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PAPER

Revisited version of Weyl's limit-point limit-circle criterion for essential self-adjointness

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Abstract

The principal aim of this paper is to present a new proof of Weyl's criterion in which it is shown that the natural framework for the associated Sturm-Liouville operators is $W^{2,1} \cap L^2$ -i.e.- the intersection of a particular Sobolev space and of the L^2 space. Indeed, we will deal with the special case of the radial operator $\left(-\frac{d^2}{dx^2} + q(x)\right)$ on a real line segment (either bounded or unbounded) that often occurs in the study of quantum systems in central potentials. We also derive from first principles the functional behaviour of the coefficients for a general second-order Sturm-Liouville operator by using some extensions of a milestone Carathéodory existence theorem.

1. Introduction

Limit-Point Limit-Circle theory was first developed by the young Herman Weyl in the early 1900's in one of his first articles [1]. Since then, such methods (hereafter denoted by LP and LC, respectively) have become increasingly important thanks to their accurate predictions on the form of the potential in the applications, which can easily supply foundamental information about the solution of a great variety of singular second-order Sturm-Liouville problems. In the modern literature, the work in [2, 3] provides an enlightening introduction to the link between these singular second-order problems and functional analysis, as well as to the applications to ordinary quantum mechanics.

The world of atomic physics offers indeed a wide range of applications of ordinary quantum mechanics. This is not an exact theory, because relativity would make it necessary to use the spectral theory of pseudo-differential operators in order to develop the quantum theory of bound states [4]. Thus, one still resorts with profit to ordinary quantum mechanics, from which one can learn valuable lessons. For example, if a physical system ruled by a central potential $V(r)$ is considered in $\mathbb{R}^n$ (the choice $n = 3$ is frequent but not mandatory), one finds an effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{\rho_{nl}}{r^2}, \tag{1.1}$$

where, on denoting by l the orbital angular momentum quantum number, one finds [5]

$$\rho_{nl} = \frac{(n-1)(n-3)}{4} + l(l+n-2) = \left(l + \frac{(n-2)}{2}\right)^2 - \frac{1}{4}. \tag{1.2}$$

As one learns from [2, 3], the LP condition at the origin is achieved if

$$V(r) + \frac{\rho_{nl}}{r^2} \geqslant \frac{3}{4r^2} \text{ as } r \to 0^+. \tag{1.3}$$

In the particular case of a free particle, $V(r) = 0$ and (6.3) leads to

$$\left(l + \frac{(n-2)}{2}\right)^2 - \frac{1}{4} \geqslant \frac{3}{4} \Longrightarrow l + \frac{n}{2} \geqslant 2. \tag{1.4}$$

Interestingly, this condition is violated just once, i.e. by s-wave stationary states (for which $l = 0$) in 3 dimensions. The same holds if $V(r)$ is a Coulomb-type potential, because then the centrifugal term on the left-hand side of (1.3) dominates on the Coulomb term as r approaches 0.

Section 2 describes what is known from a Carathéodory theorem on ordinary differential equations; section 3 studies the Sobolev functional space for solutions of our singular second-order problems; Weyl's LP-LC criterion is studied with extensive and original use of Sobolev spaces in sections 4 and 5. Explicit examples of self-adjoint extensions are analyzed in section 6. Concluding remarks are made in section 7, while relevant details are given in the appendix. Throughout our paper, the reader is assumed to have some background on the LP-LC theory [6, 7] and on operator theory [2, 3].

2. Extended Carathéodory's existence Theorem

First, we want to understand the functional behaviour of the coefficients and all possible solutions of the general Sturm-Liouville eigenvalue equation

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = ly \qquad l \in \mathbb{C}. \tag{2.1}$$

In the LP-LC literature, whenever one deals with equation (2.1), it is necessary to specify the functional space to which the coefficients $p(x)$, $p'(x)$ and $q(x)$ belong, in order to develop the theory and reach the desired results.

For example, in [6–8] the coefficients p, p' and q belong to the set of real-valued continuous functions and hence the solution y must be globally of class C^2 on the interval $I \subseteq \mathbb{R}$ of interest. On the other hand, in [9, 10], weaker conditions on p^{-1}, q are given -i.e.- they are L^1_{loc} while the solution y and its derivative y' are absolutely continuous (AC_{loc}) on the interval I of interest.

In order to clarify the hypotesis made on such functions and on the solutions, we will make use of some extended Carathéodory's existence theorems:

Theorem 2.1. *Let $I \subset \mathbb{R}$ be a closed interval and let $f(x, u(x))$ be G-regular on* I *(see appendix). Then there exists at least one absolutely continuous function* u *such that*

$$u(x) = \int_{x_0}^{x} f(s, u(s))ds \qquad x \in I \tag{2.2}$$

where x_0 *is the average point of* I.

We note that if f obeys the above theorem, there exists at least one absolutely continuous function u that satisfies the equation $u' = f(x, u(x))$ almost everywhere.

Theorem 2.2. *Let $I = [a, b] \subset \mathbb{R}$ be a compact interval and let $f(x, y)$: $I \times V \to V$ satisfy the following hypothesis (where* V *is a generic* n*-dimensional space):*

(1) For every $y \in V$, $f(x, y)$ is measurable on I *and it is continuous in* V.

(2) There exists a real-valued non-negative function $M(x) \in L^1(I)$ such that $|f(x, y)| \leqslant M(x)$.

Then there exists an absolutely continuous function $u(x)$ such that $u'(x) = f(x, u(x))$ almost everywhere on I.

Theorem 2.1 is discussed and proved in [8] while theorem 2.2 in [11].

In [11] it is shown that the requirement (2) of theorem 2.2 can be replaced by the following:

(2') For every $y \in C(I)$, $f(x, y(x))$ is summable in I, and upon taking $y \in C(I)$, the functions $\int_a^x f(t, y(t))dt$ describe an absolutely equicontinuous family on I

obtaining a more general existence theorem.

Here we want to show that, if the assumptions of theorem 2.1 are verified and we also take hypothesis *(2')* instead of *(2)*, then theorem 2.2 must also be true. From this the former will be a restricted case of the latter.

As already mentioned, for the definition of *G-regularity* we remind to the appendix at the end of this paper. Here we will only give the main condition that ensures the occurrence of this property. For this purpose we need some further notions:

Definition 1. Let $I \subset \mathbb{R}$ and let $h^1,\ldots,h^m,\ k^1,\ldots,k^m \in L^1(I,\ \mathbb{R})$. We define the subsequent convex subset $G(h,\ k)$ of $L^1(I,\ \mathbb{R}^m)$ by

$$G(h,\ k) = \{g \in L^1(I,\ \mathbb{R}^m),\ \ h^j(x) \leqslant g^j(x) \leqslant k^j(x),\ \ x \in I,\ \ 1 \leqslant j \leqslant m\}. \tag{2.3}$$

Theorem 2.3. *If the function* $f:\ (x,\ y) \in I \times \mathbb{R}^m \to f(x,\ y) \in \mathbb{R}^m$ *satisfies*

$$|f(x,\ y)| \leqslant M(x)(1 + |y|), \qquad (x,\ y) \in I \times \mathbb{R}^m \tag{2.4}$$

for some $M \in L^1(I,\ \mathbb{R})$ *and it is measurable in the* x *variable for any fixed* y *and it is also continuous in the* y *variable, then there exists a* $G(h,\ k)$ *such that* f *is G-regular on* I.

Of course, form theorem 2.3 the assumptions of theorem 2.1 are satisfied and hence (2.2) is absolutely continuous.

Definition 2. Let $G = G(h,\ k)$ be given by (2.3) and x_0 be the middle point of the interval *I*. Let $f:\ I \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be such that

$$f\left(x,\ \int_{x_0}^{x} g(s)ds\right) \in G \qquad \textit{for all } g \in G \tag{2.5}$$

Then *f* is said to be *G-integrable* on *I*.

As we can see from the definition of G-regularity in our appendix, the G-integrability is necessary for the G-regularity. By using some content in [12] we can easily see that the G-integrability implies the absolute continuity of $\int_{x_0}^{x} f(t,\ y(t))dt$ in the *x* variable where *y* is taken to be absolutely continuous as in (2.4).

From this, one finds that the G-regularity makes (2.2) absolutely continuous whenever a particular absolutely continuous function $u(x)$ is chosen, and hence $\int_{x_0}^{x} f(s,\ u(s))ds$ is a family of absolutely continuous functions if we let the *u* variable run over a particular set of continuous functions.

Since under our hypothesis *f* is taken G-integrable, it is also bounded from the definition 2 and this suggests us that $\int_{x_0}^{x} f(s,\ u(s))ds$ is an equi-absolutely continuous family of functions. From this we have already proved that theorem 2.1 is a special case of theorem 2.2 when the *(2)* hypothesis is replaced with *(2')*.

3. Sobolev functional space for solutions

Now, by expressing (2.1) in the subsequent form of first-order differential system:

$$\begin{cases} \dfrac{dy}{dx} = v \\ \dfrac{dv}{dx} = -\dfrac{p'(x)}{p(x)}v + \dfrac{q(x)}{p(x)}y - \dfrac{l}{p(x)}y \end{cases} \tag{3.1}$$

it is easy to see, by applying theorem (2.2) with the *(2')* hypothesis, that the required summability in the *x* variable forces the coefficients $p(x)^{-1},\ p'(x)$ and $q(x)$ to belong to $L^1(I)$ while the solution *y* to (2.1) and its derivative y' are absolutely continuous functions.

Now we will face the fact that an absolutely continuous function must belong to a Sobolev space -i.e.- $W^{1,1}$ defined, for example, in [13]:

Definition 3.

$$W^{1,1}([a,\ b]) \equiv \left\{u \in L^1(a,\ b):\ \exists g \in L^1(a,\ b):\ \int_a^b u\phi' = -\int_a^b g\phi\right\}, \tag{3.2}$$

for all $\phi \in C_0^1(a,\ b)$. We also recall the following

Definition 4. Let $f:\ \mathbb{R} \longrightarrow \mathbb{R}$ be a function in $[a,\ b]$ such that its derivative is defined almost everywhere and

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a), \tag{3.3}$$

then *f* is said to be absolutely continuous.

It is well known that the weaker classical hypothesis that makes it possible to perform an integration by parts (Lebesgue fondamental integral theorem) such as

$$\int_a^b f\frac{d\phi}{dx} = -\int_a^b \frac{df}{dx}\phi \qquad \phi \in C_0^1(a, b) \tag{3.4}$$

is the *absolute continuity* of the function *f*.

Now, from definition 5, it follows that such functions must be continuous and have got bounded variation in the compact interval $[a, b]$. This simpy implies that every such function belongs to the $L^1(a, b)$ space. Furthermore, from the bounded variation behaviour of *f*, the integral (3.5) must be finite for every compact interval $[a, b]$, thus f' must belong to $L^1(a, b)$.

Taking into account the definition 3, jointly with the properties obtained above, we see that every absolutely continuous function belongs to the $W^{1,1}([a, b])$ Sobolev space.

4. Weyl's LP-LC criterion

Let us consider the following special case of Sturm-Liouville equation on (*a*, *b*) taking $p = 1$ in (2.1)

$$-\frac{d^2y}{dx^2} + q(x)y = ly \qquad l \in \mathbb{C}. \tag{4.1}$$

This is an eigenvalue equation whose differential operator is

$$\mathcal{L} = -\frac{d^2}{dx^2} + q(x) \tag{4.2}$$

defined on $L^2(a, b)$. The aim of the following *Weyl's Criterion* is to provide the condition on the operator (4.2) in order to ensure its self-adjointness in terms of the LP-LC property. In this way, such a theorem provides a magnificent link between operator theory on Hilbert spaces and LP-LC theory [2, 3]:

Theorem 4.1. *Let $q(x) \in L^2_{\text{loc}}$ in $I = (a, b)$ and let $\mathcal{L} = -\frac{d^2}{dx^2} + q(x)$ with domain $D(\mathcal{L}) = C_0^\infty(a, b)$. Then the closure $\bar{\mathcal{L}}$ has got deficiency indices:*

(i) $n_+(\bar{\mathcal{L}}) = n_-(\bar{\mathcal{L}}) = 2$ if $\mathcal{L}$ is in LC at both ends of the interval;

(ii) $n_+(\bar{\mathcal{L}}) = n_-(\bar{\mathcal{L}}) = 1$ if $\mathcal{L}$ is in LC at one end and LP at the other;

(iii) $n_+(\bar{\mathcal{L}}) = n_-(\bar{\mathcal{L}}) = 0$ if $\mathcal{L}$ is in LP at both ends of the interval.
Therefore, $\mathcal{L}$ is essentially Self-Adjoint if and only if it is LP at both end-points of the interval.

From the extended version of Carathéodory existence theorem we know that, if a solution to equation (4.1) exists in a compact *I*, then it must be absolutely continuous together with its first derivative and thus it belongs to $W^{1,1}(I)$.

We can introduce the following Sobolev space that will be the basic living place for our solutions:

Definition 5.

$$W^{2,1}(I) = \{u \in W^{1,1}(I) \mid u' \in W^{1,1}(I)\}. \tag{4.3}$$

Of course, we are only interested in functions which are absolutely continuous with their first derivative.

We note that the operator (4.2) must act on a Hilbert space -i.e.- $L^2(I)$ and therefore, from (4.1), the function $\frac{d^2y}{dx^2} \in L^2(I)$.

The subsequent theorem [11] shows that the hypothesis on the second derivative of our solutions to belong to L^2 is sufficient to guarantee us the *local absolute continuity* of the solutions and their first derivative:

Theorem 4.2. *Let $g \in L^1_{\text{loc}}(I)$ and take for some $y_0 \in I$ the following expression:*

$$v(x) = \int_{y_0}^x g(t)dt \qquad x \in I. \tag{4.4}$$

Then, v(x) *is continuous in* I *and*

$$\int_I v\phi' = -\int_I g\phi \qquad \forall \phi \in C_0^1(I). \tag{4.5}$$

By applying recursively the above theorem one finds that, under the hypothesis of square summability of its second derivative, *y* and its first derivative are *locally absolutelly continuous* on *I* whatever *I* is. It is also clear that, in the case of a compact real interval, y and y' are absolutely continuous functions of $W^{1,1}(I)$ and hence y belongs to the (4.3) set.

Now, if I is not bounded or half-bounded, we are dealing with functions belonging to $W^{2,1}_{\rm loc}(I)$ and taking the square summability required for operator (4.2), the basic functional space to which our solution belongs is $W^{2,1}_{\rm loc}(I) \cap L^2(I)$.

Now we can summarise our results in the following theorem:

Theorem 4.3. *Whatever the* I *interval is, every solution to equation (4.1) belongs to the space* $W^{2,1}_{\rm loc}(I)$.

Of course, if I is compact then $W^{2,1}_{\rm loc}(I) \equiv W^{2,1}(I)$ while if the square summability is required, then y'' must belong to $L^2(I)$ and thus we obtain the *local absolute continuity* of y and y' and the Carathéodory's existence theorem is fullfilled.

5. Proof of Weyl's criterion

We can now proceed with the proof of theorem 4.1 by following the logical steps that can be found in [3]. Our method will make use of theorem 4.2 jointly with all the information obtained in the previous section.

Proof of statement (i):

If the operator $\mathcal{L}$ is LC at both ends of the interval I, then every solution to the equation $\mathcal{L}y = ly, \forall\, l \in \mathbb{C}$ for which $\Im l \neq 0$, belongs to $L^2(I)$. This means that there exist two linearly independent solutions to each of the equations $\mathcal{L}y = iy$ and $\mathcal{L}y = -iy$, and therefore the deficiency indices are $n_+(\bar{\mathcal{L}}) = n_-(\bar{\mathcal{L}}) = 2$.

Proof of statement (ii):

Suppose that $\mathcal{L}$ is LP at a and LC at b.

Let us consider a restriction $\mathcal{L}_0$ of the operator $\mathcal{L}$ acting on the subsequent linear domain:

$$D(\mathcal{L}_0) = \{\phi \in W^{2,1}_{\rm loc}([c,\, d]) \colon \phi(c) = \phi(d) = \phi'(c) = 0,\;\; \phi'' \in L^2(c,\, d)\} \tag{5.1}$$

where $[c,\, d] \subset I$. From theorem 4.2 and the comment below, we have that $D(\mathcal{L}_0) \subset (W^{2,1}_{\rm loc}([c,\, d]) \cap L^2(c,\, d))$, hence $D(\mathcal{L}_0)$ is a Hilbert sub-space of $L^2(c,\, d)$. It is easily seen that operator $\mathcal{L}_0$ is symmetric and that the domain of its adjoint is

$$D(\mathcal{L}_0^*) = \{\psi \in W^{2,1}_{loc}([c,\, d]) \mid \psi(d) = 0,\;\; \psi'' \in L^2([c,\, d])\}, \tag{5.2}$$

because the equations

$$\mathcal{L}_0\phi = \pm i\phi \tag{5.3}$$

have at most two linearly independent solutions in $L^2(c,\, d)$ and hence $n_+(\mathcal{L}_0) \leqslant 2$ and $n_-(\mathcal{L}_0) \leqslant 2$. We must rule out the case $n_+(\mathcal{L}_0) = n_-(\mathcal{L}_0) = 0$ because it is the self-adjoint one and this is not the case because $D(\mathcal{L}_0) \subset D(\mathcal{L}_0^*)$.

Now we can show that each of equations (5.3) has only one solution in $L^2(c,\, d)$.

It is indeed well known that the adjoint domain for a linear operator on a Hilbert space admits the following decomposition:

$$D(\mathcal{A}^*) = D(\mathcal{A}) \oplus K_+(\mathcal{A}) \oplus K_-(\mathcal{A}) \tag{5.4}$$

where $K_+(\mathcal{A})$ and $K_-(\mathcal{A})$ are the deficiency spaces of the operator under consideration. Let us define the operator $\mathcal{P}^2 = -\frac{d^2}{dx^2}$ on the domain (5.1) and let us denote it by $\mathcal{P}_0^2$. Of course $\mathcal{P}_0^2$ is symmetric and its adjoint has domain (5.2), thus we can certainly say that $K_+(\mathcal{L}_0) \oplus K_-(\mathcal{L}_0) = K_+(\mathcal{P}_0^2) \oplus K_-(\mathcal{P}_0^2)$. By solving the equations

$$\mathcal{P}^2\phi = \pm i\phi \tag{5.5}$$

with the condition imposed by (5.2), we obtain two one-dimensional deficiency spaces of the form

$$K_+(\mathcal{P}_0^2) = \left\{\xi \in L^2(c,\, d) \mid \xi(x) = \lambda\left(\zeta_1(x) - \frac{\zeta_2(x)}{e^{\sqrt{2}(i+1)d}}\right) \mid \lambda \in \mathbb{C}\right\}.$$

$$K_-(\mathcal{P}_0^2) = \left\{\eta \in L^2(c,\, d) \mid \eta(x) = \lambda\left(\rho_1(x) - \frac{\rho_2(x)}{e^{\sqrt{2}(i-1)d}}\right) \mid \lambda \in \mathbb{C}\right\}. \tag{5.6}$$

where ζ_1, ζ_2, ρ_1 and ρ_2 are locally square integrable on the real line. Equations (5.6) show that $K_+(\mathcal{P}_0^2) \oplus K_-(\mathcal{P}_0^2)$ is a two-dimensional linear space and hence the same holds for $K_+(\mathcal{L}_0) \oplus K_-(\mathcal{L}_0)$. Now, taking into account the fact that [6, 7] if, for some complex $l_0 \in \mathbb{C}$ all solutions to $\mathcal{L}_0 y = l_0 y$ are square integrable, than for every complex $l \in \mathbb{C}$ every solution to $\mathcal{L}_0 y = ly$ is square integrable as well, we must rule out the cases

$\dim(K_+(\mathcal{L}_0)) = 2$, $\dim(K_-(\mathcal{L}_0)) = 0$ and $\dim(K_+(\mathcal{L}_0)) = 0$, $\dim(K_-(\mathcal{L}_0)) = 2$. From this, the only case left is $\dim(K_+(\mathcal{L}_0)) = \dim(K_-(\mathcal{L}_0)) = 1$ and therefore $n_+(\mathcal{L}_0) = n_-(\mathcal{L}_0) = 1$. This also means that there exists a non-vanishing function $\tilde{u}$ that does not belong to $\mathrm{Ran}(\mathcal{L}_0 - i\mathcal{I})$. At this stage, let $I_- = (a, d]$ and let $\mathcal{L}_1$ be a second restriction of $\mathcal{L}$ defined on

$$D(\mathcal{L}_1) = \{\phi \in W^{2,1}_{\mathrm{loc}}(I_-) \cap L^2(I_-) \mid \phi(x) = \phi(d) = 0,\ x \in (a, a + \epsilon),\ \epsilon \in (a, d),\ \phi'' \in L^2(c, d)\}. \tag{5.7}$$

We note that in the case in which the end-point a is at finite distance from the origin, the basic space in (5.7) can always be taken to be $W^{2,1}(I_-)$ instead of $W^{2,1}(I_-) \cap L^2(I_-)$.

All our reasoning on $\mathcal{L}_0$ can be repeated on $\mathcal{L}_1$, leading us to the same conclusions: $\mathcal{L}_1$ is symmetric on its domain and has got the same deficiency indices of $\mathcal{L}_0$. By using some arguments that can be found in [2], we can state that there exists at most one self-adjoint extension of $\mathcal{L}_1$ by virtue of the equality of its deficiency indices. Let us denote by $\mathcal{L}_2$ such a self-adjoint extension. It is clear that, if the domain is taken to be

$$D(\mathcal{L}_2) = \{\phi \in W^{2,1}_{\mathrm{loc}}(I_-) \cap L^2(I_-); \mid \phi(a) = \phi(d) = 0,\ \phi'' \in L^2(I_-)\} \tag{5.8}$$

we are dealing with a self-adjoint extension of $\mathcal{L}_1$ and we also note that $D(\mathcal{L}_2) \subset D(\bar{\mathcal{L}}_1)$.

Take now a function $\chi \in D(\mathcal{L}_2)$ for which $\mathcal{L}_2\chi - i\mathcal{I}\chi = u$ where u is chosen in such a way that its restriction $\tilde{u}$ to the interval $[c, d]$ is not in $\mathrm{Ran}(\mathcal{L}_0 - i\mathcal{I})$. We easily see that χ cannot be equal to zero on (a, c). To see this we have to take into account that $\chi \in C^1$ on $(a, d]$ (we refer to the previous section); if this were possible, then the restriction $\tilde{\chi}$ of χ to the interval $[c, d]$ would belong to $D(\mathcal{L}_0)$ because we would have $\chi(c) = \chi(d) = \chi'(a) = 0$ and one would find that $\mathcal{L}_2\tilde{\chi} - i\mathcal{I}\tilde{\chi} = \tilde{u}$. But this contradicts our previous hypothesis, thus χ cannot be equal to zero on (a, c).

Last, since χ is L^2 near the a end-point in LP and the operator $\mathcal{L}$ is LC near the b end-point, the continuous extension $\hat{\chi}$ of χ that solves the equation $\mathcal{L}y = iy$ over the whole (a, b) is the only $L^2(a, b)$ solution. The same holds for the equation $\mathcal{L}y = -iy$ and hence we have $n_+(\bar{\mathcal{L}}) = n_-(\bar{\mathcal{L}}) = 1$.

Proof of statement (iii):

Suppose that $\mathcal{L}$ is LP at both end-points a and b. From the (ii) statement we know that there exists only one square-integrable function near a and one near b but we do not know whether they can be related in some way. Instead of showing that this is not the case, we will show that the deficiency indices of $\bar{\mathcal{L}}$ are $n_+(\bar{\mathcal{L}}) = n_-(\bar{\mathcal{L}}) = 0$.

In order to do this, we first need some arguments on the Wronskian function

$$\mathcal{W}(x; \phi, \psi) = (\phi\psi' - \phi'\psi)(x) \qquad x \in I, \quad \phi, \psi \in D(\mathcal{W}) \tag{5.9}$$

where $D(\mathcal{W})$ is defined according to

$$D(\mathcal{W}) = \{\varphi \in W^{2,1}_{\mathrm{loc}}(I) \mid \varphi'' \in L^1_{\mathrm{loc}}(I)\}, \tag{5.10}$$

and on the *regular points* of the operator $\mathcal{L}$. We say that the point a is a *regular point* (the same for b) for $\mathcal{L}$ if and only if it is finite and the subsequent condition holds:

$$\int_a^d |q(x)|^2 < \infty \qquad \forall\, d \in I. \tag{5.11}$$

First we want to show that:

(a) $\mathcal{W}(x; \phi, \psi)$ is a locally absolutely continuous function in the x variable.

(b) If $\phi, \psi \in D(\mathcal{L}^*)$ then there exist the limits $\lim_{x\to a}\mathcal{W}(x; \phi, \psi)$ and $\lim_{x\to b}\mathcal{W}(x; \phi, \psi)$ and hence

$$\mathcal{W}(b; \bar{\phi}, \psi) - \mathcal{W}(a; \bar{\phi}, \psi) = <\phi, \mathcal{L}^*\psi> - <\mathcal{L}^*\phi, \psi> \tag{5.12}$$

(c) The operator $\mathcal{L}$ is in LC at its regular point, and if $\phi \in D(\mathcal{L}^*)$ then the limits

$\lim_{x\to a}\phi(x)$ and $\lim_{x\to a}\phi'(x)$ exist and are zero if $\phi \in D(\bar{\mathcal{L}})$.

In order to prove the (a) property, we only have to take into account theorem 4.2 that ensures the local absolute continuity of the functions in $D(\mathcal{W})$ and of their first derivative. From the definition (5.9) it is easily seen that the product and summation of locally absolutely continuous functions lead to the local abolute continuity of the Wronskian.

For point (b) we can use the fact that the function $q(x)\phi\psi \in L^2_{\mathrm{loc}}(I)$ because $\phi, \psi \in D(\mathcal{W})$, while $q(x) \in L^2_{\mathrm{loc}}(I)$ from the hypothesis. Now, taking into account theorem 4.3 and the fact that the functions of $D(\mathcal{L}^*)$ need the square integrability of their second derivatives, from the Hölder inequality, such derivatives are also locally integrable and our starting functions belong to $D(\mathcal{W})$. From this we have that $D(\mathcal{L}^*) \subseteq (D(\mathcal{W}) \cap L^2(I))$.

Using the local absolute continuity of $\mathcal{W}$, we have for all $[c, d] \subset I$ that

$$\mathcal{W}(d; \bar{\phi}, \psi) - \mathcal{W}(c; \bar{\phi}, \psi) = \int_c^d (\bar{\phi}\psi'' - \bar{\phi}''\psi), \tag{5.13}$$

and on adding and subtracting the $\int_c^d q(x)\phi\psi$ term in (5.13), under the hypothesis of $\phi, \psi \in D(\mathcal{L}^*)$, we have

$$\mathcal{W}(d; \bar{\phi}, \psi) - \mathcal{W}(c; \bar{\phi}, \psi) = \int_c^d (\bar{\phi}\mathcal{L}^*\psi - \overline{\mathcal{L}^*\phi}\psi). \tag{5.14}$$

Since the functions in $D(\mathcal{L}^*)$ are locally absolutely continuous and globally square integrable with their second derivative on I, in (5.14) the limits $\lim_{c\to a}\mathcal{W}(x; \phi, \psi)$ and $\lim_{d\to b}\mathcal{W}(x; \phi, \psi)$ exist and equation (5.12) holds. This completely proves the (b) Wronskian statement.

As far as the property (c) is concerned, under the hypothesis of regularity for point a, if $\phi \in D(\mathcal{L}^*)$, then it is in $D(\mathcal{W})$ and therefore ϕ and ϕ' are locally absolutely continuous. This ensures us that the limits $\lim_{x\to a}\phi(x)$ and $\lim_{x\to a}\phi'(x)$ must exist.

Now, since $D(\mathcal{L}) \equiv C_0^\infty(I)$ and the map $\phi \to (\phi(a), \phi'(a))$ is continuous in the norm $||\phi|| + ||\mathcal{L}^*\phi||$, then $\lim_{x\to a}\phi(x) = \lim_{x\to a}\phi'(x) = 0$ for all $\phi \in D(\bar{\mathcal{L}})$. Since ϕ is locally absolutely continuous, it is bounded near a and thus it belongs to L^2. This shows the LC case.

We can now proceed with the proof of the (iii) statement, in which we will make use of the results obtained above.

Suppose that the end-point a is regular while b is LP. It is easily seen that $\mathcal{L}$ has got self-adjoint extensions because it is symmetric on $C_0^\infty(I)$ and it has deficiency indices $n_+(\mathcal{L}) = n_-(\mathcal{L}) = 1$ like the operator $\mathcal{L}_1$ defined in the proof of the (ii) statement. Among all conceivable self-adjoint extensions, we want to choose that one for which there exists some $(\alpha, \beta) \in \mathbb{R}^2 \backslash (0, 0)$ such that

$$\alpha\phi(a) + \beta\phi'(a) = 0 \tag{5.15}$$

and call it $\hat{\mathcal{L}}$. In order to do this we will use a theorem in [14], known under the name of *von Neumann's extension Theorem*[3] that provides the esplicit expression of all possible domains of the closed symmetric extensions for a closed symmetric operator, by using partial isometries between the deficiency spaces $K_+(\mathcal{L})$ and $K_-(\mathcal{L})$. Of course $\mathcal{L}$ is closable, thus we can make use of von Neumann's Theorem.

Let us define the following unitary operator between the deficiency spaces that acts like a complex conjugation:

$$\mathcal{U}: \xi \in K_+(\mathcal{L}) \longrightarrow \mathcal{U}\xi = \bar{\xi} \in K_-(\mathcal{L}). \tag{5.16}$$

Certainly $\mathcal{U}$ is an isometry, hence it is bijective.

From von Neumann's theorem we know that the self-adjoint extension related to the unitary operator (5.16) has got the following domain:

$$D(\hat{\mathcal{L}}) = \{\eta + \xi + \mathcal{U}\xi \mid \eta \in D(\bar{\mathcal{L}}),\ \xi \in K_+(\mathcal{L})\}. \tag{5.17}$$

It is straightforward that the function in $D(\hat{\mathcal{L}})$ satifies the relation (5.15) for same α and β.

[3] Let A be a closed Hermitian operator with domain $D(A)$ dense in a Hilbert space $\mathcal{H}$, and let us define the spaces [2, 3, 14]

$$\mathcal{K}_\pm = \mathrm{Ker}(A^* \mp iI),$$

with dimension denoted by $d_\pm$. For any closed symmetric operator B, we denote by U_B its Cayley transform

$$U_B \equiv (B - iI)(B + iI)^{-1},$$

extended to $\mathcal{H}$ by setting it to 0 on $\mathrm{Ran}(B + iI)^\perp$. The von Neumann Extension theorem can be stated as follows [2, 3, 14]:If A is a closed Hermitian operator, there exists a $1 - 1$ correspondence between closed symmetric extensions B of A, and partial isometries V, with initial space $\mathcal{H}_I(V) \subset \mathcal{K}_+$ and final space $\mathcal{H}_F(V) \subset \mathcal{K}_-$. This correspondence is expressed by

$$U_B = U_A + V,$$

or by

$$D(B) = \{\varphi + \psi + V\psi: \varphi \in D(A), \psi \in \mathcal{H}_I(V)\},$$

where B equals the restriction of the adjoint A^* to the domain $D(B)$. The operator B is self-adjoint if and only if

$$\mathcal{H}_I(V) = \mathcal{K}_+,\ \mathcal{H}_F(V) = \mathcal{K}_-.$$

In particular, the operator A has self-adjoint extensions if and only if $d_+ = d_-$ and, in that case, if $d_+ < \infty$, the set of self-adjoint extensions is a d_+^2-dimensional real topological manifold, in the topology of norm-resolvent convergence.

Eventually, if we show that

$$\mathcal{W}(x;\, \bar{\phi},\, \psi) = 0 \qquad \forall\, \phi,\, \psi \in D(\mathcal{L}^*) \tag{5.18}$$

at the LP end-points, under our hypothesis of LP at both ends of I, from (5.12) we get the symmetry of the operator $\mathcal{L}^*$, and because we must have $D(\mathcal{L}^*) \subseteq D(\mathcal{L}^{**})$ but at the same time is easily seen that $\mathcal{L} = \mathcal{L}^{**}$ and $D(\mathcal{L}) \equiv D(\mathcal{L}^{**})$, we must have $D(\mathcal{L}) = D(\mathcal{L}^*)$ and there follows that $\mathcal{L}$ is self-adjoint, thus its closure $\bar{\mathcal{L}}$ is symmetric from propositions (c) and (b) and has got deficiency indeces $n_+(\bar{\mathcal{L}}) = n_-(\bar{\mathcal{L}}) = 0$.

In order to show that $\mathcal{W}(x;\, \bar{\phi},\, \psi) = 0$ at the LP extremes, suppose b in LP. First of all we see that $\mathcal{W}(b;\, \bar{\phi},\, \psi) = 0$ for all $\phi,\, \psi \in D(\hat{\mathcal{L}})$ because from (5.15) we have $\mathcal{W}(a;\, \bar{\phi},\, \psi) = 0$ and from the self-adjoint behaviour of $\hat{\mathcal{L}}$ we must have $\mathcal{W}(b;\, \bar{\phi},\, \psi) = 0$ by relation (5.12). Now, if some $\eta_0 \in C_0^\infty(I)$ is chosen in such a way that it equals zero on $[c,\, b)$ for some $c \in I$, and for which (5.15) is not verified, such a function must belong to $D(\mathcal{L}^*)\backslash D(\hat{\mathcal{L}})$. From the fact that $\mathcal{L}$ has deficiency indices equal to one, there must be $\dim(D(\mathcal{L}^*) - D(\hat{\mathcal{L}})) = 1$ and therefore every function $\phi \in D(\mathcal{L}^*)$ can be written in the form

$$\phi = \phi_0 + \lambda\eta_0 \qquad \lambda \in \mathbb{C}, \quad \phi_0 \in D(\hat{\mathcal{L}}). \tag{5.19}$$

Since $\eta_0(x) = \eta'_0(x) = 0$ on $[c,\, b)$, then $\lim_{x\to b}\mathcal{W}(x;\, \bar{\phi},\, \psi) = \lim_{x\to b}\mathcal{W}(x;\, \bar{\phi}_0,\, \bar{\psi}_0) = 0$ for all $\phi,\, \psi \in D(\mathcal{L}^*)$ of the form (5.19).

This completely shows that (5.18) holds, and the desired proof is completed.

6. Examples of self-adjoint extensions

Here we want to show that indeed, the operator $-\frac{d^2}{dx^2}$ has got more that one self-adjoint extension, and these correspond to the Dirichlet and Neumann conditions at the origin. We can proceed in the following way:

(1) First we consider a particular class of domains -i.e.- $D_{\{\epsilon,\mu\}}$, that let our operator be closed and symmetric and from this, using some arguments contained in [2], we are ensuring the existence of self-adjoint extensions for such closed and symmetric restrictions.

(2) We use the von Neumann's theorem [2, 3, 14] to obtain explicitly all domains of closed and symmetric extensions -i.e- $D_{\{\epsilon,\mu\}}(c)$ (where c runs over $[0,\, 2\pi[$).

(3) We derive the form of the domains of the adjoint -i.e.- $D^*_{\{\epsilon,\mu\}}(c)$.

(4) We use the Self-Adjointness condition $D_{\{\epsilon,\mu\}}(c) = D^*_{\{\epsilon,\mu\}}(c)$ to find which of the $D_{\{\epsilon,\mu\}}(c)$ domains is of self-adjointness.

Let us define the following two-parameter domains of symmetry for $-\frac{d^2}{dx^2}$:

$$D_{\{\epsilon,\mu\}} = \{\phi \in C_0^\infty(\mathbb{R}^+)\colon \phi(x) = 0\ \forall\, x \in (0,\, \epsilon),\ \phi(\epsilon) = \mu,\ \mu \in \mathbb{C}\}. \tag{6.1}$$

It is easy to see that such domains are closed and on them our operator is symmetric, and from the fact that $n_+ = n_- = 1$, there exist self-adjoint extensions for each fixed admissible pair $(\epsilon,\, \mu)$.

In order to use von Neumann's Theorem, we need the expression of the deficiency spaces, and we easily find that

$$\mathcal{K}_+ = \{\phi \in L^2(\mathbb{R}^+)\,|\,\phi = c_+\, e^{\frac{i-1}{\sqrt{2}}x}, \qquad c_+ \in \mathbb{C}\} \tag{6.2}$$

$$\mathcal{K}_- = \{\phi \in L^2(\mathbb{R}^+)\,|\,\phi = c_-\, e^{-\frac{i+1}{\sqrt{2}}x}, \qquad c_- \in \mathbb{C}\} \tag{6.3}$$

We see that such spaces are one-dimensional linear spaces and, from von Neumann's theorem we know that all possible symmetric extensions for each of $D_{\{\epsilon,\mu\}}$, are in bijection with the isometries between the deficiency spaces. From the fact that the deficiency spaces are one-dimensional, the isometries required can only be phase factors of the form $e^{i\theta(x,c_+,c_-)}$ and therein, following the statement of the Theorem, we must have

$$e^{i\,\theta(x,c_+,c_-)}e^{\frac{i-1}{\sqrt{2}}x} = \frac{c_-}{c_+}\, e^{-\frac{i+1}{\sqrt{2}}x}, \tag{6.4}$$

and since it follows that $\left|\frac{c_-}{c_+}\right| = 1$, we set $\frac{c_-}{c_+} = e^{ic}$ with $c \in [0,\, 2\pi[$, and this shows that $\theta(x,\, c) = -\sqrt{2}\,x + c$.

From the von Neumann's criterion [2, 3, 14] we can give the explicit form to the domains of symmetric extensions that we will call $D_{\{\epsilon,\mu\}}(c)$:

$$D_{\{\epsilon,\mu\}}(c) = \{\psi \in L^2(\mathbb{R}^+) \mid \psi = \phi + z(e^{\frac{i-1}{\sqrt{2}}x} + e^{ic}e^{-\frac{i+1}{\sqrt{2}}x})\}, \tag{6.5}$$

where $\phi \in D_{\{\epsilon,\mu\}}$, $z \in \mathbb{C}$. Taking into account the symmetry relation

$$\left\langle \xi, \frac{d^2}{dx^2}\psi \right\rangle = \left\langle \frac{d^2}{dx^2}\xi, \psi \right\rangle,$$

where $\psi \in D_{\{\eta,\mu\}}(c)$ and $\xi \in D^*_{\{\eta,\mu\}}(c)$, we obtain the following equation:

$$\xi'(0)(e^{ic} + 1) + \xi(0)\left(\frac{i+1}{\sqrt{2}}e^{ic} - \frac{i-1}{\sqrt{2}}\right) = 0 \tag{6.6}$$

that defines the two following kinds of adjoint domains:

$$D_1^*(c) = \left\{\xi \in L^2(\mathbb{R}^+) \mid \frac{\xi(0)}{\xi'(0)} = -\frac{\sqrt{2}(e^{ic} + 1)}{1 + e^{ic} + i(e^{ic} - 1)}\right\} \tag{6.7}$$

with $c \neq \frac{\pi}{2}$, and

$$D_2^*(c) = \left\{\xi \in L^2(\mathbb{R}^+) \mid \frac{\xi'(0)}{\xi(0)} = -\left(\frac{\sqrt{2}(e^{ic} + 1)}{1 + e^{ic} + i(e^{ic} - 1)}\right)^{-1}\right\}, \tag{6.8}$$

with $c \neq \pi$, in which we have ruled out $\frac{\pi}{2}$ and π values that lead to singular ratios $\frac{\xi(0)}{\xi'(0)}$ and $\frac{\xi'(0)}{\xi(0)}$, respectively. First of all, it is interesting to note that (6.7) and (6.8) are independent of the (ϵ, μ) pair. In this way we can certainly say that (6.7) and (6.8) cover *all possible domains for the adjoints of the closed and symmetric extensions for any of the possible closed and symmetric realizations of* $-\frac{d^2}{dx^2}$ *over the real half-line.*

Now, by using the self-adjointness relations $D_{\{\epsilon,\mu\}}(c) = D_1^*(c)$ and $D_{\{\epsilon,\mu\}}(c) = D_2^*(c)$ we easily get the following self-adjointness domains:

$$D_1^*(\pi) = D_1(\pi) = \{\xi \in L^2(\mathbb{R}) \mid \xi(0) = 0\}, \tag{6.9}$$

$$D_2^*\left(\frac{\pi}{2}\right) = D_2\left(\frac{\pi}{2}\right) = \{\xi \in L^2(\mathbb{R}) \mid \xi'(0) = 0\}, \tag{6.10}$$

that correspond to the Dirichlet and Neumann condition at the origin.

The last thing that we want to note is that the sets (6.9) and (6.10) are both closed and open.

For example, by using the following sequence in $D_1(\pi) \cap D_2\left(\frac{\pi}{2}\right)$:

$$f_n(x) = \begin{cases} \sqrt[2]{x^3} & x \in \left[0, \frac{1}{n}\right[\\ \frac{1}{\sqrt[3]{x}} & x \in \left]\frac{1}{n}, a\right[\\ 0 & x \in [a, \infty[\end{cases} \qquad a \in (1, \infty) \tag{6.11}$$

which converges in L^2 but not in the intersection of $D_1(\pi)$ and $D_2\left(\frac{\pi}{2}\right)$, we realize that (6.9) and (6.10) are open sets. On the other hand, if we choose the following:

$$g_n(x) = \begin{cases} \frac{1}{n} - \left(x - \frac{1}{n}\right)^2 & x \in [0, a[\\ 0 & x \in [a, \infty) \end{cases} \qquad a \in \mathbb{R}^+ \tag{6.12}$$

we see that $g_n \in \left(L^2(\mathbb{R}^+) - \left(D_1(\pi) \cup D_2\left(\frac{\pi}{2}\right)\right)\right)$, hence it belongs to the complement of each $D_1(\pi)$ and $D_2\left(\frac{\pi}{2}\right)$ for every $n \in \mathbb{N}$ but its limit belongs $D_1(\pi) \cap D_2\left(\frac{\pi}{2}\right)$. This shows that the complement of (6.9) and (6.10) is an open set and therefore (6.9) and (6.10) must be closed sets.

The fact that they are closed sets also results from von Neumann's Theorem. In this way, the sequence (6.12) confirms the validity of such a Theorem. Eventually, we have obtained that the operator $-\frac{d^2}{dx^2}$ is Self-Adjoint only on domains (6.13) and (6.14), which are simultaneously closed and open.

7. Concluding remarks

In the first part of our paper we have derived two peculiar aspects of the general Sturm-Liouville operators. First over all, starting from very general and fundamental theorems, we have shown which are the weakest assumptions on the coefficients in order to obtain solutions of the eigenvalue problem that are sufficiently regular, i.e. -absolutely continuous- to be used whenever needed. A second remarkable aspect is essentially seen in the possibility of considering such regular solutions embedded in a hightly non-regular space such as (4.3). It is from this latter aspect that our proof of Weyl's theorem takes the moves. Other proofs of this theorem can be found in [15, 16], under the natural assumption of differential operators acting on a suitable Hilbert space. As far as this last pair of references are concerned, we want to mention some further functional-analytical methods which are up to date with the current developments of this subject. An example is the *maximal operator* and *minimal operator* related to a differential expression like (2.1) or (4.2). In [15], the domain of definition for the maximal operator related to a *nth*-order differential expression τ on a real line segment is

$$D_M = \{f \in \mathcal{L}^2(a,\, b):\ f^{(0)}, f^{(1)},\ \ldots, f^{(n-1)} \in AC(a,\, b);\ \ \tau f \in \mathcal{L}^2(a,\, b)\},$$

and it is shown that such a domain is densely defined and closed in the Hilbert space. Here we also define the domain for the minimal operator in the form

$$D_m = \{f \in \mathcal{L}^2(a,\, b):\ f \in C_0^2(a,\, b)\}.$$

Upon focusing on our differential operator (4.2), it is evident that the minimal operator occurs instead under the hypothesis of theorem 4.1. With the language of our paper and by means of theorem 4.2, the maximal set D_M is basically

$$D_M = \{f \in W_{\rm loc}^{2,1}([a,\, b]):\ f'' \in \mathcal{L}^2(a,\, b)\},$$

hence we derive the Hilbert-space nature of D_M by relying only upon the square summability of highest derivatives in it. In our proof we decided to use the *Von Neumann's extension theorem* in order to reach all possible self-adjoint extensions for the specific differential operator under consideration. We suggest reading [15] for a the general theory about *nth*-order differential operators's self-adjoint extensions in terms of boundary conditions of which our (5.18) represents a specific case. We remark the fact that, with the language used here, the expression (5.18) cannot be untied from the functional space (5.10). Following the conceptual behaviour of Von Neumann's theorem, we must bring to the attention of the reader another functional method that makes use of a tool called *boundary triples*. Given any Hilbert space and a symmetric operator T, it is always useful to define a sesquilinear map Γ

$$\Gamma_T(\phi,\, \psi) = \langle T^*\phi,\, \psi\rangle - \langle \phi,\, T^*\psi\rangle$$

for $\phi,\ \psi \in D(T^*)$ in order to find the closure of T. We also refer to [16] for a recent redefinition of essential self-adjointness in terms of the map Γ. By the way, a boundary triple is a triplet $(h,\ \rho_1,\ \rho_2)$ where h is a suitable Hilbert space and $\rho_1,\ \rho_2$ are $D(T^*) \longrightarrow h$ maps that satisfy the rule

$$a\Gamma_{T^*}(\phi,\, \psi) = \langle \rho_1(\phi),\, \rho_1(\psi)\rangle - \langle \rho_2(\phi),\, \rho_2(\psi)\rangle$$

for some complex a constant. In terms of boundary triples, one finds that $\Gamma(\phi,\ \psi)$ vanishes identically when we restrict the inputs $\phi,\ \psi$ to the domain of some self-adjoint extension. One can therefore say that the domain of any self-adjoint extension of a symmetric operator T is of the form

$$D(T_{\mathcal{U}}) = \{\phi \in T^*:\ \ \rho_2(\phi) = \mathcal{U}\rho_1(\phi)\}$$

where $\mathcal{U}$ is a unitary operator. In terms of boundary triples, one can obtain a more general statement and expression for the self-adjoint extensions of a symmetric operator. The role played by the sesquilinear map Γ is the same as the role played by the domain (5.10). Such a theory is described in detail in [16].

To sum up, we have indeed shown that the operator (4.2) can clearly act over a sort of spaces like (5.1), and that the Hilbert-space nature of such sets, resides only in the square summability of second derivatives of their functions by virtue of theorem 4.2. In this fashion, whenever needed, one can look for weaker solutions of the eigenvalue equation.

Similar techniques have been applied, over the years, to a wide range of topics. For example, the work in [17] studied essential self-ajointness in 1-loop quantum cosmology, the work in [18] has provided enlightening examples of boundary conditions for self-adjoint extensions of linear operators, whereas the work in [19] has suggested that a profound link might exist between the formalism for asymptotically flat space-times and the limit-point condition for singular Sturm-Liouville problems in ordinary quantum mechanics. Last, but not least, the parameter $\lambda_{nl} \equiv l + \frac{(n-2)}{2}$ in equation (1.2) is neatly related to the parameter L used in large-N quantum mechanics [20], i.e.

$$\lambda_{nl} = \frac{L}{2} - 1.$$

Moreover, since the Schrödinger stationary states are even functions of λ_{nl}, this suggests exploiting the complex-λ_{nl} plane in the analysis of scattering problems [21]. If n is kept arbitrary, this means complexifying a linear combination of l and n [5], including the particular case where l remains real while the dimension n is complexified.

Thus, there is encouraging evidence that Sobolev-space methods and yet other concepts of functional and complex analysis may provide the appropriate tool for investigating classical and quantum physics as well as correspondences among such frameworks.

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Appendix. G-regularity

For the definition of G-regularity we need to endow the $\mathbb{R}^n$ space with the norm

$$||y|| = \sum_{i=1}^{n} |y_i|. \tag{A1}$$

Here we make use of the topology induced by the *uniform convergence* in order to give to the $C(I, \mathbb{R}^n)$ vector space a Banach structure. Let us call it $\mathcal{K}$.

We will say that the sequence $u_n \in \mathcal{K} \longrightarrow u \in \mathcal{K}$ if and only if, for every $\epsilon \in \mathbb{R}^+$, there exists some $\nu \in \mathbb{N}$ such that, for all $n \geqslant \nu$ the following majorization is verified:

$$\sup_I |u_n(x) - u(x)| < \epsilon. \tag{A2}$$

Since $\mathcal{K}$ is also a metric space, we can use in (A2) the following notation for the distance between u_n and u:

$$d_n = \sup_I |u_n(x) - u(x)|. \tag{A3}$$

At this stage we have only to define the following particular sequence on the I segment:

$$x^n = x - d_n \text{sign}(x) \tag{A4}$$

and we note that $x^n \longrightarrow x$ on I only if $u_n \longrightarrow u$ on $\mathcal{K}$.

We are now in a position to define the concept of G-regularity, while we refer to definition 2 for the G-integrability.

Definition 6. Let f be G-integrable on the interval I whose middle point is x_0, and let $g \in G$ be defined in equation (2.3). Let the sequence $u_n(x) = \int_{x_0}^{x} g_n(s)ds \in G$ be such that u_n tends uniformly to $u(x) = \int_{x_0}^{x} g(s)ds$ on I. Let x^n be defined by (A4). If now every such sequence of functions verifies the condition

$$f(x^n, u_n(x^n)) \to f(x, u(x)), \tag{A5}$$

then we say that f is *G-regular* on I.

For more insights on the G-regularity property of functions, we refer the reader to [12].

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