time. Hence, twistor space P^6 parametrises all totally null three-planes of space-time M^6.

As explained in detail in [97], the twistor space P^6 admits various dimensional reductions. It has not be a surprise that, upon reducing to four space-time dimensions, P^6 can be reduced to the Penrose' twistor space, the space of all totally null 2-planes in four dimensions, as well as to to the ambitwistor space, the space of all null rays in four dimensions and the hyperplane twistor space in three dimensions, the space of all hyperplanes in four dimensions.

Having recalled the basic ingredients, in the next Section we will see how both Abelian and non-Abelian self-dual tensor fields on M^6 can be described on twistor space P^6. First, however, we shall review the reality conditions we employ to descend to Euclidean space-time, that we will make extensive use of in Chapter 7.

6.3.2. Reality structures

Up to now, we have only dealt with complex manifolds. However, in order to deal with gauge theories, one needs to put certain real structures on twistor space. These, in turn, induce real structures on all the manifolds appearing in the double fibration (6.22). In the next Chapter, we shall be interested in six-dimensional Euclidean space-time and its associated twistor space.

[1] These totally null self-dual three planes are known as α-*planes* (β-*planes* being the anti-self-dual ones).

[2] Transformations of the form $\mu_C \to \mu_C + \rho\lambda_C$, $\rho \in \mathbb{C}$, leave x invariant, so the space of solutions is three-dimensional. Also, $\varepsilon^{ABCD}\mu_C\lambda_D$ denotes a generic null-vector in six dimensions.

The twistor correspondence in Euclidean signature. Explicitly, an Euclidean signature real slice is realised by defining an anti-linear involution on twistor space $\tau : \mathsf{P}^6 \to \mathsf{P}^6$, acting on spinors as

$$\tau(\lambda_A) = (\lambda_A)^* := \hat{\lambda}^A \;, \qquad \tau(z^A) = (z^A)^* := \hat{z}_A \tag{6.20}$$

and such that $\tau^2 = 1$. By virtue of the incidence relation (6.18) we obtain an induced involution on space-time[1] explicitly given by

$$\tau(x^{AB}) = (x^{AB})^* := \hat{x}_{AB} \;. \tag{6.21}$$

This, in turn, induces a reality structure on M^6. The fixed point set of this involution, that is, $\tau(x) = x$ for $x \in \mathsf{M}^6$, defines Euclidean space-time $\mathbb{R}^6$ inside M^6. In this setting, the double fibration (6.15) reduces to the non-holomorphic fibration[2]

$$\pi \;:\; \mathsf{P}^6 \to \mathbb{R}^6 \;, \tag{6.22}$$

since $\pi_2^{-1}(\mathbb{R}^6) \cong \mathsf{P}^6$ and therefore π_1 becomes a bijection. Hence, there is no need for the double fibration picture in Euclidean signature. Explicitly, this $\mathbb{P}^3$ fibration is given by

$$(z^A, \lambda_A) \mapsto x^{AB} = \frac{2z^{[A}\hat{\lambda}^{B]} + \varepsilon^{ABCD}\hat{z}_C\lambda_D}{\lambda_A\hat{\lambda}^A} \;, \tag{6.23}$$

associating a unique point x to every pair (z, λ). In other words, the Euclidean twistor space is isomorphic to $\mathbb{R}^6 \times \mathbb{P}^3$. Here, x^{AB} is real and $\det(x^{AB}) = \delta(x, x)$, with δ being the standard Euclidean metric on $\mathbb{R}^6$.

This allows us to work with the non-holomorphic coordinates (x^{AB}, λ_A), with the projective scaling carried now exclusively by λ_A, the homogeneous coordinates on the fibres $\mathbb{P}^3$. This helps avoiding a number of technical difficulties. In terms of these coordinates we may define[3] the following differential $(0,1)$-forms on twistor space P^6

$$\bar{e}_A := \frac{\hat{\lambda}^B \mathrm{d}x_{AB}}{[\lambda\hat{\lambda}]} \qquad \text{and} \qquad \tilde{e}^{AB} := \frac{2\hat{\lambda}^{[A}\mathrm{d}\hat{\lambda}^{B]}}{[\lambda\hat{\lambda}]^2} \;, \tag{6.24}$$

[1] In the following, with a slight abuse of notation, we shall use the same notation τ for the anti-holomorphic involution induced on the different manifolds in the twistor correspondence.

[2] In Euclidean signature a point on P^6 is specified by fixing a point on $\mathbb{R}^6$ and a point on the correspondent $\mathbb{P}^3$, giving the identification of Euclidean twistor space with $\mathbb{R}^6 \times \mathbb{P}^3$. However, the correspondence remains non-local, as a full projective space $\mathbb{P}^3$ corresponds to the same point on $\mathbb{R}^6$.

[3] The analogous basis of $(0,1)$ forms on twistor space corresponding to Euclidean four-dimensional space-time were introduced in [84, 257].

such that $\Omega^{0,1}(\mathsf{P}^6) =$ span $\{\bar{e}_A, \tilde{e}^{AB}\}$. Here, we have chosen to normalise these differential forms with powers of $[\lambda\hat{\lambda}] := \lambda_A\hat{\lambda}^A(>0)$ for later convenience and ensuring that they have holomorphic weight only, i.e. they are independent of rescalings of $\hat{\lambda}$. In particular the differential forms (6.24) have homogeneity $(-1,0)$ and $(-2,0)$, respectively. Analogously, the vector fields

$$\overline{V}^A := \lambda_B\partial^{AB} \qquad \text{and} \qquad \widetilde{V}_{AB} := [\lambda\hat{\lambda}]\lambda_{[A}\frac{\partial}{\partial\hat{\lambda}^{B]}} \ , \tag{6.25}$$

generate the space of anti-holomorphic vectors, i.e. $T^{0,1}\mathsf{P}^6 =$ span $\{\overline{V}^A, \widetilde{V}_{AB}\}$, determining the twistor complex structure on Euclidean twistor space. Notice that these have been chosen to have homogeneity $(1,0)$ and $(2,0)$, respectively. Also, we observe that the vector fields (6.14) are $(0,1)$-vector field in this real setting.

Along this basis, a one-form field $A^{0,1}$ on twistor space may be expanded as

$$A^{0,1} = \bar{e}_{[A}\lambda_{B]}A^{AB} + \tilde{e}^{AB}A_{AB} \ , \tag{6.26}$$

in a way which is adapted to the description on $\mathsf{P}^6 \cong \mathbb{R}^6 \times \mathbb{P}^3$. Here, we have introduced coefficients $A_{AB} = -A_{BA}$ and $A^{AB} = -A^{BA}$, which are generally functions of $(x, \lambda, \hat{\lambda})$. In particular, $\bar{e}_{[A}\lambda_{B]}A^{AB}$ denotes the horizontal component along the base $\mathbb{R}^6$, while $\tilde{e}^{AB}A_{AB}$ is the vertical component along the fibres of the fibration (6.22). It is crucial to observe that, since $\lambda_A\overline{V}^A = 0$, the $\bar{e}_A$ are defined modulo terms proportional to λ_A[1]. As we will see, this property is reflected in the expansions of the form fields appearing throughout the rest of the Thesis.

In addition, the fibration (6.22) allows to express the integrable complex structure on P^6 in terms of coordinates on $\mathbb{R}^6$. This is given by the nilpotency of the anti-holomorphic Dolbeault operator $\bar{\partial} : \Omega^{p,q}(\mathsf{P}^6) \to \Omega^{p,q+1}(\mathsf{P}^6)$, that can be expressed as[2]

$$\bar{\partial} := 2\bar{e}_A\overline{V}^A + \tilde{e}^{AB}\widetilde{V}_{AB} \ . \tag{6.28}$$

[1]It is important to stress that the redundancy introduced by using homogeneous coordinates is reflected in the fact that the sets (6.24)-(6.25) do not define dual frames.

[2]The compatibility with the twistor correspondence easily follows from the incidence relation as

$$\bar{\partial} = \mathrm{d}\hat{z}_A P^A_B\frac{\partial}{\partial\hat{z}^B} + \mathrm{d}\hat{\pi}^A\frac{\partial}{\partial\hat{\pi}^A} \ , \tag{6.27}$$

where (z^A, π_A) are homogeneous coordinates on P^6 and we have introduced the operator $P^A_B := \delta^A_B - \frac{\pi_B\hat{\pi}^A}{[\pi\hat{\pi}]}$. The easiest way of showing this is to use local coordinates on both P^6 and $\mathbb{R}^6 \times \mathbb{P}^3$. In general, this allows to shred some lights on relations that may not be manifest by using homogeneous coordinates. See Appendix F for more details.

We want to already point out that the choice of Euclidean signature simplifies the mapping from twistor space quantities to space-time quantities. As we shall see, the basis (6.24) turns out to be helpful in relating twistor and space-time self-dual actions. We will take advantage of this fact at several points later on.

6.4. Penrose transform

The essence of twistor theory in any number of dimensions is to encode differentially constrained data (solutions to field equations) on space-time in terms of differentially unconstrained complex analytic data on twistor space. In four space-time dimensions the basic tool in this regard is the Penrose transform [48, 258], which is an integral geometric method relating solutions of the zero-rest-mass equations to certain cohomology classes on Penrose's twistor space. The Penrose transform also extends to any number of dimensions [96]. Generally, in higher dimensions it involves higher degree cohomology classes on the relevant twistor space, encoding solutions to certain field equations on space-time. In particular, in [97, 98] the Penrose transform on the twistor space P^6 has been analysed. Explicitly, it manifests itself in the following manner [97]:

Theorem 6.1. *Let $U \subset \mathsf{M}^6$ be an open and convex subset and $\hat{U} := \pi_1(\pi_2^{-1}(U)) \subset \mathsf{P}^6$ the corresponding open subset*[1] *under the double fibration* (6.15)*. For $h \in \frac{1}{2}\mathbb{N}_0$, the canonical isomorphism*

$$\mathcal{P} : H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4)) \longrightarrow H^0(U, \mathcal{Z}_h) \ , \tag{6.29}$$

is called Penrose transform.

This tells us that fields of non-negative helicity h, satisfying zero-rest-mass field equations on a suitable region of space-time U, correspond to classes of the Čech cohomology group $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$, with values in certain holomorphic line bundles over the corresponding region $\hat{U}$. We refer to the papers [97, 98] for a detailed and thorough proof. This basically consists in introducing the *relative de Rham complex* on the correspondence space of the double fibration $\mathsf{P}^6 \xleftarrow{\pi_1} \mathsf{F}^9 \xrightarrow{\pi_2} \mathsf{M}^6$ and then computing its cohomology by applying the *direct image functor* with respect to the fibration π_2.[2] It is also shown that, for $h < 0$, the cohomology groups $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ yield nothing but trivial fields on space-time, see also [96]. We

[1] Since the maps $\pi_{1,2}$ are continuous and open.

[2] See Appendix E for some general definitions and results in this respect.

invite the interested reader to consult Appendix E for an overview of the mathematical tools behind the twistor correspondence in any dimensions.

6.4.1. Integral formulas

The Penrose transform is most easily realised in terms of contour integral formulæ. In dimension six, such formulæ first appeared in the works by Hughston [248–251].

Čech representation. Let $U \subset \mathsf{M}^6$ be open and convex and $\hat{U} := \pi_1(\pi_2^{-1}(x)) \subset \mathsf{P}^6$ the correspondent subset in twistor space. Consider an open Stein covering $\hat{\mathfrak{U}} = \{\hat{U}_a\}$ of $\hat{U}$, $a = 1, \cdots, 4$, obtained by lifting the standard 4-patches covering of $\mathbb{P}^3$ to $\hat{U}$. Then an element of the Čech cohomology group $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ can be represented[1] by a holomorphic function $f_{-2h-4} = f_{-2h-4}(z, \lambda)$ on the only quadruple overlap $\hat{U}_{1234} := \hat{U}_1 \cap \hat{U}_2 \cap \hat{U}_3 \cap \hat{U}_4$. Let us consider $h \geq 0$. In terms of the Čech cocycle f_{-2h-4}, a zero-rest-mass field $\psi \in H^0(U, \mathcal{Z}_h)$ is given by the contour integral formula[2]

$$\psi_{A_1 \dots A_{2h}} := \oint_{\Gamma} \Omega^{3,0} \lambda_{A_1} \cdots \lambda_{A_{2h}} f_{-2h-4}|_{\mathsf{Y}} \ . \tag{6.30}$$

Here,

$$\Omega^{3,0} = \frac{1}{4!} \varepsilon^{ABCD} \lambda_A \mathrm{d}\lambda_B \wedge \mathrm{d}\lambda_C \wedge \mathrm{d}\lambda_D \ , \tag{6.31}$$

is the natural $\mathsf{SL}(4, \mathbb{C})$-invariant, holomorphic measure on $\mathbb{P}^3$, with values in $\mathcal{O}_{\mathbb{P}^3}(4)$, while Γ is topologically a three-torus[3] in $\hat{U}_{1234} \subset \mathbb{P}^3$, specified by the cohomology class. Moreover, the restriction $f_{-2h-4}|_{\mathsf{Y}}$ to $\mathsf{Y} \cong \mathbb{P}^3 \subset \mathsf{P}^6$, defined via twistor correspondence, is accomplished simply by imposing the incidence relation (6.18): $f_{-2h-4}|_{\mathsf{Y}} = f_{-2h-4}(x \cdot \lambda, \lambda)$ on $\hat{U}_{1234} \cap \mathsf{Y}$. We stress out that this realises the pull-back part of the Penrose transform as explained in Appendix E. The second and final step, namely integrating out the fibre coordinates, is

[1] Note that this is not the general way to represent an element of $H^3(\hat{U}, \mathcal{O}_{\hat{U}})$, for which one needs to use branched contour integrals, cf. [242].

[2] Integral formluæ for the cohomology groups $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(2h-4))$, with $h > 0$, require to thicken (via infinitesimal neighbourhood) the twistor space P^6 into its ambient space $\mathbb{P}^7_\circ := \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4$ in order to make sense. See the discussion in [97].

[3] The open sets $\hat{U}_a$ are chosen so that for any submanifold $\mathsf{Y}_x = \pi_1(\pi_2^{-1}(x))$ in $\hat{U}$ the intersections $\hat{U}_{ax} := \hat{U}_i \cap \mathsf{Y}_x$ overlap in a region $\hat{U}_{1x} \cap \hat{U}_{2x} \cap \hat{U}_{3x} \cap \hat{U}_{4x} \cong \mathbb{T}^3 \times \mathbb{R}$. The contour Γ is a three-torus winding around such region.

nothing but the contour integration[1]. One can immediately check that the integral is well defined since the respective weights cancel out[2]. The spinor field $\psi_{A_1\ldots A_{2h}} = \psi_{(A_1\ldots A_{2h})}$ forms the representation $(\mathbf{2h+1}, \mathbf{1})$ of the little group and satisfies (6.3), as shown in Appendix F.

Altogether, this shows how the role of twistor theory is indeed transforming the differential equations (6.4) that on-shell fields satisfy in space-time into pure holomorphy in twistor space.

Dolbeault representation. Alternatively, thanks to the Dolbeault isomorphism (see e.g. Appendix E), a Dolbeault representative[3] of the cohomology group $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ can be chosen. Consequently, space-time fields are represented by equivalence classes of certain $(0,3)$-forms on twistor space. Recall that the Dolbeault representative in question is a $(0,3)$-form η on $\hat{U}$ taking values in the holomorphic line bundle $\mathcal{O}_{\hat{U}}(-2h-4)$ and such that $\bar{\partial}\eta = 0$, where η is defined modulo $\bar{\partial}\Xi$, with $\Xi \in \Omega^{0,2}(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$. A space-time function of helicity h, for $h \geq 0$, is then defined by

$$\psi_{A_1\ldots A_{2h}} := \int_{\mathsf{Y}} \Omega^{3,0} \wedge \lambda_{A_1} \cdots \lambda_{A_{2h}} \eta|_{\mathsf{Y}} \ . \tag{6.32}$$

As before, $\eta|_{\mathsf{Y}}$ indicates the restriction to Y using the incidence relation and the integral depends only on the cohomology class of η. It can be easily shown that space-time field $\psi_{A_1\ldots A_{2h}}$ is a $2h$-index symmetric spinor field satisfying the zero-rest-mass equations (6.3). Hence, the integral formula (6.32) realises the isomorphism (6.29) from the left to the right. Again, we invite the reader to consult Appendix F for a proof. The choice of the Dolbeault representation will lead to considerably simpler computations in many cases.[4]

6.5. Penrose–Ward transform

For gauge theories, the right tool encoding solutions to certain field equations on space-time in terms of holomorphic data on twistor space is the *Penrose–Ward transform*. On the other

[1]Here, the contour integral corresponds to the functor of direct image for sheaves. See Appendices D, E for details.

[2]Note that this requirement determines the relationship between homogeneity and helicity.

[3] Note that working with Dolbeault cohomology allows us to avoid to use the combinatorics of Leray covers and there is no contour ambiguity involved, unlike in the Čech cohomology setup, most widely used by twistor theorists, see e.g. [48, 258, 66]. This choice leads to considerably simpler computations in many cases.

[4]Given a z.r.m. field on complex space-time M there is not, in general, a canonical way to reconstruct the twistor representative. However, in Euclidean signature there is a canonical way for some z.r.m. fields due to Woodhouse [259].

hand, the field equations in question and the associated gauge transformations can be naturally derived starting from algebraic data on twistor space.

6.5.1. Abelian fields

The Penrose–Ward transform discussed in [97, 98] provides a way to associate $\mathfrak{u}(1)$-valued self-dual 3-forms on M^6 to certain algebraic data over P^6. See also [96] for an earlier account and [99] for a supersymmetric extension. The algebraic data in question are represented by topologically trivial, holomorphic one-gerbes $\hat{\Gamma} \to \mathrm{P}^6$ over the twistor space P^6. Moreover, these objects are required to be holomorphically trivial when restricted to any complex projective 3-space $\pi_1(\pi_2^{-1}(x)) \hookrightarrow \mathrm{P}^6$, for all $x \in \mathrm{M}^6$.[1] Following Manin's terminology [65] in the principal 1-bundle case, we shall refer to such gerbes as M^6-*trivial*. As we will see, these gerbes on twistor space are parametrised by elements of the Čech cohomology group $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(2h-2))$ for $h > 0$, see e.g. [43]. In a way, such a cohomology group replaces $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ for $h < 0$,[2] that, as we have seen, yields trivial space-time fields. Besides these two cohomology groups, all the other (higher) cohomology groups, namely for $\mathcal{O}_{\hat{U}}(k)$ with $k \in \mathbb{Z}$, appear to give trivial fields on space-time, [96, 97].[3] First, we set $h = 1$ and consider $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(0)) := H^2(\hat{U}, \mathcal{O}_{\hat{U}})$. To analyse this cohomology group we consider the short exact exponential sequence of sheaves[4] on $\hat{U}$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\hat{U}} \xrightarrow{\exp} \mathcal{O}^*_{\hat{U}} \longrightarrow 0 \,. \tag{6.33}$$

Here $\mathcal{O}^*_{\hat{U}}$ denotes the subsheaf of non-vanishing holomorphic functions on $\hat{U}$, while $\exp : \mathcal{O}_{\hat{U}} \to \mathcal{O}^*_{\hat{U}}$ is the exponential map defined by $\exp(f) := e^{2\pi i f}$, for $f \in \mathcal{O}_{\hat{U}}$. The corresponding long exact sequence of cohomology groups yields

$$H^1(\hat{U}, \mathcal{O}^*_{\hat{U}}) \xrightarrow{c_1} H^2(\hat{U}, \mathbb{Z}) \longrightarrow H^2(\hat{U}, \mathcal{O}_{\hat{U}}) \longrightarrow H^2(\hat{U}, \mathcal{O}^*_{\hat{U}}) \xrightarrow{\mathrm{DD}} H^3(\hat{U}, \mathbb{Z}) \,. \tag{6.34}$$

[1] Alternatively (see Appendix E for details on the subject), this means that the pull-back gerbe $\pi_1^*\hat{\Gamma}$ is trivial on each fibre of π_2.

[2] This is in contradistinction to four-dimensional twistor theory, where fields of any integer helicity can be obtained from twistor space. Note that the case $h = 0$ is somewhat exceptional and needs to be treated differently in both the Penrose and Penrose–Ward transforms. See [97] for a full discussion of the subject, based on spectral sequence arguments.

[3] In particular, $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-2))$, for $h < 0$, yields trivial fields on space-time as the appropriate direct images vanish.

[4] See the the discussion Appendix D.

The group $H^1(\hat{U}, \mathcal{O}^*_{\hat{U}})$ topologically classifies isomorphism classes of line bundles on $\hat{U}$, while the map c_1 is the first Chern class map[1] which gives the characteristic class of line bundles (zero-gerbes). Similarly, DD is the Dixmier–Duady class and $H^2(\hat{U}, \mathcal{O}^*_{\hat{U}})$ is the moduli space of holomorphic one-gerbes over $\hat{U}$. Hence, Abelian holomorphic gerbes provide a geometric realisation of classes in $H^3(\hat{U}, \mathbb{Z})$ via their Dixmier-Douady class, analogous to the way that line bundles provide, via their Chern class, a geometric realisation of the elements of $H^2(\hat{U}, \mathbb{Z})$ in lower degree. Since the map c_1 is surjective and[2] $H^3(\hat{U}, \mathbb{Z}) = 0$ the above sequence reduces to

$$H^2(\hat{U}, \mathcal{O}_{\hat{U}}) \cong H^2(\hat{U}, \mathcal{O}^*_{\hat{U}}) \ . \tag{6.35}$$

Note that in the discussion of the more general case of $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(2h-2))$, with $h > 0$, one generally loosely speaks of holomorphic one-gerbes also in this case. Since the restriction of $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(2h-2))$ to $\pi_1(\pi_2^{-1}(x)) \cong \mathbb{P}^3 \hookrightarrow \hat{U}$, vanishes[3] for all points $x \in U$, these holomorphic one-gerbes are naturally M^6-trivial. As mentioned, the Penrose–Ward transforms provides then a space-time interpretation of such geometrical objects. Explicitly, we have the following result [97, 98]:

Theorem 6.2. *Let $U \subset \mathsf{M}^6$ be an open and convex subset and $\hat{U} := \pi_1(\pi_2^{-1}(U)) \subset \mathsf{P}^6$ the corresponding open subset under the double fibration (6.15). For $h \in \frac{1}{2}\mathbb{N}_0$, there is a canonical isomorphism*

$$\mathcal{P} : H^2(\hat{U}, \mathcal{O}_{\hat{U}}(2h-2)) \longrightarrow H^0(U, \mathcal{Z}_h) \ , \tag{6.36}$$

called Penrose–Ward transform.

In other words, this result establishes a bijection between equivalence classes of M^6-trivial holomorphic Abelian one-gerbes over twistor space P^6 and gauge equivalence classes of solutions of chiral zero-rest-mass field equations on space-time M^6 . Hence, space-time fields of helicity h satisfying (6.3) may be also described by $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(2h-2))$ on twistor space. For the particular case $h = 1$, the group $H^2(\hat{U}, \mathcal{O}^*_{\hat{U}})$ describes a self-dual 3-form curvature on

[1] Explicitly, the image $c_1(E) \in H^2(\hat{U}, \mathbb{Z})$ of a line bundle $E \in H^1(\hat{U}, \mathcal{O}^*_{\hat{U}})$ is the first Chern class.

[2] This may be seen via singular homology considerations, as $\mathbb{P}^3$ and more in general $\mathbb{P}^n$ have no odd-dimensional cells. Hence $H^{2k}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$, for $k = 0, 1, \ldots, n$ and the remaining cohomology groups are zero.

[3] The only non-vanishing cohomology groups over $\mathbb{P}^3$ with coefficients in the sheaf $\mathcal{O}(k)$ are $H^0(\mathbb{P}^3, \mathcal{O}(k))$ and $H^3(\mathbb{P}^3, \mathcal{O}(-4-k))$, for $k \geq 0$.

$U \subset \mathsf{M}^6$. Here, we are not interested in going into detail about the technicalities of the transform and we remind the interested reader to the relevant papers [97, 98] and to Appendix E for some mathematical constructions. However, we briefly review some of the main steps, that highlight some aspects relevant for our future discussions.[1] As mentioned, the starting point is the cohomological datum $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$, describing an holomorphic one-gerbe $\hat{\Gamma}$ on P^6, subject to certain triviality conditions. First, we note that a *Stein* open cover on $\mathbb{P}^3$ induces good covers $\hat{\mathfrak{U}} = \{\hat{U}_a\}$ on the holomorphic vector bundle $\mathsf{P}^6 \to \mathbb{P}^3$ as well as $\mathfrak{U}' := \{U'_a := \pi_1^{-1}(\hat{U}_a)\}$ on F^9 (see Appendix E for details). By the Dolbeault isomorphism, the cohomology group $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$ can be represented by a globally defined $(0,2)$-form $\hat{B}^{0,2}$ on $\hat{U}$, that is $\bar{\partial}$-closed [2]. The condition of M^6-triviality ensures that the pulled-back gerbe $\pi_1^*\hat{\Gamma}$ will be holomorphically trivial on the correspondence space F^9. Despite this triviality, this object contains non-trivial information. One can perform a gauge transformation yielding to a holomorphic gerbe on F^9 with trivial transition functions and flat, but non-vanishing, relative connective structure. The final step of the transform consists of pushing the gerbe $\pi_1^*\hat{\Gamma} \to \mathsf{F}^9$, together with its relative connective structure, down to space-time M^6. As rigorously shown in [97], this is realised by computing the zeroth direct images of the sheaf $\Omega^p_{\pi_1}$.[3] Concretely, this consists in 'integrating out' the λ-dependence in the $B^{0,2}_{\pi_1}$ stemming from the fibres of $\pi_2 : \mathsf{F}^9 \to \mathsf{M}^6$. Explicitly, one considers the *holomorphic relative de Rham complex*, defined as the following exact sequence of sheaves on F^9, see e.g. [97]

$$0 \longrightarrow \pi_1^{-1}\mathcal{O}_{\mathsf{P}^6} \longrightarrow \mathcal{O}_{\mathsf{F}^9} \xrightarrow{\mathrm{d}_{\pi_1}} \Omega^1_{\pi_1} \xrightarrow{\mathrm{d}_{\pi_1}} \Omega^2_{\pi_1} \xrightarrow{\mathrm{d}_{\pi_1}} \Omega^3_{\pi_1} \longrightarrow 0 \ . \tag{6.37}$$

Here, the holomorphic relative exterior derivative $\mathrm{d}_{\pi_1} : \Omega^p_{\pi_1} \to \Omega^{p+1}_{\pi_1}$ is defined as[4]

$$\mathrm{d}_{\pi_1} := 2\bar{e}_A \overline{V}^A \ . \tag{6.38}$$

We observe that we are using vector fields and differential forms adapted to the non-holomorphic coordinates (x^{AB}, λ_A). This choice is justified by the fact that, in this Euclidean setting, $\overline{V}^A$

[1] To deal with the $h = 0$ case one needs some further considerations, as explained in [97].

[2] In particular, the equation $\hat{H}^{0,3} = \bar{\partial}\hat{B}^{0,2} = 0$, where $\hat{H}^{0,3}$ denotes the $(0,3)$-part of the three-form curvature, is the same as the equation of motion of (Abelian) holomorphic CS theory for ordinary holomorphic vector bundles, but translated in the context of holomorphic one-gerbes.

[3] See and Appendix E for some definitions.

[4] We want to point out the different pre-factor in (6.38) with respect to the papers [97, 100–103]. The reason is purely conventional and has to do with the fact that our notation is already adapted to calculations on Euclidean twistor space. This will become more clear in the next Chapter.

and $\bar{e}_A$ are the generators of the twistor distribution and its dual, respectively. The relative differential form $B_{\pi_1} \in H^0(\mathsf{F}^9, \Omega^2_{\pi_1})$ can be expanded in λ_A according to

$$B_{\pi_1} = \bar{e}_A \wedge \bar{e}_B \lambda_C \varepsilon^{ABCD} B_D^{\ E} \lambda_E \,, \tag{6.39}$$

where the coefficient field $B_A^{\ B}$ depends only on $x^{AB} \in \mathsf{M}^6$ and it is trace-less. The relative flatness condition implies

$$\partial^{C(A} B_C^{B)} = 0 \,. \tag{6.40}$$

Moreover, the self-dual part of the three-form curvature, that is $H_{AB} := \partial_{C(A} B_{B)}^{\ C}$, obeys the chiral zero-rest-mass field equation

$$\partial^{AB} H_{BC} = 0 \,. \tag{6.41}$$

Similarly, the gauge freedom in defining the relative connection form $B_{\pi_1} \mapsto B_{\pi_1} + \mathrm{d}_{\pi_1}\Lambda_{\pi_1}$, for

$$\Lambda_{\pi_1} := 2\bar{e}_{[A}\lambda_{B]}\Lambda^{AB} \ \in \ H^0(\mathsf{F}^9, \Omega^1_{\pi_1}) \,, \tag{6.42}$$

corresponds to the familiar space-time gauge transformations for two-form potentials

$$B_A^{\ B} \mapsto B_A^{\ B} + \partial^{BC}\Lambda_{CA} - \partial_{AC}\Lambda^{CB} \,, \tag{6.43}$$

with space-time gauge parameter $\Lambda_{AB} = \frac{1}{2}\varepsilon_{ABCD}\Lambda^{CD}$. Hence, via the Penrose–Ward transform (6.36), M^6-trivial holomorphic one-gerbes over twistor space are bijectively mapped to Abelian 2-form potentials with self-dual 3-form field strength on space-time.

6.5.2. Non-Abelian fields

The previous result can be extended to describe non-Abelian fields. Since the Abelian tensor multiplet in six dimensions contains a 2-form gauge potential described by a $U(1)$-gerbe, it is natural to expect that the non-Abelian case requires for a non-Abelian generalisation of gerbes[1], see e.g. [21, 22, 262]. The cohomology group $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$ and its identification with the moduli space of solutions to certain field equations, was generalised to the cohomology set of principal $\mathscr{G}$-bundles for $\mathscr{G}$ a Lie quasi-group. This, in turn, can be understood as a

[1]Recall that these objects, unlike their Abelian counterparts, are not completely understood. In particular, in their usual formulation their connections are locally gauge equivalent to Abelian ones, rendering the construction of local non-trivially interacting Lagrangians 'problematic'. The interested reader may consult [25, 260, 7, 261], where solutions to this problems are provided, making use of the so-called *string structures*.

direct generalisation of the Penrose–Ward transform to higher principal bundles. Explicitly, a Penrose–Ward transform have been established between certain non-Abelian self-dual tensor fields in six dimensions[1] and holomorphic principal $\mathscr{G}$-bundles over P^6, for $\mathscr{G}$ a strict Lie 2-group [100], a semistrict Lie 2-group [102] and a Lie 3-group [101]. Finally, in [103], this has been generalised to Lie quasi-groupoids, providing perhaps the most general and concise framework for formulating self-dual non-Abelian tensor field theories within twistorial higher gauge theory. The interested reader is invited to consult Appendix D, where an introduction to (Lie) quasi groupoids by using the language of simplicial geometry is provided. Moreover, a formulation of higher principal bundles having such groupoids as their structure groups is given. In particular, we shall consider some reduced (Lie) quasi groupoids, called *Lie quasi k-groups*[2] [140, 264, 265]. The correspondence in question is provided by the following result [103]:

Theorem 6.3. *Let $\mathscr{G}$ be a Lie k-quasi-group and consider the double fibration* (6.15)*. There are bijections between*

(i) equivalence classes of M^6-trivial principal $\mathscr{G}$-bundles over P^6,

(ii) equivalence classes of holomorphically trivial holomorphic principal $\mathscr{G}$-bundles over F^9 that are equipped with a flat holomorphic relative connective structure and

(iii) gauge equivalence classes of complex holomorphic solutions to constraint equations on M^6 containing a 3-form curvature H that is self-dual, that is, $H^{AB} = 0$.

For concreteness, we specialise ourselves to the case $k = 2$, that is the one analysed in [111]. A detailed proof of the theorem can be found in [103]. Let $\mathscr{G}$ denote a Lie 2-quasi-group. The associated L_∞-algebra (L, μ_i), with graded vector space $\mathsf{L} = \mathsf{L}_{-1} \oplus \mathsf{L}_0$ and higher products μ_i, is obtained by computing the 1-jet of $\mathscr{G}$ (see Appendix E). A holomorphic principal $\mathscr{G}$-bundle over P^6, subordinate to a Stein open cover $\hat{\mathfrak{U}} := \cup_{a\in A}\hat{U}_a$, is characterised by a holomorphic simplicial map[3] $\hat{g} : N(\check{\mathscr{C}}(\hat{\mathfrak{U}} \to \mathsf{P}^6)) \to \mathscr{G}$ [266, 263]. Concretely, it comes with transition functions $\hat{g}_{ab} : \hat{U}_a \cap \hat{U}_b \to \mathscr{G}_1$ and $\hat{g}_{abc} : \hat{U}_a \cap \hat{U}_b \cap \hat{U}_c \to \mathscr{G}_2$, subject to the

[1] More generally, the correspondence also holds for some superconformal non-Abelian self-dual tensor field theories once one considers twistor space $\mathsf{P}^{6|2n}$ of chiral superspace $\mathsf{M}^{6|2n}$, for $n = 0, 1, 2$ [100–103]. We shall limit ourselves to the bosonic case.

[2] We just recall here that the categories of Lie quasi-groups and Lie simplicial groups are equivalent, as was shown in Nikolaus *et al.* [263, Proposition 3.35].

[3] Here, $N(\check{\mathscr{C}}(\mathsf{Y} \to \mathsf{X}))$ denotes the nerve of the Čech groupoid $\check{\mathscr{C}}(\mathsf{Y} \to \mathsf{X})$, known also as the Čech nerve. See definition (D.32).

certain simplicial identities. In Appendix D we list the cocycle and coboundary conditions in the case of a strict Lie 2-quasi-group. The pull-back bundle on correspondence space is the holomorphic simplicial map $g' = \pi_1^*(\hat{g}) : N(\check{\mathscr{C}}(\mathfrak{U}' \to \mathsf{F}^9)) \to N(\mathscr{G})$, where $\mathfrak{U}'$ is the induced open cover on F^9. The component maps of g' are annihilated by the twistor distribution, that is $\mathrm{d}_{\pi_1} g'_{ab} = \mathrm{d}_{\pi_1} g'_{abc} = 0$, by the definition of a pull-back.

Relative forms. Furthermore, since the pull-back bundle is trivial[1] (topologically as well as holomorphically) on all of F^9, we may introduce a globally defined holomorphic relative connective structure on such bundle

$$(A_{\pi_1}, B_{\pi_1}) \ \in \ H^0(\mathsf{F}^9, \Omega^1_{\pi_1} \otimes \mathsf{L}_0) \oplus H^0(\mathsf{F}^9, \Omega^2_{\pi_1} \otimes \mathsf{L}_{-1}) \ , \tag{6.44}$$

which is constrained to be flat. Explicitly, the relative curvatures defined by

$$\begin{aligned} F_{\pi_1} &:= \mathrm{d}_{\pi_1} A_{\pi_1} + \tfrac{1}{2}\mu_2(A_{\pi_1}, A_{\pi_1}) \ , \\ H_{\pi_1} &:= \mathrm{d}_{\pi_1} B_{\pi_1} + \mu_2(A_{\pi_1}, B_{\pi_1}) - \frac{1}{3!}\mu_3(A_{\pi_1}, A_{\pi_1}, A_{\pi_1}) \end{aligned} \tag{6.45}$$

satisfy the conditions

$$\mathcal{F}_{\pi_1} := F_{\pi_1} + \mu_1(B_{\pi_1}) = 0 \quad \text{and} \quad H_{\pi_1} = 0 \ . \tag{6.46}$$

If we now recall our discussion in Chapter 3, we immediately notice that these equations are nothing but the equations (3.32c) and we recognise $\mathcal{F}_{\pi_1}$ as the relative 2-form fake curvature. As final step one has to push the relative connective structure (6.44) down to space-time. Concretely, taking the direct images of $\Omega^p_{\pi_1}$ under the projection π_2 we may expand relative connective structure (A_{π_1}, B_{π_1}) as [97][2]

$$\begin{aligned} A_{\pi_1} &= 2\bar{e}_{[A}\lambda_{B]}A^{AB} \ , \\ B_{\pi_1} &= -\bar{e}_A \wedge \bar{e}_B \lambda_C \varepsilon^{ABCD} B_D{}^E \lambda_E \ , \end{aligned} \tag{6.47}$$

while the corresponding relative curvatures (F_{π_1}, H_{π_1}) read

$$\begin{aligned} F_{\pi_1} &= -\bar{e}_A \wedge \bar{e}_B \lambda_C \varepsilon^{ABCD} F_D{}^E \lambda_E \ , \\ H_{\pi_1} &= \frac{2}{3}\bar{e}_A \wedge \bar{e}_B \wedge \bar{e}_C \lambda_D \varepsilon^{ABCD} H^{EF} \lambda_E \lambda_F \ . \end{aligned} \tag{6.48}$$

[1] Thus, there exists a coboundary transformation trivialising g'. See [103] for details.

[2] Here, the numerical coefficients are chosen for further convenience. These calculations are suited for the discussion in [111].

Since all λ-dependence has been made explicit in the above expansions, the component fields in (6.47) and (6.48) are fields defined on the space-time M^6. Hence, we recall that $A^{AB} = \frac{1}{2}\varepsilon^{ABCD}A_{CD}$, while $B_A{}^B$ and $F_A{}^B$ are trace-less and $H^{AB} = H^{BA}$. We stress here that these expansions reflect the property that the differential forms $\bar{e}_A$ are defined modulo terms proportional to λ_A. The conditions

$$\mathcal{F}_{\pi_1} = 0 \quad \text{and} \quad H_{\pi_1} = 0 \tag{6.49}$$

determine some constraint equations on space-time. Explicitly, upon using $\mathrm{d}_{\pi_1} = 2\bar{e}_A\overline{V}^A$ and the expansions (6.47)-(6.48), we arrive after a few algebraic manipulations at the equations

$$\mathcal{F}_A{}^B = 0 \quad \text{and} \quad H^{AB} = 0 \tag{6.50}$$

on M^6. Here, the component fake curvature $\mathcal{F}$ and the anti-self-dual part of the three-form curvature H on space-time are given explicitly by

$$\begin{aligned} \mathcal{F}_A{}^B &:= \partial^{BC}A_{CA} - \partial_{CA}A^{BC} + \mu_2(A^{BC}, A_{CA}) + \mu_1(B_A{}^B) \ , \\ H^{AB} &:= \partial^{C(A}B_C{}^{B)} + \mu_2(A^{C(A}, B_C{}^{B)}) + \mu_3(A^{C(A}, A_{CD}, A^{B)D}) \ . \end{aligned} \tag{6.51}$$

Summarising, the theorem 6.3 tells us that the moduli space of solution to the equations of motion[1]

$$\mathcal{F} = 0 \quad \text{and} \quad H = \star H \tag{6.52}$$

on $U \subseteq \mathsf{M}^6$, with H and $\mathcal{F}$ given by (3.32c), is equivalent to the moduli space of holomorphic principal $\mathscr{G}$-bundles over $\hat{U} \subseteq \mathsf{P}^6$ which are homomorphically trivial when restricted to $\pi_1(\pi_2^{-1}(x)) \hookrightarrow \mathsf{P}^6$, for all $x \in \mathsf{M}^6$ [100–103]. In the next Chapter we shall see how to lift such a correspondence off-shell.

[1] For the explicit gauge transformations see e.g. [100, 101].

7

Self-dual actions from twistor space

7.1. Motivation

As we have seen in the last Chapter, self-dual higher gauge theory in six dimensions can be formulated by using categorified principal bundles in the framework of twistor geometry. Particularly interesting is that a Lagrangian description of the theory exists in this setting. Twistor space actions on P^6 for six-dimensional Abelian self-dual tensor fields has been constructed [97–99]. It appears rather natural to wonder whether these actions are equivalent to the actions describing self-dual three-forms in six dimensional space-time, such as the PST action [104–107] or Sen's action [108, 109]. Along this Chapter we shall make some progress in this direction. By imposing appropriate reality conditions and partially fixing a gauge we shall see how the twistor action reduces to its six-dimensional space-time counterpart. We believe that different space-time actions might be obtained by simple operations at the level of twistor space, such as redefining fields or imposing different gauge-fixings. This idea has been reinforced by a recent result [159], where it was shown how Sen's action can be naturally interpreted by using the language of homotopy Maurer–Cartan theory, developed along the Thesis. We shall come back to these considerations again in Chapter 8.

In addition, a non-Abelian generalisation of these twistor actions has been proposed in [43] in terms of higher holomorphic Chern–Simons theory. At least at the classical level, via the Penrose–Ward transform 6.3, such a theory reproduces non-Abelian self-dual three form fields on six-dimensional flat space-time. In Section 7.4.2. we shall analyse how such an action descends on Euclidean space-time. The computations in this Chapter are based on the ongoing work [111]. Moreover, Sen's formalism has been recently used [267] to construct a Lagrangian whose equations of motion are the one given in [268] for the non-Abelian six-dimensional $(2,0)$ tensor multiplet. This may suggest us that a homotopy algebraic analysis on twistor space could be useful also in the non-Abelian case. Along this Chapter, we shall lay the basis for this kind of discussions.

7.2. Self-dual actions on six-dimensional space-time

For the sake of completeness we give a brief overview to the problem of constructing a variational principle for the dynamics of chiral p-forms.

Introduction. p-form gauge potentials naturally appear in theories of fundamental extended objects. Of particular interest are p-forms whose $(p+1)$-form field strength is constrained to be self-dual[1]. These fields, usually called *chiral p-forms*, define the so-called self-dual gauge field theory or chiral gauge field theory. Such theories have been studied for quite a long time because of their appearence in various interesting contexts. For instance, they play a central role in dealing with six-dimensional and ten-dimensional type IIB supergravity, they show up in heterotic strings as well as in M-theory-five-branes. Unfortunately, the analysis of the dynamics of chiral fields is a long-standing problem, due to the absence of a natural action principle. This makes even the construction of a Lagrangian describing the low-energy dynamics of a single M5-brane a non-trivial problem. The main difficulty lies in the implementation of both Lorentz invariance and self-duality conditions. Rather extensive literature has been devoted to the problem of writing down an action for theories of $2n$-form fields in $4n+2$ dimensions. The initial approach consisted in imposing the self-duality constraint by hand, after deriving the equations of motion from the action. Various alternative formulations have been suggested, where self-duality condition arises on-shell. However, any of these option present some trade-offs. In fact, many of these theories break manifest Lorentz invariance[2] [270–272] while others introduce an infinite number of (typically massless) auxiliary fields, [273–280] or require to add an extra dimension [281, 282].

[1]This reduces the physical degrees of freedom in a theory to half of the case without the self-duality condition.

[2]The fact that non-manifestly Lorentz-invariant actions may yield to Lorentz-invariant dynamics was first realised by Floreanini & Jackiw [269].

Pasti–Sorokin–Tonin formalism. There exists a further, more economic and efficient approach to this problem, called the Pasti-Sorokin-Tonin (PST) formalism [104, 105, 283, 106, 107]. This method provides a manifestly Lorentz-invariant formulation of self-dual gauge field theory. This is realised by introducing a single scalar auxiliary field that, however, enters the action in a non-polynomial way (only through its derivatives).[1] It is naturally compatible with supersymmetry and kappa symmetry, with the coupling to gravity, and with the functional integral approach. Moreover, the non-dynamical nature of the auxiliary field is ensured by gauge fixing one of the extra symmetries of the theory (shift symmetry).

7.2.1. Sen's mechanism

Inspired by string field theory considerations, Sen [108, 109], see also [290], has recently proposed an alternative method to incorporate self-duality by an action principle. This approach relies on an action for self-dual Abelian fields in $4n+2$ dimensions involving a second auxiliary $2n$-form field, which eventually decouples from the physical degrees of freedom. In this approach, the self-duality condition holds off-shell. Such a construction has the feature that the coupling to gravity is somewhat non-standard, so that diffeomorphisms act differently from usual[2]. Hence, the action is not manifestly invariant under general diffeomorphisms. Furthermore, Lambert [291] has shown how to implement supersymmetry within this method, constructing the action of a $(2,0)$ tensor multiplet in six dimensions.[3]

Keeping in mind our motivation, we shall review the action proposed in [108, 109] in some detail. For definiteness, we shall consider the case of a chiral two-form in six dimensions. Let M^6 be six-dimensional space-time, coordinatised by x^M, $M = 1, \ldots, 6$. Given a two-form field B and a three form field H subject to the self-duality condition $H = \star_\eta H$, the Sen action reads[4]

$$S_{\mathsf{Sen}} = \int_{M^6} \left[\frac{1}{4} \mathrm{d}B \wedge \star_\eta \mathrm{d}B - H \wedge \mathrm{d}B + \mathcal{L}_{\mathsf{int}}(g, H) \right] , \tag{7.1}$$

where $\mathcal{L}_{\mathsf{int}}(g, H)$ is some Lagrangian density which denotes the coupling of H to the metric

[1] See also [284–288] for others actions of the same kind. See also [289] for a recent variant of the PST method.

[2] See [267] for a detailed analysis of this aspect.

[3] Recently, an action for describing the dynamics of an M5-brane in the eleven-dimensional supergravity background was proposed [292].

[4] In writing the action (7.1) we are not following the original notation of Sen, but the one of [291], to be closer to the standard $(2,0)$ literature.

field g and to external sources. For our purposes, we are not interested in the matter field part of the Lagrangian. Here, the notation points out that, although in Sen's approach the background is generically curved, the Hodge star operator $\star_\eta$ is defined with respect to the flat (Minkowski) metric.[1] Varying the action (7.1) one finds that the equations of motion for H and B are given by

$$\begin{aligned} \mathrm{d}\Big(\frac{1}{2}\star_\eta \mathrm{d}B + H\Big) = 0 \ , \\ \frac{1}{2}(\mathrm{d}B - \star_\eta \mathrm{d}B) + R = 0 \ , \end{aligned} \tag{7.2}$$

where R is defined by

$$\delta_H \mathcal{L}_{\text{int}} := -\int \delta H \wedge R \ . \tag{7.3}$$

Note that the quantity R is a $\star_\eta$ anti self-dual 3-form, by construction. It follows that there are two self-dual three forms: H and $\frac{1}{2}(\mathrm{d}B + \star_\eta \mathrm{d}B)$. The combination

$$H_s := H + \frac{1}{2}(\mathrm{d}B + \star_\eta \mathrm{d}B) \tag{7.4}$$

is free by the first equation of motion and does not contain physical degrees of freedom, due to the wrong sign of its kinetic term[2]. Sen considered an interaction term of the form

$$\mathcal{L}_{\text{int}} := H \wedge \Phi(H) \ , \tag{7.5}$$

where Φ is a linear map from self-dual three-forms to anti-self-dual three forms, that is,

$$\Phi(H) := -\star_\eta \Phi(H) \ . \tag{7.6}$$

This is determined by the following requirements, which are all satisfied at the level of action itself:

- Φ annihilates anti-self-dual forms

$$\Phi(H') = 0 \ , \quad \text{for} \quad H' = -\star_\eta H' \tag{7.7}$$

- It is symmetric in the sense that the condition

$$H_1 \wedge \Phi(H_2) = H_2 \wedge \Phi(H_1) \tag{7.8}$$

  holds for any two self-dual three-forms H_1, H_2.

[1]Similarly to the PST models, Sen's actions are available only in spaces of Minkowski signature.

[2]Moreover, after extending the theory to an abelian $(2,0)$ theory, the field H_s is a singlet under supersymmetry transformations [291].

- Moreover, for a $\star_\eta$-self-dual form H, the following linear combination is a self-dual form on (M, g)
$$H - \Phi(H) = \star_g (H - \Phi(H)) \; . \tag{7.9}$$

There exists an explicit construction of a map with the above properties, which can be found in [108, 109, 267]. However, one may immediately notice that the map Φ is necessarily aware of the background metric, even though the (pseudo-)forms themselves are not.

Equations (7.2), now read

$$\begin{aligned} \mathrm{d}\left(\frac{1}{2}\star_\eta \mathrm{d}B + H\right) &= 0 \; , \\ (1 - \star_\eta)\left(\mathrm{d}B - \Phi(H)\right) &= 0 \; . \end{aligned} \tag{7.10}$$

Moreover, in terms of the field

$$H_g := H - \Phi(H) \; , \tag{7.11}$$

the equations of motion can be recast into

$$\mathrm{d}H_s = 0 \quad \text{and} \quad \mathrm{d}H_g = 0 \; . \tag{7.12}$$

Notice that (7.9) can be rewritten as $H_g = \star_g H_g$. The field H_g describes the physics of the desired free chiral two-form on (M^6, g).[1] On the other hand, the field H_s is also free, but decouples, at least at the level of equations of motion, from the physical degree of freedom carried by H_g. A detailed analysis[2] then shows [109] that this decoupling holds also at the quantum level. Moreover, H_s also decouples from gravity (recall that only Φ "sees" g). Hence, it is completely decoupled from the physical sector and can be properly be considered as an auxiliary field (it is physically inaccessible). However, there is a minor drawback. One may notice that the presence of the Hodge star operator $\star_\eta$ already suggests that the action (7.2) is not manifestly diffeomorphism invariant. It was shown [108, 267], that the fields B and H do not behave in standard way under diffeomorphisms[3] and the non-standard transformations

[1] Hence, it is self-dual with respect (physical) metric g. It also possesses on-shell the standard transformation properties under diffeomorphisms [267].

[2] This is done in the Hamiltonian formulation, where the fully quantum mechanical analysis of the action is performed. It carries the degrees of field Π^+ in Sen's formulation [109]. For our purposes, it is sufficient to consider the classical aspects of this action.

[3] They are not standard differential forms and sometimes dubbed as pseudo-forms. The analysis of (7.2) under diffeomeorphisms is performed in [267].

which leave these fields invariant are determined. However, one can verify that the action is indeed invariant under general coordinate transformations, as it should be.

A better understanding of the mathematical origin of Sen's mechanism has been recently given in [159], adopting the language of homotopy Maurer–Cartan theory. In particular, by using that quasi-isomorphisms of L_∞-algebras define classically equivalent field theories, the unphysical field H_s has been identified as pure gauge, also in presence of coupling to matter fields.

7.3. Twistor action: Abelian case

In this Section, we shall consider the twistor space action for Abelian self-dual tensor fields as in [97–99] and we show that it reduces to its six-dimensional space-time counterpart on $\mathbb{R}^6$.

7.3.1. Action and equations of motion

In the previous Chapter we have seen how both the Penrose and the Penrose–Ward transforms identify self-dual tensor fields on M^6 with certain cohomology groups on twistor space P^6. These two isomorphisms, that we recall here for convenience

$$H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4)) \cong H^0(U, \mathcal{Z}_h) \cong H^2(\hat{U}, \mathcal{O}_{\hat{U}}(2h-2)) \ , \tag{7.13}$$

allow[1] for a twistor space action for chiral zero-rest-mass fields. In order to construct such an action, we must allow the fields to go off-shell. This is most straightforwardly done in the Dolbeault setting. Hence, we consider the Dolbeault representatives of the cohomology groups $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ and $H^2(\hat{U}, \mathcal{O}_{\hat{U}}(2h-2))$ and we lift them to off-shell fields on twistor space, assuming they are not holomorphic. Then, the twistor action takes the form [97–99]

$$S = \int_{\hat{U}} \Omega^{6,0} \wedge B^{0,2}_{2h-2} \wedge \bar{\partial} C^{0,3}_{-2h-4} \ , \tag{7.14}$$

for the fields $B^{0,2}_{2h-2} \in \Omega^{0,2}(\hat{U}, \mathcal{O}_{\hat{U}}(2h-2))$ and $C^{0,3}_{-2h-4} \in \Omega^{0,3}(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$, where the subscript denote the respective homogeneity. Here, $\Omega^{6,0}$ is the top degree holomorphic form of weight $(6,0)$ on P^6, defined as the contour integral[2]

$$\Omega^{6,0} := \frac{1}{2\pi i} \oint_{\mathscr{C}} \frac{\Omega^{4,0}(z) \wedge \Omega^{3,0}(\lambda)}{z^A \lambda_A} \ , \tag{7.15}$$

[1] We stress that this is a peculiar feature of the six-dimensional case.

[2] The contour integral localising the measure to the quadric $\mathsf{P}^6 \subset \mathbb{P}^7 \setminus \mathbb{P}^3$ can be written equivalently in terms of a holomorphic delta function as $\Omega^{6,0} = \int \Omega^{4,0}(z) \wedge \Omega^{3,0}(\lambda)\, \bar{\delta}(z \cdot \lambda)$, see e.g. [98, 293]

where $\mathscr{C}$ is any contour encircling P^6 inside $\mathcal{O}_{\mathbb{P}^3}(1)\otimes\mathbb{C}^4 \to \mathbb{P}^3$, while $\Omega^{3,0}(\lambda)$ is given by (6.31) and

$$\Omega^{4,0}(z) := \frac{1}{4!}\varepsilon_{ABCD}\mathrm{d}z^A \wedge \mathrm{d}z^B \wedge \mathrm{d}z^C \wedge \mathrm{d}z^D \ . \tag{7.16}$$

We observe that the action (7.14) is globally defined as the respective weights cancel out. We shall focus on the case $h = 1$, that is, the action for the differential forms $B^{0,2} \in \Omega^{0,2}(\hat{U}, \mathcal{O}_{\hat{U}})$ and $C^{0,3}_{-6} \in \Omega^{0,3}(\hat{U}, \mathcal{O}_{\hat{U}}(-6))$, of holomorphic weight 0 and -6, respectively. Such action is invariant under the obvious (higher) gauge transformations

$$\begin{gathered} B^{0,2} \to B^{0,2} + \bar\partial\Lambda^{0,1} \ , \qquad \Lambda^{0,1} \to \Lambda^{0,1} + \bar\partial f \ , \\ C^{0,3} \to C^{0,3}_{-6} + \bar\partial\Xi^{0,2} \ , \qquad \Xi^{0,2} \to \Xi^{0,2} + \bar\partial\Upsilon^{0,1} \ , \qquad \Upsilon^{0,1} \to \Upsilon^{0,1} + \bar\partial g \ , \end{gathered} \tag{7.17}$$

where the gauge parameters are differential forms on twistor space of appropriate weight. Explicitly, $\Lambda^{0,1} \in \Omega^{0,1}(\hat{U}, \mathcal{O}_{\hat{U}})$, $f \in C^\infty(\hat{U}, \mathcal{O}_{\hat{U}})$ and $\Xi^{0,2} \in \Omega^{0,2}(\hat{U}, \mathcal{O}_{\hat{U}}(-6))$, $\Upsilon^{0,1} \in \Omega^{0,1}(\hat{U}, \mathcal{O}_{\hat{U}}(-6))$, $g \in C^\infty(\hat{U}, \mathcal{O}_{\hat{U}}(-6))$. The Euler–Lagrange equations arising from the action (7.14) imply the holomorphicity conditions

$$\bar\partial B^{0,2} = 0 = \bar\partial C^{0,3}_{-6} \ . \tag{7.18}$$

Consequently, by the Čech–Dolbeault correspondence, the differential forms $B^{0,2}$, $C^{0,3}_{-6}$ correspond, on-shell, to representatives of the Čech cohomology groups $H^2(\hat{U}, \mathcal{O}_{\hat{U}})$ and $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-6))$, respectively. Thus, they describe self-dual three-form fields on space-time M^6. Notice that the action for $h = 1$ may be considered as the six-dimensional counterpart to holomorphic Chern–Simons theory (3.38) in the twistor description of self-dual fields on four-dimensional space-time.

7.3.2. Equivalence to space-time action

Here, we explicitly show the off-shell equivalence of the action

$$S = \int_{\hat{U}} \Omega^{6,0} \wedge B^{0,2} \wedge \bar\partial C^{0,3}_{-6} \tag{7.19}$$

with its space-time counterpart on $\mathbb{R}^6$. To validate this claim, we need a three step procedure. First, one needs to expand the field content $(B^{0,2}, C^{0,3}_{-6})$ along the basis (6.24) adapted to the description $\mathrm{P}^6 \cong \mathbb{R}^6 \times \mathbb{P}^3$. Subsequently, one has to (partially) gauge-fix the fields to remove the extra symmetry beyond space-time gauge group. Finally, the space-time action is obtained by integrating along the fibres of the twistor fibration. We shall be deliberately pedantic here as the following calculations also apply to the computationally more involved non-Abelian case. This derivation follows that of the twistor action for Yang-Mills theory in four-dimensions [84, 85, 95] and shares many of its features.[1]

Field expansions. We start by picking Euclidean signature, as discussed in 6.3.2.. Along the line of (6.26), the $(0,3)$-form field $C^{0,3}_{-6}$ admits the expansion

$$\begin{aligned} C^{0,3}_{-6} &= \varepsilon_{ABCD}\frac{\hat\lambda^A \mathrm{d}\hat\lambda^B\wedge \mathrm{d}\hat\lambda^C\wedge \mathrm{d}\hat\lambda^D}{[\lambda\hat\lambda]^4}\,C_0 + \varepsilon_{CDEF}\frac{\bar e_{[A}\lambda_{B]}\wedge\hat\lambda^C \mathrm{d}\hat\lambda^D\wedge \mathrm{d}\hat\lambda^E}{[\lambda\hat\lambda]^3}\,\widetilde C^{AB,F}\\ &\quad + \varepsilon^{ABCD}\frac{\bar e_A\wedge\bar e_B\lambda_C\wedge\hat\lambda^{[E}\mathrm{d}\hat\lambda^{F]}}{[\lambda\hat\lambda]^2}\,\mathring{C}_{D,EF} + \varepsilon^{ABCD}\bar e_A\wedge\bar e_B\wedge\bar e_C\lambda_D\,\widehat C\ . \end{aligned} \tag{7.20}$$

Here, the coefficient fields are smooth functions of $(x,\lambda,\hat\lambda)$ of appropriate homogeneity as shown in the table below. In particular, $\widetilde C^{AB,F} = -\widetilde C^{BA,F}$ and $\mathring{C}_{D,EF} = -\mathring{C}_{D,FE}$. Analogously, for the $(0,2)$-form field $B^{0,2}$ we may write

$$\begin{aligned} B^{0,2} &= \varepsilon^{ABCD}\bar e_A\wedge\bar e_B\lambda_C B_D + \frac{\bar e_{[A}\lambda_{B]}\wedge\hat\lambda^{[C}\mathrm{d}\hat\lambda^{D]}}{[\lambda\hat\lambda]^2}\mathring{B}_{CD}{}^{AB}\\ &\quad+\varepsilon_{ABCD}\frac{\hat\lambda^A\mathrm{d}\hat\lambda^B\wedge\mathrm{d}\hat\lambda^C}{[\lambda\hat\lambda]^3}\widehat B^D\ , \end{aligned} \tag{7.21}$$

where $\mathring{B}_{CD}{}^{AB}$ is anti-symmetric in both pairs of indices and again the coefficient fields depend on $(x,\lambda,\hat\lambda)$. Moreover, we recall the expression (6.28) for the Dolbeault operator.

	B_A	$\mathring{B}_{[CD]}{}^{AB}$	$\widehat B^A$	C_0	$\widetilde C^{AB,C}$	$\mathring{C}_{A,BC}$	$\widehat C$
Homogeneity	$(1,0)$	$(2,0)$	$(3,0)$	$(-2,0)$	(-3.0)	(-3.0)	(-4.0)

Figure 7.1: Coefficient fields weights

Gauge fixing. We immediately notice that the coefficient field $\hat B^D$ in (7.21) may be set to zero thanks to cohomological considerations. Recall that, here, $B^{0,2}$ is part of the connective structure of a complex (but not necessarily holomorphic) one-gerbe $\Gamma\to \mathsf{P}^6$, which is homomorphically trivial upon restriction to submanifolds $\mathsf{Y}\hookrightarrow \mathsf{P}^6$. Such a condition amounts to considering the connection $(0,2)$-form field $B^{0,2}$ as being in a gauge in which the restrictions $B^{0,2}|_{\mathsf{Y}}$ vanish. Note that a similar argument does not apply to the field $C^{0,3}_{-6}$, because of its

[1] See also [94] where a twistor space action for Yang–Mill–Higgs theory in three dimensions is discussed.

different nature. In fact, it comes from the Penrose transform 6.1 and it carries an holomorphic weight.[1]

In addition, following [259, 85], we may exploit the gauge redundancy of the action (7.14) by restricting to a gauge in which

$$\bar{\partial}^{\dagger}_{\mathsf{Y}} B^{0,2}|_{\mathsf{Y}} = 0 \ , \qquad \bar{\partial}^{\dagger}_{\mathsf{Y}} C^{0,3}_{-6}|_{\mathsf{Y}} = 0 \ . \tag{7.22}$$

This gauge will be referred to as *space-time gauge*.[2] Here, $\bar{\partial}_{\mathsf{Y}}$ is the $\bar{\partial}$-operator restricted to all the fibres $\mathsf{Y} \cong \mathbb{P}^3$ and $\bar{\partial}^{\dagger}_{\mathsf{Y}} = - * \bar{\partial}_{\mathsf{Y}} *$ denotes the adjoint operator of $\bar{\partial}_{\mathsf{Y}}$, with respect to the Fubini-Study metric on $\mathbb{P}^3$. In other words, we require the components of the field content along $\mathbb{P}^3$ to be fiberwise coclosed on each $\mathsf{Y}_x := \pi_1(\pi_2^{-1}(x)) \hookrightarrow \mathsf{P}^6$, corresponding to a space-time point $x \in \mathsf{M}^6$. This gauge is not a complete gauge fixing on Euclidean twistor space. Hence, there is still some residual (higher) gauge freedom. In particular transformations of the form

$$\bar{\partial}^{\dagger}_{\mathsf{Y}} \bar{\partial}_{\mathsf{Y}} \Lambda^{0,1}|_{\mathsf{Y}} = 0 = \bar{\partial}^{\dagger}_{\mathsf{Y}} \bar{\partial}_{\mathsf{Y}} \Xi^{0,2}|_{\mathsf{Y}} \tag{7.23}$$

for all fibres Y, and analogous conditions for all the (higher) gauge parameters, leave (7.22) unchanged. Let us consider the connection form $B^{0,2}$. On top of (7.22) we also require similar conditions to apply for the gauge parameters, that is

$$\bar{\partial}^{\dagger}_{\mathsf{Y}} \Lambda^{0,1}|_{\mathsf{Y}} = 0 \tag{7.24}$$

together with the requirement

$$\frac{\partial}{\partial \hat{\lambda}} \lrcorner \, \bar{\partial} \Lambda^{0,1} = 0 \ . \tag{7.25}$$

This determines for the field $\Lambda^{0,1}$ an expansion of the form

$$\Lambda^{0,1} = \bar{e}_{[A} \lambda_{B]} \Lambda^{AB} \ , \tag{7.26}$$

where Λ^{AB} denotes a space-time parameter. Residual gauge transformations with parameter f obey

$$\bar{\partial}^{\dagger}_{\mathsf{Y}} \bar{\partial}_{\mathsf{Y}} f(x, \lambda, \hat{\lambda}) = 0 \ . \tag{7.27}$$

This tells that f is harmonic along the fibres and $f(x, \lambda, \hat{\lambda}) = f(x)$ by the maximum modulus principle. Hence, in space-time gauge (7.22) on P^6, the residual gauge freedom reduces

[1] It does not admit a potential formulation.

[2] Sometimes it is also called *Woodhouse gauge* or harmonic gauge.

precisely to that of ordinary gauge transformations on $\mathbb{R}^6$.[1]

Because of trivial dimensional reasons ($\dim_{\mathbb{C}} \mathsf{Y} = 3$) the condition (7.22) tells us that in this gauge the restriction $C^{0,3}_{-6}|_{\mathsf{Y}}$ is required to be harmonic along the fibres of $\mathsf{P}^6 \to \mathbb{R}^6$. Moreover, since $\mathbb{P}^3$ is compact, the Hodge and Dolbeault theorems[2] imply that the restricted component field $C_0|_{\mathsf{Y}}$ is fixed in space-time gauge to be a harmonic representative of $H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-6))$. Computing the Čech cohomology group $H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-6))$ gives the index structure of C_0, which turns out to have ten symmetric space-time indices [97]. Thus, the field C_0 takes the form

$$C_0(x, \lambda, \hat{\lambda}) = \alpha_1 C_{AB} \frac{\hat{\lambda}^A \hat{\lambda}^B}{[\lambda\hat{\lambda}]^2} \ , \tag{7.28}$$

where C_{AB} is a symmetric field depending only on the space-time coordinates x^{AB} and the numerical coefficient α_1 is inserted for future convenience. However, the other components of the field $C^{0,3}_{-6}$ are not constrained by our gauge condition.[3]

Integration along the fibres. To integrate, we first observe that the holomorphic volume form (7.15) on Euclidean twistor space takes the form

$$\Omega^{6,0} = 2\pi i \frac{[\lambda\hat{\lambda}]^2}{4!4!} \varepsilon_{ABCD} e^A \wedge e^B \wedge e^C \hat{\lambda}^D \wedge \lambda_E \mathrm{d}\lambda_F \wedge \mathrm{d}\lambda_G \wedge \mathrm{d}\lambda_H \varepsilon^{EFGH} \ , \tag{7.29}$$

in terms of the complex conjugates of the differential forms in (6.24).[4] This is obtained by performing the contour integration as shown in Appendix F. Moreover, we recall that the Fubini-Study metric ω_{FS} on $\mathbb{P}^3$ in Euclidean signature has a canonical form, induced by spinor conjugation, and its Kähler form reads

$$K = \frac{1}{4!} \frac{1}{4!} \varepsilon^{ABCD} \frac{\lambda_A \mathrm{d}\lambda_B \wedge \mathrm{d}\lambda_C \wedge \mathrm{d}\lambda_D \wedge \hat{\lambda}^E \mathrm{d}\hat{\lambda}^F \wedge \mathrm{d}\hat{\lambda}^G \wedge \mathrm{d}\hat{\lambda}^H}{[\lambda\hat{\lambda}]^4} \varepsilon_{EFGH} \ . \tag{7.30}$$

[1]Moreover, by using the Dolbeault representation (6.32) for $h = 1$ one can show that the gauge parameter g in (7.17) is a holomorphic function of homogeneity $(-6, 0)$ and, hence, must vanish. This argument shall be elaborated in more detail in [111]. See also Appendix E for some related geometric notion.

[2]See e.g. [294].

[3]We just recall here that in Euclidean signature the Penrose transform provides a self-dual three form on space-time via the Woodhouse construction [259] as shown in Appendix F.

[4]Explicitly, we have defined $e^A := \frac{\lambda_B \mathrm{d}x^{AB}}{[\lambda\hat{\lambda}]}$.

The action (7.19) takes the form

$$S = \frac{2}{3}\beta\int_{U\times\mathbb{P}^3} \mathrm{d}^6x \wedge K\Bigg[B_A\lambda_B\Bigg(\alpha_1\frac{\hat\lambda^C\hat\lambda^D}{[\lambda\hat\lambda]^2}\partial^{AB}C_{CD} - \frac{1}{3}\frac{\hat\lambda^C\widetilde{V}_{CD}}{[\lambda\hat\lambda]}\widetilde{C}^{AB,D}\Bigg) + \\ + 2\mathring{B}_{[AB]}^{CD}\Bigg(\varepsilon_{DEFG}\frac{\hat\lambda^A\lambda_C}{[\lambda\hat\lambda]}\overline{V}^E\widetilde{C}^{FG,B} + \varepsilon^{ABEF}\frac{\lambda_C\hat\lambda^H}{[\lambda\hat\lambda]^2}\hat\lambda^G\widetilde{V}_{GE}\mathring{C}_{D,FH}\Bigg)\Bigg] , \tag{7.31}$$

as the other components wedge to zero against $\Omega^{6,0}$. Here, we observe that the expression inside the brackets is weightless. In writing (7.31) we have used the relation[1]

$$\beta\mathrm{d}^6x = [\lambda\hat\lambda]^2\varepsilon_{ABCD}e^A\wedge e^B\wedge e^C\hat\lambda^D\wedge\bar{e}_I\wedge\bar{e}_J\wedge\bar{e}_K\lambda_L\varepsilon^{IJKL} \tag{7.32}$$

as follows from (6.24)-(6.25) and (7.29). The field $\mathring{B}_{AB}^{CD}$ appears only linearly, so it can be considered as a Lagrange multiplier field. Integrating it out enforces the second line in (7.31) to vanish. Moreover, integrating by parts the second term in the first line, enforces the condition $\hat\lambda^C\widetilde{V}_{CD}B_A = 0$. Hence,[2] the field B_A must be holomorphic in λ. Accounting for its homogeneity with respect to λ (and $\hat\lambda$) this constraint necessary implies

$$B_A(x,\lambda,\hat\lambda) = B_A{}^B\lambda_B , \tag{7.33}$$

The field B_A^B is trace-less and depend only on x^{AB}. This provides the required object on space-time. Having solved this constraint leaves us with the following integral

$$S = \frac{2}{3}\beta\int_{U\times\mathbb{P}^3}\mathrm{d}^6x\wedge K\alpha_1\frac{\lambda_B\lambda_C\hat\lambda^I\hat\lambda^J}{[\lambda\hat\lambda]^2}B_A^C\partial^{AB}C_{IJ} . \tag{7.34}$$

Finally, we need to integrate over the $\mathbb{P}^3$-fibres of the Euclidean twistor fibration, leaving only Euclidean space integration. This is done by using the general formula

$$\int_{\mathbb{P}^3}\frac{K}{[\lambda\hat\lambda]^{2h}}\lambda_{A_1}\cdots\lambda_{A_{2h}}\hat\lambda^{B_1}\cdots\hat\lambda^{B_{2h}} = \frac{1}{\alpha_h}\delta^{B_1}_{(A_1}\cdots\delta^{B_{2h}}_{A_{2h})} , \tag{7.35}$$

where $\alpha_h := (2h+3)!/(2h)!3!$. This is just a consequence of Serre duality on projective space $\mathbb{P}^3$, as observed in Appendix F. Therefore, the twistor action (7.19) reduces to the space-time action

$$S = \frac{2}{3}\beta\int_U\mathrm{d}^6xB_A^C\partial^{AB}C_{BC} , \tag{7.36}$$

as desired.

[1] The numerical coefficient β is yet to be determined [111].

[2] In fact $\hat\lambda^C\widetilde{V}_{CD}B_A = 0$ implies $\frac{\partial}{\partial\hat\lambda^D}B_A = 0$, since $\hat\Upsilon := \hat\lambda^A\frac{\partial}{\partial\hat\lambda^A}$ is the homogeneity operator which counts the homogeneity w.r.t $\hat\lambda$.

7.3.3. An interpretation from homotopy algebras

Here, we just briefly comment on some work in progress. In order to apply the L_∞-perspective we need to employ the BV formalism at the level of twistor space.

Batalin-Vilkovisky field theory on twistor space. Recall that the classical BRST setup corresponds to a Chevalley–Eilenberg resolution taking care of the gauge equivalences. Hence, the BRST field content $\mathfrak{F}_{\mathrm{BRST}}$ on P^6 of the theory is given by the field $B^{0,2}$ and $C^{0,3}_{-6}$, together with gauge parameters defined in (7.17). We consider the differential complex[1]

$$\mathsf{L} = \left(\begin{array}{ccccccc} \overset{g}{\Omega^{0,0}_{-6}} & \xrightarrow{\bar\partial} & \overset{\Upsilon^{0,1}}{\Omega^{0,1}_{-6}} & \xrightarrow{\bar\partial} & \overset{\Xi^{0,2}}{\Omega^{0,2}_{-6}} & \xrightarrow{\bar\partial} & \overset{C^{0,3}}{\Omega^{0,3}_{-6}} \\ & & \oplus & & \oplus & & \oplus \\ & & \overset{f}{\Omega^{0,0}} & \xrightarrow{\bar\partial} & \overset{\Lambda^{0,1}}{\Omega^{0,1}} & \xrightarrow{\bar\partial} & \overset{B^{0,2}}{\Omega^{0,2}} \\ \underbrace{\qquad}_{\mathsf{L}_{-2}} & & \underbrace{\qquad}_{\mathsf{L}_{-1}} & & \underbrace{\qquad}_{\mathsf{L}_{0}} & & \underbrace{\qquad}_{\mathsf{L}_{1}} \end{array} \right) . \tag{7.37}$$

The Batalin–Vilkovisky formalism consists of doubling the field space, that is introducing the anti-fields for each field, ghost, ghost for ghosts, etc. This extends the L_∞-algebra to one that encodes the full dynamical information. Hence, the BV space of fields is given by

$$\mathfrak{F}_{\mathrm{BV}} = \{a, a^+, \Xi^{0,2}, \Xi^{0,2+}, \Upsilon^{0,1}, \Upsilon^{0,1+}, \Lambda^{0,1}, \Lambda^{0,1+}, f, f^+, g, g^+\} , \tag{7.38}$$

where we have defined $a := B^{0,2} + C^{0,3}_{-6}$. The corresponding L_∞-degrees and ghost degrees are summarised in the table below for clarity.

field	g^+	$f^+ + \Upsilon^\dagger$	$\Lambda^+ + \Xi^+$	a^+	a	$\Lambda + \Xi$	$f + \Upsilon$	g
L_∞-degree	5	4	3	2	1	0	-1	-2
ghost-degree	-4	-3	-2	-1	0	1	2	3

Figure 7.2: BV space of fields of self-dual field theory on twistor space P^6

The BV complex is obtained by the shifting the cotangent fibres by 3, that is $\mathsf{L}' = T^*[-3]\mathsf{L}$.

[1] Here, the notation implies that $\Omega^{0,\bullet} = \Omega^{0,\bullet}(\hat U, \mathcal{O}_{\hat U})$ and the subscript -6 denotes forms taking values on $\mathcal{O}_{\hat U}(-6)$.

Explicitly, we define our relevant L_∞-algebra via the differential complex

$$
\mathsf{L}' = \left(
\begin{array}{ccccccccccccccc}
\overset{g}{\Omega^{0,0}_{-6}} & \xrightarrow{\bar\partial} & \overset{\Upsilon^{0,1}}{\Omega^{0,1}_{-6}} & \xrightarrow{\bar\partial} & \overset{\Xi^{0,2}}{\Omega^{0,2}_{-6}} & \xrightarrow{\bar\partial} & \overset{C^{0,3}}{\Omega^{0,3}_{-6}} & \xrightarrow{\bar\partial} & \overset{B^{0,2+}}{\Omega^{0,4}_{-6}} & \xrightarrow{\bar\partial} & \overset{\Xi^{0,2+}}{\Omega^{0,5}_{-6}} & \xrightarrow{\bar\partial} & \overset{\Upsilon^{0,1+}}{\Omega^{0,6}} & & \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\underbrace{}_{\mathsf{L}'_{-2}} & & \underbrace{\overset{f}{\Omega^{0,0}}}_{\mathsf{L}'_{-1}} & \xrightarrow{\bar\partial} & \underbrace{\overset{\Lambda^{0,1}}{\Omega^{0,1}}}_{\mathsf{L}'_{0}} & \xrightarrow{\bar\partial} & \underbrace{\overset{B^{0,2}}{\Omega^{0,2}}}_{\mathsf{L}_{1}} & \xrightarrow{\bar\partial} & \underbrace{\overset{C^{0,3+}}{\Omega^{0,3}}}_{\mathsf{L}'_{2}} & \xrightarrow{\bar\partial} & \underbrace{\overset{\Lambda^{0,1+}}{\Omega^{0,3}}}_{\mathsf{L}'_{3}} & \xrightarrow{\bar\partial} & \underbrace{\overset{f^+}{\Omega^{0,5}}}_{\mathsf{L}'_{4}} & \xrightarrow{\bar\partial} & \underbrace{\overset{g^+}{\Omega^{0,6}}}_{\mathsf{L}'_{5}}
\end{array}
\right) . \tag{7.39}
$$

The BV space of fields comes with the canonical symplectic structure

$$
\omega_{\mathrm{BV}} = \int_{\hat U} \Omega^{6,0} \wedge \left(\delta B^{0,2} \wedge \delta B^{0,2+} + \delta C^{0,3} \wedge \delta C^{0,3+} - \delta\Lambda^{0,1} \wedge \delta\Lambda^{0,1+} + \cdots \right) \tag{7.40}
$$

of degree -1. Defining the superfield

$$
\mathsf{a} := a + a^+ + \Xi^{0,2} + \Xi^{0,2+} + \Upsilon^{0,1} + \Upsilon^{0,1+} + \cdots \tag{7.41}
$$

the BV action reads

$$
S_{\mathrm{BV}} := \tfrac{1}{2}\langle \mathsf{a}, \bar\partial \mathsf{a} \rangle_{\mathsf{L}'} , \tag{7.42}
$$

where $\langle -,- \rangle_{\mathsf{L}'}$ is the cyclic structure of degree -3 given by (2.52c). Hence,

$$
Q_{\mathrm{BV}} := \{ S_{\mathrm{BV}}, - \}_{\mathrm{BV}} \tag{7.43}
$$

and the nilpotency condition $Q_{\mathrm{BV}}^2 = 0$ is equivalent to the classical master equation

$$
\{ S_{\mathrm{BV}}, S_{\mathrm{BV}} \}_{\mathrm{BV}} = 0 . \tag{7.44}
$$

7.4. Twistor action: Non-Abelian case

In the following, we address the question of constructing a non-Abelian action on twistor space P^6. Inspired by homotopy Maurer–Cartan theory, a non-Abelian analogue of the twistor space action (7.14) has been proposed in [43].

7.4.1. A candidate action from higher gauge theory

Let $\mathscr{G}$ be a Lie 2-quasi-group and $(\mathsf{L}\ , \mu_i)$ be the associated 2-term L_∞-algebra, with $\mathsf{L} := \mathsf{Lie}(\mathscr{G})$. We remind the reader that L is the graded vector space $\mathsf{L} = \mathsf{L}_{-1} \oplus \mathsf{L}_0$, together with totally antisymmetric multilinear higher products $\mu_i : \mathsf{L}^{\wedge i} \to \mathsf{L}$, $i = 1, \ldots, 3$, of degree $2 - i$ satisfying (2.44a). Moreover, in order to have an action principle, we assume that L comes equipped with a cyclic inner product $\langle -, - \rangle$ of L_∞-degree $+1$, encoding the dynamics. We want to extend the action (7.19) to the setting of Section 6.5.2.. As in the Abelian case, to write down an action principle on twistor space we need an off-shell approach to holomorphic principal $\mathscr{G}$-bundles. We exploit the fact that the Čech–Dolbeault correspondence extends to higher principal bundles [97, 100–102, 295, 103].[1] Consequently, a holomorphic principal $\mathscr{G}$-bundle can be equivalently described by a complex principal $\mathscr{G}$-bundle with a connective structure locally given by[2]

$$A^{0,1} \in \Omega^{0,1}(\hat{U}, \mathsf{L}_0) \ , \quad B^{0,2} \in \Omega^{0,2}(\hat{U}, \mathsf{L}_{-1}) \ , \tag{7.45}$$

and subject to the constraint equations

$$\mathcal{F}^{0,2} = 0 \quad \text{and} \quad H^{0,3} = 0 \ , \tag{7.46}$$

where

$$\begin{aligned} \mathcal{F}^{0,2} &:= \bar{\partial} A^{0,1} + \frac{1}{2}\mu_2(A^{0,1}, A^{0,1}) + \mu_1(B^{0,2}) \ \in \ \Omega^{0,2}(\hat{U}, \mathsf{L}_0) \ , \\ H^{0,3} &:= \bar{\partial} B^{0,2} + \mu_2(A^{0,1}, B^{0,2}) - \frac{1}{3!}\mu_3(A^{0,1}, A^{0,1}, A^{0,1}) \ \in \ \Omega^{0,3}(\hat{U}, \mathsf{L}_{-1}) \ . \end{aligned} \tag{7.47}$$

Besides the connective structure (7.45), to write down the action we add to the kinematical data the fields[3]

$$C^{0,3} \in \Omega^{0,3}(\hat{U}, \mathcal{O}_{\hat{U}}(-6) \otimes \mathsf{L}_0) \quad \text{and} \quad D^{0,4} \in \Omega^{0,4}(\hat{U}, \mathcal{O}_{\hat{U}}(-6) \otimes \mathsf{L}_{-1}) \ . \tag{7.48}$$

Twistor action. Hence, with these ingredients we consider the following twistor action [43]

$$S = \frac{1}{2\pi i} \int_{\hat{U}} \Omega^{6,0} \wedge \mathscr{L}_{\mathsf{hhCS}} \ , \tag{7.49}$$

[1] In these papers, a version of the Čech–Dolbeault correspondence for a relative exterior derivative along a fibration is proven. However, the arguments are completely analogous for the $\bar{\partial}$ operator on a complex manifold X. See also Appendix E.

[2] Recall that given an L_∞-algebra one may construct the kinematical data of a corresponding higher gauge theory. That means, specifying notions of gauge potentials, curvatures, gauge transformations, Bianchi identities, etc.

[3] Note that here and in the following we omit to explicitly denote the holomorphic weight of the fields to ease the notation.

where

$$
\begin{aligned}
\mathscr{L}_{\mathsf{hhCS}} := {} & \langle B^{0,2}, \bar{\partial} C^{0,3} \rangle_{\mathsf{L}} + \langle D^{0,4}, \bar{\partial} A^{0,1} \rangle_{\mathsf{L}} + \frac{1}{2} \langle D^{0,4}, \mu_2(A^{0,1}, A^{0,1}) \rangle_{\mathsf{L}} \\
& + \langle D^{0,4}, \mu_1(B^{0,2}) \rangle_{\mathsf{L}} - \langle \mu_2(A^{0,1}, B^{0,2}), C^{0,3} \rangle_{\mathsf{L}} \\
& + \frac{1}{3!} \langle \mu_3(A^{0,1}, A^{0,1}, A^{0,1}), C^{0,3} \rangle_{\mathsf{L}} \ .
\end{aligned}
\tag{7.50}
$$

This is the most general holomorphic higher Chern–Simons action one could write down in this setting. As a first consistency check we observe that, since $H^1(\hat{U}, \mathcal{O}_{\hat{U}}) = 0$ and $H^4(\hat{U}, \mathcal{O}_{\hat{U}}(-6)) = 0$, this action reduces to (7.14) in the Abelian case. At the infinitesimal level, gauge transformations are parametrised by a degree zero element $\eta \in \Omega^{(0,\bullet)}(\hat{U}, \mathsf{L})$, which decomposes as

$$
\eta := X^{0,0} + \Lambda^{0,1} \quad \text{with} \quad X^{0,0} \in C^\infty(\hat{U}, \mathsf{L}_0) \ , \quad \Lambda^{0,1} \in \Omega^{0,1}(\hat{U}, \mathsf{L}_{-1}) \ . \tag{7.51}
$$

Then, the infinitesimal gauge transformations of $(A^{0,1}, B^{0,2})$ are read off (3.10). The action is invariant under the transformations

$$
\begin{aligned}
\delta A^{0,1} &= \bar{\partial} X^{0,0} - \mu_1(\Lambda^{0,1}) + \mu_2(A^{0,1}, X^{0,0}) \ , \\
\delta B^{0,2} &= \bar{\partial} \Lambda^{0,1} + \mu_2(B^{0,2}, X^{0,0}) + \mu_2(A^{0,1}, \Lambda^{0,1}) \ , \\
\delta C^{0,3} &= \mu_2(C^{0,3}, X^{0,0}) \ , \\
\delta D^{0,4} &= \mu_2(D^{0,4}, X^{0,0}) + \mu_2(C^{0,3}, \Lambda^{0,1})
\end{aligned}
\tag{7.52}
$$

The curvatures (7.47) satisfy each a Bianchi identity and induced gauge transformations. In particular, the latter read[1]

$$
\begin{aligned}
\delta \mathcal{F}^{0,2} &= \mu_2(\mathcal{F}^{0,2}, X^{0,0}) \ , \\
\delta H^{0,3} &= -\mu_2(H^{0,3}, X^{0,0}) - \mu_2(\mathcal{F}^{0,2}, \Lambda^{0,1}) \ .
\end{aligned}
\tag{7.53}
$$

The corresponding equations of motion for $(A^{0,1}, B^{0,2})$ are nothing but the equations (7.46), as desired. Modulo gauge transformations, this is equivalent to the statement that the complex principal $\mathscr{G}$-bundle in question is holomorphic on-shell. Assuming M^6-triviality, solutions to the Euler-Lagrange equations of the twistor action (7.49) are in one-to-one correspondence to moduli space of solutions of solutions to higher gauge theory for a self-dual 3-form curvature, via Penrose–Ward transform. In the next section we obtain the stronger result that such a correspondence also holds off-shell. In addition, the action (7.49) gives the equations

$$
\begin{aligned}
& \bar{\partial} C^{0,3} + \mu_1(D^{0,4}) - \mu_2(A^{0,1}, C^{0,3}) = 0 \ , \\
& \bar{\partial} D^{0,4} + \mu_2(A^{0,1}, D^{0,4}) + \mu_2(B^{0,2}, C^{0,3}) - \frac{1}{2} \mu_3(C^{0,3}, A^{0,1}, A^{0,1}) = 0 \ .
\end{aligned}
\tag{7.54}
$$

[1]Note that the action (7.49) can be re-written as $S = \int_{\hat{U}} \Omega^{6,0} \wedge \big\{ \langle D^{0,4}, \mathcal{F}^{0,2} \rangle_{\mathsf{L}} - \langle H^{0,3}, C^{0,3} \rangle_{\mathsf{L}} \big\}$.

for the fields $C^{0,3}$ and $D^{0,4}$ in the background of $A^{0,1}$ and $B^{0,2}$, respectively.

7.4.2. Equivalence to space-time action

Here, we follow the same steps as in 7.3.2. to reduce the twistor action to the desired space-time action on $\mathbb{R}^6$.

Fields expansion. The starting point of the analysis consists of considering the Euclidean structure given in 6.3.2.. Together with (7.20) and (7.21), we consider the following expansion

$$A^{0,1} = \bar{e}_{[A}\lambda_{B]}A^{AB} + \frac{\hat{\lambda}^{[A}\mathrm{d}\hat{\lambda}^{B]}}{[\lambda\hat{\lambda}]^2}\widetilde{A}_{AB} \ , \tag{7.55}$$

for the $(0,1)$ connection form and similarly

$$\begin{aligned} D^{0,4} = \varepsilon_{CDEF}\frac{\bar{e}_{[A}\lambda_{B]} \wedge \hat{\lambda}^C\mathrm{d}\hat{\lambda}^D \wedge \mathrm{d}\hat{\lambda}^E \wedge \mathrm{d}\hat{\lambda}^F}{[\lambda\hat{\lambda}]^4}D^{AB} + \varepsilon^{ABCD}\bar{e}_A \wedge \bar{e}_B\lambda_C \wedge \\ \wedge\, \varepsilon_{EFGH}\frac{\hat{\lambda}^E\mathrm{d}\hat{\lambda}^F \wedge \mathrm{d}\hat{\lambda}^G}{[\lambda\hat{\lambda}]^3}\widetilde{D}_D^{\ H} + \varepsilon^{ABCD}\frac{\bar{e}_A \wedge \bar{e}_B \wedge \bar{e}_C\lambda_D \wedge \hat{\lambda}^E\mathrm{d}\hat{\lambda}^F}{[\lambda\hat{\lambda}]^2}\widehat{D}_{EF} \ , \end{aligned} \tag{7.56}$$

for the remaining kinematical field. Here, we have defined the skew-symmetric coefficients fields A^{AB} and $\widetilde{A}_{AB}$ of holomorphic weight 0 and $+2$, respectively. Moreover, D^{AB}, $\widetilde{D}_A^B$ and $\hat{D}_{AB}$ have all homogeneity $(-2,0)$ and satisfy $D^{AB} = -D^{BA}$ and $\widehat{D}_{AB} = -\widehat{D}_{BA}$. Recall that at this points the coefficients are functions of $(x, \lambda, \hat{\lambda})$.

Gauge fixing. It is crucial here to remark the different geometric role played by the connective structure (7.45) and the kinematic fields $(C^{0,3}, D^{0,4})$. The M^6-triviality of the principal $\mathscr{G}$-bundle $\hat{g} : N(\check{\mathscr{C}}(\hat{\mathfrak{U}} \to \mathsf{P}^6)) \to \mathscr{G}$ is encoded in the assumption of the existence of a gauge in which the differential forms both $A^{0,1}$ and $B^{0,2}$ have no components along the submanifolds $\mathbb{P}^3 \hookrightarrow \mathsf{P}^6$. On top of that, as in the Abelian case, we shall employ the space-time gauge[1]

$$\bar{\partial}^{\dagger}_{\mathsf{Y}} A^{0,1}|_{\mathsf{Y}} = 0 \ , \quad \bar{\partial}^{\dagger}_{\mathsf{Y}} B^{0,2}|_{\mathsf{Y}} = 0 \ , \quad \bar{\partial}^{\dagger}_{\mathsf{Y}} C^{0,3}|_{\mathsf{Y}} = 0 \tag{7.57}$$

and similarly for all the gauge parameters. We remind the this tells us that the field $C^{0,3}$ is harmonic upon restriction to the fibres of the Euclidean twistor fibration. This means that the λ-dependence of the coefficient field C_0 is fixed by (7.28). Moreover, this choice restricts

[1]Obviously, there are no $(0,4)$-forms on Y.

the remaining gauge freedom to that of the standard space-time gauge transformations, that is, the freedom of performing gauge transformations on $\mathbb{R}^6 \times \mathbb{P}^3$ that are the pull-back of space-time gauge transformations on $\mathbb{R}^6$.[1]

Integration along the fibres. By wedging the gauge-fixed holomorphic higher Chern–Simons form against the volume form $\Omega^{6,0}$, the non-Abelian action (7.49) takes the form

$$S := S_1 + S_2 \ , \tag{7.58}$$

where

$$\begin{aligned} S_1 :=& \int_{\hat{U}} \Omega^{6,0} \wedge \Big[\langle B^{0,2}, \bar{\partial} C^{0,3} \rangle_{\mathsf{L}} - \langle \mu_2(A^{0,1}, B^{0,2}), C^{0,3} \rangle_{\mathsf{L}} \\ &+ \frac{1}{3!} \langle \mu_3(A^{0,1}, A^{0,1}, A^{0,1}), C^{0,3} \rangle_{\mathsf{L}} \Big] \\ =& \beta \int_{U \times \mathbb{P}^3} \mathrm{d}^6 x \wedge K \Big[-\frac{2}{3} \langle \overline{V}^A B_A, C_0 \rangle_{\mathsf{L}} - \frac{1}{3} \langle \mu_2(A^{AB}, B_A), C_0 \rangle_{\mathsf{L}} \lambda_B \\ &- \frac{1}{12} \langle \mu_3(A^{AB}, A_{AD}, A^{CD}), C_0 \rangle_{\mathsf{L}} \lambda_B \lambda_C - \frac{1}{3} \langle B_A, \frac{\hat{\lambda}^C \widetilde{V}_{CD}}{[\lambda \hat{\lambda}]} \widetilde{C}^{AB,D} \rangle_{\mathsf{L}} \lambda_B \Big] \ , \end{aligned} \tag{7.59a}$$

and

$$\begin{aligned} S_2 :=& \int_{\hat{U}} \Omega^{6,0} \wedge \Big[\langle D^{0,4}, \bar{\partial} A^{0,1} \rangle_{\mathsf{L}} + \frac{1}{2} \langle D^{0,4}, \mu_2(A^{0,1}, A^{0,1}) \rangle_{\mathsf{L}} + \langle D^{0,4}, \mu_1(B^{0,2}) \rangle_{\mathsf{L}} \Big] \\ =& \beta \int_{U \times \mathbb{P}^3} \mathrm{d}^6 x \wedge K \Big[-\frac{1}{6} \langle D^{AB}, \partial^{CE} A_{EA} - \partial_{EA} A^{CE} \rangle_{\mathsf{L}} \lambda_B \lambda_C \\ &- \frac{1}{12} \langle D^{AB}, \mu_2(A^{CD}, A_{DA}) \rangle_{\mathsf{L}} \lambda_B \lambda_C + \frac{1}{3} \langle D^{AB}, \mu_1(B_A) \rangle_{\mathsf{L}} \lambda_B + \frac{2}{9} \langle \widetilde{D}_A{}^B, \frac{\hat{\lambda}^C \widetilde{V}_{CB}}{[\lambda \hat{\lambda}]} A^{AD} \rangle_{\mathsf{L}} \lambda_D \Big] \end{aligned} \tag{7.59b}$$

In deriving (7.59a) and (7.59b) we have made repeatedly use of the cyclicity equation[2] (2.47b) together with (7.30) and (7.32). We further observe that the component fields $\widetilde{C}^{AB,D}$ and $\widetilde{D}_A{}^B$ can be treated as Lagrange multiplier fields. Integrating them out enforces the constraints

$$\hat{\lambda}^C \widetilde{V}_{CB} B_A = 0 \ , \qquad \hat{\lambda}^C \widetilde{V}_{CB} A^{AD} = 0 \ , \tag{7.61}$$

[1] The residual gauge symmetry still needs to be fixed further in order to quantise the theory. Any ordinary gauge choice will do the trick.

[2] Recall that a cyclic structure on L is a graded symmetric map. Hence, $\langle -, - \rangle_{\mathsf{L}}$ satisfies

$$\langle X, Y \rangle_{\mathsf{L}} = -(-)^{(|X|_{\mathsf{L}}-1)} (-)^{(|Y|_{\mathsf{L}}-1)} \langle Y, X \rangle_{\mathsf{L}} \ . \tag{7.60}$$

which, because of homogeneity reasons, yield

$$B_A(x,\lambda,\hat{\lambda}) = -B_A{}^B\lambda_B \quad \text{and} \quad A^{AB}(x,\lambda,\hat{\lambda}) = 2A^{AB} \ . \tag{7.62}$$

Here, the fields A^{AB} and $B_A{}^B$ denote space-time functions and the numerical coefficients have been chosen for future convenience. Also, recall that $A_{AB} = \frac{1}{2}\varepsilon_{ABCD}A^{CD}$ and $B_A{}^B$ is trace-less. As regards the field $D^{0,4}$, the only component surviving in this gauge is D^{AB}. Taking into account its homogeneity, it admits the general power series expansion

$$D^{AB}(x,\lambda,\hat{\lambda}) = D_{CD}^{AB}(x)\frac{\hat{\lambda}^C\hat{\lambda}^D}{[\lambda\hat{\lambda}]^2} + D_{DEF}^{ABC}(x)\frac{\lambda_C\hat{\lambda}^D\hat{\lambda}^E\hat{\lambda}^F}{[\lambda\hat{\lambda}]^3} + \cdots \ , \tag{7.63}$$

where each of the space-time coefficients is skew-symmetric in AB. Now that the λ-dependence is completely determined we can directly integrate out the $\mathbb{P}^3$ degrees of freedom, reducing the twistor action (7.49) to a space-time action on $\mathbb{R}^6$. As we shall see, the fibre integration will also fix the space-time index structure of the field D^{AB}. By using relations (7.62), (7.63) and integrating along the $\mathbb{P}^3$-fibres we get

$$S_1 = \frac{2}{3}\beta\int_U \mathrm{d}^6x\ \langle C_{BC}, \partial^{A(B}B_A{}^{C)} + \mu_2(A^{A(B}, B_A{}^{C)}) + \mu_3(A^{A(B}, A_{AD}, A^{C)D})\rangle_{\mathsf{L}} \tag{7.64}$$

together with

$$S_2 = -\frac{1}{3}\alpha_D\beta\int_U \mathrm{d}^6x\ \langle D_A{}^B, (\partial^{AC}A_{CB} - \partial_{CB}A^{AC}) + \mu_2(A^{AC}, A_{CB}) + \mu_1(B_B{}^A)\rangle_{\mathsf{L}} \ . \tag{7.65}$$

Here, in the first equation we have used again (7.28), while in deriving S_2 we have defined the coefficient field $D_A{}^B$ as

$$\alpha_D D_A{}^B := \frac{1}{\alpha_1}D_{CA}^{BC} + \frac{1}{\alpha_{3/2}}D_{CDA}^{BCD} + \cdots \ , \tag{7.66}$$

depending only on space-time coordinates. Here, the coefficient α_D is yet to be determined, while the object $D_A{}^B$ is trace-less. Finally, by recalling the space-time curvatures (6.51) and taking $\alpha_D := 2$ we obtain the expression

$$S = \frac{2}{3}\beta\int_U \mathrm{d}^6x\left[\langle C_{AB}, H^{AB}\rangle_{\mathsf{L}} - \langle D_A{}^B, F_B{}^A\rangle_{\mathsf{L}}\right] \ . \tag{7.67}$$

This is consistent with our expectation about the higher gauge theory action (7.49) on twistor space descending on space-time $\mathbb{R}^6$. In the next Chapter we shall discuss further investigations we plan to conduct in the next future.

8 Conclusions and outlook

8.1. Outcomes

We devoted the first part of the Thesis to deepen our understanding of the intimate relation between L_∞-algebras and classical field theory. Explicitly, in Chapter 4 we explained how L_∞-algebras necessarily arise from the classical master equation of the BV formalism and why quasi-isomorphisms constitute the correct and very useful notion of equivalence, corresponding to the classical equivalence of field theories. The aspiration was to present an accessible, self-contained review of the complete picture with the aim of increasing the interest in the subject and allowing for further investigation of higher structures in both classical and quantum field theories. We also presented some new results and observations. In Chapter 3 we observed that the homotopy Maurer–Cartan theory always allows for a supersymmetric extension by auxiliary fields, just as ordinary Chern–Simons theory. We also noticed that homotopy Maurer–Cartan action, at least formally, satisfies the quantum master equation of the BV formalism, which can simplify the computation of the quantum master equation. Most importantly, our perspective is adapted to an application towards higher gauge theory from the outset. As examples, in Chapter 5 we have explored Yang–Mills theory and higher Chern–Simons theory in this formalism. In this regard, we have provided a quasi-isomorphism between first and second order formulations of Yang–Mills theory in the Q-manifolds language. We considered higher Chern–Simons theory for a Lie n-algebra or n-term L_∞-algebra in dimension $2+n$, and we constructed the minimal model of the corresponding gauge L_∞-algebra.

In the second part of the Thesis we discussed an application coming from the interplay between higher homotopy structures and twistor theory. In particular, in Chapter 7 we have explicitly showed how the twistor space actions of [97–99] descend to Euclidean six-dimensional space-time. To do so, we have imposed Euclidean reality conditions on twistor space P^6, we have expanded the fields in an adapted basis by using non-holomorphic coordinates (x^{AB}, λ_A) and finally we have integrated along the $\mathbb{P}^3$-fibres of the twistor fibration. We have also proposed a twistor space action for non-Abelian fields in terms of higher holomorphic Chern–Simons theory. We have performed similar calculations to lift the twistor correspondence off-shell. Finally, we have commented about the application of the L_∞-perspective in this very context.

8.2. Future directions

We conclude the Thesis with a brief survey of some of the possible future directions for research in this field. We point out that this is by no means an exhaustive list and there are numerous projects that one could embark on. We only consider those that, by the opinions and interests of the author, are the most compelling.

Self-dual actions from twistor space. In order not to interrupt the flow, we start from the discussion in Chapter 7. As already mentioned, the calculations here are based on the work in progress [111]. Our objective in this context is to adopt the L_∞-perspective via the BV formalism on twistor space P^6 to see if space-time actions such as the PST action or Sen's action naturally descend from the same twistor space action. Hopefully, these actions would emerge by considering L_∞-quasi isomorphisms at the level of twistor space, such as field redefinitions or integrating in and out additional fields. Possibly, the greater freedom in choosing a gauge on twistor space could allow us to obtain different space-time actions from different gauge choices.

Always in this setting, it would be interesting to employ the formalism used recently by Popov in four dimensions [296] to construct a graded extension of P^6, by considering a non-integrable almost complex structure on it.

MSYM theory as Higher Chern-Simons theory. The twistor space of $\mathcal{N} = 3$ supersymmetric Yang–Mills theory is a five-dimensional complex manifold and therefore the relevant holomorphic bundles cannot be described in terms of an ordinary holomorphic Chern–Simons action functional. Recently, Saemann & Wolf [110] have proposed twistor space action by using a higher holomorphic Chern–Simons theory for a Lie 3-algebra $\mathsf{L} = \mathsf{L}_{-2} \oplus \mathsf{L}_{-1} \oplus \mathsf{L}_0$, see also [111]. It would be very interesting to show the off-shell correspondence of this twistor action to the MSYM action on space-time. This would allow to analyse higher holomorphic CS theory within twistor string theory and in the computation of the $\mathcal{N} = 4$ SYM scattering amplitudes. In this case, the calculations turn out to be quite involved because of the six-fermionic coordinates on ambitwistor space. We have started to tackle this problem in the past, but most of the work still needs to be done.

Renormalisation group and quasi-isomorphisms. This Thesis is intended also as ground work for future research on higher structures in the context of classical and quantum BV formulations of field theories. A particularly interesting topic here is the relation between quasi-isomorphism and renormalisation group flow, which has been established in some special cases, cf. [297]. This should be a quasi-isomorphism of quantum homotopy algebras. For some recent work on the renormalisation in the BV context, see [298,299] and in particular [219,171].

Supersymmetric localisation in higher gauge theory. We have seen how to supersymmetrise homotopy Maurer–Cartan theory by adding auxiliary fields, similarly to ordinary Chern–Simons theory. Hence, it would be interesting to apply (higher) supersymmetric localisation techniques to try and evaluate exactly the path integral of supersymmetric higher Chern-Simons theory on compact manifolds. Here, besides the computational task, there is still some conceptual gap to fill. However this definitely represents an interesting topic that is worth some further analysis.

Appendices

$\mathcal{A}$

L_∞-algebras and L_∞-morphisms: a coalgebra perspective

Below we shall explain the relation between L_∞-algebras and codifferential coalgebras. In this perspective, all higher products are packaged in a single codifferential. We also derive the structure equation for morphisms of L_∞-algebras from morphisms of codifferential coalgebras. The following calculations can be found in [42]. The relevant original reference for this material is [134], helpful may also be the detailed discussions in [300–302].

A.1. Codifferential graded coalgebras

Preliminaries. Given a real graded vector space V, we define the following associative algebras:

$$\begin{aligned}
\text{tensor algebra}: \quad & \bigotimes^\bullet \mathsf{V} := \mathbb{R} \oplus \mathsf{V} \oplus (\mathsf{V} \otimes \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 0} \bigotimes^k \mathsf{V}\,,\\
\text{symmetric tensor algebra}: \quad & \bigodot^\bullet \mathsf{V} := \mathbb{R} \oplus \mathsf{V} \oplus (\mathsf{V} \odot \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 0} \bigodot^k \mathsf{V}\,,\\
\text{antisymmetric tensor algebra}: \quad & \bigwedge^\bullet \mathsf{V} := \mathbb{R} \oplus \mathsf{V} \oplus (\mathsf{V} \wedge \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 0} \bigwedge^k \mathsf{V}\,,\\
\text{reduced tensor algebra}: \quad & \bigotimes^\bullet_0 \mathsf{V} := \mathsf{V} \oplus (\mathsf{V} \otimes \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 1} \bigotimes^k \mathsf{V}\,,\\
\text{reduced symmetric tensor algebra}: \quad & \bigodot^\bullet_0 \mathsf{V} := \mathsf{V} \oplus (\mathsf{V} \odot \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 1} \bigodot^k \mathsf{V}\,,\\
\text{reduced antisymmetric tensor algebra}: \quad & \bigwedge^\bullet_0 \mathsf{V} := \mathsf{V} \oplus (\mathsf{V} \wedge \mathsf{V}) \oplus \cdots = \bigoplus_{k\geq 1} \bigwedge^k \mathsf{V}\,,
\end{aligned} \tag{A.1}$$

cf. Section 2.2.. Here, $\odot$ and $\wedge$ denote the graded symmetric and antisymmetric tensor products, with weight one, e.g.

$$v_1 \odot v_2 \ := \ v_1 \otimes v_2 + (-1)^{|v_1|\,|v_2|} v_2 \otimes v_1 \ . \tag{A.2}$$

These tensor products yield embeddings of $\bigodot^\bullet \mathsf{V}$ and $\bigwedge^\bullet \mathsf{V}$ into $\bigotimes^\bullet \mathsf{V}$ as well as $\bigodot_0^\bullet \mathsf{V}$ and $\bigwedge_0^\bullet \mathsf{V}$ into $\bigotimes_0^\bullet \mathsf{V}$. We also have projectors from the reduced tensor algebra $\bigotimes_0^\bullet V$ to both reduced symmetric and antisymmetric algebras:

$$\begin{aligned} \mathrm{pr}_\odot(v_1 \otimes \cdots \otimes v_i) \ &:= \ \sum_{\sigma \in S_i} \varepsilon(\sigma; v_1, \ldots, v_i) v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)} \ , \\ \mathrm{pr}_\wedge(v_1 \otimes \cdots \otimes v_i) \ &:= \ \sum_{\sigma \in S_i} \chi(\sigma; v_1, \ldots, v_i) v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(i)} \ , \end{aligned} \tag{A.3}$$

where $\varepsilon(\sigma; v_1, \ldots, v_i)$ and $\chi(\sigma; v_1, \ldots, v_i)$ are the symmetric and antisymmetric Koszul signs of a permutation σ. Explicitly, we have

$$v_1 \odot \cdots \odot v_i \ = \ \varepsilon(\sigma; v_1, \ldots, v_i) v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)} \tag{A.4}$$

and

$$\ell_1 \wedge \ldots \wedge \ell_i \ = \ \chi(\sigma; \ell_1, \ldots, \ell_i) \ell_{\sigma(1)} \wedge \ldots \wedge \ell_{\sigma(i)} \tag{A.5}$$

for $\ell_1, \ldots, \ell_i \in \mathsf{V}$. Using the shift isomorphism $s^\bullet$ defined in (2.13), we obtain the identity

$$\chi(\sigma; \ell_1, \ldots, \ell_i) \ = \ (-1)^{\sum_{j=1}^{i-1}(i-j)(|\ell_j| + |\ell_{\sigma(j)}|)} \varepsilon(\sigma; s\ell_1, \ldots, s\ell_i) \ , \tag{A.6}$$

which we shall use later.

Reduced symmetric coalgebra. Consider now the reduced algebras $\bigwedge_0^\bullet \mathsf{V}$ and $\bigodot_0^\bullet \mathsf{V}$ as introduced in (2.11c). Together with the reduced comultiplication,

$$\begin{aligned} \Delta_0 \ &: \ \textstyle\bigodot_0^\bullet \mathsf{V} \ \to \ \bigodot_0^\bullet \mathsf{V} \otimes \bigodot_0^\bullet \mathsf{V} \ , \\ v_1 \odot \cdots \odot v_i \ &\mapsto \ \sum_{j+k=i} \sum_{\sigma \in \mathrm{Sh}(j;k)} \varepsilon(\sigma; v_1, \ldots, v_i) (v_{\sigma(1)} \odot \cdots \odot v_{\sigma(j)}) \otimes (v_{\sigma(j+1)} \odot \cdots \odot v_{\sigma(i)}) \ , \end{aligned} \tag{A.7}$$

$\bigodot_0^\bullet \mathsf{V}$ becomes a cocommutative coalgebra.

Given functions $f_1, f_2 : \bigodot_0^\bullet V \to \bigodot_0^\bullet V$, we define the symmetrised tensor product

$$f_1 \odot f_2 \ := \ m_\odot \circ (f_1 \otimes f_2) \circ \Delta_0 \ , \tag{A.8}$$

where $m_\odot(v_1 \otimes v_2) := v_1 \odot v_2$ for $v_{1,2} \in V$. Explicitly,

$$\begin{aligned}(f_1 \odot f_2)(v_1 \odot \cdots \odot v_i) \;&=\\ =\sum_{j+k=i}\sum_{\sigma\in\mathrm{Sh}(j;k)} \varepsilon(\sigma; v_1,\ldots,v_i) f_1(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(j)}) &\odot f_2(v_{\sigma(j+1)}\odot\cdots\odot v_{\sigma(i)})\;.\end{aligned} \tag{A.9}$$

Formula (A.8) generalises to expressions $f_1 \odot \cdots \odot f_i$ using shuffles with i ordered subsets.

Codifferentials. A *codifferential* on a coalgebra $(\bigodot_0^\bullet \mathsf{V}, \Delta_0)$ is a linear map $D : \bigodot_0^\bullet \mathsf{V} \to \bigodot_0^\bullet \mathsf{V}$ of degree 1, which is a nilquadratic coderivation,

$$\Delta_0 D \;=\; (D\otimes 1)\Delta_0 + (1\otimes D)\Delta_0 \quad \text{and} \quad D^2 \;=\; 0\;. \tag{A.10}$$

The first equation is the coalgebra analogue of the Leibniz rule. The second equation is equivalent to

$$\sum_{i=1}^{\infty}\sum_{j+k=i} D_{k+1}(D_j \odot \mathrm{id}^{\odot k}) \;=\; 0 \tag{A.11}$$

for all $i \geq 1$, where $D_k : \bigodot_0^k \mathsf{V} \to \mathsf{V}$ is the restriction of the codifferential,

$$D_k \;:=\; \mathrm{pr}_{\mathsf{V}} \circ D \circ \iota_{\bigodot_0^k \mathsf{V}}\;. \tag{A.12}$$

Note that we can pull back D_k along the projection $\mathrm{pr}_\odot$ defined in (A.3) to a map $D_k : \bigotimes_0^k \mathsf{V} \to \mathsf{V}$. The condition $D^2 = 0$ then simply translates to

$$\sum_{j+k=i} D_{k+1}(D_j \otimes \mathrm{id}^{\otimes k}) \circ \sum_{\sigma\in\mathrm{Sh}(j;i)} \varepsilon(\sigma,-)\sigma(-) \;=\; 0 \tag{A.13}$$

for every $i > 0$, where the sum is taken over all $(j;i)$-shuffles, $\varepsilon(\sigma,-)$ is the Koszul sign of the shuffle and $\sigma(-)$ is the application of the shuffle to elements of $\bigotimes_0^i \mathsf{V}$.

A.2. L_∞-algebras from codifferentials

The homological vector field Q dualises to the codifferential D of homogeneous degree 1 on $\bigodot_0^\bullet \mathsf{V} \cong \bigwedge_0^\bullet \mathsf{L}$, where the isomorphism is the shift isomorphism $s^\bullet$ defined in (2.13). First, we note that the codifferential decomposes into a sum

$$D \;=\; \sum_{i\in\mathbb{N}} \tilde{D}_i\;. \tag{A.14}$$

The restricted codifferentials on the coalgebra $(\bigodot_0^\bullet \mathsf{V}, \Delta_0)$ now induce a set of totally antisymmetric, multilinear products μ_i on $\bigwedge_0^\bullet \mathsf{L}$ with $\mathsf{L} = \mathsf{V}[-1]$. We define *higher products*

$$\mu_i \;:=\; (-1)^{\frac{1}{2}i(i-1)+1} s^{-1} \circ D_i \circ s^{\otimes i} \tag{A.15}$$

with $s^{\otimes i}$ as in (2.13). Here, we note that the homotopy Jacobi identities follow from $Q^2 = 0$ $\Leftrightarrow D^2 = 0$. Inserting

$$\mathrm{id}^{\otimes i} \;=\; (-1)^{\frac{1}{2}i(i-1)} s^{\otimes i} \circ (s^{-1})^{\otimes i} \tag{A.16}$$

into the conditions (A.11) equivalent to $D^2 = 0$ and concatenate with s^{-1} and $s^{\otimes i}$ to obtain

$$\begin{aligned}
0 \;&=\; \sum_{j+k=i} s^{-1} \circ D_{k+1} \circ (D_j \otimes \mathrm{id}^{\otimes k}) \circ \sum_{\sigma\in \mathrm{Sh}(j;i)} \varepsilon(\sigma,-)\sigma(-) \circ s^{\otimes i} \\
&=\; \sum_{j+k=i} (-1)^{\frac{1}{2}k(k+1)+\frac{1}{2}i(i-1)} s^{-1} \circ D_{k+1} \circ s^{\otimes k+1} \\
&\qquad \circ (s^{-1})^{\otimes (k+1)} \circ (D_j \otimes \mathrm{id}^{\otimes k}) \circ s^{\otimes i} \circ (s^{-1})^{\otimes i} \circ \sum_{\sigma\in \mathrm{Sh}(j;i)} \varepsilon(\sigma,-)\sigma(-) \circ s^{\otimes i} \\
&=\; \sum_{j+k=i} (-1)^{\frac{1}{2}k(k+1)+k(j+1)+\frac{1}{2}i(i-1)} s^{-1} \circ D_{k+1} \circ s^{\otimes k+1} \\
&\qquad \circ \left((s^{-1} \circ D_j \circ s^{\otimes j}) \otimes (s^{-1})^{\otimes k}\right) \circ (\mathrm{id}^{\otimes j} \otimes s^{\otimes k}) \circ (s^{-1})^{\otimes i} \circ \sum_{\sigma\in \mathrm{Sh}(j;i)} \varepsilon(\sigma,-)\sigma(-) \circ s^{\otimes i} \\
&=\; \sum_{j+k=i} (-1)^{\frac{1}{2}i(i-1)+\frac{1}{2}j(j-1)+k(j+1)+\frac{1}{2}k(k-1)} \mu_{k+1} \circ \\
&\qquad (\mu_j \otimes \mathrm{id}^{\otimes k}) \circ (s^{-1})^{\otimes i} \circ \sum_{\sigma\in \mathrm{Sh}(j;i)} \varepsilon(\sigma,-)\sigma(-) \circ s^{\otimes i} \\
&=\; \sum_{j+k=i} (-1)^{\frac{1}{2}i(i-1)+k} \mu_{k+1} \circ (\mu_j \otimes \mathrm{id}^{\otimes k}) \circ \sum_{\sigma\in \mathrm{Sh}(j;i)} \chi(\sigma,-)\sigma(-) \;,
\end{aligned} \tag{A.17}$$

where we used the identity

$$(s^{-1})^{\otimes i} \circ \sum_{\sigma\in \mathrm{Sh}(j;i)} \varepsilon(\sigma,-)\sigma(-) \circ s^{\otimes i} \;=\; (-1)^{\frac{1}{2}i(i-1)} \sum_{\sigma\in \mathrm{Sh}(j;i)} \chi(\sigma,-)\sigma(-) \;. \tag{A.18}$$

Here, $\sigma(-)$ is the application of the $(j;i)$-shuffle. Hence, the formula

$$\sum_{j+k=i} (-1)^k \mu_{k+1} \circ (\mu_j \otimes \mathrm{id}^{\otimes k}) \circ \sum_{\sigma\in \mathrm{Sh}(j;i)} \chi(\sigma,-)\sigma(-) \;=\; 0 \;. \tag{A.19}$$

is nothing but an alternative form of the higher homotopy Jacobi identity (2.44c). In fact, evaluating it $\ell_1 \otimes \cdots \otimes \ell_n \in \bigotimes^n \mathsf{L}$, we obtain

$$\sum_{j+k=i} \sum_{\sigma\in \mathrm{Sh}(j;i)} (-1)^k \chi(\sigma;\ell_1,\ldots,\ell_i) \mu_{k+1}(\mu_j(\ell_{\sigma(1)},\ldots,v_{\sigma(j)}),\ell_{\sigma(j+1)},\ldots,\ell_{\sigma(i)}) \;=\; 0 \;. \tag{A.20}$$

Higher products and the differential graded algebra picture. Let us briefly link the higher products μ_i and the differential graded algebra picture of an L_∞-algebra. Using the notation of Section 2.4. we consider a graded vector space L with basis τ_α and corresponding coordinate functions ξ^α on $\mathsf{L}[1]$ with respect to $s\tau_\alpha$. The choice of basis defines structure constants $f_{\beta_1\cdots\beta_k}{}^\alpha$ via the equation

$$Q\xi^\alpha \;=\; \sum_{k\geq 1}\frac{(-1)^{|\alpha|}}{k!}f_{\beta_1\cdots\beta_k}{}^\alpha\xi^{\beta_1}\cdots\xi^{\beta_k}\;, \tag{A.21}$$

where for each k, $|\beta_1|+\cdots+|\beta_k| = |\alpha|+1$. The coordinate functions ξ^α are now maps $\mathsf{L}[1]\to\mathbb{R}$, satisfying $\xi^\alpha(X) = \xi^\alpha(X^\beta s\tau_\beta) = X^\alpha$, where $X^\alpha\in\mathbb{R}$ are the coordinates of the vector $X\in\mathsf{L}[1]$ with respect to the basis $s\tau_\alpha$. Correspondingly, $\xi^{\beta_1}\cdots\xi^{\beta_k}$ is a function $\bigodot_0^k\mathsf{L}[1]\to\mathbb{R}$, with

$$(\xi^{\beta_1}\cdots\xi^{\beta_k})(s\tau_{\gamma_1}\odot\cdots\odot s\tau_{\gamma_k}) \;=\; (-1)^{\sum_{i=2}^k\sum_{j=1}^{i-1}|\beta_i||\beta_j|}\delta^{\beta_1}_{(\gamma_1}\cdots\delta^{\beta_k}_{\gamma_k)}\;, \tag{A.22}$$

where the symmetrisation of the indices $\gamma_1,\ldots,\gamma_k$ is evidently graded. Note that by precomposition with the projection $\mathrm{pr}_{\bigodot_0^k\mathsf{L}[1]}$, $\xi^{\beta_1}\cdots\xi^{\beta_k}$ becomes a map $\bigodot_0^\bullet\mathsf{L}[1]\to\mathbb{R}$.

We now contract both sides of (A.21) by τ_α from the left and apply the result to $(s\tau_{\gamma_1}\odot\cdots\odot s\tau_{\gamma_k})$ to obtain

$$\begin{aligned}
\tau_\alpha(Q\xi^\alpha)(s\tau_{\gamma_1}\odot\cdots\odot s\tau_{\gamma_k}) \;&=\; \sum_{k\geq 1}\frac{(-1)^{|\alpha|}}{k!}\tau_\alpha f_{\beta_1\cdots\beta_k}{}^\alpha(\xi^{\beta_1}\cdots\xi^{\beta_k})(s\tau_{\gamma_1}\odot\cdots\odot s\tau_{\gamma_k})\\
\tau_\alpha\xi^\alpha D(s\tau_{\gamma_1}\odot\cdots\odot s\tau_{\gamma_k}) \;&=\; (-1)^{\sum_{i=2}^k\sum_{j=1}^{i-1}|\gamma_i||\gamma_j|}\tau_\alpha f_{\gamma_1\cdots\gamma_k}{}^\alpha\\
s^{-1}(D(s\tau_{\gamma_1}\odot\cdots\odot s\tau_{\gamma_k})) \;&=\; (-1)^{\sum_{i=2}^k\sum_{j=1}^{i-1}|\gamma_i||\gamma_j|}\tau_\alpha f_{\gamma_1\cdots\gamma_k}{}^\alpha\;.
\end{aligned} \tag{A.23}$$

Using (A.15), we now compute

$$\begin{aligned}
\mu_i(\tau_{\alpha_1},\ldots,\tau_{\alpha_i}) \;&=\; ((-1)^{\frac12 i(i-1)}s^{-1}\circ D_i\circ s^{\odot i})(\tau_{\alpha_1},\ldots,\tau_{\alpha_k})\\
&=\; ((-1)^{\frac12 i(i-1)+\sum_{j=1}^i(i-j)|\tau_{\alpha_j}|})(s^{-1}(D_i(s\tau_{\alpha_1},\ldots,s\tau_{\alpha_i}))\\
&=\; (-1)^{\frac12 i(i-1)+\sum_{j=2}^i\sum_{k=1}^{j-1}(|\tau_{\alpha_j}|+1)(|\tau_{\alpha_k}|+1)+\sum_{j=1}^i(i-j)|\tau_{\alpha_j}|}\;\tau_\beta f_{\alpha_1\cdots\alpha_i}{}^\beta\\
&=\; (-1)^{\sum_{j=2}^i\sum_{k=1}^{j-1}|\tau_{\alpha_j}||\tau_{\alpha_k}|+\sum_{j=1}^i(j+1)|\tau_{\alpha_j}|}\;\tau_\beta f_{\alpha_1\cdots\alpha_i}{}^\beta\;.
\end{aligned} \tag{A.24}$$

Similarly, we have

$$
\begin{aligned}
\tau_\alpha(Q\xi^\alpha) &= \sum_{i\geq 1}\frac{(-1)^{|\alpha|}}{i!}\tau_\alpha f_{\beta_1\cdots\beta_i}{}^\alpha(\xi^{\beta_1}\cdots\xi^{\beta_i})\ ,\\
(Q\xi) &= -\sum_{i\geq 1}\frac{(-1)^{\sum_{j=2}^i\sum_{k=1}^{j-1}|\tau_{\beta_j}||\tau_{\beta_k}|+\sum_{j=1}^i(j+1)|\tau_{\beta_j}|}}{i!}\mu_i(\tau_{\beta_1},\ldots,\tau_{\beta_i})(\xi^{\beta_1}\cdots\xi^{\beta_i})\\
&= -\sum_{i\geq 1}\frac{(-1)^{\sum_{j=2}^i\sum_{k=1}^{j-1}|\tau_{\beta_j}|(2|\tau_{\beta_k}|+1)+\sum_{j=1}^i(j+1)|\tau_{\beta_j}|}}{i!}\hat\mu_i(\xi,\ldots,\xi)\\
&= -\sum_{i\geq 1}\frac{1}{i!}\hat\mu_i(\xi,\ldots,\xi)\ ,
\end{aligned}
\tag{A.25}
$$

where we used the higher products $\hat\mu_i$ on $\mathsf{L}_{\mathscr{C}}$ as defined in (2.52b).

A.3. L_∞-morphisms from coalgebra morphisms

An advantage of the coalgebra formulation is that the notion of an L_∞-morphism results more natural and transparent. Recall that a *morphism of coalgebras* from $(\bigodot_0^\bullet \mathsf{V},\Delta_0)$ to $(\bigodot_0^\bullet \mathsf{V}',\Delta_0')$ is a map $\Phi:\bigodot_0^\bullet \mathsf{V}\to\bigodot_0^\bullet \mathsf{V}'$ of degree 0 which satisfies

$$\Delta_0'\circ\Phi \;=\; (\Phi\otimes\Phi)\circ\Delta_0\ . \tag{A.26}$$

If $\bigodot_0^\bullet \mathsf{V}$ and $\bigodot_0^\bullet \mathsf{V}'$ are both endowed with a codifferential D and D', respectively, we also demand that

$$\Phi\circ D \;=\; D'\circ\Phi\ . \tag{A.27}$$

Consider now two codifferential coalgebras $(\bigodot_0^\bullet \mathsf{V},\Delta_0,D)$ and $(\bigodot_0^\bullet \mathsf{V}',\Delta_0',D')$ corresponding to two L_∞-algebras. We can restrict a morphism

$$\Phi:(\textstyle\bigodot_0^\bullet\mathsf{V},\Delta_0,D)\;\to\;(\bigodot_0^\bullet\mathsf{V}',\Delta_0',D') \tag{A.28}$$

to the maps

$$\Phi^j \;:=\; \mathrm{pr}_{\bigodot^j\mathsf{V}'}\circ\Phi\ . \tag{A.29}$$

Note that Φ is uniquely reconstructed from Φ^1 since Φ is a morphism of coalgebras, cf. e.g. [300, Prop. 1.2]. In particular, using the further decomposition

$$\Phi_i^j \;:=\; \mathrm{pr}_{\bigodot^j\mathsf{V}'}\circ\Phi|_{\bigodot^i\mathsf{V}}\ , \tag{A.30}$$

we have explicitly

$$\Phi_i^j \;=\; \sum_{k_1+\cdots+k_j=i} \frac{1}{k_1!\cdots k_j!}\,\frac{1}{j!}\,(\Phi^1_{k_1}\odot\cdots\odot\Phi^1_{k_j})\;. \tag{A.31}$$

Note that the maps $\Phi^1_{k_1}\odot\cdots\odot\Phi^1_{k_j}$ act on the totally graded symmetrised elements of $\bigodot^i \mathsf{V}$ and the result is then projected onto $\bigodot_0^j \mathsf{V}'$. The degrees of all the maps Φ, Φ_i, Φ_i^j and $\Phi|_{\odot^i\mathsf{V}}$ are zero, since Φ is of degree zero, and all other maps originate from restriction and concatenation with projectors.

Condition (A.27) decomposed into the maps Φ_i^1 and D_i reads as

$$\sum_{j+k=i}\Phi^1_{k+1}\left(D_j\otimes\mathrm{id}^{\otimes k}\right)\;=\;\sum_{j=1}^{i}\;\sum_{k_1+\cdots+k_j=i} D'_j\circ\frac{1}{k_1!\cdots k_j!}\,\frac{1}{j!}\,(\Phi^1_{k_1}\odot\cdots\odot\Phi^1_{k_j})\;. \tag{A.32}$$

We multiply this equation by factors of s and s^{-1}, restrict it and rewrite both sides, using the shifted morphisms[1]

$$\phi_i\;:=\;(-1)^{\frac12 i(i-1)}s^{-1}\circ\Phi^1_i\circ s^{\otimes i}\quad\text{with}\quad|\phi_i|\;=\;i-1\;, \tag{A.33}$$

and the postcomposition with shuffles, cf. (A.13), as

$$\begin{aligned}
&\sum_{j+k=i}(-1)^{\frac12 k(k+1)+\frac12 i(i-1)}s^{-1}\circ\Phi^1_{k+1}\circ s^{\otimes(k+1)}\circ(s^{-1})^{\otimes(k+1)}\circ\left(D_j\otimes\mathrm{id}^{\otimes k}\right)\circ s^{\otimes i}\circ\\
&\qquad\circ(s^{-1})^{\otimes i}\circ\sum_{\sigma\in\mathrm{Sh}(j;i)}\varepsilon(\sigma,-)\sigma(-)\circ s^{\otimes i}\\
&\quad=\sum_{j+k=i}(-1)^{\frac12 k(k-1)+k(j+1)+\frac12 j(j-1)}\phi_{k+1}\circ\left(\mu_j\otimes\mathrm{id}^{\otimes k}\right)\circ\sum_{\sigma\in\mathrm{Sh}(j;i)}\chi(\sigma,-)\sigma(-)
\end{aligned} \tag{A.34}$$

[1]The degree of the map ϕ_i follows from its definition together with $|\Phi_i^1|=0$ and $|s|=1$.

and

$$\begin{aligned}
&\sum_{j=1}^{i} \sum_{k_1+\cdots+k_j=i} \frac{(-1)^{\frac{1}{2}i(i-1)+\frac{1}{2}j(j-1)}}{j!}\, s^{-1}\circ D'_j \circ s^{\otimes j}\circ (s^{-1})^{\otimes j}\circ\ (\Phi^1_{k_1}\otimes\cdots\otimes\Phi^1_{k_j})\circ s^{\otimes i}\circ\\
&\quad\circ (s^{-1})^{\otimes i}\circ \sum_{\sigma\in\mathrm{Sh}(j;i)} \varepsilon(\sigma,-)\sigma(-)\circ s^{\otimes i}\\
&= \sum_{j=1}^{i} \sum_{k_1+\cdots+k_j=i} \frac{1}{j!}\,\mu'_j\circ (s^{-1})^{\otimes j}\circ\ (\Phi^1_{k_1}\otimes\cdots\otimes\Phi^1_{k_j})\circ s^{\otimes i}\circ\\
&\quad\circ \sum_{\sigma\in\mathrm{Sh}(j;i)} \chi(\sigma,-)\sigma(-)\\
&= \sum_{j=1}^{i} \sum_{k_1+\cdots+k_j=i} \frac{(-1)^{\sum_{m=1}^{j-1}k_m(j-m)}}{j!}\,\mu'_j\circ\ (s^{-1}\circ\Phi_{k_1}\circ s^{k_1}\otimes\cdots\otimes s^{-1}\circ\Phi^1_{k_j}\circ s^{k_j})\circ\\
&\quad\circ \sum_{\sigma\in\mathrm{Sh}(j;i)} \chi(\sigma,-)\sigma(-)\\
&= \sum_{j=1}^{i} \sum_{k_1+\cdots+k_j=i} \frac{(-1)^{\frac{1}{2}(k_1+1)k_1+\cdots+\frac{1}{2}(k_j+1)k_j+\sum_{m=1}^{j-1}k_m(j-m)}}{j!}\,\mu'_j\circ\ (\phi_{k_1}\otimes\cdots\otimes\phi_{k_j})\circ\\
&\quad\circ \sum_{\sigma\in\mathrm{Sh}(k_1,\ldots,k_{j-1};i)} \chi(\sigma,-)\sigma(-)\ .
\end{aligned}\tag{A.35}$$

We obtain

$$\begin{aligned}
&\sum_{j+k=i} (-1)^{\frac{1}{2}i(i-1)+k}\phi_{k+1}\circ\left(\mu_j\otimes\mathrm{id}^{\otimes k}\right)\circ \sum_{\sigma\in\mathrm{Sh}(j;i)}\chi(\sigma,-)\sigma(-)\\
&\quad= \sum_{j=1}^{i} \sum_{k_1+\cdots+k_j=i} \frac{(-1)^{\frac{1}{2}(k_1+1)k_1+\cdots+\frac{1}{2}(k_j+1)k_j+\sum_{m=1}^{j-1}k_m(j-m)}}{j!}\,\mu'_j\circ\\
&\qquad\circ\ (\phi_{k_1}\otimes\cdots\otimes\phi_{k_j})\circ \sum_{\sigma\in\mathrm{Sh}(k_1,\ldots,k_{j-1};i)}\chi(\sigma,-)\sigma(-)\ .
\end{aligned}\tag{A.36}$$

Applied to $\ell_1\otimes\cdots\otimes\ell_i\in\mathsf{L}^{\otimes i}$, we find

$$\begin{aligned}
&\sum_{j+k=i}\sum_{\sigma\in\mathrm{Sh}(j;i)} (-1)^k\chi(\sigma,-)\phi_{k+1}(\mu_j(\ell_{\sigma(1)},\ldots,\ell_{\sigma(k)}),\ell_{\sigma(k)+1},\ldots,\ell_{\sigma_i})\\
&= \sum_{j=1}^{i} \sum_{k_1+\cdots+k_j=i} \frac{(-1)^{\sum_{1\le m<n\le j}k_mk_n+\sum_{m=1}^{j-1}k_m(j-m)+\sum_{m=2}^{j}(1-k_m)\sum_{k=1}^{k_1+\cdots+k_{m-1}}|\ell_{\sigma(k)}|_{\mathsf{L}}}}{j!}\times\\
&\quad\times \sum_{\sigma\in\mathrm{Sh}(k_1,\ldots,k_{j-1};i)} \chi(\sigma,-)\mu'_j(\phi_{k_1}(\ell_{\sigma(1)},\ldots,\ell_{\sigma(k_1)}),\ldots,\phi_{k_j}(\ell_{\sigma(i-k_j+1)},\ldots,\ell_{\sigma(i)}))\ ,
\end{aligned}\tag{A.37}$$

where Koszul signs are inserted to account for the permutation of the $\ell_{\sigma(m)}$ past the ϕ_{k_n}. We also used

$$(-1)^{\frac{1}{2}i(i-1)} = (-1)^{\sum_{m=1}^{j} \frac{1}{2}k_m(k_m-1)+\sum_{1\leq m<n\leq j} k_m k_n} \tag{A.38}$$

for $k_1 + \cdots + k_j = i$.

$\mathcal{B}$ Cochain complexes and Hodge–Kodaira decomposition

An L_∞-algebra L has an underlying cochain complex (L, μ_1). Morphisms and quasi-isomorphisms of L_∞-algebras are specialisations of cochain maps and quasi-isomorphisms between the underlying cochain complexes. Here, we merely report the content of [42, Appendix B], a good reference of which is [223].

A comment on our nomenclature: usually, one would work with the homology of chain complexes with a differential of degree -1. To avoid as much confusion as possible, and due to the prominence of the de Rham complex as key example of cochain complexes, we will use the terms *cochains* and *cohomology* with differentials of degree 1.

Cochain complexes. A *cochain complex* (C, d) over an associative ring R is a family of R-modules C^k for $k \in \mathbb{Z}$ and morphisms $\mathrm{d}_k : C^k \to C^{k+1}$ of degree 1 satisfying $\mathrm{d}_{k+1} \circ \mathrm{d}_k = 0$. The k*-cocycles* of the cochain complex are defined as $Z^k := \ker(\mathrm{d}_k) \subseteq C^k$ while the k-*coboundaries* of the cochain complex are defined as $B^k := \mathrm{im}(\mathrm{d}_{k-1}) \subseteq C^k$. The cohomology of the cochain complex is then $H^k_{\mathrm{d}}(C) := Z^k/B^k$.

A *morphism* of cochain complexes $\phi : (C, \mathrm{d}) \to (C', \mathrm{d}')$ is a family of maps $\phi_k : C^k \to C'^k$ of degree 0 such that $\phi_{k+1} \circ \mathrm{d}_k = \mathrm{d}'_k \circ \phi_k$. A *quasi-isomorphism* of cochain complexes is a morphism of cochain complexes which induces an isomorphism on the cohomologies.

A *cochain homotopy* between two morphisms $\phi : (C, \mathrm{d}) \to (C', \mathrm{d}')$ and $\psi : (C, \mathrm{d}) \to (C', \mathrm{d}')$ of cochain complexes is a family of morphisms $h_k : C^{k+1} \to C'^k$ of degree -1 such that $\phi_k - \psi_k = h_k \circ \mathrm{d}_k + \mathrm{d}'_{k-1} \circ h_k$. If such a cochain homotopy exists, we say that ϕ and ψ are homotopic. A cochain homotopy is the correct notion of a *2-morphism* of cochain complexes. A *homotopy equivalence* between two cochain complexes C and C' is a pair of morphisms

$\phi : C \to C'$ and $\psi : C' \to C$ such that $\psi \circ \phi$ and $\phi \circ \psi$ are homotopic to the respective identity maps. In the special case when $\psi \circ \phi = 1$ and $\phi \circ \psi$ is homotopic to the identity, we call this a *contracting homotopy* of $\phi \circ \psi$.[1]

Split cochain complexes and Hodge–Kodaira decomposition. Suppose (C, d) is a cochain complex of vector spaces. Then, we can always choose decompositions

$$\begin{gathered} C^k/\ker(\mathrm{d}_k) \;\cong\; \mathrm{im}(\mathrm{d}_k) \quad\Longrightarrow\quad C^k/Z^k \;\cong\; B^{k+1} \quad\Longrightarrow\quad C^k \;\cong\; Z^k \oplus Z^k_c \;, \\ H^k_{\mathrm{d}}(C) \;=\; Z^k/B^k \quad\Longrightarrow\quad Z^k \;\cong\; B^k \oplus B^k_c \;, \\ Z^k_c \;\cong\; B^{k+1} \quad \text{and} \quad B^k_c \;\cong\; H^k_{\mathrm{d}}(C) \end{gathered} \tag{B.1a}$$

since short exact vector space sequences always split[2]. We thus have

$$C^k \;\cong\; H^k_{\mathrm{d}}(C) \oplus B^k \oplus B^{k+1} \;. \tag{B.1b}$$

Consequently, we can define maps $h_k : C^k \to C^{k-1}$ of degree -1 by the compositions

$$h_k \,:\, C^k \;\twoheadrightarrow\; Z^k \;\twoheadrightarrow\; B^k \;\cong\; Z^{k-1}_c \;\hookrightarrow\; C^{k-1} \tag{B.2}$$

of the projections $C^k \twoheadrightarrow Z^k$ and $Z^k \twoheadrightarrow B^k$, the isomorphism $B^k \cong Z^{k-1}_c$, and the inclusion $Z^{k-1}_c \hookrightarrow C^{k-1}$. Since $h_{k+1}(B^{k+1}) \cong Z^k_c$, these maps satisfy $\mathrm{d}_k = \mathrm{d}_k \circ h_{k+1} \circ \mathrm{d}_k$.

This motivates the following definition. A general cochain complex (C, d) is called *split* whenever there is a family of morphisms $h_k : C^k \to C^{k-1}$ of degree -1, called the *splitting maps*, such that $\mathrm{d}_k = \mathrm{d}_k \circ h_{k+1} \circ \mathrm{d}_k$.

In this case, we then automatically have[3]

$$\begin{gathered} (h_{k+1} \circ \mathrm{d}_k)^2 \;=\; h_{k+1} \circ \mathrm{d}_k \;, \quad (\mathrm{d}_{k-1} \circ h_k)^2 \;=\; \mathrm{d}_{k-1} \circ h_k \;, \\ (h_{k+1} \circ \mathrm{d}_k) \circ (\mathrm{d}_{k-1} \circ h_k) \;=\; (\mathrm{d}_{k-1} \circ h_k) \circ (h_{k+1} \circ \mathrm{d}_k) \;=\; 0 \;, \end{gathered} \tag{B.3}$$

which, in turn, yield the decomposition

$$1 \;=\; P_k + h_{k+1} \circ \mathrm{d}_k + \mathrm{d}_{k-1} \circ h_k \tag{B.4a}$$

[1] This is a particular case of a *strong deformation retract*.

[2] Note that the same holds evidently true for tensor products of split cochain complexes.

[3] Whilst $(h_{k+1} \circ \mathrm{d}_k) \circ (\mathrm{d}_{k-1} \circ h_k) = 0$ follows trivially since $\mathrm{d}_k \circ \mathrm{d}_{k-1} = 0$, if $(\mathrm{d}_{k-1} \circ h_k) \circ (h_{k+1} \circ \mathrm{d}_k) \neq 0$ one may always re-define h_k and set $\tilde{h}_k := h_k - h_k \circ h_{k+1} \circ \mathrm{d}_k$. Then, $\mathrm{d}_k = \mathrm{d}_k \circ \tilde{h}_{k+1} \circ \mathrm{d}_k$ as well, and it is easy to check that $(\mathrm{d}_{k-1} \circ \tilde{h}_k) \circ (\tilde{h}_{k+1} \circ \mathrm{d}_k) = 0$ and also all other identities hold with $\tilde{h}_k$ instead of h_k. Note that we then have the more strict relation $\tilde{h}_k \circ \tilde{h}_{k+1} \circ \mathrm{d}_k = 0$.

with

$$P_k^2 = P_k , \quad P_k \circ \mathrm{d}_{k-1} = \mathrm{d}_k \circ P_k = 0 ,$$
$$P_k \circ (\mathrm{d}_{k-1} \circ h_k) = (\mathrm{d}_{k-1} \circ h_k) \circ P_k = P_k \circ (h_{k+1} \circ \mathrm{d}_k) = (h_{k+1} \circ \mathrm{d}_k) \circ P_k = 0 . \tag{B.4b}$$

Consequently,

$$C^k \cong \underbrace{\mathrm{im}(P_k)}_{\cong H^k_{\mathrm{d}}(C)} \oplus \underbrace{\mathrm{im}(\mathrm{d}_{k-1} \circ h_k)}_{\cong B^k} \oplus \underbrace{\mathrm{im}(h_{k+1} \circ \mathrm{d}_k)}_{=: Z^k_c} \cong H^k_{\mathrm{d}}(C) \oplus B^k \oplus Z^k_c . \tag{B.5}$$

This decomposition is known as the *abstract Hodge–Kodaira decomposition*. Furthermore, the projector P_k induces a surjection $p_k : C^k \twoheadrightarrow H^k_{\mathrm{d}}(C)$ and an injection $e_k : H^k_{\mathrm{d}}(C) \hookrightarrow C^k$ with $p_k \circ e_k = 1$ by means of $P_k = e_k \circ p_k$. Thus, we obtain the diagram

$$h \circlearrowright C \underset{e}{\overset{p}{\rightleftarrows}} H^\bullet_{\mathrm{d}}(C) . \tag{B.6}$$

Put differently, a splitting of a cochain complex (C, d) is equivalent to having morphisms of cochain complexes $p_k : C^k \twoheadrightarrow H^k_{\mathrm{d}}(C)$ and $e_k : H^k_{\mathrm{d}}(C) \hookrightarrow C^k$ such that $p_k \circ e_k = 1$ and $1 - e_k \circ p_k = h_{k+1} \circ \mathrm{d}_k + \mathrm{d}_{k-1} \circ h_k$. In other words, such a splitting is equivalent to a contracting homotopy $h_k : C^k \to C^{k-1}$ of $P_k = e_k \circ p_k$. Note that p_k is a quasi-isomorphisms of cochain complexes between C and $H^\bullet_{\mathrm{d}}(C)$ and so is e_k between $H^\bullet_{\mathrm{d}}(C)$ and C.

Extension to L_∞-algebras. Consider the cochain complex (L, μ_1) underlying an L_∞-algebra (L, μ_i) with a choice of decomposition (B.1)

$$\mathsf{L} \cong B \oplus B_c \oplus Z_c \quad \text{with} \quad B_c \cong H^\bullet_{\mu_1}(\mathsf{L}) . \tag{B.7}$$

We can use the inverse of the isomorphism between B_c and $H^\bullet_{\mu_1}(\mathsf{L})$ to define a strict L_∞-morphism to B_c. Composition with an L_∞-quasi-isomorphism between L and $H^\bullet_{\mu_1}(\mathsf{L})$ then yields an L_∞-quasi-isomorphism between L to B_c. Thus, any projection $\mathsf{L} \twoheadrightarrow B_c$ can be extended to an L_∞-quasi-isomorphism $\mathsf{L} \to B_c$. Consequently, we can use the abstract Hodge–Kodaira decomposition to find a minimal model. See Section 2.5. for explicit formulas.

Example. Let L be an L_∞-algebra together with decompositions $\mathsf{L} = \ker(\mu_1) \oplus V = \mathrm{im}(\mu_1) \oplus W$. We have

$$\mu_1 : \ker(\mu_1) \oplus V \to \mathrm{im}(\mu_1) \oplus W \tag{B.8}$$

and μ_1 is invertible as a map from $V \to \mathrm{im}(\mu_1)$ with inverse $\mu_1^{-1}|_{\mathrm{im}(\mu_1)}$. Define $h : \mathsf{L} \to \mathsf{L}$ as the map of degree -1

$$h \ := \ \mu_1^{-1}|_{\mathrm{im}(\mu_1)} \circ \mathrm{pr}_{\mathrm{im}(\mu_1)} \ , \tag{B.9}$$

where $\mathrm{pr}_{\mathrm{im}(\mu_1)} : \mathsf{L} \to \mathrm{im}(\mu_1)$ is the orthogonal projection. Note that the map h satisfies $\mu_1 \circ h \circ \mu_1 = \mu_1$ and we can use it as a starting point for the abstract Hodge–Kodaira decomposition.

C
Lemmata

We are in debt of a few computational proofs of results presented in Chapter 3. In this Appendix, we aim to settle the debt, proving various formulas involving L_∞-algebras. The following material is contained in [42, Appendix C]. In some cases, such results are presented in more than one of the three possible descriptions of L_∞-algebras: higher brackets, differential coalgebra, differential graded algebra. While some of the calculations are slightly involved, they are illuminating in one sense or another.

Throughout this Appendix, let L be an L_∞-algebra with higher products μ_i. We shall occasionally assume that L is cyclic with an inner product $\langle -, - \rangle_{\mathsf{L}}$.

Cauchy product. We start by proving a result we will make extensive use of along this Appendix. Recall that the Cauchy product of two (absolutely convergent) series $\sum_{i\geq 0} a_i$ and $\sum_{i\geq 0} b_i$ is

$$\sum_{i\geq 0} a_i \sum_{j\geq 0} b_j \;=\; \sum_{i,j\geq 0} a_i b_j \;=\; \sum_{i\geq 0}\sum_{j=0}^{i} a_j b_{i-j} \;=\; \sum_{i\geq 0}\sum_{j+k=i} a_j b_k \;. \tag{C.1}$$

Hence,

$$\sum_{i,j\geq 0} \frac{1}{i!j!} a_i b_j \;=\; \sum_{i\geq 0} \frac{1}{i!} \sum_{j+k=i} \binom{i}{j} a_j b_k \;. \tag{C.2}$$

Furthermore, note that (C.1) generalises to

$$\sum_{k_1\geq 0} a^{(1)}_{k_1} \cdots \sum_{k_j\geq 0} a^{(j)}_{k_j} \;=\; \sum_{i\geq 0} \sum_{k_1+\cdots+k_j=i} a^{(1)}_{k_1} \cdots a^{(j)}_{k_j} \tag{C.3}$$

for the product of j (absolutely convergent) series and so,

$$
\begin{aligned}
\sum_{k_1\geq 0}\frac{1}{k_1!}a^{(1)}_{k_1}\cdots\sum_{k_j\geq 0}\frac{1}{k_j!}a^{(j)}_{k_j} &= \\
&= \sum_{i\geq 0}\frac{1}{i!}\sum_{k_1+\cdots+k_j=i}\binom{i-k_1}{k_2}\cdots\binom{i-k_1-\cdots-k_{j-2}}{k_{j-1}}a^{(1)}_{k_1}\cdots a^{(j)}_{k_j}\ .
\end{aligned}
\tag{C.4}
$$

C.1. Tensor product L_∞-algebras

Let A be a dg-algebra. We wish to verify the higher homotopy Jacobi identities (2.44c) of the higher products μ'_i defined in (2.51) as well as the cyclicity (2.51b) of the tensor product L_∞-algebra L_A. To this end, let us use the abbreviation $\mathsf{l}_i := a_i\otimes\ell_i$ for homogeneous $a_i\in\mathsf{A}$ and $\ell_i\in\mathsf{L}$ together with $|\mathsf{l}_i|_\mathsf{A} := |a_i|_\mathsf{A}$ and $|\mathsf{l}_i|_\mathsf{L} := |\ell_i|_\mathsf{L}$. It is sufficient to consider only those elements since the result for general elements follows from linearity.

The higher homotopy identities (2.44c) of the higher products (2.51) are

$$
\sum_{j+k=i}\sum_\sigma \chi(\sigma;\mathsf{l}_1,\ldots,\mathsf{l}_{j+k})(-1)^k\mu'_{k+1}(\mu'_j(\mathsf{l}_{\sigma(1)},\ldots,\mathsf{l}_{\sigma(j)}),\mathsf{l}_{\sigma(j+1)},\ldots,\mathsf{l}_{\sigma(j+k)}) \ = \ 0\ . \tag{C.5}
$$

We have

$$
\mu_i(\ell_{\sigma(1)},\ldots,\ell_{\sigma(i)}) \ = \ \chi(\sigma;\ell_1,\ldots,\ell_i)\mu_i(\ell_1,\ldots,\ell_i) \tag{C.6a}
$$

and

$$
\begin{aligned}
\chi(\sigma;\mathsf{l}_1,\ldots,\mathsf{l}_i) \ &= \ \chi(\sigma;a_1,\ldots,a_i)\chi(\sigma;\ell_1,\ldots,\ell_n)\times \\
&\qquad \times(-1)^{\sum_{j=2}^i|a_{\sigma(j)}|_\mathsf{A}\sum_{k=1}^{j-1}|\ell_{\sigma(k)}|_\mathsf{L}+\sum_{j=2}^i|a_j|_\mathsf{A}\sum_{k=1}^{j-1}|\ell_k|_\mathsf{L}}\ .
\end{aligned}
\tag{C.6b}
$$

To prove (C.5), we first focus on the terms containing the differential d. For $i=1$, we have

$$
\begin{aligned}
\mu'_1(\mu'_1(\mathsf{l}_1)) \ &= \ \mu'_1(\mathrm{d}a_1\otimes\ell_1+(-1)^{|a_1|_\mathsf{A}}a_1\otimes\mu_1(\ell_1)) \\
&= \ (-1)^{|a_1|_\mathsf{A}+1}\mathrm{d}a_1\otimes\mu_1(\ell_1)+(-1)^{|a_1|_\mathsf{A}}\mathrm{d}a_1\otimes\mu_1(\ell_1) \\
&= \ 0\ .
\end{aligned}
\tag{C.7}
$$

For $i > 1$, the relevant terms are

$$\begin{aligned}
&\mathrm{d}(\mu_i'(\mathsf{l}_1,\dots,\mathsf{l}_i)) + \sum_\sigma \chi(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i)(-1)^{i-1}\mu_i'(\mathrm{d}\big(\mathsf{l}_{\sigma(1)}\big),\mathsf{l}_{\sigma(2)},\dots,\mathsf{l}_{\sigma(i)}) \;=\\
&=\; s_1\,\mathrm{d}(a_1\cdots a_i)\otimes\mu_i(\ell_1,\dots,\ell_i)+\\
&\qquad+\sum_\sigma \chi(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i)\,s_2(\sigma)\left[(\mathrm{d}a_{\sigma(1)})a_{\sigma(2)}\cdots a_{\sigma(i)}\right]\otimes\mu_i(\ell_{\sigma(1)},\dots,\ell_{\sigma(i)})\\
&=\; s_1\sum_\sigma \chi(\sigma;a_1,\dots,a_i)\chi(\sigma;\ell_1,\dots,\ell_i)\left[(\mathrm{d}a_{\sigma(1)})a_{\sigma(2)}\cdots a_{\sigma(i)}\right]\otimes\mu_i(\ell_{\sigma(1)},\dots,\ell_{\sigma(i)})+\\
&\qquad+\sum_\sigma \chi(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i)\,s_2(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i)\left[(\mathrm{d}a_{\sigma(1)})a_{\sigma(2)}\cdots a_{\sigma(i)}\right]\otimes\mu_i(\ell_{\sigma(1)},\dots,\ell_{\sigma(i)})
\end{aligned}\tag{C.8a}$$

with signs

$$\begin{aligned}
s_1 \;&:=\; (-1)^{i\sum_{j=1}^i|a_j|_\mathsf{A}+\sum_{j=2}^i|a_j|_\mathsf{A}\sum_{k=1}^{j-1}|\ell_k|_\mathsf{L}}\,,\\
s_2(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i) \;&:=\; (-1)^{i-1}(-1)^{i(|a_{\sigma(1)}|_\mathsf{A}+1+\sum_{j=2}^i|a_{\sigma(j)}|_\mathsf{A})+\sum_{j=2}^i|a_{\sigma(j)}|_\mathsf{A}\sum_{k=1}^{j-1}|\ell_{\sigma(k)}|_\mathsf{L}}\\
&=\; -(-1)^{i\sum_{j=1}^i|a_{\sigma(j)}|_\mathsf{A}+\sum_{j=2}^i|a_{\sigma(j)}|_\mathsf{A}\sum_{k=1}^{j-1}|\ell_{\sigma(k)}|_\mathsf{L}}\,.
\end{aligned}\tag{C.8b}$$

The expression (C.8a) clearly vanishes since

$$\chi(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i)\,s_2(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i) \;=\; -s_1\,\chi(\sigma;a_1,\dots,a_i)\chi(\sigma;\ell_1,\dots,\ell_i)\tag{C.9}$$

due to (C.6b).

The remaining terms in (C.5) combine to

$$\begin{aligned}
&\sum_{j+k=i}\sum_\sigma s_3(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i)\times\\
&\qquad\times(a_1\cdots a_{j+k})\otimes\mu_{k+1}(\mu_j(\ell_{\sigma(1)},\dots,\ell_{\sigma(j)}),\ell_{\sigma(j+1)},\dots,\ell_{\sigma(j+k)}) \;=\; 0
\end{aligned}\tag{C.10}$$

with the sign $s_3(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i)$ given by

$$\begin{aligned}
&s_3(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_i) \;:=\\
&\quad:=\; \chi(\sigma;\mathsf{l}_1,\dots,\mathsf{l}_{j+k})(-1)^k\chi(\sigma;a_1,\dots a_{j+k})\times\\
&\qquad\times(-1)^{j\sum_{m=1}^j|a_{\sigma(m)}|_\mathsf{A}+\sum_{m=2}^j|a_{\sigma(m)}|_\mathsf{A}\sum_{n=1}^{m-1}|\ell_{\sigma(n)}|_\mathsf{L}}\times\\
&\qquad\times(-1)^{(k+1)\sum_{m=1}^{j+k}|a_m|_\mathsf{A}+\sum_{m=1}^k|a_{\sigma(j+m)}|_\mathsf{A}\,(j+\sum_{n=1}^{j+m-1}|\ell_{\sigma(n)}|_\mathsf{L})}\\
&\quad=\; (-1)^k\chi(\sigma;\ell_1,\dots,\ell_{j+k})(-1)^{\sum_{m=2}^{j+k}|a_{\sigma(m)}|_\mathsf{A}\sum_{n=1}^{m-1}|\ell_{\sigma(n)}|_\mathsf{L}+\sum_{m=2}^{j+k}|a_m|_\mathsf{A}\sum_{n=1}^{m-1}|\ell_n|_\mathsf{L}}\times\\
&\qquad\times(-1)^{j\sum_{m=1}^j|a_{\sigma(m)}|_\mathsf{A}+\sum_{m=2}^j|a_{\sigma(m)}|_\mathsf{A}\sum_{n=1}^{m-1}|\ell_{\sigma(n)}|_\mathsf{L}}\times\\
&\qquad\times(-1)^{(k+1)\sum_{m=1}^{j+k}|a_m|_\mathsf{A}+\sum_{m=1}^k|a_{\sigma(j+m)}|_\mathsf{A}\,(j+\sum_{n=1}^{j+m-1}|\ell_{\sigma(n)}|_\mathsf{L})}\\
&\quad=\; (-1)^k\chi(\sigma;\ell_1,\dots,\ell_{j+k})\,\underbrace{(-1)^{\sum_{m=2}^{j+k}|a_m|_\mathsf{A}\sum_{n=1}^{m-1}|\ell_n|_\mathsf{L}+(j+k+1)\sum_{m=1}^{j+k}|a_m|_\mathsf{A}}}_{=:\,s_4}\,,
\end{aligned}\tag{C.11}$$

where we used again (C.6b). Note that s_4 contributes an overall sign so that (C.5) reduces to the homotopy Jacobi identity (2.44c) on L.

Next suppose that L and A are equipped with inner products $\langle -,-\rangle_\mathsf{L}$ and $\langle -,-\rangle_\mathsf{A}$. Let us verify the cyclicity of the inner product $\langle -,-\rangle_{\mathsf{L}_\mathsf{A}}$ defined in (2.51b). Using the definitions of the higher products μ_i' given (2.51) we obtain for $i=1$

$$
\begin{aligned}
\langle \mathsf{l}_1, \mu_1'(\mathsf{l}_2)\rangle_{\mathsf{L}_\mathsf{A}} &= \\
&= \langle a_1\otimes \ell_1, \mathrm{d}a_2\otimes\ell_2 + (-1)^{|\mathsf{l}_2|_\mathsf{A}} a_2\otimes \mu_1(\ell_2)\rangle_{\mathsf{L}_\mathsf{A}}\\
&= (-1)^{|\mathsf{l}_1|_\mathsf{L}(|\mathsf{l}_2|_\mathsf{A}+1)}\langle a_1,\mathrm{d}a_2\rangle_\mathsf{A}\ \langle \ell_1,\ell_2\rangle_\mathsf{L} + (-1)^{(|\mathsf{l}_1|_\mathsf{L}+1)|\mathsf{l}_2|_\mathsf{A}}\langle a_1,a_2\rangle_\mathsf{A}\ \langle\ell_1,\mu_1(\ell_2)\rangle_\mathsf{L}\\
&= -(-1)^{|\mathsf{l}_1|_\mathsf{L}(|\mathsf{l}_2|_\mathsf{A}+1)+|\mathsf{l}_2|_\mathsf{A}(|\mathsf{l}_1|_\mathsf{A}+1)+|\mathsf{l}_1|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{L}|\mathsf{l}_2|_\mathsf{L}}\langle a_2,\mathrm{d}a_1\rangle_\mathsf{A}\ \langle\ell_2,\ell_1\rangle_\mathsf{L} -\\
&\quad - (-1)^{(|\mathsf{l}_1|_\mathsf{L}+1)|\mathsf{l}_2|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{A}|\mathsf{l}_2|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{L}+|\mathsf{l}_2|_\mathsf{L}+|\mathsf{l}_1|_\mathsf{L}|\mathsf{l}_2|_\mathsf{L}}\langle a_2,a_1\rangle_\mathsf{A}\ \langle\ell_2,\mu_1(\ell_1)\rangle_\mathsf{L}\\
&= -(-1)^{|\mathsf{l}_1|_\mathsf{L}(|\mathsf{l}_2|_\mathsf{A}+1)+|\mathsf{l}_2|_\mathsf{A}(|\mathsf{l}_1|_\mathsf{A}+1)+|\mathsf{l}_1|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{L}|\mathsf{l}_2|_\mathsf{L}+|\mathsf{l}_2|_\mathsf{L}(|\mathsf{l}_1|_\mathsf{A}+1)}\langle a_2\otimes\ell_2,\mathrm{d}a_1\otimes\ell_1\rangle_{\mathsf{L}_\mathsf{A}} -\\
&\quad - (-1)^{(|\mathsf{l}_1|_\mathsf{L}+1)|\mathsf{l}_2|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{A}|\mathsf{l}_2|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{L}+|\mathsf{l}_2|_\mathsf{L}+|\mathsf{l}_1|_\mathsf{L}|\mathsf{l}_2|_\mathsf{L}+|\mathsf{l}_2|_\mathsf{L}|\mathsf{l}_1|_\mathsf{A}}\langle a_2\otimes\ell_2,a_1\otimes\mu_1(\ell_1)\rangle_{\mathsf{L}_\mathsf{A}}\\
&= (-1)^{1+|\mathsf{l}_1|_{\mathsf{L}_\mathsf{A}}|\mathsf{l}_2|_{\mathsf{L}_\mathsf{A}}+|\mathsf{l}_1|_{\mathsf{L}_\mathsf{A}}|\mathsf{l}_2|_{\mathsf{L}_\mathsf{A}}}\langle\mathsf{l}_2,\mu_1'(\mathsf{l}_1)\rangle_{\mathsf{L}_\mathsf{A}}\ ,
\end{aligned}
\tag{C.12}
$$

while for $i\geq 2$ we find

$$
\begin{aligned}
\langle \mathsf{l}_1, \mu_i'(\mathsf{l}_2,\ldots,\mathsf{l}_{i+1})\rangle_{\mathsf{L}_\mathsf{A}} &= \\
&= (-1)^{i\sum_{j=2}^{i+1}|\mathsf{l}_j|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{L}\sum_{j=2}^{i+1}|\mathsf{l}_j|_\mathsf{A}+\sum_{j=3}^{i+1}|\mathsf{l}_j|_\mathsf{A}\sum_{k=2}^{j-1}|\mathsf{l}_k|_\mathsf{L}}\times\\
&\quad\times\langle a_1,a_2\cdots a_{i+1}\rangle_\mathsf{A}\ \langle\ell_1,\mu_i(\ell_2,\ldots,\ell_{i+1})\rangle_\mathsf{L}\\
&= (-1)^{i\sum_{j=2}^{i+1}|\mathsf{l}_j|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{L}\sum_{j=2}^{i+1}|\mathsf{l}_j|_\mathsf{A}+\sum_{j=3}^{i+1}|\mathsf{l}_j|_\mathsf{A}\sum_{k=2}^{j-1}|\mathsf{l}_k|_\mathsf{L}}\times\\
&\quad\times(-1)^{i+i(|\mathsf{l}_1|_\mathsf{L}+|\mathsf{l}_{i+1}|_\mathsf{L})+|\mathsf{l}_{i+1}|_\mathsf{L}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{L}+|\mathsf{l}_{i+1}|_\mathsf{A}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}}\times\\
&\quad\times\langle a_{i+1},a_1\cdots a_i\rangle_\mathsf{A}\ \langle\ell_{i+1},\mu_i(\ell_1,\ldots,\ell_i)\rangle_\mathsf{L}\\
&= (-1)^{i\sum_{j=2}^{i+1}|\mathsf{l}_j|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{L}\sum_{j=2}^{i+1}|\mathsf{l}_j|_\mathsf{A}+\sum_{j=3}^{i+1}|\mathsf{l}_j|_\mathsf{A}\sum_{k=2}^{j-1}|\mathsf{l}_k|_\mathsf{L}}\times\\
&\quad\times(-1)^{i+i(|\mathsf{l}_1|_\mathsf{L}+|\mathsf{l}_{i+1}|_\mathsf{L})+|\mathsf{l}_{i+1}|_\mathsf{L}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{L}+|\mathsf{l}_{i+1}|_\mathsf{A}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}}\times\\
&\quad\times(-1)^{i\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}+|\mathsf{l}_{i+1}|_\mathsf{L}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}+\sum_{j=2}^{i}|\mathsf{l}_j|_\mathsf{A}\sum_{k=1}^{j-1}|\mathsf{l}_k|_\mathsf{L}}\times\\
&\quad\times\langle\mathsf{l}_{i+1},\mu_i'(\mathsf{l}_1,\ldots,\mathsf{l}_i)\rangle_{\mathsf{L}_\mathsf{A}}\ ,
\end{aligned}
\tag{C.13}
$$

and therefore

$$
\begin{aligned}
&(-1)^{i\sum_{j=2}^{i+1}|\mathsf{l}_j|_\mathsf{A}+|\mathsf{l}_1|_\mathsf{L}\sum_{j=2}^{i+1}|\mathsf{l}_j|_\mathsf{A}+\sum_{j=3}^{i+1}|\mathsf{l}_j|_\mathsf{A}\sum_{k=2}^{j-1}|\mathsf{l}_k|_\mathsf{L}}\times\\
&\qquad\times(-1)^{i+i(|\mathsf{l}_1|_\mathsf{L}+|\mathsf{l}_{i+1}|_\mathsf{L})+|\mathsf{l}_{i+1}|_\mathsf{L}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{L}+|\mathsf{l}_{i+1}|_\mathsf{A}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}}\times\\
&\qquad\times(-1)^{i\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}+|\mathsf{l}_{i+1}|_\mathsf{L}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}+\sum_{j=2}^{i}|\mathsf{l}_j|_\mathsf{A}\sum_{k=1}^{j-1}|\mathsf{l}_k|_\mathsf{L}}\ =\\
&=\ (-1)^{i+i(|\mathsf{l}_1|_{\mathsf{L}_\mathsf{A}}+|\mathsf{l}_{i+1}|_{\mathsf{L}_\mathsf{A}})+|\mathsf{l}_{i+1}|_\mathsf{L}\sum_{j=1}^{i}|\mathsf{l}_j|_{\mathsf{L}_\mathsf{A}}+|\mathsf{l}_{i+1}|_\mathsf{A}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}}\times\\
&\qquad\times(-1)^{|\mathsf{l}_1|_\mathsf{L}\sum_{j=2}^{i+1}|\mathsf{l}_j|_\mathsf{A}+\sum_{j=2}^{i}|\mathsf{l}_j|_\mathsf{A}\sum_{k=1}^{j-1}|\mathsf{l}_k|_\mathsf{L}+\sum_{j=3}^{i+1}|\mathsf{l}_j|_\mathsf{A}\sum_{k=2}^{j-1}|\mathsf{l}_k|_\mathsf{L}}\\
&=\ (-1)^{i+i(|\mathsf{l}_1|_{\mathsf{L}_\mathsf{A}}+|\mathsf{l}_{i+1}|_{\mathsf{L}_\mathsf{A}})+|\mathsf{l}_{i+1}|_\mathsf{L}\sum_{j=1}^{i}|\mathsf{l}_j|_{\mathsf{L}_\mathsf{A}}+|\mathsf{l}_{i+1}|_\mathsf{A}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}}\times\\
&\qquad\times(-1)^{|\mathsf{l}_1|_\mathsf{L}(|\mathsf{l}_2|_\mathsf{A}+\cdots+|\mathsf{l}_i|_\mathsf{A}+|\mathsf{l}_{i+1}|_\mathsf{A})}\times\\
&\qquad\times(-1)^{|\mathsf{l}_1|_\mathsf{L}|\mathsf{l}_2|_\mathsf{A}+(|\mathsf{l}_1|_\mathsf{L}+|\mathsf{l}_2|_\mathsf{L})|\mathsf{l}_3|_\mathsf{A}+\cdots+(|\mathsf{l}_1|_\mathsf{L}+\cdots+|\mathsf{l}_{i-1}|_\mathsf{L})|\mathsf{l}_i|_\mathsf{A}}\times\\
&\qquad\times(-1)^{|\mathsf{l}_2|_\mathsf{L}|\mathsf{l}_3|_\mathsf{A}+(|\mathsf{l}_2|_\mathsf{L}+|\mathsf{l}_3|_\mathsf{L})|\mathsf{l}_4|_\mathsf{A}+\cdots+(|\mathsf{l}_2|_\mathsf{L}+\cdots+|\mathsf{l}_{i-1}|_\mathsf{L})|\mathsf{l}_i|_\mathsf{A}+(|\mathsf{l}_2|_\mathsf{L}+\cdots+|\mathsf{l}_i|_\mathsf{L})|\mathsf{l}_{i+1}|_\mathsf{A}}\\
&=\ (-1)^{i+i(|\mathsf{l}_1|_{\mathsf{L}_\mathsf{A}}+|\mathsf{l}_{i+1}|_{\mathsf{L}_\mathsf{A}})+|\mathsf{l}_{i+1}|_\mathsf{L}\sum_{j=1}^{i}|\mathsf{l}_j|_{\mathsf{L}_\mathsf{A}}+|\mathsf{l}_{i+1}|_\mathsf{A}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{A}}\times\\
&\qquad\times(-1)^{|\mathsf{l}_{i+1}|_\mathsf{A}\sum_{j=1}^{i}|\mathsf{l}_j|_\mathsf{L}+2\sum_{j=1}^{i-1}|\mathsf{l}_j|_\mathsf{L}\sum_{k=j+1}^{i}|\mathsf{l}_k|_\mathsf{A}}\\
&=\ (-1)^{i+i(|\mathsf{l}_1|_{\mathsf{L}_\mathsf{A}}+|\mathsf{l}_{i+1}|_{\mathsf{L}_\mathsf{A}})+|\mathsf{l}_{i+1}|_{\mathsf{L}_\mathsf{A}}\sum_{j=1}^{i}|\mathsf{l}_j|_{\mathsf{L}_\mathsf{A}}}\ .
\end{aligned}
\tag{C.14}
$$

Altogether, we obtain the cyclicity

$$
\begin{aligned}
&\langle\mathsf{l}_1,\mu_i'(\mathsf{l}_2,\ldots,\mathsf{l}_{i+1})\rangle_{\mathsf{L}_\mathsf{A}}\ =\\
&\qquad=\ (-1)^{i+i(|\mathsf{l}_1|_{\mathsf{L}_\mathsf{A}}+|\mathsf{l}_{i+1}|_{\mathsf{L}_\mathsf{A}})+|\mathsf{l}_{i+1}|_{\mathsf{L}_\mathsf{A}}\sum_{j=1}^{i}|\mathsf{l}_j|_{\mathsf{L}_\mathsf{A}}}\langle\mathsf{l}_{i+1},\mu_i'(\mathsf{l}_1,\ldots,\mathsf{l}_i)\rangle_{\mathsf{L}_\mathsf{A}}
\end{aligned}
\tag{C.15}
$$

for $i\in\mathbb{N}$, as claimed.

C.2. Homotopy Maurer–Cartan theory

Bianchi identity. Let us verify the Bianchi identity (3.6) for the curvature f defined in (3.4). To this end, recall the homotopy Jacobi identity (2.44c) for a gauge potential $a\in\mathsf{L}_1$,

$$
\sum_{j+k=i}(-1)^j\binom{i}{j}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a)\ =\ 0\ . \tag{C.16}
$$

Making use of (C.2), we rewrite

$$
\begin{aligned}
\sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+1}(f,a,\ldots,a)\ &=\ \sum_{i\geq 0}\frac{(-1)^i}{i!j!}\mu_{i+1}(\mu_j(a,\ldots,a),a,\ldots,a)\\
&=\ \sum_{i\geq 0}\frac{1}{i!}\sum_{j+k=i}(-1)^j\binom{i}{j}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a)\ =\ 0\ .
\end{aligned}
\tag{C.17}
$$

Let us prove the same statement using the Q-manifold morphism language, and in particular formula (2.53),

$$Q\xi \;=\; -\sum_{i\geq 1}\frac{1}{i!}\mu_i(\xi,\ldots,\xi) \tag{C.18}$$

for $\xi=\xi^\alpha\otimes\tau_\alpha\in(\mathsf{L}[1])^*\otimes\mathsf{L}$. We can evaluate this function on

$$\begin{aligned} \mathrm{e}^{sa} \;&:=\; 1+\mathrm{e}_0^{sa}\\ &=\; 1+sa+sa\otimes sa+sa\otimes sa\otimes sa+\cdots\\ &=\; 1+sa+\tfrac{1}{2}sa\odot sa+\tfrac{1}{3!}sa\odot sa\odot sa+\cdots\;, \end{aligned} \tag{C.19}$$

for $a\in\mathsf{L}_1$. Since sa is even, no Koszul signs appear and we obtain

$$(Q\xi)(\mathrm{e}_0^{sa}) \;=\; -\sum_{i\geq 1}\frac{1}{i!}\mu_i(a,\ldots,a) \;=\; -f\;. \tag{C.20}$$

This equation will prove very helpful. It also shows why the definition of f is so natural.

A first advantage of this alternative perspective is a trivial derivation of the Bianchi identity. We compute

$$\begin{aligned} 0 \;&=\; (-Q^2\xi)(\mathrm{e}_0^{sa})\\ &=\; (Qf(\xi)(\mathrm{e}_0^{sa})\\ &=\; \left(Q\sum_{i\geq 1}\frac{1}{i!}\mu_i(\xi,\ldots,\xi)\right)(\mathrm{e}_0^{sa})\\ &=\; \left(\sum_{i\geq 0}\frac{(-1)^{2-(i+1)}}{i!}\mu_{i+1}(Q\xi,\xi,\ldots,\xi)\right)(\mathrm{e}_0^{sa})\\ &=\; \left(\sum_{i\geq 0}\frac{(-1)^{i}}{i!}\mu_{i+1}(f(\xi),\xi,\ldots,\xi)\right)(\mathrm{e}_0^{sa}),\\ &=\; \sum_{i\geq 0}\frac{(-1)^{i}}{i!}\mu_{i+1}(f,a,\ldots,a)\;. \end{aligned} \tag{C.21}$$

Let us also give the coalgebra picture description, which shall be useful later on. Here, we have

$$D(\mathrm{e}_0^{sa}) \;=\; \sum_{i\geq 1}\tfrac{1}{i!}D_i(sa,\ldots,sa)\odot\mathrm{e}^{sa} \;=\; sf\odot\mathrm{e}^{sa}\;, \tag{C.22}$$

because

$$\begin{aligned} sf \;&=\; s\sum_{i\geq 1}\tfrac{1}{i!}\mu_i(a,\ldots,a)\\ &=\; s\sum_{i\geq 1}\tfrac{(-1)^{\frac{1}{2}i(i-1)}}{i!}s^{-1}\circ D_i\circ s^{\otimes i}(a,\ldots,a)\\ &=\; \sum_{i\geq 1}\tfrac{1}{i!}D_i(sa,\ldots,sa)\;. \end{aligned} \tag{C.23}$$

The Bianchi identity follows here from D^2 as follows:

$$
\begin{aligned}
0 &= (s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]}\circ D\circ D)(\mathrm{e}_0^{sa}) \\
&= (s^{-1}\circ \mathrm{pr}_{\odot^1 \mathsf{L}[1]}\circ D)(sf\odot \mathrm{e}^{sa}) \\
&= \sum_{i\geq 1}\frac{(-1)^i}{i!}\mu_{i+1}(f,a,\dots,a)\ .
\end{aligned}
\tag{C.24}
$$

Commutator of gauge transformations. We wish to prove (3.12). Using (C.2), we obtain

$$
\begin{aligned}
\delta_{c_0}\delta_{c_0'}a &= \sum_{i,j\geq 0}\frac{1}{i!j!}\mu_{i+2}(\mu_{j+1}(a,\dots,a,c_0'),a,\dots,a,c_0) \\
&= \sum_{i\geq 0}\frac{1}{i!}\sum_{j+k=i}\binom{i}{j}\mu_{j+2}(\mu_{k+1}(a,\dots,a,c_0'),a,\dots,a,c_0) \\
&= \sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}\binom{i-2}{j-1}\mu_{j+1}(\mu_k(a,\dots,a,c_0'),a,\dots,a,c_0) \\
&= \sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}\binom{i-2}{k-1}\mu_{j+1}(\mu_k(a,\dots,a,c_0'),a,\dots,a,c_0) \\
&= \delta_{c_0'}\delta_{c_0}a+ \\
&\qquad +\sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}(-1)^j\binom{i-2}{j-2}\mu_{j+1}(\mu_k(a,\dots,a),a,\dots,a,c_0,c_0')+ \\
&\qquad +\sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}(-1)^j\binom{i-2}{k-2}\mu_{j+1}(\mu_k(a,\dots,a,c_0,c_0'),a,\dots,a)
\end{aligned}
$$

$$
\begin{aligned}
&= \delta_{c_0'}\delta_{c_0} a + \\
&\quad + \sum_{i,j\geq 0} \frac{1}{i!j!}(-1)^i \mu_{i+3}(\mu_j(a,\ldots,a),a,\ldots,a,c_0,c_0') + \\
&\quad + \sum_{i,j\geq 0} \frac{1}{i!j!}\mu_{i+1}(a,\ldots,a,\mu_{j+2}(a,\ldots,a,c_0,c_0'))\ ,
\end{aligned}
\tag{C.25}
$$

where in the fifth step we have used the homotopy Jacobi identity (2.44c) for the elements $(a,\ldots,a,c_0,c_0')$,

$$
\begin{aligned}
&\sum_{j+k=i}(-1)^j\binom{i-2}{j-2}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a,c_0,c_0') + \\
&\quad - \sum_{j+k=i}\binom{i-2}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_0),a,\ldots,a,c_0') + \\
&\quad + \sum_{j+k=i}\binom{i-2}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_0'),a,\ldots,a,c_0) + \\
&\quad + \sum_{j+k=i}(-1)^j\binom{i-2}{k-2}\mu_{j+1}(\mu_k(a,\ldots,a,c_0,c_0'),a,\ldots,a) \;=\; 0\ .
\end{aligned}
\tag{C.26}
$$

Hence, using the expression (3.4) of the curvature, we find (3.12):

$$
[\delta_{c_0},\delta_{c_0'}]a \;=\; \delta_{c_0''}a + \sum_{i\geq 0}\frac{1}{i!}(-1)^i\mu_{i+3}(f,a,\ldots,a,c_0,c_0') \tag{C.27a}
$$

with

$$
c_0'' \;:=\; \sum_{i\geq 0}\frac{1}{i!}\mu_{i+2}(a,\ldots,a,c_0,c_0')\ . \tag{C.27b}
$$

A proof via the Q-manifold evaluation map ξ is possible, but rather technical and not very enlightening.

Gauge transformation of the curvature. The gauge transformation of the curvature is derived using the Bianchi identity (3.6) on $\Omega^\bullet(I,\mathsf{L})$,

$$
\begin{aligned}
0 \;&=\; \sum_{i\geq 0}\frac{(-1)^i}{i!}\hat\mu_{i+1}(\mathsf{f}(t),\mathsf{a}(t),\ldots,\mathsf{a}(t)) \\
&=\; \sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+1}(f(t),a(t),\ldots,a(t)) + \\
&\qquad + \mathrm{d}t\otimes\left\{\frac{\partial}{\partial t}f(t) - \sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+2}(f(t),a(t),\ldots,a(t),c(t))\right\} \\
&=\; \mathrm{d}t\otimes\left\{\frac{\partial}{\partial t}f(t) - \sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+2}(f(t),a(t),\ldots,a(t),c(t))\right\}\ ,
\end{aligned}
\tag{C.28}
$$

from which we read off the gauge transformation of the curvature,

$$\delta_{c_0} f \;:=\; \frac{\partial}{\partial t}\Big|_{t=0} f(t) \;=\; \sum_{i\geq 0} \frac{(-1)^i}{i!}\mu_{i+2}(f, a, \ldots, a, c_0) \;. \tag{C.29}$$

Alternatively, we can perform the direct computation using brackets. Upon making use of (C.2), we find

$$\begin{aligned}
\delta_{c_0} f \;&=\; \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(\delta_{c_0}a, a, \ldots, a)\\
&=\; \sum_{i,j\geq 0}\frac{1}{i!j!}\mu_{i+1}(\mu_{j+1}(a,\ldots,a,c_0), a, \ldots, a)\\
&=\; \sum_{i\geq 0}\frac{1}{i!}\sum_{j+k=i}\binom{i}{j}\mu_{j+1}(\mu_{k+1}(a,\ldots,a,c_0), a, \ldots, a)\\
&=\; \sum_{i\geq 1}\frac{1}{(i-1)!}\sum_{j+k=i}\binom{i-1}{j}\mu_{j+1}(\mu_{k}(a,\ldots,a,c_0), a, \ldots, a)\\
&=\; \sum_{i\geq 1}\frac{1}{(i-1)!}\sum_{j+k=i}\binom{i-1}{k-1}\mu_{j+1}(\mu_{k}(a,\ldots,a,c_0), a, \ldots, a)\\
&=\; \sum_{i\geq 1}\frac{1}{(i-1)!}\sum_{j+k=i}(-1)^{j-1}\binom{i-1}{j-1}\mu_{j+1}(\mu_{k}(a,\ldots,a), a, \ldots, a, c_0)\\
&=\; \sum_{i\geq 0}\frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{i-j}\mu_{i-j+2}(\mu_{j}(a,\ldots,a), a, \ldots, a, c_0)\\
&=\; \sum_{i,j\geq 0}\frac{(-1)^i}{i!j!}\mu_{i+2}(\mu_{j}(a,\ldots,a), a, \ldots, a, c_0)\\
&=\; \sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+2}(f, a, \ldots, a, c_0)\;,
\end{aligned} \tag{C.30}$$

where in the sixth step have used the homotopy Jacobi identity (2.44c) for $(a, \ldots, a, c_0)$,

$$\begin{aligned}
&\sum_{j+k=i}(-1)^j\binom{i-1}{j-1}\mu_{j+1}(\mu_k(a,\ldots,a), a, \ldots, a, c_0) + \\
&\qquad + \sum_{j+k=i}\binom{i-1}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_0), a, \ldots, a) \;=\; 0\;.
\end{aligned} \tag{C.31}$$

Altogether we recover (3.11).

Higher gauge transformations. In particular, consider a level 1 gauge transformation between two level 0 gauge transformations. These are captured by gauge potentials on the tensor product $\Omega^\bullet(I^2, \mathsf{L})$ with $I^2 := I \times I$ and $I := [0,1] \subseteq \mathbb{R}$. This time, we have the decomposition $\Omega^\bullet_1(I^2,\mathsf{L}) \cong \mathscr{C}^\infty(I^2,\mathsf{L}_1) \oplus \Omega^1(I^2,\mathsf{L}_0) \oplus \Omega^2(I^2,\mathsf{L}_{-1})$ and hence, $\mathsf{a} \in \Omega^\bullet_1(I^2,\mathsf{L})$ takes the form

$$\mathsf{a}(t,s) \;=\; a(t,s) + \mathrm{d}t \otimes c^{(1)}(t,s) + \mathrm{d}s \otimes c^{(2)}(t,s)\mathrm{d}s + (\mathrm{d}t \wedge \mathrm{d}s) \otimes \sigma(t,s)\;, \tag{C.32a}$$

where $(t,s) \in I^2$, $a(t,s) \in \mathscr{C}^\infty(I^2,\mathsf{L}_1)$, $c^{(1,2)}(t,s) \in \mathscr{C}^\infty(I^2,\mathsf{L}_0)$, and $\sigma(t,s) \in \mathscr{C}^\infty(I^2,\mathsf{L}_{-1})$. The fact that a is a homotopy between homotopies is reflected in the boundary conditions

$$a(0,s) \;=\; a \quad \text{and} \quad c^{(1)}(t,0) \;=\; c(t)\;. \tag{C.32b}$$

The geometric shape underlying this homotopy between homotopies is not a square but a bigon with a coordinate degeneracy in s at $t = 0$ and $t = 1$. Therefore, we have to supplement the above boundary conditions by

$$c^{(2)}(0,s) \;=\; c^{(2)}(1,s) \;=\; 0\;. \tag{C.32c}$$

Moreover, $\Omega^\bullet_2(I^2,\mathsf{L})$ decomposes as $\Omega^\bullet_2(I^2,\mathsf{L}) \cong \mathscr{C}^\infty(I^2,\mathsf{L}_2) \oplus \Omega^1(I^2,\mathsf{L}_1) \oplus \Omega^2(I^2,\mathsf{L}_0)$, and upon imposing the partial flatness condition $\mathsf{f} \in \mathscr{C}^\infty(I^2,\mathsf{L}_2)$, we obtain the level 0 gauge transformation (3.10) with the gauge parameter $c_0 := c^{(1)}(0,0) \in \mathsf{L}_0$ together with the level 1 gauge transformation

$$\delta_{c_{-1}} c_0 \;:=\; \frac{\partial}{\partial s}\bigg|_{t=s=0} c^{(1)}(t,s) \;=\; \sum_{i\geq 0} \frac{1}{i!}\mu_{i+1}(a,\ldots,a,c_{-1})\;, \tag{C.33}$$

where $c_{-1} := \sigma(0,0) \in \mathsf{L}_{-1}$.

The derivation of level k gauge transformations from certain components of a partially flat curvature on I^{k+1} makes it clear that this can be iterated further by considering $\Omega^\bullet(I^{k+1},\mathsf{L})$ for $k \geq 0$. Ultimately, we obtain the level $k+1$ gauge transformation,

$$\delta_{c_{-k-1}} c_{-k} \;=\; \sum_{i\geq 0} \frac{1}{i!}\mu_{i+1}(a,\ldots,a,c_{-k-1})\;, \tag{C.34}$$

for $c_{-k} \in \mathsf{L}_{-k}$.

(Higher) gauge-of-gauge transformations Let us verify (3.16). Firstly, using (C.2), we find

$$
\begin{aligned}
\delta_{c_{-1}}(\delta_{c_0}a) &= \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a,\ldots,a,\delta_{c_{-1}}c_0)\\
&= \sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+1}(\delta_{c_{-1}}c_0,a,\ldots,a)\\
&= \sum_{i,j\geq 0}\frac{(-1)^i}{i!j!}\mu_{i+1}(\mu_{j+1}(a,\ldots,a,c_{-1}),a,\ldots,a)\\
&= \sum_{i\geq 0}\frac{1}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\mu_{j+1}(\mu_{i-j+1}(a,\ldots,a,c_{-1}),a,\ldots,a)\\
&= \sum_{i\geq 1}\frac{1}{(i-1)!}\sum_{j+k=i}(-1)^j\binom{i-1}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_{-1}),a,\ldots,a)\\
&= \sum_{i\geq 1}\frac{1}{(i-1)!}\sum_{j+k=i}(-1)^{j-1}\binom{i-1}{j-1}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a,c_{-1})\\
&= \sum_{i\geq 0}\frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{i-j}\mu_{i-j+2}(\mu_j(a,\ldots,a),a,\ldots,a,c_{-1})\\
&= \sum_{i,j\geq 0}\frac{(-1)^i}{i!j!}\mu_{i+2}(\mu_j(a,\ldots,a),a,\ldots,a,c_{-1})\\
&= \sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+2}(f,a,\ldots,a,c_{-1})~,
\end{aligned}
\tag{C.35}
$$

where we have used the homotopy Jacobi identity (2.44c) for $(a,\ldots,a,c_{-1})$ in the sixth step,

$$
\begin{aligned}
&\sum_{j+k=i}(-1)^j\binom{i-1}{j-1}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a,c_{-1})+\\
&\qquad+\sum_{j+k=i}(-1)^j\binom{i-1}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_{-1}),a,\ldots,a)~=~0~.
\end{aligned}
\tag{C.36}
$$

This establishes the first part of (3.16).

As for the second part, the gauge transformation (3.10) of a (odd degree) and the gauge-of-gauge transformation of c_0 (even degree) make it clear how to extend this to c_{-k} for all $k\in\mathbb{N}$. Indeed, a straightforward calculation shows that

$$
\delta_{c_{-k-2}}(\delta_{c_{-k-1}}c_{-k})~=~\sum_{i\geq 0}\frac{(-1)^i}{i!}\mu_{i+2}(f,a,\ldots,a,c_{-k-2})~.
\tag{C.37}
$$

Covariant derivative. Next, we verify (3.19) and (3.20). For (3.19), we make use of the definitions (3.10), (3.17), and (3.18) together with (C.2) to obtain

$$
\begin{aligned}
\delta_{c_0}(\nabla\phi) &= \sum_{i\geq 0}\frac{1}{i!}\Big[\mu_{i+2}(\delta_{c_0}a,a,\ldots,a,\phi)+(-1)^{i(|\phi|_{\mathsf{L}}+1)}\mu_{i+1}(\delta_{c_0}\phi,a,\ldots,a)\Big]\\
&= \sum_{i,j\geq 0}\frac{1}{i!j!}\Big[\mu_{i+2}(\mu_{j+1}(a,\ldots,a,c_0),a,\ldots,a,\phi)-\\
&\qquad -(-1)^{i(|\phi|_{\mathsf{L}}+1)}\mu_{i+1}(\mu_{j+2}(a,\ldots,a,c_0,\phi),a,\ldots,a)\Big]\\
&= \sum_{i\geq 0}\frac{1}{i!}\sum_{j+k=i}\binom{i}{j}\Big[\mu_{j+2}(\mu_{k+1}(a,\ldots,a,c_0),a,\ldots,a,\phi)-\\
&\qquad -(-1)^{j(|\phi|_{\mathsf{L}}+1)}\mu_{j+1}(\mu_{k+2}(a,\ldots,a,c_0,\phi),a,\ldots,a)\Big]\\
&= \sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}\binom{i-2}{j-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_0),a,\ldots,a,\phi)-\\
&\qquad -\sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}(-1)^{j(|\phi|_{\mathsf{L}}+1)}\binom{i-2}{j}\mu_{j+1}(\mu_k(a,\ldots,a,c_0,\phi),a,\ldots,a)\\
&= \sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}\binom{i-2}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_0),a,\ldots,a,\phi)-\\
&\qquad -\sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}(-1)^{j(|\phi|_{\mathsf{L}}+1)}\binom{i-2}{k-2}\mu_{j+1}(\mu_k(a,\ldots,a,c_0,\phi),a,\ldots,a)\\
&= \sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}(-1)^j\binom{i-2}{j-2}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a,c_0,\phi)+\\
&\qquad +\sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}(-1)^{(j+1)|\phi|_{\mathsf{L}}}\binom{i-2}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,\phi),a,\ldots,a,c_0)\\
&= \sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}(-1)^j\binom{i-2}{j-2}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a,c_0,\phi)+\\
&\qquad +\sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}(-1)^{(j+1)|\phi|_{\mathsf{L}}}\binom{i-2}{j-1}\mu_{j+1}(\mu_k(a,\ldots,a,\phi),a,\ldots,a,c_0)\\
&= \sum_{i\geq 0}\frac{1}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\Big[\mu_{j+3}(\mu_{i-j}(a,\ldots,a),a,\ldots,a,c_0,\phi)+\\
&\qquad +(-1)^{(j+1)|\phi|_{\mathsf{L}}}\mu_{j+2}(\mu_{i-j+1}(a,\ldots,a,\phi),a,\ldots,a,c_0)\Big]\\
&= \sum_{i,j\geq 0}\frac{1}{i!j!}\Big[(-1)^i\mu_{i+3}(\mu_j(a,\ldots,a),a,\ldots,a,c_0,\phi)+\\
&\qquad +(-1)^{i|\phi|_{\mathsf{L}}}\mu_{i+2}(\mu_{j+1}(a,\ldots,a,\phi),a,\ldots,a,c_0)\Big]\,,\\
&= \sum_{i,j\geq 0}\frac{1}{i!j!}\Big[(-1)^i\mu_{i+3}(\mu_j(a,\ldots,a),a,\ldots,a,c_0,\phi)-\\
&\qquad -\mu_{i+2}(a,\ldots,a,c_0,\mu_{j+1}(a,\ldots,a,\phi))\Big]\,,
\end{aligned}
\tag{C.38}
$$

where in the sixth step we have used the homotopy Jacobi identity (2.44c) for the elements $(a, \ldots, a, c_0, \phi)$,

$$\begin{aligned}
&\sum_{j+k=i} (-1)^j \binom{i-2}{j-2} \mu_{j+1}(\mu_k(a, \ldots, a), a, \ldots, a, c_0, \phi) - \\
&\qquad - \sum_{j+k=i} \binom{i-2}{k-1} \mu_{j+1}(\mu_k(a, \ldots, a, c_0), a, \ldots, a, \phi) + \\
&\qquad + \sum_{j+k=i} (-1)^{(j+1)|\phi|_{\mathsf{L}}} \binom{i-2}{k-1} \mu_{j+1}(\mu_k(a, \ldots, a, \phi), a, \ldots, a, c_0) + \\
&\qquad + \sum_{j+k=i} (-1)^{j(|\phi|_{\mathsf{L}}+1)} \binom{i-2}{k-2} \mu_{j+1}(\mu_k(a, \ldots, a, c_0, \phi), a, \ldots, a) \;=\; 0\ .
\end{aligned} \tag{C.39}$$

Hence, using (3.4) and (3.18), we obtain

$$\delta_{c_0}(\nabla\phi) \;=\; -\sum_{i\geq 0} \frac{1}{i!} \mu_{i+2}(a, \ldots, a, c_0, \nabla\phi) + \sum_{i\geq 0} \frac{(-1)^i}{i!} \mu_{i+3}(f, a, \ldots, a, c_0, \phi)\ , \tag{C.40}$$

as required.

To verify (3.20), consider

$$\begin{aligned}
\nabla^2\phi \;&=\; \sum_{i\geq 0} \frac{1}{i!} \mu_{i+1}(a, \ldots, a, \nabla\phi) \\
&=\; \sum_{i,j\geq 0} \frac{(-1)^{i|\phi|_{\mathsf{L}}}}{i!j!} \mu_{i+1}(\mu_{j+1}(a, \ldots, a, \phi), a, \ldots, a) \\
&=\; \sum_{i\geq 0} \frac{1}{i!} \sum_{j+k=i} (-1)^{j|\phi|_{\mathsf{L}}} \binom{i}{j} \mu_{j+1}(\mu_{k+1}(a, \ldots, a, \phi), a, \ldots, a) \\
&=\; \sum_{i\geq 1} \frac{1}{(i-1)!} \sum_{j+k=i} (-1)^{j|\phi|_{\mathsf{L}}} \binom{i-1}{j} \mu_{j+1}(\mu_k(a, \ldots, a, \phi), a, \ldots, a) \\
&=\; \sum_{i\geq 1} \frac{1}{(i-1)!} \sum_{j+k=i} (-1)^{j|\phi|_{\mathsf{L}}} \binom{i-1}{k-1} \mu_{j+1}(\mu_k(a, \ldots, a, \phi), a, \ldots, a) \\
&=\; \sum_{i\geq 1} \frac{1}{(i-1)!} \sum_{j+k=i} (-1)^{j-1} \binom{i-1}{j-1} \mu_{j+1}(\mu_k(a, \ldots, a), a, \ldots, a, \phi) \\
&=\; \sum_{i\geq 0} \frac{1}{i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{i-j} \mu_{i-j+2}(\mu_j(a, \ldots, a), a, \ldots, a, \phi) \\
&=\; \sum_{i,j\geq 0} \frac{(-1)^i}{i!j!} \mu_{i+2}(\mu_j(a, \ldots, a), a, \ldots, a, \phi) \\
&=\; \sum_{i\geq 0} \frac{(-1)^i}{i!} \mu_{i+2}(f, a, \ldots, a, \phi)\ ,
\end{aligned} \tag{C.41}$$

where we have used (C.2) in the third step and the homotopy Jacobi identity (2.44c) for $(a, \ldots, a, \phi)$,

$$\begin{gathered}\sum_{j+k=i} (-1)^j \binom{i-1}{j-1} \mu_{j+1}(\mu_k(a, \ldots, a), a, \ldots, a, \phi) + \\ + \sum_{j+k=i} (-1)^{j|\phi|_{\mathsf{L}}} \binom{i-1}{k-1} \mu_{j+1}(\mu_k(a, \ldots, a, \phi), a, \ldots, a) \;=\; 0 \;,\end{gathered} \tag{C.42}$$

in the sixth step. Altogether, we arrive at (3.20),

$$\nabla^2 \phi \;=\; \sum_{i \geq 0} \frac{(-1)^i}{i!} \mu_{i+2}(f, a, \ldots, a, \phi) \;. \tag{C.43}$$

***L_∞*-morphisms and Maurer–Cartan elements.** In the following, we explain formulas (3.24a), (3.24b), and (3.25) in detail. As mentioned, one may be led to assume that ϕ should act on a gauge potential a as $a' = \phi_1(a)$. However, this does not give the desired compatibility with the L_∞-algebra structures. Instead, one should either regard the shifted exponential[1] $\mathrm{e}_0^{sa} := sa + \frac{1}{2} sa \odot sa + \frac{1}{3!} sa \odot sa \odot sa + \cdots$ in the coalgebra picture as the natural invariant object, or, equivalently in the L_∞-picture, evaluate (2.57) at $\ell_1 = \cdots = \ell_i = a \in \mathsf{L}_1$. Both approaches eventually lead to the following.

Coalgebra picture. Recall that a morphism $\phi : \mathsf{L} \to \mathsf{L}'$ corresponds to a morphism of coalgebras $\Phi : \bigodot_0^\bullet \mathsf{L}[1] \to \bigodot_0^\bullet \mathsf{L}'[1]$ and satisfies $D \circ \Phi = \Phi \circ D$. In the dual, dga-picture, we have a morphism $\Phi^* : \mathscr{C}^\infty(\mathsf{L}'[1]) \to \mathscr{C}^\infty(\mathsf{L}[1])$ satisfying $\Phi^* \circ Q = Q' \circ \Phi^*$.

From equation (A.31), it follows that

$$\begin{aligned}\Phi(\mathrm{e}_0^{sa}) \;&=\; \Phi(sa + \tfrac{1}{2} sa \odot sa + \tfrac{1}{3!} sa \odot sa \odot sa + \cdots) \\ &=\; \Phi_1^1(sa) + \tfrac{1}{2}\Phi_2^1(sa \odot sa) + \tfrac{1}{2}\Phi_1^1(sa) \odot \Phi_1^1(sa) + \tfrac{1}{3!}\Phi_3^1(sa \odot sa \odot sa) + \\ &\qquad + \tfrac{1}{2}\Phi_2^1(sa \odot sa) \odot \Phi_1^1(sa) + \tfrac{1}{3!}\Phi_1^1(sa) \odot \Phi_1^1(sa) \odot \Phi_1^1(sa) + \cdots \\ &=\; \mathrm{e}_0^{sa'} \;,\end{aligned} \tag{C.44a}$$

where

$$a' \;:=\; \sum_{i \geq 1} \frac{1}{i!} \phi_i(a, \ldots, a) \;. \tag{C.44b}$$

[1] This expression is used in the proofs in Appendix C.

We can then use equation (C.22) to compute the curvature of a' as

$$\begin{aligned} f' &= (s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]}\circ D')(\mathrm{e}_0^{sa'}) \\ &= (s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]}\circ D'\circ\Phi)(\mathrm{e}_0^{sa}) \\ &= (s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]}\circ \Phi\circ D)(\mathrm{e}_0^{sa}) \\ &= (s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]}\circ \Phi)(sf\odot \mathrm{e}^{sa}) \\ &= s^{-1}(\Phi^1_1(sf)+\tfrac{1}{2}\Phi^1_2(sf\odot sa)+\ldots) \\ &= \sum_{i\geq 0}\frac{(-1)^i}{i!}\phi_{i+1}(f,a,\ldots,a)\ . \end{aligned} \tag{C.45}$$

Furthermore, using equation (C.22), we can write gauge transformations as follows in the coalgebra picture,

$$\delta_{c_0} a = (s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]}\circ D)(sc_0\odot \mathrm{d}\mathrm{e}^{sa})\ . \tag{C.46}$$

This allows us to compare different gauge orbits,

$$\begin{aligned} \delta_{c_0'} a' &= s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]} D'(sc_0'\odot \mathrm{e}^{sa'}) \\ &= s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]}\circ D'\circ\Phi(sc_0\odot \mathrm{e}^{sa}) \\ &= s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]}\circ \Phi\circ D(sc_0\odot \mathrm{e}^{sa}) \\ &= s^{-1}\circ \mathrm{pr}_{\bigodot^1 \mathsf{L}[1]}\circ \Phi(s\delta_{c_0}a\odot \mathrm{e}^{sa}+sc_0\odot sf\odot \mathrm{e}^{sa})\ , \end{aligned} \tag{C.47}$$

and we conclude that

$$c_0' = \sum_{i\geq 0}\frac{1}{i!}\phi_{i+1}(a,\ldots,a,c_0)\ . \tag{C.48}$$

***L_∞*-morphisms and Maurer–Cartan elements using brackets.** Consider the definition (2.57) of a general L_∞-morphism $(\mathsf{L},\mu_i)\to(\mathsf{L}',\mu_i')$ evaluated at $(\ell_1,\ldots,\ell_i)=(a,\ldots,a)$ for $a\in\mathsf{L}_1$. Then, the left-hand-side of (2.57) becomes

$$\begin{aligned} &\sum_{j+k=i}(-1)^k\binom{i}{k}\phi_{k+1}(\mu_j(a,\ldots,a),a,\ldots,a) = \\ &= i!\sum_{k_1+k_2=i}\frac{(-1)^{k_1}}{k_1!}\phi_{k_1+1}\left(\frac{1}{k_2!}\mu_{k_2}(a,\ldots,a),a,\ldots,a\right)\ , \end{aligned} \tag{C.49a}$$

while the right-hand-side reads as

$$\begin{aligned}&\sum_{j=1}^{i}\frac{1}{j!}\sum_{k_1+\cdots+k_j=i}\binom{i}{k_1}\binom{i-k_1}{k_2}\cdots\binom{i-k_1-\cdots-k_{j-2}}{k_{j-1}}\times\\&\qquad\times\mu'_j(\phi_{k_1}(a,\ldots,a),\ldots,\phi_{k_j}(a,\ldots,a))\\&=\ i!\sum_{j=1}^{i}\frac{1}{j!}\sum_{k_1+\cdots+k_j=i}\mu'_j\left(\frac{1}{k_1!}\phi_{k_1}(a,\ldots,a),\ldots,\frac{1}{k_j!}\phi_{k_j}(a,\ldots,a)\right).\end{aligned} \tag{C.49b}$$

Hence, upon equating (C.49a) and (C.49b), we obtain

$$\begin{aligned}&\sum_{k_1+k_2=i}\frac{(-1)^{k_1}}{k_1!}\phi_{k_1+1}\left(\frac{1}{k_2!}\mu_{k_2}(a,\ldots,a),a,\ldots,a\right)\ =\\&\qquad=\ \sum_{j=1}^{i}\frac{1}{j!}\sum_{k_1+\cdots+k_j=i}\mu'_j\left(\frac{1}{k_1!}\phi_{k_1}(a,\ldots,a),\ldots,\frac{1}{k_j!}\phi_{k_j}(a,\ldots,a)\right).\end{aligned} \tag{C.50}$$

Thus, setting

$$a'\ :=\ \sum_{i\geq 1}\frac{1}{i!}\phi_i(a,\ldots,a) \tag{C.51}$$

and using the Cauchy product formula (C.3), we obtain from (C.50) the relation

$$\sum_{i\geq 0}\frac{(-1)^i}{i!}\phi_{i+1}(f,a,\ldots,a)\ =\ f'\ , \tag{C.52a}$$

where

$$f\ :=\ \sum_{i\geq 1}\frac{1}{i!}\mu_i(a,\ldots,a)\quad\text{and}\quad f'\ :=\ \sum_{i\geq 1}\frac{1}{i!}\mu'_i(a',\ldots,a')\ . \tag{C.52b}$$

are the corresponding curvatures. Thus, we conclude that under L_∞-morphisms, MC elements are mapped to MC elements.

Recall the formula (3.10) for gauge transformations,

$$a\ \mapsto\ a+\delta_{c_0}a\quad\text{with}\quad\delta_{c_0}a\ =\ \sum_{i\geq 0}\frac{1}{i!}\mu_{i+1}(a,\ldots,a,c_0)\quad\text{for}\quad c_0\ \in\ \mathsf{L}_0\ . \tag{C.53}$$

We wish to study (C.51) under such transformations.

Generally, we have

$$\begin{aligned}\sum_{i\geq 1}\frac{1}{i!}\phi_i(a+\delta_{c_0}a,\ldots,a+\delta_{c_0}a)\ &=\ \underbrace{\sum_{i\geq 1}\frac{1}{i!}\phi_i(a,\ldots,a)}_{=a'}+\underbrace{\sum_{i\geq 0}\frac{1}{i!}\phi_{i+1}(\delta_{c_0}a,a,\ldots,a)}_{=:\Delta a'}\\&=\ a'+\Delta a'\ .\end{aligned} \tag{C.54}$$

To compute $\Delta a'$, we again consider equation (2.57) for a general L_∞-morphism, and this time we evaluate it at $(\ell_1, \ldots, \ell_i) = (a, \ldots, a, c_0)$ for $c_0 \in \mathsf{L}_0$ and $a \in \mathsf{L}_1$. The left-hand-side of that equation becomes

$$\begin{aligned}
&\sum_{j+k=i}\left[(-1)^k\binom{i-1}{k-1}\phi_{k+1}(\mu_j(a,\ldots,a),a,\ldots,a,c_0)+\right.\\
&\qquad\left.+\binom{i-1}{j-1}\phi_{k+1}(\mu_j(a,\ldots,a,c_0),a,\ldots,a)\right] =\\
&= (i-1)!\sum_{k_1+k_2=i}\left[\frac{(-1)^{k_1}}{(k_1-1)!}\phi_{k_1+1}\left(\frac{1}{k_2!}\mu_{k_2}(a,\ldots,a),a,\ldots,a,c_0\right)+\right.\\
&\qquad\left.+\frac{1}{k_1!}\phi_{k_1+1}\left(\frac{1}{(k_2-1)!}\mu_{k_2}(a,\ldots,a,c_0),a,\ldots,a\right)\right]\ ,
\end{aligned}\tag{C.55a}$$

while the right-hand-side reads as

$$\begin{aligned}
&\sum_{j=1}^{i}\frac{1}{(j-1)!}\sum_{k_1+\cdots+k_j=i}\binom{i-1}{k_1}\binom{i-1-k_1}{k_2}\cdots\binom{i-1-k_1-\cdots-k_{j-2}}{k_{j-1}}\times\\
&\qquad\times\mu'_j(\phi_{k_1}(a,\ldots,a),\ldots,\phi_{k_j}(a,\ldots,a,c_0)) =\\
&= (i-1)!\sum_{j=1}^{i}\frac{1}{(j-1)!}\sum_{k_1+\cdots+k_j=i}\times\\
&\qquad\times\mu'_j\left(\frac{1}{k_1!}\phi_{k_1}(a,\ldots,a),\ldots,\frac{1}{k_{j-1}!}\phi_{k_{j-1}}(a,\ldots,a),\frac{1}{(k_j-1)!}\phi_{k_j}(a,\ldots,a,c_0)\right).
\end{aligned}\tag{C.55b}$$

Hence, upon equating (C.55a) and (C.55b), we obtain

$$\begin{aligned}
&\sum_{k_1+k_2=i}\left[\frac{(-1)^{k_1}}{(k_1-1)!}\phi_{k_1+1}\left(\frac{1}{k_2!}\mu_{k_2}(a,\ldots,a),a,\ldots,a,c_0\right)+\right.\\
&\qquad\left.+\frac{1}{k_1!}\phi_{k_1+1}\left(\frac{1}{(k_2-1)!}\mu_{k_2}(a,\ldots,a,c_0),a,\ldots,a\right)\right] =\\
&= \sum_{j=1}^{i}\frac{1}{(j-1)!}\sum_{k_1+\cdots+k_j=i}\times\\
&\qquad\times\mu'_j\left(\frac{1}{k_1!}\phi_{k_1}(a,\ldots,a),\ldots,\frac{1}{k_{j-1}!}\phi_{k_{j-1}}(a,\ldots,a),\frac{1}{(k_j-1)!}\phi_{k_j}(a,\ldots,a,c_0)\right).
\end{aligned}\tag{C.56}$$

Next, we set

$$c'_0 \;:=\; \sum_{i\geq 0}\frac{1}{i!}\phi_{i+1}(a,\ldots,a,c_0)\tag{C.57}$$

and use (C.51) and (C.53), the definition

$$\delta_{c_0'} a' \;:=\; \sum_{i\geq 0} \frac{1}{i!} \mu'_{i+1}(a', \ldots, a', c_0') \;, \tag{C.58}$$

and the Cauchy product formula (C.3) to obtain

$$-\sum_{i\geq 0} \frac{(-1)^i}{i!} \phi_{i+2}(f, a, \ldots, a, c_0) + \sum_{i\geq 0} \frac{1}{i!} \phi_{i+1}(\delta_{c_0} a, a, \ldots, a) \;=\; \delta_{c_0'} a' \tag{C.59}$$

from (C.56). Upon comparing this with (C.54), we find

$$\Delta a' \;=\; \delta_{c_0'} a' + \sum_{i\geq 0} \frac{(-1)^i}{i!} \phi_{i+2}(f, a, \ldots, a, c_0) \;. \tag{C.60}$$

Consequently, for MC elements this reduces to

$$\Delta a' \;=\; \delta_{c_0'} a' \quad \Longrightarrow \quad \sum_{i\geq 1} \frac{1}{i!} \phi_i(a + \delta_{c_0} a, \ldots, a + \delta_{c_0} a) \;=\; a' + \delta_{c_0'} a' \;, \tag{C.61}$$

and, combining this with (C.52a), we realise that gauge equivalent MC configurations are mapped to gauge equivalent MC configurations under L_∞-morphisms.

A curvature identity. Let L be equipped with an inner product $\langle -, - \rangle_{\mathsf{L}}$ and let f be the curvature as defined in (3.4). We wish to prove that

$$\langle f, f \rangle_{\mathsf{L}} \;=\; 0 \;. \tag{C.62}$$

Firstly,

$$\begin{aligned} \langle f, f \rangle_{\mathsf{L}} \;&=\; \sum_{i,j\geq 0} \frac{1}{i!j!} \langle \mu_i(a, \ldots, a), \mu_j(a, \ldots, a) \rangle_{\mathsf{L}} \\ &=\; \sum_{i\geq 0} \frac{1}{i!} \underbrace{\sum_{j+k=i} \binom{i}{j} \langle \mu_j(a, \ldots, a), \mu_k(a, \ldots, a) \rangle_{\mathsf{L}}}_{=:\, F_i} \;, \end{aligned} \tag{C.63}$$

and therefore

$$\begin{aligned} F_i \;&=\; \sum_{j=0}^{i} \binom{i}{j} \langle \mu_j(a, \ldots, a), \mu_{i-j}(a, \ldots, a) \rangle_{\mathsf{L}} \\ &=\; -\sum_{j=0}^{i} (-1)^j \binom{i}{j} \langle a, \mu_j(\mu_{i-j}(a, \ldots, a), a, \ldots, a) \rangle_{\mathsf{L}} \\ &=\; -\sum_{j=1}^{i-1} (-1)^j \binom{i}{j} \langle a, \mu_j(\mu_{i-j}(a, \ldots, a), a, \ldots, a) \rangle_{\mathsf{L}} \\ &=\; \sum_{j=0}^{i-2} (-1)^j \binom{i}{j+1} \langle a, \mu_{j+1}(\mu_{i-j-1}(a, \ldots, a), a, \ldots, a) \rangle_{\mathsf{L}} \;. \end{aligned} \tag{C.64}$$

Hence,

$$\begin{aligned}
F_{i+1} &= \sum_{j=0}^{i-1}(-1)^j\binom{i+1}{j+1}\langle a,\mu_{j+1}(\mu_{i-j}(a,\ldots,a),a,\ldots,a)\rangle_{\mathsf{L}}\\
&= \sum_{j=0}^{i}(-1)^j\binom{i+1}{j+1}\langle a,\mu_{j+1}(\mu_{i-j}(a,\ldots,a),a,\ldots,a)\rangle_{\mathsf{L}}\\
&= \sum_{j=0}^{i}(-1)^j\left[\binom{i}{j}+\binom{i}{j+1}\right]\langle a,\mu_{j+1}(\mu_{i-j}(a,\ldots,a),a,\ldots,a)\rangle_{\mathsf{L}}\\
&= \sum_{j=0}^{i}(-1)^j\binom{i}{j+1}\langle a,\mu_{j+1}(\mu_{i-j}(a,\ldots,a),a,\ldots,a)\rangle_{\mathsf{L}}\ ,
\end{aligned} \tag{C.65}$$

where in the last step we have use the Bianchi identity (3.6). Therefore,

$$\begin{aligned}
F_{i+1} &= \sum_{j=0}^{i-1}\binom{i}{j+1}\langle\mu_{j+1}(a,\ldots,a),\mu_{i-j}(a,\ldots,a)\rangle_{\mathsf{L}}\\
&= \sum_{j=1}^{i}\binom{i}{j}\langle\mu_{j}(a,\ldots,a),\mu_{i+1-j}(a,\ldots,a)\rangle_{\mathsf{L}}\ .
\end{aligned} \tag{C.66a}$$

However, from the first line of (C.64), we also have

$$F_{i+1} = \sum_{j=1}^{i}\binom{i+1}{j}\langle\mu_{j}(a,\ldots,a),\mu_{i+1-j}(a,\ldots,a)\rangle_{\mathsf{L}}\ . \tag{C.66b}$$

Furthermore, for any $A_{ij}=A_{ji}$ we have the identity

$$\begin{aligned}
&\binom{i+1}{1}A_{1i}+\binom{i+1}{2}A_{2i-1}+\cdots+\binom{i+1}{i-1}A_{i-12}+\binom{i+1}{i}A_{i1} =\\
&\qquad = 2\left[\binom{i}{1}A_{1i}+\binom{i}{2}A_{2i-1}+\cdots+\binom{i}{i-1}A_{i-12}+\binom{i}{i}A_{i1}\right].
\end{aligned} \tag{C.67}$$

Hence, using the symmetry of the inner product we take $A_{ij}:=\langle\mu_i(a,\ldots,a),\mu_j(a,\ldots,a)\rangle_{\mathsf{L}}$, which implies that the sum in (C.66b) is twice the sum in (C.66a), that is,

$$\begin{aligned}
&\sum_{j=1}^{i}\binom{i+1}{j}\langle\mu_{j}(a,\ldots,a),\mu_{i+1-j}(a,\ldots,a)\rangle_{\mathsf{L}} =\\
&\qquad = 2\sum_{j=1}^{i}\binom{i}{j}\langle\mu_{j}(a,\ldots,a),\mu_{i+1-j}(a,\ldots,a)\rangle_{\mathsf{L}}\ .
\end{aligned} \tag{C.68}$$

Consequently, we must have $F_{i+1}=0$ and so $\langle f,f\rangle_{\mathsf{L}}=0$, as claimed.

Becchi–Rouet–Stora–Tyutin transformations. Let us verify (4.54). Firstly, we have

$$\begin{aligned}
Q_{\mathrm{BRST}} a &= \sum_{i\geq 0} \frac{1}{i!}\mu_{i+1}(a,\ldots,a,c_0)\ ,\\
Q_{\mathrm{BRST}} c_0 &= -\sum_{i\geq 0} \frac{1}{i!}\Big[\mu_{i+1}(a,\ldots,a,c_{-1}) + \frac{1}{2!}\mu_{i+2}(a,\ldots,a,c_0,c_0)\Big]
\end{aligned} \tag{C.69}$$

and so

$$\begin{aligned}
Q^2_{\mathrm{BRST}} a &= \sum_{i\geq 0} \frac{1}{i!}\Big[\mu_{i+2}(Q_{\mathrm{BRST}} a, a\ldots,a,c_0) + (-1)^i \mu_{i+1}(Q_{\mathrm{BRST}} c_0, a,\ldots,a)\Big]\\
&= \sum_{i\geq 0} \frac{1}{i!j!}\Big[\mu_{i+2}(\mu_{j+1}(a,\ldots,a,c_0),a\ldots,a,c_0) +\\
&\qquad - (-1)^i \mu_{i+1}(\mu_{j+1}(a,\ldots,a,c_{-1}),a,\ldots,a) -\\
&\qquad - \frac{(-1)^i}{2!}\mu_{i+1}(\mu_{j+2}(a,\ldots,a,c_0,c_0),a,\ldots,a)\Big]\\
&= \sum_{i\geq 0} \frac{1}{i!}\sum_{j+k=i}\binom{i}{j}\Big[\mu_{j+2}(\mu_{k+1}(a,\ldots,a,c_0),a,\ldots,a,c_0) +\\
&\qquad - (-1)^j \mu_{j+1}(\mu_{k+1}(a,\ldots,a,c_{-1}),a,\ldots,a) -\\
&\qquad - \frac{(-1)^j}{2!}\mu_{j+1}(\mu_{k+2}(a,\ldots,a,c_0,c_0),a,\ldots,a)\Big]\\
&= \sum_{i\geq 2} \frac{1}{(i-2)!}\sum_{j+k=i}\binom{i-2}{j-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_0),a,\ldots,a,c_0) +\\
&\qquad - \sum_{i\geq 1} \frac{1}{(i-1)!}\sum_{j+k=i}(-1)^j\binom{i-1}{j}\mu_{j+1}(\mu_k(a,\ldots,a,c_{-1}),a,\ldots,a) -\\
&\qquad - \sum_{i\geq 2} \frac{1}{(i-2)!}\sum_{j+k=i}\frac{(-1)^j}{2!}\binom{i-2}{j}\mu_{j+1}(\mu_k(a,\ldots,a,c_0,c_0),a,\ldots,a)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}\binom{i-2}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_0),a,\ldots,a,c_0)+\\
&\quad -\sum_{i\geq 1}\frac{1}{(i-1)!}\sum_{j+k=i}(-1)^j\binom{i-1}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_{-1}),a,\ldots,a)-\\
&\quad -\sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}\frac{(-1)^j}{2!}\binom{i-2}{k-2}\mu_{j+1}(\mu_k(a,\ldots,a,c_0,c_0),a,\ldots,a)\\
&= -\sum_{i\geq 1}\frac{1}{(i-1)!}\sum_{j+k=i}(-1)^{j-1}\binom{i-1}{j-1}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a,c_{-1})+\\
&\quad +\sum_{i\geq 2}\frac{1}{(i-2)!}\sum_{j+k=i}\frac{(-1)^j}{2!}\binom{i-2}{j-2}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a,c_0,c_0)\\
&= \sum_{i\geq 0}\frac{1}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\Big[-\mu_{j+2}(\mu_{i-j}(a,\ldots,a),a,\ldots,a,c_{-1})+\\
&\quad +\frac{1}{2!}\mu_{j+3}(\mu_{i-j}(a,\ldots,a),a,\ldots,a,c_0,c_0)\Big]\\
&= \sum_{i,j\geq 0}\frac{(-1)^i}{i!j!}\Big[-\mu_{i+2}(\mu_j(a,\ldots,a),a,\ldots,a,c_{-1})+\\
&\quad +\frac{1}{2!}\mu_{i+3}(\mu_j(a,\ldots,a),a,\ldots,a,c_0,c_0)\Big]\,,
\end{aligned}
\tag{C.70}
$$

where we used the homotopy Jacobi identity (2.44c) for $(a,\ldots,a,c_0,c_0)$ in the sixth step,

$$
\begin{aligned}
&\frac{1}{2!}\sum_{j+k=i}(-1)^j\binom{i-2}{j-2}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a,c_0,c_0)-\\
&\qquad -\sum_{j+k=i}\binom{i-2}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_0),a,\ldots,a,c_0)+\\
&\qquad +\frac{1}{2!}\sum_{j+k=i}(-1)^j\binom{i-2}{k-2}\mu_{j+1}(\mu_k(a,\ldots,a,c_0,c_0),a,\ldots,a) \;=\; 0\,,
\end{aligned}
\tag{C.71}
$$

and for $(a,\ldots,a,c_{-1})$,

$$
\begin{aligned}
&\sum_{j+k=i}(-1)^j\binom{i-1}{j-1}\mu_{j+1}(\mu_k(a,\ldots,a),a,\ldots,a,c_{-1})+\\
&\qquad +\sum_{j+k=i}(-1)^j\binom{i-1}{k-1}\mu_{j+1}(\mu_k(a,\ldots,a,c_{-1}),a,\ldots,a) \;=\; 0\,,
\end{aligned}
\tag{C.72}
$$

respectively. Altogether, using the curvature f defined in (3.5), we arrive at

$$
Q^2_{\rm BRST}a \;=\; \sum_{i\geq 0}\frac{(-1)^i}{i!}\Big[-\mu_{i+2}(f,a,\ldots,a,c_{-1})+\frac{1}{2!}\mu_{i+3}(f,a,\ldots,a,c_0,c_0)\Big]\,. \tag{C.73}
$$

D

Quasi-groups and higher principal bundles

D.1. Categories and Sheaves

Here, we just present a coincise exposition of some mathematical notions underlying many constructions discussed in this Thesis. The following literature has proven to be useful for studying this subject: [303, 304] (category theory), [294, 305, 306] (algebraic geometry).

D.1.1. Categories

As seen along the Thesis, categories appear naturally in the study of higher gauge theory. Here, we just recall the basic definitions to fix notation.

A *category* $\mathscr{C}$ consists of the following data:

- a collection $\mathsf{Ob}(\mathscr{C})$ of objects,
- for each pair $X, Y \in \mathsf{Ob}(\mathscr{C})$, a collection $\mathsf{hom}_{\mathscr{C}}(X, Y)$ of morphisms from X to Y, also called *arrows*,

such that

- for each object $X \in \mathsf{Ob}(\mathscr{C})$, there is a distinguished identity morphism $\mathrm{id}_X \in \mathsf{hom}_{\mathscr{C}}(X, X)$,
- for each $X, Y, Z \in \mathsf{Ob}(\mathscr{C})$ and each pair of morphisms $f \in \mathsf{hom}_{\mathscr{C}}(X, Y)$ and $g \in \mathsf{hom}_{\mathscr{C}}(Y, Z)$, there is a morphism function $\circ : \mathsf{hom}_{\mathscr{C}}(X, Y) \times \mathsf{hom}_{\mathscr{C}}(Y, Z) \to \mathsf{hom}_{\mathscr{C}}(X, Z)$, defined by $(f, g) \mapsto g \circ f$, called composition.

This data is subject to the following axioms:

- for any $f \in \text{hom}_{\mathscr{C}}(X,Y)$, we have $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$,
- for each morphism $f \in \text{hom}_{\mathscr{C}}(X,Y)$, $g \in \text{hom}_{\mathscr{C}}(Y,Z)$ and $h \in \text{hom}_{\mathscr{C}}(Z,W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

Given a category $\mathscr{C}$, we define the *opposite category* $\mathscr{C}^{\text{op}}$ by $\text{Ob}(\mathscr{C}^{\text{op}}) := \text{Ob}(\mathscr{C})$ and $\text{hom}_{\mathscr{C}^{\text{op}}}(X,Y) := \text{hom}_{\mathscr{C}}(Y,X)$, that is, $\mathscr{C}^{\text{op}}$ has the same objects as $\mathscr{C}$, but the directions of all morphisms are reversed. Here, for each $X \in \mathscr{C}$, the identity $\mathrm{id}_X^{\text{op}}$ is given by id_X, while for each $f^{\text{op}} : X \to Y$ and $g^{\text{op}} : Y \to Z$, the composition $g^{\text{op}} \circ f^{\text{op}}$ is given by $f \circ g$, for all $X, Y, Z \in \mathscr{C}$.

Examples. Useful examples of categories in our discussion are[1]:

- Set, with sets as its objects and functions between sets as its morphisms.
- Top, with topological spaces as objects and continuous functions as morphisms.
- Man, with smooth manifolds as objects and smooth maps as morphisms.
- The category of complex manifolds, with complex manifolds as objects and holomorphic maps as morphisms.
- The category of (complex) holomorphic vector bundles, with (complex) holomorphic vector bundles as objects and (smooth) holomorphic bundle maps as morphism.
- Grp, whose objects are groups and whose morphisms are group homomorphisms. The categories Rings of associative and unital rings and ring homomorphisms and Fields of fields and field homomorphisms are defined similarly.
- Vect$_k$, with vector spaces over a field k as objects and k-linear maps as morphisms. For our discussion, $k = \mathbb{R}, \mathbb{C}$.

A *groupoid* is defined as a category where all the arrows are invertible. For instance, a group G is a groupoid with one object, namely it is the same thing as a category BG with one object, Ob(BG) $= \{\star\}$, in which the morphisms are given by $\{\text{G}\}$. Hence $g : \star \to \star$ is considered as a morphism, the identity morphism is $e \in$ G, the composition is the composition in G and morphisms admit an inverse, $g^{-1} : \star \to \star$.

[1]The following are concrete categories, those whose objects have underlying sets and whose morphisms are structure-preserving morphisms between these sets.

Functors. Besides the notion of categories, we shall need the notion of functors encapsulating the notion of arrows between categories. A functor $F : \mathscr{C} \to \mathscr{D}$ between categories $\mathscr{C}$ and $\mathscr{D}$ consists of the following data

- An object $F(X) \in \mathscr{D}$, for each $X \in \mathscr{C}$,
- A morphism $F(f) \in \mathsf{hom}_{\mathscr{D}}(F(X), F(Y))$, for each $f \in \mathsf{hom}_{\mathscr{C}}(X, Y)$, $X, Y \in \mathscr{C}$,

subjected to the following (functoriality) axioms

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \qquad \text{and} \qquad F(g \circ f) = F(g) \circ F(f) \ ,$$

for all $f \in \mathsf{hom}_{\mathscr{C}}(X, Y)$, $g \in \mathsf{hom}_{\mathscr{C}}(Y, Z)$ and $X, Y, Z \in \mathscr{C}$. Hence, a functor consists of a mapping on objects and a mapping on morphisms that preserves all of the structure of a category. What we have just defined is a *covariant functor*. A functor $F : \mathscr{C} \to \mathscr{D}$ is called *faithful* (respectively, *full*) if the assignment $\mathsf{hom}_{\mathscr{C}}(X, Y) \to \mathsf{hom}_{\mathscr{D}}(F(X), F(Y))$, $f \mapsto F(f)$ is injective (respectively, surjective). Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *contravariant functor* from $\mathscr{C}$ to $\mathscr{D}$ is a functor $\mathscr{C}^{\mathsf{op}} \to \mathscr{D}$. For instance, given a vector space one can define the so-called dual vector space functor. Explicitly, the functor $(-)^* : \mathsf{Vect}_k^{\mathsf{op}} \to \mathsf{Vect}_k$ carries a vector space to its dual vector space $V^* = \mathsf{hom}(V, k)$. A vector in V^* is a linear functional on V, i.e., a linear map $V \to k$. Such a functor is contravariant: a linear map $\phi : V \to W$ is sent to the linear map $\phi^* : W^* \to V^*$ that pre-composes a linear functional $\psi : W \to k$ with ϕ to obtain a linear functional $\psi \circ \phi : V \to W \to k$. Another useful example is the following. Let G be a group and $\mathscr{C}$ a category. The functor $F : \mathsf{BG} \to \mathscr{C}$ defines an action[1] of the group G on an object $X \in \mathscr{C}$. Indeed, $F : \mathsf{BG} \to \mathscr{C}$ specifies an object $X \in \mathscr{C}$, together with an endomorphism $g_* : X \to X$, for each $g \in \mathsf{G}$. This assignment satisfies $h_* g_* = (hg)_*$, for all $g, h \in \mathsf{G}$ and $e_* = \mathrm{id}_X$, where $e \in \mathsf{G}$ is the identity element. Specifying $\mathscr{C} = \mathsf{Vect}_k$, the object X endowed with such an action is called a G-representation. Similarly, for $\mathscr{C} = \mathsf{Set}$ we obtain a G-set, while for $\mathscr{C} = \mathsf{Top}$ one gets a G-space. Another example is given by the functor $H^n : \mathsf{Top}^{\mathsf{op}} \to \mathsf{AbGrp}$ from the opposite category of topological spaces to the category of Abelian groups. For each $n \in \mathbb{N}$, the functor H^n assigns to a space its nth cohomology group. Its relevance will be clear in the next Section. Moreover, for every functor $F : \mathscr{C} \to \mathscr{D}$ there is a functor $\mathscr{C}^{\mathsf{op}} \to \mathscr{D}^{\mathsf{op}}$, called *opposite functor*. We note that the composition of two functors is a functor. If both are covariant or both are contravariant the composition is covariant, while if one is covariant and one is contravariant the composition is a contravariant functor.

[1]Notice that the action specified by $\mathsf{BG} \to \mathscr{C}$ is a left-action. A right-action is a functor $\mathsf{BG}^{\mathsf{op}} \to \mathscr{C}$.

D.1.2. Sheaves

Pre-sheaves and sheaves are two of the indispensable tools used in some of the more advanced parts of algebraic topology and algebraic geometry. Roughly speaking, pre-sheaves can be seen as a way of packaging local information about a topological space in a way that is mathematically useful, while sheaves allow to locally attach data to any open subset of a topological space and glue together such local data to obtain something global.

Let X be a topological space. A *pre-sheaf* $\mathcal{P}$ of Abelian groups[1] over X is the association of an Abelian group $\mathcal{P}(U)$ to every open subset $U \subset X$ together with morphisms, called the restriction maps, $r^U_V : \mathcal{P}(U) \to \mathcal{P}(V)$, for each pair of open sets U, V with $V \subset U$. These groups and morphisms satisfy $r^U_W = r^V_W \circ r^U_V$, for $W \subset V \subset U$, r^U_U is the identity map and $\mathcal{P}(\emptyset) = \{0\}$. The elements $s \in \mathcal{P}(U)$ are called *sections* of the pre-sheaf $\mathcal{P}$ on U. Put differently, a pre-sheaf is a contravariant functor from the category whose objects are the open subsets of X and the morphisms are the inclusions of open sets, to the category AbGrp of Abelian groups.

In many situations it is desirable to reconstruct global information by gluing together pieces of local data. This local-to-global property is made precise by introducing a sharper notion, that of a *sheaf*. Explicitly, let $U \subset X$ an open subset and $\{V_a\}_{a\in A}$ an open covering of U. A pre-sheaf $\mathcal{P}$ over X becomes a sheaf if the following conditions are satisfied:

(i) Sections are determined by local data: Given two sections $s, \bar{s} \in \mathcal{P}(U)$ such that $r^U_{V_a}(s) = r^U_{V_a}(\bar{s})$ for all $a \in A$, then $s = \bar{s}$.

(ii) Compatible local data can be patched together: Given any family $(s_a)_{a\in A}$ with $s_a \in \mathcal{P}(V_a)$, if $r^{V_a}_{V_a\cap V_b}(s_a) = r^{V_b}_{V_a\cap V_b}(s_b)$, then there exists $s \in \mathcal{P}(U)$ such that $r^U_{V_a}(s) = s_a$ for all $a \in A$.

A morphism of (pre-)sheaves $\phi : \mathcal{S} \to \mathcal{S}'$ over X consists of a family of morphisms of Abelian groups $\phi_U : \mathcal{S}(U) \to \mathcal{S}'(U)$, for each open $U \subset X$, commutating with the restriction

[1]For a definition involving non-Abelian groups see e.g. the discussion in [237]. Pre-sheaves of rings (or set, vector space, R-module, etc.) are defined in the same way, by requiring that the restriction maps are ring (set, vector space, R-module, etc.) morphisms.

morphisms; i.e., the following diagram commutes

$$\begin{array}{ccc} \mathcal{S}(U) & \xrightarrow{\phi_U} & \mathcal{S}'(U) \\ \downarrow r^U_V & & \downarrow r'^U_V \\ \mathcal{S}(V) & \xrightarrow{\phi_V} & \mathcal{S}'(V) \end{array} \tag{D.1}$$

Two sheaves $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic if there exist some sheaf morphisms $\phi : \mathcal{S} \to \mathcal{S}'$ and $\psi : \mathcal{S}' \to \mathcal{S}$ such that $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$.

Examples. The following examples are useful for our purposes:

- Let M be a smooth manifold[1]. Smooth functions on M form a sheaf, denoted by $C^\infty(M)$. Also, the smooth p-forms $\Omega^p(M)$ form a sheaf.
- Let X be a complex manifold. Holomorphic functions on X form the sheaf $\mathcal{O}_{\mathsf{X}}$ and smooth (p,q)-forms $\Omega^{p,q}(\mathsf{X})$ on X is a sheaf.
- Let $E \to \mathsf{X}$ be a holomorphic vector bundle over a complex manifold X. Holomorphic sections of E on X is a sheaf, denoted by $\mathcal{O}_{\mathsf{X}}(E)$.

Turning a pre-sheaf into a sheaf. There exists a well-behaved procedure for converting a pre-sheaf $\mathcal{P}$ to a sheaf $\mathcal{S}$ on a topological space X. Such a procedure is called *sheafification*, see e.g. [307] for a clear explanation.

Definition D.1. *The* stalk *of a pre-sheaf $\mathcal{P}$ at a point $x \in X$ is the Abelian group*

$$\mathcal{P}_x := \varinjlim_U \mathcal{P}(U) \ , \tag{D.2}$$

where U ranges over all open neighbourhoods, directed by inclusion.

Remark D.2. *Let us recall the definition of direct limit. Let I be a partially ordered set, such that for each pair of elements $i, j \in I$ there is a third element k such that $i < k$ and $j < k$ (I is called a directed set). Given such an I, we consider a family of Abelian groups $\{G_i\}_{i\in I}$ such that for all $i < j$ there is a group morphism $f^i_j : G_i \to G_j$, with $f^i_i = \mathrm{id}$, and $f^j_k \circ f^i_j = f^i_k$. Then, on the set $\mathfrak{G} = \bigsqcup_{i\in I} G_i$, where $\bigsqcup$ denotes the disjoint union, we consider the following*

[1] Recall that in Chapter 2, we have used the interpretation of a manifold M as a locally ringed space, which is a topological space $|M|$ together with a sheaf of commutative rings on it.

equivalence relation: for $g \in G_i$ and $h \in G_j$ we say $g \sim h$ if there exists a $k \in I$ such that $f_k^i(g) = f_k^j(h)$. The direct limit *of the system $\{G_i\}_{i\in I}$, denoted by*

$$\varinjlim_{i\in I} G_i \ , \tag{D.3}$$

is defined as the quotient $\mathfrak{G}/\sim$. For more details, the reader may consult cf. [182].

If $x \in U$ and $s \in \mathcal{P}(U)$, the image s_x of s in $\mathcal{P}_x$ via the canonical projection $\mathcal{P}(U) \to \mathcal{P}_x$ is called the germ of s at x. Two elements $s \in \mathcal{P}(U)$, $s' \in \mathcal{P}(V)$, for U, V open neighbourhoods of x, define the same germ at x, i.e. $s_x = s'_x$ if and only if there exists an open neighbourhood $W \subset U \cap V$ of x such that s and s' coincide on $s|_W = s'|_W$. One can naturally associate with $\mathcal{P}$ a sheaf having the same stalks. One considers $\hat{\mathcal{P}} := \bigsqcup_{x\in X} \mathcal{P}_x$ (called the *étalé space*) and the natural projection $\hat{\mathcal{P}} \to X$. Sections s of $\mathcal{P}$ give rise to sections $\hat{s} : U \hookrightarrow \hat{\mathcal{P}}$ of π defined by $\hat{s}(x) = s_x$. Then, we define a new pre-sheaf $\mathcal{P}^\sharp$ by taking $\mathcal{P}^\sharp(U)$, for $U \subset X$, to be the group of sections $\sigma : U \hookrightarrow \hat{\mathcal{P}}$ of π such that, for every $x \in U$, there is an open neighbourhood $V \subset U$ of x satisfying $\sigma|_V = s$, for some $s \in \mathcal{P}(V)$. $\mathcal{P}^\sharp$[1] is called the sheaf associated with the pre-sheaf $\mathcal{P}$.

A subsheaf of a sheaf $\mathcal{S}$ over a topological space X is a sheaf $\mathcal{S}'$ over X such that $\mathcal{S}'(U)$ is a subgroup of $\mathcal{S}(U)$ for any open set $U \subset X$. The restriction maps on $\mathcal{S}'$ are inherited from the ones on $\mathcal{S}$.

Let $\mathcal{A}$ be a sheaf of rings on a topological space X. A sheaf $\mathcal{S}$ of modules over $\mathcal{A}$ is said *locally free* and of rank r if $\mathcal{S}$ is locally isomorphic to $\mathcal{A}^{\oplus r}$ on a neighbourhood of every point. Hence, for every $x \in X$ one can find a neighbourhood $U \subset X$ and sections $s_1, \dots, s_r \in \mathcal{S}(U)$ such that the sheaf homomorphism

$$s : \mathcal{A}^{\oplus r}|_U \to \mathcal{S}|_U, \quad \mathcal{A}_x^{\oplus r} \ni (a_1, \dots, a_r) \mapsto \sum_{1\le j\le r} a_j s_{j,x} \in \mathcal{S}_x \tag{D.4}$$

is an isomorphism. The notion of locally free sheaf is related to the notion of vector bundle. Given a rank r vector bundle $E \to X$ over a manifold X, the sheaf of sections (which is an $\mathcal{O}_X$-module) is a locally free sheaf of rank r, $\mathcal{S}|_{U_i} \cong \mathcal{O}_{U_I}^{\oplus r}$, for $\{U_i\}$ an open cover of X. Isomorphism classes of locally free sheaves of rank r over X are in one-to-one correspondence with isomorphism classes of vector bundles of the same rank over X. Hence, these two terms often are used to indicate the same object.

[1]Notice that in general the morphism $\mathcal{P} \to \mathcal{P}^\sharp$ is neither injective nor surjective.

A sequence of morphisms of sheaves $0 \to \mathcal{S}_1 \xrightarrow{f} \mathcal{S}_2 \xrightarrow{g} \mathcal{S}_3 \to 0$ on a topological space X is exact if, for every point $x \in X$, the sequence of morphisms of stalks $0 \to \mathcal{S}_{1x} \xrightarrow{f_x} \mathcal{S}_{2x} \xrightarrow{g_x} \mathcal{S}_{3x} \to 0$ is exact, i.e. $\ker f_x = \mathrm{im} g_x$ for all $x \in X$. Notice that, if $0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_3 \to 0$ is exact for every open subset $U \subset X$, the sequence of groups $0 \to \mathcal{S}_1(U) \to \mathcal{S}_2(U) \to \mathcal{S}_3(U)$ is exact, but the last arrow may fail to be surjective. In particular, exactness implies that f is a monomorphism and g is an epimorphism and the image of f is equal to the kernel of g.

Associated to any complex manifold X[1], we have the following short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathcal{O}_{\mathsf{X}} \xrightarrow{\exp} \mathcal{O}^*_{\mathsf{X}} \longrightarrow 0 \tag{D.5}$$

called *the exponential sheaf sequence*. Here, $\mathbb{Z}$ is the constant sheaf with stalk the integers, while $\mathcal{O}_{\mathsf{X}}$ and $\mathcal{O}^*_{\mathsf{X}}$ denote the sheaf of holomorphic functions and the subsheaf of non-vanishing holomorphic functions on X, respectively. The map ι is the obvious inclusion, while the exponential map exp is defined by the homeomorphism

$$\exp : \mathcal{O}_U \to \mathcal{O}^*_U : f \mapsto e^{2\pi i f} \ , \quad \text{for} \quad f \in \mathcal{O}_U, \ U \subset \mathsf{X} \ . \tag{D.6}$$

That $\mathbb{Z}$ is the kernel of exp is immediate, and exactness of the sequence at $\mathcal{O}$ follows from the fact that X admits a cover by contractible open sets, over each of which any non-vanishing holomorphic function has a logarithm, which is unique up to a constant of the form $2\pi k i$, $k \in \mathbb{Z}$.

D.1.3. Cohomology of sheaves

We shall now describe a cohomology theory which associates cohomology groups to a sheaf on a topological space. As clearly emerges in Chapter 6,7, sheaf cohomology plays also an important role in the twistor framework.

Čech cohomology. To begin with, let $\mathcal{P}$ be a pre-sheaf over the topological space X with an open covering $\mathfrak{U}$. We assume $\mathfrak{U}$ is labelled by a totally ordered set I and we define

$$U_{i_0,\cdots,i_q} := U_{i_0} \cap \cdots \cap U_{i_q} \ . \tag{D.7}$$

[1]We shall discuss complex geometry in the Appendix.

The *Čech complex* of $\mathfrak{U}$ with coefficients in $\mathcal{P}$ is defined as the complex whose p-th term is the Abelian group

$$C^q(\mathfrak{U},\mathcal{P}) := \prod_{i_0<\cdots<i_q} \mathcal{P}(U_{i_0,\cdots,i_q}) \ . \tag{D.8}$$

Hence, a q-cochain is a collection $f = \{f_{i_0,\cdots i_q}\}$ of sections of $\mathcal{P}$, each one belonging to the space of sections over the non-empty intersection[1] of $q+1$ open sets in $\mathfrak{U}$. Then we define the coboundary map (the *Čech differential*)

$$\delta_q : C^q(\mathfrak{U},\mathcal{P}) \to C^{q+1}(\mathfrak{U},\mathcal{P}) \tag{D.9}$$

by the formula[2]

$$(\delta_q f)_{i_0,\cdots,i_q} := \sum_{k=0}^{q+1} (-1)^k f_{i_0,\cdots,\hat{i}_k,\cdots,i_{q+1}}|_{U_{i_0,\cdots,i_{q+1}}} \ . \tag{D.10}$$

It follows that $\delta_q \circ \delta_{q-1} = 0$. The corresponding cohomology groups are

$$H^q(\mathfrak{U},\mathcal{P}) := \ker(\delta_q)/\mathrm{im}(\delta_{q-1}) \ , \quad q \geq 0 \ . \tag{D.11}$$

As usual, elements of $\ker(\delta_q)$ are called q-cocycles, while elements of $\mathrm{im}(\delta_{q-1})$ are q-boundaries. The direct sum

$$H^\bullet(\mathfrak{U},\mathcal{P}) = \bigoplus_{q\geq 0} H^q(\mathfrak{U},\mathcal{P}) \ , \tag{D.12}$$

is called the *Čech cohomology* of $\mathfrak{U}$ with coefficients in $\mathcal{P}$.

The cohomology groups $H^\bullet(\mathfrak{U},\mathcal{S})$ with coefficients in a sheaf $\mathcal{S}$ over X are then defined as the cohomology groups with coefficients in the canonical pre-sheaf of $\mathcal{S}$. If $\mathcal{S}$ is a sheaf over X, the following result holds: the cohomology group $H^0(\mathfrak{U},\mathcal{S})$ is naturally isomorphic to the group $\mathcal{S}(X)$ of sections of $\mathcal{S}$ over X.

The result depends to some extent on the choice of covering $\mathfrak{U}$, but for a good covering this dependence disappears. This is achieved by defining a notion of refinement on covers and by taking direct limits, as we shall briefly show. An open covering $\mathfrak{V} = \{V_j\}_{j\in J}$ of X is a *refinement* of an open covering $\mathfrak{U} = \{U_i\}_{i\in I}$ if each V_j is contained in at least one U_i.[3] Two

[1] Notice that, since the indexes of the open sets are taken in strictly increasing order, each intersection is counted only once.

[2] Here, the roof over a symbol means that the symbol is to be omitted.

[3] The order must be fixed at the outset, since a cover may be regarded as a refinement of another in many ways.

open coverings are *cofine* if each is a refinement of the other. Given $\mathfrak{U}, \mathfrak{V}$, there is a natural group homomorphism (induced by the restriction mappings of the sheaf $\mathcal{S}$)

$$h^{\mathfrak{U}}_{\mathfrak{V}} : H^q(\mathfrak{U}, \mathcal{S}) \to H^q(\mathfrak{V}, \mathcal{S}) \tag{D.13}$$

One can define the Čech cohomology groups depending only on the pair $(X, \mathcal{S})$ and not on the covering by

$$H^q(X, \mathcal{S}) := \varinjlim_{\mathfrak{U}} H^q(\mathfrak{U}, \mathcal{S}) \ , \tag{D.14}$$

which we call the q-th Čech cohomology group of X with coefficients in $\mathcal{S}$. Here, the direct limit is taken over a cofinal subset of the directed set of all covers of X (the order is of course the refinement of covers). Notice that the groups $H^q(X, \mathcal{S})$ are well-defined, as different cofinal families give rise to the same inductive limit.

Čech cohomology is well-behaved when the base space X is paracompact. See e.g. [294,305] for details. Recall, that a topological space X is *paracompact* if it is a Hausdorff space and if every open covering of X has a locally finite refinement. Here, we just mention the following useful definitions.

Definition D.3 (Direct image sheaf). *Let X, Y be topological spaces and $f : X \to Y$ a continuous map. The* direct image *by f of a pre-sheaf $\mathcal{P}$ on X is the pre-sheaf $f_*\mathcal{P}$ on Y defined by $(f_*\mathcal{P})(V) = \mathcal{P}(f^{-1}(V))$, for every open subset $V \subset X$. If $\mathcal{S}$ is a sheaf on X, then $f_*\mathcal{S}$ turns out to be a sheaf.*

Definition D.4 (Inverse image sheaf). *Let $\mathcal{P}$ be a pre-sheaf on Y and $f : X \to Y$ a continuous map. The* inverse image *of $\mathcal{P}$ by f is the pre-sheaf on X defined by*

$$U \to \varinjlim_{U \subset f^{-1}(V)} \mathcal{P}(V) \ , \tag{D.15}$$

for any open subsets $V \subset Y$. For $\mathcal{S}$ a sheaf on Y, $f^{-1}\mathcal{S}$ is the sheaf on X associated with the inverse image pre-sheaf of $\mathcal{S}$.

Note that the stalk at a point $x \in X$ of the inverse image pre-sheaf is isomorphic to $\mathcal{P}_{f(x)}$. Hence, the sequence of sheaves $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ and the induced sequence $0 \to f^{-1}\mathcal{S}' \to f^{-1}\mathcal{S} \to f^{-1}\mathcal{S}'' \to 0$ are both exact.

Moreover,

Definition D.5 (Acyclic sheaf)**.** *A sheaf $\mathcal{S}$ of Abelian groups on a topological space X is said to be* acyclic *if $H^q(X, \mathcal{S}) = 0$, for $q > 0$.*

Let $\mathcal{S}$ be a sheaf over X. If an open cover $\mathfrak{U}$ of the topological space X is suitably chosen, the Čech cohomologies $H^\bullet(\mathfrak{U}, \mathcal{S})$ and $H^\bullet(X, \mathcal{S})$ are isomorphic. The following theorem establishes a sufficient condition for such an isomorphism to hold. We say that an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of a topological space X is *acyclic* for a sheaf $\mathcal{S}$ if $H^q(U_{i_0,\dots,i_p}, \mathcal{S}) = 0$ for all $q > 0$ and all nonvoid intersections $U_{i_0,\dots,i_p} = U_{i_0} \cap \cdots U_{i_p}$, $i_0, \dots, i_p \in I$.

Theorem D.6 (Leray's theorem)**.** *Let $\mathcal{S}$ be a sheaf on a paracompact space X, and let $\mathfrak{U}$ be an open cover of X which is acyclic for $\mathcal{S}$ and is indexed by an ordered set. Then, for all $q \geq 0$, the cohomology groups $H^q(\mathfrak{U}, \mathcal{S})$ and $H^q(X, \mathcal{S})$ are isomorphic.*

For a proof see e.g. [294, 305].

We recall that, when X is a paracompact topological space, any exact sequence of sheaves

$$0 \longrightarrow \mathcal{S}_1 \longrightarrow \mathcal{S}_2 \longrightarrow \mathcal{S}_3 \longrightarrow 0 \tag{D.16}$$

induces a corresponding long exact sequence of Čech cohomology groups according to

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{S}_1) &\longrightarrow H^0(X, \mathcal{S}_2) \longrightarrow H^0(X, \mathcal{S}_3) \\ &\longrightarrow H^1(X, \mathcal{S}_1) \longrightarrow H^1(X, \mathcal{S}_2) \longrightarrow H^1(X, \mathcal{S}_3) \\ &\quad\longrightarrow H^2(X, \mathcal{S}_1) \longrightarrow H^2(X, \mathcal{S}_2) \longrightarrow H^2(X, \mathcal{S}_3) \longrightarrow \cdots \end{aligned} \tag{D.17}$$

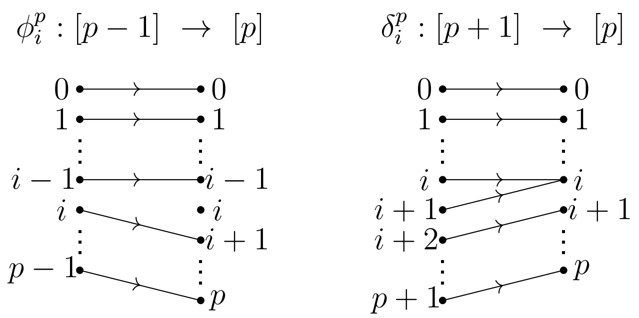
We mention that, using the notion of *flabby sheaves* one may define a cohomology theory which is well-behaved (e.g., it has long exact sequences in cohomology) on every topological space, not just on paracompact ones, cf. [308]. Moreover, regarding the comparison with other types of cohomology, it turns out that, even if in general Čech cohomology of a space can differ from singular cohomology, in case of manifolds they agree. Moreover, Čech cohomology of the constant pre-sheaf $\mathbb{R}_X$ also agrees with de Rham cohomology.

D.2. Quasi-groups

Along the thesis we have discussed L_∞-algebras as higher generalisations of Lie algebras. Now we want to face the question about their finite counter parts. As Lie algebras integrate to Lie groups and, vice-versa, Lie groups differentiate to Lie algebras, analogously the finite counter part of an L_∞-algebra is equivalent to a so-called quasi-group [140, 309, 265]. However, it turns out that this problem in the context of L_∞–algebras is rather involved. To define a quasi-group one needs the rather involved machine of simplicial geometry. Hence, we first provide a brief, elementary introduction into the rich topic of simplicial sets, which are a combinatorial substitute for topological spaces. Our treatment of the subject will be very concise and we closely follow [43]. For a more extensive and nicely illustrated introduction to simplicial sets and simplicial homotopy we refer the interested reader to [310, 103] or the text books [311–313].

D.2.1. Simplicial manifolds

Let us recall the definition of simplicial sets in terms of the *simplex category* $\mathbf{\Delta}$. This is the category whose objects are finite ordered sets $[p] := \{0, 1, \ldots, p\}$, for $p \in \mathbb{N}_0$, and whose morphisms are order-preserving maps $[p] \to [p']$. The morphisms of $\mathbf{\Delta}$ are naturally generated by the *coface maps* ϕ_i^p and *codegeneracy maps* δ_i^p, which are given by



(D.18)

Indeed, any order-preserving map $\phi : [p] \to [p']$ can be decomposed as

$$\phi = \phi_{i_m} \circ \cdots \circ \phi_{i_1} \circ \delta_{j_1} \circ \cdots \circ \delta_{j_n} \ , \tag{D.19}$$

with $p + m - n = p'$, $0 \leq i_1 < \cdots < i_m \leq p'$ and $0 \leq j_1 < \cdots < j_n < p$. Moreover, the objects in Δ have a geometric realisation in terms of the *standard topological p-simplices*

$$|\Delta^p| := \left\{ (t_0, \ldots t_p) \in \mathbb{R}^{p+1} | \sum_{i=0}^{p} t_i = 1 \quad \text{and} \quad t_i \geq 0 \right\} . \tag{D.20}$$

This is realised by means of the functor $\Delta \to \mathsf{Top}$, defined on objects by

$$[p] \mapsto |\Delta^p| \tag{D.21}$$

and on morphisms by

$$\left([p] \xrightarrow{\phi} [p']\right) \mapsto \begin{pmatrix} |\Delta^p| & \to & |\Delta^{p'}| \\ (t_0, \dots, t_p) & \mapsto & (\sum_{\phi(i)=0} t_i, \dots, \sum_{\phi(i)=p'} t_i) \end{pmatrix}, \tag{D.22}$$

with Top being be the category of topological spaces. Thus, the injection $|\Delta^p| \hookrightarrow |\Delta^{p+1}|$, induced by the coface map ϕ_i^p, inserts a 0 in the i-th coordinate $(t_0, \dots, t_p) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_p)$. Geometrically, it inserts $|\Delta^p|$ as the i-th face of $|\Delta^{p+1}|$. Analogously, the codegeneracy map δ_i^p induces the projection $|\Delta^p|$ to the i-th face of $|\Delta^{p-1}|$. Explicitly, this is given by $(t_0, \dots, t_p) \mapsto (t_0, \dots, t_i + t_{i+1}, \dots, t_p)$, collapsing together the vertices i and $i+1$.

As usual, by Δ^{op} we denote the opposite of the simplex category, which we recall is the category formed by the same objects, but reversing the morphisms, i.e. interchanging the source and target of each morphism. Moreover, let Set be the category of sets. We define a *simplicial set* as a Set-valued pre-sheaf on Δ, that is a contravariant functor $\mathscr{X} : \Delta^{\mathsf{op}} \to \mathsf{Set}$.[1] This tells us that $\mathscr{X}$ is a collection of sets $\mathscr{X} = \{\mathscr{X}_p\}_{p \in \mathbb{N}_0}$, where we have defined $\mathscr{X}_p := \mathscr{X}([p])$, called the *simplicial p-simplices*. Coface and codegeneracy maps translate into the *face maps* $\mathsf{f}_i^p := \mathscr{X}(\phi_i^p) : \mathscr{X}_p \to \mathscr{X}_{p-1}$ and the *degeneracy maps* $\mathsf{d}_i^p := \mathscr{X}(\delta_i^p) : \mathscr{X}_p \to \mathscr{X}_{p+1}$, respectively. These are subject to the simplicial identities

$$\begin{gathered} \mathsf{f}_i \circ \mathsf{f}_j = \mathsf{f}_{j-1} \circ \mathsf{f}_i \quad \text{for} \quad i < j~, \quad \mathsf{d}_i \circ \mathsf{d}_j = \mathsf{d}_{j+1} \circ \mathsf{d}_i \quad \text{for} \quad i \leq j~, \\ \mathsf{f}_i \circ \mathsf{d}_j = \mathsf{d}_{j-1} \circ \mathsf{f}_i \quad \text{for} \quad i < j~, \quad \mathsf{f}_i \circ \mathsf{d}_j = \mathsf{d}_j \circ \mathsf{f}_{i-1} \quad \text{for} \quad i > j+1~, \\ \mathsf{f}_i \circ \mathsf{d}_i = \mathrm{id} = \mathsf{f}_{j+1} \circ \mathsf{d}_i~. \end{gathered} \tag{D.23}$$

These identities follow immediately from similar identities for the coface and the codegeneracy maps. Note that the last line of (D.23) implies that the face maps are surjective and the degeneracy maps are injective. In the following, simplicial sets shall be depicted by writing arrows for the face maps as

$$\left\{ \cdots \;\substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow}\; \mathscr{X}_2 \;\substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow}\; \mathscr{X}_1 \rightrightarrows \mathscr{X}_0 \right\}~. \tag{D.24}$$

Note, that replacing Set by the category of groups Grp or the category of (Frechét) manifolds Mfd, one defines simplicial groups or simplicial manifolds, respectively.

Having defined simplicial sets, let us now move on to maps between them. Morphisms of simplicial sets, also known as *simplicial maps*, are natural transformations between the functors

[1]Notice that, unlike other (pre-)sheaves, the special pre-sheaves forming simplicial sets and simplicial manifolds are denoted by calligraphic letters $\mathscr{X}, \mathscr{Y}, \dots$.

defining the simplicial sets as pre-sheaves. Explicitly, given two simplicial sets $\mathscr{H}$ and $\mathscr{H}'$, a simplicial map $g : \mathscr{H} \to \mathscr{H}'$ is defined as the collection of maps $g^p : \mathscr{H}_p \to \mathscr{H}'_p$, commuting with the face and degeneracy maps on $\mathscr{H}$ and $\mathscr{H}'$. Simplicial sets together with simplicial maps form the category of simplicial sets sSet. It can be also defined as the functor category[1] $\mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set})$. In general, given an arbitrary category $\mathscr{C}$, a simplicial object in $\mathscr{C}$ is a functor $\Delta^{\mathsf{op}} \to \mathscr{C}$. For instance, the category of simplicial manifolds is $\mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Mfd})$.

Examples. An important example of a simplicial set is given by the *standard simplicial* p-*simplex* Δ^p, which is given by $\mathsf{hom}_\Delta(-, [p]) : \Delta^{\mathsf{op}} \to \mathsf{Set}$. We have $\Delta^p = \{\Delta^p_q\}_{q\in\mathbb{N}_0}$ where $\Delta^p_q := \mathsf{hom}_\Delta([q], [p])$ are simplicial q-simplices in Δ^p. Whenever such a map is injective it is called a non-degenerate simplicial simplex (degenerate otherwise). One can show that this is unique and given by the identity at $[p]$, while all simplicial simplices for $q > p$ in Δ^p are degenerate. Via the Yoneda lemma (see e.g. [303], Chapter 4 for a proof), there is a bijection between any simplicial map $\Delta^p \to \Delta^{p'}$ and a morphism $[p] \to [p']$ in the simplex category. The Yoneda lemma also implies the bijection

$$\mathscr{H}_p \cong \mathsf{hom}_{\mathsf{sSet}}(\Delta^p, \mathscr{H}) \tag{D.25}$$

for any simplicial set $\mathscr{H}$.

Products of simplicial sets. We observe that sSet is a symmetric monoidal category. In particular, it comes with a product. Indeed, given any two simplicial sets $\mathscr{H}$ and $\mathscr{H}'$, we define the product $\mathscr{H} \times \mathscr{H}'$ to be the simplicial set with simplicial p-simplices $(\mathscr{H} \times \mathscr{H}')_p := \mathscr{H}_p \times \mathscr{H}'_p$ together with the face and degeneracy maps acting as $\mathsf{f}_i^{\mathscr{H}\times\mathscr{H}'}(x_p, x'_p) := (\mathsf{f}_i^{\mathscr{H}} x_p, \mathsf{f}_i^{\mathscr{H}'} x'_p)$ and $\mathsf{d}_i^{\mathscr{H}\times\mathscr{H}'}(x_p, x'_p) := (\mathsf{d}_i^{\mathscr{H}} x_p, \mathsf{d}_i^{\mathscr{H}'} x'_p)$, for all $(x_p, x'_p) \in (\mathscr{H} \times \mathscr{H}')$.

Given the product, we define the following. For $\mathscr{H}$, $\mathscr{H}'$ as before, the object $\underline{\mathsf{hom}}(\mathscr{H}, \mathscr{H}')$, called the *internal hom*, is the simplicial set with simplicial p-simplices $\underline{\mathsf{hom}}_p(\mathscr{H}, \mathscr{H}') := \mathsf{hom}_{\mathsf{sSet}}(\Delta^p \times \mathscr{H}, \mathscr{H}')$ and face and degeneracy maps given by

$$\begin{aligned} \mathsf{f}_i^p &: \left(\Delta^p \times \mathscr{H} \xrightarrow{f} \mathscr{H}'\right) \longmapsto \left(\Delta^{p-1} \times \mathscr{H} \xrightarrow{\phi_i^p \times \mathrm{id}_{\mathscr{H}}} \Delta^p \times \mathscr{H} \xrightarrow{f} \mathscr{H}'\right) , \\ \mathsf{d}_i^p &: \left(\Delta^p \times \mathscr{H} \xrightarrow{f} \mathscr{H}'\right) \longmapsto \left(\Delta^{p+1} \times \mathscr{H} \xrightarrow{\delta_i^p \times \mathrm{id}_{\mathscr{H}}} \Delta^p \times \mathscr{H} \xrightarrow{f} \mathscr{H}'\right) . \end{aligned} \tag{D.26}$$

[1]Given categories $\mathscr{C}, \mathscr{D}$, the *functor category* $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ is the category whose objects are functors $F : \mathscr{C} \to \mathscr{D}$ and morphisms given by natural transformations between these functors. See e.g. [303, 304] for details.

Hence, simplicial 0-simplices $\underline{\mathsf{hom}}_0(\mathscr{H}, \mathscr{H}')$ are the simplicial maps between $\mathscr{H}$ and $\mathscr{H}'$. Invoking again the Yoneda lemma, one has the isomorphism

$$\mathsf{hom}_{\mathsf{sSet}}(\Delta^p \times \mathscr{H}, \mathscr{H}') \cong \mathsf{hom}_{\mathsf{sSet}}(\Delta^p, \underline{\mathsf{hom}}(\mathscr{H}, \mathscr{H}')) \ . \tag{D.27}$$

This can be generalised further to[1]

$$\mathsf{hom}_{\mathsf{sSet}}(\mathscr{H} \times \mathscr{H}', \mathscr{H}'') \cong \mathsf{hom}_{\mathsf{sSet}}(\mathscr{H}, \underline{\mathsf{hom}}(\mathscr{H}', \mathscr{H}'')) \ , \tag{D.28}$$

for any simplicial sets $\mathscr{H}$, $\mathscr{H}'$ and $\mathscr{H}''$.

D.2.2. Simplicial homotopy theory

Let us consider now two simplicial sets $\mathscr{H}$, $\mathscr{H}'$ and two simplicial maps g, $\tilde{g}$ between them. A *simplicial homotopy* between them is defined as an element $h \in \underline{\mathsf{hom}}_1(\mathscr{H}, \mathscr{H}') = \mathsf{hom}_{\mathsf{sSet}}(\Delta^1 \times \mathscr{H}, \mathscr{H}')$ such that the following diagram commute

$$\begin{array}{ccc} \Delta^0 \times \mathscr{H} \cong \mathscr{H} & & \\ \big\downarrow{\scriptstyle \phi^1_1 \times \mathrm{id}_{\mathscr{H}}} & \overset{g}{\searrow} & \\ \Delta^1 \times \mathscr{H} & \xrightarrow{h} & \mathscr{H}' \\ \big\uparrow{\scriptstyle \phi^1_0 \times \mathrm{id}_{\mathscr{H}}} & \underset{\tilde{g}}{\nearrow} & \\ \Delta^0 \times \mathscr{H} \cong \mathscr{H} & & \end{array} \tag{D.29}$$

An equivalent description of simplicial homotopies that will be useful to us later is the following. Using (D.28), a simplicial map $h \in \mathsf{hom}_{\mathsf{Set}}(\mathscr{H}, \mathsf{hom}(\Delta^1, \mathscr{H}'))$, which is a collection of maps $h^p = (h^p_i) : \mathscr{H}_p \to \underline{\mathsf{hom}}_p(\Delta^1, \mathscr{H}')$ with $h^p_i : \mathscr{H}_p \to \mathscr{H}'_{p+1}$ for $i = 0, \ldots, p$, is a simplicial homotopy between the simplicial maps $g^p := \mathsf{f}^{p+1}_0 \circ h^p_0 : \mathscr{H}_p \to \mathscr{H}'_p$ and $\tilde{g}_p := \mathsf{f}^{p+1}_{p+1} \circ h^p_p : \mathscr{H}_p \to \mathscr{H}'_p$. In this spirit *higher simplicial homotopies* will be elements of $\underline{\mathsf{hom}}_k(\mathscr{H} \times \mathscr{H}') \cong \mathsf{hom}_{_{\mathsf{sSet}}}(\mathscr{H}, \underline{\mathsf{hom}}(\Delta^k, \mathscr{H}'))$, for $k \geq 2$.

Particularly important simplicial sets are the horns of the standard simplicial p-simplex. Recall that, given a simplicial set $\mathscr{H}$, the simplicial simplices of $\mathscr{H}$ can be given by the standard maps (D.25) from Δ^p to $\mathscr{H}$. We call the simplicial subset of Δ^p generated by the union of all faces of Δ^p, except for the i-th one, the *(p, i)-horn* of Δ^p and we denote it by Λ^p_i. Since all the horns Λ^p_i of Δ^p arise by removing the unique non-degenerate simplicial p-simplex from Δ^p as well the i-th non-degenerate simplicial $(p-1)$-simplex, they can again

[1]Hence, hom is the right-adjoint of the product in the category sSet.

be completed to simplicial simplices. If the horns of a simplicial set can be filled, we say that certain Kan conditions are satisfied. In particular, note that the horns $\mathsf{hom}_{\mathsf{sSet}}(\Lambda^p_i, \mathscr{H})$ of a general simplicial set $\mathscr{H}$ may not always be completed to simplices $\mathsf{hom}_{\mathsf{sSet}}(\Delta^p, \mathscr{H})$. Whenever for any horn $\lambda : \Lambda^p_i \to \mathscr{H}$ there exists a simplicial map $\tilde{\delta} : \Delta^p \to \mathscr{H}$, such that

$$\begin{array}{ccc} \Lambda^p_i & \xrightarrow{\lambda} & \mathscr{H} \\ \big\downarrow & \nearrow_{\tilde{\delta}} & \\ \Delta^p & & \end{array} \tag{D.30}$$

is commutative, we call $\mathscr{H}$ a *Kan simplicial set*. This can be rephrased by saying that the natural restriction mappings

$$\mathsf{hom}_{\mathsf{sSet}}(\Delta^p, \mathscr{H}) \to \mathsf{hom}_{\mathsf{sSet}}(\Lambda^p_i, \mathscr{H}) \tag{D.31}$$

are surjective for all $p \geq 1$ and $0 \leq i \leq p$. Moreover, a *Kan simplicial manifold* is a simplicial manifold so that the above restrictions are surjective submersions. We shall denote the category of simplicial manifolds by $\mathsf{sMfd} := \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Mfd})$.[1] Also, the internal hom $\underline{\mathsf{hom}}(\mathscr{H}, \mathscr{H}')$ is Kan whenever $\mathscr{H}'$ is Kan.

Each small category[2] comes with an associated simplicial set defined as follows. The *nerve* $N(\mathscr{C})$ for a small category $\mathscr{C}$ is the simplicial set $N(\mathscr{C}) : \Delta^{\mathsf{op}} :\to \mathsf{Set}$, whose simplicial p-simplices are $N_p(\mathscr{C}) = \mathsf{Fun}([p], \mathscr{C})$. Any ordinary manifold, interpreted as a constant simplicial object, is a Kan simplicial manifold. As a further relevant example of Kan simplicial manifolds, we define the Čech groupoid. Recall that a groupoid is a small category in which every morphism is invertible. Consider two manifolds X, Y and a surjective submersion $\phi : Y \to X$. The fibre product of Y with itself over X is denoted by $Y \times_X Y := \{(y_1, y_2) \in Y \times Y | \phi(y_1) = \phi(y_2)\}$. The Čech groupoid $\check{\mathscr{C}}(Y \to X)$ is the groupoid $Y \times_X Y \rightrightarrows X$ with pairs (y_1, y_2) for $y_1, y_2 \in Y$ satisfying $\phi(y_1) = \phi(y_2)$ as its morphisms. Finally, the *Čech nerve* is defined to be the nerve of the Čech groupoid, which can be shown to be a Kan simplicial manifold. Explicitly, this is the simplicial set

$$N(\check{\mathscr{C}}(Y \to X)) = \Big\{ \cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} Y \times_X Y \times_X Y \substack{\rightarrow\\\rightarrow\\\rightarrow} Y \times_X Y \rightrightarrows Y \Big\} \tag{D.32a}$$

[1]A simplicial manifold is a sMfd-valued pre-sheaf on Δ.

[2]Recall, a category $\mathscr{C}$ is called small if the collections $\mathsf{Ob}(\mathscr{C})$ and $\mathsf{hom}(\mathscr{C})$ are sets. Otherwise, $\mathscr{C}$ is called large.

with face and degeneracy maps defined by

$$\begin{aligned} \mathsf{f}_i^p(y_0,\dots,y_p) &:= (y_0,\dots,y_{i-1},y_{i+1},\dots,y_p)\ ,\\ \mathsf{d}_i^p(y_0,\dots,y_p) &:= (y_0,\dots,y_{i-1},y_i,y_i,\dots,y_p)\ . \end{aligned} \tag{D.32b}$$

D.2.3. Quasi-groups and L_∞-algebras

We observe an important fact, which will come in handy when discussing higher principal bundles. If $\mathscr{H}'$ is a Kan simplicial set the simplicial homotopy always induces an equivalence relation on $\mathsf{hom}_{\mathsf{sSet}}(\mathscr{H},\mathscr{H}')$, even if in general that is not true.

Let us now give some definitions. Kan simplicial sets are known as *quasi-groupoids* and Kan simplicial manifolds as *Lie quasi-groupoids*, respectively. Moreover, a Kan simplicial set (manifold) is called a reduced (Lie) quasi-groupoid, if there is only one single simplicial 0-simplex. We shall see in the following how these objects, which are higher versions of groupoids, describe the gauge structure of simplicial bundles. Following the delooping hypothesis, we identify reduced (Lie) quasi-groupoids with *(Lie) quasi-groups*. It was shown by Quillen [314] that the categories of (Lie) quasi-groups and simplicial (Lie) groups are equivalent. Finally, we shall speak of *(Lie) n-quasi-groups* whenever all the (p,i)-horns for a (Lie) quasi-group can be filled uniquely for all $p>n$.

Differentiation of Lie quasi-groupoids. In this paragraph, we briefly comment about differentiation of Lie quasi-groupoids. This notion allows to endow higher principal (groupoid) bundles with connections. We follow the ideas of [140] and we invite the interested reader to consult [315, 103] for more details on the subject. We observe that, via the forgetful functor, $\mathbb{Z}$-graded manifolds may be mapped to $\mathbb{Z}_2$-graded manifolds, known as *supermanifolds*. We let SMfd be the category of (Frechét) supermanifolds.[1] Moreover, denote by SurSub the category of surjective submersions $Y\to X$ between supermanifolds Y and X as its objects and morphisms such that

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_2\\ \downarrow & & \downarrow\\ X_1 & \longrightarrow & X_2 \end{array} \tag{D.33}$$

are commutative for surjective submersions $Y_{1,2}\to X_{1,2}$. As before, we set $\mathsf{sSMfd} := \mathsf{Fun}(\boldsymbol{\Delta}^{\mathsf{op}},\mathsf{SMfd})$ and call it the category of *simplicial supermanifolds*.

[1]SMfd is the category of $\mathbb{Z}_2$-graded manifolds together with morphisms of $\mathbb{Z}_2$-graded manifolds.

Since the nerve N of the Čech groupoid of an object in SurSub is an object in sSMfd, any object $\mathscr{H} \in$ sSMfd can be used a to define a Set-valued pre-sheaf

$$\mathsf{hom}_{\mathsf{sSMfd}}(N(-), \mathscr{H}) : \mathsf{SurSub}^{\mathsf{op}} \to \mathsf{Set} \tag{D.34}$$

on SurSub. Now, in analogy to the well-known construction in differential geometry, we call *k-jets* of $\mathscr{H}$ the linearisation of this pre-sheaf. We define SurSub_k to be the subcategory of SurSub whose objects are surjective submersions of the form $X \times \mathbb{R}^{0|k} \to X$. We have the identification

$$\mathsf{hom}_{\mathsf{SurSub}_k}(X_1 \times \mathbb{R}^{0|k} \to X_1, X_2 \times \mathbb{R}^{0|k} \to X_2) \cong \mathsf{hom}_{\mathsf{sSMfd}}(X_1, X_2) \times \mathsf{hom}_{\mathsf{SMfd}}(X_1 \times \mathbb{R}^{0|k}, \mathbb{R}^{0|k}) \,. \tag{D.35}$$

Hence, a pre-sheaf on SurSub_k is equivalent to a pre-sheaf on SMfd together with an action of $\underline{\mathsf{hom}}(\mathbb{R}^{0|k}, \mathbb{R}^{0|k})$. We shall denote this by SMfd_k. For instance, sSMfd_1 is the category of Q-supermanifolds since the action of $\underline{\mathsf{hom}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ corresponds to the action of the vector field Q. Following Ševera [140], for any pre-sheaf of SurSub we may consider its restriction to SurSub_k to obtain a pre-sheaf of SMfd. We call the latter the *k-jet* of the pre-sheaf on SurSub. In addition, the k-jet of a simplicial supermanifold $\mathscr{H}$ is the k-jet of the pre-sheaf $\mathsf{hom}_{\mathsf{sSMfd}}(N(-), \mathscr{H})$.

Again, following [140, 103], one can see that taking the 1-jet of a quasi-groupoid turns out to be the appropriate higher version of the differentiation of a Lie group to a Lie algebra. In particular, the 1-jet of a Lie quasi-group is an L_∞-algebra. Explicitly, letting

$$\mathscr{G} := \left\{ \cdots \overset{\to}{\underset{\to}{\equiv}} \mathscr{G}_2 \overset{\to}{\underset{\to}{\equiv}} \mathscr{G}_1 \rightrightarrows * \right\} \tag{D.36}$$

be a Lie quasi-group with face maps f^p_i and degeneracy maps d^p_i, the 1-jet of $\mathscr{G}$ is parametrised as [103]

$$\mathsf{L}[1] = \bigoplus_{k \le 0} \mathsf{L}_k[1] \quad \text{with} \quad \mathsf{L}_k[1] := \bigcap_{i=0}^{-k} \ker\, (\mathsf{f}^{1-k}_{i\;\star})[1-k] \;, \tag{D.37}$$

where $\mathsf{f}^p_{i\;\star}$ denote the linearisation of f^p_i. Moreover, $\mu_1|_{\mathsf{L}_k[1]} = \mathsf{f}^{1-k}_{1-k\;\star}$ and the products μ_i for $i > 1$ are given in terms of the j-th order derivatives of the face maps with $j \le i$.

Proving the converse is a much more problematic as it involves topological questions. However, it can be shown that every L_∞-algebra integrates to a Lie-quasi-group [309, 265].

D.3. Higher principal bundles

Since we are interested in studying higher gauge theory we need to define connective structures on higher principal bundles with quasi-groups as their structure groups. We present a rather straightforward approach to the computation of all the kinematical ingredients to corresponding higher gauge theories given by higher Deligne cocycles and coboundaries. The subject has been extensively studied in the past and we refer to e.g. [316, 22, 5, 20, 21, 6, 317, 157, 100, 23, 263, 101, 318, 102, 103] for details.

D.3.1. Principal $\mathscr{G}$-bundles

Let us consider a Lie group G. We recall that the *Lie groupoid*[1] is defined as its delooping $\mathsf{BG} = \mathsf{G} \rightrightarrows \star$. Here, the source and the target maps are trivial, $\mathrm{id}_* = \mathbb{1}_\mathsf{G}$ and the composition is just the group multiplication in G. Because it is a category, we can consider the nerve $N(\mathsf{BG})$ of BG. This is the simplicial manifold

$$N(\mathsf{BG}) := \left\{ \cdots \overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} \mathsf{G}\times\mathsf{G} \overset{\rightarrow}{\rightrightarrows} \mathsf{G} \rightrightarrows \star \right\} \ , \tag{D.38}$$

with the obvious face and degeneracy maps. In addition, recall the Čech nerve (D.32) for a subjective submersion $\mathfrak{U} \to M$ given by a manifold M subordinate to an open cover $\mathfrak{U} := \cup_{a\in A} U_a$. In the simplicial language we have the following

Definition D.7. *Let* G *be a Lie group. A principal* G*-bundle over a manifold* M *subordinate to the cover* $\mathfrak{U} := \cup_{a\in A} U_a$ *is a simplicial map* $g : N(\check{\mathscr{C}}(\mathfrak{U} \to M)) \to N(\mathsf{BG})$.

The simplicial map g is a collection of maps $g^p : N_p(\check{\mathscr{C}}(\mathfrak{U} \to M)) \to N_p(\mathsf{BG})$ explicitly given by

$$\begin{aligned} g_a(x) &:= g^0(x,a) = \star \ , \\ g_{ab}(x) &:= g^1(x,a,b) \in \mathsf{G} \ , \\ g_{abc}(x) &:= g^2(x,a,b,c) = (g^1_{abc}(x), g^2_{abc}(x)) \in \mathsf{G}\times\mathsf{G} \ . \end{aligned} \tag{D.39}$$

Being simplicial, the g^p commute with the face and the degeneracy maps so that

$$g^1_{abc}(x) = g_{ab}(x) \ , \quad g^1_{abc}(x) g^2_{abc}(x) = g_{ac}(x) \ , \quad g^2_{abc}(x) = g_{bc}(x) \ . \tag{D.40}$$

This allows us to recover the usual description of principal G-bundles in terms of Čech cocycles, i.e. the transition functions $g_{ab} : U_a \cap U_b \to \mathsf{G}$ satisfy the standard cocycle conditions.

[1] Recall that a Lie groupoid is a groupoid internal to Mfd.

Moreover, two principal G-bundles $g, \tilde{g} : N(\check{\mathscr{C}}(\mathfrak{U} \to M)) \to N(\mathsf{BG})$ are called equivalent if and only if there is a simplicial homotopy between them. One can check that such an object, $h : \Delta^1 \times N(\check{\mathscr{C}}(\mathfrak{U} \to M)) \to N(\mathsf{BG})$, is a collection of maps $h_a : U_a \to \mathsf{G}$ satisfying the standard coboundary conditions

$$g_{ab}(x) h_b(x) = h_a(x) \tilde{g}_{ab}(x) \ . \tag{D.41}$$

This definition of principal G-bundles generalises to nerves of Lie groupoids. Generally, let us consider a Lie-quasi-group $\mathscr{G}$. Similarly, we define a *principal $\mathscr{G}$-bundle* over M subordinate to $\mathfrak{U}$ to be the simplicial map $g : N(\check{\mathscr{C}}(\mathfrak{U} \to M)) \to \mathscr{G}$ [266, 263]. Two such bundles are said to be equivalent, whenever there is a simplicial homotopy between them defining simplicial maps. It should be emphasised that this notion of equivalence is well-defined since $\mathscr{G}$ is Kan.

D.3.2. Higher non-Abelian Deligne cohomology

Besides principal bundles, we shall also need connective structures to discuss gauge theory. Recall that a connection or *connective structure* on a principal G-bundle on a manifold M subordinate to an open cover $\mathfrak{U} = \{U_a\}_{a \in A} \to M$ is a collection of $\mathfrak{g}$-valued differential 1-forms $\{A_a \in \Omega^1(U_a, \mathfrak{g})\}$, with $\mathfrak{g}$ being the Lie algebra of G, which obey

$$A_b(x) = g_{ab}^{-1}(x) A_a(x) g_{ab}(x) + g_{ab}^{-1}(x) \mathrm{d} g_{ab}(x) \tag{D.42}$$

on non-empty intersections $U_a \cap U_b \neq \varnothing$. Here, the smooth maps $g_{ab} : U_a \cap U_b \to \mathsf{G}$ are the transition functions of the principal G-bundle.

In addition, the coboundary transformations (D.41) yield the transformations

$$\tilde{A}_a(x) = h_a^{-1}(x) A_a(x) h_a(x) + h_a^{-1}(x) \mathrm{d} h_a(x) \ , \tag{D.43}$$

with $h_a : U_a \to \mathsf{G}$ smooth maps. This allows use to introduce the *Deligne cocycle* $\{A_a, g_{ab}\}$ which defines a principal G-bundle with connection and two such cocycles are called equivalent if there is a coboundary transformation of the form (D.41) and (D.43).

In this picture, the generalisation to higher principal bundles is immediate. Let $\mathscr{G}$ be a Lie-n-quasi-group and (L, μ_i), with $\mathsf{L} = \bigoplus_{k=-n+1}^{0} \mathsf{L}_k$, be the corresponding n-term L_∞-algebra obtained by computing the 1-jet of $\mathscr{G}$ (see D.2.3.). Let us consider a manifold M subordinate to an open cover $\mathfrak{U} := \cup_{a \in A} U_a$ and a higher principal bundle over M with structure group $\mathscr{G}$. This is defined by transition functions

$$g_{a_0 \cdots a_k} : U_{a_0} \cap \cdots \cap U_{a_k} \to \mathscr{G}_k \ , \qquad \text{for} \quad k = 1, \ldots, n \ , \tag{D.44a}$$

which are encoded in a simplicial map $g : N(\check{\mathscr{C}}(\mathfrak{U} \to M)) \to \mathscr{G}$, supplemented, when $n \geq 2$, by differential form-valued transition functions

$$\lambda_{a_0,\cdots a_k} \in \bigoplus_{i+j=1-k} \Omega^i(U_{a_0} \cap \cdots \cap U_{a_k}) \otimes \mathsf{L}_j \ , \qquad \text{for} \quad k = 1, \ldots, n-1 \ . \tag{D.44b}$$

A *connective structure* on the principal $\mathscr{G}$-bundle is given by a set of local L_∞-valued differential forms

$$A_a \in \bigoplus_{i+j=1} \Omega^i(U_a) \otimes \mathsf{L}_j \ . \tag{D.45}$$

Together (D.44) and (D.45) form what is known as *higher Deligne cocycle*. The general treatment of the subject, providing the complete and rather involved cocycle and coboundary conditions of such a cocycle, can be found in [266, 263, 101, 318, 102, 103]. Here, for simplicity we shall consider the cases of an ordinary Lie group and a Lie 2-quasi-group.

Principal 1-bundles. Let G be a Lie group with corresponding Lie algebra $\mathfrak{g}$. The corresponding Deligne cocycle is given by

$$\{g_{ab}, A_a\} \tag{D.46}$$

with $g_{ab} : U_a \cap U_b \to \mathsf{G}$ and $A_a \in \Omega^1(U_a, \mathfrak{g})$ subject to the cocycle conditions

$$g_{ac} = g_{ab} g_{bc} \quad \text{on} \quad U_a \cap U_b \cap U_c \ , \tag{D.47a}$$

$$A_b = g_{ab}^{-1} A_a g_{ab} + g_{ab}^{-1} \mathrm{d} g_{ab} \quad \text{on} \quad U_a \cap U_b \ . \tag{D.47b}$$

Two degree-1 cocycles $(\{g_{ab}\}, \{A_a\})$ and $(\{\tilde{g}_{ab}\}, \{\tilde{A}_a\})$ are considered equivalent if there is a degree-0 cochain $\{g_a\}$ with values in G such that

$$\tilde{A}_a = g_a^{-1} A_a g_a + g_a^{-1} \mathrm{d} g_a \quad \text{on} \quad U_a \ , \tag{D.48a}$$

$$\tilde{g}_{ab} = g_a^{-1} g_{ab} g_b \quad \text{on} \quad U_a \cap U_b \ . \tag{D.48b}$$

Principal 2-bundles. The next step in our discussion is the introduction of categorified Lie groups. Here, instead of having a fully categorified Lie group (a so-called weak Lie 2-group) as a gauge group, we focus on *strict Lie 2-quasi-groups*. These can can equivalently be described by a Lie crossed module and the corresponding strict 2-term L_∞-algebra by a differential crossed module. We would like to point out that, differential Lie crossed modules naturally describe the 3-algebras underlying the recently popular M2-brane models [319]. A *Lie crossed module* is a pair of Lie groups (G, H) together with a Lie group homomorphism

$\mathsf{t} : \mathsf{H} \to \mathsf{G}$ and an action $\rhd$ of G on H by automorphisms. The map t is G-equivariant $\mathsf{t}(g \rhd h) = g\mathsf{t}(h)g^{-1}$ and the *Peiffer identity*, $\mathsf{t}(h_1) \rhd h_2 = h_1 h_2 h_1^{-1}$, holds for all $g \in \mathsf{G}$ and $h, h_1, h_2 \in \mathsf{H}$. Furthermore, a *differential crossed module* is the 1-jet of a Lie crossed module and is given by a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g} := \mathsf{Lie}(\mathsf{G})$ and $\mathfrak{h} := \mathsf{Lie}(\mathsf{H})$ with $\mathsf{t}_* : \mathfrak{h} \to \mathfrak{g}$ such that $\mathsf{t}_*(V \rhd_* U) = [V, \mathsf{t}_*(U)]$ and $\mathsf{t}_*(U_1) \rhd_* U_2 = [U_1, U_2]$ for all $V \in \mathfrak{g}$ and $U, U_1, U_2 \in \mathfrak{h}$ where t_* and $\rhd_*$ are the linearisations of t and $\rhd$, respectively.[1]

A Deligne cocycle in the crossed module language is then given by

$$\{g_{ab}, h_{abc}, \lambda_{ab}, A_a, B_b\} \tag{D.49}$$

with $g_{ab} : U_a \cap U_b \to \mathsf{G}$, $h_{abc} : U_a \cap U_b \cap U_c \to \mathsf{H}$, $\lambda_{ab} \in \Omega^1(U_a \cap U_b, \mathfrak{h})$, $A_a \in \Omega^1(U_a, \mathfrak{g})$, and $B_a \in \Omega^2(U_a, \mathfrak{h})$ subject to the cocycle conditions

$$\begin{aligned}
\mathsf{t}(h_{abc}) g_{ab} g_{bc} &= g_{ac}\ , \\
h_{acd} h_{abc} &= h_{abd}(g_{ab} \rhd h_{bcd})\ , \\
\lambda_{ac} &= \lambda_{bc} + g_{bc}^{-1} \rhd \lambda_{ab} - g_{ac}^{-1} \rhd (h_{abc} \nabla_a h_{abc}^{-1})\ , \\
A_b &= g_{ab}^{-1} A_a g_{ab} + g_{ab}^{-1} \mathrm{d} g_{ab} - \mathsf{t}_*(\lambda_{ab})\ , \\
B_b &= g_{ab}^{-1} \rhd B_a - \nabla_b \lambda_{ab} - \tfrac{1}{2}[\lambda_{ab}, \lambda_{ab}]
\end{aligned} \tag{D.50}$$

on appropriate non-empty overlaps and $\nabla_a := \mathrm{d} + A_a \rhd_*$. Furthermore, two such cocycles $\{g_{ab}, h_{abc}, \lambda_{ab}, A_a, B_b\}$ and $\{\tilde{g}_{ab}, \tilde{h}_{abc}, \tilde{\lambda}_{ab}, \tilde{A}_a, \tilde{B}_b\}$ are equivalent whenever there is a coboundary transformation, mediated by

$$\{g_a, h_{ab}, \lambda_a\} \tag{D.51}$$

[1] Differential crossed modules and 2-term L_∞-algebras (L, μ_i) with $\mathsf{L} = \mathsf{L}_{-1} \oplus \mathsf{L}_0$ and $\mu_3 = 0$ are actually the same thing. Indeed, given such an L_∞-algebra, the corresponding differential crossed module is $\mathfrak{g} := \mathsf{L}_0$ and $\mathfrak{h} := \mathsf{L}_{-1}$, $\mathsf{t}_* := \mu_1$, $V \rhd U := \mu_2(U, V)$, $[U_1, U_2] := \mu_2(\mu_1(U_1), U_2)$, and $[V_1, V_2] := \mu_2(V_1, V_2)$ for $U, U_1, U_2 \in \mathfrak{h}$ and $V, V_1, V_2 \in \mathfrak{g}$. The antisymmetry and the Jacobi identities for the Lie brackets $[-,-]$ as well as the equivariance condition $\mathsf{t}_*(V \rhd_* U) = [V, \mathsf{t}_*(U)]$ follow from the higher Jacobi identities for μ_1 and μ_2, and the Peiffer condition $\mathsf{t}_*(U_1) \rhd_* U_2 = [U_1, U_2]$ is evidently satisfied. Obviously, the converse is also true, i.e. we can use the same identifications to construct a 2-term L_∞-algebra (L, μ_i) with $\mathsf{L} = \mathsf{L}_{-1} \oplus \mathsf{L}_0$ and $\mu_3 = 0$ from a differential crossed module, and the graded antisymmetry as well as the higher Jacobi identities for μ_1 and μ_2 follow from the Jacobi identities for the Lie brackets together with the equivariance and Peiffer conditions.

with $g_a : U_a \to \mathsf{G}$, $h_{ab} : U_a \cap U_b \to \mathsf{H}$, and $\lambda_a \in \Omega^1(U_a, \mathfrak{h})$, and explicitly given by

$$\begin{aligned} \mathsf{t}(h_{ab}) g_{ab} g_b &= g_a \tilde{g}_{ab} \\ h_{ac} h_{abc} &= (g_a \rhd \tilde{h}_{abc}) h_{ab} (g_{ab} \rhd h_{bc}) \;, \\ \lambda_a &= \tilde{\lambda}_{ab} + \lambda_b + g_b^{-1} \rhd \lambda_{ab} - g_a^{-1} \rhd (h_{ab} \nabla_a h_{ab}^{-1}) \;, \\ \tilde{A}_a &= g_a^{-1} A_a g_a + g_a^{-1} \mathrm{d} g_a - \mathsf{t}_*(\lambda_a) \;, \\ \tilde{B}_a &= g_a^{-1} \rhd B_a - \tilde{\nabla}_a \Lambda_a - \tfrac{1}{2} [\Lambda_a, \Lambda_a] \;. \end{aligned} \tag{D.52}$$

It is rather straighforward to see that with the help of these coboundary transformations, we can always set $h_{aaa} = \mathbb{1}_{\mathsf{H}}$, which, in turn, yields $g_{aa} = \mathbb{1}_{\mathsf{G}}$ and $h_{aab} = h_{abb} = \mathbb{1}_{\mathsf{H}}$. Residual coboundary transformations are then those with $h_{aa} = \mathbb{1}_{\mathsf{H}}$.

The corresponding curvatures on each patch U_a are defined by

$$F_a \;:=\; \mathrm{d} A_a + \tfrac{1}{2}[A_a, A_a] \quad \text{and} \quad H_a \;:=\; \mathrm{d} B_a + A_a \rhd B_a \;=\; \nabla_a B_a \;. \tag{D.53}$$

They satisfy the Bianchi identities

$$\mathrm{d} F_a + [A_a, F_A] = 0 \quad \text{and} \quad \mathrm{d} H_a + A_a \rhd H_a = 0 \;. \tag{D.54}$$

Notice that the Bianchi identity for H_a comes from the fake curvature condition $\mathcal{F}_a = F_a + \mathsf{t}(B_a) = 0$.

E

Geometry of twistor correspondence

In this Appendix we shall review some basic notions of complex geometry, which will be heavily used throughout this Thesis because of the intimate relation of this subject with twistor theory. Moreover, we shall look closely at the geometry behind the twistor correspondence. The following literature has proven to be useful for studying this subject: [320, 321, 306, 322] for complex geometry, [238, 323–325] for Dolbeault and Čech-description of holomorphic vector bundles and [326, 96] for the twistor transform.

E.1. Complex geometry

E.1.1. Complex manifolds and vector bundles

Homolorphic map. Let U be an open subset of $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ a continuously differential mapping. It is convenient to identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by means of the association

$$(z^1, \dots, z^n) \leftrightarrow (x^1, \dots, x^n, y^1, \dots, y^n) \ , \tag{E.1}$$

where $z^i = x^i + iy^i$, $i = 1, \dots, n$ are the coordinates in $\mathbb{C}^n$. Then, $f(x^i, y^i) = u(x^i, y^i) + iv(x^i, y^i)$ is called *holomorphic* if it satisfies the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x^i} = \frac{\partial v}{\partial y^i} \ , \quad \frac{\partial v}{\partial x^i} = -\frac{\partial u}{\partial y^i} \ . \tag{E.2}$$

for $i = 1, \dots, n$. Equivalently, using variables $z^i, \bar{z}^i$, the conditions (E.2) read

$$\frac{\partial f}{\partial \bar{z}^i} = 0 \ , \quad \text{for} \quad i = 1, \dots, n \ , \tag{E.3}$$

Indeed, Cauchy's theorem implies that f is holomorphic if and only if f is complex analytic. A map $(f^1, \cdots, , f^m) : \mathbb{C}^n \to \mathbb{C}^m$ is called holomorphic if each function f^j, for $1 \leq j \leq m$, is holomorphic.

Moreover, two open subsets $U, V \in \mathbb{C}^n$ are said to *biholomorphic* if there exists a bijective holomorphic map $f : U \to V$ whose inverse is holomorphic. The map f itself is then said to be biholomorphic.

Complex manifolds. Let X be a topological space with an open covering $\mathfrak{U} = \{U_a\}_{a\in A}$. X is called a *complex manifold* of dimension n if for every $U_a \in \mathfrak{U}$ there is a homeomorphism $h_a : U_a \to \mathbb{C}^n$ such that, for every non-empty intersection $U_a \cap U_b$, the transitions functions $g_{ab} := h_a \circ h_b^{-1} : h_b(U_a \cap U_b) \to h_a(U_a \cap U_j)$ are biholomorphisms.

The pair (U_a, h_a) is called a local complex chart and the collection $\{(U_a, h_a)\}_{a\in A}$ is called a complex atlas. If the union of two atlases is again an atlas which satisfies the previous conditions these are said to define the same complex structure. Notice that a complex manifold may carry a number of complex structures.

Example E.1 (Complex projective space)**.** *A very important family of complex manifolds are the complex projective spaces. n-dimensional complex projective space $\mathbb{P}^n$ is defined as the space of complex lines through the origin of $\mathbb{C}^{n+1}$, i.e. $\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\sim$, where we quotient by the identification $(z^1, \ldots, z^{n+1}) \sim t(z^1, \ldots, z^{n+1})$, for t any non-zero complex number. The coordinates $(z^1, \ldots, z^{n+1})$ are called* homogeneous coordinates *on $\mathbb{P}^n$. More precisely, we can show that it is a complex manifold by defining a set of coordinate charts with holomorphic transition functions. Let $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ be the projection. Since the z^i are not all zeros, we may define an open cover as follows. Let $\tilde{U}_a \subset \mathbb{C}^{n+1}$ be the open set with $z^a \neq 0$ and let $U_a = \pi(\tilde{U}_a)$. Also, we define the maps $h_a : U_a \to \mathbb{C}^n$ as*

$$h_a(z^1, \ldots, z^{n+1}) = \left(\frac{z^1}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^{n+1}}{z^i}\right) . \tag{E.4}$$

The sets U_a cover $\mathbb{P}^n$, the maps h_a are homeomorphisms and their transition functions (assuming $a < b$)

$$\begin{aligned} g_{ab} :=& h_a \circ h_b^{-1} = h_b(U_b) \to h_a(U_a) , \\ h_a \circ h_b^{-1}(w^1, \ldots, w^n) =& \left(\frac{w^1}{w^a}, \ldots, \frac{w^{a-1}}{w^a}, \frac{w^{a+1}}{w^a}, \ldots, \underbrace{\frac{1}{w^a}}_{b\text{-th argument}}, \ldots, \frac{w^n}{w^a}\right) \end{aligned} \tag{E.5}$$

are biholomorphic. Hence, (U_a, h_a) *are an holomorphic atlas of* $\mathbb{P}^n$*. The map* π *restricted to the unit sphere in* $\mathbb{C}^{n+1}$ *is surjective, so that* $\mathbb{P}^n$ *is compact. On patch* U_a *the coordinates defined by the maps* h_a*, denoted by*[1] $(w^1, \dots, \hat{w}^b, \dots, w^{n+1})$ *with* $w^a = \frac{z^a}{z^b}$*, are called* inhomogeneous or local *coordinates.*

Submanifolds. Given a complex manifold X, a submanifold of X is a pair (Y, ι), where Y is a complex manifold and $\iota : \mathsf{Y} \to \mathsf{X}$ is an injective holomorphic map whose Jacobian matrix has rank equal to the dimension of Y at any point of Y. For instance, all the closed complex submanifolds of $\mathbb{C}^n$ are also compact. In fact, there is a theorem by Chow that states that all such submanifolds of $\mathbb{C}^n$ can be defined by the zero locus of a finite number of homogeneous polynomials.[2]

Equivalence of manifolds. Two complex manifolds X and Y are *biholomorphic* if there is a biholomorphic map $f : \mathsf{X} \to \mathsf{Y}$. Equivalently, given an open cover $\mathfrak{U}$ of X and Y, one requires that on each patch $U_a \in \mathfrak{U}$ there exist biholomorphic functions h_a such that the transition functions are related by: $g^{\mathsf{X}}_{ab} = h_a \circ g^{\mathsf{Y}}_{ab} \circ h_b$ on $U_a \cap U_b \neq \varnothing$. Two complex manifolds are *diffeomorphic* if their underlying smooth manifolds are diffeomorphic.

Example E.2 (Stein manifolds)**.** *A complex manifold that can be embedded as a closed submanifold into a complex Euclidean space is called a Stein manifold.*

In the next Section, we shall see how they play an important role in making Čech cohomology on a manifold independent of the covering.

Orientation. All the complex manifolds are oriented. For simplicity, we can look at the one dimensional case. The Jacobian of a transition function $f(z) = u(x, y) + iv(x, y)$ reads

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} , \tag{E.6}$$

so that $\det J = u_x^2 + u_y^2 > 0$ and the manifold is oriented. Notice that we may always conjugate the complex structure, so that the orientation gets reversed.

[1]As usual, the hat symbol here indicates an omission.

[2]In general, a zero locus of a set of polynomials is not a manifold (due to singularities), but an algebraic variety.

Holomorphic functions. Let U be an open subset of a complex manifold X and $f : U \to \mathbb{C}$ a function. Then, f is *holomorphic* on U if, taken (U_a, h_a) such that $U \cap U_a \neq \varnothing$, the function $f \circ h_a^{-1} : h_a(U_a \cap U) \to \mathbb{C}$ is holomorphic. This definition does not depend on the choice of the coordinates. For instance, one may easily check that the projection $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ is holomorphic. Moreover, the following striking result holds: if X is a (connected) compact complex manifold and f : X $\to \mathbb{C}$ a holomorphic function, then f is constant. This is a generalisation of the maximum principle of elementary complex analysis, see e.g. [327]. The sheaf of holomorphic functions on X shall be denoted by $\mathcal{O}_{\mathsf{X}}$.

Definition E.3 (Hypersurface)**.** *Let* X *be a complex manifold. An analytic subvariety* Y *of* X *is a subset of* X *which is locally defined as the zero set of a finite collection of holomorphic functions. If* $\dim \mathsf{Y} = \dim \mathsf{X} - 1$, Y *will be called an analytic hypersurface.*

Complex structures. Let V be a real vector space. A *complex structure* on V is a $\mathbb{R}$-linear map $J : V \to V$, such that $J^2 = -\mathbb{1}_V$. A real vector space V with a complex structure J can be given the structure of a complex vector space by defining the product of a vector $v \in V$ by a complex number $z = x + iy \in \mathbb{C}$ as[1] $z \cdot v = x \cdot v + y \cdot J(v)$. A real space V with a complex structure has necessarily an even dimension, $\dim_{\mathbb{R}} V = 2n$. On the other hand, each complex vector space has a complex structure given by $J(v) = iv$.

Example E.4 (Canonical complex structure)**.** *The obvious identification of* $\mathbb{C}^n$ *with* $\mathbb{R}^{2n}$ *is obtained by equating* $z^i = x^i + iy^i$, *which induces the canonical complex structure*

$$J(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-y_1, \ldots, -y_n, x_1, \ldots, x_n)$$
$$\text{and thus} \quad J = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} . \qquad \text{(E.7)}$$

Almost complex structures. Let X be a smooth $2n$-dimensional manifold. At each point $x \in$ X the tangent space T_xX is a real $2n$-dimensional vector space. If we can endow each tangent space T_xX with a complex structure J_x so that this structure varies smoothly over the manifold, then we have an almost complex structure on X.

Definition E.5 (Almost complex structure)**.** *Let* X *be a differentiable manifold of dimension* $2n$, *and suppose* J *is a differentiable vector bundle isomorphism* $J : T\mathsf{X} \to T\mathsf{X}$ *such that*

[1] Notice that for any nonzero vector $v \in V$, the vectors v and $J(v)$ are linearly independent over $\mathbb{R}$.

$J_x : T_x\mathsf{X} \to T_x\mathsf{X}$ is a complex structure for T_xM. Then J is called an almost complex structure for the differentiable manifold X*. A manifold with a fixed almost complex structure is called an almost complex manifold.*

If X is a complex manifold then it carries a canonical almost complex structure. This is defined by transferring the almost-complex structure on $\mathbb{C}^n$ to the manifold via holomorphic charts. With respect to the coordinate system $(x^1, \dots, x^n, y^1, \dots, y^n)$ an almost complex structure on $\mathbb{C}^n$ is defined by

$$J\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial}{\partial y^i} \ , \quad J\Big(\frac{\partial}{\partial y^i}\Big) = -\frac{\partial}{\partial x^i} \ . \tag{E.8}$$

Notice that a complex manifold is naturally an almost complex manifold, but an almost-complex manifold need not be a complex manifold. The most celebrated example of an almost complex manifold which is not a complex manifold is S^6, see e.g. [328].

A mapping $f : \mathsf{X} \to \mathsf{Y}$ between almost complex manifolds is said to be holomorphic if its differential $f_* : T\mathsf{X} \to T\mathsf{Y}$ commutes with the respective almost complex structures, i.e.

$$f_* \circ J^{\mathsf{X}} = J^{\mathsf{Y}} \circ f_* \ . \tag{E.9}$$

Integrable almost complex structures. To determine whether or not a given almost-complex structure is induced by a complex structure we introduce the following object. The *Nijenhuis tensor* N of an almost-complex structure J is a tensor field of type $(1,2)$ defined by

$$N(X,Y) = \frac{1}{4}([X,Y] + J[X,JY] + J[JX,Y] - [JX,JY]) \ , \tag{E.10}$$

for any two vector fields X and Y.

Definition E.6. *An almost complex structure J is said to be* integrable *if $N \equiv 0$.*

In case an almost complex manifold is real-analytic it is not hard to show that an integrable almost-complex structure is complex as an application of the Frobenius theorem, see e.g. [329, 330]. Without the real-analyticity assumption, however, this result is highly nontrivial and is due to the following:

Theorem E.7 (Newlander–Nirenberg [331])**.** *An almost complex structure is complex if and only if it is integrable.*

Complexification. Given a real vector space V with a real scalar multiplication, we define its *complexification* as the tensor product $V^{\mathsf{c}} = V \otimes_{\mathbb{R}} \mathbb{C}$.

The complex tangent space. Let X be a complex manifold of complex dimension n. Upon using the identification[1] $\mathbb{C}^n \cong \mathbb{R}^{2n}$ one can show that X has an underlying structure of $2n$-dimensional real manifold. Let TX be the smooth tangent bundle, that is the collection of all ordinary tangent spaces to X. The complexified tangent space $T_x\mathsf{X} \otimes_{\mathbb{R}} \mathbb{C}$ admits the intrinsic basis

$$\left\{ \Big(\frac{\partial}{\partial z^1}\Big)_x, \dots, \Big(\frac{\partial}{\partial z^n}\Big)_x, \Big(\frac{\partial}{\partial \bar{z}^1}\Big)_x, \cdots \Big(\frac{\partial}{\partial \bar{z}^n}\Big)_x \right\} , \tag{E.11}$$

in terms of a set of local complex coordinates $(z^1, \dots, z^n)$ around a point $x \in \mathsf{X}$. Hence, the transition functions are holomorphic and do not mix the vectors $\frac{\partial}{\partial z^i}$ with the $\frac{\partial}{\partial \bar{z}^i}$. This yields the decomposition[2]

$$T\mathsf{X}^c := T\mathsf{X} \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}\mathsf{X} \oplus T^{0,1}\mathsf{X} \ . \tag{E.12}$$

Sections of $T^{1,0}$X and $T^{0,1}$X are called vector fields of type $(1, 0)$ and $(0, 1)$, respectively. Vector fields of type $(1, 0)$, whose action on arbitrary functions will be holomorphic, are called *holomorphic vector fields*, while *antiholomorphic vector fields* are defined analogously.

Complex differential forms. Using the decomposition of the complexified cotangent bundle, it is easy to show that the space of complex differential forms on a complex manifold admits the decomposition

$$\Omega^k(\mathsf{X})^c := \Gamma(\mathsf{X}, \Lambda^k T^*\mathsf{X} \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(\mathsf{X}) \ . \tag{E.13}$$

Moreover, there is a canonical projection of vector bundles $\pi^{p,q} : \Omega^k(\mathsf{X})^c \to \Omega^{p,q}$, for each p, q such that $p + q = k$. Elements in $\Omega^{p,q}(\mathsf{X})$ denote the space of forms of bi-degree (p, q) and are called *differential forms of type* (p, q). In particular, elements of $\Omega^{1,0}(\mathsf{X})$ and $\Omega^{1,0}(\mathsf{X})$ are dual to elements of $T^{1,0}$X and $T^{0,1}$X, respectively and local bases are dual to the ones given in (E.11) and are then given by $(\mathrm{d}z^1, \dots, \mathrm{d}z^n)$ and $(\mathrm{d}\bar{z}^1, \dots, \mathrm{d}\bar{z}^n)$.

[1] Recall that a biholomorphic map is a smooth diffeomorphism.

[2] The complex structure on X naturally extends to a map $J_x : T_x\mathsf{X} \otimes \mathbb{C} \to T_x\mathsf{X} \otimes \mathbb{C}$. Since $J^2 = -1$, the spaces $T^{1,0}$X and $T^{0,1}$X denote the eigenspaces of J_x with eigenvalue $\pm i$, respectively. Since this works at any point $x \in \mathsf{X}$ we have (E.12).

Exterior derivative. Locally, a (p,q)-form be written as

$$\omega = \frac{1}{p!q!}\omega_{i_1,\dots,i_p,j_1,\dots,j_q}(z,\bar{z})\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_p}\wedge\mathrm{d}\bar{z}^{j_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{j_q}\ . \tag{E.14}$$

The compositions[1]

$$\begin{array}{ccccc} & & & & \Omega^{p+1,q}(\mathsf{X}) \\ & & & \nearrow^{\pi^{p+1,q}} & \\ \Omega^{p,q}(\mathsf{X}) & \xrightarrow{\mathrm{d}} & \bigwedge^{p+q+1} T^*\mathsf{X} & & \\ & & & \searrow_{\pi^{p,q+1}} & \\ & & & & \Omega^{p,q+1}(\mathsf{X}) \end{array} \tag{E.15}$$

define the differential operators $\partial = \pi^{p+1,q}\circ\mathrm{d} : \Omega^{p,q}(\mathsf{X}) \to \Omega^{p+1,q}(\mathsf{X})$ and $\bar{\partial} = \pi^{p,q+1}\circ\mathrm{d} : \Omega^{p,q}(\mathsf{X}) \to \Omega^{p,q+1}(\mathsf{X})$. Locally, the exterior derivative maps a (p,q)-form to a form which is the sum of an $(p+1,q)$-form and an $(p,q+1)$-form by

$$\omega = \frac{1}{p!q!}\left(\partial_k\omega_{i_1,\dots,j_q}(z,\bar{z})\mathrm{d}z^k\wedge\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{j_q} + \bar{\partial}_k\mathrm{d}z^{i_1}\omega_{i_1,\dots,j_q}(z,\bar{z})\wedge\cdots\wedge\mathrm{d}\bar{z}^k\wedge\cdots\wedge\mathrm{d}\bar{z}^{j_q}\right)\ . \tag{E.16}$$

The identity $\mathrm{d}^2 = 0$ implies

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0\ , \tag{E.17}$$

giving $\Omega^{p,q}(\mathsf{X})$ the structure of a (bi-)graded differential algebra, recall the discussion in Section 2.2.. In particular, the operator $\bar{\partial}$ is called the *Dolbeault operator*. A holomorphic p-form on X is given by an $\omega \in \Omega^{p,0}(\mathsf{X}) =: \Omega^p_{\mathsf{X}}$ satisfying $\bar{\partial}\omega = 0$ and holomorphic 0-forms are holomorphic functions. Since $\bar{\partial}^2 = 0$ we can construct the Dolbeault cohomology groups, see later.

Homotopy lifting property. Let E, X, Y be topological spaces. A map $\pi : E \to X$ is said to have the *homotopy lifting property* with respect to the X if, given the commutative diagram[2]

$$\begin{array}{ccc} Y\times\{0\} & \xrightarrow{h} & E \\ \big\downarrow p & & \big\downarrow \pi \\ Y\times[0,1] & \xrightarrow{h_t} & X \end{array} \tag{E.18}$$

[1] The same pictorial representation can be found in [306].

[2] Here, we assume that all the maps are continuous.

there exists a map $\gamma : Y \times [0,1] \to E$ such that $\gamma(x,0) = h(x)$ and $\pi \circ \gamma(x,t) = h_t(x)$, i.e. γ gives rise to two commutative triangles.

Fibration. A *fibration* is a continuous map $E \to X$ of topological spaces which satisfies the homotopy lifting property for all topological spaces Y. We just remind that every vector bundle is a fibration see e.g. [332].

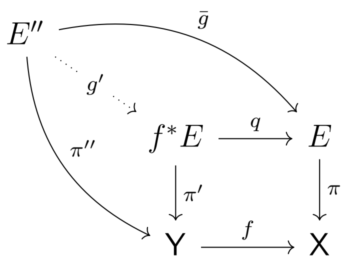
Complex vector bundles. A complex vector bundle E over a complex manifold X is a vector bundle $\pi : E \to$ X such that for each $x \in$ X, the fibre $\pi^{-1}(x)$ is a complex vector space. Every vector bundle is furthermore a fibration, see e.g. [332].

Let E and E' be two vector bundles over X. A morphism (bundle map) $\phi : E \to E'$ is a mapping such that ϕ restricted to the fibre E_x is a linear mapping to the fibre E'_x, for all $x \in$ X. We call ϕ a *monomorphism* if it is one-to-one on the fibres, an *epimorphism* if it is surjective on the fibres and an *isomorphism* if it is one-to-one and surjective on the fibres. The vector bundle E is a subbundle of E' if E is a submanifold of E' and E_x is a linear subspace of E'_x, for all $x \in X$. Another important concept is the following.

Let $\pi : E \to$ X be a complex vector bundle and $f :$ Y $\to$ X be a smooth mapping, then the *pull-back bundle* f^*E is a complex vector bundle over Y of the same rank as E such that

$$\begin{array}{ccc} f^*E & \xrightarrow{q} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathsf{Y} & \xrightarrow{f} & \mathsf{X} \end{array} \tag{E.19}$$

is commutative. Moreover, the pull-back must be universal with respect to this diagram. Explicitly, given any other bundle $E'' \to$ Y and a continuous map $\bar{g} : E'' \to E$ such that $f \circ \pi'' = \pi \circ \bar{g}$, there must exist a unique map $g' : E'' \to f^*E$ such that $\pi' \circ g' = \pi''$ and $\bar{g} = q \circ g'$



(E.20)

Hence, the total space is defined to be the set

$$f^*E = \{(y,v) \in \mathsf{Y} \times E | f(y) = \pi(v)\} \subseteq \mathsf{Y} \times E \ , \tag{E.21}$$

with projection induced by projection on the first factor. Moreover, if $\{U_a\}$ is a covering of X and g_{ab} are the transition functions of E then $\{f^{-1}(U_i)\}$ defines a covering of Y such that f^*E is locally trivial. The transition functions f^*g_{ab} of the pull-back bundle f^*E are then given by $f^*g_{ab} = g_{ab} \circ f$.

Let E_1, E_2, E_3 be three complex vector bundles over X. The sequence

$$E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3 \tag{E.22}$$

is exact at E_2, if $\ker\psi = \mathrm{im}\phi$. A short exact sequence is a sequence of the form

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0\ , \tag{E.23}$$

which is exact at E_1, E_2 and E_3. We say that the sequence splits if $E_2 = E_1 \oplus E_3$. Hence, one can understand E_2 in (E.23) as a deformation of the direct sum $E_1 \oplus E_3$.

Example E.8 (Euler sequence.). *For* X $= \mathbb{P}^n$

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(1) \otimes \mathbb{C}^{n+1} \longrightarrow T\mathbb{P}^n \longrightarrow 0 \tag{E.24}$$

is a short exact sequence called the Euler sequence.

Holomorphic vector bundles. Holomorphic vector bundles on a complex manifold X are defined in the same way as complex vector bundles, but requiring that all the maps involved are holomorphic. Explicitly, let X be a complex manifold of dimension n. A $(n+r)$-dimensional complex manifold E together with a holomorphic map $\pi : E \to$ X is a *rank-r holomorphic vector bundle* over X if there is an open covering $\{U_a\}$ of X and holomorphic maps $h_a : \pi^{-1}(U_a) \to U_a \times \mathbb{C}^r$ such that

(i) $\pi = \mathrm{pr}_1 \circ h_a$, where pr_1 is the projection onto the first factor of $U_a \times \mathbb{C}^r$ and

(ii) the map
$$\mathrm{pr}_2 \circ h_b \circ h_a^{-1}(p, \cdot) : \mathbb{C}^r \to \mathbb{C}^r \tag{E.25}$$
is a linear isomorphism, for all $x \in U_a \cap U_b$.

In Section E.1.3. we shall see the description in terms of transition functions.

A morphism between two vector bundles E, E' over X is a holomorphic map $\phi : E \to E'$ such that for every $x \in$ X one has $\phi(E_x) \subset E'_x$ and such that the resulting map $\phi_x : E_x \to E'_x$ is linear. If ϕ is a biholomorphism, it is said to be an isomorphism of vector bundles, and E and E' are said to be isomorphic.

Holomorphic line bundles. If the fibres of a holomorphic vector bundle is $\mathbb{C}$, i.e. it is of rank 1, then it is called a *holomorphic line bundle*.

Example E.9 (Tautological bundle)**.** *Let us consider the special case where* X *is* $\mathbb{P}^n$*. Recall that any point* $p \in \mathbb{P}^n$ *corresponds to a line in* $\mathbb{C}^{n+1}$*. The tautological line bundle is obtained by associating that line to each point on* $\mathbb{P}^n$*.* $\mathbb{P}^n$ *is covered by* $(n+1)$ *patches. Let* $\{U_a\}$ *be the standard cover of* $\mathbb{P}^n$*. We parametrise* $p \in U_a$ *with homogeneous coordinates* $(\lambda_1, \ldots, \lambda_{n+1})$*. Then, we define* $h_a : \pi^{-1}(U_a) \to U_a \times \mathbb{C}$ *as* $h_a(U) = (p, \lambda_a)$*, if* $p = \pi(u)$*. The tautological line bundle is then parametrised by transition functions* $g_{ab} = \lambda_a/\lambda_b$ *and we denote it by* $\mathcal{O}_{\mathbb{P}^n}(-1)$*.*

The dual of the tautological bundle, which we denote by $\mathcal{O}_{\mathbb{P}^n}(1)$, is called the *hyperplane line bundle*. Since the tensor product of two holomorphic line bundles is always a holomorphic line bundle, one can construct holomorphic line bundles $\mathcal{O}_{\mathbb{P}^n}(k)$ of $\mathbb{P}^n$, for any $k \in \mathbb{Z}$. Given coordinate functions λ_a, $a = 1, \ldots, n$ on $\mathbb{P}^n$ the line bundle $\mathcal{O}(k)$ is defined by transition functions $g_{ab} = (\lambda_a/\lambda_b)^k$. Notice that for $k \geq 0$ the vector space of holomorphic sections of $\mathcal{O}(k)$ is canonically identified with the set of homogeneous polynomials of degree k in $\mathbb{P}^n$. Hence, homogeneous coordinates of $\mathbb{P}^n$ are sections of the hyperplane line bundle. For instance, let Y be a smooth hypersurface in $\mathbb{P}^n$, defined as a zero locus of a degree k polynomial. Such a polynomial can be seen as a section of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^n}(k)$.

The *canonical bundle* K_{X} on a complex manifold X of complex dimension n is the highest exterior power of the holomorphic cotangent bundle to X, i.e. $K_{\mathsf{X}} = \bigwedge^{n,0} T^*\mathsf{X}$. Its sections are holomorphic volume forms on X, i.e $\mathcal{O}(K_{\mathsf{X}}) \cong \Omega^n_{\mathsf{X}}$. The dual of K_{X}, denoted as K^*_{X} is the *anti-canonical line bundle*. Moreover, it can be shown that every holomorphic line bundle over projective space is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-k)$, for some $k \in \mathbb{Z}$. In particular, the canonical bundle over $\mathbb{P}^n$ is isomorphic to the $(n+1)$-th exterior power of the tautological line bundle, i.e. $\mathcal{O}_{\mathbb{P}^n}(-n-1)$.

Also, given any holomorphic vector bundle E of rank r one can form the exterior product bundle as $\Lambda^k E$, for $k = 0, \ldots, r$. When $k = r$, the holomorphic line bundle $\Lambda^r E$ is given by the special symbol $\det E$ and is called the *determinant line bundle*, since its transition functions are given by the determinants of the transition function of E. If E is the cotangent bundle $T^*\mathsf{X}$ it reduces to the canonical bundle.

E.1.2. Dolbeault cohomology

Let us consider now the Dolbeault cohomology associated to a complex manifold X. We first need an analytic result, analogous to the Poincaré lemma, sometimes called $\bar{\partial}$-Poincaré lemma or Dolbeault lemma:[1]

Lemma E.10. *Let D be an open polydisk in $\mathbb{C}^n$ (that is, the cartesian product of disks in $\mathbb{C}$). Then*

$$H^{p,q}_{\bar{\partial}}(D) = 0 \ , \quad \text{for} \quad q \geq 1 \ . \tag{E.26}$$

See e.g. [294].

The kernel of the morphism $\bar{\partial} : \Omega^{p,0}(\mathsf{X}) \to \Omega^{p,1}(\mathsf{X})$ is the sheaf of holomorphic p-forms $\Omega^p(\mathsf{X})$[2]. By the $\bar{\partial}$-Poincaré lemma, the complex $\Omega^{p,q}(\mathsf{X})$ gives a fine resolution of the sheaf of holomorphic p-forms on X. Namely, the sheaf sequence

$$0 \longrightarrow \Omega^p_{\mathsf{X}} \xrightarrow{\bar{\partial}} \Omega^{p,1}(\mathsf{X}) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{p,n}(\mathsf{X}) \longrightarrow 0 \ , \tag{E.27}$$

is exact, for all $p = 0, \dots, n = \dim_{\mathbb{C}}$. The *Dolbeault cohomology* $H^{p,q}_{\bar{\partial}}(\mathsf{X})$ of X of type (p, q) is the q-th cohomology of the complex $(\Omega^{p,\bullet}(\mathsf{X}), \bar{\partial})$. The following result holds (see e.g. [294])

Proposition E.11 (Dolbeault theorem)**.** *Let X be a complex manifold. For all $p, q \geq 0$ the cohomology groups $H^{p,q}_{\bar{\partial}}(\mathsf{X})$ and $H^q(\mathsf{X}, \Omega^p)$ are isomorphic.*

Using the argument in the proof of the Dolbeault Theorem, it is possible to establish a version of the Leray theorem for the sheaf of germs of holomorphic p-forms.

Proposition E.12. *Let $\mathfrak{U}$ be an acyclic cover for the structure sheaf Ω^p. Then $H^q(\mathfrak{U}, \Omega^p) \cong H^p(\mathsf{X}, \Omega^p)$.*

One can also prove the isomorphism (see e.g. [294])

$$H^{p,q}_{\bar{\partial}}(\mathsf{X}, E) \cong H^q(\mathsf{X}, \Omega^p_{\mathsf{X}} \otimes \mathcal{E}) \ , \tag{E.28}$$

[1] There exists a stronger version of the lemma E.10 that shows that the higher Dolbeault cohomology groups vanish on arbitrary finite products of polydisks with punctured polydisks (i.e. polydisks minus the origin).

[2] Notice that $\Omega^p(\mathsf{X})$ is a subsheaf of $\bigwedge^{p,0}(\mathsf{X})$.

called *Dolbeault isomorphism*, where E denotes a holomorphic vector bundle over X, while $\mathcal{E}$ is the sheaf of its holomorphic sections.[1] From the pairing $H^{p,q}_{\bar\partial}(\mathsf{X}) \otimes H^{n-p,n-q}_{\bar\partial}(\mathsf{X}) \to \mathbb{C}$, on a n-dimensional compact complex manifold one obtains the isomorphism [294, 321]

$$H^p(\mathsf{X}, \Omega^q \otimes \mathcal{E})^* \cong H^{n-p}(\mathsf{X}, \Omega^{n-q} \otimes \mathcal{E}^*) \tag{E.29}$$

This is *Kodaira-Serre duality*. In particular, for $q = 0$ we get

$$H^p(\mathsf{X}, \mathcal{E})^* \cong H^{n-p}(\mathsf{X}, K_{\mathsf{X}} \otimes \mathcal{E}^*) \ , \tag{E.30}$$

which is usually called *Serre duality*.

E.1.3. Čech-Dolbeault correspondence

Here, we briefly recall how (holomorphic) principal bundles over a complex manifold can be equivalently described in terms of Čech cohomology and in the dual Dolbeault picture. In fact, the twistor construction requires switching between one another. Moreover, such a correspondence generalises to higher principal bundles. We remind the reader of the discussion about Čech cohomology in Appendix D. In the following we shall closely follow [237, 238, 324, 325]

Čech approach. To this end, let X be a complex manifold and let $\mathfrak{U} := \{U_a\}_{a\in A}$ be an open covering of X. By $\mathfrak{G}$ we denote a sheaf of smooth maps from open subsets of X into G. A q-cochain of the covering $\mathfrak{U}$ with values in $\mathfrak{G}$ is a collection $\psi = \{\psi_{a_0\cdots a_q}\}$ of sections of the sheaf $\mathfrak{G}$ over non-empty intersections $U_{a_0} \cap \cdots \cap U_{a_q}$. We will denote the set of such q-cochains by $C^q(\mathfrak{U}, \mathfrak{G})$.[2]

Furthermore, we define the subsets of Čech cocycles $Z^q(\mathfrak{U}, \mathfrak{G}) \subset C^q(\mathfrak{U}, \mathfrak{G})$. For example, for $q = 0, 1$ they are given by

$$\begin{aligned} Z^0(\mathfrak{U}, \mathfrak{G}) &:= \{\psi \in C^0(\mathfrak{U}, \mathfrak{G}) \mid \psi_a = \psi_b \ \text{ on } \ U_a \cap U_b \neq \varnothing\} \ , \\ Z^1(\mathfrak{U}, \mathfrak{G}) &:= \{\chi \in C^1(\mathfrak{U}, \mathfrak{G}) \mid \chi_{ab} = \chi_{ba}^{-1} \ \text{ on } \ U_a \cap U_b \neq \varnothing \\ &\qquad\qquad \text{and} \ \ \chi_{ab}\chi_{bc}\chi_{ca} = 1 \ \text{ on } \ U_a \cap U_b \cap U_c \neq \varnothing\} \ . \end{aligned} \tag{E.31}$$

[1] Recall that sometimes this distinctions will not be made.

[2] We stress that it has a group structure, where the multiplication is just pointwise multiplication.

These sets will be of particular interest. We remark that from the first of these two definitions it follows that $Z^0(\mathfrak{U},\mathfrak{G})$ coincides with the group

$$H^0(\mathsf{X},\mathfrak{G}) = \mathfrak{G}(\mathsf{X}) \ , \tag{E.32}$$

which is the group of global sections of the sheaf $\mathfrak{G}$. Hence, the zeroth Čech cohomology set is independent of the covering. Note that in general the subset $Z^1(\mathfrak{U},\mathfrak{G}) \subset C^1(\mathfrak{U},\mathfrak{G})$ is not a subgroup of the group $C^1(\mathfrak{U},\mathfrak{G})$.

We say that two 1-cocycles $\chi, \chi' \in Z^1(\mathfrak{U},\mathfrak{G})$ are equivalent if $\chi'_{ab} = \psi_a^{-1}\chi_{ab}\psi_b$, for some $\psi \in C^0(\mathfrak{U},\mathfrak{G})$, on all $U_a \cap U_b \neq \varnothing$. The set of equivalence classes induced by this equivalence relation is the first Čech cohomology set, that is denoted by $H^1(\mathfrak{U},\mathfrak{G}) \cong Z^1(\mathfrak{U},\mathfrak{G})/C^0(\mathfrak{U},\mathfrak{G})$. If the patches U_a are Stein manifolds, one can show that the first Čech cohomology sets are independent of the covering and depend only on the manifold

$$H^1(\mathfrak{U},\mathfrak{G}) \ \cong \ H^1(\mathsf{X},\mathfrak{G}) \ . \tag{E.33}$$

Recall (see (D.14)) that if that is not the case one needs to take the inductive limit.

In particular, we shall be interested in holomorphic maps from open subsets of X into $\mathsf{GL}(r,\mathbb{C})$ as well as in the sheaf $\mathsf{GL}(r,\mathcal{O}_\mathsf{X})$ of such matrix-valued functions.[1] Elements of $H^1(\mathsf{X},\mathsf{GL}(r,\mathcal{O}_\mathsf{X}))$ classify rank-r locally free sheaves of $\mathcal{O}_\mathsf{X}$ modules up to isomorphisms. Hence, rank-r holomorphic vector bundles over some complex manifold X are parametrised by $H^1(X,\mathsf{GL}(r,\mathcal{O}_\mathsf{X}))$. To make contact with our notation we shall denote elements of $C^0(\mathfrak{U},\mathsf{GL}(r,\mathcal{O}_\mathsf{X}))$ by $h = \{h_a\}$ and elements of $C^1(\mathfrak{U},\mathsf{GL}(r,\mathcal{O}_\mathsf{X}))$ by $g = \{g_{ab}\}$.

Dolbeault approach. Another approach to holomorphic vector bundles makes use of Dolbeault cohomology. Here, we follow [69]. Let us consider a rank-r complex vector bundle $E \to \mathsf{X}$ over a complex manifold X. Furthermore, we let $\Omega^{p,q}(\mathsf{X})$ be the smooth differential (p,q)-forms on X and $\bar\partial : \Omega^{p,q}(\mathsf{X}) \to \Omega^{p,q+1}(\mathsf{X})$ be the anti-holomorphic exterior derivative. First, recall that a connection ∇ is a $\mathbb{C}$-linear sheaf morphism

$$\nabla : \mathcal{E} \to \Omega^1_\mathsf{X} \otimes \mathcal{E} \ , \tag{E.34}$$

where $\mathcal{E}$ is the sheaf of sections of E and satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + \mathrm{d}f \otimes s \ , \tag{E.35}$$

[1]Notice that $\mathsf{GL}(r,\mathcal{O}_\mathsf{X})$ is a non-Abelian sheaf contrary to the Abelian sheaves considered so far.

for every section s of E and every functions f on X. Let $\nabla^{0,1} : \mathcal{E} \to \Omega^{0,1}(\mathsf{X}) \otimes \mathcal{E}$ be a $(0,1)$-connection on E. Locally, it is of the form

$$\nabla^{0,1} = \bar{\partial} + A^{0,1} \ , \tag{E.36}$$

where the connection $(0,1)$-form $A^{0,1}$ takes values in $\operatorname{End} E$. The complex vector bundle E is said to be holomorphic if the $(0,1)$-connection is flat, that is, if the corresponding curvature vanishes,

$$F^{0,2} \ = \ (\nabla^{0,1})^2 \ = \ \bar{\partial}A^{0,1} + A^{0,1} \wedge A^{0,1} \ = \ 0 \ . \tag{E.37}$$

The reader may recognise the equation of motion of holomorphic Chern-Simons theory. We say that $\nabla^{0,1}$ defines a holomorphic structure on E. We denote by $\mathfrak{Z}$ the sheaf of solutions to (E.37). The group $H^0(X, \mathsf{GL}(r, \mathcal{S}_\mathsf{X}))$, where $\mathcal{S}_\mathsf{X}$ is the sheaf of smooth functions on X, acts on $\mathfrak{Z}(\mathsf{X})$ by gauge transformations,

$$A^{0,1} \ \mapsto \ g^{-1}A^{0,1}g + g^{-1}\bar{\partial}g \ , \quad \text{with} \quad g \ \in \ H^0(X, \mathsf{GL}(r, \mathcal{S}_\mathsf{X})) \ . \tag{E.38}$$

These transformations preserve the holomorphic structure on E, as they leave (E.37) invariant. Hence the Dolbeault cohomology set

$$H^1_{\nabla^{0,1}}(\mathsf{X}, E) := \mathfrak{Z}(\mathsf{X})/\mathsf{GL}(r, \mathcal{S}_\mathsf{X})(\mathsf{X}) \tag{E.39}$$

parametrises all different holomorphic structures on the complex vector bundle E.

Equivalence of the Čech and Dolbeault descriptions. It can be shown that these two approaches to holomorphic vector bundles are equivalent.[1]

Theorem E.13. *Let* X *be a complex manifold with an open Stein covering* $\mathfrak{U} = \{U_a\}$ *and* $E \to \mathsf{X}$ *be a rank-*r *complex vector bundle over* X*. Then there is a map* $\rho : H^1(\mathsf{X}, \mathsf{GL}(r, \mathcal{O}_\mathsf{X})) \to H^1(\mathsf{X}, \mathsf{GL}(r, \mathcal{S}_\mathsf{X}))$ *of cohomology sets, such that* $H^1_{\nabla^{0,1}}(\mathsf{X}, E) \cong \ker \rho$*.*

For a proof see e.g. [238, 325]. Therefore, the moduli spaces of both descriptions are bijective and we have the equivalence

$$(E, g = \{g_{ab}\}, \nabla^{0,1}) \ \sim \ (\tilde{E}, \tilde{f} = \{\tilde{g_{ab}}\}, \bar{\partial}) \ , \tag{E.40}$$

with $\tilde{g}_{ab} = h_a^{-1} g_{ab} h_b$ for some $h = \{h_a\} \in C^0(\mathfrak{U}, \mathsf{GL}(r, \mathcal{S}_\mathsf{X}))$. Given holomorphic vector bundle $E \to \mathsf{X}$ in the Dolbeault picture, we can always find a holomorphic vector bundle $\tilde{E} \to \mathsf{X}$

[1] This may be understood as a non-Abelian generalisation of Dolbeault's theorem, see e.g. [294, 321].

in the Čech picture and vice versa, such that E and $\tilde{E}$ are equivalent as complex vector bundles. This is called *Čech–Dolbeault correspondence*. See [237, 238, 325, 333] for detailed explanations.

The following observation is at the heart of the Penrose–Ward transform [69]. Given the triple $(E, g = \{g_{ab}\}, \nabla^{0,1})$, then any solution $A^{0,1}$ of the equation $F^{0,2} = 0$ is of the form

$$A^{0,1}|_{U_a} = A_a = h_a \bar{\partial} h_a^{-1} \qquad \text{(E.41)}$$

for some $h = \{h_a\} \in C^0(\mathfrak{U}, \mathsf{GL}(r, \mathcal{S}_\mathsf{X}))$ with

$$A_b \;=\; g_{ab}^{-1} \bar{\partial} g_{ab} + g_{ab}^{-1} A_a f_{ab} \;, \qquad \text{(E.42)}$$

as patching conditions. Plugging (E.41) into (E.42) we find

$$\tilde{g}_{ab} := h_a \, \bar{\partial} h_b^{-1} \quad \text{with} \quad \bar{\partial} \tilde{g}_{ab} = 0 \;. \qquad \text{(E.43)}$$

Conversely, starting from $\tilde{g}_{ab} = h_a^{-1} g_{ab} h_b$ with $\bar{\partial}\tilde{g}_{ab} = 0$ and $\bar{\partial} g_{ab} \neq 0$, one can recover an $A^{0,1}|_{U_a} = A_a = h_a \bar{\partial} h_a^{-1}$ which obeys the patching conditions.

In the Penrose–Ward transform one relates the transition function of a holomorphic vector bundle on twistor space to those of the pull-back on the correspondence space. By the definition of the pull-back, the transition function is annihilated by the vector fields that span the tangent spaces of the leaves of π_1 (see later). Then, these transition functions are related to a Lie algebra-valued (differential) one-form with components only along π_1. More precisely, this one-form is the connection one-form of the so-called relative connection along π_1 which by our very construction turned out to be flat. In this picture, the transition function is the Čech representative while the Lie algebra-valued one-form is the Dolbeault representative of a relatively flat bundle on the correspondence space. In this setting, it was also shown a version of Čech–Dolbeault correspondence for higher principal bundles. See e.g. [100–103]. In this case, one considers a correspondence for a relative exterior derivative along a fibration. However, higher cohomology groups with values in the sheaf of smooth differential $(0, q)$-forms $\Omega^{0,q}$, which are needed to verify the correspondence for relative differential forms, vanish trivially as this is a fine sheaf when working in the smooth category. Hence, the arguments for a $\bar{\partial}$ operator on a complex manifold are completely analogous.

E.2. Generalities of twistor correspondence

We first introduce some very useful mathematical notions.

Normal bundle. Let X be a manifold and $Y \subset X$ a submanifold. The normal bundle $\mathsf{N}_{\mathsf{Y|X}}$ of Y inside X is defined via the short exact sequence

$$0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow N_{Y|X} \longrightarrow 0\ . \tag{E.44}$$

Hence, $N_{Y|X} = \frac{TX|_Y}{TY}$. For complex manifolds, it is understood that one considers holomorphic tangent spaces $\mathsf{T}^{1,0}\mathsf{X}$ and $\mathsf{T}^{1,0}\mathsf{Y}$.

Relative de Rham complex. The classical de Rham sequence (see Section 2.2.) on a (smooth) manifold provides a connection between solutions of certain differential equations and the topology of the manifold. Similarly, the *relative de Rham sequence* for a certain mapping between manifolds establishes a connection between the topology of the fibres of the mapping and solutions of differential equations [334]. This turns out to be a very fundamental tool in relating objects in twistor space to space-time objects. Let Z be a smooth manifold with covering $\mathfrak{U} = \{U_a\}$ and let $U \subset Z$ be an open subset of Z. Furthermore, Ω^p_Z denotes the sheaf of smooth differential p-forms on Z and $\Omega^\bullet_Z := \bigoplus_{p\geq 0} \Omega^p_Z$. We consider a fibration $\pi_1 : Y \to Z$, for a smooth manifold Y . The sheaf of relative 1-forms $\Omega^1_{\pi_1}$, with respect to the fibration π_1, is defined by the exact sequence

$$0 \longrightarrow \pi_1^*\Omega^1_Z \longrightarrow \Omega^1_Y \xrightarrow{pr_{\pi_1}} \Omega^1_{\pi_1} \longrightarrow 0\ , \tag{E.45}$$

that is, $\Omega^1_{\pi_1}$ is the quotient sheaf and pr_{π_1} denotes the usual quotient map. This is dual to the sequence

$$0 \longrightarrow T_{\pi_1} \longrightarrow T_Y \longrightarrow \pi_1^*T_Z \longrightarrow 0\ , \tag{E.46}$$

defining the relative tangent bundle T_{π_1} on Y (sheaf of vertical vector fields). Hence, relative one-forms are dual to those vectors which are tangent to the fibres of π_1.

Remark E.14 (Twistor space P^6)**.** *Let us specialise the sequence* (E.46) *to* $\mathsf{X} = \mathbb{P}^7$*, with coordinates* (z^A, λ_A) *and* $\mathsf{Y} = \mathbb{P}^3$ *and consider the inclusion* $\mathbb{P}^3 \hookrightarrow \mathbb{P}^7$*, given by* $z^A\lambda_A = 0$ *and* $\lambda_A \neq 0$*. The normal bundle* $\mathsf{N}_{\mathbb{P}^3|\mathbb{P}^7}$ *of* $\mathbb{P}^3$ *inside* $\mathbb{P}^7$ *is given by [97]*

$$0 \longrightarrow \mathsf{T}\mathbb{P}^3 \longrightarrow i^*\mathsf{T}\mathbb{P}^7 \longrightarrow N_{\mathbb{P}^3|\mathbb{P}^7} \longrightarrow 0\ , \tag{E.47}$$

where $\mathsf{T}\mathbb{P}^3 = \langle\frac{\partial}{\partial\lambda_A}\rangle$ *and* $\mathsf{T}\mathbb{P}^7 = \langle\frac{\partial}{\partial z^A}, \frac{\partial}{\partial\lambda_A}\rangle$*. This implies that* $\mathsf{N}_{\mathbb{P}^3|\mathbb{P}^7} \cong \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \to \mathbb{P}^3$ *Hence, one may alternatively define twistor space* P^6 *via the exact sequence*

$$0 \longrightarrow \mathsf{P}^6 \longrightarrow i^*N_{\mathbb{P}^3|\mathbb{P}^7} \xrightarrow{\kappa} \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow 0\ , \tag{E.48}$$

where $\kappa : (z^A, \lambda_A) \mapsto z^A\lambda_A$*.*

The sequence (E.45) easily extends this to higher order forms and to vector-bundle-valued differential forms. In general, relative p-forms with respect to the fibration $\pi_1 : Y \to Z$ are defined by

$$0 \longrightarrow \pi_1^*\Omega^1_Z \wedge \Omega^{p-1}_Y \longrightarrow \Omega^p_Y \xrightarrow{pr_{\pi_1}} \Omega^p_{\pi_1} \longrightarrow 0 \ , \tag{E.49}$$

for $p = 1, \ldots, n := (\dim Y - \dim Z)$. The ordinary exterior derivative $\mathrm{d} : \Omega^0_{\mathsf{Y}} \to \Omega^1_Y$ on Y may be composed with the canonical projection $pr_{\pi_1} : \Omega^p_{\mathsf{Y}} \to \Omega^p_{\pi_1}$ to define the induced relative exterior derivative

$$d_{\pi_1} := pr_{\pi_1} \circ \mathrm{d} : \Omega^p_{\pi_1} \to \Omega^{p+1}_{\pi_1} \ , \tag{E.50}$$

whose kernel is the smooth functions on Y which are locally constant along the fibres of π_1. The relative differential d_{π_1} induces a complex, known as the *relative de Rham complex*. In fact, the sequence

$$0 \longrightarrow \pi_1^{-1}\mathcal{O}_Z \longrightarrow \mathcal{O}_Y \xrightarrow{\mathrm{d}_{\pi_1}} \Omega^1_{\pi_1} \xrightarrow{\mathrm{d}_{\pi_1}} \Omega^2_{\pi_1} \xrightarrow{\mathrm{d}_{\pi_1}} \ldots \xrightarrow{\mathrm{d}_{\pi_1}} \Omega^n_{\pi_1} \longrightarrow 0 \tag{E.51}$$

is an exact sequence of sheaves on Y. Here, $\pi_1^{-1}\mathcal{O}_Z$ denotes the topological inverse image of the sheaf $\mathcal{O}_Z$ on Y, namely the subsheaf of $\mathcal{O}_Y$ consisting of those functions locally constant on the fibres of π_1. Let E denote a vector bundle on Z. One may obtain a resolution of $\pi_1^{-1}\mathcal{O}_Z(E)$ by tensoring (E.51) with $\otimes_{\pi_1^{-1}\mathcal{O}_Z}\pi_1^{-1}\mathcal{O}_Z(E)$, where $\pi_1^{-1}\mathcal{O}_Z(E)$ is the sheaf of sections of π_1^*E that are constant along π_1. The following (twisted) sequence

$$0 \longrightarrow \pi_1^{-1}\mathcal{O}_Z(E) \longrightarrow \mathcal{O}_Y(\pi_1^*E) \xrightarrow{\mathrm{d}_{\pi_1}} \Omega^1_{\pi_1}(E) \xrightarrow{\mathrm{d}_{\pi_1}} \ldots \xrightarrow{\mathrm{d}_{\pi_1}} \Omega^n_{\pi_1}(E) \longrightarrow 0 \ , \tag{E.52}$$

is exact. Here and in the following $\Omega^p_X(E)$ denotes sheaves of germs of differential E-valued p-forms on X, with $\mathcal{O}_X(E) := \Omega^0_X(E)$ and we have defined $\Omega^p_{\pi_1}(E) := \Omega^p_{\pi_1} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(\pi_1^*E)$. We have also used that $\pi_1^*\mathcal{O}(E) = \mathcal{O}(\pi_1^*E)$, for the sheaf of germs of sections of a vector bundle E, where π_1^*E is the usual pull-back vector bundle.

Before going into the details of the twistor transform, let us recall a result about differential forms.

Proposition E.15. *Let $\pi : Y \to X$ be a surjective submersion with connected fibres and $T_\pi :=$ ker $(\mathrm{d}\pi)$ the tangent bundle relative to π. Then, the pull-back map $\pi^* : \Omega^\bullet(X) \to \Omega^\bullet(Y)$ is injective and identifies $\Omega^\bullet(X)$ as the graded subalgebra of basic forms*

$$\{\alpha \in \Omega^\bullet(Y) : \iota_V\alpha = \mathcal{L}_V\alpha = 0 \ , \quad \textit{for all} \ \ V \in \Gamma(Y, T_\pi)\} \subset \Omega(Y) \ . \tag{E.53}$$

Namely, a differential form in the image of π^* is a basic form, that is, a differential form on Y which is horizontal ($\iota_V \alpha = 0$) and invariant ($\mathcal{L}_V \alpha = 0$). This tells us when differential forms on Y descend to honest differential forms on X. We shall make use of this notion in gauge fixing the twistor action in Chapter 7.

Leray spectral sequence. Let $\pi : Y \to X$ be a continuous map of topological spaces and let $\mathcal{S}$ be a sheaf on Y. The q-th direct image sheaf is defined as the sheaf generated by the pre-sheaf $U \to H^q(\pi^{-1}(U), \mathcal{S})$, for $U \subset X$ open. If π is a proper surjective map there is a spectral sequence

$$E_2^{p,q} = H^p(X, \pi_*^q \mathcal{S}) \Longrightarrow H^{p+q}(Y, \mathcal{S}) \ , \tag{E.54}$$

called Leray spectral sequence. Moreover, if $\mathcal{S}$ is coherent, the sheaf $\pi_*^q \mathcal{S}$ is also coherent, by the Direct Image Theorem. Assuming X to be a Stein manifold one has the following

$$H^q(\mathsf{Y}, \mathcal{S}) \cong \Gamma(\mathsf{X}, \pi_*^q \mathcal{S}) \ . \tag{E.55}$$

E.2.1. Twistor transforms

For completeness, we shall establish the twistor correspondences on quite generic ground, as considered, e.g., by Eastwood [326]. We start by defining the twistor correspondence in the holomorphic category. The central objects of the twistor technology are double fibrations of the form[1]

$$\begin{array}{ccc} & \mathsf{Y} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathsf{Z} & & \mathsf{X} \end{array} \tag{E.56}$$

Here, X, Y and Z are complex manifolds together with two suitable[2] holomorphic projections $\pi_{1,2}$. We are interested in moving geometrical objects along this double fibration. In the following, we shall work with complex analytic objects. Such a double fibration induces a correspondence between X and Z, given by the assignment

$$\begin{array}{ccc} \text{points } p \text{ in Z} & \longleftrightarrow & \text{subspaces } \pi_1(\pi_2^{-1}(x)) \text{ in X} \\ \text{subspaces } \pi_2(\pi_1^{-1}(z)) \text{ in Z} & \longleftrightarrow & \text{points } x \text{ in X} \, , \end{array}$$

[1] Recall that such a framework can be easily generalised for studying certain supersymmetric field theories, just replacing the manifolds in double fibration by supermanifolds.

[2] Following [326], these are considered to be surjective holomorphic mappings of maximal rank, i.e. submersions.

known as *twistor correspondence*. In particular, using this correspondence one starts with data given on Z (such as sheaf cohomology groups, holomorphic vector bundles, contact forms, etc.) and ends up with solutions to differential equations on X. The main idea here is indeed to derive the equations of physics by using the rigidity of complex geometry. This is a two-step procedure. First, one needs to pull-back data on Z to data on Y and then one descends to X by integrating along the fibres of π_2. By virtue of this process the derived data on X should possess some differential property reflecting the fact that the intermediary object on Y is in some sense constant along the fibres of π_1. Under certain conditions, the transform is invertible. Hence, such a construction provides a way of studying physical objects on X (space-time), obeying differential equations, in terms of unconstrained objects on a totally different manifold Z (twistor space). We shall refer to such a map as *Penrose–Ward transform*.

Let us see how this works. The discussion here follows the one in [326, 96]. Let us start from the fibration $\pi_1 : \mathsf{Y} \to \mathsf{Z}$ and consider a holomorphic vector bundle $\hat{E}$ on Z. As we have seen, any fibration comes with an associated relative exterior derivative, which is the projection of the exterior derivative onto 1-forms along the fibres. Provided the fibres are connected, a function is constant along the fibres if and only if its relative exterior derivative is zero. We pull-back $\hat{E}$ via π_1 to $E := \pi_1^*\hat{E}$ on Y. By construction, E is described by transition functions which are constant along the fibres of π_1. Explicitly, they are annihilated by the vector fields spanning the twistor distribution and hence by d_{π_1}. Therefore, one can show that the pull-back bundle E comes with a non-trivial relative connection[1]. This is defined as the differential operator [326]

$$\nabla_{\pi_1} : \mathcal{O}(E) \to \Omega^1_{\pi_1}(\hat{E}) \; , \tag{E.57}$$

satisfying the relative Leibniz rule

$$\nabla_{\pi_1}(f\omega) = f\nabla_{\pi_1}\omega + (\mathrm{d}_{\pi_1} f) \otimes \omega \; , \tag{E.58}$$

for $f \in \mathcal{O}_{\mathsf{Y}}$ and $\omega \in \mathcal{O}(E)$. Moreover, if some topological conditions apply, such a connection is flat, namely the corresponding curvature $\nabla^2_{\pi_1} \in \Gamma(\mathsf{Y}, \Omega^2_{\pi_1}(\mathsf{End}(E)))$ vanishes. Explicitly the following result holds:

Proposition E.16 (Eastwood [326]). *Let π_1 be a surjective holomorphic map of maximal rank. If the fibres of π_1 are connected and simply connected then holomorphic vector bundles*

[1] Generally, a relative connective structure is composed by differential forms that have components only along the fibration in question.

on Z *are in one-to-one correspondence to holomorphic vector bundles on* Y *with a flat relative connection.*

For every integer $q \geq 0$ there is a natural map on cohomology[1]

$$\pi_1^{-1} : H^q(\mathsf{Z}, \mathcal{O}(\hat{E})) \to H^q(\mathsf{Y}, \pi_1^{-1}\mathcal{O}(\hat{E})) \ , \tag{E.59}$$

in terms of the inverse image sheaf $\pi_1^{-1}\mathcal{O}(\hat{E})$. If the fibres of π_1 are connected, the following isomorphism

$$H^0(\mathsf{Z}, \mathcal{O}(\hat{E})) \cong H^0(\mathsf{Y}, \pi_1^{-1}\mathcal{O}(\hat{E})) \ , \tag{E.60}$$

holds [96]. To deal with higher cohomology the following result comes to help:

Proposition E.17 (Buchdahl [335])**.** *Let* π_1 *as above. Suppose that* π_1 *has connected fibres and there is an* $n_0 > 0$ *such that* $H^q(\pi_1^{-1}(z), \mathbb{C}) = 0$ *for* $q = 1, \dots, n_0$ *and for all* $z \in \mathsf{Z}$*. Then, the homomorphism* (E.59) *is an isomorphism for* $q = 0, 1, \dots, n_0$ *and a monomorphism for* $q = n_0 + 1$*.*

Moreover, in the previous hypothesis, if the fibers of π_1 are cohomologically acyclic (i.e. contractible) then $\pi_1^{-1} : H^q(\mathsf{Z}, \mathcal{O}(\hat{E})) \to H^q(\mathsf{Y}, \pi_1^{-1}\mathcal{O}(\hat{E}))$ is an isomorphism for all q [326]. Recalling that these objects originate from sheaves on Z, then one uses the result E.17, to connect the cohomology groups of both kinds of sheaves.

The second stage consists of pushing down cohomological data $H^q(\mathsf{Y}, \pi_1^{-1}\mathcal{O}(\hat{E}))$ onto X under $\pi_2 : \mathsf{Y} \to \mathsf{X}$. As we shall see, this is realised by taking direct images of $\Omega_{\pi_1}(\hat{E})$ with respect to the fibration π_2 to obtain certain sheaves on (space-time) X, and using the tool of Leray spectral sequence for computing cohomology groups. In particular it can be shown, via spectral sequences argument, that the fundamental object to study is the cohomology of the complex $\Omega^\bullet_{\pi_1}(\hat{E})$.

Penrose transform. As we have seen, the twisted relative de Rham sequence of sheaves (E.52)

$$0 \longrightarrow \pi_1^{-1}\mathcal{O}_{\mathsf{Z}}(\hat{E}) \longrightarrow \Omega^0_{\pi_1}(\hat{E}) \xrightarrow{\mathrm{d}_{\pi_1}} \cdots \xrightarrow{\mathrm{d}_{\pi_1}} \Omega^n_{\pi_1}(\hat{E}) \longrightarrow 0 \ , \tag{E.61}$$

is exact. Hence, there exists a spectral sequence associated with the resolution [326]

$$E_1^{p,q} = H^q(\mathsf{Y}, \Omega^p_{\pi_1}(\hat{E})) \Longrightarrow H^{p+q}(\mathsf{Y}, \pi_1^{-1}\mathcal{O}(\hat{E})) \ , \tag{E.62}$$

[1] This can be realised by the Dolbeault resolution for $\mathcal{O}(\hat{E})$. See [96] for an explanation.

with differentials induced by ∇_{π_1}. In particular, this gives us a way of computing $H^{p+q}(\mathsf{Y}, \pi_1^{-1}\mathcal{O}(\hat{E}))$ in terms of the cohomology groups $H^q(\mathsf{Y}, \Omega^p_{\pi_1}(\hat{E}))$. If the complex manifold X is Stein, the Leray spectral sequence collapses and gives

$$H^q(\mathsf{Y}, \Omega^p_{\pi_1}(E)) \cong \Gamma(\mathsf{X}, \pi^q_{2*}(\Omega^p_{\pi_1}(E))) \tag{E.63}$$

Summarising, we have the following

Theorem E.18 (Eastwood [326]). *Let us consider the correspondence* (E.56) *with* $\hat{E}$ *a holomorphic vector bundle over* Z. *In the hypotheses of the theorem E.17 there is a spectral sequence*

$$E_1^{p,q} = \Gamma(\mathsf{X}, \pi^q_{2*}\Omega^p_{\pi_1}(E)) \Longrightarrow H^{p+q}(\mathsf{Z}, \mathcal{O}(\hat{E})) \ . \tag{E.64}$$

We shall refer to this as the *Penrose transform*.

Ward correspondence. We start by requiring the holomorphic vector bundle $\hat{E}$ to be X-trivial or, in other words, the pull-back bundle E to be trivial on each fibre of π_2. Now, we are interested in considering what extra information on X is supplied by the relative connection (E.57) on E. Again, this involves computing direct images. Under these assumptions, the direct image

$$E' := \pi_{2*}\pi_1^*\hat{E} \tag{E.65}$$

is an holomorphic vector bundle on X of the same rank as $\hat{E}$. By pushing down ∇_{π_1} one obtains a first order differential operator [96]

$$D := \pi_{2*}\nabla_{\pi_1} : E' \to \pi_{2*}\Omega^1_{\pi_1} \otimes E' \tag{E.66}$$

satisfying a Leibnitz-type rule

$$D(fs) = fD(s) + \partial f \otimes s \tag{E.67}$$

where $\partial := \pi_{2*}\mathrm{d}_{\pi_1}$. If the canonical[1] map $\Omega_{\mathsf{X}} \to \pi_{2*}\Omega^1_{\pi_1}$ is an isomorphism, then D descends to a connection

$$\nabla : \mathcal{O}(E') \to \Omega^1(E') \ , \tag{E.68}$$

on X. The flatness of ∇ on certain submanifolds of X is then related to the flatness of the relative connection (E.57). This is contained in the following result:

[1]The idea is that something is canonical if it transforms correctly when the whole system is subjected to a morphism of the category in which one is working.

Proposition E.19 (Eastwood [326])**.** *We are in the hypothesis of E.16. Let us assume* $\pi_1(\pi_2^{-1}(x))$ *is compact and connected for all* $x \in \mathsf{X}$*. Moreover,* $\Omega_1(\mathsf{X}) \cong \pi_{2*}\Omega^1_{\pi_1}(\mathsf{Y})$*. Then, there is a bijection between* X*-trivial holomorphic vector bundles on* Z *and holomorphic vector bundles on* X *equipped with a connection flat on each* $\pi_2(\pi_1^{-1}(z))$*, for all* $z \in \mathsf{Z}$*.*

This establishes the connection with E.16 and close the circle. The equations $\nabla^2|_{\pi_2(\pi_1^{-1}(z))} = 0$ yield a system of nonlinear integrable differential equations on space-time X.

To conclude we just observe that the requirements needed above are always satisfied in our setting, i.e. for the double fibration $\mathsf{P}^6 \leftarrow \mathsf{F}^9 \rightarrow \mathsf{M}^6$.

F Calculations on twistor space

The aim of this Appendix is to present a few useful tools for performing computations on twistor space P^6. We also prove some results that we have used in Chapters 6 and 7.

F.1. Homogeneous–local coordinates: a dictionary

Here, we shall introduce local coordinates on twistor space P^6 and re-write many relevant objects accordingly. Although all the calculations in the body of the Thesis are presented, more elegantly, in homogeneous coordinates, using local coordinates sometimes comes in handy to avoid some subtleties that homogeneous coordinates may induce. For instance, we recall that the differential one-forms $\bar{e}_A$, spanning the dual of the twistor distribution, are defined up to terms proportional to λ_A, as $\lambda_A \overline{V}^A = 0$. Similarly, because of the invariance under rescaling of homogeneous coordinates, local sections of $\mathcal{O}_{\mathbb{P}^3}$ fulfil the equation $\lambda_A \frac{\partial}{\partial \lambda_A} f = 0$, where $\lambda_A \frac{\partial}{\partial \lambda_A}$ is the Euler vector field on $\mathbb{P}^3$.

Coordinates. Let (z^A, π_A), for $A = 0, \ldots 3$, be homogeneous coordinates on $\mathbb{P}^7$. Recall that the the invariance under $\mathbb{C}_0$ rescalings of homogeneous coordinates means that they only contain seven complex degrees of freedom. We have defined twistor space P^6 to be a quadric hypersurface in $\mathbb{P}^7_0 \setminus \mathbb{P}^3$ in which $(\lambda_A) \neq 0$. A standard open covering of P^6 is a lift of the standard cover of $\mathbb{P}^3$ and consists of four coordinate patches $\mathfrak{U} = \{U_i\}$, $i = 1, \ldots, 4$. We define the inhomogeneous or local coordinates on a patch $U_1 \subset \mathsf{P}^6 = \{[\pi_A] \mid \pi_0 \neq 0\}$ by

$$\left(\zeta^a := \frac{z^a}{\pi_0},\ \eta_a := \frac{\pi_a}{\pi_0}\right), \quad \text{for} \quad a = 1, 2, 3\ . \tag{F.1}$$

Coordinates on the other three patches are defined accordingly.

Analogously, let $\mathfrak{V} = \{V_i\}$, $1 = 1, \ldots, 4$ be an open covering of Euclidean twistor space. Recall that in this setting $\mathsf{P}^6 \cong \mathbb{R}^6 \times \mathbb{P}^3$. It can be equivalently coordinatised by global coordinates (x^{AB}, λ_A), where λ_A are homogeneous coordinates on $\mathbb{P}^3$. On a patch $V_1 \subset \mathbb{R}^6 \times \mathbb{P}^3 = \{[(x^{AB}, \lambda_A)] \mid \lambda_0 \neq 0\}$, we introduce local coordinates

$$\Big(x^{ab}, x^{0a}, \rho_a := \frac{\lambda_a}{\lambda_0}\Big) \ , \quad \text{for} \quad a = 1, 2, 3 \ , \tag{F.2}$$

where ρ_a are local fibre coordinates. Using the Euclidean reality conditions defined in 6.3.2. one may invert the incidence relation $z^A = x^{AB}\lambda_B$, obtaining

$$x_{0a} = \frac{1}{1 + [\eta\bar{\eta}]}\Big[\bar{\eta}^b\bar{\zeta}_b\eta_a - \bar{\zeta}_a + \varepsilon_{abc}\zeta^b\bar{\eta}^c\Big] \ , \tag{F.3}$$

$$x_{ab} = \varepsilon_{abc}x^{0c} = \frac{1}{1 + [\eta\bar{\eta}]}\Big[2\bar{\zeta}_{[a}\eta_{b]} - \varepsilon_{abc}\zeta^c - \varepsilon_{abc}\bar{\eta}^c\zeta^d\eta_d\Big] \ , \tag{F.4}$$

where $[\eta\bar{\eta}] := \sum_{p=1}^{3} |\eta_p|^2$. Here, three dimensional indices are raised and lowered by the rules

$$x_{ab} = \varepsilon_{abc}x^{0c} \quad \text{and} \quad x^{0a} = \frac{1}{2}\varepsilon^{abc}x_{bc} \ . \tag{F.5}$$

Vector fields. In Section 6.3.2. we have also introduced the $(0,1)$-vector fields $\{\overline{V}^A, \widetilde{V}_{AB}\}$ on Euclidean twistor space. In terms of homogeneous coordinates (z^A, π_A) the space $T^{0,1}\mathsf{P}^6$ is generated by

$$\Big\{\frac{\partial}{\partial \hat{z}_A}, \pi_{[A}\frac{\partial}{\partial \hat{\pi}^{B]}}\Big\} \ , \tag{F.6}$$

with

$$\frac{\partial}{\partial \hat{z}_A} = \frac{2}{[\lambda\hat{\lambda}]}\overline{V}^A \quad \text{and} \quad \pi_{[A}\frac{\partial}{\partial \hat{\pi}^{B]}} = -\frac{1}{[\lambda\hat{\lambda}]}\varepsilon_{ABCD}x^{CE}\lambda_E\overline{V}^D + \lambda_{[A}\frac{\partial}{\partial \hat{\lambda}^{B]}} \ , \tag{F.7}$$

where we have used $\hat{z}_A = x_{AB}\hat{\lambda}^B$. Similarly for the $(0,1)$-differential forms $\{\bar{e}_A, \tilde{e}^{AB}\}$ we have

$$\bar{e}_A + \tfrac{1}{2}\varepsilon_{ABCD}z^B\tilde{e}^{CD} = \frac{1}{[\pi\hat{\pi}]}\Big(\mathrm{d}\hat{z}_A - \frac{2}{[\pi\hat{\pi}]}\hat{z}_{[A}\pi_{B]}\mathrm{d}\hat{\pi}^B\Big) \quad \text{and} \quad \tilde{e}^{AB} = \frac{2}{[\pi\hat{\pi}]^2}\hat{\pi}^{[A}\mathrm{d}\hat{\pi}^{B]} \tag{F.8}$$

Locally, on Euclidean twistor space[1], we may introduce the following sets of $(0,1)$-vector fields

$$\Big\{\overline{E}^a := \partial^{a0} + \rho_b\partial^{ab} \ , \ \frac{\partial}{\partial \bar{\rho}^a}\Big\} \tag{F.9}$$

[1]Here and in the following, local quantities are always considered on a patch even if not explicitly specified.

and $(0,1)$-differential forms

$$\Big\{\bar{e}_a := \frac{1}{1+[\rho\bar{\rho}]}\Big[\mathrm{d}x_{a0} + \bar{\rho}^b(\rho_a \mathrm{d}x_{b0} + \mathrm{d}x_{ab})\Big]\ ,\ \mathrm{d}\bar{\rho}^a\Big\}\ , \tag{F.10}$$

such that

$$T^{0,1}\mathsf{P}^6 = \mathsf{span}\Big\{\overline{E}^a; \frac{\partial}{\partial\bar{\rho}^a}\Big\} \quad \text{and} \quad \Omega^{0,1}(\mathsf{P}^6) = \mathsf{span}\{\bar{e}_a; \mathrm{d}\bar{\rho}^a\}\ , \tag{F.11}$$

Differently from the situation in Section 6.3.2., here (F.9) and (F.10) define actual frames of $(0,1)$-vectors and $(0,1)$-forms, respectively. By using conventions

$$\partial^{ab} \lrcorner\, \mathrm{d}x_{cd} = \delta_c^{[a}\delta_d^{b]}\ , \quad \partial^{ab} \lrcorner\, \mathrm{d}x_{c0} = 0\ , \quad \partial^{a0} \lrcorner\, \mathrm{d}x_{c0} = \frac{1}{2}\delta_c^a \quad \text{and} \quad \partial^{a0} \lrcorner\, \mathrm{d}x_{bc} = 0\ , \tag{F.12}$$

one can immediately see that these bases are dual in the sense that

$$\frac{\partial}{\partial\bar{\rho}^a} \lrcorner\, \mathrm{d}\bar{\rho}^b = \delta_a^b\ , \quad \frac{\partial}{\partial\bar{\rho}^a} \lrcorner\, \bar{e}_b = 0\ , \quad \overline{E}^a \lrcorner\, \bar{e}_b = \frac{1}{2}\delta_b^a\ , \quad \overline{E}^a \lrcorner\, \mathrm{d}\bar{\rho}^b = 0\ . \tag{F.13}$$

Then, due to the diffeomorphism $\mathsf{P}^6 \cong \mathbb{R}^6 \times \mathbb{P}^3$, we have the following transformation laws between the coordinate anti-holomorphic vector fields:

$$\begin{aligned}
\frac{\partial}{\partial\bar{\eta}^a} &= \frac{\partial}{\partial\bar{\rho}^a} + 2x_{ab}\overline{E}^b + \frac{2}{1+[\rho\bar{\rho}]}\bar{\rho}^c x_{ac}\rho_b\overline{E}^b\ ,\\
\frac{\partial}{\partial\bar{\zeta}_a} &= \frac{2}{1+[\rho\bar{\rho}]}(\overline{E}^a + \bar{\rho}^a\rho_b\overline{E}^b)\ .
\end{aligned} \tag{F.14}$$

Dolbeault operator. Let (z^A, π_A) and (x^{AB}, λ_A) be homogeneous coordinates on P^6 and coordinates on Euclidean twistor space, respectively. As regards the anti-holomorphic exterior derivate, in Section 6.3.2. we claimed that

$$\bar{\partial} = 2\bar{e}_A\overline{V}^A + \tilde{e}^{AB}\widetilde{V}_{AB} = \mathrm{d}\hat{z}_A P_B^A\frac{\partial}{\partial\hat{z}^B} + \mathrm{d}\hat{\pi}^A\frac{\partial}{\partial\hat{\pi}^A}\ , \tag{F.15}$$

in terms of the projector $P_B^A := \delta_B^A + \frac{\pi_B\hat{\pi}^A}{[\pi\hat{\pi}]}$. This is an example where using local coordinates turns out to be quite useful. Indeed, a lengthy, but straightforward, calculation shows that

$$\bar{\partial} = \mathrm{d}\bar{\eta}^a\frac{\partial}{\partial\bar{\eta}^a} + \mathrm{d}\bar{\zeta}_a\frac{\partial}{\partial\bar{\zeta}_a} = 2\bar{e}_a\overline{E}^a + \mathrm{d}\bar{\rho}^a\frac{\partial}{\partial\bar{\rho}^a}\ . \tag{F.16}$$

Here, we have used the definitions (F.9)-(F.10) and (F.14) together with the relations

$$\bar{\eta}^b x_{ab} = x_{0a} + \bar{\zeta}_a\ , \qquad \bar{\eta}^b x_{b0} = \bar{\eta}^b\bar{\zeta}_b \tag{F.17}$$

and the symmetry relation $T_{[ab]c} = 3T_{[abc]} + T_{[ac]b} + T_{[cb]a}$, for some general tensor T.

Differential forms. We have seen that in the Abelian case the relevant fields are $(0,2)$- and $(0,3)$-differential forms on twistor space. In local coordinates, their expansions read

$$B^{0,2} = \varepsilon^{abc}\bar{e}_a \wedge \bar{e}_b(B_c - \rho_c B_0) + \bar{e}_a \wedge \mathrm{d}\bar{\rho}^b \Big[\bar{\rho}^c \mathring{B}^{[a0]}_{[cb]} + \mathring{B}^{[a0]}_{[0b]} + \rho_c \mathring{B}^{[ac]}_{[0b]} + \rho_c \bar{\rho}^d \mathring{B}^{[ac]}_{[db]}\Big] \ , \quad \text{(F.18)}$$

and

$$\begin{aligned} C^{0,3} &= \varepsilon_{abc}\mathrm{d}\bar{\rho}^a \wedge \mathrm{d}\bar{\rho}^b \wedge \mathrm{d}\bar{\rho}^c \, C_0 + \varepsilon_{cdf}\bar{e}_a \wedge \mathrm{d}\bar{\rho}^c \wedge \mathrm{d}\bar{\rho}^d \Big[\rho_b \widetilde{C}^{[ab],f} - \rho_b \bar{\rho}^f \widetilde{C}^{[ab],0} \\ &\quad + \widetilde{C}^{[a0],f} - \bar{\rho}^f \widetilde{C}^{[a0],0}\Big] + \varepsilon^{abc}\bar{e}_a \wedge \bar{e}_b \wedge \mathrm{d}\bar{\rho}^d \Big[\mathring{C}_{c,[0d]} - \rho_c \mathring{C}_{0,[0d]} + \bar{\rho}^f \mathring{C}_{c,[fd]} - \rho_c \mathring{C}_{0,[fd]}\Big] \\ &\quad - \varepsilon^{abc}\bar{e}_a \wedge \bar{e}_b \wedge \bar{e}_c \, \widehat{C} \ . \end{aligned} \quad \text{(F.19)}$$

Here, the coefficient fields are weightless smooth functions of $(x_{ab}, x_{0a}, \rho_a, \bar{\rho}^a)$. Similar expansions hold for the other form fields entering the non-Abelian case.

Holomorphic measure. Moreover, the holomorphic volume form (7.15) may be re-written as

$$\Omega^{6,0} = \oint_{\mathscr{C}} \varepsilon_{abc} e^a \wedge e^b \wedge e^c \wedge \mathrm{d}\rho_l \wedge \mathrm{d}\rho_m \wedge \mathrm{d}\rho_n \varepsilon^{lmn} \ . \quad \text{(F.20)}$$

in terms of the complex conjugate basis of (F.10)

$$\Big\{e^a := \frac{1}{1 + [\rho\bar{\rho}]}\Big[\mathrm{d}x^{a0} + \rho_b(\mathrm{d}x^{ab} + \bar{\rho}^a \mathrm{d}x^{b0})\Big], \ \mathrm{d}\rho_a\Big\} \ . \quad \text{(F.21)}$$

This allow us to check calculations without dealing with homogeneous coordinates, and their corresponding subtleties.

F.2. Integral formulas

In the following we give a proof of the integral formulæ presented in Section 6.4.. We borrow some ideas from [257]. See also [97].

Čech representation. Let us consider the integral[1]

$$\psi_{A_1 \ldots A_{2h}} := \oint_{\Gamma} \Omega^{3,0} \lambda_{A_1} \cdots \lambda_{A_{2h}} f_{-2h-4}|_{\mathsf{Y}} \ . \quad \text{(F.22)}$$

We recall that the open convex subset $\hat{U} \subset \mathsf{P}^6$ is Stein, while $f_{-2h-4} \in H^3(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$ is a Čech cocycle and $\mathsf{Y} \cong \mathbb{P}^3$. One may immediately realise that this integral formula gives

[1]Recall that the contour Γ is, for any $x \in U \subset \mathsf{M}^6$, $S^1 \times S^1 \times S^1$.

rise to solutions of the massless field equations by directly differentiating under the integral sign and using the fact that

$$\begin{aligned}\partial^{AB} f_{-2h-4}\Big|_X &= \frac{1}{2}\varepsilon^{ABCD}\partial_{CD} f_{-2h-4}(x^{EF}\lambda_F,\lambda_E) = \frac{1}{2}\varepsilon^{ABCD}\delta^E_{[C}\delta^F_{D]}\,\lambda_F\frac{\partial f_{-2h-4}}{\partial z^E}\bigg|_{\mathsf{Y}} = \\ &= \frac{1}{2}\varepsilon^{ABCD}\lambda_{[D}\frac{\partial f_{-2h-4}}{\partial z^{C]}}\bigg|_{\mathsf{Y}} .\end{aligned} \tag{F.23}$$

Explicitly,

$$\begin{aligned}\partial^{AB}\psi_{BA_1\dots A_{2h-1}} &= \oint_\Gamma \Omega^{3,0}\lambda_B\lambda_{A_1}\cdots\lambda_{A_{2h-1}}\partial^{AB} f_{-2h-4}\big|_{\mathsf{Y}} \\ &= \oint_\Gamma \Omega^{3,0}\lambda_{A_1}\cdots\lambda_{A_{2h-1}}\frac{1}{4}\varepsilon^{ABCD}\left(\lambda_B\lambda_D\frac{\partial f_{-2h-4}}{\partial z^C}\bigg|_{\mathsf{Y}} - \lambda_B\lambda_C\frac{\partial f_{-2h-4}}{\partial z^D}\bigg|_{\mathsf{Y}}\right) = 0\,,\end{aligned} \tag{F.24}$$

by symmetry reasons. Hence, the integral formula (F.22) realises this isomorphism (6.29) from the left to the right. In particular, for $h = 1$ (case of interest) one has the expression

$$C_{AB}(x) = \oint_\Gamma \Omega^{(3,0)}\lambda_A\lambda_B f_{-6}|_{\mathsf{Y}} \ , \tag{F.25}$$

with $f_{-6} \in H^3(\mathsf{P}^6, \mathcal{O}_{\mathsf{P}^6}(-6))$ and $\partial^{AB}C_{BC} = 0$. We notice that the contour integral formula (F.22) can be recognised as a representation of a special instance of Serre duality. Considering the Abelian sheaf $\mathcal{O}(k)$, Serre duality on projective space $\mathbb{P}^3$ reads

$$H^q(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k)) \cong H^{3-q}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-k-4)) \ . \tag{F.26}$$

This follows from that fact that the canonical line bundle on $\mathbb{P}^3$ is $K := \det T^*\mathbb{P}^3 \cong \mathcal{O}_{\mathbb{P}^3}(-4)$.

Dolbeault representation. Alternatively, we have defined the integral formula

$$\psi_{A_1\dots A_{2h}} := \int_{\mathsf{Y}} \Omega^{3,0}\wedge\lambda_{A_1}\cdots\lambda_{A_{2h}}\eta|_{\mathsf{Y}} \ , \tag{F.27}$$

for $\eta = \eta(z,\lambda)$ a Dolbeault representative of the cohomology group $H^{0,3}_{\bar\partial}(\hat U, \mathcal{O}_{\hat U}(-2h-4))$. This integral depends only on the cohomology class of η. Once again, the fact that the z.r.m. equations are indeed obeyed follows directly by differentiating under the integral sign and using the fact that

$$\frac{\partial}{\partial x^{AB}} = \lambda_{[B}\frac{\partial}{\partial z^{A]}} \ , \tag{F.28}$$

thanks to the incidence relation. Here, we have used the holomorphicity condition, $\bar\partial\eta = 0$, i.e. the Dolbeault representative η does not depend on the complex conjugated twistor variables.

To be precise about anti-holomorphic dependence on twistor space, we must specify some reality conditions on it. Let us consider Euclidean reality conditions (6.22), providing twistor space with the non-holomorphic fibration $\pi : \mathsf{P}^6 \to \mathbb{R}^6$. In this real setting we consider $\eta \in \Omega^{(0,3)}(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$. For convenience, we recall the expansion for a $(0,3)$-form field

$$\begin{aligned}\eta = \varepsilon_{ABCD}\frac{\hat{\lambda}^A \mathrm{d}\hat{\lambda}^B \wedge \mathrm{d}\hat{\lambda}^C \wedge \mathrm{d}\hat{\lambda}^D}{[\lambda\hat{\lambda}]^4}\,\eta_0 + \varepsilon^{ABCD}\bar{e}_A \wedge \bar{e}_B \wedge \bar{e}_C \lambda_D\,\hat{\eta} \\ + \varepsilon_{CDEF}\frac{\bar{e}_{[A}\lambda_{B]} \wedge \hat{\lambda}^C \mathrm{d}\hat{\lambda}^D \wedge \mathrm{d}\hat{\lambda}^E}{[\lambda\hat{\lambda}]^3}\,\tilde{\eta}^{AB,F} + \varepsilon^{ABCD}\frac{\bar{e}_A \wedge \bar{e}_B \lambda_C \wedge \hat{\lambda}^E \mathrm{d}\hat{\lambda}^F}{[\lambda\hat{\lambda}]^2}\,\mathring{\eta}_{D,EF}\ ,\end{aligned} \tag{F.29}$$

in terms of the coefficient fields $\{\eta_0, \hat{\eta}, \tilde{\eta}^{AB,F}, \mathring{\eta}^{EF,D}\}$ of appropriate weights. Hence, (F.27) reduces to

$$\psi_{A_1\ldots A_{2h}} = \int_{\mathsf{Y}} \Omega^{3,0} \wedge \lambda_{A_1} \cdots \lambda_{A_{2h}} \eta_0|_{\mathsf{Y}}\ \Omega^{0,3}\ , \tag{F.30}$$

as the only component that appear in the integral is the one that points along the $\mathbb{P}^3$ direction of the Euclidean twistor space. Again, $\eta_0|_{\mathsf{Y}}$ denotes that the incidence relations have been imposed. Differentiating (F.27) under the integral sign gives

$$\partial^{AB}\psi_{BA_1\ldots A_{2h-1}} = \int_Y \Omega^{3,0}\lambda_{A_1}\cdots\lambda_{A_{2h-1}}(\lambda_B\partial^{AB}\eta_0|_{\mathsf{Y}}) \wedge \Omega^{0,3} = \int_Y K\lambda_{A_1}\cdots\lambda_{A_{2h-1}}\overline{V}^A\eta_0|_{\mathsf{Y}}\ , \tag{F.31}$$

where K is the Kähler metric (7.30), on each copy of the $\mathbb{P}^3$ fibre of the Euclidean twistor fibration. It is more convenient to work with local coordinates (F.2). For the sake of the argument, let us consider the simplest case

$$\psi_B = \int_{\mathsf{Y}} \Omega^{3,0} \wedge \lambda_B \eta|_{\mathsf{Y}} = \int_{\mathsf{Y}} K\lambda_B\eta_0|_{\mathsf{Y}}\ .$$

On a patch V_1, by using the expansion

$$\begin{aligned}\eta = \varepsilon_{abc}\mathrm{d}\bar{\rho}^a \wedge \mathrm{d}\bar{\rho}^b \wedge \mathrm{d}\bar{\rho}^c\ \eta_0 + \varepsilon_{cdf}\bar{e}_a \wedge \mathrm{d}\bar{\rho}^c \wedge \mathrm{d}\bar{\rho}^d\Big[\rho_b\tilde{\eta}^{[ab],f} - \rho_b\bar{\rho}^f\tilde{\eta}^{[ab],0} \\ + \tilde{\eta}^{[a0],f} - \bar{\rho}^f\tilde{\eta}^{[a0],0}\Big] + \ \varepsilon^{abc}\bar{e}_a \wedge \bar{e}_b \wedge \mathrm{d}\bar{\rho}^d\Big[\mathring{\eta}_{c,[0d]} - \rho_c\mathring{\eta}_{0,[0d]} + \bar{\rho}^f\mathring{\eta}_{c,[fd]} - \rho_c\mathring{\eta}_{0,[fd]}\Big] \\ - \varepsilon^{abc}\bar{e}_a \wedge \bar{e}_b \wedge \bar{e}_c\ \hat{\eta}\end{aligned} \tag{F.32}$$

and differentiating, one obtains[1]

$$\begin{aligned}\partial^{AB}\psi_B &= \int_{\mathfrak{Y}} K^{(1)}\overline{V}^A_{(1)}\eta_0^{(1)}|_{\mathsf{Y}} = \int_{\mathfrak{Y}} \mathrm{d}\rho_1 \wedge \mathrm{d}\rho_2 \wedge \mathrm{d}\rho_3 \wedge \mathrm{d}\bar{\rho}^1 \wedge \mathrm{d}\bar{\rho}^2 \wedge \mathrm{d}\bar{\rho}^3\overline{E}^a\eta_0^{(1)}|_{\mathsf{Y}} \\ &= \frac{1}{6}\int_{\mathfrak{Y}} \mathrm{d}\rho_1 \wedge \mathrm{d}\rho_2 \wedge \mathrm{d}\rho_3 \wedge \mathrm{d}\bar{\rho}^1 \wedge \mathrm{d}\bar{\rho}^2 \wedge \mathrm{d}\bar{\rho}^3\frac{\partial}{\partial\bar{\rho}^c}\Big[\rho_b\tilde{\eta}^{ab,c} - \rho_b\bar{\rho}^c\tilde{\eta}^{[ab],0} + \tilde{\eta}^{[a0],c} - \bar{\rho}^c\tilde{\eta}^{[a0],0}\Big]\ ,\end{aligned}$$

which vanishes by Stoke's theorem on the projective space Y.

[1] Here, the suffix (1) denotes the restriction to the patch V_1. Note that, in the second line the incidence relation is assumed to be taken into account.

Woodhouse method. On the other side, we are also interested in knowing how, given a z.r.m. field on space-time M^6, one can construct a twistorial cohomology class which manifests the requested space-time degrees of freedom. For this purpose, again, the additional structure induced by Euclidean reality conditions turns out to be very useful. We follow a method due to Woodhouse [259]. Let $\psi_{A_1\cdots A_{2h}} \in H^0(U, \mathcal{Z}_h)$ and η^ψ be the corresponding element of $H^{0,3}(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4))$. Recalling expansion (F.29), the component along $\mathbb{P}^3$ is given by

$$\eta_0^\psi(x, \lambda, \hat{\lambda}) := \alpha_h \psi_{A_1\ldots A_{2h}}(x) \frac{\hat{\lambda}^{A_1} \ldots \hat{\lambda}^{A_{2h}}}{[\lambda\hat{\lambda}]^{2h}} , \tag{F.33}$$

as can be easily seen by considering the integral formula (F.27) and using the identity

$$\int_{\mathbb{P}^3} \frac{K}{[\lambda\hat{\lambda}]^{2h}} \lambda_{A_1} \cdots \lambda_{A_{2h}} \hat{\lambda}^{B_1} \ldots \hat{\lambda}^{B_{2h}} = \frac{1}{\alpha_h} \delta^{B_1}_{(A_1} \cdots \delta^{B_{2h}}_{A_{2h})} . \tag{F.34}$$

The remaining components fields of η^ψ follow by exploiting the $\bar{\partial}$-closure condition. Explicitly,

$$\begin{aligned}
\bar{\partial}\eta^\psi = {} & 2\frac{\overline{e}^A \wedge \Omega^{(0,3)}}{[\lambda\hat{\lambda}]^4} \left[\overline{V}^A \eta_0^\psi - \frac{1}{3} \frac{\lambda_B \hat{\lambda}^C}{[\lambda\hat{\lambda}]} \widetilde{V}^\psi_{CD} \widetilde{\eta}^{AB,D} \right] + \\
& - 2\frac{\overline{e}^A \wedge \overline{e}^B \wedge \overline{e}^C \lambda_D \wedge \mathrm{d}\hat{\lambda}^F}{[\lambda\hat{\lambda}]^4} \left[\hat{\lambda}^E \widetilde{V}^\psi_{EF} \hat{\eta} + \overline{V}^M \hat{\lambda}^E \mathring{\eta}^\psi_{M,EF} \right] + \\
& - 3\frac{1}{[\lambda\hat{\lambda}]^3} \Bigg[\overline{e}_{[A} \lambda_B \wedge \overline{e}_{M]} \wedge \hat{\lambda}^C \mathrm{d}\hat{\lambda}^D \wedge \mathrm{d}\hat{\lambda}^E \varepsilon_{CDEF} \overline{V}^M \widetilde{\eta}_\psi^{AB,F} + \\
& \varepsilon^{ABCD} \overline{e}^A \wedge \overline{e}^B \lambda_C \wedge \mathrm{d}\hat{\lambda}^{[F} \hat{\lambda}^E \wedge \mathrm{d}\hat{\lambda}^{M]} \frac{\hat{\lambda}^L \widetilde{V}_{LM}}{[\lambda\hat{\lambda}]} \mathring{\eta}^\psi_{D,EF} \Bigg] = 0 ,
\end{aligned} \tag{F.35}$$

imposes relations between the coefficients fields as the distinct $(0,4)$-forms on twistor space have to vanish independently.

F.3. Integration over the fibres

In this Section we show some results the have been useful in performing integration over twistor space.

Holomorphic measure. Here, we are interested in showing how the holomorphic measure defined in (7.15) may be rewritten in the form (7.29), by using non-holomorphic coordinates (x^{AB}, λ_A). We perform a (local) change of variable by solving the quadric constraint $z^A\lambda_A = 0$ for z as

$$z^A = \xi^A w + x^{AB} \lambda_B . \tag{F.36}$$

Here, we have introduced coordinates ξ^A, satisfying the constraint $\xi^A\lambda_A = 1$ and of homogeneity such that the combination $\xi^A w$ has holomorphic weight $+1$. Hence, $z^A\lambda_A = w$. Moreover, we set $\mathrm{d}z^A = \alpha^A(w) + \xi^A \mathrm{d}w + x^{AB}\mathrm{d}\lambda_B$, were the $(1,0)$-form α is defined by $\alpha^A(w) := \mathrm{d}\xi^A w + \mathrm{d}x^{AB}\lambda_B$. Noting that $\alpha^A\lambda_A = -w\xi^A\mathrm{d}\lambda_A$, the holomorphic form is given by

$$\begin{aligned}\Omega^{6,0} &= \oint_{\mathscr{C}} \frac{1}{4!}\varepsilon_{ABCD}\frac{\mathrm{d}z^A\wedge\mathrm{d}z^B\wedge\mathrm{d}z^C\wedge\mathrm{d}z^D}{z^A\lambda_A}\wedge\Omega^{3,0}(\lambda)\\ &= \oint_{\mathscr{C}} \frac{1}{4!}\varepsilon_{ABCD}\frac{\alpha^A(w)\wedge\alpha^B(w)\wedge\alpha^C(w)\xi^D\mathrm{d}w}{w}\wedge\Omega^{3,0}(\lambda)\ ,\end{aligned} \tag{F.37}$$

as the other terms vanish when wedging to $\Omega^{3,0}(\lambda)$. Performing the contour integral in w gives

$$\Omega^{6,0} = 2\pi i\frac{1}{4!}\varepsilon_{ABCD}\mathrm{d}x^{AI}\lambda_I\wedge\mathrm{d}x^{BJ}\lambda_J\wedge\mathrm{d}x^{CK}\lambda_K\xi^D\wedge\Omega^{3,0}(\lambda)\ . \tag{F.38}$$

Finally, choosing $\xi^A := \frac{\hat\lambda^A}{[\lambda\hat\lambda]}$, equation (7.29) follows immediately from the definition (6.24), as $\alpha^A(0) = [\lambda\hat\lambda]e^A$.

Useful relations. Before integrating along the fibres of the twistor fibration one needs to properly rearrange the indices in order to factorise the space-time measure and the Fubini-Study measure on $\mathbb{P}^3$. This translates into straightforward, but lengthy calculations. The following results products of differential forms are helpful in this regard. Just to give the idea, let $A_1^{0,1} = \bar{e}_{[A}\lambda_{B]}A_1^{AB}$ and $A_2^{0,1} = \bar{e}_{[C}\lambda_{D]}A_2^{CD}$. The wedge product reads

$$A_1^{0,1}\wedge A_2^{0,1} = \frac{1}{4}\varepsilon^{ABCD}\bar{e}_A\wedge\bar{e}_B\lambda_C\Big(A_1^{FE}A_{2ED} - A_{1ED}A_2^{FE}\Big)\lambda_F\ , \tag{F.39}$$

where we have used

$$\bar{e}_{[A}\lambda_{B]}\wedge\bar{e}_{[C}\lambda_{D]} = \frac{3}{2}\left[\bar{e}_{[A}\lambda_B\wedge\bar{e}_{C]}\lambda_D - \bar{e}_{[A}\lambda_B\wedge\bar{e}_{D]}\lambda_C\right]\ . \tag{F.40}$$

Similarly,

$$\begin{aligned}\bar{e}_{[A}\lambda_{B]}\wedge\bar{e}_{[C}\lambda_{D]}\wedge\bar{e}_{[E}\lambda_{F]} = \frac{1}{2}\Big[&\bar{e}_{[A}\lambda_B\wedge\bar{e}_C\wedge\bar{e}_{E]}\lambda_D\lambda_F - \bar{e}_{[A}\lambda_B\wedge\bar{e}_C\wedge\bar{e}_{F]}\lambda_D\lambda_E\\ &- \bar{e}_{[A}\lambda_B\wedge\bar{e}_D\wedge\bar{e}_{E]}\lambda_C\lambda_F + \bar{e}_{[A}\lambda_B\wedge\bar{e}_D\wedge\bar{e}_{F]}\lambda_C\lambda_E\Big]\ .\end{aligned} \tag{F.41}$$

Serre duality. In showing how both Abelian and non-Abelian twistor actions reduces to space-time actions it is necessary to integrate along the $\mathbb{P}^3$-fibres of $\mathrm{P}^6 \to \mathbb{R}^6$. In particular, we have integrals of the form

$$\int_U \mathrm{d}^6x\int_{\mathbb{P}^3} K\phi^{AB\cdots}\psi_{CD\ldots}\frac{\lambda_A\lambda_B\cdots\hat\lambda^C\hat\lambda^D\cdots}{[\lambda\hat\lambda]^{2h}}\ , \tag{F.42}$$

where K is the Kähler form on $\mathbb{P}^3$ given in (7.30), while $\phi^{AB\cdots}$, $\psi_{CD\ldots}$ denote space-time dependent symmetric tensors with $2h$ indices each. Following the discussion in [84] we define the quantities

$$\Phi := \phi^{AB\ldots}\lambda_A\lambda_B\cdots \in H^{0,0}_{\bar\partial}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2h)) \tag{F.43}$$

and

$$\Psi := \psi_{CD\ldots}\frac{\hat\lambda^C\hat\lambda^D\cdots}{[\lambda\hat\lambda]^{2h+4}}\,\Omega^{0,3} \in H^{0,3}_{\bar\partial}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2h-4))\ , \tag{F.44}$$

where $\Omega^{0,3} := \frac{1}{4!}\varepsilon_{ABCD}\hat\lambda^A \mathrm{d}\hat\lambda^B \wedge \mathrm{d}\hat\lambda^c \wedge \mathrm{d}\hat\lambda^d$. Here, we have used the fact that Φ is annihilated by the $\bar\partial$ on $\mathbb{P}^3$ and Ψ is $\bar\partial$-closed because $\dim_{\mathbb{C}}\mathbb{P}^3 = 3$. By using the Dolbeault isomorphism (E.28) one finds

$$\begin{aligned} H^{0,0}_{\bar\partial}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2h)) &\cong H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2h))\ ,\\ H^{0,3}_{\bar\partial}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2h-4)) &\cong H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2h-4)) \end{aligned} \tag{F.45}$$

Hence, the duality pairing is

$$\int_{\mathbb{P}^3} \Omega^{3,0}\Phi \wedge \Psi = \alpha_h \phi^{AB\cdots}\psi_{AB\ldots}\ . \tag{F.46}$$

This, together with the coefficient α_h, follow from the following lemma for complex projective space.

Let $\omega^{n-1}_{\mathsf{FS}}$ be the Standard Fubini-Study Kähler metric on $\mathbb{P}^{n-1}$, with K_{n-1} its correspondent Kähler form and let $[\lambda_1, \cdots, \lambda_n]$ denote homogeneous coordinates on $\mathbb{P}^{n-1}$. Given the $2h$-tuples of integers $\mathbf{A} = (A_1, \cdots, A_{2h})$ and $\mathbf{B} = (B_1, \cdots, B_{2h})$, for $h \in \frac{1}{2}\mathbb{N}_0$, we define the generalised Kronecker δ by

$$\delta_{\mathbf{AB}} := \sum_{\tau\in S_{2h}} \prod_{j=1}^{2h} \delta_{A_{\tau(j)}B_{\tau(j)}}\ , \tag{F.47}$$

where S_{2h} is the permutation group in $2h$ symbols. The following linear algebraic lemma, see e.g. [336, 337] makes Serre duality on a (complex) projective space explicit.[1]

Lemma F.1. *If* $V_{\mathbf{A}} = \lambda_{A_1}\cdots\lambda_{A_{2h}}$ *and* $V_{\mathbf{B}} = \lambda_{B_1}\cdots\lambda_{B_{2h}}$ *then*

$$\int_{\mathbb{P}^{n-1}} K_{n-1}\frac{V_{\mathbf{A}}\overline{V}_{\mathbf{B}}}{[\lambda\bar\lambda]^{2h}} = \frac{(n-1)!}{(n+2h-1)!}\delta_{\mathbf{AB}}\ . \tag{F.48}$$

[1] Here, the bar denotes complex conjugation.

We are interested in the case $n = 4$. Assuming we are choosing Euclidean reality conditions, as explained in 6.3., the formulas (F.48)-(F.47) translate to the following integrals

$$\int_{\mathbb{P}^3} K \frac{\lambda_A \hat{\lambda}^B}{[\lambda\hat{\lambda}]} = \frac{1}{4}\delta_A^B \quad \text{and} \quad \int_{\mathbb{P}^3} K \frac{\lambda_A \lambda_B \hat{\lambda}^C \hat{\lambda}^D}{[\lambda\hat{\lambda}]^2} = \frac{1}{10}\delta^C_{(A}\delta^D_{B)} \ , \tag{F.49}$$

for the relevant cases $h = 1/2$ and $h = 1$, respectively. Here, we have used K as in (7.30).

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