Numerical Methods in Superstring Field Theory*

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Abstract

A descent relation $\langle V_3|V_1\rangle = \tilde{Z}_3\langle V_2|$ in the fermionic NS string with the vertices in the standard oscillator basis is presented.

The main object in the String Field Theory (SFT) is the string functional $\Psi[X(\sigma)]$. One can relate with a string functional $\Psi[X(\sigma)]$ a state $|\Psi\rangle$ in the Fock space

$$\Psi[X(\sigma)] \equiv \langle X(\sigma)|\Psi\rangle \tag{1}$$

where $\langle X(\sigma)|$ is

$$\langle X(\sigma)| = \prod_{n=0}^{\infty}\langle x_n| = \langle 0|\exp\left\{-\sum_{n\geq 1}\left(\frac{1}{2}\,nx_n^2 - i\sqrt{2n}\,x_n\hat{\alpha}_n - \frac{1}{2n}\,\hat{\alpha}_n\hat{\alpha}_n\right)\right\}. \tag{2}$$

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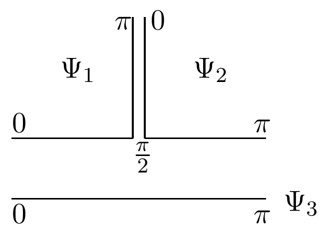
Here α_n are the usual modes of the string,

$$\langle x_n|\,\hat{x}_n = \langle x_n|\,x_n\,, \qquad \hat{x}_n = i(\hat{\alpha}_n - \hat{\alpha}_{-n})/\sqrt{2}n$$

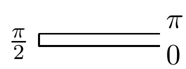
and $\langle 0|$ is the Fock space vacuum annihilated by all α_{-n}.
One of the main ingredient of SFT is a multiplication of string functionals. In the Witten covariant SFT [1] the definition of string functionals multiplication $\Psi_1 \star \Psi_2 = \Psi_3$ is given by gluing a right half ($\sigma \in [0, \frac{\pi}{2}]$) of one string to a left half ($\sigma \in [\frac{\pi}{2}, \pi]$) of the other one. This multiplication can be drawn as



This picture has an analytic form in terms of the path integral

$$(\Psi_1 \star \Psi_2)[X(\sigma)] \equiv \int \prod_{0\leqslant\sigma\leqslant\pi} dX^1(\sigma)dX^2(\pi-\sigma) \prod_{0\leqslant\sigma\leqslant\frac{\pi}{2}} \delta(X^1(\pi-\sigma) - X^2(\pi))$$
$$\times\, \delta(X^1(\sigma) - X(\sigma))\delta(X^2(\pi-\sigma) - X(\pi-\sigma))\Psi_1[X^1(\sigma)]\Psi_2[X^2(\sigma)]. \quad (3)$$

In the Witten covariant SFT the action is defined via an integral $\int \Psi : \Psi \to \mathbb{C}$. The integral is given by folding the string and identifying the two sides:



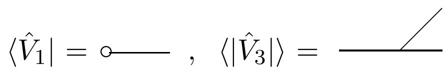
An analytic form of the integral reads

$$\int \Psi = \int \prod_{0\leqslant\sigma\leqslant\pi} dX(\sigma) \prod_{0\leqslant\sigma\leqslant\frac{\pi}{2}} \delta(X(\sigma) - X(\pi-\sigma))\Psi[X(\sigma)]. \quad (4)$$

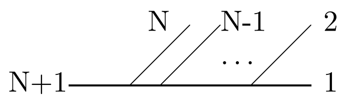
The string field multiplication (3) and the integral (4) can be presented in terms of vertices $\langle|\hat{V}_3|\rangle$ and $\langle\hat{V}_1|$ acting in the Fock space. These vertices have been constructed explicitly by Gross and Jevicki [2].

$$|\Psi_1 \star \Psi_2\rangle = {}_{12}\langle|\hat{V}_3|\rangle_3|\Psi\rangle_1|\Psi\rangle_2, \qquad \int \Psi = {}_1\langle\hat{V}_1|\Psi\rangle_1. \quad (5)$$

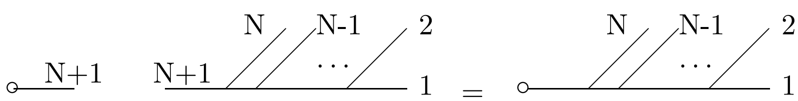
See [1, 2] and reviews [3] for more details and references. It is useful to use the graphical representation of these vertices:



One can build an infinite tower of vertices ${}_{1\ldots N}\langle|\hat{V}_{N+1}|\rangle_{N+1}$ by gluing of $N-1$ vertices V_3. This tower graphically has the form of a tree graph with $N+1$ legs. Inside this tree we can glue free legs of the vertices $\langle|\hat{V}_3|\rangle$ arbitrary, all of these gluing are equivalent due to the associativity of the Witten multiplication.



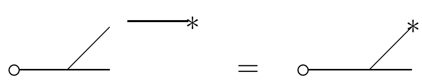
One can also build an infinite tower of vertices ${}_{1\ldots N}\langle\hat{V}_N|$ associated with vertices ${}_{1\ldots N}\langle|\hat{V}_{N+1}|\rangle_{N+1}$ by adding one more $\langle\hat{V}_1|$ as shown below:



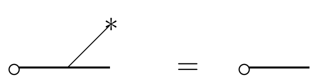
One can define also a ket $|\widehat{V}_1\rangle$ as a solution of the "descent relation" [2]

$$\langle\widehat{V}_2|\widehat{V}_1\rangle = \langle\widehat{V}_1|. \tag{6}$$

It is important to note that solution (6) is unique and $|\hat{V}_1\rangle$ satisfies the overlap condition. This defining equation for $|\widehat{V}_1\rangle$ can be represented graphically. We display a vertex corresponding to $|\widehat{V}_1\rangle$ by a line outgoing from $*$ to the left. According to Wick's theorem the LHS of (6) can be presented as



In this notations the equation (6) looks like

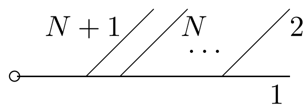


With $|\widehat{V}_1\rangle$ subject to (6) we are going to prove the descent relations

$$\langle\widehat{V}_{N+1}|\widehat{V}_1\rangle = \langle\widehat{V}_N|. \tag{7}$$

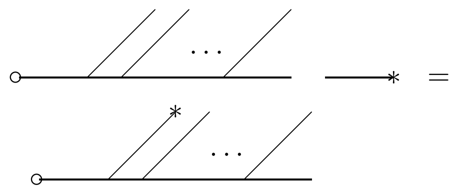
To prove (7) we have to use our construction for the tower of the vertices $\langle\widehat{V}_{N+1}|$. Taking a simplest graph representing $\langle\hat{V}_{N+1}|$

$\langle \widehat{V}_{N+1}| \quad =$

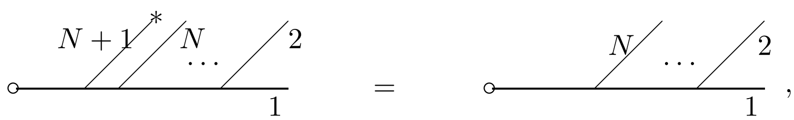


we have

$\langle \widehat{V}_{N+1}|\widehat{V}_1\rangle \quad =$



Taking into account (6) we get that



that exactly gives $\langle \widehat{V}_N|$.

However, the bra vertices $\langle V_i|$ are usually found as the solutions of the overlap equations [2] and not by gluing V_3-s. For instance, the overlap equation for the matter fields X^μ reads:

$$\langle V_N|(X^r(\sigma) - X^{r-1}(\pi-\sigma)) = 0, \quad r = 1,\dots,N \quad \sigma \in [0, \frac{\pi}{2}]. \tag{8}$$

The vertices $\langle V_N|$ from these equations are defined up to numerical factors, i.e. the vertex $\langle \widehat{V}_N|$ and the vertex $\langle V_N|$ can be different but are related by a factor

$$\langle \widehat{V}_N| = Z_N \langle V_N| \text{ and } |\widehat{V}_1\rangle = Z_{-1}|V_1\rangle.$$

Hence, the descent relations for $\langle V_N|$ look like

$$\langle V_{N+1}|V_1\rangle = Z_N Z_{-1}^{-1} Z_{N+1}^{-1}\langle V_N| \equiv \widetilde{Z}_{N+1}\langle V_N|. \tag{9}$$

We can use the similar scheme to build Witten's tower of ghost vertices $\langle \hat{V}_N^{gh}|$ in terms of the vertex $\langle|\hat{V}_3^{gh}|\rangle$. Similar to the matter sector the ghost sector descent relation has the form

$$\langle \hat{V}_{N+1}^{gh}|\hat{V}_1^{gh}\rangle = \langle \hat{V}_N^{gh}|\,. \tag{10}$$

There are several papers [4, 5] devoted to a checking of descent relations in the bosonic SFT. The main difficulty in these calculations comes from an infinite dimensionality of Neumann matrices defining vertices $\langle V_{N+1}|$. There are two methods to perform these calculations.

The first method is known as a level truncation method [6] [1]. One truncates infinite Neumann matrices up to some level M and performs the calculations. At each level M one compares the resulting (numerically calculated) matrices with Neumann matrices in $\langle V_{N+1}|$ and calculates $\widetilde{Z}_{N+1}(M)$.

The actual calculations were done for $N=2$ [4, 5, 8] and it was found that the Neumann matrices of $\langle V_2|$ are reproduced with an accuracy growing as $M\to\infty$. It happened that the situation with $\widetilde{Z}_3(M)=Z_3^X(M)Z_3^{gh}(M)$ brought a surprise. The coefficient Z_3^X calculated in the matter sector is gone to zero as $M\to\infty$, but the coefficient Z_3^{gh} calculated in the ghost sector goes to infinity as $M\to\infty$. If one multiplies these two coefficients $\widetilde{Z}_3=Z_3^X Z_3^{gh}$ and puts $M\to\infty$ the coefficient $\widetilde{Z}_3$ tends to a constant. We used two different schemes of calculation $\widetilde{Z}_3$ and got two different answers for $\widetilde{Z}_3$ which, in its turn, different from results of [4, 5]. In fact, it is not surprising since we deal with divergent quantities and two schemes of calculations deal with two different regularizations.

The second method uses the κ-basis [9]. In the κ-basis the Neumann matrices are diagonal. This basis is related with K_n symmetries of the vertices

$$\langle V_3|(K_n^{(1)}+K_n^{(2)}+K_n^{(3)})=0, \qquad (11)$$

where $K_n=L_n-(-)^n L_{-n}$ and L_n are the Virasoro generators. It is important that for $n=odd$ the vertices are invariant in the matter and ghost sectors [2, 9] separately, but for $n=even$ only the full vertex (11) is K_n-invariant. The K_n invariance of a vertex means that the Neumann matrices and the matrix corresponding to the operator K_n commute (choose $n=1$):

$$[K_{1,nm},V_{nm}]=0. \qquad (12)$$

Therefore, if one finds the eigenvectors of the matrix $K_{1,nm}$ and chooses them as basis vectors then the Neumann matrices are reduced to the diagonal form. The calculation are greatly simplified in this basis [9, 10]. One can perform a check of the descent relation analytically and the proper structure of the descent relation is deduced at [11, 12, 13]. It is important to stress that divergences also arise in the κ-basis calculation via normalization constants Z_3^X and Z_3^{gh}. The theory can be consistently regularized to give finite results. Note that in [11, 12, 13, 8] there are used slightly different regularizations. Thus we conclude that a regularization is a essential element of the descent relation calculation. This fact also clarifies the problem of different $\tilde{Z}$-s in the level truncation calculation, the numerical value of $\tilde{Z}_3$ is regularization dependent. In this lecture the use the level truncation method.

We have tested the descent relation for the fermionic NS string where in addition to the matter fields X^μ and the ghosts b,c there are the fermionic matters ψ^μ and fermionic ghosts β,γ.

[1] This method is different from the field level truncation proposed by V. A. Kostelecky and S. Samuel [7] .

We can also build the Witten fermionic tower of the vertices $\langle\hat{V}_N|$ from the vertices $\langle|\hat{V}_3|\rangle$ and $\langle\hat{V}_1|$. The descent relation also takes place for these vertices. Just as in the bosonic case the actual vertices $\langle V_N|$ are obtained as the solutions of the overlap equations that states the issue of checking of the descent relation for NS string.

The fermionic vertices $\langle V_N|$ are the solutions of the following overlap equations

$$\begin{aligned} &\langle V_N^{\psi}|(\psi^r(\sigma) - i\psi^{r-1}(\pi-\sigma)) = 0,\\ &\langle V_N^{\beta\gamma}|(\beta^r(\sigma) + i\beta^{r-1}(\pi-\sigma)) = 0, \quad r = 1,\dots,N, \quad \sigma\in[0,\frac{\pi}{2}],\\ &\langle V_N^{\beta\gamma}|(\gamma^r(\sigma) + i\gamma^{r-1}(\pi-\sigma)) = 0. \end{aligned} \tag{13}$$

For example, the vertices $\langle V_3|$ for fermionic matter and its ghosts are [2]

$$\langle V_3^{\psi}| = {}_{321}\langle 0|\exp\left\{-\frac{1}{2}\sum_{a,b=1}^{3}\sum_{r,s\geq 1/2}^{\infty}\psi_r^a K_{rs}^{ab}\psi_s^b\right\}, \tag{14}$$

$$\langle V_3^{\beta\gamma}| = {}_{123}\langle -1|\exp\left\{-\sum_{a,b=1}^{3}\sum_{\substack{r\geq 1/2\\ s\geq 1/2}}\beta_r^a\bar{K}_{rs}^{ab}\gamma_s^b\right\}e^{-\phi(\frac{\pi}{2})}Y^2\left(\frac{\pi}{2}\right)Y^3\left(-\frac{\pi}{2}\right).$$

The vacuum $\langle 0|$ in the matter sector is defined as $\langle 0|\psi_r = 0,\ r\leqslant\frac{1}{2}$. In the ghost sector the situation is more complicated. Here Y is the picture changing operator [14]. This is an essentially new object which appears in the superstring field theory [15, 16, 17]. In the superghost sector all vacua are non-equivalent and the picture changing operator converts them from one to another. The vacuum with picture q is defined as

$$\langle q|\beta_r = 0,\ r\leqslant -\frac{3}{2}-q,\quad \langle q|\gamma_s = 0,\quad s\leqslant\frac{1}{2}+q.$$

The picture changing operators do not enter explicitly in our calculations of $\langle V_3|V_1\rangle$. We can do the calculations by closing the brackets of the string number "1" while Y-s are sitting on the string number "2" and "3". However, in the simplest descent relation $\langle V_2|V_1\rangle = \langle V_1|$ in the superghost sector one needs to use them. We cannot check this descent relation unlike in the bosonic string.

The Neumann matrices K_{rs}^{ab} and $\bar{K}_{rs}^{ab}$ are infinite dimensional ones and they have very complicated form [2]. However, there is simple representation for the Neumann matrices [18, 19]. Using this representation the structure of the vertex $\langle V_2|$ is easily reproduced. As in the case of the bosonic SFT we got coefficient $\tilde{Z}_3$ which is not equal to one. The coefficient Z_3^{ψ} which was got in fermionic matter sector vanishes as $M\to\infty$, but in the superghost sector the coefficient $Z_3^{\beta\gamma}$ diverges as $M\to\infty$. It is very non-trivial that the result coefficient $\tilde{Z}_3 = Z_3^{\psi} Z_3^{\beta\gamma}$ is constant as in the bosonic case. Note that this factor depends on the regularization.

However, the full coefficient Z_3 contains as the bosonic as the fermionic contributions. In the bosonic sector of the NS string the coefficient $\widetilde{Z}_3^B = Z_3^X Z_3^{gh}$ differs from the coefficient $\widetilde{Z}_3$ in the pure bosonic string origin to change of the d. That gives the linear grow of $\widetilde{Z}_3^B$ in M. Unfortunately $\widetilde{Z}_3^F$ in the fermionic sector goes to zero not so quickly to cancel M and the overall $Z_3 = \widetilde{Z}_3^B \widetilde{Z}_3^F$ appears to be divergent. This issue demands further investigations.

Acknowledgments

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