

POISSON AND INTEGRABLE SYSTEMS THROUGH THE NAMBU BRACKET AND ITS JACOBI MULTIPLIER

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ABSTRACT. Poisson and integrable systems are orbitally equivalent through the Nambu bracket. Namely, we show that every completely integrable system of differential equations may be expressed into the Poisson-Hamiltonian formalism by means of the Nambu-Hamilton equations of motion and a reparametrisation related by the Jacobian multiplier. The equations of motion provide a natural way for finding the Jacobian multiplier. As a consequence, we partially give an alternative proof of a recent theorem in [13]. We complete this work presenting some features associated to Hamiltonian maximally superintegrable systems.

1. Introduction and main statement. In this note we connect Nambu-Poisson structures, integrable systems and Jacobian multipliers showing that every integrable system is expressed as a set of Nambu-Poisson equations of motion after a reparametrisation of the independent variable. The interested reader could see Appendix A for a brief review of the Nambu formalism. For our own surprise, we found that the reformulation of an integrable system in the Nambu-Poisson scenario is carried out in a straightforward way, since it is obtained by computing a determinant of the gradients of the associated first integrals. Furthermore, we obtain an extra benefit out of this process, the computation of the Nambu-Poisson equations provides a vector field proportional to the vector field associated to the integrable system and the ratio function relating both of them is the Jacobian multiplier, which is obtained without extra computations.

Before going into details we set terminology and notation. Along this note we consider an integrable C^k autonomous system of ordinary differential equations of first order given by

$$\dot{\omega} = f(\omega), \quad \omega \in A \subset \mathbb{R}^N, \quad (1)$$

where the dot denotes derivative with respect to the independent variable s , A is an open subset of \mathbb{R}^N , and $f(\omega) = (f_1(\omega), \dots, f_N(\omega)) \in C^\infty(A)$. We precise that by *completely integrable system* we mean a system having $N - 1$ functionally

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independent first integrals. Observe that versus other papers related to integrability, we do not make any assumption on the functional type of integrals (Darboux, polynomial, etc.) and we are not concern on the search of the named integrals [7]. To conclude this paragraph devoted to set terminology, we recall the Jacobi multiplier definition. Let \mathcal{X} denotes the vector field associated to system (1) and $\operatorname{div} \mathcal{X} = \operatorname{div} f$ denotes the divergence of the vector field \mathcal{X} or f . Thus, $\mathcal{X}(\phi)$ gives the variation of a function ϕ along the flow of system (1): $\mathcal{X}(\phi) = f \cdot \nabla \phi$, and the following relation holds for any $f \in \mathcal{C}^\infty(A)$, where $A \subset \mathbb{R}^N$

$$\operatorname{div}(Jf) = \mathcal{X}(J) + J \operatorname{div}(f). \quad (2)$$

A function J of class C^1 is called a *Jacobian multiplier* of system (1) if it is defined in a full Lebesgue measure subset $A^* \subset A$, and satisfies

$$\operatorname{div}(Jf) \equiv 0, \quad \text{i.e.,} \quad \mathcal{X}(J) = -J \operatorname{div} \mathcal{X}. \quad (3)$$

A Jacobian multiplier is an integrating factor for the two dimensional case of system (1). In addition, it plays a central role in the linearisation of completely integrable systems proposed in [13]. Due to the connection with this paper we maintain the notation given there for the benefit of the reader.

The following result is the main contribution of this note. The proof is provided in Section 2.

Theorem 1.1. *Let us consider the completely integrable system (1) provided with the set of integrals $\mathcal{I} = \{I_1, \dots, I_{N-1}\}$. Then the following claims hold:*

- (i) *System (1) is expressed into the Nambu formalism in a full Lebesgue measure subset of A after a reparametrisation of the independent variable. Therefore, because of the Nambu nested structures, (1) is expressed as a Poisson system, i.e., integrable and Poisson systems are orbitally equivalent. More precisely, the independent variable s is replaced by τ by means of the relation*

$$\frac{ds}{d\tau} = \lambda_{\mathcal{I}}(\omega), \quad (4)$$

the function $\lambda_{\mathcal{I}}$ is explicitly constructed in the proof of the theorem and it depends on the integrals and the order in which we pick them.

- (ii) *If $\operatorname{div}(f) \neq 0$ we have $\lambda_{\mathcal{I}}$ is a Jacobi multiplier. Otherwise, $\lambda_{\mathcal{I}}$ is a first integral of (1).*
- (iii) *There are just $N - 1$ functionally independent Jacobi multipliers.*

Recently in [23] the authors managed to realise the integrable Rössler system (see [21]) as a Poisson-Hamiltonian system. Here we show that this task could be carried out generically for every completely integrable system. Hamiltonian formalism is usually formulated on a symplectic manifold environment. Nevertheless, this approach is generalised to the case of a Poisson manifold by keeping just enough of the properties of Poisson brackets to describe Hamiltonian systems [15]. It gives place to the extension of the notion of a Hamiltonian vector field from symplectic to Poisson context. Namely, given a Poisson bracket $\{\cdot, \cdot\}$, the triplet $(M, \{\cdot, \cdot\}, \mathcal{H})$ is called a Poisson dynamical system and the Hamilton's equations of motion read as follows

$$\dot{\omega}_i = \{\omega_i, \mathcal{H}\}.$$

In order to define the appropriate Poisson brackets that allows us to express an integrable system as a realisation of the Hamilton's equations, we consider the standard Nambu structure in \mathbb{R}^N and we show in part (i) of Theorem 1.1 that the

equations of system (1) are obtained, up to a reparametrisation of the independent variable, as the Nambu-Hamilton equations of motion, where the $N - 1$ associated Hamiltonians are given by the first integrals of (1). This fact plays a key role in the Poisson formulation of system (1) given in Theorem 1.1, since the existence of, at least, $N - 1$ first integrals allows to define a Poisson structure in which one of them plays the role of the Hamiltonian function and the others are the Casimirs. This process generates automatically, and without extra computation, the Jacobian multiplier (a generalisation of an integrating factor) associated to the original system of differential equation (1). Therefore, we partially obtain in Theorem 1.1, part (ii), an alternative proof of a recent theorem in [13], that provides a linearisation of completely integrable systems. Note that in the case of considering several sets of functionally independent integrals, the theorem leaves open which one might be more convenient in each application, since the corresponding Jacobian multiplier defines the reparametrisation given in (4).

For the completeness of this work we recall in the Appendix A the Nambu formalism introduced in [19]. It is a generalisation of classical Hamiltonian dynamics, which initially was tailored for the case of the rigid body system keeping in mind the importance of the conservation of the Liouville theorem. Two decades after the original paper of Nambu was extended to the $N \geq 3$ case in [22] and [2] shows that other classical system, as the Kepler or the harmonic oscillator, fit into the Nambu formalism. More examples and several applications can be found in [14, 16, 20, 3]. The Nambu brackets has been related to the treatment of constrained Hamiltonian systems in previous works [5, 1, 18, 10]. Furthermore, the Marsden-Ratiu geometrical reduction for Poisson manifolds was also extended to Nambu-Poisson manifolds [12].

The note is organised as follows. In Section 2 we give the proofs of the main results summarised above and in Section 3 we study several applications illustrating the geometrical interpretation of the Nambu-Hamilton equations and showing how the results may be applied in finding the Jacobian multiplier of the Rössler system or expressing several integrable system in the Nambu formalism. Finally Appendix A review some of the basic features of the Nambu structures.

2. Integrable systems, their Poisson-Hamiltonian structure and Jacobian multipliers. The following Theorem 1.1 contains the main results of this work, it shows in a constructive way that every integrable system is orbitally equivalent to a Poisson one. Conditions to guarantee that an ordinary differential equation may be written exactly (without reparametrisation) in the Nambu form are given in [17]. We also provide the Jacobi multiplier in case the original system has non zero divergence, which in addition is an alternative for the claim (a) in Theorem 2 of [13]. For an application example see Section 3.2.

Before we proceed with the proof of Theorem 1.1, we provide a Lemma giving some properties relative to the Jacobian multipliers.

Lemma 2.1. *Let the system (1) such that $\text{div}(f) \neq 0$. Then, the following claims hold*

- (i) *If J_1 and J_2 are non vanishing Jacobian multipliers. Then, the quotient J_1/J_2 is a first integral.*
- (ii) *Let K be a first integral and let J be a Jacobian multiplier. Then, KJ is also a Jacobian multiplier.*

Proof. (i) Let J_1 and J_2 two Jacobian multipliers for system (1). Thus, both of them satisfy the relation given in (3). After an easy algebraic manipulation making use of (3) we obtain that

$$\frac{\mathcal{X}(J_1)}{J_1} = \frac{\mathcal{X}(J_2)}{J_2} = -\text{div}(f), \quad (5)$$

hence

$$\mathcal{X}(\ln(J_1)) = \mathcal{X}(\ln(J_2)), \quad (6)$$

which implies that $\ln(J_1)$ and $\ln(J_2)$ differs in a function \mathcal{K} satisfying that $\mathcal{X}(\mathcal{K}) = 0$. That is to say, a first integral of the system (1). Then, we can write

$$\ln(J_1) = \ln(J_2) + \mathcal{K}, \quad (7)$$

and finally J_1 and J_2 are related as follows

$$J_1 = J_2 e^{\mathcal{K}}. \quad (8)$$

(ii) Let J be a Jacobian multiplier and K a first integral. Then, relation (2) yields as follows

$$\text{div}(KJf) = \mathcal{X}(KJ) + KJ\text{div}(f) = K\mathcal{X}(J) + J\mathcal{X}(K) + KJ\text{div}(f),$$

since K is an integral we have that $\mathcal{X}(K) = 0$, which leads to

$$\text{div}((KJ)f) = K\mathcal{X}(J) + KJ\text{div}(f) = K(\mathcal{X}(J) + J\text{div}(f)) = 0.$$

Therefore, KJ is a Jacobian multiplier. \square

Next we give the proof of Theorem 1.1.

Proof. (i) Let us consider the standard Nambu structure $\{\dots\}_{\text{Nambu}}$ on \mathbb{R}^N . As a first step, we are going to show that the Nambu-Hamilton equations of motion with Hamiltonians given by the set of integrals $\mathcal{I} = \{I_1, \dots, I_{N-1}\}$ are essentially the original system. That is to say, the following system of differential equations

$$\dot{\omega}_i = \{\omega_i, I_1, \dots, I_{N-1}\}_{\text{Nambu}} = \det(\nabla\omega_i, \nabla I_1, \dots, \nabla I_{N-1}), \quad (9)$$

and system (1) are orbitally equivalent. For this purpose, let us write (9) in the more compact way

$$\dot{\omega} = h(\omega), \quad (10)$$

where $h_i(\omega) = \{\omega_i, I_1, \dots, I_{N-1}\}$. Thus, since I_1, \dots, I_{N-1} are integrals of systems (1) and (10), we have that solutions of both systems satisfy the identity

$$\nabla I_i(\omega) \dot{\omega} = 0,$$

which implies that $f(\omega)$ and $h(\omega)$ satisfy

$$\nabla I_i(\omega) f(\omega) = 0 = \nabla I_i(\omega) h(\omega),$$

for $i = 1, \dots, N-1$. These relations imply that $f(\omega)$ and $h(\omega)$ belong to the orthogonal complement of the space generated by the $N-1$ independent vectors

$$\nabla \mathcal{I} = \{\nabla I_1(\omega), \dots, \nabla I_{N-1}(\omega)\}.$$

Therefore, h and f are collinear in the subset $A^* \subseteq A$ of full Lebesgue measure where $\text{rank}(\nabla \mathcal{I}) = N-1$ and $f(\omega) \neq 0 \neq h(\omega)$. That is to say,

$$f(\omega) = \lambda_{\mathcal{I}}(\omega)h(\omega) \quad (11)$$

and $\lambda_{\mathcal{I}}(\omega) \neq 0$ for $\omega \in A^*$. It implies that systems (1) and (10) are orbitally equivalent by mean of the reparametrisation of the independent variable s in (1) given by

$$\frac{ds}{d\tau} = \lambda_{\mathcal{I}}(\omega). \quad (12)$$

Finally we show that (1) is expressed, up to the above reparametrisation, into the Poisson-Hamiltonian formalism. Without lost of generality, let us define the Poisson bracket $\{ , \}_{\mathcal{I}}$ obtained after fixing the first $N - 2$ integrals of the set \mathcal{I} . Namely,

$$\{ , \}_{\mathcal{I}} : \mathcal{F}(\mathbb{R}^N) \otimes \mathcal{F}(\mathbb{R}^N) \longrightarrow \mathcal{F}(\mathbb{R}^N),$$

which is given by

$$\{F, G\}_{\mathcal{I}} = \{F, G, I_1, \dots, I_{N-2}\}_{Nambu}, \quad \forall F, G \in \mathcal{F}(\mathbb{R}^N).$$

We consider $\mathcal{H} = I_{N-1}$ as the “Hamiltonian function”. Thus, we obtain the following Hamiltonian equations of motion

$$\dot{\omega}_i = \{\omega_i, \mathcal{H}\}_{\mathcal{I}}, \quad (13)$$

which correspond with the Nambu-Hamilton equations (10) and the reparametrised system (1).

(ii) Let us consider the case $\text{div}(f) \neq 0$. Then, in the preceding part of this theorem we show that system (1) is orbitally equivalent to system (10). That is to say, $f(\omega) = \lambda_{\mathcal{I}}(\omega)h(\omega)$ and taking into account that h is divergenceless, $\lambda_{\mathcal{I}} \neq 0$ by construction and $\text{div}((\lambda_{\mathcal{I}}(\omega))^{-1}f) = \text{div}(h) = 0$, we have that $J = (\lambda_{\mathcal{I}}(\omega))^{-1}$ is the Jacobian multiplier. For the case in which system (1) is such that $\text{div}(f) = 0$, we have also that the function J satisfies $\text{div}(Jf) = 0$. Thus, by using relation (2) we obtain

$$\text{div}(Jf) = \mathcal{X}(J) + J \text{div}(f),$$

which in this case reduces to $\mathcal{X}(J) = 0$ and J is a first integral of (1).

(iii) Is a direct consequence of Lemma 2.1. \square

3. Applications. The applications puts into light several aspect of the Nambu dynamical approach. In Section 3.1, we illustrate the geometric interpretation of the Nambu-Hamilton equations of motion. In part 3.2 we express a completely integrable system into the Poisson formalism (up to reparametrisation) and also gives and example of how the Jacobian multiplier is obtained through the Nambu approach. Finally in Section 3.3 we present some basic features associated to Hamiltonian maximally superintegrable systems.

3.1. The N -extended Euler system. Given $N - 1$ hyper-manifolds in \mathbb{R}^N , the Nambu-Hamilton equations of motion are interpreted as a parametrisation of their intersection curves. The N -extended Euler system illustrates this geometric interpretation. Let us consider the trajectory \mathcal{C} described by a particle constrained to be in the intersection of the following hyper-quadrics in \mathbb{R}^N

$$C_{ij}(v) = \frac{1}{2}(\alpha_i \omega_j(v)^2 - \alpha_j \omega_i(v)^2). \quad (14)$$

This is a elliptic curve in the 3-dimensional case. Generally, hyper-elliptic for the N dimensional case. The set of hyper-quadrics given by $C_1 = \{C_{11}, \dots, C_{1N-1}\}$

is functionally independent and their intersection gives \mathcal{C} . Therefore, the hyperelliptic curves are the trajectories of the following Nambu-Hamilton equations of motion

$$\dot{\omega}_i = \{C_{11}, \dots, C_{1N-1}, \omega_i\}, \quad i = 1, \dots, N, \quad (15)$$

After some straightforward computations, these equations are given in the following way

$$\frac{d\omega_i}{dv} = \alpha_i \prod_{j \neq i} \omega_j, \quad (1 \leq i, j \leq N). \quad (16)$$

This system of differential equations is named as the N -extended Euler system (N -EES) and has been studied before in [6, 8]. We have shown above that the N -EES is precisely a set of Nambu-Hamilton equations of motion and therefore it is also a Poisson-Hamiltonian system.

For the case in which $\sum \alpha_i = 0$ and $N = 3$ we have the Euler system for the rigid body. It was first expressed into the Nambu formalism in [19]. One of the features of the system (16) is that it allows, from a dynamical system point of view, dealing with a large family of functions in the real domain in a unified way. It ranges from trigonometric functions (harmonic oscillator) to elliptic functions (pendulum and free rigid body), including also rational functions (for unbounded trajectories), etc.

3.2. The completely integrable Rössler system. In the recent paper of Llibre, Valls and Zhang, [13] it is shown that a completely integrable system of differential equations is orbitally equivalent to the linear differential system $\dot{\omega} = \omega$. This linearisation rely on the construction of the corresponding Jacobian multiplier. We give an alternative way of finding the Jacobian multiplier that we apply to the Rössler system, which is the only completely integrable system of differential equations constructed by Rössler [21] and was also studied in [13] as an application example. This system is given by

$$\dot{x} = -y - z, \quad \dot{y} = x, \quad \dot{z} = xz. \quad (17)$$

In the cited paper [13], the authors give the linearising change of variables for system (17) by means of the following pair of integrals

$$H_1 = \frac{1}{2}(x^2 + y^2) + z, \quad H_2 = e^{-y}z, \quad (18)$$

and the Jacobian multiplier $J = e^{-y}$. More precisely, the named change of variables reads as follows

$$y_1 = J H_1, \quad y_2 = J H_2, \quad y_3 = J,$$

which leads to the linear system $\dot{y}_i = (\dot{J}/J)y_i$, for $i = 1, 2, 3$.

We obtain here J in an alternative way. Since H_1 and H_2 are functionally independent, system (17) is orbitally equivalent to the following Nambu-Hamilton equations of motion

$$\dot{x} = \{H_1, H_2, x\}, \quad \dot{y} = \{H_1, H_2, y\}, \quad \dot{z} = \{H_1, H_2, z\}. \quad (19)$$

where $\{, \}$ is the standard Nambu bracket in \mathbb{R}^3 . Thus, taking into account that

$$\nabla H_1 = (x, y, 1), \quad \nabla H_2 = (0, -e^{-y}yz, e^{-y}),$$

we have that the Nambu-Hamilton equations of motion (29) are given by

$$\dot{x} = e^{-y}(-y - z), \quad \dot{y} = e^{-y}x, \quad \dot{z} = e^{-y}xz. \quad (20)$$

Thus, since the vector field associated to system (20) is divergenceless, we have that e^{-y} is the Jacobian multiplier for (17).

3.3. The harmonic oscillator. The aim of this example is to show that a Hamiltonian maximally superintegrable system can be expressed in the Hamiltonian-Poisson formalism in several different ways. That is to say, there are several Poisson brackets and several candidates for the Hamiltonian functions that lead to the original system up to the independent variable reparametrisation. Furthermore, the named reparametrisations are always given by first integrals. For this purpose, we study the planar harmonic oscillator in resonance 1:1. The phase space is \mathbb{R}^4 and the Hamiltonian function is given by

$$\mathcal{H}_O = \frac{1}{2}(|p|^2 + |q|^2). \quad (21)$$

There are 2^2 quadratic polynomials in the variables (q, p) , where $p, q \in \mathbb{R}^2$, that generate the space of functions invariant with respect to the action given by the flow of H_O

$$\begin{aligned} \pi_1 &= p_1^2 + q_1^2, & \pi_3 &= q_1 q_2 + p_1 p_2, \\ \pi_2 &= p_2^2 + q_2^2, & \pi_4 &= q_1 p_2 - q_2 p_1. \end{aligned} \quad (22)$$

The following are some examples (without the aim of being exhaustive) of the Nambu-Hamilton equations of motion for different choices of the integrals. Namely, the Nambu-Hamilton equations read as follows

$$\begin{aligned} \dot{q}_i &= \{q_i, \pi_j, \pi_k, \pi_l\}, \\ \dot{p}_i &= \{p_i, \pi_j, \pi_k, \pi_l\}, \end{aligned} \quad (23)$$

and for example, let $j = 1, k = 2, l = 4$. Then, we obtain

$$\begin{aligned} \dot{q}_1 &= \{q_1, \pi_1, \pi_2, \pi_4\} = (2\pi_4) p_1, \\ \dot{q}_2 &= \{q_2, \pi_1, \pi_2, \pi_4\} = (2\pi_4) p_2, \\ \dot{p}_1 &= \{p_1, \pi_1, \pi_2, \pi_4\} = -(2\pi_4) q_1, \\ \dot{p}_2 &= \{p_2, \pi_1, \pi_2, \pi_4\} = -(2\pi_4) q_2. \end{aligned} \quad (24)$$

Note that by fixing two of the integrals $\{\pi_1, \pi_2, \pi_4\}$ several Poisson brackets are obtained

$$\{F, G\}_{12} = \{F, \pi_1, \pi_2, G\}, \quad \{F, G\}_{14} = \{F, \pi_1, G, \pi_4\}, \quad \{F, G\}_{24} = \{F, G, \pi_2, \pi_4\},$$

all of them lead to exactly the same equations of motion (24) when we choose the Hamiltonian function to be π_4 , π_2 or π_1 respectively. Other choices produce proportional equations, for example $j = 1, k = 3, l = 4$ leads to

$$\begin{aligned} \dot{q}_1 &= \{q_1, \pi_1, \pi_3, \pi_4\} = (4\pi_2) p_1, \\ \dot{q}_2 &= \{q_2, \pi_1, \pi_3, \pi_4\} = (4\pi_2) p_2, \\ \dot{p}_1 &= \{p_1, \pi_1, \pi_3, \pi_4\} = -(4\pi_2) q_1, \\ \dot{p}_2 &= \{p_2, \pi_1, \pi_3, \pi_4\} = -(4\pi_2) q_2. \end{aligned} \quad (25)$$

Therefore, we obtain a time-reparametrisation of the Hamilton equations of the harmonic oscillator for each choice of the integrals π_i . In addition, the function relating the independent variables of the pure oscillator system and every example given above is a first integral.

Appendix A. A précis on generalized Nambu dynamics. With the aim of making this work as self contained as possible, we review here, without giving a proof, some of the basics concepts and facts relative to Nambu dynamics. All of them and further details may be found in [19, 22, 24] and the references therein.

In 1973 Yoichiro Nambu introduced a generalization of the Hamiltonian dynamics by introducing a new bracket called the Nambu bracket [19]. This new structure generalises the Poisson bracket and enables to define the Nambu-Poisson manifolds (see [22]). Here we recall some of the basic ideas about this generalization.

Hamiltonian dynamics takes place on a Poisson manifold, *i.e.*, a pair made of a smooth manifold M endowed with a bilinear operation $\{ , \}$ on $\mathcal{F}(M) = C^\infty(M)$ satisfying skew-symmetry, the Jacobi identity and the Leibniz rule. Nambu's generalization hinges on the introduction of the Nambu-Poisson manifold of order N . It is a smooth manifold M endowed with a N -multilinear operation on $\mathcal{F}(M)$ called the *Nambu bracket of order N* , satisfying the following properties.

- (i): $\{ , \} : \mathcal{F}(M)^{\otimes N} \rightarrow \mathcal{F}(M)$ is a N -multilinear operation.
- (ii): Skew-symmetry, $\{f_1, \dots, f_N\} = \epsilon(i_1, \dots, i_N)\{f_{i_1}, \dots, f_{i_N}\}$. Where ϵ is the N -dimensional Levi-Civita symbol.
- (iii): The fundamental identity holds

$$\begin{aligned} & \{ \{f_1, \dots, f_N\}, f_{N+1}, \dots, f_{2N-1} \} + \{f_N, \{f_1, \dots, f_{N-1}, f_{N+1}\}, f_{N+2}, \dots, f_{2N-1} \} + \\ & \dots + \{f_N, \dots, f_{2N-2}, \{f_1, \dots, f_{N-1}, f_{2N-1}\} \} = \{f_1, \dots, f_{N-1}, \{f_N, \dots, f_{2N-1}\}\}. \end{aligned}$$

- (iv): $\{ , \}$ Leibniz rule is satisfied in each factor

$$\{f_1 f_2, \dots, f_{N+1}\} = f_1 \{f_2, \dots, f_{N+1}\} + f_2 \{f_1, \dots, f_{N+1}\}, \quad \forall f_1, \dots, f_{N+1} \in \mathcal{F}(M).$$

According to the above properties, Nambu bracket may be rendered in a geometrical sense as a smooth section of the vector space of exterior N -forms on the tangent bundle of M , *i.e.*, $\bigwedge^N TM$. In other words, the Nambu bracket is realised as the N -contravariant tensor

$$\{f_1, \dots, f_N\} \stackrel{\text{def}}{=} \beta(df_1, \dots, df_N), \quad (26)$$

where $\beta \in \bigwedge^N TM$ is called the Nambu tensor. It is given, in local coordinates $x = (\omega_1, \dots, \omega_N)$, by the following expression

$$\beta = \sum_{i_1, \dots, i_N=1}^N \beta_{i_1, \dots, i_N}(x) \frac{\partial}{\partial \omega_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial \omega_{i_N}}, \quad (27)$$

Notice that β may be expressed, for suitable local coordinates, by $\beta_{i_1, \dots, i_N}(x) = \epsilon(i_1, \dots, i_N)$, where ϵ is the Levi-Civita tensor. In what follows, we name the N -contravariant tensor given by Levi-Civita tensor as the *standard Nambu bracket* on $\mathcal{F}(M)$, which will be denoted by the usual bracket $\{ , \}$. That is to say, the standard Nambu bracket is given by the determinant of the gradients of the functions involved.

Even though Nambu is a generalisation of Hamiltonian dynamics, there are also fundamental differences between them. For example, in [11, 9], it is proven that every Nambu-Poisson bracket with $N \geq 3$ is in essentially a determinant. This is not true for Poisson ones.

This bracket allows to study the variation of $f \in \mathcal{F}(M)$ on Nambu-Poisson manifolds when it is restricted to be in the intersection of $N - 1$ hyper-manifolds

H_i

$$\frac{df}{dt} = \{f, H_1, \dots, H_{N-1}\}. \quad (28)$$

In this vein, the Nambu formulation has been applied to Hamiltonian systems with constraints (see [10] and the references therein). It is straightforward to extend the above formalism to the dynamics of points $\omega = (\omega_1, \dots, \omega_N)$ in the phase space M by means of the Nambu-Hamilton equations of motion as they were first given in [19]

$$\frac{d\omega_i}{dt} = \{\omega_i, H_1, \dots, H_{N-1}\} = \sum_{i_1 \dots i_{N-1} \neq i} \epsilon(i, i_1, \dots, i_{N-1}) \frac{\partial H_1}{\partial \omega_{i_1}} \dots \frac{\partial H_{N-1}}{\partial \omega_{i_{N-1}}}, \quad (29)$$

where H_i are called the Hamiltonian functions.

Next we gather below some basic features of the Nambu structures, which will be of high relevance in the subsequent development. All of them, except the Remark 1, can be found in [22].

Theorem A.1 (Nambu nested structure). *The set of all possible Nambu structures on M is isomorphic to the Grassmann algebra $\bigwedge TM$. Since it is a graded associative algebra, every Nambu structure is considered as an N -degree element of $\bigwedge TM$ and by fixing f_1, \dots, f_k in (26), with $k \leq N-2$ we are left with a Nambu structure of order $N-k$. More precisely for the case $k = N-2$ the Nambu structure obtained is a Poisson structure and the fixed integrals f_1, \dots, f_{N-2} are the Casimirs.*

Theorem A.2 ($SL(N, \mathbb{R})$ Nambu bracket invariance). *The Nambu bracket is invariant under the action (left or right) of the special linear group $SL(N, \mathbb{R})$. That is to say, let ϕ be the action of $SL(N, \mathbb{R})$ on $\mathcal{F}(M)^{\otimes N}$ given by*

$$\phi : SL(N, \mathbb{R}) \times \mathcal{F}(M)^{\otimes N} \longrightarrow \mathcal{F}(M)^{\otimes N}, (A, F) \mapsto F', \quad (30)$$

where $A \in SL(N, \mathbb{R})$ and $F, F' \in \mathcal{F}(M)^{\otimes N}$ are the N -tuples given by $F = (f_1, \dots, f_N)$ and $F' = AF = (f'_1, \dots, f'_N)$. Thus, the following identity holds

$$\{f_1, \dots, f_N\} = \{f'_1, \dots, f'_N\} \quad (31)$$

Theorem A.3 (Liouville Condition). *The corresponding phase flow on the phase space of the Nambu-Hamilton equations of motion is divergence-free and preserves the standard volume form $d\omega_1 \wedge \dots \wedge d\omega_N$.*

However, the reciprocal of Theorem A.3 is not true, *i.e.*, divergence-free systems can be written into the Nambu formalism. Such a statement was made in [9], but the error is shown in [4], see [17] for further details.

Remark 1 (Geometric interpretation). Let us consider \mathbb{R}^N together with the standard Nambu bracket and $N-1$ hyper-manifolds $M_i^{h_i}$ given by the level sets $H_i = h_i$ of the functions $H_i \in \mathcal{F}(M)$ for $i = 1, \dots, N-1$. Then, the Nambu-Hamilton equations of motion given in (29) may be interpreted as a parametrisation of the intersection curves resulting from $\bigcap_{i=1}^{N-1} M_i^{h_i}$.

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