Low Energy Unimodular Gravity

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SERGIO GONZÁLEZ MARTÍN

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Tesis Doctoral dirigida por

ENRIQUE ÁLVAREZ VÁZQUEZ

Catedrático de Física Teórica
de la Universidad Autónoma de Madrid

Cincuenta Aniversario 1968–2018 UAM Universidad Autónoma de Madrid

Instituto de Física Teórica UAM-CSIC

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Publications

During the developing of the Ph.D. several themes were studied and the followings papers were published, some of them unrelated to the topic of this thesis

- Enrique Álvarez, Sergio González-Martin, and Mario Herrero-Valea. "Some Cosmological Consequences of Weyl Invariance". In: *JCAP* 1503.03 (2015), p. 035. DOI: 10.1088/1475-7516/2015/03/035. arXiv: 1501.07819 [hep-th].
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ABSTRACT

It has been known for a century that the theory of General Relativity proposed by Albert Einstein describes accurately gravitational phenomena at low energies. This theory, when considered classically, is capable of reproducing the right results for the tests carried on the Solar system, as well as many other cosmological observations. However, there are several problems when we try to compute quantum effects. Specifically, one of the biggest mysteries of today's theoretical physics is the so-called *cosmological constant problem*. This problem is sometimes split in two, on the one hand there is the matter of why it has its particular value (very close to zero but not exactly). On the other hand, quantum corrections to this constant are about 120 orders of magnitude bigger than the observed one.

Although there are several proposals to solve this problem, some of them by studying the non-perturbative sector of the theory, there is a natural solution to the "second" problem within perturbation theory. In this regard, the theory of Unimodular Gravity provides a natural solution to the cosmological constant problem because in this theory the cosmological constant appears as an integration constant at the classical level. Moreover, this theory is compatible with all the current observations. In this work we demonstrate that there are no quantum corrections to first order in perturbation theory to the cosmological constant value, and we argue that this result holds to any loop order as long as the Ward identities are respected.

Further, since Unimodular Gravity yields the same classical predictions than General Relativity, we look for differences between both theories at the quantum level by adding matter. As part of this study, we show that (for both theories) the beta functions, despite being used as the motivation for proposals such as Asymptotic Safety, lack any physical meaning when matter is coupled to gravity.

RESUMEN

Sabemos que la teoría de la Relatividad General propuesta por Albert Einstein describe correctamente los fenómenos gravitatorios a bajas energías. Esta teoría, cuando se trata clásicamente, es capaz de calcular correctamente los test llevados a cabo en el sistema solar, así como muchas otras observaciones cosmológicas. Aparecen sin embargo varios problemas cuando se intenta implementar el formalismo de una Teoría Cuántica de Campos. En particular, uno de los mayores misterios en la física teórica actual es el llamado *problema de la constante cosmológica*. Éste puede dividirse en dos; por un lado, está el problema de explicar por qué tiene un valor tan particular (muy próximo a cero) si no hay ninguna simetría que lo fuerce a ser distinto de cero. Por otro, las correcciones cuánticas a esta constante cuando se añade materia, predicen un valor 120 órdenes de magnitud mayor que el valor observado.

Aunque las soluciones propuestas son varias, muchas de ellas estudiando el sector no perturbativo, existe una solución natural al "segundo" problema dentro de la teoría de perturbaciones. En la teoría de Gravedad Unimodular la constante cosmológica aparece a nivel clásico únicamente como una constante de integración. Esta teoría es además compatible con todas las observaciones actuales. En este trabajo demostramos explícitamente que no hay correcciones al valor de la constante cosmológica a primer orden en teoría de perturbaciones y argumentamos que, debido a la simetría Weyl que presenta la teoría, esto se cumple para todos los órdenes siempre que no haya una anomalía Weyl.

Además, puesto que la teoría de Gravedad Unimodular da las mismas predicciones que la Relatividad General a nivel clásico, buscamos diferencias a nivel cuántico mediante el estudio del acoplo de materia a la teoría. Como parte de este estudio, demostramos que (en ambas teorías) las funciones beta, a pesar de ser utilizadas como pilar fundamental en las propuestas de Seguridad Asintótica, carecen de significado físico cuando se acopla materia a gravedad.

Contents

Abstract . . . viii
Resumen . . . ix

I INTRODUCTION

II PRELIMINARIES

1 NOTATION AND CONVENTIONS 9

2 THE WHAT AND THE WHY OF UNIMODULAR GRAVITY 11

3 ABOUT THE GAUGE SYMMETRIES OF UNIMODULAR GRAVITY 17

3.1 Tdiff invariance of the unimodular action . . . 19

4 HEAT KERNEL IN A NUTSHELL 25

4.1 Background field method . . . 25
4.2 Short time expansion . . . 29
4.2.1 Heat Kernel coefficients of a Laplace-type operator 32

III ONE-LOOP QUANTUM CORRECTIONS

5 ONE-LOOP QUANTUM CORRECTIONS 37

5.1 BRST quantization of Unimodular Gravity . . . 37
5.1.1 Fixing the gauge freedom . . . 40
5.1.2 The TDiff sector . . . 43
5.1.3 The Weyl sector . . . 47
5.2 The one-loop effective action of Unimodular Gravity . . . 48
5.2.1 The Barvinsky-Vilkovisky technique . . . 51
5.2.2 The final result . . . 55

IV UNIMODULAR GRAVITY VERSUS GENERAL RELATIVITY

6 GRAVITATIONAL CORRECTIONS TO THE BETA FUNCTION 61

6.1 The setting . . . 63

6.2 Beta functions 66
6.3 Final discussion 72
7 S MATRIX CORRECTIONS 75
7.1 UV divergent contributions to $\lambda\phi^4$ 75
7.1.1 Gravity coupled to $\lambda\phi^4$ 76
7.1.1.1 General Relativity coupled to $\lambda\phi^4$ 76
7.1.1.2 Unimodular Gravity coupled to $\lambda\phi^4$ . . . 77
7.1.2 The $\phi + \phi \to \phi + \phi$ scattering at one-loop and at order $\kappa^2 m_\phi^2$ 79
7.1.2.1 General Relativity contributions 79
7.1.2.2 Unimodular Gravity contributions . . . 83
7.2 Scattering of fermions in the Yukawa theory 84
7.2.1 Yukawa theory coupled to Gravity 84
7.2.1.1 Yukawa theory coupled to General Relativity 85
7.2.1.2 Yukawa theory coupled to Unimodular Gravity 86
7.2.2 The $fermion + fermion \to fermion + fermion$ scattering at one-loop and $y^2\kappa^2$ order 88
7.2.2.1 The General Relativity case 88
7.2.2.2 The Unimodular Gravity case 97

V CONCLUSIONS
Summary and discussion 105
Resumen y discusión 109

VI APPENDIX
A HEAT KERNEL COEFFICIENTS OF A QUARTIC OPERATOR 115
B HEAT KERNEL CONTRIBUTIONS OF THE DIFFERENT OPERATORS INVOLVED 117
The contribution of S_{bc} 117
The contribution of S_π 118
The contribution of $S_{\bar{c}\phi}$ 119
The contribution of S_W 119

C FUNCTIONAL TRACES 121

BIBLIOGRAPHY 126

Part I

Introduction

Introduction

The advances in theoretical physics during the beginning of the XX century settled the foundations of modern theories upon which we have achieved a deep understanding of the fundamental structure of our universe.

On the one hand, the development of Quantum Field Theory, opened a whole new field that allowed us to describe –and compute– the behaviour of subatomic particles. In particular, the Standard Model of Particle Physics constitutes the summit of this framework and provides us with the most complete theory to describe Nature. Although incomplete, it accurately predicts most of the particles and interactions that rule the subatomic world. On the other hand, the theory of General Relativity [1] improved Newton's Theory of Universal Gravitation –solving some open issues and opening a new field of research, achieving a very precise description of classical gravity.

However, there are well-known problems to describe gravity. If one tries to get General Relativity into the successful framework of Quantum Field Theory it turns out to be non-renormalizable [2], that is, when computing quantum corrections, a new operator appear –with a new coupling– in the action via renormalization for each order. Moreover, the coupling to matter does not improve this behavior [3–6]. This means that for the complete theory there will be an infinite number of constants, each of them to be fixed through an experiment. Hence, it cannot be a complete theory.

There are sundry attempts to solve this problem, from the possibility of an asymptotic safety scenario, or the assumption of a particular quantization of the metric as in Loop Quantum Gravity to the modification of the UV degrees of freedom in String Theory (see [7] for a more complete summary) or more radical proposals such as breaking Lorentz invariance in Horava-Liftsitz gravity. Despite all the proposals, nowadays there is still no agreement in which one –if any– is the right one,

and it seems that we are still far from an ultraviolet-complete theory of gravity.
What is accepted however, is that although General Relativity is not a complete theory, it can be thought as an effective field theory below the Planck scale (10^{19}GeV) [8, 9]. Thus we can treat it perturbatively as any other Quantum Field Theory and it is a valid description for all the possible phenomena that we are able to detect at this time. This means that any measurement that we can carry out is accurately described by the theory.

Still, there is a second problem that arises in this scheme, more profound in a sense, that is the so-called *cosmological constant problem* [10]. First proposed by Einstein [11] to allow static solutions of the field equations, it was later rejected by himself after observations indicating that the universe was expanding. However, the measurements of an accelerated expansion in the 90's, that led to the Nobel prize in 2011 brought the cosmological constant back into the game. This cosmological constant is the simplest way to explain this accelerated expansion. It can be interpreted as a vacuum energy, and it is hypothesized that there is a *dark energy* that produces it. However it also constitutes one of the biggest conundrums in Physics nowadays. First of all, while the measurements point lo a low –but non-zero– value, there is not a natural way to. The problem arises when one adds matter and tries to quantize the theory; there are different contributions to the vacuum energy coming from quantum effects [12] that modify its classical value, and these contributions are around 120 orders of magnitude bigger than the measured value.

Albeit this thesis does not attempt to find any solution to the first problem, that is, to find a renormalizable theory of gravity –therefore, we will use the effective field theory approach, it is devoted to a possible solution of the cosmological constant one. This solution comes through the Unimodular Gravity theory, that offers an easy way of solving the discrepancy between the theoretical and observational value of the cosmological constant.

Let us just finish this introduction by being forthright with the cosmological constant problem. Despite dark energy is the most accepted explanation of the accelerated expansion of the universe, it is not the only one. Although there are many proposals let us just mention two which are among the most prominent ones like the models of *modified gravity*; since the existence of dark energy relies on the truthfulness of General Relativity, there are different theories proposed –or modifications of General Relativity– that can account for the accelerated expansion without its existence it is worth saying nonetheless that the new measurements of gravitational waves have put strong constraints on these models [13]. Another popular –at least for a time– proposal was that of the *Quintessence* model where the presence of a scalar field could account for the acceleration.

The thesis is organized as follows. In Part II the theory of Unimodular Gravity is defined and the motivations for its study introduced. Also an extended analysis on the classical symmetries of the theory as can be found in [14] is carried out. Finally, there is a brief description of two common techniques in QFT that are widely used later; the background field formalism and the Heat Kernel –or Schwinger-DeWitt– method. In Part III we compute the one-loop quantum corrections to the theory – cf. [15, 16]– showing that there is no renormalization of the cosmological constant (an argument that can be extended to further loops). Part IV is devoted to the comparison between physical effects in General Relativity and Unimodular Gravity. First, we show that there is a difference when computing beta functions of gravity coupled to matter, although it lacks any physical interpretation. Secondly, we compute some S matrix elements showing that the two theories are equivalent up to one-loop order. Finally, a summary of all the findings through the thesis is presented in Part V.

Part II

Preliminaries

1

Notation and conventions

This first chapter is devoted to settle down the notation and conventions that are used throughout the thesis. First of all, everything is given in terms of natural units, that is, $c = \hbar = 1$. With respect to the gravitational conventions, we follow those of [17], that is, the *mostly minus* Minkowski metric is given by

$$\eta_{\mu\nu} \equiv \{+,-,-,-\}, \tag{1.1}$$

and the Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\rho}, \tag{1.2}$$

while the Ricci tensor is

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}. \tag{1.3}$$

With these, the commutator of covariant derivatives with our conventions is

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] V^\lambda &= R^{\lambda}{}_{\rho\mu\nu} V^\rho, \\ [\nabla_\mu, \nabla_\nu] h^{\alpha\beta} &= h^{\beta\lambda} R^{\alpha}{}_{\lambda\mu\nu} + h^{\alpha\lambda} R^{\beta}{}_{\lambda\mu\nu}. \end{aligned} \tag{1.4}$$

The Einstein-Hilbert (EH) action principle thus reads

$$S_{\text{EH}} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|g|} R, \tag{1.5}$$

where $\kappa^2 = 8\pi G$, and G is the Newton's gravitational constant.

$g_{\mu\nu}$ denotes the space-time metric in general, while $\tilde{g}_{\mu\nu}$ will be used for the unimodular one.

2

The What and the why of Unimodular Gravity

The purpose of this chapter is to define what is Unimodular Gravity (UG) as well as motivate its study as a low energy candidate of quantum gravity.

It is not generally known (there is however a footnote in [18]), that after the first publication General Relativity (GR), Einstein proposed a different set of equations [19] also known as Einstein trace-free equations and that have been label *Unimodular Gravity* afterwards. Although the original purpose of these equations –an alternative proposal to Mie's theory on the stability of the electron– was proved wrong, the equations have gotten a further interest in relation with the cosmological constant problem [20–25].

In particular, and for an arbitrary dimension n, the UG equations read

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = \kappa^{n-2} \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right). \tag{2.1}$$

Although one may be tempted to say that there is less information here that in the usual GR equations –indeed, the trace has been left out– there is a further identity, namely the second Bianchi identity

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R, \tag{2.2}$$

which is geometrical, and therefore valid also for UG. By virtue of this identity, we can apply a derivative on (2.1) to get

$$\frac{n-2}{2n}\ \nabla_\nu R = -\frac{\kappa^{n-2}}{n}\ \nabla_\nu T. \tag{2.3}$$

This is a first integral, that integrates to

$$\frac{n-2}{2n}\ R + \frac{\kappa^{n-2}}{n}\ T = C, \tag{2.4}$$

with C an arbitrary constant, that will depend on boundary conditions. Going back to (2.1) the full Einstein equations with a cosmological constant term can be recovered

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + C g_{\mu\nu} = \kappa^{n-2} T_{\mu\nu}. \tag{2.5}$$

This means, that any classical effect predicted by GR is also predicted in UG [26–29], so the theory fulfills all the classical tests of GR and there is also room for inflation [30, 31]. There has been some research in the quantum properties of the theory [32–36] but there is still much to uncover. The purpose of this thesis is to deepen in the quantum theory. Moreover, since the original equations couple to the traceless part of the energy-momentum tensor, any possible vacuum energy, or dynamical cosmological constant coming from a non-trivial minimum in the potential of a scalar field, is absorbed into the still arbitrary constant. This claim of the equivalence of the classical tests is the usual motivation for working with UG, but it is not the only one. It has been long believed that any "second order" theory that propagates a free massless spin-two field at the linear level has to be Einstein theory (that is, the massless limit of Fierz-Pauli theory [37]). However, as it was proved in [38, 39], it is not the only one.

The most general action principle built out of dimension four operators for a spin two field $h_{\mu\nu}$ can be written in terms of four local operators

$$L \equiv \sum_{i=1}^{4} C_i\ \mathcal{O}^{(i)}, \tag{2.6}$$

where the operators read

$$
\begin{aligned}
\mathcal{O}^{(1)} &\equiv \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial^\mu h^{\rho\sigma}, \\
\mathcal{O}^{(2)} &\equiv -\frac{1}{2}\partial^\rho h_{\rho\sigma}\partial_\mu h^{\mu\sigma}, \\
\mathcal{O}^{(3)} &\equiv \frac{1}{2}\partial_\mu h\partial_\lambda h^{\mu\lambda}, \\
\mathcal{O}^{(4)} &\equiv -\frac{1}{4}\partial_\mu h\partial^\mu h,
\end{aligned}
\tag{2.7}
$$

and all indices are raised and lowered with the flat space metric $\eta_{\mu\nu}$, and $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. The constant $C_1 = 1$ can be fixed as a global normalization. The result of the work in [39] was that imposing only *transverse diffeomorphisms* (*TDiff*) invariance is enough. At the linear level (*LTDiff*) this forces

$$C_2 = 1, \tag{2.8}$$

where *LTDiff* invariance is just

$$
\begin{aligned}
\delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \\
\partial_\mu \xi^\mu &= 0.
\end{aligned}
\tag{2.9}
$$

The most important result was however the following. Amongst all the *TDiff* invariant theories obtained for arbitrary values of C_3 and C_4 there are only two that propagate spin two only, without any admixture of spin zero. Those are, first

$$C_3 = C_4 = 1, \tag{2.10}$$

which has an enhanced symmetry under linearized diffeomorphisms (without the transversality restriction). This is the Fierz-Pauli theory.

The other one corresponds to

$$
\begin{aligned}
C_3 &= \frac{2}{n}, \\
C_4 &= \frac{n+2}{n^2}.
\end{aligned}
\tag{2.11}
$$

This is actually a truncation of the Fierz-Pauli one obtained by

$$h_{\mu\nu} \to h_{\mu\nu} - \frac{1}{n}h\eta_{\mu\nu}, \tag{2.12}$$

(notice that it is not a field redefinition, because it is not invertible).

This theory was dubbed *Weyl transverse diffeomorphisms* (*WTDiff*) and is actually the linear limit of Unimodular Gravity. This will motivate our definition for an action principle of UG.

Let us now turn our attention to the full nonlinear theory. The naive way of defining an action principle for UG is just to set

$$\det\tilde{g}_{\mu\nu} \equiv \tilde{g} = -1, \tag{2.13}$$

in the EH action (1.5). This yields just (let us forget for the time being of the $-\frac{1}{2\kappa^2}$ factor, since it is global an will not change any of the following)

$$S = \int d^n x R[\tilde{g}], \tag{2.14}$$

where as expected, if we were to add a matter field, for example

$$S_{\mathrm{m}} = \int d^n x \left(\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - V(\psi) \right), \tag{2.15}$$

the potential does not couple to the graviton.

However, this is more intricate than it seems. The action (2.15) is invariant only under *TDiff* (2.9). That means that in order to get the equations of motion the allowed variations (δ^T) are constrained by

$$\delta^T \sqrt{|g|} = 0, \tag{2.16}$$

which implies

$$g^{\mu\nu} \delta^T g_{\mu\nu} = 0. \tag{2.17}$$

In addition to the technical complications produced by this, when considering the quantum theory this means that we need to integrate over constrained functional variables $\mathcal{D}\tilde{g}_{\mu\nu}$.

In order to solve this problem, and motivated by the linear analysis in [39], we shall follow the idea of defining the theory starting from General Relativity by a non-invertible field redefinition

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = |g|^{-\frac{1}{n}} g_{\mu\nu}, \tag{2.18}$$

generalizing the way the flat space *WTDiff* theory is defined starting from the Fierz-Pauli theory (2.12). Through this definition, we obtain the UG action as

$$S_{UG} = -\frac{1}{\kappa^{(n-2)}} \int d^n x \ |g|^{\frac{1}{n}} \ \left\{ \left(R + \frac{(n-1)(n-2)}{4n^2} \frac{\nabla_\mu g \nabla^\mu g}{g^2} \right) \right\}. \qquad (2.19)$$

For this unconstrained metric, the equations of motion (EM) are given [39] by the manifestly traceless expression

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} - \frac{(n-2)(2n-1)}{4n^2} \left(\frac{\nabla_\mu g \nabla_\nu g}{g^2} - \frac{1}{n} \frac{(\nabla g)^2}{g^2} g_{\mu\nu} \right) + \\ + \frac{n-2}{2n} \left(\frac{\nabla_\mu \nabla_\nu g}{g} - \frac{1}{n} \frac{\nabla^2 g}{g} g_{\mu\nu} \right) = \kappa^{n-2} \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right), \qquad (2.20)$$

that reduces to (2.1) when $|g| = 1$.

There is a further reason why UG seems a promising low energy level for quantum gravity. Van der Bij, van Dam and Ng [40] showed a long time ago that *TDiff* is enough to make gauge artifacts of the three excess gauge polarizations when going to the massless limit in a spin two flat space theory (there are five polarizations in the massive case and only two in the massless limit). This intuitively means that we only need three arbitrary gauge parameters, which is exactly what we have in *LTDiff*(M).
Moreover, it is worth remarking that a recent reanalysis of the classical Deser [41] argument for the non-linear completion of the linear Fierz-Pauli theory in [42] gets the result that both Unimodular Gravity and General Relativity (and only these two) are the allowed possibilities.

It has been made clear that Unimodular Gravity seems a promising theory of gravity. Due to the facts that they are classically equivalent, but UG seems to partially solve the cosmological constant problem, it seems that there is no good reason to not consider UG instead of GR. The purpose of this thesis is double. First, to show that the nature of the cosmological constant keeps unchanged at the quantum level, *i.e.* it does not get quantum corrections and does not couple to the vacuum energy. Secondly, we try to see if there is any (quantum) physical observable that can tell UG from GR at the quantum level.

3

About the gauge symmetries of Unimodular Gravity

In the previous chapter it has been said that UG does not have the full *Diff* symmetry of GR but a subgroup that we call *TDiff*, (2.9), that is, those that keeps the volume fixed. Now, we want to somewhat clarify what this symmetry implies. Following [14], we define the *unimodular reduction* mapping, given in (2.18) as

$$\text{UR}: \ g_{\mu\nu} \to \tilde{g}_{\alpha\beta} = |g|^{-\frac{1}{n}} g_{\mu\nu} \tag{3.1}$$

As was said before, it is not invertible, since there is no way to reconstruct $g_{\alpha\beta}$ from $\tilde{g}_{\alpha\beta}$.

On the other hand, once we restrict the theory to unimodular metrics, the ensuing theory (UG) is not invariant under the full diffeomorphism group of the manifold. *Diff*(M), but only under the subgroup that preserves the unimodularity condition, which we have dubbed *TDiff*(M). This is essentially what mathematicians call the volume preserving subgroup [43]. It has been pointed out that this symmetry is enough to kill the three unwanted polarizations when defining the massless theory from a massive theory in flat space [40]. At any rate, as was written before, under unimodular reduction Einstein-Hilbert action gets transformed into

$$\begin{aligned} \text{UR}: S_{\text{GR}} &\equiv -\frac{1}{2\kappa^{n-2}} \int d^n x \ \sqrt{|g|} \ R\left[g_{\alpha\beta}\right] \longrightarrow \\ \longrightarrow \quad S_{\text{UG}} &\equiv -\frac{1}{2\kappa^{n-2}} \int d^n x \ R\left[\tilde{g}_{\alpha\beta}\right], \end{aligned} \tag{3.2}$$

and the unimodular action in terms of unconstrained variables reads

$$S_{\text{UG}} = -\frac{1}{2\kappa^{n-2}} \int d^n x \; |g|^{\frac{1}{n}} \; \left(R + \frac{(n-1)(n-2)}{4n^2} \frac{\nabla_\mu g \nabla^\mu g}{g^2} \right). \tag{3.3}$$

We always represent covariant derivatives associated to the Levi-Civita connection of a given metric with the same symbol; id est,

$$\begin{aligned} \nabla_\mu g_{\alpha\beta} &= 0, \\ \widetilde{\nabla}_\mu \tilde{g}_{\alpha\beta} &= 0, \end{aligned} \tag{3.4}$$

and so on.

The covariant derivative acting on $g(x)$ is defined as if $g(x)$ were a true scalar,

$$\begin{aligned} \nabla_\mu g(x) &\equiv \partial_\mu g(x), \\ \nabla_\sigma \nabla_\mu g(x) &\equiv \partial_\sigma \partial_\mu g(x) - \Gamma^\lambda_{\sigma\mu} \partial_\lambda g(x). \end{aligned} \tag{3.5}$$

What we mean is that with the ordinary covariant derivative (which is the only one we use in this computation) the derivative of a density is *not* a density of the same weight. This is not a problem for us, but is something we have to take into account. Once here one can never go back to the Einstein frame as this action is Weyl invariant.

In terms of this unconstrained metric, the equations of motion (EM) are given by the manifestly traceless expression [15]

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = \Theta_{\mu\nu}, \\ & \Theta_{\mu\nu} \equiv \frac{(n-2)(2n-1)}{4n^2} \left(\frac{\nabla_\mu g \nabla_\nu g}{g^2} - \frac{1}{n} \frac{(\nabla g)^2}{g^2} \; g_{\mu\nu} \right) - \\ & \qquad - \frac{n-2}{2n} \left(\frac{\nabla_\mu \nabla_\nu g}{g} - \frac{1}{n} \frac{\nabla^2 g}{g} g_{\mu\nu} \right). \end{aligned} \tag{3.6}$$

The explicit presence of the determinant of the metric, g clearly indicates the EM are not *Diff* invariant.

Now given the fact that the EM are Weyl invariant, we can always transform from $g_{\mu\nu}$ to $\bar{g}_{\mu\nu}$ such that

$$\bar{g} = 1, \tag{3.7}$$

where the EM simply read

$$\bar{R}_{\mu\nu} = \frac{1}{n}\bar{R}\bar{g}_{\mu\nu}. \tag{3.8}$$

The solution of these equations are by definition *Einstein spaces* [44]. The Bianchi identities in the absence of torsion do imply then $\nabla_\mu R = 0$.

Given an unimodular Einstein space, $\bar{g}_{\mu\nu}$, all its Weyl rescalings

$$g_{\mu\nu} \equiv \Omega^2(x)\ \bar{g}_{\mu\nu} \tag{3.9}$$

are also solutions of the equations (3.6). They span a *Weyl orbit* of solutions. In four dimensions it is well known that the necessary and sufficient condition [45] for a space to be conformally Einstein is for it to be Bach-flat

$$B_{\mu\nu} \equiv \nabla^\alpha\nabla^\beta W_{\alpha\mu\nu\beta} - \frac{1}{2}R^{\alpha\beta}\ W_{\alpha\mu\nu\beta} = 0, \tag{3.10}$$

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor. We are not aware of a similar statement in arbitrary dimension.

The full symmetry group of this action is quite large though, incorporating Weyl transformations of the metric. This means that in the process of unimodular reduction of Einstein-Hilbert, the symmetry group changes, namely

$$\text{UR}: \quad \mathit{Diff}(M) \longrightarrow \mathit{TDiff}(M) \ltimes \mathit{Weyl}(M) \tag{3.11}$$

Let us examine this process of symmetry reduction in more detail. We shall be cavalier about domains of definition of the transformations, and all of our reasoning will be purely local.

3.1 TDIFF INVARIANCE OF THE UNIMODULAR ACTION

It is not immediately obvious in which reference systems are the EM (3.6) valid.

Let us first start with the analysis of the already mentioned change of the symmetry group in the process of unimodular reduction.

We can represent a linearized element of *Diff*$_0(M)$ (the subgroup of *Diff*(M) connected with the identity) as

$$x \to x' \equiv x + \xi. \tag{3.12}$$

The corresponding jacobian matrix is

$$J^{\alpha}_{\beta'}(x) \equiv \frac{\partial x^{\alpha}}{\partial x^{\beta'}}, \tag{3.13}$$

and its determinant will be denoted by the letter J.

The determinant of the metric then transforms as

$$g(x) \to g^{\xi}\,(x+\xi) = J^2(x)\; g(x), \tag{3.14}$$

and for the case of a volume preserving diffeomorphisms, it is transverse in the sense that

$$\partial_{\lambda}\xi^{\lambda}_{T} = 0, \tag{3.15}$$

and the jacobian matrix is itself unimodular

$$J_T = 1. \tag{3.16}$$

Let us examine what happens with the action of *TDiff*$_0 \ltimes$ *Weyl*(M). Clearly

$$g^{\xi}_{\mu\nu}(x) \equiv J^{\alpha}_{\mu} J^{\beta}_{\nu} g_{\alpha\beta}(x-\xi), \tag{3.17}$$

and consequently

$$g^{\xi\Omega}_{\mu\nu} \equiv \Omega^2(x)\; J^{\alpha}_{\mu} J^{\beta}_{\nu} g_{\alpha\beta}(x-\xi). \tag{3.18}$$

On the other hand, the other way around yields

$$g^{\Omega\xi}_{\mu\nu}(x) = \Omega^2(x-\xi)\; J^{\alpha}_{\mu} J^{\beta}_{\nu} g_{\alpha\beta}(x-\xi) \tag{3.19}$$

This corresponds to the non-commutativity of the diagram

$$\begin{array}{ccc} g_{\mu\nu} & \xrightarrow{TDiff} & g^{\xi}_{\mu\nu} \\ \Big\downarrow{\scriptstyle Weyl} & & \Big\downarrow{\scriptstyle Weyl} \\ g^{\Omega}_{\mu\nu} & \xrightarrow{TDiff} & g^{\xi\Omega}_{\mu\nu} \neq g^{\Omega\xi}_{\mu\nu} \end{array} \tag{3.20}$$

thus being the reason why the symmetry group is a semidirect product.

We can move now to answer the question of the validity of the equation of motion of UG. The two possible paths when going from GR to UG are shown in the following diagram.

$$\begin{array}{ccc} GR & \xrightarrow{Diff} & GR \\ \big\downarrow{\scriptstyle UR} & & \big\downarrow{\scriptstyle UR} \\ UG & \xrightarrow{Diff} & UG \end{array} \tag{3.21}$$

The rightmost path correspond to, first perform a diffeomorphism

$$g^{\xi}_{\mu\nu}(x) \equiv \left(T_{\xi} g\right)_{\mu\nu}(x) \equiv J^{\alpha}_{\mu}(x-\xi) J^{\nu}_{\nu}(x-\xi) g_{\alpha\beta}(x-\xi), \tag{3.22}$$

and unimodularly reduce afterwards. The corresponding unimodular metric is then

$$\left(T_U T_{\xi} g\right)_{\mu\nu}(x) = J^{-\frac{2}{n}}(x)\ g^{-\frac{1}{n}}(x)\ \ \left(T_{\xi} g\right)_{\mu\nu}. \tag{3.23}$$

The left path in the diagram corresponds to perform an arbitrary diffeomorphism after the unimodular reduction. The result is

$$\left(T_{\xi} T_U g\right)_{\alpha\beta}(x) = J^{-\frac{2}{n}}(x-\xi) g^{-\frac{1}{n}}(x-\xi)\ \ \left(T_{\xi} g\right)_{\alpha\beta}. \tag{3.24}$$

This means again that the diagram above is not commutative.

Indeed, we find particularly clarifying to examine what happens in this latter case

If we perform a *Diff* in (3.6) the determinant $g(x)$ transforms as

$$\begin{aligned} \nabla_{\lambda'} g^{\xi}(x') &= J^{\alpha}_{\lambda'} \nabla_{\alpha}\left(J^2(x)\ g(x)\right) = \\ &= J^{\alpha}_{\lambda'}\left(J^2\ \nabla_{\alpha} g(x) + 2g(x)\ J(x) \nabla_{\alpha} J\right). \end{aligned} \tag{3.25}$$

This conveys the fact that the first monomial in the EM transforms as

$$\begin{aligned} \frac{\nabla_{\mu'} g^{\xi}(x') \nabla_{\nu'} g^{\xi}(x')}{g'(x')^2} &= \frac{J^{\alpha}_{\mu'} J^{\beta}_{\nu'}}{J^4 g^2}\left(\nabla_{\alpha} g J^2 + 2gJ\nabla_{\alpha} J\right)\left(J^2 \nabla_{\beta} g + 2gJ\nabla_{\beta} J\right) = \\ &= J^{\alpha}_{\mu'} J^{\beta}_{\nu'}\left\{\frac{\nabla_{\alpha} g \nabla_{\beta} g}{g^2} + 2\frac{\nabla_{\alpha} J \nabla_{\beta} g + \nabla_{\alpha} g \nabla_{\beta} J}{Jg} + 4\frac{\nabla_{\alpha} J \nabla_{\beta} J}{J^2}\right\}, \end{aligned} \tag{3.26}$$

and its trace, which is the one subtracted from it in (3.6), is just

$$\left(\frac{\nabla_\alpha g}{g}+2\frac{\nabla_\alpha J}{J}\right)^2. \tag{3.27}$$

The second monomial transforms in turn as

$$\begin{aligned}\frac{\nabla_{\mu'}\nabla_{\nu'}g^\xi\,(x+\xi)}{g^\xi(x+\xi)}&=\frac{J^\rho_{\mu'}\;J^\alpha_{\nu'}\nabla_\rho}{J^2 g}\left(J^2\;\nabla_\alpha g(x)+2g(x)\;J(x)\nabla_\alpha J\right)=\\ =J^\rho_{\mu'}\;J^\alpha_{\nu'}&\left\{2\frac{\nabla_\rho J}{J}\frac{\nabla_\alpha g}{g}+\frac{\nabla_\rho\nabla_\alpha g}{g}+2\frac{\nabla_\rho g}{g}\frac{\nabla_\alpha J}{J}+2\frac{\nabla_\alpha J}{J}\frac{\nabla_\rho J}{J}+2\frac{\nabla_\rho\nabla_\alpha J}{J}\right\},\end{aligned} \tag{3.28}$$

being its trace now

$$4\frac{\nabla_\alpha J\nabla^\alpha g}{gJ}+\frac{\nabla^2 g}{g}+2\frac{\nabla_\alpha J\nabla^\alpha J}{J^2}+2\;\frac{\nabla^2 J}{J}. \tag{3.29}$$

The conclusion is that when performing a general $Diff_0(M)$ transformation in the unimodular equations of motion the extra terms generated are

$$\begin{aligned}\mathrm{EoM}\left[g^\xi_{\mu\nu}\right]_{\alpha'\beta'}&=J^\alpha_{\alpha'}J^\beta_{\beta'}\Bigg\{\mathrm{EoM}\left[g_{\mu\nu}\right]_{\alpha\beta}+\\ &+\frac{n-2}{2n}\left(\frac{1}{n}\frac{\nabla_\alpha J\nabla_\beta g+\nabla_\alpha g\nabla_\beta J}{Jg}+2\frac{1-n}{n}\frac{\nabla_\alpha J\nabla_\beta J}{J^2}++2\frac{\nabla_\alpha\nabla_\beta J}{J}\right)-\\ &-\frac{n-2}{n^2}\left(\frac{1}{n}\frac{\nabla_\mu J\nabla^\mu g}{gJ}+\frac{1-n}{n}\frac{\nabla_\mu J\nabla^\mu J}{J^2}+\frac{\nabla^2 J}{J}\right)\;g_{\alpha\beta}\Bigg\}.\end{aligned} \tag{3.30}$$

To be specific: the fact that a given metric $g_{\mu\nu}$ is a solution of the unimodular equations of motion does not imply that it remains a solution after an arbitrary diffeomorphism $\xi\in Diff_0(M)$ unless of course this happens to be transverse, $\xi\in TDiff_0(M)$. Certainly there is no problem with performing the *Diff* before the unimodular reduction, since GR is invariant as shown in the last diagram.

In other words, the assertion that a given metric is a solution of the UG equations of motion is not Diff(M) invariant, only TDiff(M)$\ltimes$Weyl(M) invariant. The remaining question is if there is a coordinate system which is not attainable through a symmetry transformation. It could be thought that there is none, by the following argument. An arbitrary diffeomorphism acts as

$$g^{\xi}_{\mu\nu}(x) \equiv J^{\alpha}_{\mu} J^{\beta}_{\nu} g_{\alpha\beta}(x-\xi), \tag{3.31}$$

which has the same number of parameters as the action of a volume preserving diffeomorphism composed with a Weyl transformation. However, there is a subtlety here, since one should have solutions to the equation

$$\frac{J^{\alpha}_{\mu}}{J} = \frac{\partial y^{\alpha}}{\partial x^{\mu}} \tag{3.32}$$

and this is possible only when

$$\partial_{\nu}\left(\frac{J^{\alpha}_{\mu}}{J}\right) = \partial_{\mu}\left(\frac{J^{\alpha}_{\nu}}{J}\right) \tag{3.33}$$

which will not be, in general, true. Therefore, not all coordinate systems are reachable in UG.

However, let us finally remark that given any metric, it can be made unimodular through a diffeormophism.

All we have to do is to find a solution of the equation

$$J(x) = \frac{1}{g(x)^2}. \tag{3.34}$$

At the linear level this is

$$\partial_{\mu}\xi^{\mu} = \frac{1}{g(x)^2} - 1, \tag{3.35}$$

which is trivially solved in a formal way by

$$\xi^{\mu}(x) = \partial^{\mu}\, \Box^{-1}\, \left(\frac{1}{g(x)^2} - 1\right), \tag{3.36}$$

whose solution is unique under essentially the same conditions as the corresponding solution of the wave equation [46].

4

Heat Kernel in a nutshell

We need a suitable technique to compute loop corrections, in particular, the effective action in Part III. Of course, there is always the possibility of using diagrammatic techniques, but they can be quite involved and, moreover, explicit gauge invariance is lost throughout the computations. Because of this, we introduce in this section the Heat Kernel (also known as Schwinger-DeWitt) technique [47–51].

4.1 Background field method

A first tool for computing the heat kernel is the background field method, [52], which allows us to quantize gauge field theories without losing explicit gauge invariance.

Since we are interested in obtaining the effective action, we start with an action $S(A)$ that depends on a number of fields A (that also include the metric in our case). The quantum dynamics of our theory are described by the path integral

$$\mathcal{Z}[\mathcal{J}] = \int \mathcal{D}\mathcal{A}\ \rceil^{\rangle(\mathcal{S}[\mathcal{A}]+\mathcal{J}\mathcal{A})}. \tag{4.1}$$

The S matrix can be obtained from the Green's functions of the theory by LSZ reduction. These are then defined as the time ordered n-point functions

$$G^{(n)}(A) = \langle 0|T\{\overbrace{A\ldots A}^{\text{n times}}\}|0\rangle = \left(\frac{1}{i}\frac{\delta}{\delta J}\right) Z[J]|_{J=0} \tag{4.2}$$

That is, the Green's functions are determined by taking functional derivatives with respect to the source function J of the generating functional.

However only connected Green's functions contribute to the S matrix, so we define the generator of connected Green functions as

$$W[J] = -i \ln Z[J] \ , \tag{4.3}$$

where W is called the free energy. The presence of the logarithm implies that when taking functional derivatives of the free energy, only the connected pieces are obtained.

The connected Green's functions can be further simplified by expressing them in terms of one-particle-irreducible (1PI) pieces (1 PI diagrams are those which cannot be split in two by cutting an internal line). These diagrams can then be stringed together to recover the full connected diagrams that one wants to compute. The 1PI Green's functions are generated by a functional called the *effective action* $\Gamma[\bar{A}]$. It is defined as

$$\Gamma[\bar{A}] = W[J] - J\bar{A} \tag{4.4}$$

where the *mean field*

$$\bar{A} \equiv \frac{\delta W}{\delta J} \tag{4.5}$$

can be seen as the vacuum expectation value of A in the presence of the source.

In order to get the S-matrix, we can now expand this effective action to the desired order in a loop expansion and get the connected functions from there. Therefore, it is an easy way to access to study the quantum theory.

The tool that we use for computing this is the background field method. It works as follows.

We take the path integral in (4.1) and shift the fields in the classical action $S(A)$ writing them as a background field $\bar{\phi}$ plus a perturbation, so now $\tilde{A} = \bar{\phi} + A$ and

$$\widetilde{\mathcal{Z}} = \int \mathcal{D}A e^{i(S[\bar{\phi}+A]+JA)} \tag{4.6}$$

The *background* field $\bar{\phi}$ obeys

$$\left.\frac{\delta S}{\delta \tilde{A}_\mu}\right|_{\bar{\phi}_\mu} = 0 \ . \tag{4.7}$$

With this new path integral we define a new effective energy

$$\widetilde{W}[J,\bar{\phi}] = -i \log \tilde{\mathcal{Z}}[J,\bar{\phi}] \tag{4.8}$$

and a new mean field

$$\widetilde{A} = \frac{\delta \widetilde{W}}{\delta J} \tag{4.9}$$

so the background field effective action is

$$\widetilde{\Gamma} = \widetilde{W}[J,\bar{\phi}] - J\widetilde{A} \tag{4.10}$$

It is easy now to find a relation between (4.10) and (4.4). For doing this, we can rewrite (4.6) by shifting the integration variable $A \to A - \bar{\phi}$ getting

$$\widetilde{\mathcal{Z}}[J,\bar{\phi}] = Z[J]e^{-iJ\bar{\phi}} \tag{4.11}$$

therefore, taking the logarithm

$$\widetilde{W}[J,\bar{\phi}] = W[J] - J\bar{\phi} \tag{4.12}$$

taking now the functional derivative $\frac{\delta}{\delta J}$ and looking at the previous definitions of the mean fields, we finally get the relation

$$\widetilde{A} = \bar{A} - \bar{\phi} \tag{4.13}$$

and we get, as the main result of the background field procedure the relation

$$\widetilde{\Gamma}[\widetilde{A},\bar{\phi}] = W[J] - J\bar{\phi} - J\bar{A} + J\bar{\phi} = \Gamma[\widetilde{A} + \bar{\phi}] \tag{4.14}$$

As an special case of this equation, we can take $\widetilde{A} = 0$ so

$$\widetilde{\Gamma}[0,\bar{\phi}] = \Gamma[\bar{\phi}] \tag{4.15}$$

That is, in order to get the effective action it is enough to compute $\widetilde{\Gamma}[0,\bar{\phi}]$.

The background field effective action $\widetilde{\Gamma}[\widetilde{A},\bar{\phi}]$ generates all the 1PI graphs contributing to Green's functions in the presence of the background field $\bar{\phi}$. Now, since Green's functions are generated by taking derivatives of the effective action, $\widetilde{\Gamma}[\widetilde{A},\bar{\phi}]$ generates functions in presence of a background field, while $\widetilde{\Gamma}[0,\bar{\phi}]$ –which has no dependence on $\widetilde{A}$– is the sum of all 1PI vacuum diagrams. Therefore, the background field method, by using (4.15), allow us to compute the effective action of a QFT ($\Gamma[\bar{\phi}=\bar{A}]$) by just summing over vacuum diagrams ($\widetilde{\Gamma}[0,\bar{\phi}=\bar{A}]$).

Let us now particularize this method for the one-loop computations. Starting from the path integral

$$\widetilde{\mathcal{Z}}[J,\bar{\phi}] = \int \mathcal{D}A e^{i(S[\bar{\phi}+A]+JA)} \tag{4.16}$$

We expand the field A perturbatively, and, by virtue of (4.15), we are only interested in the vacuum diagrams. At one loop order, there is only a single contribution, and it is enough to expand the action to second order

$$S[\bar{\phi}+A] = S[\bar{\phi}] + \left.\frac{\delta S}{\delta A}\right|_{A=\bar{\phi}} A + \frac{1}{2} A \left.\frac{\delta^2 S}{\delta A^2}\right|_{A=\bar{\phi}} A + \mathcal{O}(A^3) \tag{4.17}$$

We are going to set $\bar{\phi} = \bar{A}$. This, together with the fact that at the lowest order $W \approx S$ implies that the linear term vanishes by the definition (4.5).

Defining

$$\mathcal{D} = \left.\frac{\delta^2 S}{\delta A^2}\right|_{A=\bar{\phi}} \tag{4.18}$$

the path integral up to one-loop order is just

$$\widetilde{\mathcal{Z}}[J,\bar{\phi}] = \int \mathcal{D}A e^{iS[\bar{\phi}] + i\frac{1}{2}A\mathcal{D}A + iJA} \tag{4.19}$$

The first factor can be absorbed in the overall normalization, so in the absence of sources[1] we have

$$\widetilde{\mathcal{Z}}[0,\bar{\phi}] = \det{}^{-\frac{1}{2}}(\mathcal{D}) \tag{4.20}$$

and the free energy

$$\widetilde{W}[0,\bar{\phi}] = \frac{i}{2}\log(\det\mathcal{D}) \tag{4.21}$$

and the effective action, using (4.10) reads

$$\Gamma[\bar{\phi}] = \widetilde{\Gamma}[0,\bar{\phi}] = \widetilde{W}[0,\bar{\phi}] = \frac{i}{2}\log(\det\mathcal{D}) \tag{4.22}$$

Summing up, the use of the background field method allow us to reduce the computation of the effective action to a determinant of some operator, and in doing so, gauge invariance is preserved explicitly along the process [53–55].

4.2 SHORT TIME EXPANSION

For the actual computation of the determinant, we will use the Schwinger-DeWitt technique, more known as Heat kernel technique.

The heat kernel, for a given operator $\mathcal{D}$ is defined as

$$K(t,x,y,\mathcal{D}) = \langle x|e^{-t\mathcal{D}}|y\rangle \tag{4.23}$$

and receives its name because it satisfies the heat kernel equation

$$(\partial_t + \mathcal{D}_x)K(t,x,y,\mathcal{D}) = 0 \tag{4.24}$$

with $K(t=0,x,y,\mathcal{D}) = \delta^{n)}(x-y)$.

We should now relate the heat kernel with the one-loop effective action (4.21). In doing that, we follow the arguments of [56]. Let us turn in the following to Euclidean signature by performing a Wick rotation in order to avoid subtleties with the path integral. For each non-zero eigenvalue

1 Sources are nevertheless irrelevant for computing vacuum diagrams

of the operator[2] λ we can relate this integral to the logarithm of the eigenvalue

$$\ln\lambda = -\int_0^\infty \frac{dt}{t} e^{-t\lambda} \ . \tag{4.25}$$

Using now the identity $\ln\det(\mathcal{D}) = \mathrm{Tr}\log\mathcal{D}$ we find

$$W = -\frac{1}{2}\int_0^\infty \frac{dt}{t} K(t,\mathcal{D}), \tag{4.26}$$

where $K(t,\mathcal{D}) = \mathrm{Tr}(e^{-t\mathcal{D}}) = \int d^n x\sqrt{g}K(t,x,y,\mathcal{D})$.

The ultraviolet divergences are related to the lower limit of the integral, so we need to regularize it by introducing a cutoff Λ^{-2}

$$W = \lim_{\Lambda\to\infty} W_\Lambda = \lim_{\Lambda\to\infty} -\frac{1}{2}\int_{\Lambda^{-2}}^\infty \frac{dt}{t} K(t,\mathcal{D}) \ . \tag{4.27}$$

Since the UV divergences are encoded in the small proper time region, we can expand the heat kernel there as

$$K(t,x,y,\mathcal{D}) = K_0(t,x,y)\sum_{p=0}^\infty a_p(x,y)t^{\frac{p}{2}} \tag{4.28}$$

Here $K_0(t,x,y)$ is the heat kernel of the laplacian operator,

$$K_0(t,x,y) = \frac{1}{(4\pi t)^{t\frac{n}{2}}} e^{-\frac{\sigma(x,y)}{2t}} \tag{4.29}$$

where $\sigma(x,y)$ is Synge's world function [57]. The coefficients a_p, that get the name of *heat kernel coefficients*, encode the UV divergences. Odd coefficients vanish, $a_{2j+1} = 0$, and at coinciding arguments, $a_p(x,x)$ they are local polynomials of the background field and its derivatives, starting by $a_0(x,x) = 1$.

With this expansion, the logarithm of the determinant is just

2 A zero eigenvalue means the presence of a zero mode, thus a gauge symmetry. We suppose that gauge fixing terms are included in the operator $\mathcal{D}$.

$$\begin{aligned}\log\det\mathcal{D} = &-\int\frac{d\tau}{\tau}\mathrm{Tr}(K(\tau,x,y,\mathcal{D})) = \\ &= -\lim_{\sigma\to 0}\int\frac{d\tau}{\tau}\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int d^nx\sqrt{g}\sum_{p=0}^{\infty}\tau^{\frac{p}{2}}\mathrm{Tr}\left(a_p(x,y)e^{-\frac{\sigma(x,y)}{2t}}\right) = \\ &= -\int d^nx\sqrt{g}\lim_{\sigma\to 0}\sum_{p=0}^{\infty}\frac{\sigma(x,y)^{\frac{p-n}{2}}}{4^{\frac{n+p}{4}}\pi^{\frac{n}{2}}}\Gamma\left(\frac{n-p}{2}\right)\mathrm{Tr}(a_p(x,y)) \end{aligned} \quad (4.30)$$

Before going any further to the actual computation of the divergence and the heat kernel coefficients, let us just comment that this expansion is only valid for second order *minimal* operators [58]. That is, those whose highest order term is proportional to the laplacian ($\Box$). That is

$$\mathcal{D} = \hat{G}\Box + \hat{J}\nabla + \hat{M}, \quad (4.31)$$

where $\hat{G}, \hat{J}$ and $\hat{M}$ are matrices that account for the possible index structure. It is possible however –and quite straightforward– to extend the expansion to higher order operators, as long as they are minimal [59].

Going back to (4.30), in the case of $n = 4$ we can use dimensional regularization to obtain the logarithmic divergence –and therefore the one-loop effective action. In this case, there is a divergence $\frac{1}{\sigma^2}$ for $p = 0$, but it does not depend on the particular operator. The next divergent term corresponds to $p = 2$. Defining $\epsilon = 4 - n$, the we have

$$\Gamma\left(-\frac{\epsilon}{2}\right) = -\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon^2), \quad (4.32)$$

$$\sigma(x,y)^{\frac{\epsilon}{2}} = 1 + \frac{\epsilon}{2}\log\sigma(x,y) + \mathcal{O}(\epsilon^2), \quad (4.33)$$

therefore the effective action is given by

$$W = \frac{1}{2}\log\det\mathcal{D} = -\frac{1}{(4\pi)^2}\frac{1}{n-4}\int d^4x\sqrt{g}\ \mathrm{Tr}a_4(x,x). \quad (4.34)$$

This means that, in $n = 4$, the one-loop effective energy is given by the a_4 coefficient in the heat kernel expansion.

4.2.1 *Heat Kernel coefficients of a Laplace-type operator*

Most of the operators involved in the computation of the effective action of Unimodular Gravity are second order minimal (Laplace-type) operators of the form

$$D = -G_{AB}\Box + N^{\mu}_{AB}\nabla_{\mu} + M_{AB}, \tag{4.35}$$

where the capital indices refer to some possible gauge bundle. It can be always taken to a simple form after redefining the covariant derivative as $\mathcal{D} = \nabla + \omega$, so that

$$D = -G_{AB}\mathcal{D}^2 - E_{AB}, \tag{4.36}$$

where

$$\omega^{A}_{\mu\ B} = \frac{1}{2}G^{AC}N_{\mu CB}, \tag{4.37}$$

$$E^{A}_{B} = G^{AC}(-M_{CB} - \omega_{\mu CF}\omega^{\mu\ F}_{B} - \nabla_{\mu}\omega^{\mu}_{CB}). \tag{4.38}$$

Once in this form, the Heat Kernel coefficients for such type of operators are well-known and were first computed in [51, 60]. We are not going to describe the process of obtaining the coefficients in detail, but a exhaustive review can be found in [61]. Let us just mention that there are two main techniques for this. In the way that they were originally introduced by Schwinger and DeWitt [47–51], that is, recursively solving the heat equation (4.24) with the ansatz (4.28). This is a general method valid for many types of operators, but it is quite tedious. The other option is to rely on the universality of the coefficients, and use the properties introduced by Gilkey [62].

Let us just state here the result; the coefficients for $n = 4$ read

$$a_0 = \frac{1}{(4\pi)^2}\int_{\mathcal{M}} d^n x\ \sqrt{g}\ \mathrm{Tr}\left\{f\right\}\ , \tag{4.39}$$

$$a_2 = \frac{1}{(4\pi)^2}\frac{1}{6}\ \int_{\mathcal{M}} d^n x\ \sqrt{g}\ \mathrm{Tr}\left\{f(6E+R)\right\}\ , \tag{4.40}$$

$$a_4 = \frac{1}{(4\pi)^2}\frac{1}{360}\int d^n x\sqrt{|g|}\ \mathrm{Tr}\{60\Box E + 60RE + 180E^2 + 12\Box R + \\ + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 30\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu}\} \tag{4.41}$$

We have included total derivatives since they contribute if one is interested in the phenomenon of conformal anomalies. Here $\hat{\mathcal{R}}_{\mu\nu}$ refers to the field strength defined by

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]V_A = \hat{\mathcal{R}}_{\mu\nu A}{}^{B} V_B, \tag{4.42}$$

In the case of the graviton fluctuations this field strength is given by the Ricci identity.

$$[\nabla_\mu, \nabla_\nu]h^{\rho\sigma} = \hat{\mathcal{R}}^{\rho\sigma}_{\mu\nu\alpha\beta} h^{\alpha\beta}, \tag{4.43}$$

$$\hat{\mathcal{R}}^{\rho\sigma}_{\mu\nu\alpha\beta} = \frac{1}{2}\left(R^{\rho}_{\alpha\mu\nu}\delta^{\sigma}_{\beta} + R^{\sigma}_{\alpha\mu\nu}\delta^{\rho}_{\beta} + R^{\rho}_{\beta\mu\nu}\delta^{\sigma}_{\alpha} + R^{\sigma}_{\beta\mu\nu}\delta^{\rho}_{\alpha}\right). \tag{4.44}$$

Let us just mention that also the a_6 coefficient is computed in higher dimensions [58] as well as some others for particular cases [63–67].

As a special case to be considered later is just to consider the simplest operator

$$D = -\Box, \tag{4.45}$$

acting onto a scalar field. Here both N^μ and M are zero and the field strength vanishes. This means that its Heat Kernel a_4 coefficient is just

$$a_4(\Box) = \frac{1}{(4\pi)^2}\frac{1}{360}\int d^n x\sqrt{|g|}\ \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{4.46}$$

Part III

One-loop quantum corrections

5

One-loop quantum corrections

5.1 BRST quantization of unimodular gravity

In order to compute quantum corrections to Unimodular Gravity, we are going to rely on two main techniques of common use in the path integral approach to gauge theories and, in particular, to quantum gravity: the background field method in conjunction with the BRST formalism and the Schwinger-De Witt technique. Using a combination of both, we will be able to compute the quantum effective action of the theory for arbitrary background solutions of the classical fields. We should like to stress that the application of BRST formalism to the case at hand is quite involved since, as we shall see below, the theory has first-stage reducible gauge transformations in the language of Ref. [68]. Further, the choice of gauge-fixing terms has to be made with care; otherwise one ends up dealing with very complicated differential operators, which puts the feasibility of the one-loop computation in jeopardy.

To quantize the classical Unimodular Gravity theory defined by the action in (2.19) within the background field formalism, one splits the metric $g_{\mu\nu}$ into two parts: one contains the background metric $\bar{g}_{\mu\nu}$ and the other the quantum fluctuations $h_{\mu\nu}$. We shall find it advantageous to use the following splitting

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + |\bar{g}|^{\frac{1}{n}}\, h_{\mu\nu} \tag{5.1}$$

rather than usual splitting $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Notice that we can convert the splitting in (5.1) into the usual splitting by performing a Weyl transformation of the quantum field. This is supported by the claim in [69, 70] that there is no conformal anomaly in Unimodular Gravity.

Let us also express the background metric $\bar{g}_{\mu\nu}$ in terms of a metric $\tilde{g}_{\mu\nu}$ such that $|\tilde{g}| = 1$ as follows

$$\bar{g}_{\mu\nu} = |\bar{g}|^{\frac{1}{n}}\, \tilde{g}_{\mu\nu}. \tag{5.2}$$

Then, we have the following equality which stems from the Weyl invariance of the classical action

$$S_{UG}[\, g_{\mu\nu} = \bar{g}_{\mu\nu} + |\bar{g}|^{\frac{1}{n}}\, h_{\mu\nu}] = S_{UG}[\, g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}], \tag{5.3}$$

where $S_{UG}[\, g_{\mu\nu}]$ is given in (2.19).

Let us warn the reader that from now on the covariant derivative will be defined with respect to the metric $\tilde{g}_{\mu\nu}$ and that, unless explicitly said, we are dropping the tilde over background quantities in order to get cleaner formulas. However, it is important to keep in mind that there are two different classical metrics so far; $\bar{g}_{\mu\nu}$ representing an *arbitrary* background metric and $\tilde{g}_{\mu\nu}$ a unimodular ($\tilde{g} = -1$) background metric.

Thus, from now we will write

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}. \tag{5.4}$$

To quantize the theory defined by S_{UG} in (5.3), one has to identify first the gauge symmetries of it and then fix them. We already now that S_{UG} is invariant under *TDiff* and *Weyl* gauge transformations, and they can be written in BRST form

$$\begin{aligned}
&s_D g_{\mu\nu} = s_W g_{\mu\nu} = 0, \\
&s_D h_{\mu\nu} = \nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^{T\rho}\nabla_\rho h_{\mu\nu} + \nabla_\mu c^{T\rho} h_{\rho\nu} + \nabla_\nu c^{T\rho} h_{\rho\mu}, \\
&s_W h_{\mu\nu} = 2c\left(g_{\mu\nu} + h_{\mu\nu}\right),
\end{aligned} \tag{5.5}$$

where c and $c^{T\mu}$ are the anticommuting ghost fields for Weyl invariance and transverse diffeomorphisms, respectively. In this language, the transverse condition is satisfied by imposing $\nabla_\mu c^{T\mu} = 0$ on the ghost field. The superscript T thus means that the vector satisfies this condition. The gauge fixing procedure of these gauge symmetries will be discussed next.

The partition function of the theory is now

$$\mathcal{Z}[J] = \int \mathcal{D}h_{\mu\nu}\; e^{-S_2 - \int d^n x\; J^{\mu\nu} h_{\mu\nu}}, \tag{5.6}$$

and when the sources vanish this defines the quantum effective action to be

$$W = -\frac{1}{2}\log\left(\det \mathcal{D}\right), \tag{5.7}$$

where $\mathcal{D}$ is the operator driving the one-loop quantum fluctuations, defined by the quadratic term in the expansion of the action around the background metric

$$S_2 = \int d^n x\; \mathcal{L}_2^{\text{UG}} = \int d^n x\; h^{\mu\nu}\mathcal{D}_{\mu\nu\rho\sigma}h^{\rho\sigma}. \tag{5.8}$$

It is useful to write down the expression as it would stand *before* the background metric is assumed to be unimodular

$$\begin{aligned}
\mathcal{L}_2^{\text{UG}} =& \frac{1}{4}h^{\mu\nu}\bar{\Box}h_{\mu\nu} - \frac{n+2}{4n^2}h\bar{\Box}h + \frac{1}{2}\left(\bar{\nabla}_\mu h^{\mu\alpha}\right)\left(\bar{\nabla}_\nu h^\nu_\alpha\right) - \frac{1}{n}\left(\bar{\nabla}_\mu h\right)\left(\bar{\nabla}_\nu h^{\mu\nu}\right) + \\
&+ \frac{2-n}{2n}\left(\bar{\nabla}_\alpha \log \bar{g}\right)\left(\frac{1}{2}h^{\beta\lambda}\bar{\nabla}_\lambda h^\alpha_\beta + \frac{3}{2}h^{\alpha\beta}\bar{\nabla}_\lambda h^\lambda_\beta - \frac{1}{n}h\bar{\nabla}_\lambda h^{\alpha\lambda}\right) + \\
&+ \frac{(n-2)^2}{8n^3}\left(\bar{\nabla}^\alpha \log\bar{g}\right)\; h\bar{\nabla}_\alpha h + \frac{n-2}{2n^2}\left(\bar{\nabla}_\beta \log\bar{g}\right)\; h^{\alpha\beta}\bar{\nabla}_\alpha h + \\
&+ (n^2-3n+2)\left(\frac{1}{8n^4}h^2(\bar{\nabla}\log\bar{g})^2 - \frac{1}{8n^3}h_{\mu\nu}h^{\mu\nu}(\bar{\nabla}\log\bar{g})^2 + \right.\\
&\left. + \frac{1}{4n^2}h^\lambda_\alpha h_{\beta\lambda}\left(\bar{\nabla}^\alpha\log\bar{g}\right)\left(\bar{\nabla}^\beta\log\bar{g}\right) - \frac{1}{4n^3}hh_{\alpha\beta}\left(\bar{\nabla}^\alpha\log\bar{g}\right)\left(\bar{\nabla}^\beta\log\bar{g}\right)\right) + \\
&- \frac{1}{n}hh^{\mu\nu}\bar{R}_{\mu\nu} - \frac{8-6n+n^2}{8n^2}\left(\bar{\nabla}^\alpha\log\bar{g}\right)\; h^{\mu\nu}\bar{\nabla}_\alpha h_{\mu\nu} + \frac{1}{2}h^{\mu\nu}h^{\alpha\beta}\bar{R}_{\mu\alpha\nu\beta} + \\
&- \frac{1}{2n}h^{\mu\nu}h_{\mu\nu}\bar{R} + \frac{1}{2n^2}h^2\bar{R} + \frac{1}{2}h^{\alpha\beta}h^\mu_\beta\bar{R}_{\mu\alpha}\,.
\end{aligned} \tag{5.9}$$

Of course, $\mathcal{D}$ will contain in principle zero modes coming from the gauge symmetries of the theory translated to the linear level which will make its determinant singular. This is solved by constructing an appropriate gauge fixing term using the BRST quantization method.

Finally, since we are using the splitting (5.1), the action for the one-loop quantum fluctuations simplifies somewhat, since all terms depending on $\nabla_\mu g$ now vanish. Thus, we end up with

$$\mathcal{L}_2^{\mathrm{UG}} = \frac{1}{4}h^{\mu\nu}\bar{\Box}h_{\mu\nu} - \frac{n+2}{4n^2}h\bar{\Box}h + \frac{1}{2}\left(\bar{\nabla}_\mu h^{\mu\alpha}\right)\left(\bar{\nabla}_\nu h^\nu_\alpha\right) - \frac{1}{n}\left(\bar{\nabla}_\mu h\right)\left(\bar{\nabla}_\nu h^{\mu\nu}\right) + \\ + \frac{1}{2}h^{\alpha\beta}h^\mu_\beta R_{\mu\alpha} - \frac{1}{n}hh^{\mu\nu}R_{\mu\nu} + \frac{1}{2}h^{\mu\nu}h^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{2n}h^{\mu\nu}h_{\mu\nu}R + \frac{1}{2n^2}h^2 R\,. \tag{5.10}$$

After computing the quantum effective action and owing to the already mentioned fact that no conformal anomaly is present in the theory, one can just undo the transformation (5.2) and recover the expression for arbitrary background metrics by performing a conformal transformation away from the Einstein frame.

5.1.1 *Fixing the gauge freedom*

To gauge-fix the gauge symmetries in (5.5), we shall use the BRST technique in a similar way as in [71] and introduce the following nilpotent BRST operator

$$s = s_D + s_W, \tag{5.11}$$

where s_D and s_W are defined in (5.5).

The path integral over the ghost fields must be restricted to the subspace of transverse vectors. However, the definition of such a measure $[\mathcal{D}c^{T\mu}]$ over transverse vectors is a notorious problem [72, 73]. The way to come to grips with it chosen in this thesis is to parametrize this subspace in terms of unconstrained fields so that we can then integrate over the full space of c^μ, whose integration measure is well-defined. This we do by introducing an operator $\Theta_{\mu\nu}$[1]

$$c^T_\mu = \Theta_{\mu\nu}c^\nu = \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu - R_{\mu\nu}\right)c^\nu = \left(Q_{\mu\nu} - \nabla_\mu\nabla_\nu\right)c^\nu, \tag{5.12}$$

which maps vectors into transverse vectors. In this way, the transversality condition over c^T_μ translates into a gauge symmetry for c_ν

$$c_\nu \to \nabla_\nu f, \tag{5.13}$$

1 One can easily check that $\Theta_{\mu\nu}$ is indeed an endomorphism in the space spanned by transverse vectors.

with f an arbitrary function. Indeed, this transformation takes c_ν into a longitudinal vector, so that the $\Theta_{\mu\nu}$ operator annihilates it. Of course, in order to perform now the functional integration over c^μ we must gauge fix this new gauge symmetry by introducing a non-trivial stairway of ghost levels with BRST transformations defined in such a way that the BRST algebra closes

$$\begin{aligned} s_D^2 &= s_W^2, \\ \{s_D, s_W\} &= 0, \end{aligned} \tag{5.14}$$

on all the different fields considered.

The systematic way to obtain this field content together with the appropriate BRST transformations is by using the Batalin-Vilkovisky [68] formalism. However, in our case, things are easy enough as to allow us to guess what the BRST transformations read, once the field content of the theory is chosen as done in [68] for first-stage reducible and irreducible gauge transformations. Notice that the gauge transformations in (5.5) generated by s_D, with c_μ in (5.12), are first-stage reducible due to the gauge symmetry in (5.13). However, the gauge symmetries in (5.5) generated by s_W are irreducible. We introduce the following set of fields:

$$\begin{aligned} &h_{\mu\nu}^{(0,0)},\ c_\mu^{(1,1)},\ b_\mu^{(1,-1)},\ f_\mu^{(0,0)},\ \phi^{(0,2)}, \\ &\pi^{(1,-1)},\ \pi'^{(1,1)},\ \bar{c}^{(0,-2)},\ c'^{(0,0)}, \\ &c^{(1,1)},\ b^{(1,-1)},\ f^{(0,0)}, \end{aligned} \tag{5.15}$$

where $c_\mu^{(1,1)}$ denotes c_μ, $h_{\mu\nu}^{(0,0)}$ stands for $h_{\mu\nu}$ and the superscript (n, m) carries the Grassmann number, n, (defined modulo two) and ghost number, m. In this language, the BRST operators s_D and s_W enjoy Grassmann number 1 and ghost number 1, each.

Here we have three families –displayed in three different lines– of fields. The first line includes the physical graviton field together with the usual ghost field content that would be naively necessary in order to gauge fix an unrestricted *Diff* symmetry. In addition, there is a ϕ field which accounts for the transformation in (5.13). The second line represents the field content introduced to gauge fix the gauge symmetry in (5.13), together with the one that will be induced on $b_\mu^{(1,-1)}$. Finally, the third line is the field content due to Weyl invariance.

Field	s_D	s_W
$g_{\mu\nu}$	0	0
$h_{\mu\nu}$	$\nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^{\rho T}\nabla_\rho h_{\mu\nu} + \nabla_\mu c^{\rho T} h_{\rho\nu} + \nabla_\nu c^{\rho T} h_{\rho\mu}$	$2c^{(1,1)}\left(g_{\mu\nu} + h_{\mu\nu}\right)$
$c^{(1,1)\mu}$	$\left(Q^{-1}\right)^\mu_\nu \left(c^{\rho T}\nabla_\rho c^{T\nu}\right) + \nabla^\mu \phi^{(0,2)}$	0
$\phi^{(0,2)}$	0	0
$b_\mu^{(1,-1)}$	$f_\mu^{(0,0)}$	0
$f_\mu^{(0,0)}$	0	0
$\bar{c}^{(0,-2)}$	$\pi^{(1,-1)}$	0
$\pi^{(1,-1)}$	0	0
$c'^{\,(0,0)}$	$\pi'^{\,(1,1)}$	0
$\pi'^{\,(1,1)}$	0	0
$c^{(1,1)}$	$c^{T\rho}\nabla_\rho c^{(1,1)}$	0
$b^{(1,-1)}$	$c^{T\rho}\nabla_\rho b^{(1,-1)}$	$f^{(0,0)}$
$f^{(0,0)}$	$c^{T\rho}\nabla_\rho f^{(0,0)}$	0

Figure 1: BRST transformations of the fields involved in the path integral.

Now, we define the action of s_D and s_W on the fields as shown in Table 1, where the $(Q^{-1})^\mu_\nu$ denotes the inverse of the operator $Q_{\mu\nu} = g_{\mu\nu}\Box - R_{\mu\nu}$, which exists provided $Det(Q) \neq 0$. This is our case since $Q_{\mu\nu}$ is just a standard Laplacian-type operator acting on vector fields.

With these definitions, it can be readily shown that the equations in (5.14) hold. In doing so, it is advisable to show first that

$$s_D c^{T\mu} = c^{T\rho}\nabla_\rho c^{T\nu}, \tag{5.16}$$

if $c^{T\mu}$ is defined as in (5.12). This can be done by using the following results

$$\nabla_\mu(c^{\rho T}\nabla_\rho c^{T\mu}) = 0, \quad \nabla_\mu\left[(Q^{-1})^\mu_\nu\left(c^{\rho T}\nabla_\rho c^{T\nu}\right)\right] = 0. \tag{5.17}$$

The path integral quantization of the theory is accomplished now by adding to the classical action the gauge-fixing action, $S_{gauge-fixing}$, which is an appropriate BRST-exact term:

$$S_{gauge-fixing} = \int d^n x \; s\left(X_{TD} + X_W\right), \tag{5.18}$$

X_{TD} and X_W are polynomials of the quantum fields with ghost number -1 and Grassmann number equal to 1 and such that they give rise to free-kinetic terms that are invertible. Since we are only interested in one-loop computations, we shall further assume that X_{TD} and X_W are quadratic in the quantum fields. In the next sections we will construct the terms X_{TD} and X_W and derive the differential operators involved in the path integral whose contribution to the quantum effective action needs to be computed.

5.1.2 *The TDiff sector*

Let us start with the function X_{TD} performing the gauge fixing of the *TDiff* symmetry. With the field content introduced above and with the BRST transformations as given in table 1, one has the following general quadratic polynomial in the quantum fields associated to the gauge-fixing of the *TDiff* symmetry

$$X_{TD} = b^{(1,-1)}_\mu\left[F^\mu + \rho_1 f^{\mu(0,0)}\right] + \bar{c}^{(0,-2)}\left[F^\mu_2 c_\mu + \rho_2 \pi'^{\,(1,1)}\right] + \\ + c'^{\,(0,0)}\left[F^\mu_1 b^{(1,-1)}_\mu + \rho_3 \pi^{(1,-1)}\right], \tag{5.19}$$

where F_μ is a function containing the graviton field that can be identified with the usual gauge fixing condition in the Faddeev-Poppov technique and F^μ_1, F^μ_2 and the three ρ_i can be freely chosen. This is enough to fix the *TDiff* symmetry with the minimal possible content of fields.

After applying the s operator, this gives a term in the action

$$\int d^n x\, sX_{TD} = \int d^n x\; f_\mu^{(0,0)} \left(F^\mu + \rho_1 f^{\mu(0,0)}\right) - b_\mu^{(1,-1)} sF^\mu + \\ + \pi^{(1,-1)} \left(F_2^\mu c_\mu^{(1,1)} + \rho_2 \pi'^{\,(1,1)}\right) + \bar{c}^{(0,-2)} F_2^\mu \nabla_\mu \phi^{(0,2)} + \\ + \pi'^{\,(1,1)} \left(F_{1\mu} b^{\mu(1,-1)} + \rho_3 \pi^{(1,-1)}\right) + c'^{\,(0,0)} F_1^\mu f_\mu^{(0,0)}. \tag{5.20}$$

where we have already taken into account that in the expansion (5.1) the metric is unimodular.

Now, there are some simplifications that can be done. First, let us take the terms containing $f_\mu^{(0,0)}$

$$f_\mu^{(0,0)} \left(F^\mu + \rho_1 f^{\mu(0,0)}\right) + f_\mu^{(0,0)} \bar{F}_1^\mu c'^{\,(0,0)}, \tag{5.21}$$

where we have introduced $\bar{F}_1^\mu$ using integration by parts as

$$\int d^n x\; a F_1^\mu b = \int d^n x\; b \bar{F}_1^\mu a. \tag{5.22}$$

These can be rewritten completing the square as

$$\rho_1 \left(f_\mu^{(0,0)} + \frac{1}{2\rho_1}(F_\mu + \bar{F}_{1\mu} c'^{\,(0,0)})\right)^2 - \frac{1}{4\rho_1}(F_\mu + \bar{F}_{1\mu} c'^{\,(0,0)})^2, \tag{5.23}$$

and shifting the variable $f_\mu^{(0,0)}$ the first term does not contribute to the effective action and we are left with the gauge fixing action

$$S_{hc'} = -\frac{1}{4\rho_1} \int d^n x\; (F_\mu + \bar{F}_{1\mu} c'^{\,(0,0)})^2, \tag{5.24}$$

where ρ_1 has been chosen to be a constant. This would be the outcome of a standard Faddeev-Poppov procedure.

Now let us focus on the terms containing the fermionic π fields. Those read

$$\pi^{(1,-1)} \left(F_2^\mu c_\mu^{(1,1)} + \rho_2 \pi'^{\,(1,1)}\right) + \pi'^{\,(1,1)} \left(F_1^\mu b_\mu^{(1,-1)} + \rho_3 \pi^{(1,-1)}\right) = \\ = \left(\pi^{(1,-1)} - F_1^\mu b_\mu^{(1,-1)} (\rho_2 - \rho_3)^{-1}\right) (\rho_2 - \rho_3) \left(\pi'^{(1,1)} + (\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)}\right) + \\ + F_1^\mu b_\mu^{(1,-1)} (\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)}, \tag{5.25}$$

and, again, by shifting the π fields we are left with a gauge fixing term plus an extra path integral depending on how we choose the operators ρ_2 and ρ_3

$$S_\pi + S_{gf}^{bc} = \int d^n x \, \left(\pi^{(1,-1)}(\rho_2 - \rho_3)\pi'^{\,(1,1)} + F_1^\mu b_\mu^{(1,-1)}(\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)} \right). \tag{5.26}$$

So that the BRST action for the *TDiff* sector is further simplified to

$$\int d^n x \; sX_{TD} = \int d^n x \, \Big(-b_\mu^{(1,-1)} sF^\mu + \bar{c}^{(0,-2)} F_2^\mu \nabla_\mu \phi^{(0,2)} + \\ +\pi^{(1,-1)}(\rho_2 - \rho_3)\pi'^{(1,1)} + F_1^\mu b_\mu^{(1,-1)}(\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)} \\ -\frac{1}{4\rho_1}(F_\mu + \bar{F}_{1\mu} c'^{\,(0,0)})^2 \Big). \tag{5.27}$$

As a next step, the function F_μ is chosen with two requirements in mind. First, that the term $F_\mu F^\mu$ is able to cancel the non-diagonal pieces of the operators in the original lagrangian for the graviton fluctuations and also that it is Weyl invariant so both gauge fixing sectors decouple and their ghost fields do not interact. With these two requirements, the choice is almost unique

$$F_\mu = \nabla^\nu h_{\mu\nu} - \frac{1}{n}\nabla_\mu h, \tag{5.28}$$

and its variation under a transverse diffeomorphism is the equivalent to the application of the s operator

$$sF_\mu = \Box c_\mu^T + \nabla^\nu \nabla_\mu c_\nu^T = \Box c_\mu^T + R_\mu^\nu c_\nu^T, \tag{5.29}$$

where in the second step we have used Ricci identity $[\nabla_\nu, \nabla_\mu] c^\nu = R_{\mu\nu} c^\nu$ and the fact that, since we are performing a transverse diffeomorphism, c_μ^T satisfies $\nabla^\mu c_\mu^T = 0$.

Now, we have to rewrite c_μ^T in terms of an unconstrained field as explained before. We do this by introducing the operator $\Theta_{\mu\nu}$ defined in (5.12),

$$sF_\mu = \left(g_\mu^\alpha \Box + R_\mu^\alpha \right) \left(g_{\alpha\nu} \Box - \nabla_\alpha \nabla_\nu - R_{\alpha\nu} \right) c^{\nu(1,1)} = \\ = \Box^2 c_\mu^{(1,1)} - \nabla_\mu \Box \nabla_\nu c^{\nu(1,1)} - 2R_{\mu\rho} \nabla^\rho \nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \\ - 2\nabla_\sigma R_{\mu\rho} \nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho} R^{\rho\nu} c_\nu^{(1,1)}. \tag{5.30}$$

The terms in the action for $b_\mu^{(1,-1)}$ and $c_\mu^{(1,1)}$ is then

$$S_{bc} = -\int d^n x\, b^{\mu\,(1,-1)} \Big(\Box^2 c_\mu^{(1,1)} - \nabla_\mu \Box \nabla_\nu c^{\nu(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \\ -\Box R_{\mu\rho} c^{\rho(1,1)} - 2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu} c_\nu^{(1,1)} + \\ + F_1^\mu b_\mu^{(1,-1)} (\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)}\Big)\,. \qquad (5.31)$$

The non-diagonal term with four derivatives can be canceled by an appropriate choice of the functions F_1^μ, F_2^μ, ρ_2 and ρ_3. We choose them to be

$$\begin{aligned} F_1^\mu b_\mu^{(1,-1)} &= -\nabla^\alpha b_\alpha^{(1,-1)}, \\ F_2^\mu c_\mu^{(1,1)} &= \nabla^\mu c_\mu^{(1,1)}, \\ (\rho_2 - \rho_3)^{-1} &= -\Box\,. \end{aligned} \qquad (5.32)$$

Thus

$$F_1^\mu b_\mu^{(1,-1)} (\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)} = \left(\nabla^\nu b_\nu^{(1,-1)}\right) \Box \nabla^\mu c_\mu^{(1,1)} = \\ = -b_\nu^{(1,-1)} \nabla^\nu \Box \nabla^\mu c_\mu^{(1,1)}, \qquad (5.33)$$

where in the second step we have performed an integration by parts keeping in mind that we are always under an integral sign. The final action term for $b_\mu^{(1,-1)}$ and $c_\mu^{(1,1)}$ is then

$$S_{bc} = \int d^n x\, b^{\mu\,(1,-1)} \Big(\Box^2 c_\mu^{(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \\ -2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu} c_\nu^{(1,1)}\Big)\,. \qquad (5.34)$$

And with the choice (5.32) for $(\rho_2 - \rho_3)$, the integration over the π fields is given just by

$$S_\pi = \int d^n x\, \pi^{(1,-1)} \Box^{-1} \pi'^{\,(1,1)}. \qquad (5.35)$$

The operator involving $c'^{\,(0,0)}$, and induced by this choice of fixing functions is

$$S_{hc'} = -\int d^n x\, \frac{1}{4\rho_1} \left[\bar{F}_1^\mu c'^{(0,0)} \bar{F}_{1\mu} c'^{(0,0)} + 2F_\mu \bar{F}_1^\mu c'^{\,(0,0)} + F_\mu F^\mu\right] = \\ = -\int d^n x\, \frac{1}{4\rho_1} \left[\nabla_\mu c'^{\,(0,0)} \nabla^\mu c'^{\,(0,0)} + 2F_\mu \nabla^\mu c'^{\,(0,0)} + F_\mu F^\mu\right]. \qquad (5.36)$$

which mixes with the operator of the graviton fluctuation due to the term containing F_μ and $c'^{\,(0,0)}$.

Finally, the operator for $\bar{c}^{(0,-2)}$ and $\phi^{(0,2)}$ is

$$S_{\bar{c}\phi} = \int d^n x \; \bar{c}^{(0,-2)} \Box \phi^{(0,2)}. \tag{5.37}$$

Summarizing, the BRST exact action for the TDiff symmetry is reduced to

$$\begin{aligned} S^{TDiff}_{BRST} = \int d^n x \; b^\mu \Big(& \Box^2 c^{(1,1)}_\mu - 2R_{\mu\rho}\nabla^\rho \nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \\ & -2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu}c^{(1,1)}_\nu \Big) + \pi^{(1,-1)}\Box^{-1}\pi'^{(1,1)} + \\ & + \bar{c}^{(0,-2)}\Box\phi^{(0,2)} - \frac{1}{4\rho_1}\Big(F_\mu F^\mu + \nabla_\mu c'^{\,(0,0)}\nabla^\mu c'^{\,(0,0)} + 2F_\mu \nabla^\mu c'^{\,(0,0)}\Big) = \\ & = S_{bc} + S^{bc}_{gf} + S_{\bar{c}\phi} + S_\pi + S_{hc'}. \end{aligned} \tag{5.38}$$

The contribution of all these pieces to the quantum effective action will be computed in section 5.2.

5.1.3 *The Weyl sector*

Now we turn our attention to the second part of the gauge fixing sector, corresponding to the Weyl invariance of the theory. We choose the function X_W to be

$$X_W = \nabla_\mu b^{(1,-1)} \nabla^\mu \left(f^{(0,0)} - \alpha \, g(h) \right), \tag{5.39}$$

with $g(h)$ being some function of the trace of the graviton fluctuation only, to ensure that it is invariant under a *TDiff* transformation. The parameter α we mean to keep arbitrary all along the computation. The on shell effective action should be independent of α (because it appears in a BRST exact piece), and this will be used as a nice partial check of our results.

After the application of s, the BRST exact action is

$$S^{Weyl}_{BRST} = \int d^n x \; \left[\nabla_\mu f^{(0,0)} \nabla^\mu \left(f^{(0,0)} - \alpha \, g(h) \right) - \alpha \nabla_\mu b^{(1,-1)} \nabla^\mu \left(s g(h) \right) \right], \tag{5.40}$$

and we choose $g(h)$ to be the simplest choice

$$g(h) = h. \tag{5.41}$$

The BRST term piece is then

$$\begin{aligned} S_{Weyl} &= \int d^n x \; \nabla_\mu f^{(0,0)} \nabla^\mu \left(f^{(0,0)} - \alpha\, h \right) - 2n\alpha \nabla_\mu b^{(1,-1)} \nabla^\mu c^{(1,1)} = \\ &= \int d^n x \; \left(-f^{(0,0)} \Box f^{(0,0)} + \frac{\alpha}{2} f^{(0,0)} \Box h + \frac{\alpha}{2} h \Box f^{(0,0)} \right) + \\ &+ 2n\alpha\, b^{(1,-1)} \Box c^{(1,1)} = S_W + S_{hf}. \end{aligned} \tag{5.42}$$

This gives two contributions to the one-loop effective action. The first part needs to be added to the original action of Unimodular Gravity. The second piece is the corresponding ghost action.

5.2 THE ONE-LOOP EFFECTIVE ACTION OF UNIMODULAR GRAVITY

Once the gauge freedom is fixed completely, the computation of the one-loop counterterm of Unimodular Gravity is reduced to a computation of a set of determinants. By collecting all the terms defined in the previous sections, the pole part of the one-loop effective action will be given, as explained in chapter 4, by

$$W_\infty = W_\infty^{UG} + W_\infty^{bc} + W_\infty^{\pi} + W_\infty^{\bar{c}\phi} + W_\infty^{W}, \tag{5.43}$$

where each W_∞^i refers to the contribution to the pole given by the action labeled as S_i in the previous sections, with the only exception of W_∞^{UG} which is given by

$$S_{UG} = S_2 + S_{hc'} + S_{hf}. \tag{5.44}$$

Each of this action terms have the general structure

$$S = \int d^n x \; \Psi^A F_{AB} \Psi^B, \tag{5.45}$$

where Ψ^A will be a vector containing different fields and F_{AB} a differential operator action over the fields. For instance, if we take S_W we identify

$$\Psi^A = \begin{pmatrix} b \\ c \end{pmatrix} \tag{5.46}$$

and the operator to be

$$F_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times n\alpha\Box\,. \tag{5.47}$$

All but one of the operators involved in our computation are *minimal* operators, meaning that their principal symbol is diagonal and they are of the form

$$F_{AB} = \gamma_{AB}\Box^m + K_{AB}. \tag{5.48}$$

where γ_{AB} is a metric in configuration space (this includes the spacetime metric as well as a metric defined on whatever space in which the indices carried by the fields live) and K_{AB} is a differential operator of order $m-1$ as most. The contribution of an operator of this kind to the quantum effective action is quite standard and their computation was reviewed in [74] by using the Schwinger-DeWitt technique. Some details are given in chapter 4 as well as in Appendix A for quartic operators. Let us explain here the main points of the computation pertaining to the only non-minimal operator, namely the one contained in S_{UG}

$$S^{(1)}_{UG} = S_2 + S_{hc'} + S_{hf} = \int d^n x\, \mathcal{L}, \tag{5.49}$$

and

$$\begin{aligned}\mathcal{L} = &\frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{4n}h\Box h + \frac{1}{2}h^{\alpha\beta}h^{\mu}_{\beta}R_{\mu\alpha} + \frac{1}{2}h^{\mu\nu}h^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{n}hh^{\mu\nu}R_{\mu\nu} - \\ &- \frac{1}{2n}h^{\mu\nu}h_{\mu\nu}R - \frac{1}{2}\left(\nabla_\mu c'^{\,(0,0)}\nabla^\mu c'^{\,(0,0)} + 2\left(\nabla_\nu h^{\nu}_{\mu} - \frac{1}{n}\nabla_\mu h\right)\nabla^\mu c'^{\,(0,0)}\right) + \\ &+ \left(-f\Box f + \frac{\alpha}{2}f\Box h + \frac{\alpha}{2}h\Box f\right) + \frac{1}{2n^2}h^2 R\,.\end{aligned} \tag{5.50}$$

where $\rho_1 = \frac{1}{2}$ in order to cancel the non-diagonal parts in the kinetic term for $h_{\mu\nu}$.

To write it in the form (5.45), we identify

$$\Psi^A = \begin{pmatrix} h^{\mu\nu} \\ f \\ c' \end{pmatrix} \tag{5.51}$$

and the differential operator takes the form

$$F_{AB} = \gamma_{AB}\Box + J^{\mu\nu}_{AB}\nabla_\mu\nabla_\nu + M_{AB}\,. \tag{5.52}$$

where the different matrices involved read

$$\gamma_{AB} = \begin{pmatrix} -\frac{1}{4}\left(\frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) g_{\alpha\beta} & \frac{\alpha}{2} g_{\mu\nu} & -\frac{1}{8} g_{\mu\nu} \\ \frac{\alpha}{2} g_{\rho\sigma} & -1 & 0 \\ -\frac{1}{8} g_{\rho\sigma} & 0 & \frac{1}{2} \end{pmatrix} \tag{5.53}$$

$$J^{\alpha\beta}_{AB} = \begin{pmatrix} 0 & 0 & \frac{1}{4}\left(g^\alpha_\mu g^\beta_\nu + g^\alpha_\nu g^\beta_\mu\right) \\ 0 & 0 & 0 \\ \frac{1}{4}\left(g^\alpha_\rho g^\beta_\sigma + g^\alpha_\sigma g^\beta_\rho\right) & 0 & 0 \end{pmatrix} \tag{5.54}$$

$$M_{AB} = \begin{pmatrix} M_{hh} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.55}$$

with

$$M_{hh} = \left(\frac{1}{2}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) R_{\alpha\beta} - \frac{1}{8}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\gamma_{\alpha\beta} R + \frac{1}{2} R_{(\mu\rho\nu\sigma)}\,. \tag{5.56}$$

The round parenthesis for us mean complete symmetrization in all the enclosed indices unless otherwise stated. We have introduced the tensors

$$\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{4}\left(g_{\mu\rho}\delta^{(\alpha}_\nu\delta^{\beta)}_\sigma + g_{\mu\sigma}\delta^{(\alpha}_\nu\delta^{\beta)}_\rho + g_{\nu\rho}\delta^{(\alpha}_\mu\delta^{\beta)}_\sigma + g_{\nu\sigma}\delta^{(\alpha}_\mu\delta^{\beta)}_\rho\right), \tag{5.57}$$

$$\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\nu}\delta^{(\alpha}_\rho\delta^{\beta)}_\sigma + g_{\rho\sigma}\delta^{(\alpha}_\mu\delta^{\beta)}_\nu\right). \tag{5.58}$$

Since the principal symbol of this operator (the highest order in derivatives) is not diagonal but it contains a non-minimal term given by the matrix $J^{\alpha\beta}_{AB}$, the application of the Schwinger-DeWitt technique requires extra work.

5.2.1 *The Barvinsky-Vilkovisky technique*

There is a useful technique developed in [74] to compute the contribution to the quantum effective action of non-minimal operators. It is our aim to apply it to the case of S_{UG} as given in (5.44).

Let us concentrate on the highest derivative term of the operator

$$D_{AB}(\nabla) = \gamma_{AB}\Box + J^{\alpha\beta}_{AB}\nabla_\alpha\nabla_\beta\,. \tag{5.59}$$

Furthermore, assume that the full operator F_{AB} can be included in a one-parameter family

$$F_{AB}(\nabla|\lambda) = D_{AB}(\nabla|\lambda) + M_{AB}, \qquad 0 \leq \lambda \leq 1, \tag{5.60}$$

so that F_{AB} is minimal at $\lambda = 0$. In our case, we can simply choose

$$F_{AB}(\nabla|\lambda) = \gamma_{AB}\Box + \lambda J^{\alpha\beta}_{AB}\nabla_\alpha\nabla_\beta + M_{AB}. \tag{5.61}$$

where λ parametrizes the introduction of the non-minimal term in such a way that for $\lambda = 0$ the operator is minimal and for $\lambda = 1$ the operator whose determinant is desired is obtained.

Following Schwinger the effective action can be obtained by differentiating in λ and integrating afterwards, arriving to[2]

$$W(\lambda) = W(0) - \frac{1}{2}\int_0^\lambda d\lambda' Tr\left[\frac{d\widehat{F}(\lambda')}{d\lambda'}\widehat{G}(\lambda')\right]. \tag{5.62}$$

where $\widehat{G}(\lambda)$ is the Green function of the operator F_{AB}, defined by $\widehat{F}(\lambda)\widehat{G}(\lambda) = \mathbb{I}$, and we are dismissing ultralocal contributions and keeping only the

2 In order to simplify the notation, we are going to use a hat symbol for matrix operators carrying mixed capital indices. Thus, things like the following are assumed

$$\hat{A} \equiv A = A^B_A,$$
$$\hat{A}\hat{B} \equiv AB = A^B_A B^C_B,$$
$$Tr(\hat{A}) = tr\left(\gamma^B_A A^A_B\right),$$

and so on. Here tr denotes the usual matrix trace (i.e. sum of the elements of the diagonal).

pole part of $W(\lambda)$. The effective action for our original operator corresponds to $\lambda = 1$. Here, $W(0)$ is the effective action of the corresponding minimal operator, obtained by setting $\lambda = 0$.

Many of the technical difficulties appear already in flat space. It is useful to consider the ordinary matrix in (euclidean) momentum space

$$\hat{D}(k) = D^A_B(k) = \gamma^{AC} D_{CB}(k), \tag{5.63}$$

with k_μ a constant vector. Its inverse has the form,

$$\hat{D}^{-1}(k) = \frac{\hat{K}(k)}{(k^2)^m}, \tag{5.64}$$

with m being an integer.

From this, it is clear that

$$\hat{D}(k)\hat{K}(k) = (k^2)^m \mathbb{I}. \tag{5.65}$$

Were we to trade the vector k_μ for the covariant derivative, and owing to the non-commutative character of the latter, a remainder appears

$$\hat{D}(\nabla)\hat{K}(\nabla) = \Box^m + \hat{K}_1(\nabla), \tag{5.66}$$

and going to the full operator, we get

$$\hat{F}(\nabla)\hat{K}(\nabla) = \Box^m + \hat{M}(\nabla), \tag{5.67}$$

with $\hat{K}_1$ and $\hat{M}$ being now operators of as much order $2m-1$ in derivatives.

The last equation allows us to expand the Green function of $\hat{F}$ in powers of $\hat{M}$ as follows

$$\hat{G} = -\hat{K}\frac{\mathbb{I}}{\Box^m}\sum_{p=0}^{4}\left(-\hat{M}\frac{\mathbb{I}}{\Box^m}\right)^p + O\left(\mathfrak{m}^5\right). \tag{5.68}$$

The notation $O\left(\mathfrak{m}^5\right)$ means that we are keeping only terms up to background dimension four.

May be this is a good point to comment of power divergences [75, 76] in the heat kernel formalism. As was stated in chapter 4, it is possible to regularize the proper time integral by introducing both an ultraviolet Λ^{-2} and infrared η^{-2} cutoff, that is

$$\int_0^\infty d\tau \to \int_{\Lambda^{-2}}^{\eta^{-2}} \tag{5.69}$$

In that way we get

$$W_\infty = \frac{1}{2} a_0 \Lambda^4 + a_2 \Lambda^2 + a_4 \log \left(\frac{\Lambda}{\eta} \right). \tag{5.70}$$

It is to be stressed that those are gauge invariant *proper time cutoffs*, not to be confused with momentum cutoffs.

The first term in this expansion yields a universal (that is, independent of the form of the action) quartic renormalization of the cosmological constant. This term is non-dynamical in that it does not depend on any of the fields present in the theory.

The second term yields quadratic divergences, and the last term yields the physically most interesting ones, namely the logarithmic divergences. In pure gravity the quadratic divergences are necessarily proportional to

$$\int R d\mu(g), \tag{5.71}$$

which is the only dimension two invariant. The measure depends on whether full *Diff*(M) invariance is implemented (as in GR, $d\mu(g) \equiv \sqrt{|g|} d^n x$) or else only the subgroup *TDiff*(M) (as in UG $d\mu(g) \equiv d^n x$). There has been some discussion going on in the literature on the physical relevance of those quadratic divergences *confer* [77–80].

We shall concentrate on the logarithmic ones (which are the only ones seen in dimensional regularization). In order to compute this Green function it is useful to commute the $\frac{\mathbb{I}}{\Box^m}$ to the right

$$\hat{G} = -\hat{K} \sum_{p=0}^{4} (-1)^p \hat{M}_p \frac{\mathbb{I}}{\Box^{m(p+1)}} + O\left(\mathfrak{m}^5\right), \tag{5.72}$$

with the operators M_p given recursively by

$$\hat{M}_0 = \mathbb{I}, \tag{5.73}$$

$$\hat{M}_{p+1} = \hat{M}\hat{M}_p + [\Box^m, \hat{M}_p]. \tag{5.74}$$

Furthermore, it can be proven [74] that if the coefficient of the highest derivative term (of order $2d$) is covariantly conserved and there is no term of order $2d-1$, as it is the case for (5.61), then $M_4 = M_3 = 0$ and $M_2 = M^2 + m[\Box, M]\Box^{m-1}$.

Turning now our attention to this explicit case and computing the inverse of the operator $\hat{D}(k)$ in the sense (5.64) we get $m = 3$ and the calculation of the effective action then reduces just to

$$W(\lambda) = W(0) - \\ -\frac{1}{2}\int_0^\lambda d\lambda' Tr\left[\hat{J}^{\alpha\beta}\nabla_\alpha\nabla_\beta\left\{\hat{K}\left(-\frac{\mathbb{I}}{\Box^3} + \hat{M}\frac{\mathbb{I}}{\Box^6} - 3[\Box,\hat{M}]\frac{\mathbb{I}}{\Box^7} - \hat{M}^2\frac{\mathbb{I}}{\Box^9}\right)\right\}\right]. \tag{5.75}$$

The computation of $W(0)$ is just the one of a minimal second order operator cf.[74]

$$W(0) = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\left\{\frac{16}{15}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \left(\frac{2}{8\alpha^2-1} - \frac{46}{15}\right)R_{\mu\nu}R^{\mu\nu} + \\ + \left(\frac{13}{24} + \frac{1}{2-16\alpha^2}\right)R^2\right\}. \tag{5.76}$$

The rest of the pieces in (5.75) are obtained following the steps outlined above. The number of terms grows enormously after applying successive derivatives through Leibniz's rule. The computation has been performed with the help of the Mathematica software *xAct* [81]. A fair amount of computing time has been necessary in order to simplify the resulting expressions.

There is a last non-trivial issue that we have to take care of. After computing the terms in (5.75), the output will be a collection of terms of the schematic form

$$Tr\left(\mathcal{O}_{\nu_1\nu_2\ldots\nu_j}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_p}\frac{\mathbb{I}}{\Box^n}\right), \tag{5.77}$$

with $p \leq 2n-4$.

These functional traces can be computed by introducing the formal representation of $\frac{\mathbb{I}}{\Box^n}$ through a Laplace transform and performing a dimensional regularization afterwards, keeping only the logarithmic divergent terms as explained in [74] and summarized in Appendix C.

After doing all this and computing the functional traces we are finally left with a simple result for the perturbation to $W(0)$

$$-\frac{1}{2}\int_0^1 d\lambda' Tr\left[\hat{J}^{\alpha\beta}\nabla_\alpha\nabla_\beta\left\{\hat{K}\left(-\frac{\mathbb{I}}{\Box^3}+\hat{M}\frac{\mathbb{I}}{\Box^6}-3[\Box,\hat{M}]\frac{\mathbb{I}}{\Box^7}-\hat{M}^2\frac{\mathbb{I}}{\Box^9}\right)\right\}\right]=$$
$$=\frac{1}{16\pi^2}\frac{1}{n-4}\int d^nx\left\{\left(\frac{1}{6\alpha^2}+\frac{2}{1-8\alpha^2}\right)R_{\mu\nu}R^{\mu\nu}+\frac{1}{24}\left(\frac{12}{8\alpha^2-1}-\frac{1}{\alpha^2}-5\right)R^2\right\} \tag{5.78}$$

And finally, putting all together we find that the contribution to the pole part of the effective action of S_{UG} is

$$W_\infty^{UG}=\frac{1}{16\pi^2}\frac{1}{n-4}\int d^nx\left\{\frac{16}{15}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}+\left(\frac{1}{6\alpha^2}-\frac{46}{15}\right)R_{\mu\nu}R^{\mu\nu}+\left(\frac{1}{3}-\frac{1}{24\alpha^2}\right)R^2\right\}, \tag{5.79}$$

where we have neglected total derivatives in the integrand.

As has been already advertised, all the dependence on the gauge fixing, represented by the presence of the parameter α in the final result, disappears when we use the background equations of motion $R_{\mu\nu}=\frac{1}{4}Rg_{\mu\nu}$. This is as it should be because all gauge fixing is BRST exact.

5.2.2 *The final result*

After computing the contribution of the non-minimal operator, we are finally ready to write the pole part of the effective action of Unimodular Gravity, which reads

$$W_\infty=W_\infty^{UG}+W_\infty^{bc}+W_\infty^{\pi}+W_\infty^{\bar{c}\phi}+W_\infty^{W}. \tag{5.80}$$

Here W_∞^{UG} is the contribution we have computed in the last section while the rest of the contributions are given in Appendix B. Adding everything, we find that the final result is

$$W_\infty = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x \left(\frac{119}{90}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \left(\frac{1}{6\alpha^2} - \frac{359}{90}\right)R_{\mu\nu}R^{\mu\nu} + \right.$$
$$\left. + \frac{1}{72}\left(22 - \frac{3}{\alpha^2}\right)R^2\right). \tag{5.81}$$

Now we would like to focus on the issue of on-shell renormalizability. It is known that although General Relativity is one-loop finite in the absence of a cosmological constant, this property is lost in its presence. The on-shell counterterm in this case was obtained in [82] and it amounts to a renormalization of the cosmological constant and is proportional to

$$W_\infty^{GR} \equiv \frac{1}{16\pi^2(n-4)}\int \sqrt{|g|}d^4x \, \left(\frac{53}{45}\,W_4 - \frac{1142}{135}\Lambda^2\right). \tag{5.82}$$

Since the main attractive feature of Unimodular Gravity is precisely the different rôle that the cosmological constant plays with respect to GR, we would like to see what happens here with the renormalization group flow when we take the counterterm to be on-shell so that all external legs correspond to physical states. In that case, the equations of motion for the $|g| = 1$ fixed background are the traceless Einstein equations

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = 0, \tag{5.83}$$

which, altogether with Bianchi identities, imply the following for the operators appearing in the effective action

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = E_4, \tag{5.84}$$
$$R_{\mu\nu}R^{\mu\nu} = \frac{1}{4}R^2, \tag{5.85}$$
$$R = \text{constant}. \tag{5.86}$$

The first line is nothing more than the statement of the Gauss-Bonet theorem when we take into account the equations of motion. E_4 is thus Euler density, whose integral gives the Euler characteristic of the manifold.

By using these, we find that the on-shell effective action takes then the form

$$W_{\infty}^{\text{on-shell}} = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x \left(\frac{119}{90}E_4 - \frac{83}{120}R^2\right). \tag{5.87}$$

The contribution of the cosmological constant to the effective action happens to be a non-dynamical quantity, since it does not couple to the metric because the $\sqrt{g}$ factor in the integration measure is absent. This implies that we can disregard this term since it will not contribute to any correlator involving physical fields. We conclude, therefore, that in this case there is no renormalization of the cosmological constant and its peculiar status in Unimodular Gravity is preserved through quantum corrections.

Indeed, this effect is not specific to one-loop computations. We then conclude that the bare value of the cosmological constant is protected and quantum corrections do not modify it.

It could be thought that this effect is just a gauge artifact of our background choice $|\tilde{g}| = 1$. However, it can be easily argued that this is not the case. As we have commented before in this work, if we now want to obtain the effective action for an arbitrary background from the one with unimodular background metric, it is enough to make a change of variables so that

$$\tilde{g}_{\mu\nu} = g^{-\frac{1}{n}} g_{\mu\nu}. \tag{5.88}$$

This transformation is available as long as there is no conformal anomaly in the theory. This is indeed the case, since there exists a regularization in which the anomaly vanishes [69, 83] When doing this, we can see that the real reason of the cosmological constant not being renormalized is indeed the presence of Weyl invariance in our formalism, which protects the appearance of any mass scale in the effective action and, as a consequence, in the expectation value of the equations of motion. Therefore, our argument holds and the cosmological constant is protected and fixed to its bare value all along the renormalization group flow and at any loop order.

Part IV

Unimodular Gravity versus General Relativity

6

Gravitational corrections to the beta function

We study the beta functions of the quartic and Yukawa couplings of General Relativity and Unimodular Gravity coupled to the $\lambda\phi^4$ and Yukawa theories with masses. We show that the General Relativity corrections to those beta functions as obtained from the 1PI functional by using the standard MS multiplicative renormalization scheme of Dimensional Regularization are gauge dependent and, further, that they can be removed by a non-multiplicative, though local, field redefinition. An analogous analysis is carried out when General Relativity is replaced with Unimodular Gravity. Thus we show that any claim made about the change in the asymptotic behaviour of the quartic and Yukawa couplings made by General Relativity and Unimodular Gravity lack intrinsic physical meaning.

Robinson and Wilczek suggested that when coupled to a Yang-Mills theory, it improves the behaviour of the theory regarding asymptotic freedom [84]; but it was proved later that this result is gauge dependent [85, 86]. Further, it is also known that a non-multiplicative renormalization can be used to eliminate some of the contributions to the beta functions in the Yang-Mills case [87].

The beta functions of the $\lambda\phi^4$ (quartic) and Yukawa couplings, and the logarithmic UV divergences that contribute to them, enter the quantitative analysis of a large number of High Energy Physics topics. Two chief topics among these are *a)* the study of the vacuum stability by using the renormalization-group-improved effective potential, with its implications in the study of the Physics of the Early Universe and the physics of the Standard Model and beyond -see [88] and references therein, and *b)* the construction of Asymptotically Safe theories of Quantum Gravity coupled to matter along the lines laid out in Reference [89]. Thus it is plain that the computation of the logarithmic UV divergent contributions which may lead to a change in the value of beta functions in question due to the interaction of the corresponding matter fields with gravitons is needed. In view of the fate of the corrections found in Reference [84], it is necessary to see whether or not these gravitational corrections are gauge independent and invariant under non-multiplicative field renormalization so that an intrinsic physical meaning can be ascribed to them.

In Reference [90], it was shown that the General Relativity contributions to beta functions of the quartic and Yukawa couplings obtained by using the multiplicative MS scheme of Dimensional Regularization applied to the 1PI functional do not vanish in the de Donder gauge of the graviton field. The contributions obtained lead to asymptotic freedom for appropriate values of the masses involved –among these values are masses of the real Higgs and top quark.

The first aim of the present section is to show that the General Relativity corrections to the beta functions of the quartic and Yukawa couplings as computed in the de Donder gauge in [90] are gauge dependent artifacts and that, besides, they can be removed by appropriate non-multiplicative field redefinitions. Thus, we conclude that the General Relativity corrections to the beta functions in question obtained by using the multiplicative MS scheme of Dimensional Regularization applied to the 1PI functional have no intrinsic physical meaning and, that, therefore, any physical conclusion derived from them cannot be trusted. The second aim is to show that this same situation is reproduced when Unimodular Gravity is used instead of General Relativity. We shall actually see that in the gauge we shall use the Unimodular Gravity corrections to the beta function of the quartic coupling vanish in the Multiplicative MS scheme of Dimensional Regularization.

One word of caution: when, in the following, we talk about gravity corrections –either from General Relativity or from Unimodular Gravity– we refer to corrections that are of order κ^2.

6.1 THE SETTING

We start from the well known Einstein-Hilbert Lagrangian coupled to a massive real scalar ϕ via a ϕ^4 interaction and a Dirac fermion ψ via a Yukawa interaction. This is

$$\mathcal{L}_{\rm GR} =\sqrt{-g}\left\{-\frac{2}{\kappa^2}R+\bar{\psi}(i\not{D}-m_\psi)\psi+\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+ -\frac{1}{2}m_\phi^2\phi^2-g\phi\bar{\psi}\psi-\frac{\lambda}{4!}\phi^4\right\}, \tag{6.1}$$

while for unimodular gravity

$$\begin{aligned}\mathcal{L}_{\rm UG} =& -\frac{2}{\kappa^2}(-g)^{\frac{1}{4}}\ \left(R+\frac{3}{32}\frac{\nabla_\mu g\nabla^\mu g}{g^2}\right)+\bar{\psi}(i\not{D}-m_\psi)\psi+\\ &+\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m_\phi^2\phi^2-g\phi\bar{\psi}\psi-\frac{\lambda}{4!}\phi^4,\end{aligned} \tag{6.2}$$

where $\kappa = 32\pi G$, and g, λ are -respectively- the Yukawa and the ϕ^4 coupling constants.

In order to keep explicit the gauge dependence, we use a generalized gauge condition for general relativity:

$$\mathcal{L}_{\rm GR} = \alpha\Big(\partial^\mu h_{\mu\nu}-\frac{1}{2}\partial_\nu h\Big)^2, \tag{6.3}$$

where α is an arbitrary gauge parameter. This yields a propagator

$$\begin{aligned}\langle h_{\mu\nu}(k)h_{r\sigma}(-k)\rangle_{\rm GR} =& \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\rho\sigma}\right)-\\ &-i\left(\frac{1}{2}+\alpha\right)\left(\eta_{\mu\rho}k_\nu k_\sigma+\eta_{\mu\sigma}k_\nu k_\rho+\eta_{\nu\rho}k_\mu k_\sigma+\eta_{\nu\sigma}k_\mu k_\rho\right).\end{aligned} \tag{6.4}$$

The gauge fixing and propagator of unimodular gravity are found in [15, 38, 91] and read

$$\begin{aligned}\langle h_{\mu\nu}(k)h_{\rho\sigma}(-k)\rangle =& \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}\right)-\frac{i}{k^2}\frac{\alpha^2n^2-n+2}{\alpha^2n^2(n-2)}\eta_{\mu\nu}\eta_{\rho\sigma}\\ &+\frac{2i}{n-2}\left(\frac{k_\rho k_\sigma\eta_{\mu\nu}}{(k^2)^2}+\frac{k_\mu k_\nu\eta_{\rho\sigma}}{(k^2)^2}\right)-\frac{2in}{n-2}\frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2)^3}.\end{aligned} \tag{6.5}$$

And it reduces in 4 dimensions to

$$\langle h_{\mu\nu}(k)h_{r\sigma}(-k)\rangle_{\rm UG} = \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}\right) - \frac{i}{k^2}\frac{8\alpha^2-1}{16\alpha^2}\eta_{\mu\nu}\eta_{\rho\sigma} + \\ + i\left(\frac{k_\rho k_\sigma \eta_{\mu\nu}}{k^4}+\frac{k_\mu k_\nu \eta_{\rho\sigma}}{k^4}\right) - 4i\frac{k_\mu k_\nu k_\rho k_\sigma}{k^6}. \qquad (6.6)$$

Let us remark that in the case of unimodular gravity the interaction comes from $h_{\mu\nu}\widehat{T}^{\mu\nu} = \widehat{h}_{\mu\nu}T^{\mu\nu}$ with $T^{\mu\nu}$ the energy-momentum tensor and the hat quantities the traceless ones. Therefore one can work with the traceless propagator $\langle\widehat{h}_{\mu\nu}(k)\widehat{h}_{r\sigma}(-k)\rangle$ (which can be trivially obtained from (6.5)) and the full energy-momentum tensor, therefore using the same Feynman rules for the vertices as in general relativity, or use (6.5) coupled to $\widehat{T}^{\mu\nu}$.

In order to compute the beta functions, the first step is to find the 1PI gravitational corrections to the scalar and fermion propagators. These are shown in figures 1a and 1b.

(a) Scalar propagator P_S

(b) Fermion propagator P_Y

Figure 1: Corrections to the scalar and fermion propagators.

Using the propagators listed above, and computing divergences in dimensional regularization ($D = 4 + 2\epsilon$) these are,

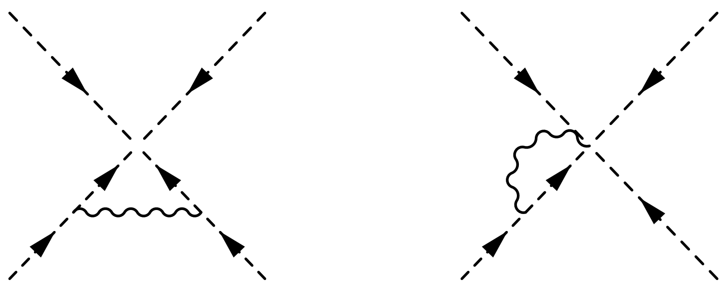
$$P_S^{\mathrm{GR}} = \kappa^2 \left(-\frac{i}{16\pi^2\epsilon}\right) m_\phi^2 \left[1 + \left(\frac{1}{2} + \alpha\right)\right](p^2 - m_\phi^2), \tag{6.7}$$

$$P_S^{\mathrm{UG}} = 0, \tag{6.8}$$

$$P_Y^{\mathrm{GR}} = \kappa^2 \left(-\frac{i}{16\pi^2\epsilon}\right) \left\{\frac{3}{8} m_\psi p^2 - \frac{1}{8} p^2 \not{p} + \frac{1}{4} m_\psi^2 (\not{p} - m_\psi) + \right.$$
$$\left. + \left(\frac{1}{2} + \alpha\right) \left[\frac{3}{4} m_\psi p^2 - \not{p}\left(\frac{15}{32} p^2 + \frac{29}{32} m_\psi^2\right) - \frac{19}{16} m_\psi^3\right]\right\}, \tag{6.9}$$

$$P_Y^{\mathrm{UG}} = \kappa^2 \left(-\frac{i}{16\pi^2\epsilon}\right) \left\{\not{p}\left(\frac{3}{16} m_\psi^2 - \frac{5}{16} p^2\right) + \frac{3}{8} m_\psi p^2\right\}. \tag{6.10}$$

The corrections to the ϕ^4 (1PI) vertex (figure 2) read



(a) + 5 permutations (b) + 4 permutations

Figure 2: ϕ^4 vertices.

$$V_\phi^{\mathrm{GR}} = \kappa^2 \lambda \left(-\frac{i}{16\pi^2\epsilon}\right) \left(\frac{3}{2} + \alpha\right) \left[\frac{1}{2} \sum_{i=1}^{4} p_i^2 - 4 m_\phi^2\right], \tag{6.11}$$

$$V_\phi^{\mathrm{UG}} = 0. \tag{6.12}$$

Finally, we compute the divergences of the (1PI) Yukawa vertices listed in figure 3. These ones read

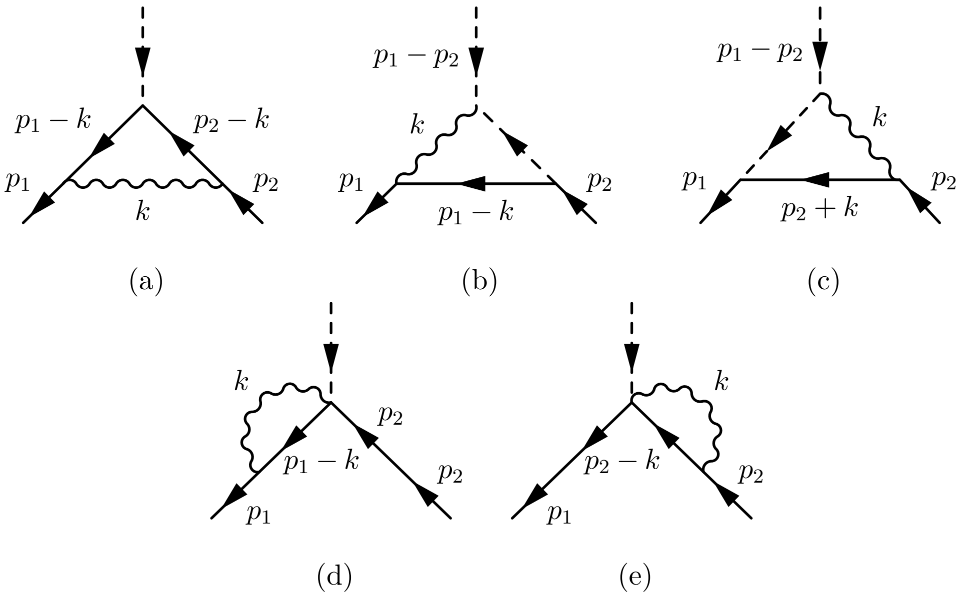


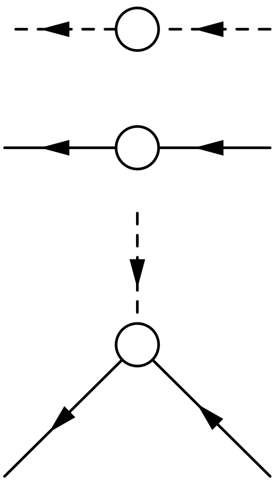
Figure 3: Contributions to the Yukawa vertex.

$$
\begin{aligned}
V_\psi^{\mathrm{GR}} &= g\kappa^2\left(-\frac{i}{16\pi^2\epsilon}\right)\left[-\frac{1}{4}m_\phi^2-\frac{3}{4}m_\psi^2+\frac{1}{16}(p_1+p_2)^2+\frac{1}{4}m_\psi(\not{p}_1+\not{p}_2)+\right.\\
&\quad\left.+\frac{1}{8}\not{p}_1\not{p}_2\right]+g\kappa^2\left(-\frac{i}{16\pi^2\epsilon}\right)\left(\frac{1}{2}+\alpha\right)\left[-m_\phi^2-\frac{57}{16}m_\psi^2+\frac{47}{32}(p_1^2+p_2^2)-\right.\\
&\quad\left.-\frac{13}{8}p_1\cdot p_2+m_\psi(\not{p}_1+\not{p}_2)-\frac{9}{16}\not{p}_1\not{p}_2\right], \qquad (6.13)\\
V_\psi^{\mathrm{UG}} &= g\kappa^2\left(-\frac{i}{16\pi^2\epsilon}\right)\left[\frac{9}{16}(p_1^2+p_2^2)-\frac{3}{8}p_1\cdot p_2+\frac{3}{16}m_\psi(\not{p}_1+\not{p}_2)-\frac{3}{8}\not{p}_1\not{p}_2\right]. \qquad (6.14)
\end{aligned}
$$

6.2 BETA FUNCTIONS

We shall proceed now to the computation of the Yukawa and quartic coupling beta function gravitational corrections coming from General Relativity and Unimodular Gravity. To use the well known multiplicative MS renormalization scheme of Dimensional Regularization, we define

$$g_0 = \mu^{-\epsilon} Z_g Z_\psi^{-1} Z_\phi^{-1/2} g, \qquad Z_g = 1 + \delta Z_g, \tag{6.15}$$
$$\Psi_0 = Z_\Psi^{1/2} \Psi, \qquad Z_\Psi, = 1 + \delta Z_\Psi, \tag{6.16}$$
$$\bar{\Psi}_0 = Z_\Psi^{1/2} \bar{\Psi}, \qquad Z_\phi = 1 + \delta Z_\phi, \tag{6.17}$$
$$m_{\Psi_0} = Z_m Z_\Psi^{-1} m_\Psi, \qquad Z_{m_\Psi} = 1 + \delta Z_{m_\Psi}, \tag{6.18}$$
$$m_{\phi_0} = Z_m Z_\phi^{-1} m_\phi, \qquad Z_{m_\phi} = 1 + \delta Z_{m_\phi}. \tag{6.19}$$



$$= \quad i(\delta Z_\Psi \not{p} - \delta Z_{m_\Psi} m_\Psi),$$

$$= \quad i(\delta Z_\phi p^2 - \delta Z_{m_\phi} m_\phi^2),$$

$$= \quad -ig\mu^{-\epsilon} \delta Z_g.$$

Figure 4: Counterterms

The counterterms obtained from the previous definitions are given in figure 4.

Following the standard MS procedure, the wave function renormalizations (δZ_Ψ and δZ_Ψ) are obtained by imposing that the contributions proportional to $\not{p}$ in the sum given in figure 5 are finite as $\epsilon \to 0$. This yields the values

$$\delta Z_\phi = \frac{1}{16\pi^2\epsilon}\kappa^2 m_\phi^2\left[1+\left(\frac{1}{2}+\alpha\right)\right], \tag{6.20}$$

$$\delta Z_\Psi = \frac{1}{16\pi^2\epsilon}\kappa^2 m_\Psi^2\left[\frac{1}{4}+\left(\frac{1}{2}+\alpha\right)\frac{29}{32}\right]. \tag{6.21}$$

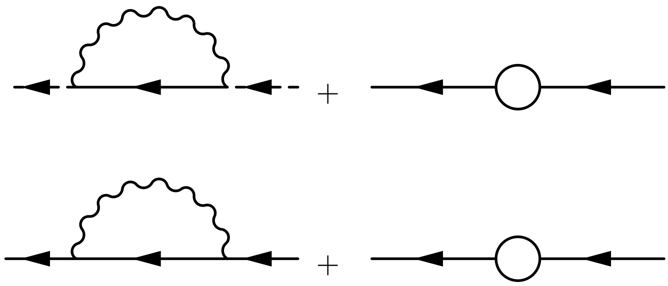


Figure 5: Wave function renormalization.

For δZ_g, we demand that there is no singularity independent of the external momenta at $\epsilon \to 0$ in the sum of figure 6; hence

$$\delta Z_g = \frac{1}{16\pi^2\epsilon}\kappa^2\left\{m_\phi^2\left[\frac{1}{4}+\left(\frac{1}{2}+\alpha\right)\right]+m_\Psi^2\left[\frac{3}{4}\left(\frac{1}{2}+\alpha\right)\frac{57}{16}\right]\right\}. \tag{6.22}$$

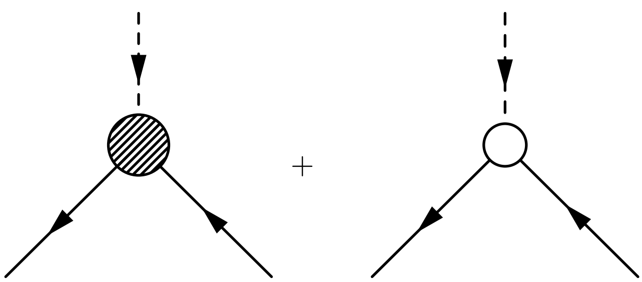


Figure 6: Yukawa vertex renormalization.

Defining $\beta_g = \mu \frac{dg(\mu)}{d\mu}$, and using standard techniques, one obtains the General Relativity contribution, $\beta_g^{\rm GR}$, to β_g, at order κ^2, from $\delta \widetilde{Z}_g = \delta Z_g - \delta Z_\Psi - \frac{1}{2}\delta Z_\phi$:

$$\beta_g^{\rm GR} = \frac{1}{16\pi^2}\kappa^2\Big\{m_\phi^2\Big[\frac{1}{2} - \Big(\frac{1}{2}+\alpha\Big)\Big] + m_\Psi^2\Big[-1-\Big(\frac{1}{2}+\alpha\Big)\frac{85}{16}\Big]. \tag{6.23}$$

The explicit dependence on the parameter α shows the gauge-dependent nature of this beta function in presence of gravity. Insofar as no physical observables can depend on the gauge, no physical consequences should be extracted from here.

We follow the same procedure for unimodular gravity to find

$$\delta Z_\Psi^{\rm UG} = \frac{1}{16\pi^2\epsilon}\kappa^2 m_\Psi^2 \frac{3}{16}, \tag{6.24}$$

$$\delta Z_\phi^{\rm UG} = 0, \tag{6.25}$$

$$\delta Z_g^{\rm UG} = 0, \tag{6.26}$$

so that

$$\beta_g^{\rm UG} = \frac{1}{16\pi^2}\kappa^2 m_\Psi^2 \frac{3}{16}. \tag{6.27}$$

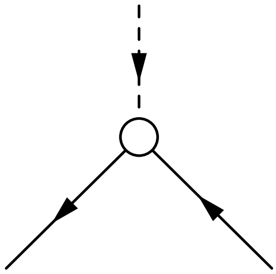
We can see that we get a difference between general relativity and unimodular gravity by comparing (6.23) and (6.27). However we will see in the sequel that we can get rid of these beta functions by using a *non-multiplicative* renormalization, this is, by performing a field redefinition.

Let us now define

$$g_0 = \mu^{-\epsilon} Z_g Z_\psi^{-1} Z_\phi^{-1/2} g, \qquad \phi_0 = \phi + \frac{1}{2}\delta Z_\phi \phi, \tag{6.28}$$

$$\Psi_0 = \Psi + \frac{1}{2}\delta Z_\Psi \Psi + \frac{1}{2}a_1\kappa^2 m_\Psi^2 \Psi + \frac{1}{2}b_1 \kappa^2 m_\phi^2 \Psi, \quad m_{\Psi_0} = (1+\delta Z_{m_\Psi}) m_\Psi, \tag{6.29}$$

$$\bar\Psi_0 = \bar\Psi + \frac{1}{2}\delta Z_\Psi \bar\Psi + \frac{1}{2}a_1\kappa^2 m_\Psi^2 \bar\Psi + \frac{1}{2}b_1 \kappa^2 m_\phi^2 \bar\Psi, \quad m_{\phi_0} = (1+\delta Z_{m_\phi}) m_\phi. \tag{6.30}$$



$$= \quad -ig(\mu^{-\epsilon}\delta Z_g + a_1\kappa^2 m_\Psi^2 + b_1\kappa^2 m_\phi^2).$$

Figure 7: Counterterm.

Therefore the matter lagrangian can be written as

$$\begin{aligned}
\mathcal{L}_{\rm m} &= \bar{\Psi}_0(i\partial\!\!\!/ - m_{\Psi_0})\Psi_0 + \frac{1}{2}(\partial_\mu\phi_0\partial^\mu\phi_0 - m_{\phi_0}^2\phi_0^2) - g_0\Psi_0\bar{\phi}_0\Psi_0 = \\
&= \bar{\Psi}(i\partial\!\!\!/ - m_\Psi)\Psi + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_\phi^2\phi^2) - g\mu^{-\epsilon}\Psi\bar{\phi}\Psi + \\
&+ \left\{\delta Z_\Psi\bar{\Psi}i\partial\!\!\!/\Psi + \delta Z_\phi\partial_\mu\phi\partial^\mu\phi - m_\Psi\delta Z_{m_\Psi}\bar{\Psi}\Psi - \frac{1}{2}m_\phi^2\delta Z_{m_\phi}\right\} - \\
&- g\mu^{-\epsilon}\left\{\delta Z_g + a_1\kappa^2 m_\Psi^2 + b_1\kappa^2 m_\phi^2\right\}\bar{\Psi}\phi\Psi + \\
&+ \frac{1}{2}\left(a_1\kappa^2 m_\Psi^2 + b_1\kappa^2 m_\phi^2\right)\left[i\bar{\Psi}\phi\partial\!\!\!/\Psi + i\partial\!\!\!/\phi\Psi\right].
\end{aligned} \tag{6.31}$$

While the counterterms for the scalar and fermion field propagator remain unchanged with respect to the multiplicative renormalization, the counterterm for the vertex is now given by the expression in figure 7.

Imposing again that the sum in figure 6 is zero (plus terms depending on the external momenta) when $\epsilon \to 0$, we find

$$\delta Z_\phi = \frac{1}{16\pi^2\epsilon}\kappa^2 m_\phi^2\left[1 + \left(\frac{1}{2} + \alpha\right)\right], \tag{6.32}$$

$$\delta Z_\Psi = \frac{1}{16\pi^2\epsilon}\kappa^2 m_\Psi^2\left[\frac{1}{4} + \frac{29}{32}\left(\frac{1}{2} + \alpha\right)\right], \tag{6.33}$$

$$\begin{aligned}
\delta\widetilde{Z}_g &= \delta Z_g - \delta Z_\Psi - \frac{1}{2}\delta Z_\phi = \\
&= \frac{1}{16\pi^2\epsilon}\kappa^2 m_\phi^2\left[-\frac{1}{4} + \frac{1}{2}\left(\frac{1}{2} + \alpha\right)\right] + \frac{1}{16\pi^2\epsilon}\kappa^2 m_\Psi^2\left[\frac{1}{2} + \frac{85}{32}\left(\frac{1}{2} + \alpha\right)\right] - \\
&- a_1\kappa^2 m_\Psi^2 - b_1\kappa^2 m_\phi^2.
\end{aligned} \tag{6.34}$$

It is clear that by choosing

$$a_1 = \frac{1}{16\pi^2\epsilon}\left[\frac{1}{2} + \frac{85}{32}\left(\frac{1}{2} + \alpha\right)\right], \tag{6.35}$$

$$b_1 = \frac{1}{16\pi^2\epsilon}\left[-\frac{1}{4} + \frac{1}{2}\left(\frac{1}{2} + \alpha\right)\right], \tag{6.36}$$

we shall wipe out the gravitational correction to $\delta\widetilde{Z}_g$ so the gravitational corrections to the beta function of the Yukawa coupling is now given by

$$\beta_g^{\mathrm{GR}}\Big|_{\mathrm{gravitational}} = 0. \tag{6.37}$$

We have seen here that the gravitational contribution to β_g can be brushed away by carrying out a field redefinition. Therefore, it is an *inessential* [89] contribution. However, notice that one cannot do the same with the contributions in absence of gravity, which show that they are *essential* contributions.

Finally, we can perform the same non-multiplicative renormalization for unimodular gravity finding

$$\delta Z_\phi^{\mathrm{UG}} = 0, \tag{6.38}$$

$$\delta Z_\Psi^{\mathrm{UG}} = \frac{1}{16\pi^2\epsilon}\kappa^2 m_\Psi^2 \frac{3}{16}, \tag{6.39}$$

$$\delta \widetilde{Z}_g^{\mathrm{UG}} = \delta Z_g^{\mathrm{UG}} - \delta Z_\Psi^{\mathrm{UG}} - \frac{1}{2}\delta Z_{phi}^{\mathrm{UG}} = -\frac{1}{16\pi^2\epsilon}\kappa^2 m_\Psi^2 \frac{3}{16} - a_1\kappa^2 m_\Psi^2 - b_1\kappa^2 m_\phi^2. \tag{6.40}$$

Accordingly, we can set

$$a_1 = -\frac{1}{16\pi^2\epsilon}\frac{3}{16}, \tag{6.41}$$

$$b_1 = 0, \tag{6.42}$$

to make again $\delta\widetilde{Z}_g = 0$ and

$$\beta_g^{\mathrm{UG}}\Big|_{\mathrm{gravitational}} = 0. \tag{6.43}$$

The computation of the gravitational corrections of the beta function of the ϕ^4 interaction is done by following an akin the process. Defining

$$\lambda_0 = \lambda\mu^{-2\epsilon} Z_\lambda Z_\phi^{-2}, \quad Z_\lambda = 1 + \delta Z_\lambda, \tag{6.44}$$

we have obtained

$$\delta Z_\lambda^{\text{GR}} = \frac{4}{16\pi^2\epsilon}\left(\frac{3}{2}+\alpha\right), \tag{6.45}$$

$$\delta Z_\lambda^{\text{UG}} = 0. \tag{6.46}$$

Hence, one can compute the gravitational corrections to the beta functions of the quartic coupling, $\lambda\phi^4$, to be

$$\beta_\lambda^{\text{GR}} = -\frac{1}{4\pi^2}\kappa^2 m_\phi^2\left(\frac{3}{2}+\alpha\right)\lambda, \tag{6.47}$$

$$\beta_\lambda^{\text{UG}} = 0. \tag{6.48}$$

In this case the beta function of unimodular gravity is directly zero for this particular gauge. For general relativity, as we did with the Yukawa coupling, we can reabsorb this discrepancy by means of a non-multiplicative renormalization. In this case, we can carry out the following field redefinition

$$\phi_0 = \phi + \omega_1\phi + \omega_2\kappa^2\partial^2\phi + w_3\kappa^2\mu^{-2\epsilon}\phi^3\delta Z_\phi\phi, \tag{6.49}$$

and we can set $\beta_\lambda^{\text{GR}} = 0$ by choosing

$$\omega_1 = -\frac{1}{16\pi^2\epsilon}\kappa^2 m_\phi^2, \tag{6.50}$$

$$\omega_2 = 0, \tag{6.51}$$

$$\omega_3 = \frac{1}{16\pi^2\epsilon}\frac{1}{4!}2\lambda. \tag{6.52}$$

6.3 FINAL DISCUSSION

The knowledge of beta functions of the $\lambda\phi^4$ (quartic) and Yukawa couplings, and the logarithmic UV divergences that contribute to them, is needed in the quantitative analysis of such important issues as the vacuum stability by using the renormalization-group-improved effective potential –see [88] and references therein–, and the construction of Asymptotically Safe theories of Quantum Gravity coupled to matter as put forward in Reference [89]. Therefore, a computation of the logarithmic UV divergent contributions which may yield a change of the value of beta functions in question due to the interaction of the corresponding matter fields with gravitons is much needed. However, the fact that the corrections computed in Reference [84] for the gauge coupling constants turned out to lack any intrinsic physical meaning –see References [85, 87]– makes necessary to ascertain whether or not these gravitational corrections to the quartic and Yukawa couplings are gauge independent and invariant under non-multiplicative field renormalizations so that an intrinsic physical meaning can be assigned to them.

In this section, we have computed the General Relativity corrections to the beta functions of the Yukawa and $\lambda\phi^4$ theory as obtained from the 1PI functional by using the standard multiplicative MS dimensional regularization scheme. We have shown that they are gauge dependent and that, besides, they can be set to zero by appropriate, non-multiplicative, field redefinitions: they are *inessential* corrections [89]. We thus conclude that these corrections do not have any intrinsic physical meaning and, therefore, the statements about asymptotic freedom made in reference [90] are not physically meaningful. Of course, the gauge dependence of the gravitational corrections to the beta function can be avoided by using the DeWitt-Vilkovisky action instead of the 1PI functional –as done in reference [92] for the $\lambda\phi^4$ theory–, but it is plain that those gauge-independent contributions can still be removed by appropriate non-multiplicative, but local, field redefinitions such as the ones –with different coefficients, of course– introduced in this section. The use of the DeWitt-Vilkovisky effective action does not give the gravitational corrections in question any intrinsic physical meaning, so that any conclusion drawn from them also lack intrinsic physical content.

For the sake of comparison, we have carried out a similar computation for the case of Unimodular Gravity –for a gauge-fixing choice which yields no free parameters: the computations are hard enough already– and found that the corresponding gravitational corrections to the beta functions do not agree with those from General Relativity –curiously enough the corrections to the beta function of the $\lambda\phi^4$ vanish for Unimodular Gravity, and, that they can also be set to zero by appropriate local non-multiplicative field redefinitions. So one cannot use these gravitational corrections to the beta functions in question to distinguish between General Relativity and Unimodular Gravity. In fact, they behave in the same manner from the physical point of view: they are not *essential* in either case, for they correspond to field redefinitions.

Several final comments are in order. First, we would like to point out that our conclusions are quite in keeping with the conclusions –ie, the inclusion of gravitational effects into the running coupling constants has not a universal meaning– in Reference [93] in the massless case, but our approach to the problem is not the same and, besides, our theories are massive. Notice that the contributions computed before —and in [90, 92]-– vanish if the masses are sent to zero. Secondly, that our analysis is in complete harmony with the discussion carried out in reference [87] for the Yang-Mills coupling constant. Thirdly, the results that we have presented are to be taken into account unavoidable when developing the asymptotic safety program as applied to Gravity interacting with matter, with the proviso that the UV divergences computed here correspond to logarithmic divergences when a cutoff is used.

7

S MATRIX CORRECTIONS

7.1 UV DIVERGENT CONTRIBUTIONS TO THE SCATTERING OF MASSIVE SCALAR PARTICLES

In the previos chapter the coupling of Unimodular Gravity to a massive $\lambda\phi^4$ theory was introduced and the corrections to the beta function of the coupling λ coming from Unimodular Gravity were computed. The results obtained point in the direction that, when coupled to the $\lambda\phi^4$ theory, Unimodular Gravity and General Relativity are equivalent at the quantum level, at least when the Cosmological Constant can be dropped and for the one-loop UV divergent behaviour considered. However, this conclusion regarding the UV behaviour of these theories –General Relativity plus $\lambda\phi^4$ and Unimodular Gravity coupled to $\lambda\phi^4$– cannot be considered as final since, as shown in Ref. [94], the gravitational corrections to the beta function of the coupling λ have a very dubious physical meaning. To settle this issue for once and all is important since it has been argued [90, 92] that the General Relativity corrections to the beta function of the coupling λ gives rise to asymptotic freedom, with obvious implications on the Higgs physics.

The purpose of this section is to compute the one-loop and order $\kappa^2 m_\phi^2$ UV divergent contributions to the S matrix element of the scattering process $\phi+\phi\rightarrow\phi+\phi$ in a massive –with mass m_ϕ, $\lambda\phi^4$ theory coupled either to General Relativity or to Unimodular Gravity, both in the vanishing cosmological constant situation. We shall show that such UV divergent behaviour is the same in UG case as in the GR instance and this is in spite of the fact this equivalence does not hold

Feynman diagram by Feynman diagram. This result is not trivial since UG does not couple to the scalar potential in the classical action and it provides further evidence that UG and GR are equivalent at the quantum level and for zero Cosmological Constant. As a side result, we shall show that the UV divergent contributions which give rise to the non-vanishing gravitational corrections to the beta function of the coupling λ computed in [90, 92, 94] are completely useless for characterizing UV divergent behaviour S matrix element of the $\phi + \phi \to \phi + \phi$ scattering. This is completely at odds with the non-gravitational corrections to the beta function of λ and it shows beyond the shadow of a doubt that the gravitational corrections to the beta function of the coupling constants lack, in general, any intrinsic physical meaning. This also applies to the physical implications of a beta function turning negative due to the gravitational corrections.

7.1.1 *Gravity coupled to $\lambda\phi^4$*

In this section we shall just display the classical actions of General Relativity and Unimodular Gravity coupled to $\lambda\phi^4$ and the graviton free propagator in each case.

7.1.1.1 *General Relativity coupled to $\lambda\phi^4$*

It goes without saying that the classical action of General relativity coupled to $\lambda\phi^4$ reads

$$\begin{aligned} S_{GR\phi} &= S_{EH} + S^{(GR)}_{\lambda\phi^4} \\ S_{EH} &= -\frac{2}{\kappa^2}\int d^nx\,\sqrt{-g}R[g_{\mu\nu}] \\ S^{(GR)}_{\lambda\phi^4} &= \int d^nx\,\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4\right] \end{aligned} \tag{7.1}$$

where $\kappa^2 = 32\pi G$, $R[g_{\mu\nu}]$ is the scalar curvature for the metric $g_{\mu\nu}$.

Using the standard splitting

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}; \tag{7.2}$$

and the generalized de Donder gauge-fixing term

$$\int d^n x\, \alpha(\partial^\mu h_{\mu\nu} - \partial_\nu h)^2, \quad h = h_{\mu\nu}\eta^{\mu\nu},$$

which depends on the gauge parameter α, one obtains the following free propagator of the graviton field $h_{\mu\nu}$:

$$\begin{aligned}\langle h_{\mu\nu}(k)h_{\rho\sigma}(-k)\rangle = \frac{i}{2k^2}\,(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma}) - \\ - \frac{i}{(k^2)^2}\left(\frac{1}{2}+\alpha\right)(\eta_{\mu\rho}k_\nu k_\sigma + \eta_{\mu\rho}k_\nu k_\sigma + \eta_{\nu\rho}k_\mu k_\sigma + \eta_{\nu\sigma}k_\mu k_\rho),\end{aligned} \tag{7.3}$$

where $\eta^{\mu\nu}$ denotes the Minkowski, $(+,-,-,-)$, metric.

Up to first order in κ, $S^{(GR)}_{\lambda\phi^4}$ in (7.1) is given by

$$S^{(GR)}_{\lambda\phi^4} = \int d^n x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{\kappa}{2}T^{\mu\nu}h_{\mu\nu}\right] + O(\kappa^2), \tag{7.4}$$

where

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\Big(\frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4\Big).$$

In (7.4), contractions are carried out with $\eta_{\mu\nu}$, the Minkowski metric.

7.1.1.2 *Unimodular Gravity coupled to $\lambda\phi^4$*

Let $\hat{g}_{\mu\nu}$ denote the Unimodular –ie, with determinant equal to (–1)– metric of the n dimensional spacetime manifold. We shall assume the mostly minus signature for the metric. Then, the classical action of Unimodular gravity coupled to $\lambda\phi^4$ reads

$$\begin{aligned} S_{UG\phi} &= S_{UG} + S^{(UG)}_{\lambda\phi^4} \\ S_{UG} &= -\frac{2}{\kappa^2}\int d^n x\; R[\hat{g}_{\mu\nu}] \\ S^{(UG)}_{\lambda\phi^4} &= \int d^n x \left[\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4\right] \end{aligned} \tag{7.5}$$

where $\kappa^2 = 32\pi G$, $R[\hat{g}_{\mu\nu}]$ is the scalar curvature for the unimodular metric.

To quantize the theory we shall proceed as in Refs. [15, 38, 39] and introduce the unconstrained fictitious metric, $g_{\mu\nu}$, thus

$$\hat{g}_{\mu\nu} = (-g)^{-1/n}\, g_{\mu\nu}, \tag{7.6}$$

where g is the determinant of $g_{\mu\nu}$. Then, we shall express the action in (7.1) in terms of the fictitious metric $g_{\mu\nu}$ by using (7.6). Next, we shall split $g_{\mu\nu}$ as in (7.2)

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{7.7}$$

and, finally, we shall defined the path integral by integration over $h_{\mu\nu}$ and the matter fields, once an appropriate BRST invariant action has been constructed.

Since our computations will always involve the scalar field ϕ and will be of order κ^2, we only need –as will become clear in the sequel– the free propagator of $h_{\mu\nu}$, $\langle h_{\mu\nu}(k)h_{\rho\sigma}(-k)\rangle$, and the expansion of $S_{\lambda\phi^4}$ up to first order in κ. Remember that we have obtained the propagator in (6.5), and it reads (in $n = 4$)

$$\begin{aligned}\langle h_{\mu\nu}(k)h_{\rho\sigma}(-k)\rangle = {} & \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma}\right) - \frac{i}{k^2}\frac{\alpha^2 n^2 - n + 2}{\alpha^2 n^2 (n-2)}\eta_{\mu\nu}\eta_{\rho\sigma} \\ & + \frac{2i}{n-2}\left(\frac{k_\rho k_\sigma \eta_{\mu\nu}}{(k^2)^2} + \frac{k_\mu k_\nu \eta_{\rho\sigma}}{(k^2)^2}\right) - \frac{2in}{n-2}\frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2)^3}.\end{aligned} \tag{7.8}$$

The expansion of $S_{\lambda\phi^4}$ in powers of κ reads

$$S^{(UG)}_{\lambda\phi^4} = \int d^n x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{\kappa}{2}\, T^{\mu\nu}\hat{h}_{\mu\nu}\right] + O(\kappa^2), \tag{7.9}$$

where $\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{n}h$, with $h = \eta_{\mu\nu}h^{\mu\nu}$, is the traceless part of $h_{\mu\nu}$ and

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\Big(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4\Big). \tag{7.10}$$

Again, the contractions in (7.9) are carried out with the Minkowski metric $\eta_{\mu\nu}$.

Notice that the summand in $T^{\mu\nu}$ which is proportional to $\eta^{\mu\nu}$ does not actually contribute to $T^{\mu\nu}\hat{h}_{\mu\nu}$, since $\hat{h}_{\mu\nu}$ is traceless. In terms of Feynman diagrams, this amounts to saying that the $\eta^{\mu\nu}$ part of $T^{\mu\nu}$ will never contribute to a given diagram since it will always be contracted with a free propagator involving $\hat{h}_{\mu\nu}$. This as opposed to the case of General Relativity and makes the agreement between General Relativity coupled to matter and Unimodular Gravity coupled to matter quite surprising already at the one-loop level.

It is the free propagator of $\hat{h}_{\mu\nu}$, $\langle\hat{h}_{\mu\nu}(k)\hat{h}_{\rho\sigma}(-k)\rangle$, and not the full graviton propagator in (6.5), the correlation function that will enter the computations carried out in this section. From (6.5) one readily obtains that

$$\langle\hat{h}_{\mu\nu}(k)\hat{h}_{\rho\sigma}(-k)\rangle = \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{2}{n-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right) + \frac{2i}{n-2}\frac{k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}}{(k^2)^2} - \frac{2in}{n-2}\frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2)^3}. \tag{7.11}$$

7.1.2 *The $\phi+\phi\rightarrow\phi+\phi$ scattering at one-loop and at order $\kappa^2 m_\phi^2$*

The purpose of this section is to work out the one-loop and order $\kappa^2 m_\phi^2$ UV divergent contribution, coming from General Relativity and Unimodular Gravity, to the dimensionally regularized S-matrix element of the $\phi+\phi\rightarrow\phi+\phi$ scattering process and discuss the meaning of the results we shall obtain.

7.1.2.1 *General Relativity contributions*

Let us consider the General Relativity case in the first place. To define the S-matrix of the $\phi+\phi\rightarrow\phi+\phi$ scattering at one-loop, we need the one-loop propagator of the scalar field ϕ to have simple pole at the physical mass, m_ϕ with residue i. This is achieved by introducing the following mass and wave function renormalizations

$$\begin{aligned} &m_\phi^2 = M^2 + i\Gamma_{\phi\phi}(p^2 = m_\phi^2, \kappa) \\ &\phi = \phi_R\Big[1 - i\Gamma'_{\phi\phi}(p^2 = m_\phi^2, \kappa)\Big]^{-1/2}, \quad \Gamma'_{\phi\phi}(p^2,\kappa) = \frac{\partial\Gamma_{\phi\phi}(p^2,\kappa)}{\partial p^2}, \end{aligned} \tag{7.12}$$

where M^2 and ϕ are the bare objects in the action in (7.1). In the previous equation, the symbol $\Gamma_{\phi\phi}(p^2)$ denotes the one-loop contribution to the 1PI two-point function of the scalar field. The General Relativity contribution, $i\Gamma^{(GR)}_{\phi\phi}(p^2,\kappa)$ –the non-gravitational ones can be found in standard textbooks– to $i\Gamma_{\phi\phi}(p^2)$ is given by the diagram in Figure 1a and it reads

$$i\Gamma^{(GR)}_{\phi\phi}(p^2,\kappa) \;=\; \left(\frac{1}{16\pi^2\epsilon}\right)\left[1+\left(\frac{1}{2}+\alpha\right)\right]\kappa^2 M^2(p^2-M^2)+ \quad (7.13)$$
$$+\,\text{UV finite contributions}, \quad (7.14)$$

where $n = 4+2\epsilon$ is the spacetime dimension. The wavy line in Figure 1a denotes the free propagator in (7.3).

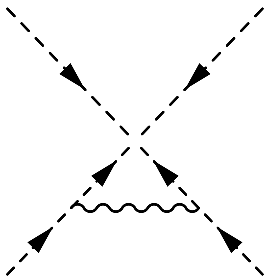
Now, in terms of the m_ϕ and ϕ_r defined in (7.12), the one-loop and order $\kappa^2 m_\phi^2$ General Relativity contribution to the dimensionally regularized S-matrix element of the scattering process $\phi+\phi\to\phi+\phi$ is given by the sum the diagrams in Figures 8, 9 and 10 –bear in mind that the wavy lines represent free propagator in (7.3). Notice that the diagram in Figure 9 comes from the wave function renormalization in (7.12), which guarantees that asymptotically ϕ_r is the free field at $t=\pm\infty$. It can be shown that the sum of all the diagrams in Fig. 8 is given by

$$\begin{aligned} &i\Gamma^{(GR)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa)|_{p_i^2=m_\phi^2} = \\ &= \left(\frac{-1}{16\pi^2\epsilon}\right)\left[1+\left(\frac{1}{2}+\alpha\right)\right]\kappa^2\lambda\Big(\sum_{i<j} p_i\cdot p_j|_{p_i^2=m_\phi^2}+4m_\phi^2\Big) + \text{UV finite contributions} = \\ &= \left(\frac{-1}{16\pi^2\epsilon}\right)\left[1+\left(\frac{1}{2}+\alpha\right)\right]\kappa^2 m_\phi^2\lambda\Big[2\Big] + \text{UV finite contributions.} \end{aligned} \quad (7.15)$$

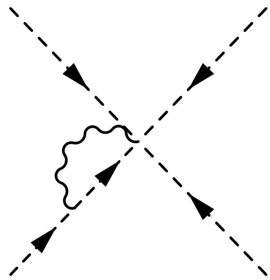
Note that $i,j = 1,2,3$ and 4.

Taking into account (7.12) and (7.14), one concludes that contribution to the dimensionally regularized S-matrix coming from the diagram in Figure 9 reads

$$\begin{aligned} &i\Gamma^{(GR,ct)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa) = \lambda\left[1-i\frac{\partial\Gamma^{(GR)}_{\phi\phi}}{\partial p^2}(p^2=m_\phi^2,\kappa)\right]^{-2} - \lambda = \\ &= \left(\frac{1}{16\pi^2\epsilon}\right)\left[1+\left(\frac{1}{2}+\alpha\right)\right]\kappa^2 m_\phi^2\lambda\Big[2\Big] + \text{UV finite contributions.} \end{aligned} \quad (7.16)$$



(a) + 5 permutations



(b) + 4 permutations

Figure 8: 1 loop scalar four-point function: $i\Gamma^{(GR)\,\text{or}\,(UG)}_{\phi\phi\phi\phi}(p_1, p_2, p_3, p_4; \kappa)$

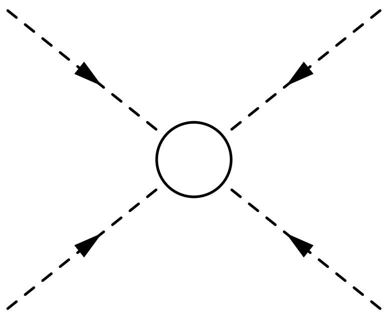


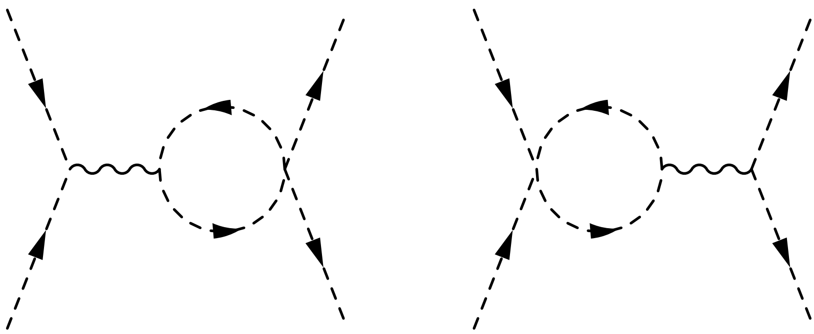
Figure 9: Onshell counterterm

From (7.15) and (7.16), one immediately realizes that the

$$i\Gamma^{(GR)}_{\phi\phi\phi\phi}(p_1, p_2, p_3, p_4; \kappa)|_{p_i^2=m_\phi^2} + i\Gamma^{(GR,ct)}_{\phi\phi\phi\phi}(p_1, p_2, p_3, p_4; \kappa) = \text{UV finite contributions,} \tag{7.17}$$

so that the General Relativity one-loop and order $\kappa^2 m_\phi^2$ UV divergent contributions to the S-matrix of the process $\phi + \phi \to \phi + \phi$ may only come from the non-1PI diagrams in Figure 10. This sum reads

$$\begin{aligned}
&i\mathrm{N}\Gamma^{(GR)}_{\phi\phi\phi\phi}(p_1, p_2, p_3, p_4; \kappa)|_{p_i^2 = m_\phi^2} = \\
&= \left(\frac{-1}{16\pi^2\epsilon}\right)\left(\frac{1}{12}\right)\kappa^2\lambda\left[\frac{1}{2}(s+t+u)|_{p_i^2=m_\phi^2} + m_\phi^2\right] + \text{UV finite contributions} = \\
&= \left(\frac{-1}{16\pi^2\epsilon}\right)\left(\frac{1}{2}\right)\kappa^2 m_\phi^2\lambda + \text{UV finite contributions.}
\end{aligned} \tag{7.18}$$



u and t channels

Figure 10: Non-1PI diagrams

We are now ready to display the one-loop and order $\kappa^2 m_\phi^2$ UV contribution to the dimensional regularized S matrix element of the scattering process $\phi + \phi \to \phi + \phi$ coming from General Relativity. The contribution in question is obtained by adding the UV divergent contributions in (7.15), (7.16) and (7.18) and it reads

$$\left(\frac{-1}{16\pi^2\epsilon}\right)\left(\frac{1}{2}\right)\kappa^2 m_\phi^2\lambda. \tag{7.19}$$

Let us insist on the fact that the contribution in (7.19) only comes from the diagrams in Figure 10, which are one particle reducible, for the contribution coming from the 1PI diagram in Figure 9 cancels the contributions coming from the diagrams in Figure 8, as seen in (7.17).

7.1.2.2 *Unimodular Gravity contributions*

To compute the one-loop and order $\kappa^2 m_\phi^2$ Unimodular Gravity contributions to the S matrix element giving the $\phi+\phi\to\phi+\phi$ scattering, one proceeds as in the previous subsection,taking into account that now the wavy lines in the Feynman diagrams in Figures 1a, 8, and 10 stand for the traceless free correlation function in (7.11) and that the diagrams in Figure 8b are zero since they come from the contraction of the $\eta^{\mu\nu}$ bit of $T_{\mu\nu}$ in (7.10) and the traceless $\hat{h}_{\mu\nu}$ field. Our computations yield the following results

$$\begin{aligned}
&i\Gamma^{(UG)}_{\phi\phi}(p^2,\kappa) = 0 + \text{UV finite contributions}\\
&i\Gamma^{(UG)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa)|_{p_i^2=m_\phi^2} = 0 + \text{UV finite contributions}\\
&i\text{N}\Gamma^{(UG)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa)|_{p_i^2=m_\phi^2} =\\
&= \left(\frac{-1}{16\pi^2\epsilon}\right)\left(\frac{1}{12}\right)\kappa^2\lambda\left[\frac{1}{2}(s+t+u)|_{p_i^2=m_\phi^2} + m_\phi^2\right] + \text{UV finite contributions} =\\
&= \left(\frac{-1}{16\pi^2\epsilon}\right)\left(\frac{1}{2}\right)\kappa^2 m_\phi^2\lambda + \text{UV finite contributions,}
\end{aligned} \tag{7.20}$$

where $i\Gamma^{(UG)}_{\phi\phi}(p^2,\kappa)$ is give by the diagram in Figure 1a, $i\Gamma^{(UG)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa)$ is the sum of all the diagrams –which are not trivially zero– in Figure 8 and $i\text{N}\Gamma^{(UG)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa)$ is the sum of all the diagrams in Figure 10 and m_ϕ is the physical mass of the scalar field ϕ.

By applying the on-shell definitions in (7.12) –ie, now M^2 and ϕ are the bare objects in the action in (7.5)– to our case, one concludes that for Unimodular Gravity the diagram in Fig. 9 is given by

$$\begin{aligned}
i\Gamma^{(UG,ct)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa) &= \lambda\left[1 - i\frac{\partial\Gamma^{(GR)}_{\phi\phi}}{\partial p^2}(p^2=m_\phi^2, M^2)\right]^{-2} - \lambda =\\
&= 0 + \text{UV finite contributions.}
\end{aligned} \tag{7.21}$$

Taking into account the results in (7.20) and (7.21) and adding the UV divergent contributions to $i\Gamma^{(UG)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa)|_{p_i^2=m_\phi^2}$, $i\Gamma^{(UG,ct)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa)$ and $i\text{N}\Gamma^{(UG)}_{\phi\phi\phi\phi}(p_1,p_2,p_3,p_4;\kappa)|_{p_i^2=m_\phi^2}$, one obtains the one-loop and order $\kappa^2 m_\phi^2$ UV contribution to the dimensional regularized S matrix element of the scattering process $\phi + \phi \to \phi + \phi$ coming from Unimodular Gravity, which runs thus

$$\left(\frac{-1}{16\pi^2\epsilon}\right)\left(\frac{1}{2}\right)\kappa^2 m_\phi^2 \lambda. \tag{7.22}$$

This is the same contribution that we obtained in the General Relativity case –see (7.19). Notice, however, that both $i\Gamma^{(UG)}_{\phi\phi\phi\phi}(p_1, p_2, p_3, p_4; \kappa)|_{p_i^2=m_\phi^2}$ and $i\Gamma^{(UG,ct)}_{\phi\phi\phi\phi}(p_1, p_2, p_3, p_4; \kappa)$ are UV finite, which is at odds with their General Relativity counterparts in (7.15), (7.16).

7.2 SCATTERING OF FERMIONS IN THE YUKAWA THEORY COUPLED TO GRAVITY

In this section we compute the lowest order gravitational UV divergent radiative corrections to the S matrix element of the $fermion + fermion \to fermion + fermion$ scattering process in the massive Yukawa theory, coupled either to Unimodular Gravity or to General Relativity. We show that both Unimodular Gravity and General Relativity give rise to the same UV divergent contribution in Dimensional Regularization. This is a nontrivial result, since in the classical action of Unimodular Gravity coupled to the Yukawa theory, the graviton field does not couple neither to the mass operator nor to the Yukawa operator. This is unlike the General Relativity case. The agreement found points in the direction that UG and GR give rise to the same quantum theory when coupled to matter, as long as the cosmological constant vanishes. Along the way we have come across another unexpected cancellation of UV divergences for both UG and GR, resulting in the UV finiteness of the one-loop and κy^2 order of the vertex involving two fermions and one graviton only.

7.2.1 *Yukawa theory coupled to Gravity*

First of all, we shall just display the classical actions of the Yukawa theory coupled to a gravitational field as described by GR and UG. We shall also display the free graviton propagator in each case.

7.2.1.1 *Yukawa theory coupled to General Relativity*

Let e_a^μ be the vielbein, $e_a^\mu e_b^\nu g_{\mu\nu} = \eta_{ab}$, for the Lorentzian metric $g_{\mu\nu}$, where as always $\eta_{ab} = (+,-,-,-)$. Let γ^a denote the Dirac matrices: $[\gamma^a, \gamma^b] = \eta^{ab}$. The torsion-free spin connection ω_μ is defined,

$$\omega_\mu = \frac{1}{8}[\gamma^b, \gamma^c] e_{\nu b} \bigtriangledown_\mu e_c^\nu,$$

where $\bigtriangledown_\mu$ is the covariant derivative as given by the Christoffel symbols. Let ψ, denote a spinor field in spacetime, its covariant derivative being given by

$$D_\mu \psi = (\partial_\mu + \omega_\mu)\psi.$$

The classical action of General relativity coupled to the Yukawa theory reads

$$\begin{aligned} S_{\text{GR-Yukawa}} &= S_{\text{EH}} + S^{(\text{GR})}_{\text{Yukawa}}, \\ S_{\text{EH}} &= -\frac{2}{\kappa^2}\int d^n x\, \sqrt{-g} R[g_{\mu\nu}], \\ S^{(\text{GR})}_{\text{Yukawa}} &= \int d^n x \sqrt{-g}\left[\bar{\psi}\left(i e_a^\mu\, \gamma^a D_\mu \psi - m\psi\right) + \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2 - y\,\bar{\psi}\psi\phi\right]. \end{aligned} \tag{7.23}$$

where $\kappa^2 = 32\pi G$ and $R[g_{\mu\nu}]$ is the scalar curvature for the metric $g_{\mu\nu}$.

Let us remind that when using the standard splitting

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{7.24}$$

and the generalized de Donder gauge-fixing term

$$\int d^n x\, \alpha(\partial^\mu h_{\mu\nu} - \partial_\nu h)^2, \quad h = h_{\mu\nu}\eta^{\mu\nu},$$

which depends on the gauge parameter α, one obtains the free propagator of the graviton field $h_{\mu\nu}$ in (7.3)

$$\begin{aligned} \langle h_{\mu\nu}(k) h_{\rho\sigma}(-k)\rangle = &\frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma}\right) - \\ &\frac{i}{(k^2)^2}\left(\frac{1}{2} + \alpha\right)\left(\eta_{\mu\rho}k_\nu k_\sigma + \eta_{\mu\sigma}k_\nu k_\rho + \eta_{\nu\rho}k_\mu k_\sigma + \eta_{\nu\sigma}k_\mu k_\rho\right). \end{aligned} \tag{7.25}$$

where $\eta^{\mu\nu}$ denotes the Minkowski, $(+,-,-,-)$, metric.

Up to first order in κ, $S^{(GR)}_{Yukawa}$ in (7.23) is given by

$$S^{(\mathrm{GR})}_{\mathrm{Yukawa}} = \int d^n x \left[\bar{\psi}\Big(i\partial\!\!\!/ - m\Big)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 - y\bar{\psi}\psi\phi - \frac{\kappa}{2}T^{\mu\nu}h_{\mu\nu}\right] + O(\kappa^2), \tag{7.26}$$

where

$$T^{\mu\nu} = \frac{i}{4}\bar{\psi}\Big(\gamma^\mu\overrightarrow{\partial}^\nu + \gamma^\nu\overrightarrow{\partial}^\mu\Big)\psi - \frac{i}{4}\bar{\psi}\Big(\gamma^\mu\overleftarrow{\partial}^\nu + \gamma^\nu\overleftarrow{\partial}^\mu\Big)\psi + \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\Big(\frac{1}{2}\bar{\psi}(i\overrightarrow{\partial\!\!\!/} - m)\psi - \frac{1}{2}\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m)\psi + \frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - \frac{1}{2}M^2\phi^2 - y\bar{\psi}\psi\phi\Big). \tag{7.27}$$

In (7.26), contractions are carried out with $\eta_{\mu\nu}$.

7.2.1.2 *Yukawa theory coupled to Unimodular Gravity*

Let $\hat{g}_{\mu\nu}$ denote the Unimodular –ie, with determinant equal to -1– metric of the n dimensional spacetime manifold. We shall assume the mostly minus signature for the metric. Then, the classical action of the Yukawa theory coupled to Unimodular Gravity reads

$$\begin{aligned} S_{\text{UG-Yukawa}} &= S_{UG} + S^{(\mathrm{UG})}_{\mathrm{Yukawa}}, \\ S_{UG} &= -\frac{2}{\kappa^2}\int d^n x\, R[\hat{g}_{\mu\nu}], \\ S^{(\mathrm{UG})}_{\mathrm{Yukawa}} &= \int d^n x \left[\bar{\psi}\Big(i\hat{e}^\mu_a\gamma^a\hat{D}_\mu\psi - m\psi\Big) + \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2 - g\,\bar{\psi}\psi\phi,\right. \end{aligned} \tag{7.28}$$

where $\kappa^2 = 32\pi G$, $R[\hat{g}_{\mu\nu}]$ is the scalar curvature for the unimodular metric, $\hat{e}^\mu_a$ is the vielbein, $\hat{e}^\mu_a\hat{e}^\nu_b\hat{g}_{\mu\nu} = \eta_{ab}$ for the metric $\hat{g}_{\mu\nu}$, $\eta_{ab} = (+,-,-,-)$, γ^a denote the Dirac matrices: $[\gamma^a,\gamma^b] = \eta^{ab}$ and $\hat{D}_\mu = \partial_\mu + \hat{\omega}_\mu$ is the Dirac operator for the torsion-free spin connection

$$\hat{\omega}_\mu = \frac{1}{8}[\gamma^b,\gamma^c]\hat{e}_{\nu b}\hat{\nabla}_\mu\hat{e}^\nu_c.$$

$\hat{\nabla}_\mu$ is the covariant derivative as given by the Christoffel symbols of $\hat{g}_{\mu\nu}$

To quantize the theory we shall follow Refs. [15, 38, 39] and introduce an unconstrained fictitious metric in (7.6), $\hat{g}_{\mu\nu}$, thus

$$\hat{g}_{\mu\nu} = (-g)^{-1/n}\, g_{\mu\nu}, \tag{7.29}$$

where g is the determinant of $g_{\mu\nu}$. Next, we shall express the action in (7.28) in terms of the fictitious metric $\hat{g}_{\mu\nu}$ by using (7.6), then, we shall split $g_{\mu\nu}$ as in (7.2)

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}; \tag{7.30}$$

and, finally, we shall define the path integral by integration over $h_{\mu\nu}$ and the matter fields, once an appropriate BRS invariant action has been constructed.

Since our computations will always involve the matter fields $\bar{\psi}$, ψ and ϕ, and will be of order κ^2, we shall only need –as will become clear in the sequel– the free propagator of $h_{\mu\nu}$, $\langle h_{\mu\nu}(k)h_{\rho\sigma}(-k)\rangle$, and the expansion of $S^{(\mathrm{UG})}_{\mathrm{Yukawa}}$ up to first order in κ. Using the gauge-fixing procedure previously discussed , we obtained the unimodular propagator (6.5)

$$\begin{aligned}\langle h_{\mu\nu}(k)h_{\rho\sigma}(-k)\rangle = &\frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}\right) - \frac{i}{k^2}\frac{\alpha^2 n^2 - n + 2}{\alpha^2 n^2 (n-2)}\eta_{\mu\nu}\eta_{\rho\sigma} + \\ &+ \frac{2i}{n-2}\left(\frac{k_\rho k_\sigma \eta_{\mu\nu}}{(k^2)^2} + \frac{k_\mu k_\nu \eta_{\rho\sigma}}{(k^2)^2}\right) - \frac{2in}{n-2}\frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2)^3}.\end{aligned} \tag{7.31}$$

The expansion of $S^{(\mathrm{UG})}_{\mathrm{Yukawa}}$ in powers of κ reads

$$\begin{aligned}S^{(\mathrm{UG})}_{\mathrm{Yukawa}} = &\int d^n x \left[\bar{\psi}\Big(i\partial\!\!\!/ - m\Big)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 - y\bar{\psi}\psi\phi - \frac{\kappa}{2}T^{\mu\nu}\hat{h}_{\mu\nu}\right] + \\ &+ O(\kappa^2),\end{aligned} \tag{7.32}$$

where $\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{n}h$ –with $h = \eta_{\mu\nu}h^{\mu\nu}$– is the traceless part of $h_{\mu\nu}$, and $T^{\mu\nu}$ is given in (7.27). Again, the contractions in (7.32) are carried out with the help of $\eta_{\mu\nu}$.

Let us point out that the term in $T^{\mu\nu}$ which is proportional to $\eta^{\mu\nu}$ does not actually contribute to $T^{\mu\nu}\hat{h}_{\mu\nu}$, since $\hat{h}_{\mu\nu}$ is traceless. In terms of Feynman diagrams, this can be stated by saying that the $\eta^{\mu\nu}$ part of $T^{\mu\nu}$ will never contribute to a given diagram since it will always be contracted with a free propagator involving $\hat{h}_{\mu\nu}$. This is not what happens in the General Relativity case and makes the agreement between General Relativity coupled to the Yukawa theory and Unimodular Gravity coupled to the latter a non-trivial issue already at one-loop.

We shall use the correlation function $\langle \hat{h}_{\mu\nu}(k)\hat{h}_{\rho\sigma}(-k)\rangle$, which can be easily obtained from (6.5) and is written in (7.11),

$$\begin{aligned}\langle \hat{h}_{\mu\nu}(k)\hat{h}_{\rho\sigma}(-k)\rangle = &\frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}-\frac{2}{n-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)+\\ &+\frac{2i}{n-2}\frac{k_\mu k_\nu\eta_{\rho\sigma}+k_\rho k_\sigma\eta_{\mu\nu}}{(k^2)^2}-\frac{2in}{n-2}\frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2)^3}.\end{aligned} \tag{7.33}$$

7.2.2 *The fermion + fermion → fermion + fermion scattering at one-loop and $y^2\kappa^2$ order*

The purpose of this section is to work out the one-loop, and $y^2\kappa^2$ order, UV divergent contribution, coming from General Relativity and Unimodular Gravity, to the dimensionally regularized S matrix element of the *fermion* + *fermion* → *fermion* + *fermion* scattering process.

7.2.2.1 *The General Relativity case*

To work out the UV divergent contribution in question to *fermion* + *fermion* → *fermion* + *fermion*, we shall need the UV divergent contributions coming from the 1PI diagramas in Figures 1a to 12. These contributions read in the General Relativity case

$$\begin{aligned}
i\Gamma^{(GR)}_{\phi\phi}(p^2,\kappa) &= \left(\frac{1}{16\pi^2\epsilon}\right)\left[1+\left(\frac{1}{2}+\alpha\right)\right]\kappa^2M^2(p^2-M^2),\\
i\Gamma^{(GR)}_{\psi\bar\psi}(p;\kappa) &= \left(\frac{-i}{16\pi^2\epsilon}\right)\kappa^2\Bigg[\left(\frac{3}{8}p^2m-\frac{1}{8}p^2\not{p}+\frac{1}{4}m^2(\not{p}-m)\right)+\\
&\quad+\left(\frac{1}{2}+\alpha\right)\left(\frac{3}{4}p^2m-\frac{19}{16}m^3+\left(\frac{29}{32}m^2-\frac{15}{32}p^2\right)\not{p}\right)\Bigg],\\
i\Gamma^{(GR)}_{\psi\bar\psi\phi}(p_1,p_2;\kappa) &= y\kappa^2\left(-\frac{i}{16\pi^2\epsilon}\right)\Bigg\{\left(-\frac{1}{4}M^2-\frac{3}{4}m^2+\frac{1}{16}(p_1+p_2)^2+\right.\\
&\quad\left.+\frac{1}{4}m(\not{p}_1+\not{p}_2)+\frac{1}{8}\not{p}_1\not{p}_2\right)+\left(\frac{1}{2}+\alpha\right)\left(-m_\phi^2-\frac{57}{16}m^2+\right.\\
&\quad\left.+\frac{47}{32}(p_1^2+p_2^2)-\frac{13}{8}p_1\cdot p_2+m(\not{p}_1+\not{p}_2)-\frac{9}{16}\not{p}_1\not{p}_2\right)\Bigg\},\\
i\Gamma^{(GR)}_{\psi\bar\psi h_{\mu\nu}}(p_1,p_2;\kappa) &= \kappa y^2\left(\frac{-i}{16\pi^2\epsilon}\right)\left\{\left[\frac{1}{8}(\not{p}_1+\not{p}_2)+\frac{1}{2}m\right]\eta^{\mu\nu}-\frac{1}{16}(p_1+p_2)^\mu\gamma^\nu\right.\\
&\quad\left.-\frac{1}{16}(p_1+p_2)^\nu\gamma^\mu\right\}.
\end{aligned} \tag{7.34}$$

where $i\Gamma^{(GR)}_{\phi\phi}(p^2, \kappa)$ and $i\Gamma^{(GR)}_{\psi\bar{\psi}}(p; \kappa)$ are given by the diagrams in Figure 1a and Figure 1b , respectively. The wavy line stands for the graviton propagator in (7.3). $i\Gamma^{(GR)}_{\psi\bar{\psi}\phi}(p_1, p_2; \kappa)$ is obtained by adding up all the contributions the diagrams in Figure 11 give rise to. The sum of the diagrams in Figure 12 yields $i\Gamma^{(GR)}_{\psi\bar{\psi}h_{\mu\nu}}(p_1, p_2; \kappa)$.

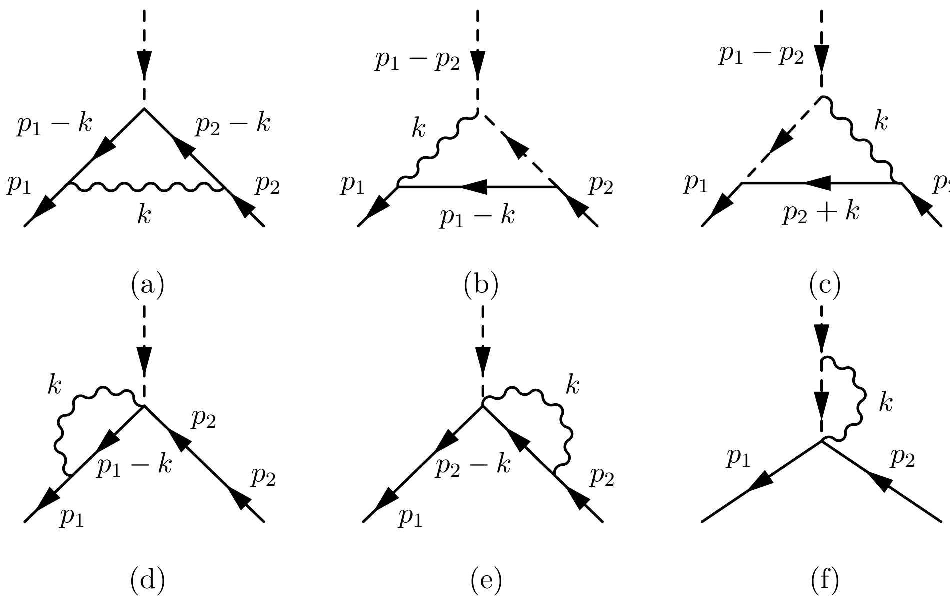


Figure 11: Vertices of order $y\kappa^2$.

To define the S matrix elements it is necessary to express the bare masses M and m in terms of the corresponding physical masses –ie, the

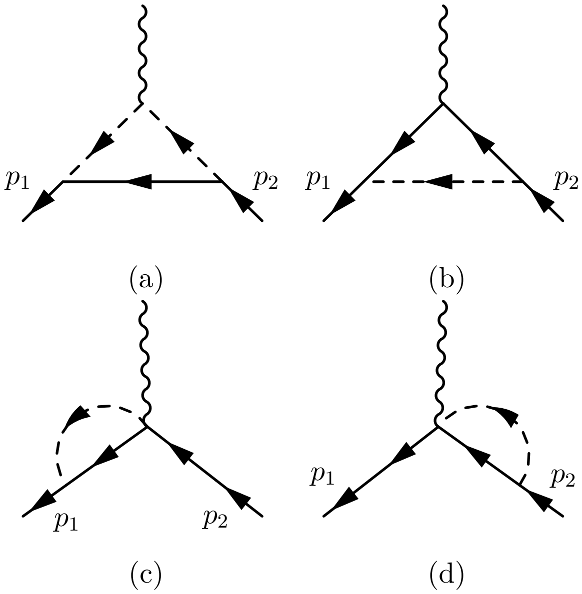


Figure 12: Vertices of order $y^2\kappa$.

poles of the propagators– M_ϕ and m_ψ. This is accomplished by using the following formulae

$$\begin{aligned}
&m = Z_m\, m_\psi, \quad M^2 = Z_M\, M_\phi^2,\\
&Z_m = 1 - \Sigma_1(p^2 = m_\psi^2) - \Sigma_2(p^2 = m_\psi^2), \quad Z_M = 1 - \frac{i}{M_\phi^2}\Gamma_{\phi\phi}(p^2 = M_\phi^2),\\
&i\Gamma_{\psi\bar\psi}(p,\kappa) = i\Gamma^{(NG)}_{\psi\bar\psi}(p) + i\Gamma^{(G)}_{\psi\bar\psi,}(p;\kappa),\\
&i\Gamma_{\phi\phi}(p;\kappa) = i\Gamma^{(NG)}_{\phi\phi}(p) + i\Gamma^{(G)}_{\phi\phi}(p;\kappa),\\
&i\Gamma^{(NG)}_{\psi\bar\psi}(p,\kappa) = (\not p - m)\Sigma_2^{(NG)}(p^2) + m\Sigma_1^{(NG)}(p^2),\\
&i\Gamma^{(G)}_{\psi\bar\psi}(p,\kappa) = (\not p - m)\Sigma_2^{(G)}(p^2) + m\Sigma_1^{(G)}(p^2),
\end{aligned} \tag{7.35}$$

where the superscript G stand for gravitational –those from General Relativity in the current subsection– contributions, given in (7.34), and the superscript NG denote the corresponding contributions in absence of gravity, whose actual values are not needed in this computation.

To obtain the S matrix elements from the Green functions of the fields, it is also convenient to introduce renormalized fields ϕ_R and ψ_R, so that the Laurent expansion of their propagators, $\langle\phi_R(p)\phi_R(-p)\rangle$ and $\langle\psi_R(p)\bar{\psi}_R(p)\rangle$, around the mass shell read

$$\langle\psi_R(p)\bar{\psi}_R(p)\rangle = \frac{i(\not{p}+m_\psi)}{p^2-m_\psi^2} + \text{Regular terms},$$
$$\langle\phi_R(p)\phi_R(-p)\rangle = \frac{i}{p^2-M_\phi^2} + \text{Regular terms}.$$

The fields ψ_R and ϕ_R are obtained from the bare fields, ψ and ϕ –the fields in (7.23)– by introducing the following wave function renormalizations

$$\begin{aligned}
&\psi = Z_\psi^{1/2}\psi_R, && \phi = Z_\phi^{1/2}\phi_R,\\
&Z_\psi = 1+\delta Z_\psi, && \delta Z_\psi = \Sigma_2(p^2=m_\psi^2) + 2m_\psi^2\Sigma_1'(p^2=m_\psi^2),\\
&Z_\phi = 1+\delta Z_\phi, && \delta Z_\phi = i\Gamma_{\phi\phi}'(p^2=M_\phi^2),
\end{aligned} \tag{7.36}$$

with

$$\Sigma_1'(p^2) \equiv \frac{d}{dp^2}\Sigma_1(p^2)\text{and} \qquad \Gamma_{\phi\phi}'(p^2) \equiv \frac{d}{dp^2}\Gamma_{\phi\phi}(p^2). \tag{7.37}$$

The reader should bear in mind that in the defining Z_m, Z_ψ, Z_M and Z_ϕ in terms of $i\Gamma_{\psi\bar{\psi}}(p,\kappa)$ and $i\Gamma_{\phi\phi}(p;\kappa)$, we have taken into account that we are working at the one-loop level.

Considering (7.34), (7.36) and the definitions in (7.35), one obtains

$$\begin{aligned}
\delta Z_\psi &= \frac{1}{16\pi^2\epsilon}\left[\frac{1}{2}y^2+\kappa^2 m_\psi^2\left(\frac{5}{8}+\left[\frac{1}{2}+\alpha\right]\right)\right] + \text{UV finite contributions},\\
\delta Z_\phi &= \frac{1}{16\pi^2\epsilon}\left[2y^2+\kappa^2 M_\phi^2\left(1+\left[\frac{1}{2}+\alpha\right]\right)\right] + \text{UV finite contributions},
\end{aligned} \tag{7.38}$$

where $n=4+2\epsilon$, n being the dimension of spacetime in Dimensional Regularization. The bits of δZ_ψ and δZ_ϕ in (7.38) that are independent of κ are the usual ones that can be found in textbooks.

The wave function renormalizations in (7.36) and (7.38) give rise to a vertex counterterm diagrammatically represented by the diagram in Figure 13, whose value reads

$$-iy(\delta Z_\psi + \frac{1}{2}\delta Z_\phi) = -\frac{ig}{16\pi^2\epsilon}\left[\frac{3}{2}y^2 + \kappa^2 m_\psi^2\left(\frac{5}{8} + \left[\frac{1}{2}+\alpha\right]\right)\right.$$
$$\left. + \kappa^2 M_\phi^2\left(1 + \left[\frac{1}{2}+\alpha\right]\right)\right] + \text{UV finite contributions.} \tag{7.39}$$

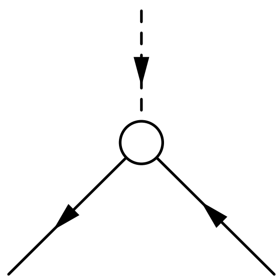


Figure 13: Counterterm $y\kappa^2$.

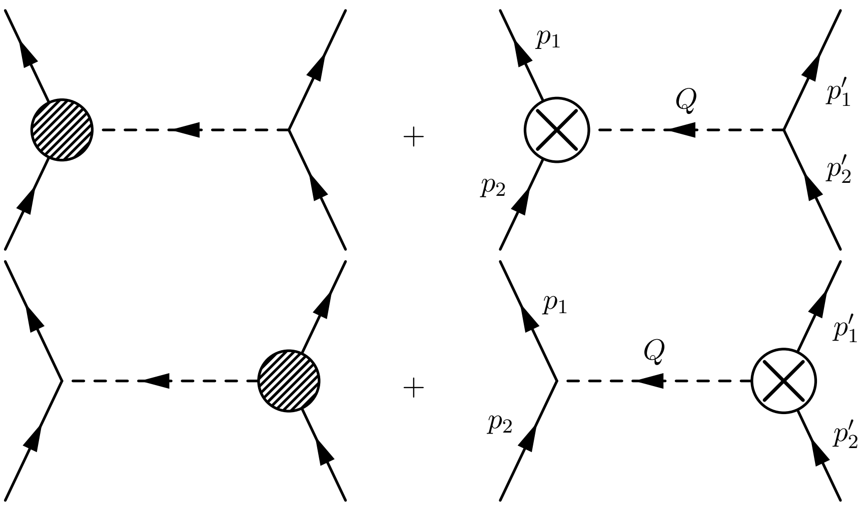


Figure 14: S matrix contributions.

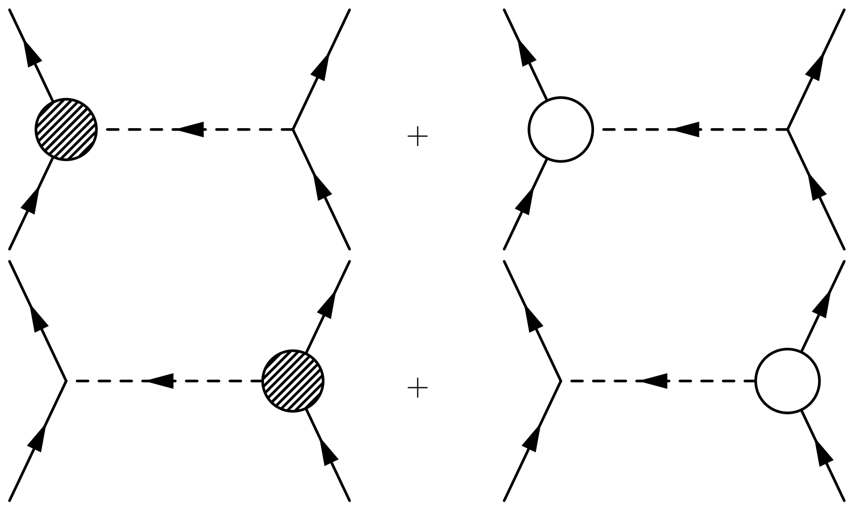


Figure 15: Crossing S matrix contributions.

Using the value of $i\Gamma^{(GR)}_{\psi\bar{\psi}\phi}(p_1, p_2; \kappa)$, displayed in (7.34), and the counterterm in (7.39), one obtains the following expression for the UV divergent General Relativity contribution to the S matrix coming from the sum of all the diagrams in Figure 14:

$$(\bar{u}(p_1) \cdot u(p_2))(\bar{u}(p_1^{'}) \cdot u(p_2^{'}))\times \\ \times\left(\frac{-i}{16\pi^2\epsilon}\right) y^2\kappa^2 \left[\frac{1}{2}M_\phi^2 + \frac{3}{2}m_\psi^2 - \frac{1}{8}Q^2 + \left[\tfrac{1}{2} + \alpha\right]\left(Q^2 - M_\phi^2\right)\right] \frac{i}{Q^2 - M_\phi^2}, \tag{7.40}$$

where $Q = p_1 - p_2$. Bear in mind that the blob with slanted lines in diagrams of Figure 14 represents the sum of all the diagrams in Figure 11, i.e. $i\Gamma^{(GR)}_{\psi\bar{\psi}\phi}(p_1, p_2; \kappa)$.

The crossing diagrams in Figure 15 yields the following UV divergent General Relativity contribution to the S matrix:

$$-(\bar{u}(p_1) \cdot u(p_2^{'}))(\bar{u}(p_1^{'}) \cdot u(p_2))\times \\ \left(\frac{-i}{16\pi^2\epsilon}\right) y^2\kappa^2 \left[\frac{1}{2}M_\phi^2 + \frac{3}{2}m_\psi^2 - \frac{1}{8}\tilde{Q}^2 + \left[\frac{1}{2} + \alpha\right](\tilde{Q}^2 - M_\phi^2)\right] \frac{i}{\tilde{Q}^2 - M_\phi^2}, \tag{7.41}$$

where $\tilde{Q} = p_1^{'} - p_2$.

Let us denote by Box8a, Box8b and Box8c the UV divergent General Relativity UV divergent contributions to the S matrix coming from the diagrams a), b) an c) in Figure 16, respectively. A lengthy computation yields at the following simple expressions

$$\begin{aligned}
\text{Box8a} &= \left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2 y^2\left(-\frac{3}{8}\right)(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1')\cdot u(p_2')),\\
\text{Box8b} &= \left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2 y^2\left(\frac{3}{2}\right)(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1')\cdot u(p_2')),\\
\text{Box8c} &= \left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2 y^2\Big[-1-\Big[\frac{1}{2}+\alpha\Big]\Big](\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1')\cdot u(p_2')).
\end{aligned}$$

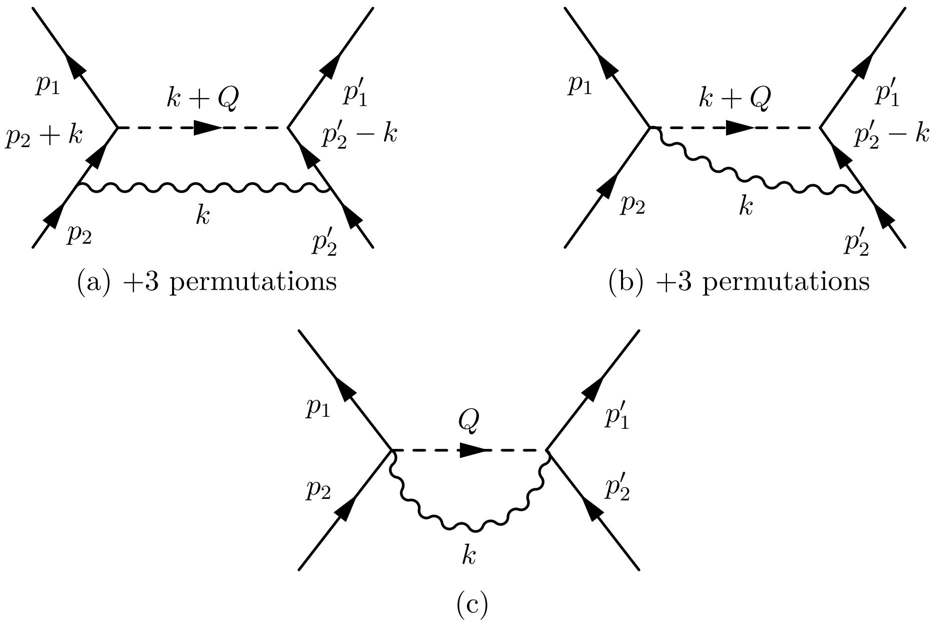


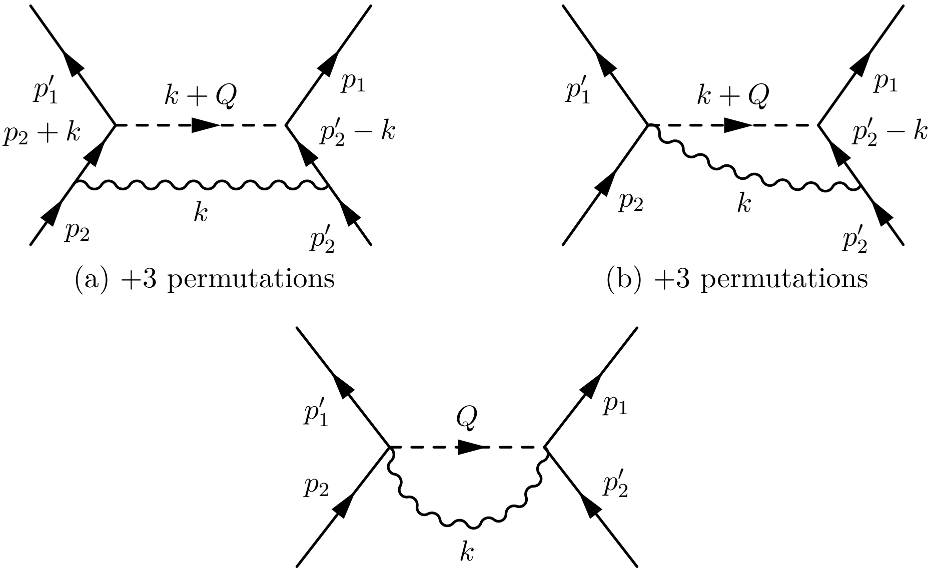
Figure 16: Box diagrams.

By adding Box8a, Box8b and Box8c above, one obtains

$$\left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2 y^2\left(\frac{1}{8}-\left(\frac{1}{2}+\alpha\right)\right)(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1')\cdot u(p_2')). \tag{7.42}$$

Obviously, the sum of the crossing diagrams in Figure 17 yields the following UV contribution to the S matrix for General Relativity

$$-\left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2 y^2\left(\frac{1}{8}-\left(\frac{1}{2}+\alpha\right)\right)(\bar{u}(p_1')\cdot u(p_2))(\bar{u}(p_1)\cdot u(p_2')). \tag{7.43}$$



(c)

Figure 17: Box diagrams (crossing).

Adding the expressions in (7.40), (7.41), (7.42) and (7.43), one obtains the following result

$$(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1^{'})\cdot u(p_2^{'}))\left(\frac{1}{16\pi^2\epsilon}\right)y^2\kappa^2\left\{\left[\frac{3}{8}M_\phi^2+\frac{3}{2}m_\psi^2\right]\frac{1}{Q^2-M_\phi^2}\right\}-$$
$$(\bar{u}(p_1)\cdot u(p_2^{'}))(\bar{u}(p_1^{'})\cdot u(p_2))\left(\frac{1}{16\pi^2\epsilon}\right)y^2\kappa^2\left\{\left[\frac{3}{8}M_\phi^2+\frac{3}{2}m_\psi^2\right]\frac{1}{\tilde{Q}^2-M_\phi^2}\right\}, \tag{7.44}$$

where $Q = p_1 - p_2$ and $\tilde{Q} = p_1^{'} - p_2$. Let us stress that the dependence on the gauge parameter α goes away upon summing over all diagrams in Figures 14 to 17.

Let us introduce now the counterterm vertex in Figure 18, which comes from renormalization produced by the constants Z_ψ and Z_m in Minkowski spacetime applied to the energy-momentum tensor in (7.27):

$$i\Gamma^{(ct)}_{\psi\bar{\psi}h_{\mu\nu}}(p_1,p_2;\kappa) = -\,i\frac{\kappa}{2}\Big\{\delta Z^{(Mink)}_{\psi}\Big[\frac{1}{4}\Big(\gamma^{\mu}(p_1+p_2)^{\nu}+\gamma^{\nu}(p_1+p_2)^{\mu}\Big)- \\ -\frac{1}{2}\eta^{\mu\nu}\Big((\not{p}_1+\not{p}_2)-2m\Big)\Big]+\delta Z^{(Mink)}_{m}\eta^{\mu\nu}m\Big\}, \quad (7.45)$$

where

$$\delta Z^{(Mink)}_{\psi} = \frac{1}{2}y^2\Big(\frac{1}{16\pi\epsilon}\Big), \quad \delta Z^{(Mink)}_{m} = -\frac{3}{2}y^2\Big(\frac{1}{16\pi\epsilon}\Big).$$

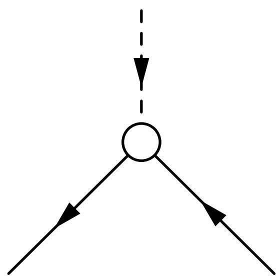


Figure 18: Counterterm $y^2\kappa$.

Taking into account the expressions presented in (7.34) and (7.45), one concludes that

$$i\Gamma^{(GR)}_{\psi\bar{\psi}h_{\mu\nu}}(p_1,p_2;\kappa)+i\Gamma^{(ct)}_{\psi\bar{\psi}h_{\mu\nu}}(p_1,p_2;\kappa) \;=\; 0. \quad (7.46)$$

Hence, at one-loop and $y^2\kappa$ order, the vertex $\bar{\psi}\psi h_{\mu\nu}$ is UV finite and, as a result, there is no UV divergent contribution to the S matrix coming from the sum of the diagrams in Figure 19. Likewise for the sum of the diagrams in Figure 12. Note that the black blob in Figures 19 and 20 represents $i\Gamma^{(GR)}_{\psi\bar{\psi}h_{\mu\nu}}(p_1,p_2;\kappa)$ in (7.34).

The cancellation of UV divergences in (7.46) is an addition to the list of surprising UV cancellations that occur when gravity is coupled to matter: See [86, 95] and references therein.

Summarizing, the UV divergent contribution the S matrix element of the $fermion + fermion \to fermion + fermion$ scattering is given by (7.44) at one-loop and $y^2\kappa^2$ order, due to the cancellation of UV divergences in (7.46).

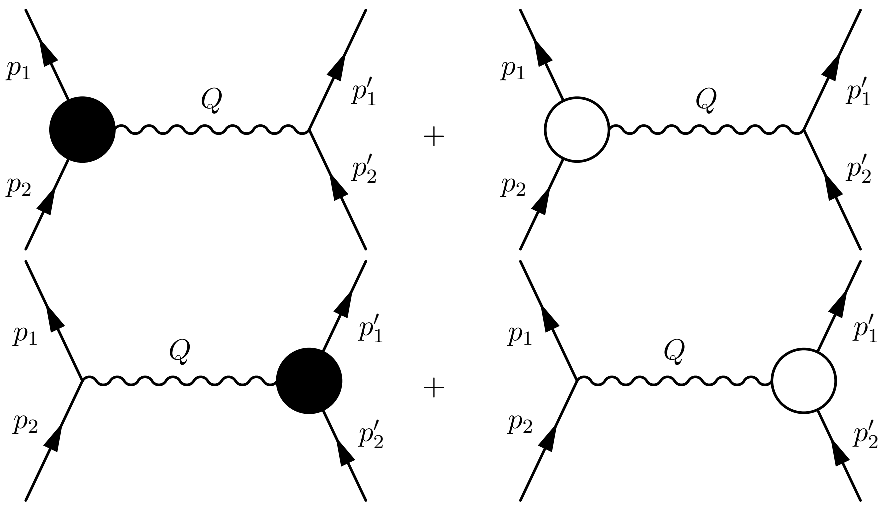


Figure 19: S matrix contributions.

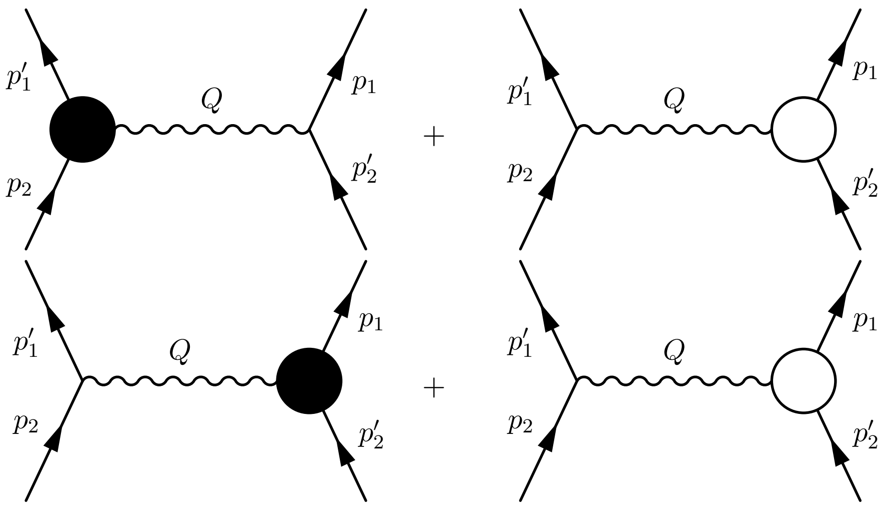


Figure 20: Crossing S matrix contributions.

7.2.2.2 *The Unimodular Gravity case*

When Unimodular Gravity is the theory of quantum gravity the diagrams to be computed are the same as those in the General Relativity case –ie, diagrams in Figures 1a to 20, with the proviso that the internal graviton line represents the propagator –given in (7.11)– of the field $\hat{h}_{\mu\nu}$ –this is the field which couples to the Energy-momentum tensor, see (7.32).

Let us denote by $i\Gamma^{(UG)}_{\phi\phi}(p^2,\kappa)$ and $i\Gamma^{(UG)}_{\psi\bar{\psi}}(p;\kappa)$ the UV divergent part of the diagrams in Figures 1a and 1b. Let $i\Gamma^{(UG)}_{\psi\bar{\psi}\phi}(p_1,p_2;\kappa)$ be UV divergent contribution coming from the sum of all diagrams in Figure 11, and, finally, let $i\Gamma^{(UG)}_{\psi\bar{\psi}\hat{h}_{\mu\nu}}(p_1,p_2;\kappa)$ stand for the UV divergent contribution obtained by summing all diagrams in Figure 12. We have

$$\begin{aligned}
&i\Gamma^{(UG)}_{\phi\phi}(p^2,\kappa)=0,\\
&i\Gamma^{(UG)}_{\psi\bar{\psi}}(p;\kappa)=\left(\frac{-i}{16\pi^2\epsilon}\right)\kappa^2\left(\frac{3}{8}p^2m-\frac{5}{16}p^2\not{p}+\frac{3}{16}m^2\not{p}\right),\\
&i\Gamma^{(UG)}_{\psi\bar{\psi}\phi}(p_1,p_2;\kappa)=y\kappa^2\left(\frac{-i}{16\pi^2\epsilon}\right)\left(\frac{9}{16}(p_1^2+p_2^2)-\frac{3}{8}p_1\cdot p_2+\right.\\
&\qquad\qquad\left.+\frac{3}{16}m(\not{p}_1+\not{p}_2)-\frac{3}{8}\not{p}_1\not{p}_2\right),\\
&i\Gamma^{(UG)}_{\psi\bar{\psi}\hat{h}_{\mu\nu}}(p_1,p_2;\kappa)=\kappa y^2\left(\frac{-i}{16\pi^2\epsilon}\right)\left\{\left[\frac{1}{8}(\not{p}_1+\not{p}_2)+\frac{1}{2}m\right]\eta^{\mu\nu}-\right.\\
&\qquad\qquad\left.-\frac{1}{16}(p_1+p_2)^\mu\gamma^\nu-\frac{1}{16}(p_1+p_2)^\nu\gamma^\mu\right\}.
\end{aligned}\tag{7.47}$$

Taking into account the definitions in (7.36) and the results in (7.47), one concludes, after a little algebra, that

$$\begin{aligned}
\delta Z_\psi&=\frac{1}{16\pi^2\epsilon}\frac{1}{2}y^2+\text{UV finite contributions},\\
\delta Z_\phi&=\frac{1}{16\pi^2\epsilon}2y^2+\text{UV finite contributions},
\end{aligned}$$

i.e. in the Unimodular Gravity case and for the gauge-fixing leading to the propagator in (7.11), there are no $y^2\kappa^2$ UV divergent contributions to the wave function renormalizations of the fermion and scalar fields. Hence, unlike in the General Relativity case, the countertem vertex in Figure 13 can be set to zero when computing the UV divergent contributions to the S matrix. This very same reasoning applies to those diagrams in Figures 14 and 15 which involve the vertex in Figure 13.

Let us point out that the blob with slanted lines in Figure 14 represents now the function $i\Gamma^{(UG)}_{\psi\bar{\psi}\phi}(p_1,p_2;\kappa)$. Next, by using the value of $i\Gamma^{(UG)}_{\psi\bar{\psi}\phi}(p_1,p_2;\kappa)$ in (7.47), one obtains that the Unimodular Gravity UV divergent contribution to the S matrix coming form the diagrams in Figure 14 reads:

$$(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1^{'})\cdot u(p_2^{'}))\times\left(\frac{1}{16\pi^2\epsilon}\right)y^2\kappa^2\left[\frac{3}{2}m_\psi^2+\frac{3}{8}Q^2\right]\frac{1}{Q^2-M_\phi^2}, \tag{7.48}$$

where $Q=p_1-p_2$.

Now, the corresponding contribution coming from the crossing diagrams in Figure 7 runs thus

$$-(\bar{u}(p_1^{'})\cdot u(p_2))(\bar{u}(p_1)\cdot u(p_2^{'}))\times\left(\frac{1}{16\pi^2\epsilon}\right)y^2\kappa^2\left[\frac{3}{2}m_\psi^2+\frac{3}{8}\tilde{Q}^2\right]\frac{1}{\tilde{Q}^2-M_\phi^2}, \tag{7.49}$$

where $\tilde{Q}=p_1^{'}-p_2$.

Let us call Box8aUG, Box8bUG and Box8cUG the sum of the diagrams in Figures 8a, 8b and 8c, respectively, where the graviton line stands for the propagator in (7.11). A long computation yields the following simple results for the UV divergent contribution to the S matrix coming from those diagrams:

$$\begin{aligned}
\text{Box8aUG} &= \left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2y^2\left(\frac{17}{8}\right)(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1^{'})\cdot u(p_2^{'})),\\
\text{Box8bUG} &= \left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2y^2\left(-\frac{3}{2}\right)(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1^{'})\cdot u(p_2^{'})),\\
\text{Box8cUG} &= \left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2y^2(-1)(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1^{'})\cdot u(p_2^{'})).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\text{Box8aUG}+\text{Box8bUG}+\text{Box8cUG}=\\
&=\left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2y^2\left(-\frac{3}{8}\right)(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1^{'})\cdot u(p_2^{'})).
\end{aligned} \tag{7.50}$$

The contribution coming from the crossing diagrams in Figure 17 reads

$$\left(\frac{1}{16\pi^2\epsilon}\right)\kappa^2y^2\left(-\frac{3}{8}\right)(\bar{u}(p_1^{'})\cdot u(p_2))(\bar{u}(p_1)\cdot u(p_2^{'})). \tag{7.51}$$

Let us add all the UV divergent contributions to the S matrix coming from the diagrams in Figures 14, 15, 16 and 17. These contributions are given in (7.48), (7.49), (7.50) and (7.51); their sum being

$$\begin{aligned}
&(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1^{'})\cdot u(p_2^{'}))\left(\frac{1}{16\pi^2\epsilon}\right)y^2\kappa^2\left\{\left[\frac{3}{2}m_\psi^2+\frac{3}{8}Q^2-\frac{3}{8}(Q^2-M_\phi^2)\right]\frac{1}{Q^2-M_\phi^2}\right\}-\\
&(\bar{u}(p_1)\cdot u(p_2^{'}))(\bar{u}(p_1^{'})\cdot u(p_2))\left(\frac{1}{16\pi^2\epsilon}\right)y^2\kappa^2\left\{\left[\frac{3}{2}m_\psi^2+\frac{3}{8}Q^2-\frac{3}{8}(Q^2-M_\phi^2)\right]\frac{1}{\tilde{Q}^2-M_\phi^2}\right\}=\\
&(\bar{u}(p_1)\cdot u(p_2))(\bar{u}(p_1^{'})\cdot u(p_2^{'}))\left(\frac{1}{16\pi^2\epsilon}\right)y^2\kappa^2\left\{\left[\frac{3}{8}M_\phi^2+\tfrac{3}{2}m_\psi^2\right]\frac{1}{Q^2-M_\phi^2}\right\}-\\
&(\bar{u}(p_1)\cdot u(p_2^{'}))(\bar{u}(p_1^{'})\cdot u(p_2))\left(\frac{1}{16\pi^2\epsilon}\right)y^2\kappa^2\left\{\left[\frac{3}{8}M_\phi^2+\tfrac{3}{2}m_\psi^2\right]\frac{1}{\tilde{Q}^2-M_\phi^2}\right\},
\end{aligned} \tag{7.52}$$

where $Q=p_1-p_2$ and $\tilde{Q}=p_1^{'}-p_2$

It is plain that (7.44) and (7.52) are equal, i.e., the sum of the UV divergent contributions to the S matrix coming from the diagrams in Figures 14, 15, 16 and 17 in General Relativity and Unimodular Gravity is the same. Notice, however, that the contribution coming from each diagram is not the same in General Relativity as in Unimodular Gravity.

It is clear –the gravitational field here is a mere spectator– that the counterterm vertex represented by the diagram in Figure 18 has the same value for Unimodular Gravity as for General Relativity, i.e. is given by the expressions in (7.45) upon replacing $h_{\mu\nu}$ with $\hat{h}_{\mu\nu}$. Next, notice that $i\Gamma^{(UG)}_{\psi\bar{\psi}\hat{h}_{\mu\nu}}(p_1,p_2;\kappa)$ in (7.47) is equal to $i\Gamma^{(GR)}_{\psi\bar{\psi}\hat{h}_{\mu\nu}}(p_1,p_2;\kappa)$ in (7.34). Hence, as in the General Relativity case -see (7.46), the following cancellation of UV divergences hold

$$i\Gamma^{(UG)}_{\psi\bar{\psi}\hat{h}_{\mu\nu}}(p_1,p_2;\kappa)+i\Gamma^{(ct)}_{\psi\bar{\psi}\hat{h}_{\mu\nu}}(p_1,p_2;\kappa)\;=\;0. \tag{7.53}$$

Thus, the one loop and $y^2\kappa$ order correction to the $\bar{\psi}\psi\hat{h}_{\mu\nu}$ is UV finite.

Using the previous result, one concludes that the sum of the diagrams in Figure 19 contains no UV divergent pieces in the Unimodular Gravity case either. Same result for the sum of the diagrams in Figure 20.

In summary, due to the UV cancellation we have just discussed, only the sum of the diagrams in Figures 14, 15, 16 and 17 gives, in the Unimodular Gravity case, a UV divergent contribution to the S matrix element of the $fermion+fermion\to fermion+fermion$ scattering at one-loop and $y^2\kappa^2$ order. Full agreement between Unimodular Gravity and General Relativity has been thus reached.

Part V

Conclusions

Summary and discussion

Throughout this thesis we have studied the theory of Unimodular Gravity. To achieve this, we have started studying if the main property of this theory at the classical level –the decoupling between the cosmological constant and the metric tensor– holds at the quantum level. And secondly, we have tried to find observable differences between Unimodular Gravity and General Relativity at the quantum level.

With respect to the cosmological constant problem, it appears in a different guise in UG with respect to GR. The corresponding EoM admit a first integral that plays the same role as the cosmological constant in GR. The novelty is however that this first integral is not related to the zero momentum piece of the potential, but is rather determined by the boundary conditions, as is the rest of the dynamics. This is an important shift of the paradigm, in the sense that it explains why a huge value for the vacuum energy does not imply a correspondingly huge value for the cosmological constant.

We have argued that (one-loop) quantum corrections do not generate a cosmological constant in UG. It would be more precise to say that the cosmological constant is generated, but it is a non-dynamical quantity, since it does not couple to the gravitational field. This is due to the absence of the $\sqrt{g}$ factor in the integration measure. The analysis is long and quite technical, but the result is simple enough. The implication is that we can disregard this term since it will not contribute to any correlator involving physical fields. Therefore, we conclude that in this case there is no renormalization of the cosmological constant and its peculiar status in UG is preserved through quantum corrections.

Moreover, due to our definition of UG –that is, in terms of unconstrained metrics, there is a Weyl symmetry that forbids operators of zero dimension,

$$S = M^n \int d^n x \, (-g)^\beta \,, \tag{7.54}$$

for any non-vanishing β.

This means that, although we have performed the explicit computation at one-loop level, the result can be extended to any loop order, since it relies in the fact that the operators contributing to the cosmological constant are non-dynamical.

Once it was clear that the theory is useful to solve the cosmological constant problem even when quantum corrections are taken into account, we focused on finding other differences between the two theories. In order to achieve that, in the second part of the thesis we studied if there is any (quantum) observable that can tell UG from GR. We obtained several results to this respect.

First of all, we computed the corrections to the beta functions of the Yukawa and $\lambda\phi^4$ theory both in GR and UG and found different values for the beta functions –actually, for UG the $\lambda\phi^4$ theory it vanishes, when performing the usual multiplicative renormalization. Although this may seem a difference between these two theories, we have shown that these beta functions are gauge dependent. The conclusion is that these corrections do not have any intrinsic physical meaning, therefore, the statements about asymptotic freedom are not physically meaningful. Moreover, we also proved that they can be set to zero by a suitable, non-multiplicative, field redefinition. This means that these gravitational corrections to the beta functions in question can not distinguish between both theories.

Of course, the gauge dependence of the gravitational corrections to the beta function can be avoided by using the DeWitt-Vilkovisky action instead of the 1PI functional –as done in reference [92] for the $\lambda\phi^4$ theory– but those gauge-independent contributions can still be removed by non-mutiplicative, but local, field redefinitions. The use of the DeWitt-Vilkovisky effective action does not give the gravitational corrections in question any intrinsic physical meaning, so that any conclusion drawn from them also lack physical content.

Finally, and motivated by this previous result, we computed the UV divergent contribution to the S matrix element for the $fermion + fermion \rightarrow fermion + fermion$ scattering process in the Yukawa theory coupled to UG (at one-loop and $y^2\kappa^2$ order). We have proved that it is the same as the corresponding S matrix element when UG is replaced with GR. We should point out that this agreement does not hold for each individual Feynman diagram but it unfolds upon adding the contributions coming from classes of the Feynman diagrams (which of course yields a result independent of the gauge parameter). This is a common result in every computation comparing UG and GR, as we have seen when computing the beta functions as well as in other works [91]. Of course, the gauge symmetries of UG are not the same –as was explained in detail in chapter 3– as those of GR, then, agreement between non gauge invariant objects computed in both theories is not to be expected and it does not occur in general.

As a final remark, these results explicitly show that the beta function of the Yukawa coupling cannot be used to draw any physically meaningful conclusion, a fact already discussed in [94] (and [96], in the case the $\lambda\phi^4$ theory).

To sum up, we conclude that UG is a compelling alternative to GR that partially solves the cosmological constant problem, even from the point of view of an effective field theory. Besides this result –and the technical difficulties that UG has when computing quantum corrections compared to GR, we have not found any physical effect up to one-loop order that can differentiate both theories. At this point, one should wonder whether GR or UG is the most reasonable low energy effective field theory for describing gravity. To find any difference at two-loop order could help to put an end to this question.

Finally, as an open question, it would be interesting to find out if UG can be derived from String Theory as GR can. We have not explicitly proved that but, since we have given proof that the S-matrix is the same for both theories, we are confident that this is te case.

Resumen y discusión

A lo largo de esta tesis hemos estudiado la teoría de Gravedad Unimodular . Para ello, empezamos estudiando si la principal propiedad de la teoría – el desacoplo entre la constante cosmológica y el tensor métrico– se mantiene a nivel cuántico. Después, intentamos ver si hay algún observable que pueda distinguir entre ambas teorías a nivel cuántico.

Con respecto a la constante cosmológica, ésta aparece de forma distinta en la Gravedad Unimodular a como lo hace en Relatividad General. En la primera, las ecuaciones de movimiento admiten una integral primera que juega el papel de la constante cosmológica en Relatividad General. Sin embargo, la novedad aquí es que esta integral primera no se relaciona con la parte constante del potencial, sino que está determinada por las condiciones de contorno. Este es un cambio importante en el paradigma, puesto que permite explicar por qué un valor muy grande de la energía de vacío no implica un valor muy grande de la constante cosmológica.

Hemos expuesto que las correcciones cuánticas (en particular a un loop) no generan un término de constante cosmológica en Gravedad Unimodular. Sería más adecuado, sin embargo, decir que se genera una constante cosmológica, pero que es no dinámica, ya que no se acopla al campo gravitatorio. Esto se debe a la ausencia del factor $\sqrt{g}$ en la acción. A pesar de que el análisis es largo y técnicamente complicado, la conclusión es sencilla: podemos ignorar ese término ya que no contribuye a ninguna cantidad física. Por lo tanto, concluimos que en este caso no hay renormalización de la constante cosmológica y que las correcciones cuánticas no modifican su valor en Gravedad Unimodular.

Además, debido a nuestra definición de la acción en términos de métricas sin ligaduras, la teoría tiene una simetría Weyl que impide que aparezcan operadores de dimensión cero en la acción

$$S = M^n \int d^n x \, (-g)^\beta \,, \tag{7.55}$$

para β distinto de cero.

Esto significa que aunque hemos presentado el resultado explícito a orden un loop, el resultado puede extenderse a cualquier orden, ya que se debe al hecho de que los operadores que contribuyen a la constante cosmológica son no dinámicos.

Una vez ha quedado claro que la teoría es interesante para resolver el problema de la constante cosmológica, incluso cuando tratamos cuánticamente la teoría, nos centramos en encontrar otras diferencias entre ambas teorías. Para ello, en la segunda parte de la tesis estudiamos si hay algún observable (a nivel cuántico) que pueda distinguir entre ambas. A este respecto obtenemos varios resultados.

En primer lugar, calculamos en ambas teorías las correcciones a las funciones beta para una interacción de Yukawa y de un campo escalar $\lambda\phi^4$. Cuando utilizamos una renormalización multiplicativa, encontramos distintos resultados para estas funciones beta –de hecho, en Gravedad Unimodular la función beta para $\lambda\phi^4$ se anula directamente. Esto puede parecer, *a priori*, una diferencia entre ambas teorías, pero también demostramos que estas funciones beta dependen del gauge escogido. La conclusión es por lo tanto que no pueden tener significado físico y, en consecuencia, que las afirmaciones sobre libertad asintótica en presencia de gravedad no tienen sentido. También probamos que pueden incluso hacerse cero (y por lo tanto iguales) cuando se utiliza una renormalización no multiplicativa. Esto significa que no podemos utilizar las funciones beta para distinguir entre Gravedad Unimodular y Relatividad General. Por supuesto, la dependencia del gauge de las correcciones gravitatorias a las funciones beta podrían evitarse usando la acción de DeWitt-Vilkovisky en lugar del funcional generador de los diagramas 1PI –tal y como se hace en la referencia [92], pero esas contribuciones, aunque independientes del gauge, pueden eliminarse por medio de redefiniciones de los campos no multiplicativas. Por lo tanto, usar la acción de DeWitt-Vilkovisky no da ningún significado físico a las correcciones gravitatorias, luego cualquier conclusión sacada a partir de ellas carece también de significado físico.

Finalmente, motivado por el resultado anterior, calculamos las contribuciones divergentes ultravioleta a la matriz S para los procesos $fermion + fermion \to fermion + fermion$ con la interacción de Yukawa acoplada a la Gravedad Unimodular (a un loop y $y^2\kappa^2$ order), y comprobamos que son las mismas cuando se acopla a Relatividad General. En este punto cabe destacar que dichas contribuciones no coinciden para cada diagrama de Feynman, sino que aparece al sumar todas las contribuciones que vienen de distintos tipos de diagramas (que es por supuesto independiente del parámetro gauge). Este es un resultado común en todos los cálculos en los que hemos comparado Relatividad General con Gravedad Unimodular, ya lo hemos visto al calcular las funciones beta, así como en otros trabajos anteriores [91]. Por supuesto, las simetrías gauge en Gravedad Unimodular no son las mismas que en Relatividad general –como se explica en el capítulo 3– por lo que no se debe esperar un acuerdo entre cantidades que dependen del gauge en ambas teorías, y de hecho no ocurre en general. Como último comentario a este respecto, nuestros resultados demuestran explícitamente que la función beta para la interacción de Yukawa no puede usarse para obtener ninguna consecuencia física, hecho que se discute en [94] (y [96] para el caso de $\lambda\phi^4$).

Para concluir, llegamos a la conclusión que la Gravedad Unimodular es una teoría muy prometedora como alternativa a Relatividad General, ya que soluciona parcialmente el problema de la constante cosmológica. Además de este resultado –y de las dificultades técnicas que tiene la Gravedad Unimodular si lo comparamos con Relatividad General, no hemos encontrado ningún efecto físico, al menos a orden un loop, que nos permita diferenciar ambas teorías. En este punto deberíamos preguntarnos cuál de las dos es más razonable como teoría efectiva a baja energía para describir la gravedad. Encontrar una diferencia a orden dos loops podría contestar esta pregunta. Sería interesante descubrir si se puede obtener la Gravedad Unimodular a partir de la teoría de

Cuerdas tal como se puede hacer con la Relatividad General. Aunque no hemos demostrado esto explícitamente,dado que hemos probado que la matriz S es igual para ambas teorías creemos que este es el caso.

Part VI

Appendix

A

Heat Kernel coefficients of a quartic operator

One of the operators appearing in our computations is an operator whose leading part contains four covariant derivatives. The Heat Kernel of these operators have been also studied by many people and fairly general formulas have been given. However, here we are only interested in the contribution to the effective action in four dimensions. This has been computed in [74, 97] for an operator of the form we are interested in

$$D = \gamma_{AB}\Box^2 + \Omega^{\mu\nu\alpha}_{AB}\nabla_\mu\nabla_\nu\nabla_\alpha + J^{\mu\nu}_{AB}\nabla_\mu\nabla_\nu + H^{\mu}_{AB}\nabla_\mu + P_{AB}. \tag{A.1}$$

The corresponding expression of this kind of operators in four dimensions is quite involved. However, when $\Omega^{\mu\nu\alpha}_{AB} = 0$ as it is in the case of our work, the resultant expression simplifies a lot and reads, with our conventions

$$\begin{aligned} W_\infty = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\sqrt{|g|}\, Tr\Bigg(&\frac{1}{90}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{1}{90}R_{\mu\nu}R^{\mu\nu} + \frac{1}{36}R^2\mathbb{I} - \hat{P} + \\ &+\frac{1}{6}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu} - \frac{1}{6}J^{(\mu\nu)}R_{\mu\nu} + \frac{1}{12}J^{\mu}_{\mu}R + \frac{1}{48}(J^{\mu}_{\mu})^2 + \\ &+\frac{1}{24}J_{(\mu\nu)}J^{(\mu\nu)} - \frac{1}{2}J^{[\mu\nu]}\hat{\mathcal{R}}_{\mu\nu}\Bigg), \end{aligned} \tag{A.2}$$

where, as usual

$$J^{(\mu\nu)} = \frac{1}{2}\left(J^{\mu\nu} + J^{\nu\mu}\right), \tag{A.3}$$

$$J^{[\mu\nu]} = \frac{1}{2}\left(J^{\mu\nu} - J^{\nu\mu}\right). \tag{A.4}$$

B

Heat Kernel contributions of the different operators involved

Here we compute the different heat kernel coefficients corresponding to each of the minimal differential operator appearing in the path integral formulation of Unimodular Gravity.

The contribution of S_{bc}

The action term for the fields $b^{\mu(1,-1)}$ and $c^{\mu(1,1)}$ was defined in equation (5.34) and reads

$$\int d^n x \; b^\mu \left(\Box^2 c_\mu^{(1,1)} - 2 R_{\mu\rho} \nabla^\rho \nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \right. \\ \left. - 2 \nabla_\sigma R_{\mu\rho} \nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho} R^{\rho\nu} c_\nu^{(1,1)} \right) . \tag{B.1}$$

This is a quartic operator of the form (A.1) if we identify

$$J^{\mu\nu}_{\alpha\beta} = -2 R^\mu_\alpha \delta^\nu_\beta , \tag{B.2}$$

$$H^\mu_{\alpha\beta} = -2 \nabla^\mu R_{\alpha\beta} , \tag{B.3}$$

$$P_{\alpha\beta} = -\Box R_{\alpha\beta} - R_{\alpha\rho} R^\rho_\beta , \tag{B.4}$$

here the bundle indices are just spacetime greek indices that we indicate with α and β.

And the field strength

$$[\nabla_\mu , \nabla_\nu] c^\alpha = \hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta} c_\beta = R_{\mu\nu}{}^{\alpha\beta} c_\beta . \tag{B.5}$$

Plugging this into (A.2) we find that the contribution of S_{bc} to the quantum effective action is

$$W_{\infty}^{bc} = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x \left(\frac{11}{45}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{41}{45}R_{\mu\nu}R^{\mu\nu} - \frac{1}{18}R^2\right), \tag{B.6}$$

where we have set $|g| = 1$ and we have multiplied by minus two in order to take into account of the fact that there are two fermionic fields.

THE CONTRIBUTION OF S_π

The action term for the dynamics of the fermionic π fields was defined in (5.35) and reads

$$S_\pi = \int d^n x\ \pi^{(1,-1)}\Box^{-1}\pi'^{\,(1,1)}. \tag{B.7}$$

Even if this is a pseudo-differential operator, its contribution to the pole part of the quantum effective action can be easily computed thanks to the fact that $\Box \times \Box^{-1} = 1$. This means that

$$\det(\Box) = \det(\Box^{-1})^{-1} \longrightarrow \log\left[\det(\Box)\right] = -\log\left[\det(\Box^{-1})\right], \tag{B.8}$$

if there is no multiplicative anomaly. This sums up into the fact that the corresponding Heat Kernel expansion of $\Box^{-1}$ will be minus the expansion of $\Box$. Therefore, by using the result of (4.46)

$$a_4(\Box) = -\frac{1}{(4\pi)^2}\frac{1}{360}\int d^n x\left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right), \tag{B.9}$$

where we have already set $g = 1$.

However, here we are integrating over two fermionic fields, which introduces another factor of minus two. Thus, we have that

$$a_4^{\pi} = \frac{1}{(4\pi)^2}\frac{1}{180}\int d^n x\ \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right), \tag{B.10}$$

and its contribution to the effective action is given by

$$W_{\infty}^{\pi} = \frac{1}{16\pi^2}\frac{1}{n-4}\frac{1}{180}\int d^n x\ \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{B.11}$$

THE CONTRIBUTION OF $S_{\bar{c}\phi}$

The action term for $\bar{c}$ and ϕ was given in equation (5.37), reading

$$\int d^n x \; \bar{c}^{(0,-2)} \Box \phi^{(0,2)} . \tag{B.12}$$

This is the simplest possible operator and its a_4 coefficient was given in (4.46). It reads

$$a_4^{\bar{c}\phi} = \frac{1}{(4\pi)^2} \frac{1}{180} \int d^n x \; \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \tag{B.13}$$

where a factor of two has been introduced to take into account that we have two fields. Again, remind that we have set $g = 1$.

Its contribution to the effective action is given by

$$W_\infty^{\bar{c}\phi} = \frac{1}{16\pi^2} \frac{1}{n-4} \frac{1}{180} \int d^n x \; \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \tag{B.14}$$

THE CONTRIBUTION OF S_W

The action term for the Weyl ghost field was given in (5.42) and reads

$$2n\alpha \int d^n x \; b\Box c \tag{B.15}$$

The global multiplicative constant will not contribute to the pole part of the quantum effective action, since it gives just an ultralocal contribution, so we can dismiss it, having just

$$\int d^n x \; b\Box c \tag{B.16}$$

Again, we are left the simplest possible operator and its a_4 coefficient was given in (4.46). It reads

$$a_4^W = -\frac{1}{(4\pi)^2} \frac{1}{180} \int d^n x \; \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) , \tag{B.17}$$

where a factor of minus two has been introduced to take into account that we have two fermionic fields and we have set again $g = 1$.

Its contribution to the effective action is given by

$$W_{\infty}^{W} = -\frac{1}{16\pi^2}\frac{1}{n-4}\frac{1}{180}\int d^n x \,\left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{B.18}$$

C

Functional traces

The functional traces

$$Tr\left(\mathcal{O}_{\nu_1\nu_2\ldots\nu_j}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_p}\frac{\mathbb{I}}{\Box^n}\right), \tag{C.1}$$

that appear in the calculation of the quantum effective action will lead to new contributions to the divergences and can be computed using the heat kernel representation of the operator.

Starting with an operator $\hat{F}(\nabla)$, it can be written as[1]

$$(\hat{F}(\nabla))^{-n} = \frac{1}{(n-1)!}\left[\left(\frac{d}{dm^2}\right)^{n-1} G(m^2)\right]_{m^2=0}. \tag{C.2}$$

Now, the heat kernel representation of the Green function is

$$G(m^2) = \int_0^\infty \exp\left(-sm^2\right)\exp\left(-s\hat{F}(\nabla)\right), \tag{C.3}$$

where

$$\exp(-s\hat{F}(\nabla))\delta(x,x') = \frac{1}{(4\pi)^{n/2}}\frac{\mathcal{D}^{1/2}(x,x')}{s^{n/2}}\exp\left(-\frac{\sigma(x,x')}{2s}\right)\hat{\Omega}(s|x,x'), \tag{C.4}$$

and with

$$\hat{\Omega}(s|x,x') = \sum_{n=0}^{\infty} s^n \hat{a}_n(x,x'). \tag{C.5}$$

1 Let us note here that while [74] are performing their computations in lorentzian signature, we are doing them in the euclidean setting. The differences account for some global signs and some factors of i in the definition of the proper time and the effective action.

Here σ is the world function, defined by the equation $\sigma = \frac{1}{2}\sigma_\mu \sigma^\mu$ and $\mathcal{D}(x, x')$ is the so-called Van-Vleck determinant

$$\mathcal{D}(x, x') = \left| det \left(-\frac{\partial \sigma}{\partial x^\mu \partial x'^\nu} \right) \right|, \tag{C.6}$$

$$\mathcal{D}(x, x') = g^{1/2}(x) g^{1/2}(x') \Delta(x, x'). \tag{C.7}$$

For the particular case of $\hat{F}(\nabla) = \hat{\Box}$ we can find the representation of the inverse Laplace operator

$$\frac{\mathbb{I}}{\Box^n} = \frac{1}{(n-1)!} \int_0^\infty ds\ s^{n-1} \exp\left(-s\hat{\Box}\right). \tag{C.8}$$

Each of the traces we find in our computation can now be computed by acting with derivatives on this representation and using the tables of coincidence limits given in [74]. Finally it is needed to integrate over s, where we find that only three types of (logarithmic) divergent integrals arise for dimension $n \to 4$

$$\int_0^\infty \frac{ds}{s^{n/2+k}}, \quad \text{with} \quad k = -1, 0, 1. \tag{C.9}$$

and whose pole part can be obtained by integrating by parts, which gives the Laurent series of the result.

All but one of the functional traces we need in our computation can be found in [74]. Here we give the value of all of them and remark that we have rederived all of them explicitly, thus checking their results.

The divergent functional traces corresponding to p=2n (C.1) that appear are

$$\begin{aligned}
\nabla_\mu \nabla_\nu \frac{\mathbb{I}}{\Box} = \frac{\sqrt{g}}{8(n-4)\pi^2} \frac{1}{2} \Bigg\{ \Bigg[&+\frac{1}{45} R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \frac{1}{45} R_{\alpha\beta\lambda\mu} R_\nu^{\alpha\beta\lambda} - \frac{2}{45} R_{\mu\alpha} R_\nu^\alpha - \\
&- g_{\mu\nu} \left(\frac{1}{180} R_{\alpha\beta\lambda\sigma} R^{\alpha\beta\lambda\sigma} - \frac{1}{180} R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{72} R^2 + \frac{1}{30} \Box R \right) \mathbb{I} + \\
&+ \frac{1}{30} \Box R_{\mu\nu} + \frac{1}{18} R R_{\mu\nu} + \frac{1}{10} \nabla_\mu \nabla_\nu R \Bigg] - \frac{1}{12} g_{\mu\nu} \hat{\mathcal{R}}_{\alpha\beta} \hat{\mathcal{R}}^{\alpha\beta} + \frac{1}{6} R \hat{\mathcal{R}}_{\mu\nu} + \\
&+ \frac{1}{6} \hat{\mathcal{R}}_{\mu\alpha} \hat{\mathcal{R}}_\nu^\alpha + \frac{1}{6} \hat{\mathcal{R}}_{\nu\alpha} \hat{\mathcal{R}}_\mu^\alpha - \frac{1}{6} \nabla_\mu \nabla^\alpha \hat{\mathcal{R}}_{\alpha\nu} - \frac{1}{6} \nabla_\nu \nabla^\alpha \hat{\mathcal{R}}_{\alpha\mu} \Bigg\}.
\end{aligned} \tag{C.10}$$

$$
\begin{aligned}
&\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta\frac{\mathbb{I}}{\Box^2} = \frac{\sqrt{g}}{8(n-4)\pi^2}\left\{\left[\frac{1}{180}\left(R^\lambda_\mu\left(11R_{\nu\alpha\beta\lambda}-R_{\beta\alpha\nu\lambda}\right)+\right.\right.\right.\\
&+R^\lambda_\nu\left(11R_{\mu\alpha\beta\lambda}-R_{\beta\alpha\mu\lambda}\right)+R^\lambda_\alpha\left(11R_{\mu\nu\beta\lambda}-R_{\beta\nu\mu\lambda}\right)+R^\lambda_\beta\left(11R_{\mu\nu\alpha\lambda}-R_{\alpha\nu\mu\lambda}\right)\Big)+\\
&+\frac{1}{90}\left(R_\mu{}^\lambda{}_\nu{}^\sigma\left(R_{\lambda\alpha\sigma\beta}+R_{\lambda\beta\sigma\alpha}\right)+R_\mu{}^\lambda{}_\alpha{}^\sigma\left(R_{\lambda\nu\sigma\beta}+R_{\lambda\beta\sigma\nu}\right)+R_\mu{}^\lambda{}_\beta{}^\sigma\left(R_{\lambda\nu\sigma\alpha}+R_{\lambda\alpha\sigma\nu}\right)\right)+\\
&+\frac{1}{20}\left(\nabla_\mu\nabla_\nu R_{\alpha\beta}+\nabla_\mu\nabla_\alpha R_{\nu\beta}+\nabla_\mu\nabla_\beta R_{\nu\alpha}+\nabla_n\nabla_a R_{\mu\beta}+\nabla_\nu\nabla_\beta R_{\mu\alpha}+\nabla_a\nabla_\beta R_{\mu\nu}\right)\Big]\mathbb{I}+\\
&+\frac{1}{12}\left[R_{\mu\nu}\hat{\mathcal{R}}_{\alpha\beta}+R_{\mu\alpha}\hat{\mathcal{R}}_{\nu\beta}+R_{\mu\beta}\hat{\mathcal{R}}_{\nu\alpha}+R_{\nu\alpha}\hat{\mathcal{R}}_{\mu\beta}+R_{\nu\beta}\hat{\mathcal{R}}_{\mu\alpha}+R_{\alpha\beta}\hat{\mathcal{R}}_{\mu\nu}\right]+\\
&+\frac{1}{2}\left[\nabla_\mu\nabla_\nu\hat{\mathcal{R}}_{\alpha\beta}+\nabla_\mu\nabla_\alpha\hat{\mathcal{R}}_{\nu,b}+\nabla_\nu\nabla_\alpha\hat{\mathcal{R}}_{\mu\beta}\right]+\frac{1}{8}\left[\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}_{\alpha\beta}+\hat{\mathcal{R}}_{\alpha\beta}\hat{\mathcal{R}}_{\mu\nu}+\hat{\mathcal{R}}_{\mu\alpha}\hat{\mathcal{R}}_{\nu\beta}+\right.\\
&\left.+\hat{\mathcal{R}}_{\nu\beta}\hat{\mathcal{R}}_{\mu\alpha}+\hat{\mathcal{R}}_{\mu\beta}\hat{\mathcal{R}}_{\nu\alpha}+\hat{\mathcal{R}}_{\nu\alpha}\hat{\mathcal{R}}_{\mu\beta}\right]-\frac{1}{12}\left[\hat{\mathcal{R}}_{\mu\lambda}\left(R^\lambda_{\alpha\nu\beta}+R^\lambda_{\beta\nu\alpha}\right)+\hat{\mathcal{R}}_{\nu\lambda}\left(R^\lambda_{\alpha\mu\beta}+R^\lambda_{\beta\mu\alpha}\right)+\right.\\
&\left.+\hat{\mathcal{R}}_{\alpha\lambda}\left(R^\lambda_{\nu\mu\beta}+R^\lambda_{\beta\mu\nu}\right)+\hat{\mathcal{R}}_{\beta\lambda}\left(R^\lambda_{\mu\nu\alpha}+R^\lambda_{\alpha\nu\mu}\right)\right]-\frac{1}{2}\left[-\frac{1}{9}\left(R_{\alpha\mu\beta\nu}+R_{\beta\mu\alpha\nu}\right)R\mathbb{I}+\right.\\
&+g_{\mu\nu}\left[\left(\frac{1}{36}R_{\alpha\beta}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\alpha\sigma\beta}+\frac{1}{90}R_{\rho\sigma\lambda\alpha}R^{\rho\sigma\lambda}{}_\beta-\frac{1}{45}R_{\alpha\lambda}R^\lambda_\beta+\frac{1}{60}\Box R_{\alpha\beta}+\frac{1}{20}\nabla_\alpha\nabla_\beta R\right)\mathbb{I}+\right.\\
&\left.+\frac{1}{12}\left(\hat{\mathcal{R}}_{\alpha\lambda}\hat{\mathcal{R}}^\lambda{}_\beta+\hat{\mathcal{R}}_{\beta\lambda}\hat{\mathcal{R}}^\lambda{}_\alpha\right)-\frac{1}{12}\left(\nabla_\alpha\nabla^\lambda\hat{\mathcal{R}}_{\lambda\beta}+\nabla_\beta\nabla^\lambda\hat{\mathcal{R}}_{\lambda\alpha}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\alpha\beta}\right]+\\
&+g_{\mu\alpha}\left[\left(\frac{1}{36}R_{\nu\beta}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\nu\sigma\beta}+\frac{1}{90}R_{\rho\sigma\lambda\nu}R^{\rho\sigma\lambda}{}_\beta-\frac{1}{45}R_{\nu\lambda}R^\lambda_\beta+\frac{1}{60}\Box R_{\nu\beta}+\frac{1}{20}\nabla_\nu\nabla_\beta R\right)\mathbb{I}+\right.\\
&\left.+\frac{1}{12}\left(\hat{\mathcal{R}}_{\nu\lambda}\hat{\mathcal{R}}^\lambda{}_\beta+\hat{\mathcal{R}}_{\beta\lambda}\hat{\mathcal{R}}^\lambda{}_\nu\right)-\frac{1}{12}\left(\nabla_\nu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\beta}+\nabla_\beta\nabla^\lambda\hat{\mathcal{R}}_{\lambda\nu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\nu\beta}\right]+\\
&+g_{\mu\beta}\left[\left(\frac{1}{36}R_{\nu\alpha}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\nu\sigma\alpha}+\frac{1}{90}R_{\rho\sigma\lambda\nu}R^{\rho\sigma\lambda}{}_\alpha-\frac{1}{45}R_{\nu\lambda}R^\lambda_\alpha+\frac{1}{60}\Box R_{\nu\alpha}+\frac{1}{20}\nabla_\nu\nabla_\alpha R\right)\mathbb{I}+\right.\\
&\left.+\frac{1}{12}\left(\hat{\mathcal{R}}_{\nu\lambda}\hat{\mathcal{R}}^\lambda{}_\alpha+\hat{\mathcal{R}}_{\alpha\lambda}\hat{\mathcal{R}}^\lambda{}_\nu\right)-\frac{1}{12}\left(\nabla_\nu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\alpha}+\nabla_\alpha\nabla^\lambda\hat{\mathcal{R}}_{\lambda\nu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\nu\alpha}\right]+\\
&+g_{\nu\alpha}\left[\left(\frac{1}{36}R_{\mu\beta}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\mu\sigma\beta}+\frac{1}{90}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_\beta-\frac{1}{45}R_{\mu\lambda}R^\lambda_\beta+\frac{1}{60}\Box R_{\mu\beta}+\frac{1}{20}\nabla_\mu\nabla_\beta R\right)\mathbb{I}+\right.\\
&\left.+\frac{1}{12}\left(\hat{\mathcal{R}}_{\mu\lambda}\hat{\mathcal{R}}^\lambda{}_\beta+\hat{\mathcal{R}}_{\beta\lambda}\hat{\mathcal{R}}^\lambda{}_\mu\right)-\frac{1}{12}\left(\nabla_\mu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\beta}+\nabla_\beta\nabla^\lambda\hat{\mathcal{R}}_{\lambda\mu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\mu\beta}\right]+\\
&+g_{\nu\beta}\left[\left(\frac{1}{36}R_{\mu\alpha}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\mu\sigma\alpha}+\frac{1}{90}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_\alpha-\frac{1}{45}R_{\mu\lambda}R^\lambda_\alpha+\frac{1}{60}\Box R_{\mu\alpha}+\frac{1}{20}\nabla_\mu\nabla_\alpha R\right)\mathbb{I}+\right.\\
&\left.+\frac{1}{12}\left(\hat{\mathcal{R}}_{\mu\lambda}\hat{\mathcal{R}}^\lambda{}_\alpha+\hat{\mathcal{R}}_{\alpha\lambda}\hat{\mathcal{R}}^\lambda{}_\mu\right)-\frac{1}{12}\left(\nabla_\mu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\alpha}+\nabla_\alpha\nabla^\lambda\hat{\mathcal{R}}_{\lambda\mu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\mu\alpha}\right]+\\
&+g_{\alpha\beta}\left[\left(\frac{1}{36}R_{\mu\nu}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\mu\sigma\nu}+\frac{1}{90}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_\nu-\frac{1}{45}R_{\mu\lambda}R^\lambda_\nu+\frac{1}{60}\Box R_{\mu\nu}+\frac{1}{20}\nabla_\mu\nabla_\nu R\right)\mathbb{I}+\right.\\
&\left.\left.+\frac{1}{12}\left(\hat{\mathcal{R}}_{\mu\lambda}\hat{\mathcal{R}}^\lambda{}_\nu+\hat{\mathcal{R}}_{\nu\lambda}\hat{\mathcal{R}}^\lambda{}_\mu\right)-\frac{1}{12}\left(\nabla_\mu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\nu}+\nabla_\nu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\mu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\mu\nu}\right]\right]+\\
&+\frac{1}{4}\left(g_{\mu\nu}g_{\alpha\beta}+g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}\right)\left[\left[\frac{1}{180}\left(R_{\lambda\sigma\rho\gamma}R^{\lambda\sigma\rho\gamma}-R_{\lambda\delta}R^{\lambda\sigma}\right)+\frac{1}{30}\Box R-\frac{1}{72}R^2\right]\mathbb{I}+\right.\\
&\left.\left.+\frac{1}{36}\left(R_{\mu\nu}R_{\alpha\beta}+R_{\mu\alpha}R_{\nu\beta}+R_{\mu\beta}R_{\nu\alpha}\right)+\frac{1}{12}\hat{\mathcal{R}}_{\lambda\sigma}\hat{\mathcal{R}}^{\lambda\sigma}\right]\right\} \qquad \text{(C.11)}
\end{aligned}
$$

For $p = 2n - 1$ just one is involved

$$\nabla_\mu \frac{\mathbb{I}}{\Box} = \frac{\sqrt{g}}{8(n-4)\pi^2}\left(\frac{1}{12}\nabla_\mu R\mathbb{I} - \frac{1}{6}\nabla^\nu \hat{\mathcal{R}}_{\nu\mu}\right). \tag{C.12}$$

The ones with $p = 2n - 2$

$$\frac{\mathbb{I}}{\Box} = \frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{6}R\mathbb{I}, \tag{C.13}$$

$$\nabla_\mu\nabla_\nu \frac{\mathbb{I}}{\Box^2} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\left[\frac{1}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\mu\nu}\right], \tag{C.14}$$

$$\begin{aligned}\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu \frac{\mathbb{I}}{\Box^3} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{4}\Big\{&-\frac{1}{12}g^{(2)}_{\mu\nu\alpha\beta}\mathbb{I} - \frac{2}{6}R_{\mu\beta\nu\alpha} - \frac{2}{6}R_{\nu\beta\mu\alpha}\mathbb{I} + \\ &+ g_{\beta\nu}\left(\frac{1}{6}R_{\alpha\mu}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\alpha\mu}\right) + g_{\alpha\nu}\left(\frac{1}{6}R_{\mu\beta}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\mu\beta}\right) + \\ &+ g_{\beta\mu}\left(\frac{1}{6}R_{\alpha\nu}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\alpha\nu}\right) + g_{\alpha\mu}\left(\frac{1}{6}R_{\beta\nu}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\beta\nu}\right) + \\ &+ g_{\alpha\beta}\left(\frac{1}{6}R_{\mu\nu}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\mu\nu}\right) + g_{\mu\nu}\left(\frac{1}{6}R_{\alpha\beta}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\alpha\beta}\right)\Big\}.\end{aligned} \tag{C.15}$$

$$\begin{aligned}\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\nabla_\sigma\nabla_\lambda \frac{\mathbb{I}}{\Box^3} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{6}\Big\{&g^{(2)}_{\mu\nu\alpha\beta}\hat{B}_{\sigma\lambda} + g^{(2)}_{\mu\nu\alpha\sigma}\hat{B}_{\beta\lambda} + g^{(2)}_{\mu\nu\beta\sigma}\hat{B}_{\alpha\lambda} + g^{(2)}_{\mu\alpha\beta\sigma}\hat{B}_{\nu\lambda} + \\ &+ g^{(2)}_{\mu\nu\alpha\lambda}\hat{B}_{\beta\sigma} + g^{(2)}_{\mu\nu\beta\lambda}\hat{B}_{\alpha\sigma} + g^{(2)}_{\mu\alpha\beta\lambda}\hat{B}_{\nu\sigma} + g^{(2)}_{\nu\alpha\beta\lambda}\hat{B}_{\mu\sigma} + g^{(2)}_{\mu\nu\sigma\lambda}\hat{B}_{\alpha\beta} + \\ &+ g^{(2)}_{\mu\alpha\sigma\lambda}\hat{B}_{\nu\beta} + g^{(2)}_{\nu\alpha\sigma\lambda}\hat{B}_{\mu\beta} + g^{(2)}_{\mu\beta\sigma\lambda}\hat{B}_{\nu\alpha} + g^{(2)}_{\nu\beta\sigma\lambda}\hat{B}_{\mu\alpha} + g^{(2)}_{\alpha\beta\sigma\lambda}\hat{B}_{\mu\nu} - \\ &- \frac{1}{12}\Big[g_{\sigma\lambda}\left(R_{\beta\nu\alpha\mu} + R_{\alpha\nu\beta\mu}\right) + g_{\beta\lambda}\left(R_{\sigma\nu\alpha\mu} + R_{\alpha\nu\sigma\mu}\right) + \\ &+ g_{\alpha\lambda}\left(R_{\sigma\nu\beta\mu} + R_{\beta\nu\sigma\mu}\right) + g_{\nu\lambda}\left(R_{\sigma\alpha\beta\mu} + R_{\beta\alpha\sigma\mu}\right) + \\ &+ g_{\mu\lambda}\left(R_{\sigma\alpha\beta\nu} + R_{\beta\alpha\sigma\nu}\right) + g_{\beta\sigma}\left(R_{\lambda\nu\alpha\mu} + R_{\alpha\nu\lambda\mu}\right) + \\ &+ g_{\alpha\sigma}\left(R_{\lambda\nu\beta\mu} + R_{\beta\nu\lambda\mu}\right) + g_{\mu\sigma}\left(R_{\lambda\alpha\beta\nu} + R_{\beta\alpha\lambda\nu}\right) + \\ &+ g_{\alpha\beta}\left(R_{\lambda\nu\sigma\mu} + R_{\sigma\nu\lambda\mu}\right) + g_{\nu\beta}\left(R_{\lambda\alpha\sigma\mu} + R_{\sigma\alpha\lambda\mu}\right) + \quad (C.16) \\ &+ g_{\mu\beta}\left(R_{\lambda\alpha\sigma\nu} + R_{\sigma\alpha\lambda\nu}\right) + g_{\nu\alpha}\left(R_{\lambda\beta\sigma\mu} + R_{\sigma\beta\lambda\mu}\right) + \\ &+ g_{\mu\alpha}\left(R_{\lambda\beta\sigma\nu} + R_{\sigma\beta\lambda\nu}\right) + g_{\mu\nu}\left(R_{\lambda\beta\sigma\alpha} + R_{\sigma\beta\lambda\alpha}\right) + \frac{1}{8}g^{(3)}_{\mu\nu\alpha\beta\sigma\lambda}R\Big]\mathbb{I}\Big\}\end{aligned} \tag{C.17}$$

where the field strength $\hat{\mathcal{R}}_{\mu\nu}$ defined as in (4.42) and

$$
\begin{aligned}
&g^{(0)} = 1,\\
&g^{(1)}_{\mu\nu} = g_{\mu\nu},\\
&g^{(2)}_{\mu\nu\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} + g_{\mu\nu}g_{\alpha\beta},\\
&g^{(3)}_{\mu\nu\alpha\beta\sigma\lambda} = g_{\mu\nu}g^{(2)}_{\alpha\beta\sigma\lambda} + g_{\mu\alpha}g^{(2)}_{\nu\beta\sigma\lambda} + g_{\mu\beta}g^{(2)}_{\nu\alpha\sigma\lambda} + g_{\mu\sigma}g^{(2)}_{\nu\alpha\beta\lambda} + g_{\mu\lambda}g^{(2)}_{\nu\alpha\beta\sigma},\\
&g^{(n+1)}_{\mu_1\dots.\mu_{2n+2}} = \sum_{i=2}^{2n+2} g_{\mu_1\mu_i}g^{(n)}_{\mu_2\dots\mu_{i-1}\mu_{i+1}\mu_{2n+2}},\\
&\hat{B}_{\alpha\beta} = \frac{1}{24}R_{\alpha\beta}\mathbb{I} + \frac{1}{8}\hat{\mathcal{R}}_{\alpha\beta}.
\end{aligned}
\tag{C.18}
$$

Finally for p=2n-4 all traces can be computed with the expression

$$
\nabla_{\mu_1}\dots.\nabla_{\mu_{2n-4}}\frac{\mathbb{I}}{\Box^n} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{g^{(n-2)}_{\mu_1\dots.\mu_{2n-4}}}{2^{n-2}(n-1)!}
\tag{C.19}
$$

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