Fermion masses and flavor mixing in modular A_4 symmetry

Mohammed Abbas*

Physics Department, Faculty of Science and Arts-Tabarjal, Jouf University, Al-Jouf 42421, Kingdom of Saudi Arabia
and Physics Department, Faculty of Sciences, Ain Shams University, Abbassiyah 11566, Cairo, Egypt

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We consider a flavor model based on the A_4 modular group to account for both lepton and quark parameters (masses and mixing). The inverse seesaw mechanism is considered to produce the light neutrino masses. Lepton masses and mixing are obtained in terms of Yukawa coupling ratios and values of the modulus τ nearby some fixed points for inverted neutrino mass hierarchy. The quark masses and mixing are arisen at the same τ values used in inverted neutrino mass hierarchy and are in agreement with the recent data.

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I. INTRODUCTION

The smallness of the neutrino mass is usually explained via the type-I seesaw mechanism [1] which is the common scenario used to account for neutrino masses and mixing. In this mechanism, the small neutrino mass is obtained by the extension of the fermion content of the Standard Model (SM) with three gauge singlets as heavy right-handed neutrinos N_i. The mass of the light neutrino can be calculated through the relation $m_\nu = -m_D M_R{}^{-1} m_D^T$, where m_D is the Dirac mass and M_R is the Majorana mass of right-handed neutrinos N_i. To account for the tiny neutrino mass of order $\mathcal{O}(10^{-2})$ eV, either the mass scale of N_i will be of order $\mathcal{O}(10^{11}$ GeV) or one should consider a very small Dirac coupling for TeV mass scale of right-handed neutrinos. The large right-handed neutrino masses are far from experimental reach. In addition, the lepton number is violated via the large scale of the right-handed neutrino masses.

On the other hand, the inverse seesaw mechanism [2–4] is an alternative mechanism used to account for the small neutrino mass by considering a small scale μ_s and making a double suppression by the new scale M_R via the relation $m_\nu = m_D M_R{}^{-1} \mu_s M_R^{T-1} m_D^T$. In the case of a type-I seesaw, the lepton number violation (LNV) takes place via the Majorana mass term of N_i, which is very large. Conversely, in the inverse seesaw, the lepton number is violated by the very small mass μ_s of the singlet S which is a very small scale compared to the electroweak scale. One can consider the lepton number as an approximate symmetry of nature, so it is convenient to break it by a small amount instead of a large mass like M_R. According to 't Hooft [5], if μ_s tends to zero, the neutrino mass m_ν goes to zero and LNV vanishes so that the symmetry is enhanced.

The flavor symmetry was proposed to account for many aspects such as the differences in mixing and mass hierarchy for lepton and quark sectors. Several models based on discrete symmetries were proposed to account for fermion masses and mixing (see [6]). Most of these models suffer from considerations of a large number of scalars (flavons), considering extra Z_N symmetries and/or fine-tuning to account for experimental data.

Recently, finite modular groups Γ_N have been proposed to explain the flavor aspects [7,8]. In such groups, the group transformations are extended to include the coupling constants which can transform nontrivially. Extra symmetries under modular weights are impeded into the group, so there is no need to impose other symmetries to match the data. Some of Γ_N are isomorphic to finite permutation groups, for instance, $\Gamma_2 \cong S_3$ [9–12], $\Gamma_3 \cong A_4$ [13–19], $\Gamma_4 \cong S_4$ [20–24], and $\Gamma_5 \cong A_5$ [25–27]. Attempts have been made to account for both leptons and quarks using a modular group with a single modulus value for both leptons and quarks [28–34]. Models with different moduli for charged lepton and neutrino sectors have been studied using the concept of modular residual symmetries [35,36]. Multiple modular symmetries with more than one modular group have been discussed in [37]. The double covering of modular groups has been investigated in [38–41]. The modular invariance combining with the generalized CP symmetry has been studied to predict CP violating phases

*maabbas@ju.edu.sa

of quarks and leptons [42–44]. Most of the above models used either the type-I seesaw mechanism or the nonrenormalizable five-dimensional operator to generate neutrino masses. The inverse seesaw mechanism has been used for some modular invariance models [18,45].

In this paper, we introduce a model based on modular A_4 symmetry to account for masses and mixing for leptons and quarks. First, we give an introduction to the modular groups and how to use them as flavor symmetries, then we explain our A_4 model in the lepton sector, and finally we study the quark masses and mixing.

II. MODULAR GROUPS

The modular group $\bar{\Gamma}$ is defined as linear fractional transformations on the upper half of the complex plan $\mathcal{H}$ and has the form [8,46–48]

$$\gamma : \tau \to \gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \tag{1}$$

where $a, b, c, d \in Z, ad - bc = 1$. The modular group $\bar{\Gamma}$ is isomorphic to the projective special linear group

$$PSL(2, Z) = SL(2, Z)/\{I, -I\}, \tag{2}$$

where

$$SL(2, Z) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in Z, ad - bc = 1 \right\}. \tag{3}$$

The generators of the group $\bar{\Gamma}$ are two matrices S and T where their action on the complex number τ is given by

$$S : \tau \to \frac{-1}{\tau}, \qquad T : \tau \to \tau + 1. \tag{4}$$

In the two by two representation, the two generators S, T can be represented as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

They should satisfy the conditions

$$S^2 = \mathbf{1}, \qquad (ST)^3 = \mathbf{1}.$$

Define the infinite modular groups $\Gamma(N), N = 1, 2, 3, \ldots$ as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}. \tag{6}$$

For $N = 1$,

$$\Gamma(1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 1 \right\}. \tag{7}$$

Since any integers can satisfy the conditions $a, d = 1 \bmod 1$ and $b, c = 0 \bmod 1$, $\Gamma(1) \equiv SL(2, Z)$. For $N = 1$, 2, we define $\bar{\Gamma}(N) = \Gamma(N)/\{I, -I\}$ whereas for $N > 2$, $\bar{\Gamma}(N) = \Gamma(N)$ because $-I \notin \Gamma(N)$ for $N > 2$. It is straightforward to notice that $\bar{\Gamma}(1) = PSL(2, Z) = \bar{\Gamma}$. The group $\bar{\Gamma}$ and its subgroup $\bar{\Gamma}(N)$ are discrete but infinite, while the quotient modular group $\Gamma_N = \bar{\Gamma}/\bar{\Gamma}(N)$ is finite. The group Γ_N is called the finite modular group and can be obtained by extending the conditions on the generators with the condition $T^N = \mathbf{1}$. For some N, the finite modular group Γ_N is isomorphic to a permutation group, for instance, $\Gamma_2 \cong S_3$, $\Gamma_3 \cong A_4$, $\Gamma_4 \cong S_4$ and, $\Gamma_5 \cong A_5$.

The modular function $f(\tau)$ of weight $2k$ is a meromorphic function of the complex variable τ which satisfies

$$f(\gamma(\tau)) = f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N), \tag{8}$$

where the integer $k \geq 0$. By using Eqs. (1) and (6), it is easy to calculate

$$\frac{d(\gamma(\tau))}{d\tau} = \frac{1}{(c\tau + d)^2}. \tag{9}$$

From Eq. (8), one can get

$$\frac{f(\gamma(\tau))}{f(\tau)} = \left(\frac{d(\gamma(\tau))}{d\tau}\right)^{-k} \qquad f(\gamma(\tau)) d(\gamma(\tau))^k = f(\tau) d\tau^k.$$

From the above equation, we conclude that the k-form $f(\tau) d\tau^k$ is invariant under $\Gamma(N)$. If the modular function is holomorphic everywhere, it is called "modular form" of weight $2k$. The modular forms of level N and weight $2k$ form a linear space of finite dimension. In the basis at which the transformation of a set of modular forms $f_i(\tau)$ is described by a unitary representation $\rho(\gamma)$, one can get

$$f_i(\gamma(\tau)) = (c\tau + d)^{2k} \rho_{ij}(\gamma) f_j(\tau), \qquad \gamma \in \Gamma(N). \tag{10}$$

Consider the superpotential $W(z,\phi)$ be written in terms of supermultiplets ϕ^I, where I refers to different sectors in the theory,

$$W(\tau,\phi) = \sum_I \sum_n Y_{I_1 I_2 \ldots I_n}(\tau)\phi^{I_1}\ldots\phi^{I_n}. \quad (11)$$

The supermultiplets ϕ^I transform under Γ_N in the representation $\rho(\gamma)$ as

$$\phi^{(I)}(\tau) \to \phi^{(I)}(\gamma(\tau)) = (c\tau + d)^{-2k}\rho^{(I)}(\gamma)\phi^{(I)}(\tau). \quad (12)$$

The invariance of the superpotential $W(z,\phi)$ under the modular transformation requires $Y_{I_1 I_2 \ldots I_n}(z)$ to be a modular form transforming in the representation

$$Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (cz+d)^{2k_Y(n)}\rho(\gamma)Y_{I_1 I_2 \ldots I_n}(\tau). \quad (13)$$

The modular invariance forces the condition

$$k_Y = k_{I_1} + k_{I_2} + \cdots + k_{I_n}. \quad (14)$$

A. Modular forms of level 3

The group A_4 has one triplet representation **3** and 3 singlets **1** **1**$'$, and **1**$''$ and is generated by two elements S and T satisfying the conditions

$$S^2 = T^3 = (ST)^3 = \mathbf{1}. \quad (15)$$

The modular form of level 3 has the form

$$f_i(\gamma(\tau)) = (c\tau + d)^{2k}\rho_{ij}(\gamma)f_j(\tau), \qquad \gamma \in \Gamma(3).$$

The modular form of weight 2 and level 3 transforms as a triplet and is given by $Y_3^{(2)} = (y_1, y_2, y_3)$, [8] where

$$y_1(\tau) = \frac{i}{2\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right],$$
$$y_2(\tau) = \frac{-i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right],$$
$$y_3(\tau) = \frac{-i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right], \quad (16)$$

where $\omega = e^{2i\pi/3}$ and the Dedekind eta-function $\eta(z)$ is defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q = e^{2\pi i\tau}. \quad (17)$$

One can construct modular forms of higher weights using the multiplication rules of A_4 [8]. Modular forms of weight 4 are constructed via multiplication of two triplets of weight 2. Using A_4 multiplication rules of two triplets, one can get one triplet and three singlets all of weight 4 as

$$Y_3^{(4)} = \begin{pmatrix} y_1^2 - y_2 y_3 \\ y_3^2 - y_2 y_1 \\ y_2^2 - y_1 y_3 \end{pmatrix}, \qquad Y_1^{(4)} = y_1^2 + 2y_2 y_3,$$
$$Y_2^{(4)} = y_3^2 + 2y_2 y_1, \qquad Y_3^{(4)} = y_2^2 + 2y_1 y_3. \quad (18)$$

The representations of the above singlets are

$$Y_1^{(4)} \sim 1, \qquad Y_2^{(4)} \sim 1', \qquad Y_3^{(4)} \sim 1''.$$

At all values of τ, the condition $Y_3^{(4)} = 0$ is satisfied.

We will use the basis where the generators of A_4 in triplet representation are

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \quad (19)$$

III. A_4 MODULAR INVARIANCE MODEL

The lepton content in the model is extended by adding a triplet of chiral supermutiplets N^c as a right-handed neutrino and three SM singlets S_i to get the neutrino masses via the inverse seesaw mechanism. We add a gauge singlet scalar χ transforming trivially under A_4 to get the masses of the singlet fermions N^c and S. Contrary to most flavor symmetric models, no more flavons are considered and no extra discrete symmetries are considered in our model. According to the modular invariance condition in Eq. (14), we chose the modular weights such that the following relations are satisfied:

$$\begin{aligned} k_L + k_{H_d} + k_E &= 2, \\ k_L + k_{H_u} + k_N &= 2, \\ 2k_S + 4k_\chi &= 0, \\ k_S + k_N + k_\chi &= 0. \end{aligned} \quad (20)$$

If we chose $k_L = 3$, $k_{H_u} = 0$, we can get the modular weights of other fields as shown in Table I. The lepton modular A_4 invariant superpotential can be written as

TABLE I. Assignment of flavors under A_4 and the modular weight k_I.

Fields	L	E_1^c	E_2^c	E_3^c	N^c	S	H_d	H_u	χ
A_4	3	1	$1''$	$1'$	3	3	1	1	1
k_I	3	−1	−1	−1	−1	2	0	0	−1

$$
\begin{aligned}
w_l = {} & \lambda_1 E_1^c H_d (L \otimes Y_3^{(2)})_1 + \lambda_2 E_2^c H_d (L \otimes Y_3^{(2)})_1' \\
& + \lambda_3 E_3^c H_d (L \otimes Y_3^{(2)})_1'' + g_1 ((N^c H_u L)_{3S} Y_3^{(2)})_1 \\
& + g_2 ((N^c H_u L)_{3A} Y_3^{(2)})_1 + h (N^c \otimes S)_1 \chi \\
& + \frac{f}{\Lambda^3} (S \otimes S)_1 \chi^4,
\end{aligned}
\tag{21}
$$

where Λ is the nonrenormalizable scale and g_1 is the coupling constant of the term of the symmetric triplet arising from the product of the two triplets L and Y, while g_2 is the coupling of the antisymmetric triplet term. After spontaneous symmetry breaking, the scalar fields H_u, H_d, and χ acquire vevs namely v_u, v_d, and v' respectively, where $v' \gg v_u, v_d$. We assume that v' satisfies the relation $\frac{v'}{\Lambda} \sim \mathcal{O}(\lambda_c)$ where $\lambda_c = 0.22$ is the Cabibbo angle. From Eq. (21), we can write the charged lepton mass matrix as

$$
m_e = v_d \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \times \begin{pmatrix} y_1 & y_3 & y_2 \\ y_2 & y_1 & y_3 \\ y_3 & y_2 & y_1 \end{pmatrix}. \tag{22}
$$

To deal only with the left-handed mixing, it is convenient to use the Hermitian matrix $M_e = m_e^\dagger m_e$ which can be diagonalized as

$$
M_e^{\text{diag}} = U_e^\dagger M_e U_e.
$$

The neutrino mass matrices are

$$
\mu_s = f v' \lambda_c^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M_R = h v' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
m_D = v_u \begin{pmatrix} 2 g_1 y_1 & (-g_1 + g_2) y_3 & (-g_1 - g_2) y_2 \\ (-g_1 - g_2) y_3 & 2 g_1 y_2 & (-g_1 + g_2) y_1 \\ (-g_1 + g_2) y_2 & (-g_1 - g_2) y_1 & 2 g_1 y_3 \end{pmatrix}. \tag{23}
$$

The neutrino mass matrix in the basis (ν_L, N^c, S) is given by

$$
M = \begin{pmatrix} 0 & m_D & 0 \\ m_D^T & 0 & M_R \\ 0 & M_R^T & \mu_s \end{pmatrix}. \tag{24}
$$

After diagonalization of this matrix, one can get three eigenvalues, one for the light neutrino and the other two for the heavy neutrino states. The masses of the light neutrino state m_ν can be obtained as

$$
m_\nu = m_D M_R^{-1} \mu_s M_R^{T-1} m_D^T. \tag{25}
$$

The overall parameter $\frac{f v_u^2 g_1^2 \lambda_c^3}{h^2 v'}$ determines the scale of light neutrino masses and can be easily chosen to achieve the desired scale. For instant, we can set $h \sim \mathcal{O}(1\ \text{GeV})$, $f \sim \mathcal{O}(0.001\ \text{GeV})$, $v' \sim \mathcal{O}(100\ \text{TeV})$, $v_u \sim \mathcal{O}(10^2\ \text{GeV})$, and $g_1 \sim \mathcal{O}(0.01\ \text{GeV})$ to get the neutrino masses of order $\mathcal{O}(10^{-1}\ \text{eV})$. The neutrino mass matrix m_ν is complex and symmetric, so it is convenient to diagonalize the Hermitian matrix $M_\nu = m_\nu^\dagger m_\nu$,

$$
M_\nu^{\text{diag}} = U_\nu^\dagger M_\nu U_\nu. \tag{26}
$$

The Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton mixing matrix U_{PMNS} is given by

$$
U_{\text{PMNS}} = U_e^\dagger U_\nu. \tag{27}
$$

The mixing angles can be calculated from the relations

$$
Sin^2(\theta_{13}) = |(U_{\text{PMNS}})_{13}|^2, \quad Sin^2(\theta_{12}) = \frac{|(U_{\text{PMNS}})_{12}|^2}{1 - |(U_{\text{PMNS}})_{13}|^2},
$$

$$
Sin^2(\theta_{23}) = \frac{|(U_{\text{PMNS}})_{23}|^2}{1 - |(U_{\text{PMNS}})_{13}|^2}. \tag{28}
$$

The mixing angles and mass ratios are determined by the ratios g_2/g_1, $\frac{\lambda_1}{\lambda_3}$, $\frac{\lambda_2}{\lambda_3}$ and the modulus τ. The parameter g_2/g_1 is complex in general, so we can write it as

$$
\frac{g_2}{g_1} = g e^{i\phi}, \tag{29}
$$

where ϕ is the relative phase of g_1 and g_2. The best fit values and 3σ ranges for the experimental results are summarized in Table II, in which the neutrino mass squared differences are defined as

TABLE II. 3 σ range for neutrino mixings and mass difference squares from [49] for inverted hierarchy.

	$\frac{\Delta m_{12}^2}{(10^{-5}\ \text{eV}^2)}$	$\frac{\lvert\Delta m_{23}^2\rvert}{(10^{-3}\ \text{eV}^2)}$	$r = \frac{\Delta m_{12}^2}{\lvert\Delta m_{23}^2\rvert}$	$\theta_{12}/^\circ$	$\theta_{23}/^\circ$	$\theta_{13}/^\circ$	δ_{CP}/π
Best fit	7.39	2.51	0.0294	33.82	49.8	8.6	1.57
3σ range	6.79–8.01	2.41–2.611	0.026–0.033	31.61–36.27	40.6–52.5	8.27–9.03	1.088–2

$$\Delta m_{12}^2 = m_2^2 - m_1^2, \qquad |\Delta m_{23}^2| = |m_3^2 - (m_2^2 + m_1^2)/2|.$$

The parameters are scanned in the upper half of the complex plane by fixing $r = \frac{\Delta m_{12}^2}{|\Delta m_{23}^2|}$ and the mixing angles with the 3σ ranges in Table II. The modulus τ is scanned in the ranges $\mathrm{Re}[\tau] \in [-0.5, 0.5]$ and $\mathrm{Im}[\tau] \in [0.4, 1]$, the coupling ratio g is scanned in the range $g \in [0.5, 3]$, and the phase $\phi \in [-\pi, \pi]$. We study the model in the case of normal and inverted hierarchies.

A. Normal hierarchy

For the normal hierarchy, we found the following benchmark: $\tau = -0.245 + 0.5236i$, $g = 2.503$, $\phi = -0.105\pi$, $\frac{\lambda_1}{\lambda_3} = 0.00031$, $\frac{\lambda_2}{\lambda_1} = 0.063$, with

$$r = \frac{\Delta m_{12}^2}{|\Delta m_{23}^2|} = 0.0293, \qquad \frac{m_e}{m_\tau} = 0.0003, \qquad \frac{m_\mu}{m_\tau} = 0.061,$$
$$\theta_{12} = 33.25°, \qquad \theta_{23} = 41.678°, \qquad \theta_{13} = 8.73°. \tag{30}$$

B. Inverted hierarchy

For inverted neutrino mass hierarchy, we found the following benchmarks:

(1) $\tau = -0.494 + 0.55i$, $g = 2.05$, $\phi = -\pi/2$, $\frac{\lambda_1}{\lambda_3} = 0.0009$, $\frac{\lambda_2}{\lambda_1} = 0.07$, with

$$r = \frac{\Delta m_{12}^2}{|\Delta m_{23}^2|} = 0.0286, \qquad \frac{m_e}{m_\tau} = 0.0003,$$
$$\frac{m_\mu}{m_\tau} = 0.061, \qquad \theta_{12} = 32.4°,$$
$$\theta_{23} = 49.26°, \qquad \theta_{13} = 8.54°. \tag{31}$$

(2) $\tau = 0.0962 + 0.984i$, $g = 2.05$, $\phi = -\pi/2$, $\frac{\lambda_1}{\lambda_3} = 0.0009$, $\frac{\lambda_2}{\lambda_1} = 0.07$, with

$$r = \frac{\Delta m_{12}^2}{|\Delta m_{23}^2|} = 0.0296, \qquad \frac{m_e}{m_\tau} = 0.0003,$$
$$\frac{m_\mu}{m_\tau} = 0.061, \qquad \theta_{12} = 32.36°,$$
$$\theta_{23} = 49.24°, \qquad \theta_{13} = 8.73°. \tag{32}$$

The two points $\tau = -0.494 + 0.55i$ and $\tau = 0.0962 + 0.984i$ are close to the fixed points $\tau_1 = -0.5 + 0.5i$ and $\tau_C = i$ respectively. The two fixed points are related to each other as $\tau_1 = ST\tau_C$. The modular group A_4 is broken to its subgroup $Z_2 = \{I, S\}$ at $\tau_C = i$ as neutrino and charged lepton mass matrices are invariant under the S transformation [35,36]. The lepton masses and mixing at τ_C are studied in [35,36] with the conclusion that τ_C cannot be used to lead to the correct lepton masses and mixing. The point $\tau_1 = -0.5 + 0.5i$ is invariant under ST^2ST transformation $\tau = \frac{-(1+\tau)}{1+2\tau}$ at which the group A_4 is broken to its subgroup $Z_2 = \{I, ST^2ST\}$. The charged lepton mass matrix M_e is invariant under unitary transformation $S_1 = ST^2ST$, $S_1^\dagger M_e S_1 = M_e$, where

$$S_1 = ST^2ST = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & -1 & 2\omega^2 \\ 2\omega^2 & 2\omega & -1 \end{pmatrix}. \tag{33}$$

One of the eigenvalues of M_e is zero since theueterment of M_e vanishes, so τ_1 cannot be used to lead to the correct charged lepton masses. The matrix M_ν is invariant under the transformation $S_1 = ST^2ST$ in Eq. (33). In this case, one of the eigenvalues of M_ν is zero and $\mathrm{Det}(M_\nu) = 0$. The mixing matrix in this case has two vanishing mixing angles and a nearly maximal angle. As we see, the observed lepton masses and mixing are consequences of breaking modular residual symmetry by deviation from τ_C or τ_1.

IV. QUARK MASSES

The embedding of the quark sector into a flavor model is a challenge due to the differences in the mass hierarchy and mixing for leptons and quarks. In this model, we extend the modular A_4 symmetry to the quark sector at the value of the modulus $\tau_1 = -0.494 + 0.55i$. All quarks transform as singlets under A_4. The assignments of the quark fields are shown in Table III.

The A_4 invariant superpotential for down quarks can be written as

$$w_d = \frac{h_{11}^d}{\Lambda^3} d_1^c H_d Q_1 \chi^3 + \frac{h_{22}^d}{\Lambda} d_2^c H_d Q_2 \chi + \frac{h_{23}^d}{\Lambda^2} Y_2^{(4)} d_3^c H_d Q_2 \chi^2 + h_{33}^d Y_1^{(4)} d_3^c H_d Q_3. \tag{34}$$

The chosen A_4 and k_I assignments prevent the other mixing terms. Without loss of generality, we assume that $h_{11}^d/h_{33}^d \sim h_{22}^d/h_{33}^d \sim 1/2$ and $h_{23}^d/h_{33}^d \sim 1$. The down quark mass matrix takes the form

$$m_d = h_{33}^d \langle H_d \rangle \begin{pmatrix} \lambda^3/2 & 0 & 0 \\ 0 & \lambda/2 & 0 \\ 0 & Y_2^{(4)}\lambda^2 & Y_1^{(4)} \end{pmatrix}. \tag{35}$$

TABLE III. Assignment of quarks under A_4 and the modular weight k_I.

Fields	Q_1	Q_2	Q_3	u_1^c	u_2^c	u_3^c	d_1^c	d_2^c	d_3^c
A_4	1	1′	1″	1″	1	1′	1	1″	1′
k_I	3	2	0	4	4	1	0	−1	4

To deal with left-handed mixing only, we construct the matrix $M_d = m_d^\dagger m_d$ which can be diagonalized by

$$V_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.490 - 0.869i & 0.0257 + 0.045i \\ 0 & 0.0524 & 0.9985 \end{pmatrix}, \quad (36)$$

with the corresponding eigenvalues

$$M_d = \mathrm{diag}(\lambda^4/2, \lambda^2/2, 1) h_{33}^d Y_1^{(4)} \langle H_d \rangle. \quad (37)$$

The hierarchical spectrum of the down quark masses is in a good agreement with the recent data for quark masses [50]:

$$m_d = 4.67^{+0.48}_{-0.17}\ \mathrm{MeV}, \qquad m_s = 93^{+11}_{-5}\ \mathrm{MeV},$$
$$m_b = 4.18^{+0.03}_{-0.02}\ \mathrm{GeV}.$$

For the up quarks, using the condition $Y_3^{(4)} = 0$, the invariant superpotential under modular A_4 can be written as

$$w_u = \frac{h_{11}^u}{\Lambda^3} Y_2^{(4)} u_1^c H_u Q_1 \chi^3 + \frac{h_{12}^u}{\Lambda^2} Y_1^{(4)} u_1^c H_u Q_2 \chi^2 + \frac{h_{21}^u}{\Lambda^3} Y_1^{(4)} u_2^c H_u Q_1 \chi^3 + h_{23}^u Y_2^{(4)} u_2^c H_u Q_3 + \frac{h_{33}^u}{\Lambda} u_3^c H_u Q_3 \chi. \quad (38)$$

The up quark mass matrix takes the form

$$m_u = \langle H_u \rangle \begin{pmatrix} h_{11}^u Y_2^{(4)} \lambda^3 & h_{12}^u Y_1^{(4)} \lambda^2 & 0 \\ h_{21}^u Y_1^{(4)} \lambda^3 & 0 & h_{23}^u Y_2^{(4)} \\ 0 & 0 & h_{33}^u \lambda \end{pmatrix}. \quad (39)$$

Assume that the couplings $h_{11}^u \sim h_{12}^u \sim h_{21}^u \sim h_{33}^u$, $h_{23}^u \sim 3h_{33}^u$, the Hermitian matrix $M_u = m_u^\dagger m_u$ is diagonalized by

$$V_u = \begin{pmatrix} -0.478 + 0.848i & -0.11 + 0.198i & 0.0017 - 0.0035i \\ -0.118 - 0.195i & 0.504 + 0.83i & 1.5 \times 10^{-7} \\ 0.00349 & 0.0008245 & 0.999942 \end{pmatrix}, \quad (40)$$

with the corresponding eigenvalues

$$M_u^{\mathrm{diag}} = h_{33}^u \langle H_u \rangle Y_1^{(4)} \mathrm{diag}(\lambda^7, \lambda^3, 1), \quad (41)$$

which are in agreement with the up quark mass ratios [50]

$$\frac{m_u}{m_t} = 0.000012, \qquad \frac{m_c}{m_t} = 0.008.$$

The Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix V_{CKM} takes the form

$$|V_{\mathrm{CKM}}| = |V_u^\dagger V_d| = \begin{pmatrix} 0.9737 & 0.224 & 0.006 \\ 0.224 & 0.9723 & 0.05 \\ 0.005 & 0.05 & 0.9986 \end{pmatrix}, \quad (42)$$

which is close to the correct V_{CKM} [50]. The same result of quark masses and mixing can be obtained at $\tau = 0.0964 + 0.984i$ while the observed masses and mixing are not achieved at $\tau = 0.2525 + 0.526i$ with the above quark model.

V. CONCLUSION

We built an A_4 modular invariance model to account for both lepton and quark masses and mixing. The model is free from large number of flavons or extra symmetries like Z_N symmetries which were considered in many models based on the flavor symmetry. The neutrino masses are obtained via the inverse seesaw mechanism. The predicted lepton mixing and mass ratios are compatible with the recent data. The neutrino mass square difference ratios and lepton mixing angles are determined in terms of the coupling ratio g_2/g_1 and the modulus τ at values near fixed points for the inverted hierarchy scenario. For the same value of $\tau = -0.494 + 0.55i$, we extend the modular A_4 symmetry to the quark sector. The calculated quark mass ratios and mixing are in quite agreement with the experimental results.

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