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Topologically quantized Schwarzschild black hole

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Abstract

We present a new version of the Schwarzschild solution that involves an intrinsically discrete structure apt for quantization. Our method is the harmonic mapping of the unit sphere (S^2) into itself. This explains the areal quantization whereas the energy quantum derives from the energy of the harmonic map. Likewise, all thermodynamical quantities are naturally quantized at lower orders. *'There is Plenty of Room at the Bottom'* R. P. Feynman [R. P. Feynman, Lecture given on December 29, 1959 at the annual meeting of the APS with the title There's Plenty of Room at the Bottom: An Invitation to Enter a New Field of Physics.

1. Introduction

When K. Schwarzschild solved the spherically symmetric field equations formulated by Einstein in 1916 [1] he had discovered in fact an infinite class of vacuum metrics involving the harmonic maps (HMs) of the form $f: S^2 \to S^2$ [2, 3]. In more appropriate terms the Schwarzschild (S) metric can be rewritten as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 f^2(\theta)(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

where

$$f(\theta) = \pm k \frac{\sin\Theta}{\sin\theta} \tag{2}$$

with

$$\tan\frac{\Theta}{2} = \alpha\left(\tan\frac{\theta}{2}\right)^{\pm k} \tag{3}$$

in which $k \in \mathbb{N}$ and $\alpha = const.$ that will be fixed as $\alpha = 1$ for the time being [3]. Substituting this gives

$$f(\theta) = \frac{2k(\sin\theta)^{k-1}}{(1-\cos\theta)^k + (1+\cos\theta)^k}. \tag{4}$$

In general, we consider two Riemannian manifolds $\mathcal{M}$ and M' described by the metrics

$$\mathcal{M}: ds^2 = g_{ab}dx^a dx^b, \quad (a, b: 1, 2,\ldots,n), \tag{5}$$

$$\mathcal{M}': ds'^2 = g'_{AB}dx^A dx^B, (A, B: 1, 2,\ldots,m), \tag{6}$$

and a map $f^A: \mathcal{M} \to \mathcal{M}', (A = 1, 2,\ldots,m)$defined by the energy functional

$$E(f^A) = \frac{1}{2}\int g'_{AB}\frac{\partial f^A}{\partial x^a}\frac{\partial f^B}{\partial x^b}g^{ab}\sqrt{g}\,d^n x, \tag{7}$$

in which g is the determinant of g_{ab}. We note that n and m in our study are 2. This expression is adopted as the action so that the variational principle $\delta E(f^A) = 0$, yields the field equations

$$\nabla^2 f^A + \Gamma'^A_{BC}\frac{\partial f^B}{\partial x^b}\frac{\partial f^C}{\partial x^c}g^{bc} = 0. \tag{8}$$

Here ∇^2 stands for the covariant Laplacian on $\mathcal{M}$ and Γ'^A_{BC} are the connection coefficients on $\mathcal{M}'$. We refer to [4–7] for the details of the f^A: $S^2 \to S^2$ HM. The topic of HMs was considered long ago as an alternative formulation of physical theories [8, 9] which will not be our aim in this Letter. Well-known classes of Einstein equations follow from (8) upon the appropriate choice of $\mathcal{M}$ and $\mathcal{M}'$ metrics which aided in obtaining new solutions [10–12].

2. The formalism

In this Letter our choice for coordinates on $\mathcal{M}'$ and $\mathcal{M}$ are $f^A = \{\Theta, \Phi\}$ and $x^a = \{\theta, \varphi\}$, respectively, so that the two metrics are

$$\mathcal{M}\colon ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 \tag{9}$$

and

$$\mathcal{M}'\colon ds'^2 = d\Theta^2 + \sin^2\Theta d\Phi^2. \tag{10}$$

The choice $\Phi(\varphi) = k\varphi$, with $k \in \mathbb{N}$ (our choice will be $k > 0$), yields from the HM prescription the results (2) and (3) as the solution of the equation (8). The integer k denotes the number of times S^2 is wrapped in the map and if k is not an integer, (1) does not solve the vacuum Einstein equations. For $k = 1$, with the (+) sign, (3) reduces to the identity map, which gives the standard S-metric. We consider $k \geqslant 2$, so that a quantum number $n = \triangle k = k_2 - k_1 = \pm1, \pm2, \pm3, \ldots$, can be introduced. We note that with any $k \geqslant 1$ ($k \in \mathbb{N}$) the metric (1) is a vacuum solution of Einstein's equations. In other words, each k represents a Schwarzschild black hole. Therefore going from one solution to another, changes k for instance from k_1 to k_2, such that $n = \triangle k$ represents this transition. Under the transform of S^2 the new Killing vectors (or angular momentum operators) take the form

$$L_x = \frac{1}{k}\cot\Theta\sin(k\varphi)\frac{\partial}{\partial\varphi} - \frac{1}{f(\theta)}\cos(k\varphi)\frac{\partial}{\partial\theta}, \tag{11}$$

$$L_y = \frac{1}{k}\cot\Theta\cos(k\varphi)\frac{\partial}{\partial\varphi} - \frac{1}{f(\theta)}\sin(k\varphi)\frac{\partial}{\partial\theta} \tag{12}$$

and

$$L_z = \frac{1}{k}\frac{\partial}{\partial\varphi} \tag{13}$$

which satisfy the original algebra $[L_i, L_j] = \epsilon_{ijk}L_k$, upon substitutions for Θ and $f(\theta)$ from equations (3) and (4) above.

Since (1) is a vacuum solution we need to add an energy term to match with any possible contribution as a quantum correction. This is done by considering a Reissner-Nordström (RN)-like term where an appropriate coefficient replaces the charge term [13]. Upon this consideration, a possible choice for a topologically quantum-corrected S-metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{\left(1 - \frac{1}{k}\right)\ell_h^2}{r^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{\left(1 - \frac{1}{k}\right)\ell_h^2}{r^2}\right)} + r^2 f^2(\theta)(d\theta^2 + \sin^2\theta d\varphi^2) \tag{14}$$

where the minimum length $\ell_{\rm h}$ for HMs is added for dimensional reasons. At this point we leave it open whether $\ell_{\rm h}$ is to be identified with the Planck length $\ell_P = \left(\frac{\hbar G}{c^3}\right)^{1/2}$, in which G is Newton's gravitational constant c is the speed of light and $\hbar$ is the Planck constant. For this argument, we refer to Misner [8].

The surface area $\mathcal{A}$ of the wrapped S^2 is given by

$$\mathcal{A} = 4\pi k r^2 \tag{15}$$

where for the area of the event horizon we take $r = r_h$. The energy of the map from (7) gives $E(f^A) = 4\pi k$, however, fixing this so that $k = 1$ corresponds to S-vacuum with the right physical dimensionality compels us to choose the energy quantum as

$$E = 4\pi(k - 1)\ell_h. \tag{16}$$

Let us note that (14) has the vacuum limit in $M = 0$ which consists entirely of quantum effects with frequency $\omega_n = \frac{4\pi \mid n \mid \ell_h}{\hbar}$. Here, the energy-momentum tensor is given by

$$T_\mu^\nu = diag(-\rho, p_r, p_\theta, p_\varphi) = \frac{\left(1-\frac{1}{k}\right)\ell_h^2}{r^4} diag(-1, -1, 1, 1) \tag{17}$$

in which $\rho = \frac{\left(1-\frac{1}{k}\right)\ell_h^2}{r^4}$ is the energy density such that the total conserved energy of the spacetime

$$E = \int \sqrt{-g}\,\rho d^3x = 2\pi\left(1-\frac{1}{k}\right)\ell_h^2 \int_0^\pi f(\theta)^2 \sin\theta d\theta \int_{\ell_h}^\infty \frac{dr}{r^2} = 4\pi(k-1)\ell_h. \tag{18}$$

Let us clarify that $g = \det g_{\mu\nu} = -f^4r^4\sin^2\theta$ in (18) is the determinant of the 3 + 1-dimensional spacetime, while $g = \det g_{ab} = f^4r^4\sin^2\theta$ in equation (7) is the determinant of the angular part of the 3 + 1-dimensional spacetime. Furthermore, the Kretschmann scalar of (14) is given by

$$\mathcal{K} = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48M^2r^2 - 96Mr\left(1-\frac{1}{k}\right)\ell_h^2 + 56\left(\left(1-\frac{1}{k}\right)\ell_h^2\right)^2}{r^8}, \tag{19}$$

where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor of the 3 + 1-dimensional spacetime. The pressure components are also evaluated under the quantum assumption that $r \geqslant \ell_h$. In this manner both A and E are quantized. The event horizon ($r_h = r_+$) and the inner horizon (r_-, which is absent in the classical S-metric) are

$$r_+ \simeq M\left(2 - \frac{\left(1-\frac{1}{k}\right)\ell_h^2}{2M^2}\right) \tag{20}$$

$$r_- \simeq \frac{\left(1-\frac{1}{k}\right)\ell_h^2}{2M}, \tag{21}$$

and in the extremal case, we have

$$r_+ = r_- = M = \sqrt{1-\frac{1}{k}}\,\ell_h \tag{22}$$

which is valid only for microscopical black holes. At this level, the mass automatically becomes quantized by the quantum of length ℓ_h. It is observed from (18) that the quantum correction to the mass/energy is

$$\triangle M = 4\pi n\ell_h,\ (n = \pm 1, \pm 2, \ldots) \tag{23}$$

which will give the emission/absorption frequency as $\omega_n = \frac{4\pi|n|\ell_h}{\hbar}$. The area change $\triangle\mathcal{A}$ gets contributions from both $\triangle k = n$ and from $\triangle M$, which sums up to give

$$\triangle\mathcal{A} \simeq 16\pi M^2 n\left\{1 + \frac{8\pi k}{M}\ell_h - \frac{\left(1-\frac{1}{k}\right)\ell_h^2}{2M^2} + \mathcal{O}(\ell_h^3)\right\}. \tag{24}$$

In the bracket, the first term corresponds to the classical area change, whereas the second and third terms correspond to the semiclassical and quantum corrections, respectively. The quantum correction to the area is $-8\pi n\left(1-\frac{1}{k}\right)\ell_h^2$, which is positive or negative depending on the absorption or emission of gravitational radiation.

The quantized Hawking temperature and the entropy of the black hole (14) are given by

$$T_H = \left(-\frac{g_{00}'}{4\pi\sqrt{-g_{00}g_{11}}}\right)_{r=r_+} = \frac{\chi}{2\pi M(1+\chi)^2}, \tag{25}$$

and

$$S = \frac{\mathcal{A}(r_+)}{4} = k\pi M^2(1+\chi)^2, \tag{26}$$

respectively, where

$$\chi = \sqrt{1 - \frac{\left(1-\frac{1}{k}\right)\ell_h^2}{M^2}}. \tag{27}$$

Using (25) and (26) we calculate the heat capacity of the black hole for fixed k which is given by

$$C_k = \left(T_H\frac{\partial S}{\partial T_H}\right)_k = \frac{4\pi km^2\chi(1+\chi)^2}{2(1-2\chi)}. \tag{28}$$

At this stage, a simple application will be in order. The circular, radial geodesics for $\theta = const.$, $\varphi = const.$, are described by the reduced Lagrangian

$$\mathcal{L} = -\frac{1}{2}\psi\dot{t}^2 + \frac{1}{2\psi}\dot{r}^2 \tag{29}$$

where

$$\psi = 1 - \frac{2M}{r} + \frac{\left(1 - \frac{1}{k}\right)\ell_h^2}{r^2} \tag{30}$$

and a dot stands for $\frac{d}{ds}$. This admits the dimensionless energy constant E_0 so that for $r = r_0 = const.$ we have

$$r_0 = \frac{M}{E_0^2 - 1}\left[\sqrt{1 + \frac{(E_0^2 - 1)\left(1 - \frac{1}{k}\right)\ell_h^2}{M^2}} - 1\right] \tag{31}$$

exactly, and

$$r_0 \simeq \frac{\left(1 - \frac{1}{k}\right)\ell_h^2}{2M}\left[1 - \frac{(E_0^2 - 1)\left(1 - \frac{1}{k}\right)}{4}\left(\frac{\ell_h^2}{M^2}\right) + \mathcal{O}\left(\left(\frac{\ell_h^2}{M^2}\right)^2\right)\right] \tag{32}$$

in expansion. This defines the possible circular radii of a *quantum Schwarzschild atom* with mass M which collapses to $r_0 = 0$, for $k = 1$ to create the classical singularity. To the order $\frac{\ell_h^2}{M}$, the radius (32) takes the orbital values

$$r_0 \simeq \frac{\ell_h^2}{M}\frac{\left(1 - \frac{1}{k}\right)}{2} = \frac{\ell_h^2}{M}\begin{cases} 1/4, & k = 2 \\ 1/3, & k = 3 \\ 3/8, & k = 4 \\ \cdots & \cdots \\ 1/2, & k \to \infty \end{cases} \tag{33}$$

and there is no need to state that as in the atomic model all states need not be filled in such a quantized Schwarzschild model.

The inclusion of angular momentum parameter ℓ can be considered in the simplest form in the equilateral plane $\theta = \frac{\pi}{2}$ where the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\psi(r)\dot{t}^2 + \frac{1}{2\psi(r)}\dot{r}^2 + \frac{1}{2}k^2r^2\dot{\varphi}^2 \tag{34}$$

which still bears the imprints of the integer k. In terms of the constants $E = \psi(r)\dot{t}$ and $\ell = k^2r^2\dot{\varphi}$, and by introducing $u = \frac{1}{r}$, the quantized Kepler orbits reduce to the form

$$\frac{k^2}{\ell}(\varphi - \varphi_0) = \int^u \frac{du}{\sqrt{E^2 - \left(1 - 2Mu + \left(1 - \frac{1}{k}\right)\ell_h^2u^2\right)\left(1 + \frac{\ell^2}{k^2}u^2\right)}} \tag{35}$$

to be fixed in terms of the elliptic functions. Our formalism can be extended to other metrics that involve S^2 sector. Inserting the same substitutions as (2) and (3) into the Kerr metric with expected complications provide a similar quantization. For the RN case, it is similar to the S-metric that the charge term in the metric will be shifted according to $\frac{Q^2}{r^2} \to \frac{Q^2 + \left(1 - \frac{1}{k}\right)\ell_h^2}{r^2}$, in which the charge Q will compete with the quantum of length. In the extremal case, we may argue about the quanta of charge, as in the case of mass. Going to higher dimensional maps such as f: $S^n \to S^n$, faces technical problems which may be overcome by resorting to the special Hopf polynomial type maps [14].

3. Conclusion

Different from other researchers [15–20], we don't define the areal quantization but derive it by appealing to the geometric HMs of $S^2 \to S^2$. Area and energy are quantized naturally in terms of the topological number k which measures at the same time the degree of the map [2, 3]. In other words, we propose the topological degree of the HMs to represent the quantum integer for a quantized S-black hole. One may naturally ask: what happens if $k \neq$ integer? For non-integer k the vacuum equations are not satisfied and then fixing the problem with a suitable source may be interesting from the topological point of view. Recalling from C. W. Misner [9], since the coupling constant is dimensional, any quantum gravity must be non-perturbative otherwise non-renormalizability will follow. Our presentation in this Letter provides one such possibility within the context of HMs. The areal change $\triangle\mathcal{A}$ consists of classical, semiclassical, and quantal terms in reducing powers. Since $r \geqslant \ell_h$, the fundamental length, and k$\geqslant$2 the $r = 0$ singularity of the S-metric is removed whereas the correspondence principle is provided by $k = 1$. Although we left the choice of $\ell_{\rm h}$ open, for the unity of quantum in the Universe $\ell_h = \ell_P$ may be a reasonable choice. Otherwise, assuming that gravity remains non-cooperative with other fields, $\ell_{\rm h}$ becomes a candidate for a new fundamental length in our universe. As a next step, we may employ the local isometry between black holes and colliding waves to study the wave structure of the quantized Schwarzschild geometry. Finally, in the Quantum Theory of fields/particles, we overcome all difficulties by reducing everything to harmonic equations/functions, in Quantum Gravity it is logical to resort to the same concept through the HMs. It is our belief that this work will prompt further applications toward a better understanding of quantum black holes.

Data availability statement

No new data were created or analysed in this study.

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