Author: Dalhuisen, Jan Willem
Title: The Robinson congruence in electrodynamics and general relativity
Issue Date: 2014-03-25

The Robinson Congruence in Electrodynamics and General Relativity

Jan Willem Dalhuisen

ISBN: 978-90-8593-000-6

The Robinson Congruence in Electrodynamics and General Relativity

PROEFSCHRIFT

ter verkrijging van
de graad van Doctor aan de Universiteit Leiden,
op gezag van Rector Magnificus prof. mr. C.J.J.M. Stolker,
volgens besluit van het College voor Promoties
te verdedigen op dinsdag 25 maart 2014
klokke 10:00 uur

door

Jan Willem Dalhuisen

geboren te Elst (Gld)
in 1966

Promotiecommissie:

Promotor:	Prof. dr. D. Bouwmeester	Universiteit Leiden en University California Santa Barbara
Leden:	Prof. dr. N. Van den Bergh	Universiteit Gent
	Prof. dr. A. Burinskii	Russian Academy of Sciences, Moscow
	Prof. dr. E.R. Eliel	Universiteit Leiden
	Prof. dr. G. Nienhuis	Universiteit Leiden
	Dr. A. Boyarsky	Universiteit Leiden
	Dr. P. J. M. Bongaarts	Universiteit Leiden

The work presented in this thesis has been made possible by financial support from the Marie-Curie Program EU-CT-2006-042580.

Casimir PhD series, Delft-Leiden 2014-2

The limits of my language are the limits of my world.

Whereof one cannot speak, thereof one must be silent.

Ludwig Wittgenstein

Contents

Introduction

This thesis is an account of the research that started with a simple observation. In 2008 the investigations of W.T.M. Irvine and D. Bouwmeester with respect to curious solutions to Maxwell's equations culminated in an article in Nature Physics [1]. These solutions, which had been described earlier but less complete by A. Rañada [2], were dubbed linked and knotted beams of light. Although they are more general than the term 'light' suggests, 'linked and knotted' describes these solutions very well: a picture of all the electric field lines at a particular instant of time would reveal that they consist of circles, any two of which are linked in the manner of two neighboring Olympic rings. The complete collection of circles fill all of space in a tangled manner. 'Knotted' refers to this tangle, but more aptly, also to the mathematical notion of knot of which a circle is a simple example.
One of the investigators, D.B., had been a post-doc in the group of Roger Penrose and knew therefore very well the similarity between the new solutions and a picture representing a certain kind of twistor, drawn by its inventor in the standard reference for twistor theory [3]. This intriguing configuration is referred to in the twistor theory literature as the Robinson congruence.
Both pictures, of the electric field lines and the twistor, exhibit a geometry that is far from commonplace and a naturally justified question was therefore born: is there any deeper connection between these? It was my initial task, as a theorist in a predominant experimental quantum optics group, to address this question.
Confronted with this task it is also natural to search for the differences. There are many. A twistor is a mathematical object, whereas Maxwell equations describe accurately an import part of nature. Apart from an overall translational movement the twistor picture does not change in time. The circles of the electric field deform in a complicated way that is very hard to describe. There are other twistors, unrelated to the linked structure. Besides the electric field, the magnetic field exhibits the same structure, only rotated with respect to the electric one. Why then should there be a deeper correspondence?

Happily, as part of present research, it was found that a neat description of the linked and knotted solutions was possible in terms of a complex combination of the electric and magnetic field. From this it could be established that there is another physical quantity of the knotted solutions, the Poynting vector, also exhibiting the same structure as the electric or magnetic field, again rotated, now with respect to both. The surprise was the observation that the time development of the Poynting vector was simple and equal to the time development in the twistor picture. This fact was an important contribution to [4,5] and led to the conclusion that further investigation was warranted.

It was realized that the investigated structures related to Robinson congruences could be of interest in other fields of physics too. Plasma physics, hydrodynamics and general relativity were possible theories among others for which the knotted solutions could be 'generalized'. The connection to general relativity very much appealed to me although it implied that I would venture deeper into terra incognita for the research group that I was embedded in. My research supervisor encouraged this branching out of the research and arranged support from additional researchers. In the second half of my PhD research I have therefore worked closely together with a bachelor, a Masters, and another PhD student. Furthermore we had fruitful discussion with Prof. Iwo Białinicky-Birula (Warsaw), Prof. Roger Penrose (Oxford, and Lorentz professor 2011) and Prof. Alexander Burinskii (Moscow, and visiting professor summer 2012).
Whereas the additional students, with my support, focused on studies of twistors and Robinson congruences in plasma physics and in the linearized version of general relativity, I set myself the ambitious goal of searching for a solution of the full non-linear Einstein equations with a source term based on the electromagnetic knot. The approach that I took is based on the Newman-Penrose tetrad formalism and by the time of writing of this thesis several insights have been obtained that reduces the large number of coupled equations to a much smaller and more manageable set of equations. However a conclusive result has not yet been obtained. It might well be that the next step would require a significant numerical effort. As this thesis will show, my interest is mainly in analytic studies and future contributions from a researcher with better computer skills might be needed to complete this ambitious project. Apart from presenting my findings in a clear and precise way I have taken it therefore as my task to present my understanding of knotted structures of light, of special topics in twistor theory, and of the Newman-Penrose formalism in a form that will bring new researchers quickly up to date in this specialized field of research.

In the following I will describe and comment on the content of my thesis chapter by chapter.

Chapter 1 introduces the basic ingredients for this thesis: tetrads, spinors and twistors. With the possible exception of twistors, every physicist is familiar with these concepts. However, the way in which spinors are introduced here, as general mathematical objects without the for physicists self-evident association

with spin-$\frac{1}{2}$ particles, is not standard. Vectors and tensors can be built from spinors, and therefore it is possible to 'rewrite' many physical theories in 'spinor-language', and this will be done in later chapters.
The introduction of a null tetrad leads in a natural way to Infeld Van Der Waerden symbols, hybrid quantities that figure as a dictionary between 'tensor-language' and 'spinor-language'. In later chapters the null tetrad will turn up again, but now as a field of basis vectors in the Newman-Penrose formalism.
Twistors are introduced in the most basic and succinct manner as a combination of two spinors. Twistor theory is covering a broad and expanding range of topics but for this thesis only one instance of the relationships between subspaces of twistor space and subspaces of Minkowski space will be relevant: a null twistor as a null geodesic.
The important Kerr theorem is given, including a sketchy proof, and will play an essential role in later chapters.

Chapter 2 is about the geometry of the key structure in this thesis, the Hopf fibration, which is intimately linked to the Robinson congruence. Modern literature deals with the Hopf fibration with the help of mathematical tools from algebraic topology and discusses the geometry in question in just a few lines. Given the importance of a detailed understanding of all aspects of the Hopf fibration, we prefer a more basic treatment, in which all features are explicitly written out.

Chapter 3 recapitulates the basic equations of Maxwell theory in different forms. The familiar vector notation is used to define the units that will be used and to list the properties that will be of importance throughout this work. The Riemann-Silberstein vector and the Faraday tensor are introduced to rewrite the equations in 'RS-form' and in manifest covariant (relativistic) form. Finally, the equations are rewritten in spinor-language. Since this will be the least familiar, some basic examples are given.
All the presented forms will be used in later chapters.

Chapter 4 together with chapter 5 are two long chapters containing most of the new results that are presented in this thesis. It starts with a summary of what can be considered the germ of present research, Rañada's attempt at a topological theory of electromagnetism. This summary contains elucidations, partly in the form of an appendix, not present in the original work of Rañada as well as explicit expressions for the fields considered to make the connection with chapter 2. Although these expressions have appeared in the literature before [6], we show that in order to have a manageable expression for all times, they should be combined in the form of the Riemann-Silberstein vector (4.1). In this form these fields were found to be given in a different context and without further ado in [7] as a specific example of Trautman-Robinson fields. The Poynting vector, which has not been considered before, exhibits the structure of a Hopf fibration for all times (4.2). Alternative forms of these fields are also presented.

Next, with the help of a theorem by Robinson we show that the 'optical analogy' for null twistors can be extended to non-null twistors. This establishes a mathematical correspondence between a non-null twistor and an electromagnetic Hopf knot. Included is a discussion of *local* duality transformations in electrodynamics, for which literature is very scarce.
The method used for the optical analogy for a non-null twistor is also applied to a null twistor, leading to what we call a degenerate Robinson congruence. We note that this congruence can also be obtained by a sequence of operations on a plane wave solution, including conformal inversion and shifts of parameters into the complex plane. This sequence was brought to our attention by work of, and discussions with, I. Białinicky-Birula [7,8].
The main results of this chapter were published in [4].
We surmise that in the later part of chapter 4 there is a strong link with the work of E. Newman [9] but acknowledge that more research has to be done in this direction.

Chapter 5 was intended to study the said sequence of operations in more detail, but grew into a more general study of conformal transformations and in particular conformal inversion, after it was observed that not much was written about this elusive symmetry of vacuum Maxwell equations.
In the literature one encounters confusion about the interpretation of conformal transformations. Sometimes, without explicitly stating this, it is assumed that the transformation is just a change of coordinates, and that tensors transform accordingly, even when only Minkowski space is considered. This is incorrect, although results derived in this manner have been found to be valid. To avoid confusion we present a careful elementary exposition of the symmetry at hand, resulting in a prescription for obtaining new solutions from old ones. To the best of our knowledge this has not been done previously in this manner.
As a new result, a nice transformation formula for the Riemann-Silberstein vector under conformal inversion is derived and applied to some familiar fields. Among these examples is an electromagnetic Hopf knot, which, surprisingly, is transformed into another Hopf knot.

Chapter 6 discusses the Penrose transform, a contour integral formula that needs a twistor function as input and the evaluation of which gives a solution to a particular zero rest mass free field equation. It is shown what twistor function is needed in order to get an electromagnetic Hopf knot as a result of the contour integral. This twistor function, which was found by my fellow students J. Swearngin and A. Thompson, adapted slightly in order to meet the requirements, is then used in the Penrose transform to arrive at a solution to linearized Einstein equations. This solution, exhibiting the Hopf structure in different ways, is discussed.
The results are also discussed in [10,11].

Chapter 7 is a review of mathematical notions from differential geometry that are needed in the general theory of relativity, with an emphasis on the tetrad formalism. In the remaining chapters we lean strongly on this chapter. The method of presentation is based on [12].

Chapter 8 presents two exact solutions to Einstein's equations. One contains a Robinson congruence, the other a degenerate Robinson congruence. Again, but different as in chapter 4, there is a relationship between the two via a complex shift. The solutions are arrived at via the method presented by Debney, Kerr and Schild [13], the relevant content of which is summarized and supplemented.
Although the solutions are not new, we do not known whether they have been published in this form. It is pointed out that there seems to be no physical interpretation attached to the curves of the congruence. Comparison with chapter 6, where the Robinson congruence appears as a linearized solution *with* interpretation suggests itself, but has not been carried out to completion.

Chapter 9 treats the electromagnetic Hopf knot in the Newman-Penrose formalism, neglecting the curvature of space-time due to the energy density of the knot. It leads to a suitable choice of tetrad fields adapted to the structure of the knot and an expression for the spin coefficients in terms of Minkowski coordinates. The knowledge of this description of the Hopf knot is needed in the next chapter, were we try to include gravity effects.
The chapter is based on many calculations that are presented in a much condensed form.

Chapter 10 presents an ambitious attempt to redo the calculation of chapter 9, but now taking into account the effect of the energy distribution on the curvature of space-time. This leads to a problem for which there is no obvious solution: in order to describe the source correctly we need the metric, but in order to calculate the metric we need to know the source. Ultimately this is related to the problem of interpretation of coordinates in general relativity. It is therefore allowed to take some freedom in the definition of the source, and here we show one attempt to do this in such a way that the relevant equations can be solved. This is "work in progress" and the assumptions that were chosen in simplifying the problem have not yet resulted in a solution although important insights have been obtained in how to address this challenging problem.

1

Tetrads, spinors and twistors

Here we introduce mathematical concepts that are needed throughout this work. Some of them are very familiar. However, the form in which they are presented is not standard.

1.1 tetrads

We consider Minkowski space (flat space-time) M^4 and orthonormal basis vectors $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) \equiv g_{ab} = \eta_{ab}\ , \quad (\eta_{ab}) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Here $\mathbf{g}$ is the metric tensor and indices $a, b, ...$ run from 0 to 3, whereas $k, l, ...$will run from 1 to 4. The collection [0,1,2,3] is used to stress the difference of the first component from the others as a time component.
It is possible to change to a basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4\}$ such that:

$$(\mathbf{g}(\mathbf{E}_k, \mathbf{E}_l)) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{1.1}$$

Since all $\mathbf{E}_k$ are null vectors, this basis is called a null tetrad. Two of them need be complex in order to satisfy the definition (1.1). An example is provided by:

$$\mathbf{E}_1 = \tfrac{1}{\sqrt{2}}(\mathbf{e}_0 + \mathbf{e}_3)$$
$$\mathbf{E}_2 = \tfrac{1}{\sqrt{2}}(\mathbf{e}_0 - \mathbf{e}_3)$$
$$\mathbf{E}_3 = \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 + i\mathbf{e}_2)$$
$$\mathbf{E}_4 = \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 - i\mathbf{e}_2).$$

If we define $(\mathbf{e}_{AX'}) = \begin{pmatrix} \mathbf{e}_{11'} & \mathbf{e}_{12'} \\ \mathbf{e}_{21'} & \mathbf{e}_{22'} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_4 \\ \mathbf{E}_3 & \mathbf{E}_2 \end{pmatrix}$, we find from the form of $(\mathbf{g}(\mathbf{E}_k, \mathbf{E}_l))$ that:

$$\mathbf{g}(\mathbf{e}_{AX'}, \mathbf{e}_{BY'}) = \epsilon_{AB}\epsilon_{X'Y'},$$

in which

$$(\epsilon_{AB}) = (\epsilon_{X'Y'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.2}$$

Indices A, B take the values 1 and 2, indices A', B' also take the values 1 and 2, but to avoid confusion this is sometimes written as $1'$ and $2'$. Both $\{\mathbf{e}_{AX'}\}$ and $\{\mathbf{e}_a\}$ form a basis and therefore coefficients exist that relate the two:

$$\mathbf{e}_{AX'} = \sigma^a{}_{AX'}\mathbf{e}_a.$$

These coefficients are called Infeld Van Der Waerden symbols and from the definition and the form of the metric tensor in both bases it follows that they satisfy:

$$\sigma^a{}_{AX'}\sigma^b{}_{BY'}g_{ab} = \epsilon_{AB}\epsilon_{X'Y'}. \tag{1.3}$$

Einstein summation convention is used throughout this thesis. In the example given above they are given by:

$$
\boxed{
\begin{aligned}
(\sigma^0{}_{AX'}) &= \tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
(\sigma^1{}_{AX'}) &= \tfrac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\
(\sigma^2{}_{AX'}) &= \tfrac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\
(\sigma^3{}_{AX'}) &= \tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}
\end{aligned}
}
\tag{1.4}
$$

1.2 spinors

For most physicists the first encounter with spinors is in a course on non-relativistic quantum mechanics when considering particles with spin. Every physicist is therefore familiar with (Schrödinger) two-spinors and possibly (Dirac) four-spinors. Lesser known is the fact that all of physics can be rewritten, sometimes to great advantage, in spinor form. In a sense a spinor can be considered as the "square root of a vector" (see for example (1.6)), and is thus more basic than a vector. Although a solid introduction to spinors should mention the group theoretical background [14], for our purposes it suffices to introduce spinors in a more elementary and direct way.

The considerations in the previous paragraph show that, in a sense, Minkowski space can be considered as a subspace of the tensor product of two complex vector spaces of dimension two:

$M^4 \subset M^4_{\mathbb{C}} \cong S \otimes S'$,

in which the tensor product is regarded as complexified Minkowski space.

$S \equiv$ spin-space $\equiv \{$1-index spinors$\} \equiv \{\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} | \psi^A \in \mathbb{C}\}$,

basis vectors: $\mathfrak{e}_1, \mathfrak{e}_2$

$S' \equiv$ conjugate spin-space$\equiv \{$1-index conjugate spinors$\} \equiv$
$\{\begin{pmatrix}\psi^{1'} & \psi^{2'}\end{pmatrix} | \psi^{X'} \in \mathbb{C}\}$,

basis vectors: $\mathfrak{e}_{1'}, \mathfrak{e}_{2'}$.

The column and row vector notation is arbitrary, but handy in a first encounter on spinors. Now we can make the following identification, relating the basis in $M^4_{\mathbb{C}}$ (*and* M^4) with those in S and S':

$$\mathbf{e}_{AX'} = \mathfrak{e}_A \otimes \mathfrak{e}_{X'}. \tag{1.5}$$

Corresponding to S and S' we define the dual spaces S^* and S'^* in which the vector components have subindices, and we can identify S with S^* via:

$\psi_A \equiv \psi^B \epsilon_{BA}$ or $\psi^A \equiv \epsilon^{AB}\psi_B$ in which $(\epsilon^{AB}) = (\epsilon_{AB})$.

The same relations hold for primed spinor indices. This shows that (ϵ_{AB}) can be considered as the metric spinor and plays the same role in S as the metric tensor in M^4, including the raising and lowering of spinor indices. The antisymmetry of the metric spinor has consequences unfamiliar in ordinary vector language, such as (for arbitrary spinors $(\psi^A), (\phi^A)$ and 2-component spinor $(\psi_{AB}) \in S^* \otimes S^*$):

$\psi^1 = \psi_2$ and $\psi^2 = -\psi_1$
$\psi^A \psi_A = 0$
$\psi^A \phi_A = -\psi_A \phi^A$
${\epsilon^A}_B = -\delta^A_B = -{\epsilon_B}^A \quad (\delta^A_B = 1$ if $A = B$, otherwise $0)$
$\psi_{AB} = -\psi_{BA} \iff \psi_{AB} = \frac{1}{2}\epsilon_{AB}{\psi_R}^R$ (and obvious generalization for many-index spinors)
$\psi_{AB} = \psi_{BA} \iff \psi_{AB} = \alpha_{(A}\beta_{B)}$ for some (α_A) and (β_B) (and obvious generalization)

Round brackets ( ) around indices means symmetrization and square brackets [ ] anti-symmetrization. In the last example (α_A) and (β_A) are called the principal spinors of (ψ_{AB}).

An arbitrary tensor T can be written $T = T^{ab...c}\mathbf{e}_a \otimes \mathbf{e}_b \otimes ... \otimes \mathbf{e}_c$, but also as $T^{AX'...BY'}\mathbf{e}_{AX'} \otimes ... \otimes \mathbf{e}_{BY'}$ and therefore:

$T^{a...c} = {\sigma^a}_{AX'}...{\sigma^c}_{BY'}T^{AX'...BY'}$ or $T_{a...c} = {\sigma_a}^{AX'}...{\sigma_c}^{BY'}T_{AX'...BY'}$

$T_{AX'...BY'}$ is called the spinor equivalent of the tensor $T_{a...c}$ and is given by:

$T_{AX'...BY'} = {\sigma^a}_{AX'}...{\sigma^c}_{BY'}T_{a...c}$.

Note that ${\sigma^a}_{AX'}$ is a mixed quantity, since the tensor index a is raised and lowered with the metric tensor, whereas the spinor indices A, X' are raised and lowered with the metric spinor. The general rule for going from tensor indices to spinor indices or vice versa with the help of the Infeld Van Der Waerden symbols can be read off from the foregoing three examples. We see that the spinor equivalent of the metric tensor is given by (1.3). With the choice (1.4) for the Infeld Van Der Waerden symbols, they satisfy $\sigma^{a*}_{AX'} \equiv (\sigma^a{}_{AX'})^* = \sigma^a{}_{XA'}$, and it can be shown that this relation holds for every possible choice in agreement with the requirement (1.3). As a consequence, the spinor equivalent of a real tensor T satisfies $T^*_{AX'\ldots BY'} = T_{XA'\ldots YB'}$. The Hermitian conjugate $\bar{T}$ of a spinor T is defined by $\bar{T}_{AX'\ldots BY'} = T^*_{XA'\ldots YB'}$ and this spinor is called Hermitian if $\bar{T} = T$. Thus the spinor equivalent of a real tensor is Hermitian. It follows that the spinor that corresponds to a real null vector, $V^a = \sigma^a{}_{AX'}V^{AX'}$, can be written

$$V^{AX'} = \xi^A \bar{\xi}^{X'}, \tag{1.6}$$

so, in this sense, it is ξ^A that corresponds to V^a. Of course, there is freedom in the choice of ξ^A. The spinor equivalent of a real anti-symmetric tensor, $T_{ab} = -T_{ba}$, can be written

$$T_{AX'BY'} = \tau_{AB}\epsilon_{X'Y'} + \bar{\tau}_{X'Y'}\epsilon_{AB}, \tag{1.7}$$

with $\tau_{AB} = \frac{1}{2}T_{AR'B}{}^{R'} = \tau_{BA}$. In this sense, it is the symmetric τ_{AB} that corresponds to the anti-symmetric T_{ab}. The principal spinors of τ define two principal null directions via (1.6) that correspond to the eigenvectors of (T_{ab}). For a totally anti-symmetric tensor with three indices, T_{abc}, we have the spinor equivalent

$$T_{AX'BY'CZ'} = t_{BX'}\epsilon_{AC}\epsilon_{Y'Z'} - t_{AY'}\epsilon_{BC}\epsilon_{X'Z'}$$

for some $t_{BX'}$, as can be seen by using previous result. This result in turn can be used to derive the spinor corresponding to a totally anti-symmetric four-index tensor, T_{abcd}, as

$$T_{AW'BX'CY'DZ'} = a(\epsilon_{BD}\epsilon_{W'Z'}\epsilon_{AC}\epsilon_{X'Y'} - \epsilon_{AD}\epsilon_{W'Y'}\epsilon_{BC}\epsilon_{X'Z'})$$

for some constant a. For the completely anti-symmetric Levi-Civita pseudotensor, ϵ_{abcd}, this leads to $a = \pm i$. We use the convention $\epsilon_{0123} = 1 = -\epsilon^{0123}$. The choice (1.4) for the Infeld Van Der Waerden symbols gives $+i$. Other choices might give $-i$! With $a = +i$ we have a spinor equivalent for the dual, $*T^{ab}$, of an anti-symmetric tensor T_{ab}, defined by $*T^{ab} = \frac{1}{2}\epsilon^{abcd}T_{cd} \Leftrightarrow *T_{ab} = \frac{1}{2}\epsilon_{abcd}T^{cd}$

$$*T_{AX'BY'} = i(-\tau_{AB}\epsilon_{X'Y'} + \bar{\tau}_{X'Y'}\epsilon_{AB}) \tag{1.8}$$

where $T_{AX'BY'}$ is given by (1.7). Therefore, the first part in (1.7), $\tau_{AB}\epsilon_{X'Y'}$ is called the anti-self-dual part of $T_{AX'BY'}$, and the last part, $\bar{\tau}_{X'Y'}\epsilon_{AB}$, its self-dual part.

Since in differential geometry the basis vectors are associated with differential operators, it is natural to extend the equivalence between tensors and their spinor form to differential operators:

$\nabla^{AX'} = \sigma_a{}^{AX'}\partial^a.$

Using (1.4) (the speed of light will be put equal to 1 in this work) we find

$$(\nabla^{AX'}) = \frac{1}{\sqrt{2}}\begin{pmatrix}\partial^t + \partial^z & \partial^x + i\partial^y \\ \partial^x - i\partial^y & \partial^t - \partial^z\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\partial_t - \partial_z & -\partial_x - i\partial_y \\ -\partial_x + i\partial_y & \partial_t + \partial_z\end{pmatrix}.$$

This enables us to rewrite differential equations in spinor form. As an example, Maxwell's equations in vacuum can be written as

$\nabla^{AX'}\phi_{AB} = 0$

and linearized Einstein's equations in vacuum are

$\nabla^{AX'}\phi_{ABCD} = 0.$

In the sequel this will be shown in more detail and we will profit from the resemblance of these two equations, which constitutes one of the advantages of the spinor formalism.

1.2.1 geodesic shear-free null congruences

From equation (1.5) we infer that in spin-space there exist a normalized dyad, $((\vartheta^A), (\iota^A))$ with $\epsilon_{AB}\vartheta^A\iota^B = 1$, from which we can construct a null tetrad $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$ in M^4 satisfying (1.1), as follows:

$l^a = \sigma^a_{AX'}\vartheta^A\bar{\vartheta}^{X'}, n^a = \sigma^a_{AX'}\iota^A\bar{\iota}^{X'}, m^a = \sigma^a_{AX'}\vartheta^A\bar{\iota}^{X'}, \bar{m}^a = \sigma^a_{AX'}\iota^A\bar{\vartheta}^{X'}.$

In these formula there is still an unnecessary reference to the basis $\{\mathbf{e}_0, ..., \mathbf{e}_3\}$. This can be avoided by writing:

$\mathbf{l} \leftrightarrow \vartheta\bar{\vartheta}, \;\; \mathbf{n} \leftrightarrow \iota\bar{\iota}, \;\; \mathbf{m} \leftrightarrow \vartheta\bar{\iota}, \;\; \bar{\mathbf{m}} \leftrightarrow \iota\bar{\vartheta}.$

It is clear that the spinors that form the normalized dyad could be spinor fields on M^4. The corresponding null tetrad now consists of four vector fields that satisfy (1.1) everywhere. This tetrad is a basis that possibly changes from point to point.

We now consider a congruence of null geodesics in M^4, a space filling family of curves with the property that all curves are geodesics and the tangent vectors to the curves are null. The collection of tangent vectors, $\{\mathbf{l}(x)|x \in M^4\}$, is a vector field that satisfies $l^a l_a = 0$ everywhere, and therefore there exists a spinor field, ϑ, such that $\mathbf{l} \leftrightarrow \vartheta\bar{\vartheta}$. We complement ϑ with a spinor field ι such that (ϑ, ι) forms a normalized dyad. The condition that the integral curves of $\mathbf{l}$ are geodesics is $l^b \nabla_b l^a = f(x^\mu) l^a$, for some arbitrary function f (see also section **7.2.4**). In terms of ϑ this becomes $\vartheta^A \vartheta^B \bar{\vartheta}^{X'} \nabla_{BX'} \vartheta_A = 0$.

The behaviour of neighbouring null geodesics can be expressed in terms of the so called optical scalars, rotation, expansion and shear (see section **7.5**). Following the time development of the geodesics that make up a circle at one instant of time, it is clear what rotation and expansion do to this circle. Shear distorts the circle into an ellipse, without changing its area. The mathematical definition of (complex) shear is $m^a m^b \nabla_b l_a$, or, equivalently $\vartheta^A \vartheta^B \bar{\iota}^{X'} \nabla_{BX'} \vartheta_A$.

If the congruence of null geodesics is shear-free, a property of great importance in this thesis as we will see, the two equations $\vartheta^A \vartheta^B \bar{\vartheta}^{X'} \nabla_{BX'} \vartheta_A = 0$ and $\vartheta^A \vartheta^B \bar{\iota}^{X'} \nabla_{BX'} \vartheta_A$ have to be fulfilled simultaneously. Since $\bar{\vartheta}$ and $\bar{\iota}$ form a basis we arrive at

$$\mathbf{l} \leftrightarrow \vartheta\bar{\vartheta} \text{ geodesic and shear-free} \iff \vartheta^A \vartheta^B \nabla_{BX'} \vartheta_A = 0 \tag{1.9}$$

1.2.2 Kerr's theorem

We can try to solve (1.9) by decomposing ϑ in the *constant* basis $(\mathfrak{e}_1, \mathfrak{e}_2)$ as $\vartheta = \lambda(-Y\mathfrak{e}_1 + \mathfrak{e}_2)$. For $\mathfrak{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathfrak{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ we have $\vartheta = \lambda\begin{pmatrix}-Y\\1\end{pmatrix}$. With

$u \equiv \frac{1}{\sqrt{2}}(t+z),\ v \equiv \frac{1}{\sqrt{2}}(t-z),\ w \equiv \frac{1}{\sqrt{2}}(x+iy)$ and $\bar{w} \equiv \frac{1}{\sqrt{2}}(x-iy)$ we find $(\nabla_{AX'}) = \begin{pmatrix} \partial_u & \partial_w \\ \partial_{\bar{w}} & \partial_v \end{pmatrix}$, and equation (1.9) therefore becomes $Y\partial_u Y - \partial_{\bar{w}} Y = 0$ and $Y\partial_w Y - \partial_v Y = 0$. Note that λ plays no role, since $\vartheta^A(-Y\mathfrak{e}_1 + \mathfrak{e}_2)_A = 0$. The solution to these two equations can be given implicitly by an arbitrary analytic function F in the following way:

$$F(Y, w+Yv, u+Y\bar{w}) \equiv 0,\ \ Y \equiv -\frac{\vartheta^1}{\vartheta^2} \iff \mathbf{l} \leftrightarrow \vartheta\bar{\vartheta} \text{ geodesic and shear-free} \tag{1.10}$$

This is known as Kerr's theorem and is of importance in chapter 4 and 8. In words: the general (analytic) geodesic shear-free null congruence in flat space-time is related to the zero set of an arbitrary analytic function. Although this theorem has been shown to work in Minkowski space, its action can be extended to the curved Kerr-Schild spaces of chapter 8.

1.3 twistors

Twistors were invented almost half a century ago by Roger Penrose [15] and, though not widely used by physicists, have proven very useful in solving some nonlinear equations. There has been a renewed interest in twistors within the physical community since E. Witten combined string theory with twistor theory [16], but it is only very recently that twistor theory is used by high energy physicists for calculational purposes in scattering processes [17]. Although twistors do appear in books for undergraduate students [18], it is considered as an advanced subject. The full body of twistor theory, with all its mathematical intricacies, its beautiful geometrical structure and its relations with group theory, certainly needs some time to digest. However, in this thesis we would like to add only one very tiny thing to that body (see chapter 4), and we do not need the full theory for that. We therefore present here only the barest essentials of twistor theory, just enough for the aforementioned purpose.

We now combine two spinors to form a twistor: twistor space consists of pairs of spinors $(\Omega^A, \bar{\Pi}_{X'}) \equiv (\mathcal{Z}^\alpha) \equiv \mathcal{Z}$ $(\alpha = 1,2,3,4)$ with corresponding conjugate $(\bar{\mathcal{Z}}_\alpha) \equiv (\Pi_A, \bar{\Omega}^{X'})$ and inner product:

$$\mathcal{Z}^\alpha \bar{\mathcal{Z}}_\alpha = \Omega^A \Pi_A + \bar{\Omega}^{X'} \bar{\Pi}_{X'} = \mathcal{Z}^1 \bar{\mathcal{Z}}^3 + \mathcal{Z}^2 \bar{\mathcal{Z}}^4 + \bar{\mathcal{Z}}^1 \mathcal{Z}^3 + \bar{\mathcal{Z}}^2 \mathcal{Z}^4. \tag{1.11}$$

A twistor for which the inner product is 0 is called null, otherwise it is non-null. Twistor space is related to Minkowski space by the incidence relation:

$$\Omega^A = iX^{AX'}\bar{\Pi}_{X'} \tag{1.12}$$

in which $X^{AX'}$ is the spinor equivalent of the space-time point x^μ:

$$(X^{AX'}) = \frac{1}{\sqrt{2}}\begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \tag{1.13}$$

It is not difficult to proof that if and only if the (fixed) twistor $(\Omega^A, \bar{\Pi}_{X'})$ in eq. (1.12) is null does there exist a real solution for x^μ: in that case the solution space is a null geodesic. (The incidence relation shows that it is really projective twistor space that should concern us, since any non zero complex multiple of a twistor gives rise to the same solution space. For reasons of succinctness, here, as elsewhere in this thesis, we will not be too concerned about technical subtleties (like analyticity, compactification, points at infinity . . .) that are needed for a complete treatment, whenever these subtleties do not influence the result that concerns us.) For future reference we give here the spinorial form of the null geodesic:

$$X^{AY'} = X_0^{AY'} + \lambda\Pi^A\bar{\Pi}^{Y'} \tag{1.14}$$

in which $(X_0^{AY'})$ corresponds to an arbitrary point on the geodesic, and λ is a real parameter. A null twistor thus corresponds to a null geodesic in M^4. For a non-null twistor the solution space is a subspace of $M^4_{\mathbb{C}}$ that does not intersect the real part M^4. There is therefore no direct correspondence between a non-null twistor and a subspace of M^4 defined via (1.12). The most natural we can do is to consider the collection of null twistors that are orthogonal to the non-null twistor $\mathcal{A}^\alpha$: $\{\mathcal{Z}^\alpha | \bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha = 0\}$. This zero set of an analytic function defines a collection of disjoint null geodesics that fill Minkowski space and is known as a Robinson congruence. Now, for $\mathcal{A}^\alpha = (0, \frac{1}{\sqrt{2}}, 0, 1)$ and constant t, the structure of the projection of the corresponding Robinson congruence onto this time slice is a Hopf fibration. As a function of time, this fibration moves with the speed of light along the z-axis.
The details of these statements will be presented in chapter 2 and chapter 4.

It is one of the goals of this work to show that the "optical analogy" between null twistors and null geodesics (light rays), can be extended or generalized to non-null twistors.

Finally, we rewrite Kerr's theorem (1.10) in terms of twistors. Kerr's theorem states that for any spinor field ϑ that defines a geodesic shear-free null congruence via (1.6), there exist a function of three variables, F, such that ϑ is given implicitly by $F(-\frac{\vartheta^1}{\vartheta^2}, w - \frac{\vartheta^1}{\vartheta^2}v, u - \frac{\vartheta^1}{\vartheta^2}\bar{w}) = 0$. We can rewrite this as a function of four variables, g, that is homogeneous to some degree, say n:

$F(-\frac{\vartheta^1}{\vartheta^2}, w - \frac{\vartheta^1}{\vartheta^2}v, u - \frac{\vartheta^1}{\vartheta^2}\bar{w}) =$

$g(-i(u - \frac{\vartheta^1}{\vartheta^2}\bar{w}), -i(w - \frac{\vartheta^1}{\vartheta^2}v), 1, -\frac{\vartheta^1}{\vartheta^2})$ (this *defines* g) $=$

$-(\frac{1}{\vartheta^2})^n g(i(u\vartheta^2 - \bar{w}\vartheta^1), i(w\vartheta^2 - v\vartheta^1), -\vartheta^2, \vartheta^1) =$

$-(\frac{1}{\vartheta^2})^n g(-i(u\vartheta_1 + \bar{w}\vartheta_2), -i(w\vartheta_1 + v\vartheta_2), \vartheta_1, \vartheta_2).$

So there exists a function, f (homogeneous with the same degree) such that:

$f(i(u\bar{\vartheta}_{1'} + w\bar{\vartheta}_{2'}), i(\bar{w}\bar{\vartheta}_{1'} + v\bar{\vartheta}_{2'}), \bar{\vartheta}_{1'}, \bar{\vartheta}_{2'}) = 0.$

But the argument $(i(u\bar{\vartheta}_{1'} + w\bar{\vartheta}_{2'}), i(\bar{w}\bar{\vartheta}_{1'} + v\bar{\vartheta}_{2'}), \bar{\vartheta}_{1'}, \bar{\vartheta}_{2'})$ defines a null twistor:

$$\mathcal{Z} = (iX^{AY'}\bar{\vartheta}_{Y'}, \bar{\vartheta}_{X'}). \tag{1.15}$$

Therefore the Kerr theorem can be stated as follows. The zero set of a homogeneous analytic twistor function $f(\mathcal{Z}^\alpha)$ defines a geodesic shear-free null congruence in M^4, or, with $\mathcal{Z}$ as in (1.15):

$$\{\mathcal{Z}^\alpha \mid f(\mathcal{Z}^\alpha) = 0 \;\wedge\; \mathcal{Z}^\alpha\bar{\mathcal{Z}}_\alpha = 0\} \iff \mathbf{l} \leftrightarrow \vartheta\bar{\vartheta} \text{ geodesic shear-free.} \tag{1.16}$$

2

The Hopf fibration

Next we present a detailed and elementary study of the leading part in this thesis, the Hopf fibration. Amazingly, this somewhat contrived structure is relevant in various situations in physics.

2.1 stereographic projection

Let S^2 be the two-sphere in $\mathbb{R}^3$: $S^2 = \{x \in \mathbb{R}^3 | \; ||x|| = 1\}$, $N = (0,0,1)$ the north-pole of S^2 and $S = (0,0,-1)$ its south-pole. $||x||$ denotes the Euclidian norm of x. Define S^2_S to be the two-sphere without its north-pole: $S^2_S = S^2 \setminus N$, and similarly $S^2_N = S^2 \setminus S$.
The stereographic projection from the north-pole, π^2_N, maps points from S^2_S to $\mathbb{R}^2$, seen as the plane through the equator of S^2, in the following way.
Let (x^1, x^2, x^3) be a point in S^2. The line through N and this point is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} x^1 \\ x^2 \\ x^3 - 1 \end{pmatrix}.$$

This line intersects the equator-plane in the point where $\lambda = \frac{1}{1-x^3}$: $(\frac{x^1}{1-x^3}, \frac{x^2}{1-x^3}, 0)$.
Now, π^2_N is defined as $\pi^2_N(x^1, x^2, x^3) = (\frac{x^1}{1-x^3}, \frac{x^2}{1-x^3})$. Projection from N projects points from the northern hemisphere outside the equator in the equator-plane and points from the southern-hemisphere inside the equator. On the equator π^2_N is the identity mapping. (Strictly speaking this is incorrect, since π^2_N maps points from one space into another.)

Stereographic projection from the south-pole can be done in a similar way: $\pi_S^2 : S_N^2 \to \mathbb{R}^2$ given by $\pi_S^2(x^1, x^2, x^3) = (\frac{x^1}{1+x^3}, \frac{x^2}{1+x^3})$. The image of the northern-hemisphere is now inside the equator and the image of the southern-hemisphere is outside the equator. On the equator π_S^2 is the identity.

In physics one often encounters continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ that approach the same value, c, in whatever direction, once the distance from the origin in $\mathbb{R}^2$ is large enough. In this case it is possible to add to $\mathbb{R}^2$ one point, ∞, and define $f(\infty) = c$. Thus f can be extended to a function $f : S^2 \to \mathbb{R}$ in a natural way. The same is true for a function $g : \mathbb{R}^3 \to \mathbb{R}$ that under similar conditions can be extended to a function $g : S^3 \to \mathbb{R}$. The domain of the function f and g has been compactified by one point: S^2 is the one point compactification of $\mathbb{R}^2$, S^3 the one point compactification of $\mathbb{R}^3$.
When we consider the inverse of the projection π_N^2, we see that here we do have a mapping from $\mathbb{R}^2$ that, although its values are not in $\mathbb{R}$, approach the same point in S^2 (when S_S^2 is considered as part of S^2) whenever we proceed away from the origin. It is then natural to add to the above definition of π_N^2 the rule $\pi_N^2(0,0,1) \equiv \infty$. We then have $\pi_N^2 : S^2 \to \mathbb{R}^2 \cup \{\infty\} \equiv \mathbb{R}_C^2$. Similar considerations hold for π_S^2.

An equivalent definition of the projection follows from the identification of $\mathbb{R}^2$ with $\mathbb{C}$ via $(x^1, x^2) \leftrightarrow x^1 + ix^2$. This gives $\pi_N^2 : S^2 \to \mathbb{C}_C$ as $\pi_N^2(x^1, x^2, x^3) = \frac{x^1+ix^2}{1-x^3}$, and we can add $\pi_N^2(0,0,1) = \infty$. Furthermore $\pi_S^2 : S^2 \to \mathbb{C}_C$ as $\pi_S^2(x^1, x^2, x^3) = \frac{x^1+ix^2}{1+x^3}$, and $\pi_N^2(0,0,-1) = \infty$.

The inverse of the projection is easily obtained by noting that the line through N and $(x^1, x^2, 0)$ is given by

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} x^1 \\ x^2 \\ -1 \end{pmatrix}.$$

This line intersects the sphere when $(\lambda x^1)^2 + (\lambda x^2)^2 + (1-\lambda)^2 = 1$. The solution for λ is $\frac{2}{1+(x^1)^2+(x^2)^2}$, so:

$$(\pi_N^2)^{-1}(x^1, x^2) = (\frac{2x^1}{1+(x^1)^2+(x^2)^2}, \frac{2x^2}{1+(x^1)^2+(x^2)^2}, \frac{(x^1)^2+(x^2)^2-1}{1+(x^1)^2+(x^2)^2}).$$

We can take $(\pi_N^2)^{-1}(\infty) = (0,0,1)$ to be included in this formula. With the identification $(x^1, x^2) \leftrightarrow x^1 + ix^2 \equiv z$ and $\bar{z} = x^1 - ix^2$ we have:

$$(\pi_N^2)^{-1}(z) = (\frac{z+\bar{z}}{1+z\bar{z}}, \frac{z-\bar{z}}{i(1+z\bar{z})}, \frac{z\bar{z}-1}{1+z\bar{z}}).$$

Similarly:

$(\pi_S^2)^{-1}(x^1,x^2) = (\frac{2x^1}{1+(x^1)^2+(x^2)^2}, \frac{2x^2}{1+(x^1)^2+(x^2)^2}, \frac{1-((x^1)^2+(x^2)^2)}{1+(x^1)^2+(x^2)^2})$, or

$(\pi_S^2)^{-1}(z) = (\frac{z+\bar{z}}{1+z\bar{z}}, \frac{z-\bar{z}}{i(1+z\bar{z})}, \frac{1-z\bar{z}}{1+z\bar{z}})$.

The "z-form" makes calculations very simple, for example $\pi_S^2((\pi_N^2)^{-1}(z)) = \frac{1}{\bar{z}}$.

Very little changes in the calculations and definitions above when we consider the stereographic projection in one dimension higher. Let S^3 be the three sphere in $\mathbb{R}^4$: $S^3 = \{x \in \mathbb{R}^4| \ ||x|| = 1\}$, $N = (0,0,0,1)$ and $S = (0,0,0,-1)$. The projection from N is:

$\pi_N^3(x^1,x^2,x^3,x^4) = (\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4})$.

The stereographic projection from S is:

$\pi_S^3(x^1,x^2,x^3,x^4) = (\frac{x^1}{1+x^4}, \frac{x^2}{1+x^4}, \frac{x^3}{1+x^4})$.

Extension of the definition and compactification of spaces work as before, for example $\pi_N^3(0,0,0,1) = \infty$ and $\pi_N^3 : S^3 \to \mathbb{R}_C^3$. There is however a difference when we would like to consider $\mathbb{C}$ instead of $\mathbb{R}^2$. This cannot be done for the target-space, but for the domain it can: $S^3 = \{(z^1,z^2) \in \mathbb{C}^2| \ ||z^1||^2 + ||z^2||^2 = 1\}$. We identify z^1 with $x^1 + ix^2$ and z^2 with $x^3 + ix^4$ and define $\pi_N^3 : \mathbb{C}^2 \to \mathbb{R}_C^3$ as $\pi_N^3(z^1,z^2) \equiv \pi_N^3(x^1,x^2,x^3,x^4)$.

2.2 the Hopf map

We are now in a position to define a very special mapping, discovered by Heinz Hopf in 1931 [19, 20], from S^3 to S^2. This map, the Hopf map, has played an important role in physics in seemingly unrelated situations ranging from qubits to Taub-NUT spaces in general relativity [21] and it will figure prominently in this thesis:

$$H : S^3 \to S^2,\ H(z^1,z^2) = (\pi_N^2)^{-1}(\frac{z^1}{z^2}). \tag{2.1}$$

Since for an arbitrary point $p \in S^2$ it is always possible to find $(z^1,z^2) \in S^3$ such that $\pi_N^2(p) = \frac{z^1}{z^2}$ and the definition of H involves the ratio $\frac{z^1}{z^2}$ only, the pre-image of (the arbitrary point) p is given by $\{(e^{i\phi}z^1, e^{i\phi}z^2) \in \mathbb{C}^2 \mid \phi \in [0,2\pi]\}$. See figure 2.1 for this and the following discussion. We now investigate the stereographic projection of this pre-image.

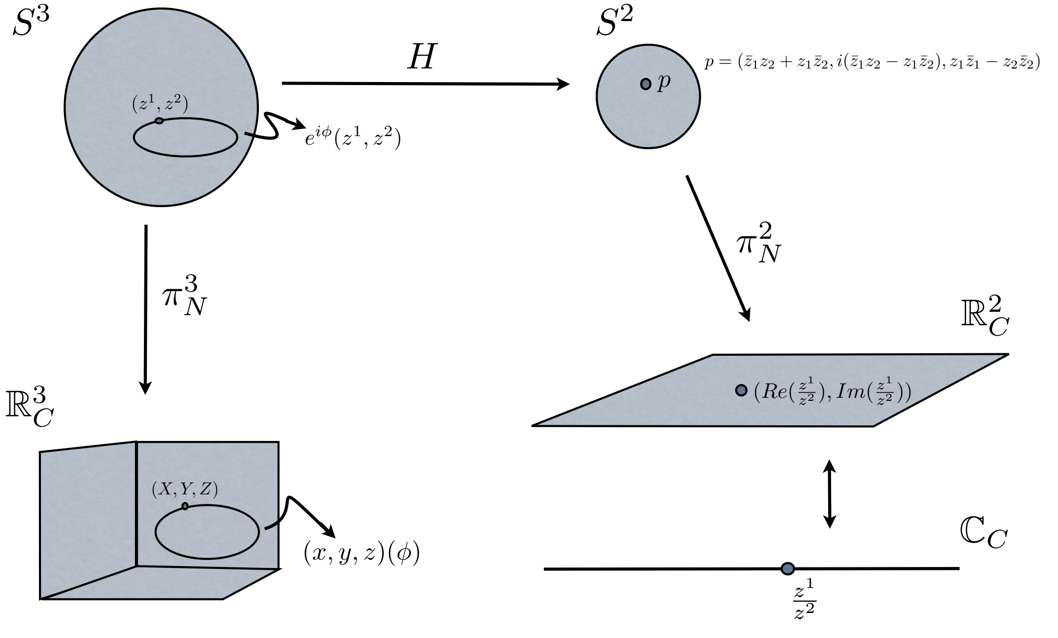

Figure 2.1: Hopf mapping and stereographic projection

We split $e^{i\phi}z^1$ into its real and imaginary parts: $e^{i\phi}z^1 = (x^1\cos\phi - x^2\sin\phi) + i(x^2\cos\phi + x^1\sin\phi)$. Also: $e^{i\phi}z^2 = (x^3\cos\phi - x^4\sin\phi) + i(x^4\cos\phi + x^3\sin\phi)$. Thus: $\pi_N^3(e^{i\phi}z^1, e^{i\phi}z^2) = \pi_N^3(x^1\cos\phi - x^2\sin\phi, x^2\cos\phi + x^1\sin\phi, x^3\cos\phi - x^4\sin\phi, x^4\cos\phi + x^3\sin\phi)$.

The result for $\phi = 0$ is $\pi_N^3(x^1, x^2, x^3, x^4) = (\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4})$ and we write this as (X, Y, Z). A short calculation shows that the result for general ϕ can be expressed in terms of X, Y and Z as follows: $\pi_N^3(e^{i\phi}z^1, e^{i\phi}z^2) = \frac{1}{N}(X\cos\phi - Y\sin\phi, Y\cos\phi + X\sin\phi, Z\cos\phi - \frac{x^4}{1-x^4}\sin\phi)$, where $N = \frac{1}{1-x^4} - \frac{x^4}{1-x^4}\cos\phi - Z\sin\phi$. We write this as $(x(\phi), y(\phi), z(\phi))$ or $(x, y, z)(\phi)$:

$$(x, y, z)(\phi) = \frac{1}{N}(X\cos\phi - Y\sin\phi, Y\cos\phi + X\sin\phi, Z\cos\phi - \frac{x^4}{1-x^4}\sin\phi).$$

This is a closed parametrized curve, say C, through the point $(X, Y, Z) \in \mathbb{R}^3_C$ and every such point will be reached by some $p \in S^2$. This implies that all these curves fill $\mathbb{R}^3_C$, or: any point in $\mathbb{R}^3_C$ is on at least one such curve. In fact there can only be one curve through every point, as is clear from the form of the pre-image of p in S^3. The tangent vector to the curve at $(x, y, z)(\phi)$ is:

$$\partial_\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} (\phi) = \begin{pmatrix} \frac{-X\sin\phi - Y\cos\phi}{N} - \frac{X\cos\phi - Y\sin\phi}{N^2}(\frac{x^4}{1-x^4}\sin\phi - Z\cos\phi) \\ \frac{-Y\sin\phi + X\cos\phi}{N} - \frac{Y\cos\phi + X\sin\phi}{N^2}(\frac{x^4}{1-x^4}\sin\phi - Z\cos\phi) \\ \frac{-Z\sin\phi - \frac{x^4}{1-x^4}\cos\phi}{N} - \frac{Z\cos\phi - \frac{x^4}{1-x^4}\sin\phi}{N^2}(\frac{x^4}{1-x^4}\sin\phi - Z\cos\phi) \end{pmatrix}.$$

It is easy to see that the first two components, when written in terms of x, y and z are $-y + xz$ and $x + yz$ respectively. The second term in the third component is also straightforward, it equals z^2. The first term in the third component is more difficult: it equals $\frac{N - \frac{1}{1-x^4}}{N}$. The third component can thus be written as $1 + z^2 - \frac{1}{N}\frac{1}{1-x^4}$. We proceed to show that it can also be written as $\frac{1}{N}\frac{1}{1-x^4} - x^2 - y^2$, and therefore $\frac{1}{N}\frac{1}{1-x^4} = \frac{1}{2}(1 + x^2 + y^2 + z^2)$:

Third component $= \frac{-Z\sin\phi - \frac{x^4}{1-x^4}\cos\phi}{N} - \frac{Z\cos\phi - \frac{x^4}{1-x^4}\sin\phi}{N^2}(\frac{x^4}{1-x^4}\sin\phi - Z\cos\phi) = \frac{-Z\sin\phi - \frac{x^4}{1-x^4}\cos\phi}{N} + z^2 = \frac{1}{N^2}[-\frac{Z}{1-x^4}\sin\phi - \frac{x^4}{(1-x^4)^2}\cos\phi + 2Z\frac{x^4}{1-x^4}\cos\phi.\sin\phi + (\frac{x^4}{1-x^4})^2\cos^2\phi + Z^2\sin^2\phi] + z^2 = \frac{1}{N^2}[-\frac{Z}{1-x^4}\sin\phi + Z^2 - \frac{x^4}{(1-x^4)^2}\cos\phi + (\frac{x^4}{1-x^4})^2] = \frac{1}{N^2}[-\frac{Z}{1-x^4}\sin\phi - \frac{x^4}{(1-x^4)^2}\cos\phi + \frac{1}{(1-x^4)^2} - X^2 - Y^2] = \frac{1}{N^2}[-\frac{Z}{1-x^4}\sin\phi - \frac{x^4}{(1-x^4)^2}\cos\phi + \frac{1}{(1-x^4)^2}] - x^2 - y^2 = \frac{1}{N}\frac{1}{1-x^4} - x^2 - y^2$.

Combining these results we have the tangent vector at (x, y, z) given by

$$\begin{pmatrix} -y + xz \\ x + yz \\ \frac{1}{2}(1 - x^2 - y^2 + z^2) \end{pmatrix}. \tag{2.2}$$

The integral curves of this vector field are the curves $(x, y, z)(\phi)$ we started with. A similar calculation shows that if instead of using stereographic projection from the north-pole we would have used projection from the south-pole, the tangent vector at (x, y, z) would have been: $\begin{pmatrix} -(y + xz) \\ x - yz \\ \frac{1}{2}(-1 + x^2 + y^2 - z^2) \end{pmatrix}$.

In the definition of H we used stereographic projection from the north-pole. It would have been possible to use projection from the south-pole. But since $\pi^2_S((\pi^2_N)^{-1}(z)) = \frac{1}{\bar{z}}$, it follows that $(\pi^2_S)^{-1}(\frac{\bar{z^2}}{\bar{z^1}}) = (\pi^2_N)^{-1}(\frac{z^1}{z^2})$ and therefore, except for a different relation between the z^i and the x^j, everything in the discussion of integral curves and tangent vector fields would stay the same.

We now investigate in closer detail the tangent vector field, or, what amounts to the same, the integral curves $(x, y, z)(\phi)$. It will be helpful to have a different view of S^3 and with it of $\mathbb{R}^3$, which will be developed first.

2.2.1 $\mathbb{R}^3$ as a collection of nested tori

There is an obvious reparametrization for $S^3 = \{(z^1, z^2) \in \mathbb{C}^2 \mid ||z^1||^2 + ||z^2||^2 = 1\}$, namely if we write $r_1 e^{i\alpha_1}$ for z^1 and $r_2 e^{i\alpha_2}$ for z^2. Of course, $r_1^2 + r_2^2 = 1$, and this implies that there is a $\delta \in [0, \frac{\pi}{2}]$ such that $r_1 = \cos\delta$ and $r_2 = \sin\delta$.

The topology of the set $T \subset S^3$, for which $r_1 = r_2 = \frac{1}{2}\sqrt{2}$ $(\delta = \frac{\pi}{4})$ is that of a torus: $T = \{(\frac{1}{2}\sqrt{2}e^{i\alpha_1}, \frac{1}{2}\sqrt{2}e^{i\alpha_2}) \mid \alpha_1, \alpha_2 \in [0, 2\pi]\}$. According to general theorems, the topology of the stereographic projection of T on $\mathbb{R}^3_C$ is a torus too. This will also become clear by performing the calculation: $T_\pi \equiv \pi^3_N(T) = \pi^3_N(\frac{1}{2}\sqrt{2}e^{i\alpha_1}, \frac{1}{2}\sqrt{2}e^{i\alpha_2}) = (\frac{\cos\alpha_1}{\sqrt{2}-\sin\alpha_2}, \frac{\sin\alpha_1}{\sqrt{2}-\sin\alpha_2}, \frac{\cos\alpha_2}{\sqrt{2}-\sin\alpha_2})$. For any fixed α_2, varying α_1 through its range makes circles around and perpendicular to the z-axis. It suffices therefore to have a closer look at one particular section with α_1 fixed, for example $\alpha_1 = 0$. Now varying α_2 gives a closed loop in the $x > 0$ part of the x-z plane: $(\frac{1}{\sqrt{2}-\sin\alpha_2}, 0, \frac{\cos\alpha_2}{\sqrt{2}-\sin\alpha_2})$. For $\alpha_2 \in [0, \pi]$, change to $\beta \equiv \alpha_2 - \frac{\pi}{2}$ $(\in [-\frac{\pi}{2}, \frac{\pi}{2}])$ and for $\alpha_2 \in [\pi, 2\pi]$ change to $\gamma \equiv \alpha_2 - \frac{3\pi}{2}$ $(\in [-\frac{\pi}{2}, \frac{\pi}{2}])$. For $\beta = 0$ and for $\gamma = 0$ we have two different points on the x-axis. Furthermore, the x-component of T_π is symmetric and the z-component of T_π anti-symmetric for $\beta \to -\beta$ and for $\gamma \to -\gamma$. This completes the picture of the loop in the x-z plane. With suitable choices for a symmetric function $r(\psi)$, it can be summarized as $(1 + r(\psi)\cos\psi, 0, r(\psi)\sin\psi)$ in which ψ is the polar angle as seen from (1,0,0). For T_π we now have $((1 + r(\psi)\cos\phi)\cos\alpha_1, (1 + r(\psi)\cos\phi)\sin\alpha_1, r(\psi)\sin\psi)$, which is the parametrization of a (when r is different from 1, deformed) torus.

For $r_1 = 0$ and $r_2 = 1$ $(\delta = \frac{\pi}{2})$ we have a circle: $\{(0, e^{i\alpha})\} \subset S^3$. Stereographic projection gives $\pi^3_N(0, e^{i\alpha}) = \pi^3_N(0, 0, \cos\alpha, \sin\alpha) = (0, 0, \frac{\cos\alpha}{1-\sin\alpha})$, and this is the z-axis in $\mathbb{R}^3$, or a circle with infinite radius in $\mathbb{R}^3_C$ when the point ∞ is included (and identified, as before, with -∞).

For $r_1 = 1$ and $r_2 = 0$ $(\delta = 0)$ we have a circle $\{(e^{i\beta}, 0)\} \subset S^3$. Stereographic projection now gives $\pi^3_N(e^{i\beta}, 0) = \pi^3_N(\cos\beta, \sin\beta, 0, 0) = (\cos\beta, \sin\beta, 0)$, the unit circle in the x-y plane. Note that this circle lies within the aforementioned torus and is identical with the degenerate torus, for which the function $r(\psi)$ is zero.

For any fixed r_1, r_2 with $r_1 > r_2$ $(\delta \in \langle 0, \frac{\pi}{4}\rangle)$, continuity demands that, after stereographic projection, we have a torus (in $\mathbb{R}^3$) contained within T_π

($\delta = \frac{\pi}{4}$) and containing the unit circle in the x-y plane ($\delta = 0$). A similar calculation as before ($\delta = \frac{\pi}{4}$) confirms this picture: for the closed loop in the x-z plane we now have $(\frac{\cos\delta}{1-\sin\delta.\sin\alpha_2}, 0, \frac{\cos\alpha_2}{\frac{1}{\sin\delta}-\sin\alpha_2})$ instead of $(\frac{1}{\sqrt{2}-\sin\alpha_2}, 0, \frac{\cos\alpha_2}{\sqrt{2}-\sin\alpha_2})$. The x-coordinate of this loop varies between $\frac{\cos\delta}{1+\sin\delta}$ and $\frac{\cos\delta}{1-\sin\delta}$. For $\delta \in \langle 0, \frac{\pi}{4}\rangle$ this interval encloses 1 and is enclosed by the previous interval for the x-coordinate, $[\frac{1}{\sqrt{2}+1}, \frac{1}{\sqrt{2}-1}]$. For every x in the new interval $[\frac{\cos\delta}{1+\sin\delta}, \frac{\cos\delta}{1-\sin\delta}]$ we want to check whether the new loop is closer to the x-axis than the corresponding loop for $\delta = \frac{\pi}{4}$. This is done as follows. A particular $x \in [\frac{\cos\delta}{1+\sin\delta}, \frac{\cos\delta}{1-\sin\delta}]$ corresponds to (possibly two) particular value(s) of α_2, say α_2^{new}, for the new loop: $x = \frac{\cos\delta}{1-\sin\delta.\sin\alpha_2^{new}}$. The same x for the ($\delta = \frac{\pi}{4}$)-loop corresponds to, say, α_2^{old}: $x = \frac{1}{\sqrt{2}-\sin\alpha_2^{old}}$. Thus: $\sin\alpha_2^{old} = \sqrt{2} - \frac{1-\sin\delta.\sin\alpha_2^{new}}{\cos\delta}$. We should now check $|z^{new}(\alpha_2^{new})| < |z^{old}(\alpha_2^{old})|$, where $z^{new/old}$ is on the new/old loop: $|\frac{\sin\delta.\cos\alpha_2^{new}}{1-\sin\delta.\sin\alpha_2^{new}}| < |\frac{\cos\alpha_2^{old}}{\sqrt{2}-\sin\alpha_2^{old}}| = |\frac{\cos\alpha_2^{old}.\cos\delta}{1-\sin\delta.\sin\alpha_2^{new}}| \Leftrightarrow |\sin\delta.\cos\alpha_2^{new}| < |\cos\delta.\cos\alpha_2^{old}|$. Squaring, using $\sin^2\alpha_2 + \cos^2\alpha_2 = 1$ and $\cos\delta > \frac{1}{2}\sqrt{2}$ leads ultimately to $|\sin\delta.\sin\alpha_2^{new}| < 1$, which is indeed satisfied.
This shows that it is possible to rewrite the new loop as $(1 + r^{new}(\psi)\cos\psi, 0, r^{new}(\psi)\sin\psi)$ in which the function r^{new}takes values between 0 and r^{old} for every ψ.

Note that for $\delta \in \langle 0, \frac{\pi}{4}\rangle$ we have $\frac{1}{\sin\delta} > \sqrt{2}$ and $|z^{new}(\alpha_2)| < |z^{old}(\alpha_2)|$ for every α_2. But this is not enough to conclude that the new loop is inside the old loop everywhere, since the same value for α_2 in both cases do correspond to different x values and also to different ψ values.

The remaining case where $r_2 > r_1$, or $\delta \in \langle \frac{\pi}{4}, \frac{\pi}{2}\rangle$, is now easy to guess. For any fixed δ the projection will be a (deformed) torus that contains torus T_π, and for $\delta \downarrow \frac{\pi}{4}$ the projection will approach T_π. For $\delta \uparrow \frac{\pi}{2}$ it will approach the z-axis (or circle in $\mathbb{R}^3_C$). With very minor changes (interchange of $<$ and $>$) the previous calculation can be repeated in order to confirm this guess.

In summary, we have seen that S^3 consists of a set of tori that after stereographic projection lead to a set of nested tori that fill all of $\mathbb{R}^3$. Starting with a degenerate torus for $\delta = 0$, the unit circle in the x-y plane, for increasing δ we get bigger and bigger tori, each containing the previous one. When δ approaches $\frac{\pi}{2}$ the tori approach the z-axis, that can be seen as a degenerate torus. This infinite or degenerate torus itself is reached for $\delta = \frac{\pi}{2}$.

2.2.2 Hopf fibration

From the foregoing it now follows that the curve C, the pre-image of the point p in S^2 for which $\pi_N^2(p) = \frac{z^1}{z^2}$ after stereographic projection, is fully contained within the torus that corresponds to $\delta = \tan^{-1}|\frac{z^2}{z^1}|$. The collection of all the curves that stem from $q \in S^2$ with $|\pi_N^2(q)| = |\frac{z^1}{z^2}|$ fill up this torus completely, since every point will at least be reached once. (Two such curves will either coincide completely or have no point in common, as we have seen before). It is obvious that these q form a circle in S^2 that is parallel to the equator. Thus the equator-parallel circles in S^2 correspond to the tori in $\mathbb{R}^3$ (or $\mathbb{R}^3_C$). It is readily verified that the degenerate circle (0,0,-1) corresponds to the z-axis, the equator to T_π and the degenerate circle (0,0,1) to the unit circle in the x-y plane. Furthermore, the (non-degenerate) equator-parallel circles in the southern hemisphere correspond to tori that contain T_π, those of the northern hemisphere to tori contained by T_π.

The curve C that contains the point (X, Y, Z) for $\phi = 0$ lies in the plane $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \mu \begin{pmatrix} Y \\ -X \\ \frac{x^4}{1-x^4} \end{pmatrix}$, as can be seen by taking $\lambda = \frac{\cos\phi}{N}$ and $\mu = -\frac{\sin\phi}{N}$. C is therefore the intersection of this plane with the torus that corresponds to $\delta = \tan^{-1}|\frac{z^1}{z^2}|$. Since C also contains the point $\frac{1-x^4}{1+x^4}(-X, -Y, -Z)$ for $\phi = \pi$ (note: $\frac{1-x^4}{1+x^4} > 0$), C must be a Villarceau circle: depending on the orientation of the plane, the intersection of a plane through the origin with a torus with centre the origin consists of two circles that are both on opposite sides of the torus-symmetry axes, two concentric circles with the origin as centre, or two Villarceau circles [22–24]. The fact that (X, Y, Z) together with $\frac{1-x^4}{1+x^4}(-X, -Y, -Z)$ are on C excludes the first two possibilities, and C must be one of the Villarceau circles. The case $x^4 = 1$ corresponds to the curve through ∞ (the z-axis) and the case $x^4 = 0$ corresponds to the curve $(x, y, z)(\phi) = \frac{1}{1-Z\sin\phi}(X\cos\phi - Y\sin\phi, Y\cos\phi + X\sin\phi, Z\cos\phi)$. For $|Z| = 1$ this again is the z-axis and for $|Z| < 1$, although (X, Y, Z) and also $(-X, -Y, -Z)$ are on the curve, the circle is not centred around the origin, since for $\phi = \frac{\pi}{2}$ we have $\frac{1}{1-Z}(-Y, X, 0)$ whereas for $\phi = \frac{3\pi}{2}$ we have $\frac{1}{1+Z}(Y, -X, 0)$ ($Z = 0$ gives the unit circle in the x-y plane).

Often one speaks of the Hopf fibration, and we can now understand where this comes from. S^3 can be thought of as base space S^2 with fibers S^1, since every point in S^3 is specified by one particular $p \in S^2$ and a $\phi \in [0, 2\pi\rangle$ and this interval can be identified with S^1. Locally we can write $S^3 = S^2 \times S^1$. Globally this is not correct.

2.2.3 linking

There is one more important property of the curves $(x, y, z)(\phi)$ that concerns the interconnection between any two of them. The analogues in S^3 (via $(\pi_N^3)^{-1}$) of the unit circle in the x-y plane and the z-axis are two circles that are linked once. The concept of linking is clear intuitively [25]. Stereographic projection does not change this linking property. Via the Hopf map and π_N^2 these two circles correspond to two specific points in $\mathbb{C}_C$ (or $\mathbb{R}^2_C$), say p and q. Two arbitrary points, not being p or q, can with the help of a suitably defined mapping be mapped to p and q respectively. This mapping induces a mapping in S^3 by which the curves (circles) that correspond to the two arbitrary points are being mapped into the analogues of the x-axis and unit circle in the x-y plane. It is intuitively clear that this mapping does not change the linking between the curves. Thus the two circles in S^3 (or $\mathbb{R}^3$) that are the pre-images of two arbitrary points in S^2 are also linked once. (These statements can be made mathematically precise with the help of algebraic topology [26].)

2.2.4 conclusion

We end this chapter by stating that a study of the Hopf map leads naturally to a picture of $\mathbb{R}^3$ as a collection of disjoined and singly linked circles, organised on nested tori, a truly remarkable and beautiful picture!

3 Electrodynamics

Among all the great theories of physics, the one completed by Maxwell about a century and a half ago takes a very special place. Maxwell's equations show that the speed of light is the same in all inertial reference frames. Whereas many tried to resolve this "problem" Einstein derived special relativity from it. It serves as a paradigm in gauge field theory and it is here that almost everyone starts learning about that. Besides being aesthetically very appealing, it is very hard to overrate its technological significance or its value for society in general (try to imagine a world without knowledge of electricity). Accordingly, the effort by engineers and scientists to solve the central equations of this theory, to understand and make use of these solutions has been huge. The more surprising it is that even now amazing new solutions are being found. [1,2]. In the next chapter we will have a closer look at one such solution in which the Hopf fibration plays an important role.

In this chapter we want to rewrite Maxwell's equations in the language of spinors that was introduced in chapter 1.

3.1 Maxwell equations

In nondimensionalized (Heaviside-Lorentz, c=1) units Maxwell's equations are:

$$\nabla \cdot \mathbf{E} = \rho \tag{3.1}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} \tag{3.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.3}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{3.4}$$

The electromagnetic field $(\mathbf{E}, \mathbf{B})$ can be characterized in terms of potentials ϕ and $\mathbf{A}$ as follows:

$\mathbf{B} = \nabla \times \mathbf{A}$
$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$.

Associated with $(\mathbf{E}, \mathbf{B})$ are the energy density w, energy flux (Poynting vector) $\mathbf{S}$, momentum density $\mathbf{p}$ and stress tensor σ^{ij} given by:

$$w = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} \tag{3.5}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} \tag{3.6}$$

$$\mathbf{p} = \mathbf{E} \times \mathbf{B} \tag{3.7}$$

$$\sigma^{ij} = -E^i E^j - B^i B^j + \delta^{ij} \frac{\mathbf{E}^2 + \mathbf{B}^2}{2}. \tag{3.8}$$

Here, as elsewhere, $i, j, k \in [1, 2, 3]$ and Greek indices will indicate an element from $[0, 1, 2, 3]$, except when it is used for a twistor (see section **1.3**).
An electromagnetic field that satisfies $|\mathbf{E}| = |\mathbf{B}|$ and $\mathbf{E} \cdot \mathbf{B} = 0$ is called null.

3.2 Riemann-Silberstein vector

A convenient way to rewrite Maxwell's equations is with the help of the Riemann-Silberstein vector $\mathbf{F}^{RS} = \mathbf{E} + i\mathbf{B}$. It's complex conjugate will be denoted by $\mathbf{F}^{RS*}$. Maxwell's equations become

$$\nabla \cdot \mathbf{F}^{RS} = \rho, \qquad \partial_t \mathbf{F}^{RS} = -i\nabla \times \mathbf{F}^{RS} - \mathbf{j}. \tag{3.9}$$

The Poynting vector and energy density of the electromagnetic field can be written as

$\mathbf{S} = \frac{i}{2}\mathbf{F}^{RS} \times \mathbf{F}^{RS*}$ and $w = \frac{1}{2}\mathbf{F}^{RS} \cdot \mathbf{F}^{RS*}$.

A null electromagnetic field satisfies $\mathbf{F}^{RS} \cdot \mathbf{F}^{RS} = 0$.

3.3 Faraday tensor

Maxwell's equations can be written in manifest Lorentz covariant form with the help of the electromagnetic field tensor (or Faraday tensor):

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{3.10}$$

and the (four-)current density $(\rho, \mathbf{j})$ as follows:

$$\partial_\nu F^{\mu\nu} = -j^\mu \tag{3.11}$$

and

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0. \tag{3.12}$$

In terms of the (four-)potential $(\phi, \mathbf{A})$ we can write $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

Other forms of the electromagnetic field tensor are:

$$(F^\mu{}_\nu) = (F^{\mu\beta}\eta_{\beta\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

$$(F_{\mu\nu}) = (\eta_{\mu\alpha}F^{\alpha\beta}\eta_{\beta\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

$$((*F)^{\mu\nu}) = (\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}) = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

It follows that $F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{E}^2 - \mathbf{B}^2)$ and $(*F)^{\mu\nu}F_{\mu\nu} = -\mathbf{E}\cdot\mathbf{B}$ and therefore we have for a null electromagnetic field: $F_{\mu\nu}F^{\mu\nu} = 0 = F_{\mu\nu}(*F)^{\mu\nu}$.

The energy density, energy flux, momentum density and stress tensor ((3.5) - (3.8)) are combined in the symmetrical energy-momentum tensor of the electromagnetic field

$$T^{\mu\nu} = -F^{\mu\alpha}F^\nu{}_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = \begin{pmatrix} w & \mathbf{S} \\ \mathbf{S} & \boldsymbol{\sigma} \end{pmatrix} \tag{3.13}$$

Note that (3.12) does not contain 4^3 independent equations, but only 4 as becomes clear by rewriting this equation in the form $\epsilon_{\alpha\beta\gamma\delta}\partial^\beta F^{\gamma\delta} = 0$. This in turn can be written in terms of the dual of $F^{\mu\nu}$: $\partial^\beta(*F)_{\alpha\beta} = 0$, or

$$\partial_\nu(*F)^{\mu\nu} = 0$$

and this is like the source-free form of (3.11). Therefore, with the definition

$$\mathcal{F}^{\mu\nu} = F^{\mu\nu} + i(*F)^{\mu\nu}$$

all of Maxwell's equations (3.1) - (3.4) can be written succinctly as

$$\partial_\nu \mathcal{F}^{\mu\nu} = -j^\mu. \tag{3.14}$$

In terms of the Riemann-Silberstein vector we have

$$(\mathcal{F}^{\mu\nu}) = \begin{pmatrix} 0 & -F_x^{RS*} & -F_y^{RS*} & -F_z^{RS*} \\ F_x^{RS*} & 0 & -iF_z^{RS*} & iF_y^{RS*} \\ F_y^{RS*} & iF_z^{RS*} & 0 & -iF_x^{RS*} \\ F_z^{RS*} & -iF_y^{RS*} & iF_x^{RS*} & 0 \end{pmatrix}. \tag{3.15}$$

3.4 Maxwell spinor

Since the electromagnetic field tensor is anti-symmetric, we know from (1.7) that its spinor equivalent can be written as

$F_{AX'BY'} = \phi_{AB}\epsilon_{X'Y'} + \bar{\phi}_{X'Y'}\epsilon_{AB}, \quad \phi_{AB} = \frac{1}{2}F_{AR'B}{}^{R'} = \phi_{BA}.$

This ϕ_{AB} is called Maxwell spinor. From the formula for the dual of an anti-symmetric tensor (1.8) we find for the spinor equivalent of $\mathcal{F}_{\mu\nu}$:

$\frac{1}{2}\mathcal{F}_{AX'BY'} = \phi_{AB}\epsilon_{X'Y'}$

and for Maxwell's equations (3.14) without source term (after some index manipulation):

$$\nabla^{AX'}\phi_{AB} = 0. \tag{3.16}$$

We see that when ϕ_{AB} is a solution, $f\phi_{AB}$ is also a solution whenever $\alpha^A\nabla_{AX'}f = 0$. It is a simple exercise to show that for a null electromagnetic field the principal spinors and thus the principal null directions of ϕ_{AB} coincide: $\phi_{AB} = \alpha_A\alpha_B$ and therefore $\phi_{AB}\phi^{AB} = 0$. In this case equation (3.16) implies $\alpha^A\alpha^B\nabla_{AX'}\alpha_B = 0$, as can be seen by substitution and contraction with α^B. Thus the congruence defined by α is geodesic and shear free (see (1.9)).

In anticipation of the Petrov classification in general relativity (section **7.6**) it may be remarked here that the Maxwell spinor gives rise to a simple classification of electromagnetic fields. For this we use the two principal null directions (PND) of the symmetric Maxwell spinor:

regular electromagnetic field: the PND are different,
null electromagnetic field: the PND coincide.

From section **1.2** we know that the principal null directions of the Maxwell spinor correspond to the eigenvectors of the Faraday tensor. The two eigenvalues turn out to be $\lambda_\pm = \pm\sqrt{-\frac{1}{4}(|\mathbf{B}|^2 - |\mathbf{E}|^2) - \frac{i}{2}\mathbf{E}\,.\,\mathbf{B}}$. As can be seen using equation (4.8) from chapter 4, this is also the solution for to the following equation: ${\phi_A}^B\beta_B = \lambda_\pm\beta_A$. Combining previous results, we see that:

field is null $\leftrightarrow$ $|\mathbf{B}| = |\mathbf{E}|,\ \mathbf{E}\,.\,\mathbf{B} = 0$ $\leftrightarrow$ $\lambda_\pm = 0$ $\leftrightarrow$ $\phi_{AB} = \alpha_A\alpha_B$.

The first arrow is the definition of section **3.1**, the second arrow is a result from this section, and the last arrow follows from section **1.2**.

3.5 simple examples of Maxwell spinors

We now give some simple examples in which the dependence on the space-time coordinates only appears in a common factor. Except in the last example, all fields are circularly polarized.

plane wave in -z direction

$(\phi_{AB}(x^\mu)) = f(x^\mu)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$:

This is a solution of (3.16) if $f(x^\mu)$ satisfies $\partial_t f = \partial_z f$ and $(\partial_x + i\partial_y)f = 0$. For continuous f the last equation implies that it is holomorphic in $x + iy$. The first equation is satisfied whenever f is a function of the combination $t + z$. For $f(x^\mu) = e^{ik_\mu x^\mu}$ and $(k_\mu) = (k, 0, 0, k)$ with $k > 0$ the electromagnetic field is a right circularly polarized plane wave travelling in the -z direction. Using $f(x^\mu) = e^{-ik_\mu x^\mu}$ with the same (k_μ) leads to a left circularly polarized plane wave propagating in the same direction.

plane wave in +z direction

When instead of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ we use $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ the function $f(x^\mu)$ must be anti-holomorphic (a function of the combination $x - iy$ and not $x + iy$) and with respect to t and z it can only depend on the combination $t - z$. Using the same $f(x^\mu)$ as before, but now with $(k_\mu) = (k, 0, 0, -k)$, leads to plane waves travelling in the +z direction. The polarization interpretation is the same as before.

plane wave in -x direction

$(\phi_{AB}(x^\mu)) = f(x^\mu) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$:

Now $f(x^\mu)$ has to satisfy $(\partial_t - \partial_z - \partial_x + i\partial_y)f = 0$ and $(-\partial_x - i\partial_y + \partial_t + \partial_z)f = 0$, which is equivalent to $\partial_t f = \partial_x f$ and $(\partial_z - i\partial_y)f = 0$. So $f(x^\mu)$ is a function of $z - iy$ and $t + x$. Again, we can use the same $f(x^\mu)$ as before, now with $(k_\mu) = (k, k, 0, 0)$ and we have a plane wave travelling in the -x direction with amplitude twice as big as before. The interpretation of polarization is the same as in the previous cases.

plane wave in +x direction

When instead of using $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ we use $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ the function $f(x^\mu)$ depends on $z - iy$ and $t - x$. Again using the same $f(x^\mu)$ leads to propagation in the x direction and polarizations as before.

plane wave in $\pm$ y direction

$(\phi_{AB}(x^\mu)) = f(x^\mu) \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$:

Now f is a function of $x + iz$ and $t + y$. The same f as before with $(k_\mu) = (k, 0, k, 0)$ leads to plane waves propagating in the -y direction and polarizations as before. Complex conjugation of the matrix entries leads to propagation in the y direction.

constant fields

Note that $(\phi_{AB}(x^\mu)) = f(x^\mu)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $(\phi_{AB}(x^\mu)) = f(x^\mu)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ are only possible for constant f and leads to a constant electric field without magnetic field and a constant magnetic field without an electric field respectively.

4

Twistors and electromagnetic knots

4.1 electromagnetic field with Hopf-structure

In 1989 A.F. Rañada published an article [2] in which he showed that solutions to source free Maxwell's equations can be obtained from functions $\phi, \theta : S^3 \to S^2$ that satisfy a duality relation. By considering the Hopf map, he arrived at an electromagnetic field where the magnetic field lines are the closed curves C from chapter 2. The electric field is everywhere perpendicular to the magnetic field and the electric field lines also form a Hopf fibration. We will refer to such a configuration as an electromagnetic knot [1,4].
Here we will shortly summarize that part of Rañada's paper that will be of importance to us in the sequel, supplemented with explicit calculations for the electromagnetic field.

Start with two complex functions, $\phi(t, \mathbf{r})$ and $\theta(t, \mathbf{r})$, defined on the three-sphere. Let

$$f_{\mu\nu}(\chi) \equiv \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu \chi^* \partial_\nu \chi - \partial_\nu \chi^* \partial_\mu \chi}{(1+\chi^* \chi)^2}.$$

Define $F_{\mu\nu} = f_{\mu\nu}(\phi)$ and $G_{\mu\nu} = f_{\mu\nu}(\theta)$ and suppose that F is the dual of G:

$$F_{\mu\nu} = (*G)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}.$$

When we apply ∂_β to $F_{\gamma\delta}$ we get sums of terms symmetrical in β and γ or δ. Thus $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0$. For the same reason we have $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta G_{\gamma\delta} = 0$.

Therefore, F and G satisfy the homogeneous Maxwell equations, equation (3.12). But $0 = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\partial^{\beta}G^{\gamma\delta} = \partial^{\beta}(*G)_{\alpha\beta} = \partial^{\beta}F_{\alpha\beta}$. Thus F also satisfies the source-free inhomogeneous Maxwell equation, equation (3.11). Interchange of the role of F and G in this argument shows that G also satisfies all of the (source-free) Maxwell equations.

Rewriting Maxwell's equations in the Lagrange formalism, using the duality condition as a constraint, shows that it is only necessary to have the functions ϕ and θ for $t = 0$, because the duality relation, which takes the form (or better: that implies) $(\nabla\phi^* \times \nabla\phi) \cdot (\nabla\theta^* \times \nabla\theta) = 0$ now, remains valid once it is for $t = 0$. See the appendix to this chapter for more details.

So all we need to make this prescription work are two functions, $\phi, \theta : S^3 \to S^2$ that satisfy $(\nabla\phi^* \times \nabla\phi) \cdot (\nabla\theta^* \times \nabla\theta) = 0$. The first function that comes to mind is the Hopf map $H(z^1, z^2) = (\pi_N^2)^{-1}(\frac{z^1}{z^2})$ (2.1). If we stereographically project (from N) S^3 onto $\mathbb{R}^3$ and S^2 onto $\mathbb{R}^2$, this function becomes $\phi_H(\mathbf{r}) = \phi_H(x, y, z) = \frac{2(x+iy)}{2z+i(r^2-1)}$. This follows from $\pi_N^3(z^1, z^2) = \pi_N^3(x^1, x^2, x^3, x^4) = (\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4}) \equiv (x, y, z)$ and $\frac{z^1}{z^2} = \frac{x^1+ix^2}{x^3+ix^4} = \frac{x+iy}{z+i\frac{x^4}{1-x^4}}$. With $r^2 = x^2 + y^2 + z^2$ we have $r^2 - 1 = \frac{2x^4}{1-x^4}$ since $||(x^1, x^2, x^3, x^4)|| = 1$. A straightforward calculation shows that $\mathbf{B}(t = 0, \mathbf{r}) \propto \nabla\phi_H^* \times \nabla\phi_H \propto \begin{pmatrix} 2(xz - y) \\ 2(x + yz) \\ 1 - x^2 - y^2 + z^2 \end{pmatrix}$. As expected the magnetic field lines coincide with the integral curves C (2.2) from chapter 2. We still need a function $\theta(0, \mathbf{r})$ that solves the duality relation. It is natural to try another Hopf structure, rotated with respect to the previous one. The level curves of the function $\theta_H(x, y, z) = \phi_H^*(y, z, x)$ certainly exhibit a Hopf structure (level curves of ϕ^* coincide with those of ϕ , but different numbers are attached to the same curves). This θ_H gives $\mathbf{E}(t = 0, \mathbf{r}) \propto \nabla\theta_H^* \times \nabla\theta_H \propto \begin{pmatrix} -1 - x^2 + y^2 + z^2 \\ 2(z - xy) \\ -2(y + xz) \end{pmatrix}$ and it follows that it solves the duality relation $(\nabla\phi_H^* \times \nabla\phi_h) \cdot (\nabla\theta_H^* \times \nabla\theta_H) = 0$. Therefore we finally have a solution to Maxwell's equations in vacuum in which at $t = 0$ the electric and magnetic field lines are perpendicular Hopf fibrations. In particular, any two electric (or magnetic) field lines are (possibly degenerate) circles that are linked once. Although the smooth time development according to Maxwell's equations deforms the circles into intricate closed curves, the linking property is not broken.

4.2 relation between non-null twistors and electromagnetic knots

A detailed calculation shows that the electric and magnetic field at $t = 0$ are given by the following expressions:

$$\mathbf{E}(0,\mathbf{r}) = \tfrac{1}{(1+r^2)^3}\begin{pmatrix} -1 - x^2 + y^2 + z^2 \\ 2(z - xy) \\ -2(y + xz) \end{pmatrix},$$

$$\mathbf{B}(0,\mathbf{r}) = \tfrac{1}{(1+r^2)^3}\begin{pmatrix} 2(xz - y) \\ 2(x + yz) \\ 1 - x^2 - y^2 + z^2 \end{pmatrix}.$$

As can be expected, the expression for both fields for arbitrary t are rather intricate, but if we combine these to form the Riemann-Silberstein vector we have a simple expression:

$$\mathbf{F}^{RS}(t,\mathbf{r}) = \tfrac{1}{((t-i)^2-r^2)^3}\begin{pmatrix} (x - iz)^2 - (t - i + y)^2 \\ 2(x - iz)(t - i + y) \\ i(x - iz)^2 + i(t - i + y)^2 \end{pmatrix}$$

(again, the speed of light has been be put equal to 1). It is easy to check that the field is null.
However, in the sequel we will often use a rotated version of this field configuration, a(n active) rotation around the x-axis, such that the direction of the z-axis goes over into the direction of the y-axis:

$$\mathbf{F}^{RS}(t,\mathbf{r}) = \frac{1}{((t-i)^2 - r^2)^3}\begin{pmatrix} (x - iy)^2 - (t - i - z)^2 \\ i(x - iy)^2 + i(t - i - z)^2 \\ -2(x - iy)(t - i - z) \end{pmatrix} \tag{4.1}$$

From this it is immediately clear that the separate expressions for the fields become simple only for $t = 0$, the instant of time when the electric-, magnetic- and also the Poynting vector field all have the structure of a Hopf fibration. The general formula for the Poynting vector in terms of the Riemann-Silberstein vector

$$\mathbf{S} = \tfrac{-i}{2}\mathbf{F}^{RS*} \times \mathbf{F}^{RS}$$

gives a tractable expression for all times:

$$\mathbf{S} = \frac{(1 + x^2 + y^2 + (t - z)^2)}{((t^2 - r^2)^2 + 1 + 2(t^2 + r^2))^3}\begin{pmatrix} 2(x(t - z) + y) \\ 2(y(t - z) - x) \\ x^2 + y^2 - (t - z)^2 - 1 \end{pmatrix} \tag{4.2}$$

This shows that the *structure* (i.e. a picture of the field lines) of this vector field moves undistorted along the z-axis with the speed of light (see figure 4.1). This closely resembles the geometrical picture of a non-null twistor described at the end of chapter 1. We now present a deeper correspondence between a non-null twistor and the knotted electromagnetic field, in which the Poynting vector plays a central role. But first we note that equation (4.1) implies that the electric field can be written as a combination of two vector fields that both have the structure of a Hopf fibration that moves undistorted along the z-axis. A different combination of these two fields yield the magnetic field (with $f = f(x^\mu)$):

$$\mathbf{E}((t,\mathbf{r}) = Re(f)\begin{pmatrix} -1-x^2+y^2+(t-z)^2 \\ -2((t-z)+xy) \\ 2(-y+x(t-z)) \end{pmatrix} - Im(f)\begin{pmatrix} 2(xy-(t-z)) \\ 1-x^2+y^2-(t-z)^2 \\ -2(x+y(t-z)) \end{pmatrix} \tag{4.3}$$

$$\mathbf{B}((t,\mathbf{r}) = Re(f)\begin{pmatrix} 2(xy-(t-z)) \\ 1-x^2+y^2-(t-z)^2 \\ -2(x+y(t-z)) \end{pmatrix} + Im(f)\begin{pmatrix} -1-x^2+y^2+(t-z)^2 \\ -2((t-z)+xy) \\ 2(-y+x(t-z)) \end{pmatrix}. \tag{4.4}$$

We therefore have three orthogonal Hopf fibrations moving with the speed of light along the z-axis, one of which corresponds with the Poynting vector, and different space and time dependent combinations of the other two make up the electric and magnetic field.

The Riemann-Silberstein vector (4.1) can also be written as a combination of five time independent vector fields with time dependent coefficients:

$$\mathbf{F}^{RS}(t,\mathbf{r}) = \frac{-(1+r^2)^3}{((t-i)^2-r^2)^3}\{\mathbf{E}_0 + i\mathbf{B}_0 - \frac{4t}{(1+r^2)}\mathbf{A}_0 + i\frac{4t}{(1+r^2)}\mathbf{C}_0 + \frac{t^2}{(1+r^2)^3}\mathbf{W}\}. \tag{4.5}$$

In this expression $\mathbf{A}$ is the vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$, and $\mathbf{C}$ is the vector potential for the electric field (since $\nabla \cdot \mathbf{E} = 0$), $\mathbf{E} = \nabla \times \mathbf{C}$:

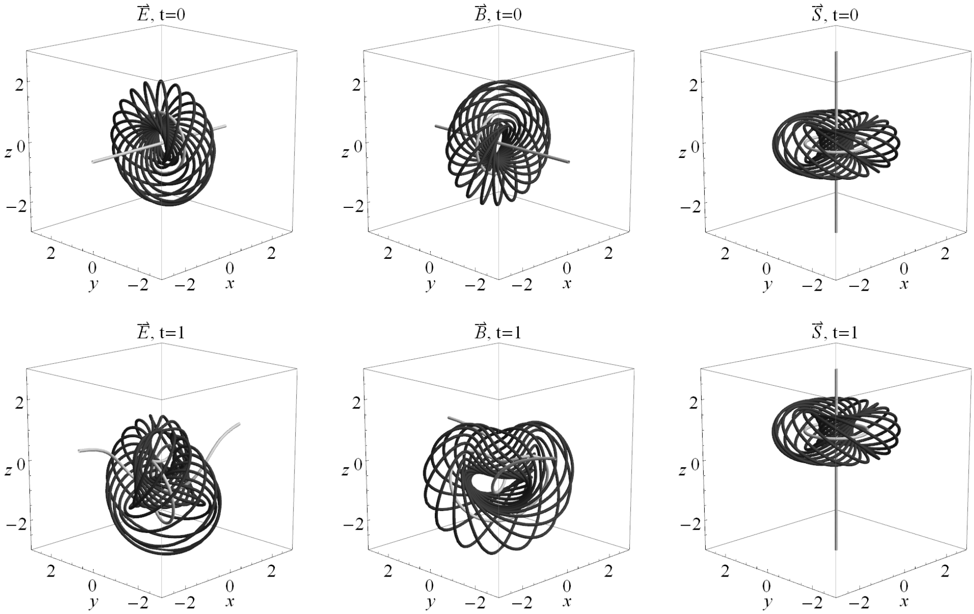

Figure 4.1: Some electric, magnetic and Poynting vector field lines of an electromagnetic knot for t=0 and t=1. At t=0 the (electric, magnetic or Poynting) field lines are singly linked circles on nested tori that fill all of space. At t=0 the structure of all three fields is identical up to rotation. Linking is conserved, but the overall structure of the electric and magnetic field lines gets strongly distorted, whereas the structure of the Poynting vector is independent of time: it moves along the central axis with the speed of light. (picture produced by A. Wickes)

$$\mathbf{A}_0 = \mathbf{A}_{t=0} = \frac{1}{2(1+r^2)^2}\begin{pmatrix} z \\ 1 \\ -x \end{pmatrix}, \mathbf{C}_0 = \mathbf{C}_{t=0} = \frac{1}{2(1+r^2)^2}\begin{pmatrix} -1 \\ z \\ -y \end{pmatrix} \text{ and } \mathbf{W} = \begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix}.$$

Alternatively, $\mathbf{F}^{RS}(t,\mathbf{r}) = \frac{-(1+r^2)^3}{((t-i)^2-r^2)^3}\{\mathbf{F}^{RS}_{t=0} + i\frac{4t}{(1+r^2)}\mathbf{D}_{t=0} + \frac{t^2}{(1+r^2)^3}\mathbf{W}\}$,

in which $\mathbf{D} = \mathbf{C} + i\mathbf{A}$ is the vector potential for $\mathbf{F}^{RS}$. Note that $\mathbf{W}$ corresponds to a constant electromagnetic field, on which we have to say more in chapter 5.

One of the first things one learns when studying twistor theory [3, 27] is that there exists an optical analogy for a null twistor. A null twistor corresponds to a null geodesic in Minkowski space, hence to a light ray. No such direct analogy is given for a non-null twistor, although we have seen that its geometrical picture resembles the Poynting vector of an electromagnetic knot. We proceed

to show that this analogy is not merely a pictorial one.
As was shown in chapter 1, the geometrical picture corresponding to the non-null twistor $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ is defined to be the congruence in M^4 build from all null geodesics (straight lines) that correspond to null twistors $\mathcal{Z}$ orthogonal to $\mathcal{A}$: $\{\mathcal{Z}^\alpha | \bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha = 0\}$. Since $\mathcal{Z}$ must be null, it can be written as $(\mathcal{Z}^\alpha) = (iX^{AY'}\bar{\Pi}_{Y'}, \bar{\Pi}_{X'})$ with $(X^{AY'})$ given by equation (1.8) and $(t,x,y,z) \in M^4 \subset M^4_{\mathbb{C}}$. From $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ it follows that $(\bar{\mathcal{A}}_\alpha) = (0, 1, 0, \frac{1}{\sqrt{2}})$. The orthogonality condition can now be written as $(x - iy)\bar{\Pi}_{1'} + (-i + t - z)\bar{\Pi}_{2'} = 0$ and is solved by

$$(\Pi_1, \Pi_2) = f(x^\mu)(-t + z - i, x + iy), \tag{4.6}$$

in which $f(x^\mu)$ is an arbitrary scalar function that does not affect the integral curves of the corresponding vector field, known as a Robinson congruence, $k^\mu = \sigma^\mu{}_{AX'}\Pi^A\bar{\Pi}^{X'}$. Note the change from $(\bar{\Pi}_{1'}, \bar{\Pi}_{2'})$ to (Π_1, Π_2). According to Kerr's theorem (1.10) this vector field is geodesic and shear-free (in the notation of chapter 1: $w + Yv + iY \equiv 0$ here). These two properties also follow directly from $\Pi^A\Pi^B\nabla_{AX'}\Pi_B = 0$ (1.9), which is easy to check for considered spinor field, or from (1.16) with $f(\mathcal{Z}^\alpha) = \bar{\mathcal{A}}_\alpha\mathcal{Z}^\alpha$. According to a theorem by Robinson [28] it is possible to construct a null electromagnetic field from any geodesic shear free null congruence: a scalar field $\psi(t,x,y,z)$ exists such that the spinor equivalent of the anti-self-dual part of the electromagnetic field tensor is given by

$$\Phi_{AB} = e^\psi \Pi_A \Pi_B. \tag{4.7}$$

(On hindsight, looking back at the discussion following (3.16) in combination with (1.9), we could ourselves have come up with Robinson's theorem, or at least (and somewhat less presumptuous), this theorem should come as no surprise. However, Robinson wrote [28] in 1961. Not much of what is needed for the theorem was known then, and besides, spinors were not used. Even just checking [28] is not so easy a task.)
We now show that an explicit solution can be obtained in the form of an electromagnetic knot. From $F_i^{RS*} = \mathcal{F}_{0i} = \sigma_0{}^{AX'}\sigma_i{}^{BY'}\mathcal{F}_{AX'BY'} = 2\sigma_0{}^{AX'}\sigma_i{}^{BY'}\phi_{AB}\epsilon_{X'Y'}$ and using $(\sigma_0{}^{AX'}) = (\sigma^0{}_{AX'})$, $(\sigma_1{}^{AX'}) = (\sigma^1{}_{AX'})$, $(\sigma_2{}^{AX'}) = -(\sigma^2{}_{AX'})$ and $(\sigma_3{}^{AX'}) = (\sigma^3{}_{AX'})$ together with equation (1.4), we find

$$\mathbf{F}^{RS*} = \begin{pmatrix} \Phi_{00} - \Phi_{11} \\ i(\Phi_{00} + \Phi_{11}) \\ -2\Phi_{01} \end{pmatrix}. \tag{4.8}$$

This leads to the following relation between the Poynting vector corresponding to Φ_{AB} and the vector field $(k^0, \mathbf{k})$ that is related to the Robinson congruence:

$$\mathbf{S} = 2e^{\psi}e^{\psi^c}k^0\mathbf{k} \tag{4.9}$$

$$k^\alpha = \sigma^\alpha{}_{AX'}\Pi^A\bar{\Pi}^{X'}. \tag{4.10}$$

Here we already see that the structure of an electromagnetic knot emerges, since $\mathbf{k}$ is a Hopf fibration that moves with the speed of light along the z-axis. For null electromagnetic fields and given Poynting vector, the only freedom left for the electric and magnetic field is a local duality transformation:

$$\mathbf{F}^{RS\prime} = e^{i\alpha}\mathbf{F}^{RS} \tag{4.11}$$

in which $\alpha(x^\mu)$ stisfies

$$\nabla\alpha = \beta(x^\mu)\mathbf{S} \tag{4.12}$$

and

$$\partial_t\alpha = -\beta(x^\mu)||\mathbf{S}|| \tag{4.13}$$

for $\beta(x^\mu)$ an arbitrary function. This follows from writing out Maxwell's equations for $\mathbf{F}^{RS\prime}$ and for $\mathbf{F}^{RS}$ (3.9). For $\beta = 0$, α is constant and the duality transformation is global. For β constant, α satisfies the homogeneous wave equation as follows from the for null fields valid equation $\nabla \cdot \mathbf{S} = -\partial_t||\mathbf{S}||$.
The general formula for the energy density, ϵ, in terms of the Riemann-Silberstein

vector

$\epsilon = \frac{1}{2}\mathbf{F}^{RS*} \cdot \mathbf{F}^{RS}$

when combined with equation (4.7) leads to

$\epsilon = 2e^{\psi}e^{\psi^c}(k^0)^2.$

Together with equation (4.9), this gives

$(E_{field}, \mathbf{P}_{field}) = 2\int d\mathbf{r} e^{\psi}e^{\psi^c}k^0(k^0, \mathbf{k})$

which for source-free fields is an (energy-momentum) four-vector [29].
Combining equations (4.6), (4.7) and (4.8) leads directly to the electromagnetic field that corresponds to the twistor $(\mathcal{A}^{\alpha})$:

$$\mathbf{F}^{RS} = g(x^{\mu}) \begin{pmatrix} (x-iy)^2 - (t-i-z)^2 \\ i(x-iy)^2 + i(t-i-z)^2 \\ -2(x-iy)(t-i-z) \end{pmatrix}$$

in which $g(x^{\mu})$ is a function built from $f(x^{\mu})$ and ψ. When compared with equation (4.1), we see that with the choice

$g(x^{\mu}) = \frac{1}{((t-i)^2 - r^2)^3},$

we indeed do get a solution to Maxwell's equations. The mathematical relation between a non-null twistor and an electromagnetic knot is hereby established. However, we still do need to investigate the freedom left in the choice of $g(x^{\mu})$.

4.2.1 relation is not unique

Any solution of Maxwell's equations with the property of having a Poynting vector that has the structure of a Hopf fibration that moves undistorted with the speed of light along the central axis will be referred to as an electromagnetic Hopf knot. From equations (4.6), (4.9) and (4.10), it is clear that any choice of $g(x^{\mu})$ that is allowed by Maxwell's equations corresponds to such a knot. We have seen that among the solutions is the particular knot given by equation (4.1), say $\mathbf{F}^{part.}$ and corresponding $\mathbf{S}^{part.}$ (4.2). From equations (4.6)-(4.10) we see that for any other solution there exist functions $\tilde{\psi}$ and $\tilde{f}$ such that

$\mathbf{F}^{RS} = h(x^{\mu})\mathbf{F}^{part.}, \quad \mathbf{S} = h(x^{\mu})h^*(x^{\mu})\mathbf{S}^{part.} \text{ and } h(x^{\mu}) = e^{\tilde{\psi}}(\tilde{f})^2.$

Again, we see explicitly that all solutions are electromagnetic Hopf knots.

Those that have $\mathbf{S} = \mathbf{S}^{part.}$ can be obtained from the particular knot by a local duality transformation, see equations (4.11), (4.12) and (4.13). The case $|h(x^\mu)| \neq 1$ might lead to different solutions, but not to different structures, since at every point the Poynting vector is multiplied by a positive number and (because $\mathbf{E} \cdot \mathbf{B} = 0$ and $||\mathbf{E}|| = ||\mathbf{B}||$) the only new consequence is that at each point $||\mathbf{E}||$ gets multiplied by the square root of this number.
Write $h = |h|e^{i\alpha}$ and consider Maxwell's equations for $\mathbf{F}^{RS} = h(x^\mu)\mathbf{F}^{part.}$:
$\nabla \cdot \mathbf{F}^{RS} = 0$ gives

$$\nabla|h| \cdot \mathbf{E}^{part.} - |h|\nabla\alpha \cdot \mathbf{B}^{part.} = 0 \tag{4.14}$$

$$\nabla|h| \cdot \mathbf{B}^{part.} + |h|\nabla\alpha \cdot \mathbf{E}^{part.} = 0 \tag{4.15}$$

$\nabla \times \mathbf{F}^{RS} = i\partial_t \mathbf{F}^{RS}$ gives

$$\nabla|h| \times \mathbf{E}^{part.} - |h|\nabla\alpha \times \mathbf{B}^{part.} = -\mathbf{B}^{part.}\partial_t|h| - |h|\mathbf{E}^{part.}\partial_t\alpha \tag{4.16}$$

$$\nabla|h| \times \mathbf{B}^{part.} + |h|\nabla\alpha \times \mathbf{E}^{part.} = \mathbf{E}^{part.}\partial_t|h| - |h|\mathbf{B}^{part.}\partial_t\alpha. \tag{4.17}$$

Now suppose α satisfies equations (4.12) and (4.13) for some β and $\mathbf{S} = \mathbf{S}^{part.}$. The equations (4.14) - (4.17) then become

$\nabla|h| \cdot \mathbf{E}^{part.} = 0$, $\nabla|h| \cdot \mathbf{B}^{part.} = 0$, $\nabla|h| \times \mathbf{E}^{part.} = -\mathbf{B}^{part.}\partial_t|h|$ and $\nabla|h| \times \mathbf{B}^{part.} = \mathbf{E}^{part.}\partial_t|h|$.

The first two equations imply $\nabla|h| = \gamma(x^\mu)\mathbf{S}^{part.}$ for some function $\gamma(x^\mu)$, and this combined with the last two equations give $\partial_t|h| = -\gamma(x^\mu)||\mathbf{S}^{part.}||$. The norm function $|h|$ thus satisfies the same equations as the phase function α, except for a (possibly) different function multiplying the Poynting vector.
We proceed to show that this exhausts the possibilities for solving Maxwell's equations with given Poynting vector $\mathbf{S} = h(x^\mu)h^*(x^\mu)\mathbf{S}^{part.}$. Suppose $\alpha(x^\mu) = \alpha_E(x^\mu) + \alpha_B(x^\mu) + \alpha_S(x^\mu)$ with $\nabla\alpha_S = s(x^\mu)\mathbf{S}^{part.}$, $\nabla\alpha_E = e(x^\mu)\mathbf{E}^{part.}$, $\nabla\alpha_B = b(x^\mu)\mathbf{B}^{part.}$ and e, b not both equal to 0. From equation (4.16) it follows that $(\nabla\alpha \times \mathbf{B}^{part.}) \cdot \mathbf{E}^{part.} = ||\mathbf{S}^{part.}||\partial_t\alpha$ and, given the assumption for α, this leads to $\partial_t\alpha = -s(x^\mu)||\mathbf{S}^{part.}||$. This equation is used in that part of equation (4.16) that is perpendicular to $\mathbf{B}^{part.}$ to obtain $e(x^\mu) = 0$ and in that part

of equation (4.17) that is perpendicular to $\mathbf{E}^{part.}$ to obtain $b(x^\mu) = 0$. This contradicts the assumption.
The requirements on $|h|$ and α define an equivalence relation and partitions the electromagnetic Hopf knots into equivalence classes. A class is defined by the structure (i.e. integral curves) of the Poynting vector and again within this class an equivalence relation is defined by the condition on $\alpha(x^\mu)$. This subclass is defined by the Poynting vector, not only its structure but also its norm. This result is obtained as follows. For $\mathbf{F}' \sim \mathbf{F} \Leftrightarrow \mathbf{F}' = ge^{i\beta}\mathbf{F}$, reflexivity is achieved by $g = 1$ and $\beta = 0$, symmetry by $g \to \frac{1}{g}$ and $\beta \to -\beta$, and multiplication of norm functions and addition of phases supplies transitivity.

The question arises whether there exist a local duality transformation satisfying (4.12) and (4.13) such that the electric and magnetic field lines also move undistorted in time, like the structure of the Poynting vector. This will be the case if there exists a function $h(x^\mu)$ such that $\frac{h(x^\mu)}{((t-i)^2-r^2)^3} \equiv G(x^\mu)$ is a real function (see (4.3) and (4.4)), and $|h(x^\mu)|$ and $\alpha \equiv Arg(h(x^\mu))$ both satisfy equations (4.12) and (4.13) with a function $\beta = \beta_1$ and $\beta = \beta_2$ respectively. This leads to:

from the condition on $|h(x^\mu)|$:

$$((t^2+1-r^2)^2+4r^2)\partial_t|G| + 6|G|(t^2+1-r^2)t = -\frac{\beta_1}{\sqrt{((t^2+1-r^2)^2+4r^2)}}||\mathbf{S}||$$

and

$$((t^2+1-r^2)^2+4r^2)\nabla|G| - 6|G|(t^2-1-r^2)\mathbf{r} = \frac{\beta_1}{\sqrt{((t^2+1-r^2)^2+4r^2)}}\mathbf{S}$$

from the condition on α, split into real and imaginary parts:

$$(t^2-1-r^2)\partial_t G + 6tG = -(2t\beta_2 + \frac{\beta_1(t^2-1-r^2)}{|G|((t^2+1-r^2)^2+4r^2)^{\frac{3}{2}}})G||\mathbf{S}||,$$

$$2t\partial_t G + 6G = ((t^2-1-r^2)\beta_2 - \frac{2t\beta_1}{|G|((t^2+1-r^2)^2+4r^2)^{\frac{3}{2}}})G||\mathbf{S}||$$

and

$$(t^2-1-r^2)\nabla G - 6G\mathbf{r} = (2t\beta_2 + \frac{\beta_1(t^2-1-r^2)}{|G|((t^2+1-r^2)^2+4r^2)^{\frac{3}{2}}})G\mathbf{S},$$

$$2t\nabla G = -((t^2-1-r^2)\beta_2 - \frac{2t\beta_1}{|G|((t^2+1-r^2)^2+4r^2)^{\frac{3}{2}}})G\mathbf{S}.$$

These are six real equations for three real functions, β_1, β_2 and G. From the structure of the Poynting vector we can conclude that $G = 0$ does not appear, so either $|G| = G$ or $|G| = -G$ everywhere. From the three equations containing time derivatives or the three vector equations, it is easy to show by elimination of β_2 that the only possibility is $|G| = G$. Elimination of β_1 from the vector equations now leads to a contradiction. So the "best" we can achieve in this respect is the decomposition of (4.3) and (4.4).

4.2.2 changing the twistor

A different choice for the non-null twistor $\mathcal{A}^\alpha$ will lead to a different class of electromagnetic Hopf knots. It is instructive to calculate the electromagnetic field that corresponds to the arbitrary non-null twistor $\mathcal{A}^\alpha = (\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}, c, d)$. The analogue of (4.6) is

$$(\Pi_1, \Pi_2) = f(x^\mu)(c(x - iy) + d(t - z) + ib, -c(t + z) - d(x + iy) - ia)$$

which leads to

$$\Phi_{00} = e^\psi f^2[c(x - iy) + d(t - z) + ib]^2 \quad (4.18)$$

$$\Phi_{11} = e^\psi f^2[c(t + z) + d(x + iy) + ia]^2 \quad (4.19)$$

$$\Phi_{01} = -e^\psi f^2[c(x - iy) + d(t - z) + ib][c(t + z) + d(x + iy) + ia]. \quad (4.20)$$

From these, remembering (4.8), interesting properties can be read off.
For example, $(\mathcal{A}^\alpha) = (0, \frac{b}{\sqrt{2}}, 0, 1)$ and $b \in \mathbb{R}\backslash\{0\}$ leads to the same solution space as before, but with x^μ replaced by $(x')^\mu = \frac{1}{b}x^\mu$. For $b > 0$ this amounts to an expansion ($b > 1$) or contraction ($b \in \langle 0, 1\rangle$) of previous solutions. For $b < 0$ there is an additional parity change and time reversal. All of these will be referred to as dilations. This interpretation of b remains valid for $c \neq 0$, but not when $a \neq 0$ unless a and b change both with the same factor. In general, for a non-null ($Re(b) \neq 0$) twistor $\mathcal{A}^\alpha = (a, b, 0, 1)$, we find that $Re(b)$ is related to dilations whenever $\frac{a}{b}$ is constant, $Im(b)$ to time translations or, equivalently, z-translations and that $Re(a)$ is related to y-translations and $Im(a)$ to translations in x direction.
When we interchange the components of the spinors that constitute the twistor and negate a, $(a, b, 0, 1) \rightarrow (b, -a, 1, 0)$, we find that the new solutions will be obtained from the old ones by a conformal inversion, $x^\mu \rightarrow \frac{x^\mu}{x_\nu x^\nu}$ (the general rule is $\mathbf{F}^{RS*} \rightarrow -\mathbf{F}^{RS}(t \rightarrow -t)$, which for Hopf knots amounts to a conformal inversion, see chapter 5). The interpretation of a and b is exactly as before.
The class of solutions of Maxwell's equations that is obtained for $a = b = c = d = 1$ is:

$$\mathbf{E} = Re(g(x^\mu)) \begin{pmatrix} 2(y - z(t+x)) \\ -2(yz + (t+x)) \\ (t+x)^2 + y^2 - z^2 - 1 \end{pmatrix} - Im(g(x^\mu)) \begin{pmatrix} 2(z + y(t+x)) \\ 1 + y^2 - z^2 - (t+x)^2 \\ 2(yz - (t+x)) \end{pmatrix},$$

$$\mathbf{B} = Re(g(x^\mu)) \begin{pmatrix} 2(z + y(t+x)) \\ 1 + y^2 - z^2 - (t+x)^2 \\ 2(yz - (t+x)) \end{pmatrix} + Im(g(x^\mu)) \begin{pmatrix} 2(y - z(t+x)) \\ -2(yz + (t+x)) \\ (t+x)^2 + y^2 - z^2 - 1 \end{pmatrix},$$

$$\mathbf{S} = (1 + (t+x)^2 + y^2 + z^2)(g(x^\mu))(g(x^\mu))^* \begin{pmatrix} 1 + (t+x)^2 - y^2 - z^2 \\ 2(y(t+x) - z) \\ 2(y + z(t+x)) \end{pmatrix}.$$

It can be seen that a representative of this class is given by a rotated version of (4.1), the axis of rotation is the y-axis and it is such that the +z-axis rotates to the +x-axis.

4.3 the method applied to a null twistor

We have seen two routes to visualize a twistor. The first is valid for a null twistor and gives a ray of light, the second for a non-null twistor and gives a class of electromagnetic Hopf knots. What happens if we use the second method for a null twistor? Applying this method to $(\mathcal{A}^\alpha) = (0,0,0,1)$ leads to the class of

$$\mathbf{F}^{RS} = \frac{1}{(t^2 - r^2)^3} \begin{pmatrix} (x - iy)^2 - (t-z)^2 \\ i(x-iy)^2 + i(t-z)^2 \\ -2(x-iy)(t-z) \end{pmatrix}, \tag{4.21}$$

with corresponding Poynting vector

$$\mathbf{S} \propto \begin{pmatrix} 2x(t-z) \\ 2y(t-z) \\ x^2 + y^2 - (t-z)^2 \end{pmatrix}. \tag{4.22}$$

Changing this null twistor into the non-null $(\mathcal{A}^\alpha) = (0, \frac{b}{\sqrt{2}}, 0, 1)$ amounts to applying an imaginary time translation $t \to t - ib$ to the field (4.21), as can be seen from (4.18)-(4.20). Since for $b = 1$, $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ corresponds to (4.1) this implies that an electromagnetic Hopf knot is obtained from (4.21) by the transformation $t \to t - i$, as can be verified by direct comparison. For this reason

we shall refer to the null congruence corresponding to (4.21) as a degenerate Robinson congruence in the sequel.
The null twistor $(\mathcal{A}^\alpha) = (0, 1, 0, 0)$ leads to (the class of) plane waves travelling in the -z direction (see the examples at the end of chapter 3), and $(\mathcal{A}^\alpha) = (1, 0, 0, 0)$ to plane waves propagating in the z direction. As will be shown in chapter 5, a conformal inversion of this last field in the limit $k = 0$ (thus a constant electromagnetic field) leads to (4.21). Therefore we have the surprising sequence:

$$\mathbf{E} = \hat{x}, \mathbf{B} = \hat{y} \rightarrow \text{(4.21)} \rightarrow \text{(4.1)}, \tag{4.23}$$

where the first arrow indicates a conformal inversion and the second an imaginary time translation. $(\mathcal{A}^\alpha) = (1, 1, 0, 0)$ leads to a plane wave propagating in the x direction.
As a final example, the null twistor $(\mathcal{A}^\alpha) = (0, 0, 1, 1)$ gives the class of solutions for which the $\frac{\pi}{2}$ rotated version (with y-axis as rotation axis) of (4.21) is a representative.
The light rays that correspond to these null twistors via the first method coincide with the unique direction defined by the Poynting vector of the electromagnetic fields that correspond to these twistors via the second method. From equations (4.9), (4.10) and (1.14) it follows that this is true in general, and the second method can therefore be considered as a generalization of the first.

4.4 shift in the complex domain

By considering Maxwell's equations for the Riemann-Silberstein vector (3.9), complex numbers have been introduced in electrodynamics. On various occasions we have seen that it is a convenient way to combine electric and magnetic fields into one vector. The complex structure here acts as a bookkeeping device, to distinguish between electric and magnetic parts. The complex time shift in the previous paragraph hints at another possible role for a complex structure. The source-free Maxwell's equations for the Riemann-Silberstein vector are invariant under all the transformations that have been obtained in the last two paragraphs (dilation, (complex) translation, rotation, conformal inversion). Let us consider these equations as being defined not only in M^4, but in $M^4_{\mathbb{C}}$. It is tempting to assume that all the aforementioned electromagnetic fields that correspond to different kinds of twistors are part of one (super)solution to these Maxwell equations in $M^4_{\mathbb{C}}$. Depending on where we agree to locate the real slice M^4 in $M^4_{\mathbb{C}}$ (and this really is a matter of choice) we do get one of the aforementioned solutions on this slice. For example, for a particular choice of real space-time we have solution (4.21). If now we shift the choice of real slice to a different part of

$M^4_{\mathbb{C}}$ related to the previous choice by $t \to t - i$, we get the solution (4.1) on this space-time. Pushing the real slice still further into the imaginary time direction, $t \to t - ib$ ($b > 1$) we get the expanded version of (4.1).
A careful analysis is needed to show whether this assumption can stand the test of mathematical scrutiny, and to disclose the possible relationship with work of E. Newman [9,30–34] and of A. Burinskii [35,36].

A somewhat related question is the following. Dilations are part of the conformal group $C(1,3)$. The transformations in twistor space that leave invariant the inner product belong to $SU(2,2)$ which is a fourfold cover of the conformal group. Does a $SU(2,2)$ transformation of a twistor always correspond to a particular conformal transformation of the corresponding electromagnetic field? Of course, the norm of $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ is in general not the same as that of $(\mathcal{A}^\alpha) = (0, \frac{b}{\sqrt{2}}, 0, 1)$, but this last twistor leads to the same class of solutions as some constant times this twistor. The constant can be adjusted so that the two twistors are related by a $SU(2,2)$ transformation.

4.5 conclusion and final remarks

In this chapter several forms to represent an electromagnetic Hopf knot have been discussed. Using a theorem by Robinson, we presented a correspondence between a non-null twistor and a class of solutions of the source-free Maxwell equations. Based on the relation (4.9) the class is characterized by the structure of the Poynting vector. All solutions within this class are electromagnetic Hopf knots. All knots that constitute one class can be parametrized by a complex function $h(x^\mu) = |h|e^{i\alpha}$ in which both $\alpha(x^\mu)$ and $|h(x^\mu)|$ satisfy equations (4.12) and (4.13): a restricted local duality transformation. A different twistor will correspond to a different class of knots only when the twistor is non-null. The method can be applied to a null twistor, but will lead to a class of solutions different from electromagnetic Hopf knots.

The main results of this chapter were published in [4]. Later, in [37], B.H. Trishin discussed part of present calculations from a different perspective. He used the Newman-Penrose formalism, the Kerr theorem and null coordinates as will be discussed in chapter 8, to arrive at the general solution to our equation (4.7), translated into the Riemann-Silberstein vector as:

$$\mathbf{F}^{RS} = (1 + \bar{w}\partial_u Y + v\partial_w Y) f(u + \bar{w}Y, w + vY) \begin{pmatrix} 1 - Y^2 \\ i(1 + Y^2) \\ -2Y \end{pmatrix},$$

in which f is an arbitrary holomorphic function. The notation is that of chapter 8. It is interesting to note that when we use the congruence that corresponds to the Schwarzschild solution of Einsteins equations (according to chapter 8 we then have $Y = -\frac{z+r}{x-it}$) we do get the following expressions for the electric, magnetic and Poynting vector field:

$$\mathbf{E} \propto \begin{pmatrix} -(y^2+z^2+zr) \\ xy \\ x(z+r) \end{pmatrix}, \quad \mathbf{B} \propto \begin{pmatrix} -xy \\ x^2+z^2+zr \\ -y(z+r) \end{pmatrix} \text{ and } \mathbf{S} \propto \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The electromagnetic torus knots described in [38] (published october 2013) all have the same Poynting vector as the electromagnetic Hopf knot presented in this chapter. These torus knots thus belong to the same equivalence class as the Hopf knot. For every torus knot, there exist therefore a local duality transformation (4.11), (4.12) and (4.13) relating it to the Hopf knot. This argument might be used to show that in the case of torus knots all the field lines are closed. In [38] this is shown only for the "core field lines" that correspond to degenerate tori.
It is also of some interest to note that the β appearing in [38] and describing part of the Bateman construction for generating null electromagnetic fields, corresponds to Y in the method of Debney, Kerr and Schild as presented here in chapter 8. Does this give a clue of how to arrive at exact solutions of Einstein equations that can be described as gravitational torus knots?

Appendix

From $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk}B_k \Leftrightarrow B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$ we have $\mathbf{E}\cdot\mathbf{B} = E_iB_i = -\frac{1}{2}\epsilon_{ijk}F_{0i}F_{jk}$. For $F_{\mu\nu} = f_{\mu\nu}(\phi)$ we see that $F_{0i}F_{jk}$ contains only factors symmetrical in i and j or k:

$\mathbf{E}\cdot\mathbf{B} = 0$

Since $\epsilon^{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta}$ we have $G_{\mu\nu} = -(*F)_{\mu\nu}$ and so $G_{0i} = -B_i$ and $G_{ij} = -\epsilon_{ijk}E_k \Leftrightarrow E_i = -\frac{1}{2}\epsilon_{ijk}G_{jk}$.

Using $E_i = F_{0i}$ we get

$\mathbf{E}(0,\mathbf{r}) = \frac{\sqrt{a}}{2\pi i} \frac{\partial_0\phi^*(0,\mathbf{r})\nabla\phi(0,\mathbf{r}) - \partial_0\phi(0,\mathbf{r})\nabla\phi^*(0,\mathbf{r})}{(1+\phi^*(0,\mathbf{r})\phi(0,\mathbf{r}))^2}$

and from $E_i = -\frac{1}{2}\epsilon_{ijk}G_{jk}$ we get

$\mathbf{E}(0,\mathbf{r}) = -\frac{1}{2}\epsilon_{ijk}\frac{\sqrt{a}}{2\pi i} \frac{\partial_j\theta^*(0,\mathbf{r})\partial_k\theta(0,\mathbf{r}) - \partial_j\theta(0,\mathbf{r})\partial_k\theta^*(0,\mathbf{r})}{(1+\theta^*(0,\mathbf{r})\theta(0,\mathbf{r}))^2}.$

Combining these expressions makes it possible to calculate $\partial_0\phi$ and a similar argument shows the possibility of calculating $\partial_0\theta$ from $\phi(0,\mathbf{r})$ and $\theta(0,\mathbf{r})$.

The points in $\mathbb{R}^3$ for which ϕ takes a constant value consists of the intersection between two surfaces, one for which the real part of ϕ is constant and the other for which the imaginary part of ϕ is constant. Very special cases aside, this intersection is a one dimensional curve. The magnetic field lines coincide with these level curves as follows from the following consideration. We have $\nabla\phi \cdot \mathbf{B} = (\partial_i\phi)B_i = -\frac{1}{2}\epsilon_{ijk}(\partial_i\phi)F_{jk} = 0$, since $(\partial_i\phi)F_{jk}$ contains only terms that are symmetrical in i and j or k. For the same reasons we also have $\nabla\phi^* \cdot \mathbf{B} = 0$, and thus $\mathbf{B} \propto \nabla\phi^* \times \nabla\phi$. A similar argument shows that the electric field lines coincide with the level curves of θ. We now see that the duality condition implies $(\nabla\phi^* \times \nabla\phi)\cdot(\nabla\theta^* \times \nabla\theta) = 0$.

5

Conformal inversion

5.1 introduction

It is well known that Maxwell's equations are invariant under Poincare transformations $\mathcal{P}(1,3)$. Less familiar is the fact that $\mathcal{P}(1,3)$ is part of a larger group, the conformal group $\mathcal{C}(1,3)$, under which the source free Maxwell equations are also invariant, as was discovered by Bateman and Cunningham in 1909 [39–42]. This extra symmetry does not lead to new (unknown) conservation laws, since in the case of vacuum Maxwell equations the generators of the conformal group can be expressed analytically in terms of the elements of the Poincare algebra, all of whose conserved quantities are already known [43,44].
Apart from electrodynamics, conformal transformations play an important role in general relativity. They are also used in a reformulation of Brans-Dicke theory as an alternative to general relativity, in which the mass acquires a space-time dependent variation [45,46]. This variation depends on the conformal factor Ω from the definition of conformally related metrics: $g' = \Omega^2 g$.
It is noteworthy to remark that in contrast with Poincare symmetry $\mathcal{P}(1,3)$, requiring conformal symmetry $\mathcal{C}(1,3)$ for a (spin 1) vector field uniquely leads to Maxwells equations [44].
Besides the Poincare transformations, the conformal group consists of dilations and special conformal transformations. Here we investigate what can be learned from this lesser known symmetry.

5.2 special conformal transformations and conformal inversion

When applying Lie's approach to symmetries of differential equations (see appendix **A**) to source free electrodynamics, you will arrive at the maximal invariance group of Maxwell equations in vacuum [44]. The elements that make up this group are duality transformations, dilations, Poincare transformations and special conformal transformations. The first three of these are very familiar, therefore we concentrate on the special conformal group. We would like to warn the reader with more than average knowledge about the conformal group that it is a mistake to interpret a special conformal transformation as a change from an inertial system to an uniformly accelerating coordinate system [43,47–49], as used to be done early in the history of conformal transformations and is still being done every now and then (see for example [50]).

A special conformal transformation ($\mathcal{SCT}(b)$) is given by

$$x^\mu \rightarrow \tilde{x}^\mu = \frac{x^\mu + b^\mu (x_\nu x^\nu)}{1+2(b_\nu x^\nu)+(x_\nu x^\nu)(b_\nu b^\nu)}$$

and can be build from a conformal inversion ($\mathcal{CI}(a)$) $x^\mu \rightarrow \tilde{x}^\mu = a\frac{x^\mu}{x_\nu x^\nu}$ and a translation ($\mathcal{T}(c)$) $x^\mu \rightarrow \tilde{x}^\mu = x^\mu + c^\mu$ as follows:

$$\mathcal{SCT}(b) = \mathcal{CI}(1) \circ \mathcal{T}(b) \circ \mathcal{CI}(1).$$

In deriving this result, use has been made of lowering indices on transformed coordinates. This is done with the original flat space-time metric tensor, η: the transformation is thus not to be regarded as a coordinate transformation! (See the sequel for more on this.) Note that $\mathcal{SCT}(b) \circ \mathcal{SCT}(a) = \mathcal{SCT}(a+b)$.

Dilations $\mathcal{D}(a) : x^\mu \rightarrow ax^\mu$ can be obtained in the same way, but with different parameters: $\mathcal{D}(a) = \mathcal{CI}(a) \circ \mathcal{T}(0) \circ \mathcal{CI}(1) = \mathcal{CI}(a) \circ \mathcal{CI}(1)$. We conclude that any conformal transformation can be built from Poincare transformations and conformal inversions, therefore we limit our discussion to $\mathcal{CI} = \mathcal{CI}(1)$, which can be regarded as a length inversion operator [43] and is in itself a conformal transformation: Maxwell's equations in vacuum are invariant under $\mathcal{CI}$.
Note that conformal inversion is ill-defined for points on the light-cone. A general discussion of the conformal group should therefore include M^4_C, the compactification of Minkowski space. However, for our discussion it will suffice to consider M^4.

5.3 prescription for obtaining new fields from old ones

It is of no use to consider a $\mathcal{SCT}$ or a $\mathcal{CI}$ as a coordinate transformation (as is often done in a different context), since Maxwell equations would not be valid in the new system. According to the approach of Lie, the physical content of the conformal symmetry of vacuum electrodynamics is that every solution can be transformed into another different solution. In order to arrive at the new transformed solution from a given one, we proceed as follows.

We start with Minkowski space M^4, coordinates (x) and metric tensor $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, a copy hereof $\tilde{M}^4$, coordinates $(\tilde{x})$ and metric tensor $\tilde{\eta} = \eta$ and a mapping between these manifolds $f : (x) \to (\tilde{x})$ given by $\tilde{x}^\mu = \frac{x^\mu}{(x \cdot x)}$.
We use the notation $x \cdot x$ for $x_\nu x^\nu$. Because $\tilde{\eta} = \eta$ we have for the inverse $f^{-1} : (\tilde{x}) \to (x)$ with $x^\mu = \frac{\tilde{x}^\mu}{(\tilde{x} \cdot \tilde{x})}$.
We use f to pull back tensor fields to M^4. In this way the metric tensor $\tilde{\eta}$ is mapped to $g_{\mu\nu}(x) := (f^*\tilde{\eta})_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x) \frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x) \tilde{\eta}_{\alpha\beta}$. Using $\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \frac{\delta^\alpha_\mu}{x \cdot x} - 2 \frac{x^\alpha x_\mu}{(x \cdot x)^2}$ we get $g_{\mu\nu}(x) = \frac{1}{(x \cdot x)^2} \eta_{\mu\nu}$ and this shows that, by definition, f is a conformal map.
We will regard tensor fields in M^4 and therefore use η to raise and lower indices, not g. We proceed with pulling back a Faraday tensor (3.10) (satisfying Maxwell equations) $(\tilde{F}_{\mu\nu})$ from $\tilde{M}^4$.

$F_{\mu\nu}(x) := (f^*\tilde{F})_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x) \frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x) \tilde{F}_{\alpha\beta}(\tilde{x}(x))$ and

$F^{\mu\nu}(x) = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta} \frac{\partial \tilde{x}^\sigma}{\partial x^\alpha} \frac{\partial \tilde{x}^\kappa}{\partial x^\beta} \eta_{\sigma\lambda}\eta_{\kappa\rho} \tilde{F}^{\lambda\rho}(\tilde{x}(x)) = |\frac{\partial(\tilde{x})}{\partial(x)}| \frac{\partial x^\mu}{\partial \tilde{x}^\lambda} \frac{\partial x^\nu}{\partial \tilde{x}^\rho} \tilde{F}^{\lambda\rho}(\tilde{x}(x))$.

See appendix **B** for details of the calculation. These are tensor fields on M^4. Since $\tilde{F}$ is a Faraday tensor on $\tilde{M}^4$, the question arises wether F is a Faraday tensor on M^4:

$\partial_\lambda F_{\mu\nu}(x) + \partial_\nu F_{\lambda\mu}(x) + \partial_\mu F_{\nu\lambda}(x) = 0$ (3.12) follows easily by reshuffling of indices, systematic cancellation of terms and using the fact that $\tilde{F}$ is a Faraday tensor on $\tilde{M}^4$. The other pair of equations (3.11) is more intricate, but the result is $\partial_\mu F^{\mu\nu}(x) = 0$ (see appendix **C**). So indeed, the pull back of a Faraday tensor is again a Faraday tensor when the mapping is a conformal inversion. With this result we end up with a prescription for constructing new Maxwell fields (vacuum solutions to Maxwell equations) from known ones:

In Minkowski space M^4, coordinates (x) and metric tensor $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,

we define functions $f^\mu(x) = \frac{x^\mu}{(x \cdot x)}$.

If F is a Faraday tensor, so:

$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$ and $\partial_\mu F^{\mu\nu} = 0$ ($F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$)

then F' is also a Faraday tensor, in which:

$F'_{\mu\nu}(x) = (\partial_\mu f^\alpha)(\partial_\nu f^\beta)F_{\alpha\beta}(\frac{x}{x \cdot x})$ and $F'^{\mu\nu}(x) = \eta^{\mu\alpha}\eta^{\nu\beta}F'_{\alpha\beta}(x)$.

A similar strategy to obtain new solutions from old with the help of conformal transformations exists in general relativity, see for a general discussion [51] and more specifically the references therein, such as [52].

5.4 transformation of Riemann-Silberstein vector

In electromagnetic calculations it can be advantageous to work with the Riemann-Silberstein vector $\mathbf{F}^{RS} = \mathbf{E} + i\mathbf{B}$. Instead of the Faraday tensor, F, we should then use the anti-self- or self-dual form, $F^{(asd)}$ or $F^{(sd)}$, of it:

$F^{(asd)}_{\mu\nu} := (F_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho})$ and $F^{(sd)}_{\mu\nu} := (F_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho})$

Note that $F^{(asd)}$ coincides with $\mathcal{F}$ from chapter 3 (3.15).
Because $F'_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)F_{\alpha\beta}(\frac{x}{x \cdot x})$ and $F'^{\mu\nu}(x) = |\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\frac{\partial x^\nu}{\partial \tilde{x}^\rho}F^{\lambda\rho}(\frac{x}{x \cdot x})$ it is not immediately clear how to calculate $F^{(asd)\prime}$ from $F^{(asd)}$. But with help from appendix **B**(4) and **B**(5) it can be shown that in the case of conformal inversions (*but not in general!*)

$F'^{\mu\nu}(x) = |\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\frac{\partial x^\nu}{\partial \tilde{x}^\rho}F^{\lambda\rho}(\frac{x}{x \cdot x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\lambda}\frac{\partial \tilde{x}^\nu}{\partial x^\rho}F^{\lambda\rho}(\frac{x}{x \cdot x})$.

Now we have: $F'^{(asd)}_{\mu\nu} = (F'_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\lambda\rho}F'^{\lambda\rho}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)F_{\alpha\beta}(\frac{x}{x.x}) + \frac{\partial \tilde{x}^\alpha}{\partial x^\kappa}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\sigma}(x)\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}F^{\kappa\sigma}(\frac{x}{x.x})$,

and this leads to (see appendix **D**):

$F'^{(asd)}_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)F^{(sd)}_{\alpha\beta}(\frac{x}{x\cdot x})$,

so the self-dual is transformed into an anti-self-dual form.

And thus: $F'^{\rho\sigma}_{(asd)}(x) = \eta^{\rho\mu}\eta^{\sigma\nu}F'^{(asd)}_{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu}\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)F^{(sd)}_{\alpha\beta} = \frac{\partial \tilde{x}_\alpha}{\partial x_\rho}(x)\frac{\partial \tilde{x}_\beta}{\partial x_\sigma}(x)F^{\alpha\beta}_{(sd)} = \frac{\partial \tilde{x}^\rho}{\partial x^\alpha}(x)\frac{\partial \tilde{x}^\sigma}{\partial x^\beta}(x)F^{\alpha\beta}_{(sd)} = |\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\rho}{\partial \tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial \tilde{x}^\beta}F^{\alpha\beta}_{(sd)}(\frac{x}{x\cdot x})$

From this the transformation of the Riemann-Silberstein vector follows:

$$\boxed{\mathbf{F}^{RS\prime}(x) = \frac{1}{(t^2-r^2)^3}[-(t^2-r^2)\mathbf{F}^{RS*}(\frac{x}{x\cdot x}) + 2\mathbf{r}\times(\mathbf{r}\times\mathbf{F}^{RS*}(\frac{x}{x\cdot x})) - 2it\mathbf{r}\times\mathbf{F}^{RS*}(\frac{x}{x\cdot x})]}$$

where we have used $x = (x^\mu) = (t, \mathbf{r})$. From this it can be shown that applying the transformation twice leads back to the original input field: $\mathcal{CI}(\mathcal{CI}(\mathbf{E},\mathbf{B})) = (\mathcal{CI})^2(\mathbf{E},\mathbf{B}) = (\mathbf{E},\mathbf{B})$. This, of course, also follows immediately from $x''^\mu = \frac{x'^\mu}{x'\cdot x'} = (x\cdot x)x'^\mu = x^\mu$. In looking for eigenstates of this conformal inversion operation, the only possible eigenvalues are +1 and -1. More generally we have: $\mathcal{CI}(b)\circ\mathcal{CI}(a) = \mathcal{D}(\frac{b}{a}) = \mathcal{CI}(\frac{1}{a})\circ\mathcal{CI}(\frac{1}{b})$.

From the transformation formula it follows that the Poynting vector transforms in a complicated way, but it can be expressed as:

$\mathbf{S}' = f_1(x^\mu)\mathbf{S}(\frac{x}{x\cdot x}) + f_2(x^\mu)\mathbf{r}$.

The precise form of the functions f_1 and f_2 is not very illuminating.

The nice formula for the transformation of the Riemann-Silberstein vector under conformal inversion is to be contrasted with the complicated transformation of this vector under a special conformal transformation. In [44] it is given as (adapted to our notation and combined to form the RS-vector):

$\mathbf{F}^{RS\prime}(x^\mu) = (1 - 2b^\mu\tilde{x}_\mu + b_\nu b^\nu\tilde{x}_\mu\tilde{x}^\mu)\{(b_\mu\tilde{x}^\mu - 1)^2\mathbf{F}^{RS}(\tilde{x}) + 2(b^\mu\tilde{x}_\mu - 1)(-ib_0\tilde{\mathbf{r}}\times\mathbf{F}^{RS}(\tilde{x}) + i\tilde{t}\mathbf{b}\times\mathbf{F}^{RS}(\tilde{x}) - \mathbf{b}(\tilde{\mathbf{r}}\cdot\mathbf{F}^{RS}(\tilde{x})) + \tilde{\mathbf{r}}(\mathbf{b}\cdot\mathbf{F}^{RS}(\tilde{x}))) + \mathbf{b}\times\tilde{\mathbf{r}}(-i\tilde{t}(\mathbf{b}\cdot\mathbf{F}^{RS}(\tilde{x})) + ib_0(\tilde{\mathbf{r}}\cdot\mathbf{F}^{RS}(\tilde{x})) + \mathbf{b}\cdot\tilde{\mathbf{r}}\times\mathbf{F}^{RS}(\tilde{x})) + (\mathbf{b}\tilde{t} - \tilde{\mathbf{r}}b_0)(-i\mathbf{b}\cdot\tilde{\mathbf{r}}\times\mathbf{F}^{RS}(\tilde{x}) - \tilde{t}\mathbf{b}\cdot\mathbf{F}^{RS}(\tilde{x}) + b_0\tilde{\mathbf{r}}\cdot\mathbf{F}^{RS}(\tilde{x}))\}$

Note that in this formula, $\tilde{x}$ still has to be expressed as function of x. For $(b^\mu) = (b, 0, 0, 0)$ this becomes the relatively simple

$\mathbf{F}^{RS\prime}(x) =$

$\frac{1}{(1+2bt+b^2(t^2-r^2))^3}\{(1+bt)^2\mathbf{F}^{RS}(\frac{x}{x\cdot x}) - b^2\mathbf{r}(\mathbf{r}\cdot\mathbf{F}^{RS}(\frac{x}{x\cdot x})) + 2ib(1+bt)\mathbf{r}\times\mathbf{F}^{RS}(\frac{x}{x\cdot x})\}$.

Since $\mathcal{SCT}(b,\mathbf{0}) = \mathcal{CI}(1)\circ\mathcal{T}(b,\mathbf{0})\circ\mathcal{CI}(1)$ and the conformal inversion of

an electromagnetic Hopf knot is again a Hopf knot (see example (**5.5.2**) below), the sequence (4.23) from chapter 4 implies that a special conformal transformation with $(b^\mu) = (b, \mathbf{0})$ applied to the field $\{\mathbf{E} = \hat{x}, \mathbf{B} = \hat{y}\}$ gives an electromagnetic Hopf knot when b is imaginary.
Substitution of $\mathbf{F}^{RS} = \hat{x} + i\hat{y}$ and $b = i$ leads to

$$\mathbf{F}^{RS\prime} = \frac{-1}{((t-i)^2-r^2)^3} \begin{pmatrix} x(x+iy) - (t-i)^2 - 2z(t-i) \\ y(x+iy) - i(t-i)^2 - 2iz(t-i) \\ (x+iy)(z+2(t-i)) \end{pmatrix}$$

whereas $\mathcal{CI}(1) \circ \mathcal{T}(i, \mathbf{0}) \circ \mathcal{CI}(1)$ applied to $\mathbf{F}^{RS} = \hat{x} + i\hat{y}$ gives (see examples (**5.5.1**) and (**5.5.2**) below)

$$\mathbf{F}^{RS\prime} = \frac{-1}{((t-i)^2-r^2)^3} \begin{pmatrix} (x+iy)^2 - (t-i+z)^2 \\ -i(x+iy)^2 - i(t-i+z)^2 \\ 2(x+iy)(t-i+z) \end{pmatrix}.$$

The difference between these is

$\frac{-1}{((t-i)^2-r^2)^3} \begin{pmatrix} iy(x+iy) - z^2 \\ -ix(x+iy) - iz^2 \\ z(x+iy) \end{pmatrix}$, which can be written as

$\frac{-b^2}{(1+2bt+b^2(t^2-r^2))^3} \mathbf{r} \times (\mathbf{r} \times \mathbf{F}^{RS})$.

We conclude that this last term, the general form of which will be

$$(1 - 2b^\mu \tilde{x}_\mu + b_\nu b^\nu \tilde{x}_\mu \tilde{x}^\mu)(\mathbf{b}\tilde{t} - \tilde{\mathbf{r}} b_0) \times (b_0 \tilde{\mathbf{r}} \times \mathbf{F}^{RS}(\tfrac{x}{x \cdot x})),$$

is missing from the expression given in [44].

With the missing term included, the transformation formula for $\mathcal{SCT}(b, \mathbf{0})$ becomes:

$$\mathbf{F}^{RS\prime}(x) = \frac{1}{(1+2bt+b^2(t^2-r^2))^3} \times$$

$$\{(1+bt)^2 \mathbf{F}^{RS}(\tfrac{x}{x \cdot x}) - b^2(\mathbf{r}(\mathbf{r} \cdot \mathbf{F}^{RS}(\tfrac{x}{x \cdot x})) + \mathbf{r} \times (\mathbf{r} \times \mathbf{F}^{RS})) + 2ib(1+bt)\mathbf{r} \times \mathbf{F}^{RS}(\tfrac{x}{x \cdot x})\},$$

the correctness of which can be checked with application of $\mathcal{SCT}(b, \mathbf{0}) = \mathcal{CI}(1) \circ \mathcal{T}(b, \mathbf{0}) \circ \mathcal{CI}(1)$ to other examples and various b.

An interesting application is the use of this formula to a constant electric field $\mathbf{E} = \hat{x}$:

$$\mathcal{SCT}(i, \mathbf{0}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -\tfrac{1}{2}\{\mathbf{F}_H + \mathbf{F}_H^*(t \to -t)\},$$

in which $\mathbf{F}_H$ is the Riemann-Silberstein vector of the Hopf knot (4.1). Using example (**5.5.2**) below we infer that the special conformal transformation with parameter $(i, \mathbf{0})$ of a constant electric field gives an eigenstate of the conformal inversion operation with eigenvalue -1. It also implies that applying an imaginary time translation $t \to t - i$ to the conformal inversion of a constant electric field in the x-direction gives a superposition of two Hopf knots that, except for a minus sign, are conformal inverted states of each other, $\frac{1}{2}\{\mathbf{F}_H + \mathbf{F}^*_H(t \to -t)\}$. At $t = 0$ this field is given by:

$$\mathbf{E}(0, \mathbf{r}) = \frac{1}{(1+r^2)^3} \begin{pmatrix} -1 - x^2 + y^2 + z^2 \\ 2(z - xy) \\ -2(y + xz) \end{pmatrix}, \quad \mathbf{B}(0, \mathbf{r}) = \mathbf{0}.$$

The electric field is a Hopf fibration, identical to the one in $\mathbf{F}_H(t = 0)$. For arbitrary t, the field can be expressed with the same definitions as in (4.5) as

$$\mathbf{F}^{RS}(t, \mathbf{r}) = \frac{-(1+r^2)^3}{((t-i)^2 - r^2)^3}\{\mathbf{E}_0 + i\frac{4t}{1+r^2}\mathbf{C}_0 + \frac{t^2}{(1+r^2)^3}Re(\mathbf{W})\}.$$

The same exercise with a constant magnetic field $\mathbf{B} = \hat{y}$ gives an eigenstate of conformal inversion with eigenvalue +1, $\frac{1}{2}\{\mathbf{F}_H - \mathbf{F}^*_H(t \to -t)\}$. At $t = 0$ this field is:

$$\mathbf{E}(0, \mathbf{r}) = \mathbf{0}, \quad \mathbf{B}(0, \mathbf{r}) = \begin{pmatrix} 2(xy + z) \\ 1 - x^2 + y^2 - z^2 \\ -2(x - yz) \end{pmatrix},$$

so the magnetic field is the same as that of $\mathbf{F}_H(t = 0)$. For arbitrary t we now have

$$\mathbf{F}^{RS}(t, \mathbf{r}) = \frac{-(1+r^2)^3}{((t-i)^2 - r^2)^3} i\{\mathbf{B}_0 + i\frac{4t}{1+r^2}\mathbf{A}_0 + \frac{t^2}{(1+r^2)^3}Im(\mathbf{W})\}.$$

In (4.5) we defined five vector fields of which only four had a clear physical interpretation. It is only now that we can say something about the interpretation of the other, $\mathbf{W}$:

$$\mathcal{SCT}(i, \mathbf{0})(\mathbf{W}) = \mathbf{F}_H,$$

the electromagnetic Hopf knot that was used to define $\mathbf{W}$.

5.5 examples of conformal inversion

The transformation formula for the Riemann-Silberstein vector will be used to investigate the conformal inversion of some specific electromagnetic fields.

5.5.1 constant electric and magnetic field

$$\mathbf{F}^{RS} = (\hat{x} + i\hat{y}) = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}.$$

This leads to

$$\mathbf{F}^{RS\prime} = \frac{1}{(t^2-r^2)^3} \begin{pmatrix} (x-iy)^2 - (t-z)^2 \\ i(x-iy)^2 + i(t-z)^2 \\ -2(x-iy)(t-z) \end{pmatrix}, \text{ or}$$

$$\mathbf{E}' = \frac{1}{(t^2-r^2)^3} \begin{pmatrix} x^2 - y^2 - (z-t)^2 \\ 2yx \\ -2x(t-z) \end{pmatrix}, \qquad \mathbf{B}' = \frac{1}{(t^2-r^2)^3} \begin{pmatrix} -2xy \\ x^2 - y^2 + (t-z)^2 \\ 2y(t-z) \end{pmatrix}.$$

This field has been considered in chapter 4, equation (4.21) (degenerate Robinson congruence). Applying the imaginary time translation $t \to t - i$ to this Riemann-Silberstein vector gives the electromagnetic Hopf knot of the next example.

5.5.2 electromagnetic Hopf knot

$$\mathbf{F}^{RS} = \frac{1}{((t-i)^2-r^2)^3} \begin{pmatrix} (x-iy)^2 - (t-i-z)^2 \\ i(x-iy)^2 + i(t-i-z)^2 \\ -2(x-iy)(t-i-z) \end{pmatrix}.$$

A lengthy calculation (see appendix **E**) results in: $\mathbf{F}^{RS\prime} = -\mathbf{F}^{RS*}(t \to -t)$.
Notice that this can also be seen as a rotation of $180°$ around the x-axis: $\mathbf{F}^{RS\prime} = -Rot_{\hat{x}}(\pi)\{\mathbf{F}^{RS}\}$.
Thus the conformal inversion of an electromagnetic Hopf knot is a different electromagnetic Hopf knot. The sum and difference of these two knots is obviously an eigenstate of conformal inversion.

5.5.3 plane wave

Example (**5.5.1**) is the $k \downarrow 0$ limit of the plane wave:

$$\mathbf{F}^{RS} = e^{ik.x}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix},\ (k^{\mu}) = (k, 0, 0, k),\ k > 0.$$

The transformed field is a local duality transformation of the field obtained in (**5.5.1**):

$$\mathbf{F}^{RS\prime} = \frac{e^{i\alpha}}{(t^2-r^2)^3}\begin{pmatrix} (x-iy)^2-(t-z)^2 \\ i(x-iy)^2+i(t-z)^2 \\ -2(x-iy)(t-z) \end{pmatrix},$$

in which $\alpha = k\frac{t-z}{t^2-r^2}$ satisfies a "generalization" of the homogeneous wave equation, (4.12) and (4.13), with $\beta(x^{\mu}) = k\frac{(t^2-r^2)^4}{(t-z)^2+x^2+y^2}$.

Of course, the field we started with in (**5.5.3**) is itself a local duality transformation of the field we started with in (**5.5.1**) and in this case α satisfies the homogeneous wave equation (see the discussion of equations (4.12) and (4.13)).

The field of a charge distribution satisfies vacuum Maxwell's equations except on the distribution's world tube. In this case we can still use the transformation formula, the resulting field of which will be the field due to a different charge distribution. Clearly, the transformation of the charge distribution is defined such that Maxwell's equations including sources are invariant under present transformation. For a conformal inversion this does not necessarily lead to a physical acceptable charge distribution. Here are a few examples.

5.5.4 Coulomb field

$$\mathbf{F}^{RS}(x) = \frac{q}{r^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$\mathbf{F}^{RS*}\left(\frac{x}{x.x}\right) = q\frac{|t^2-r^2|^3}{(t^2-r^2)r^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{q(t^2-r^2)^2\mathrm{sgn}(t^2-r^2)}{r^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This gives the dichotomic $\mathbf{E}'(x) = -q\frac{\text{sgn}(t^2-r^2)}{r^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{B}' = \mathbf{0}$.

This amounts to changing the sign of the charge of the particle within the particle's light cone ($t^2 > r^2$), or, disregarding the solution for $t < 0$, it is the field of a point particle that turns into its anti-particle at $t = 0$, the effect of which spreads out with the speed of light.
But a better way to cope with the twofold character of the transformed solution might be the following. Since the only purpose of the transformation is to find new solutions to Maxwell's equations and the transformation at hand is ill-defined on the light cone, it is allowed to change the sign of the result in any of the three parts in which the light-cone splits space-time. The resulting field will still be a solution, with the possible exception of the light-cone, where it might be ill-defined. If we apply this strategy we end up with the following physically acceptable possibilities, of which the first two can be extended smoothly across the light-cone.

- the transformed field is the same as the input field
- the transformation has the effect of charge conjugation $q \to -q$
- at $t = 0$ the charge of the particle changes sign, which has a spreading effect.

5.5.5 uniformly moving charged particle

(through origin, velocity v, direction $\hat{x}$)

$$\mathbf{F}^{RS}(x) = \frac{q}{A\|\mathbf{r}-\mathbf{v}t\|^3}\begin{pmatrix} x - vt \\ y - ivz \\ z + ivy \end{pmatrix},$$

in which $A = \gamma^2(1 - \beta^2 sin^2\psi)^{\frac{3}{2}}$ and $\psi = cos^{-1}(\mathbf{n}.\frac{\mathbf{v}}{v})$ and $\mathbf{n}$ a unit vector from the charge to fieldpoint ($\mathbf{r}$) [29].

This gives $\mathbf{F}^{RS\prime}(x) = -\text{sgn}(t^2 - r^2)\mathbf{F}^{RS}(x)$,

and again, this amounts to trading the particle for its antiparticle within the light cone (centered at the coordinate origin) and no change outside it. Exactly the same considerations concerning sign changes in part of the solution apply as in (**5**.**5**.**4**).

5.5.6 charged particle with uniform acceleration

(acceleration g, direction $\hat{z}$)
The case for uniform acceleration is much more difficult to handle, especially due to retardation effects. This touches the very interesting and conceptually difficult subject of radiation reaction for charges in uniform acceleration. A lot has been written about it for over a century, also in the context of general relativity. Some of the frequently cited authors are Bondi and Gold (1955) [53], Fulton and Rohrlich (1960) [54], DeWitt and Brehme (1960) [55], Boulware (1980) [56] and Parrott (2001) [57]. Any two authors agree on some points and disagree on others. Part of the problem is the interpretation of coordinates and related with this the correct definition of physical measurable quantities like energy.
Here we follow Lyle (Uniformly Accelerating Charged Particles: A Threat to the Equivalence Principle [58]), and take his solution (eq. (15.178)-(15.182)) adapted to our notations and conventions as our input field:

$$\mathbf{F}^{RS}(x) = \frac{q}{4\pi}\frac{g\theta(z+t)}{R^3}\begin{pmatrix} -xz+iyt \\ -yz-ixt \\ -\frac{1}{2}(z^2-t^2-x^2-y^2-\frac{1}{g^2}) \end{pmatrix} + \frac{q}{4\pi}\frac{2g^2\delta(z+t)}{1+g^2(x^2+y^2)}\begin{pmatrix} -x-iy \\ -y+ix \\ 0 \end{pmatrix},$$

with θ the Heaviside step function, δ Dirac's function, g the (constant) acceleration of the particle (taken to be in the +z direction), and $R := \frac{g}{2}[(\frac{1}{g^2}+x^2+y^2+z^2-t^2)^2 - 4(z^2-t^2)\frac{1}{g^2}]^{\frac{1}{2}}$.
The charge moves along the hyperbola $(\frac{1}{g}\text{cosh}(g\tau), 0, 0, \frac{1}{g}\text{sinh}(g\tau))$, in which the parameter τ is eigen time. It is important to note that the step function has been "put in by hand" in order not to violate causality principles. The Dirac function is needed for the field to be a solution of Maxwell's equations on all of Minkowski space.

For the calculation of the output field we need $\mathbf{F}^{RS*}(\frac{x}{x\cdot x})$:

We first calculate the transformed prefactor (skipping all the details) of the θ part

$R^3(\frac{x}{x\cdot x}) = \frac{g^3}{8}\frac{1}{|t^2-r^2|^3}[(\frac{r^2-t^2}{g^2}+1)^2 - \frac{4}{g^2}(z^2-t^2)]^{\frac{3}{2}}$ and note that this equals the original prefactor with g^{-1} substituted for g and an additional prefactor $\frac{1}{|t^2-r^2|^3}$:

$$R^3(\frac{x}{x.x}) = \frac{1}{|t^2-r^2|^3}\{R(g \to g^{-1})\}^3.$$

This is used to show that $\mathbf{E}'_\theta(x) = q\frac{\text{sgn}(t^2-r^2)}{4\pi}\frac{g^{-1}\theta(\frac{z+t}{t^2-r^2})}{(R(g^{-1}))^3}\begin{pmatrix} xz \\ yz \\ \frac{1}{2}(z^2-t^2-x^2-y^2-g^2) \end{pmatrix}$

and thus:

for $t^2 > r^2$: $\mathbf{E}'_\theta(x) = -\mathbf{E}_\theta(x; g \to g^{-1})$.

Note that in the region $z + t > 0$ and $t^2 - r^2 > 0$ (future pointing light cone) this amounts to changing the direction of the z-axis (besides the change in acceleration):

$\mathbf{E}'_\theta(x) = \mathbf{E}_\theta(\hat{z} \to -\hat{z}; g \to g^{-1})$.

For $t^2 < r^2$ we get a + sign instead of a - sign, but now the field is nonzero only for $z + t < 0$. It can be written as:

$\mathbf{E}'_\theta(x) = -\mathbf{E}_\theta(t \to -t, \hat{\mathbf{r}} \to -\hat{\mathbf{r}}; g \to g^{-1})$

A similar calculation leads to the transformed $\mathbf{B}$-fields:

$$\mathbf{B}'_\theta(x) = q\frac{\text{sgn}(t^2-r^2)}{4\pi}\frac{g^{-1}\theta(\frac{z+t}{t^2-r^2})}{(R(g^{-1}))^3}\begin{pmatrix}-yt\\ xt\\ 0\end{pmatrix}$$

and thus:

for $t^2 > r^2$: $\mathbf{B}'_\theta(x) = -\mathbf{B}_\theta(x; g \to g^{-1})$

for $t^2 < r^2$ we get a + sign instead of a - sign and it is nonzero only for $z + t < 0$.

The transformation of the δ-part:

As before, first the prefactor:

We need the following:

$\delta(\frac{z+t}{t^2-r^2}) = \delta(\frac{z+t}{(t-z)(z+t)-(x^2+y^2)}) = (x^2 + y^2)\delta(z + t)$

and frequently use the fact that $f(t^2 - r^2)\delta(z + t) = f(-(x^2 + y^2))\delta(z + t)$

in order to arrive at:

$\frac{2g^2}{1+g^2(x^2+y^2)}\delta(z + t) \to \frac{2g^2(x^2+y^2)^2}{g^2+(x^2+y^2)}\delta(z + t)$

and:

$$\mathbf{E}'_\delta(x) = \frac{q}{4\pi}\delta(z + t)\frac{2g^2}{(x^2+y^2)(g^2+x^2+y^2)}\begin{pmatrix}-x\\ -y\\ 0\end{pmatrix},$$

$$\mathbf{B}'_\delta(x) = \frac{q}{4\pi}\delta(z+t)\frac{2g^2}{(x^2+y^2)(g^2+x^2+y^2)}\begin{pmatrix}-y\\ x\\ 0\end{pmatrix}$$

This expression is unlike the θ part: it is not a simple replacement $g \to g^{-1}$. But these δ fields are defined only on a hypersurface of measure zero. They were in the first place defined in such a way that the combined θ and δ fields are a solution of Maxwell equations on the hypersurface where the δ field is nonzero, because the θ field alone is only a solution in Minkowski space with this hypersurface left out.

Note: if we really would like an expression for the δ field like that of the θ field, it would be:

$$\mathbf{E}'_\delta(x) = \frac{g^2}{x^2+y^2}\mathbf{E}_\delta(x; g \to g^{-1})\ , \quad \mathbf{B}'_\delta(x) = \frac{g^2}{x^2+y^2}\mathbf{B}_\delta(x; g \to g^{-1}).$$

Even if we disregard the δ part of the field, the resulting field is physically unacceptable. It would be due to an unrealistic charge distribution, with charges running backwards in time. Of course, this is due to the fact that conformal inversion can not be considered a valid coordinate transformation. Like before, artificial changes in the transformed field will result in acceptable solutions.

5.6 conclusion

We started this chapter in order to find out what can be learned from a symmetry in the equations of source free electrodynamics that is rarely mentioned and seems not to be investigated on a large scale: invariance under conformal transformations. We picked out an ingredient, conformal inversion, of this symmetry that together with well known Poincare transformations can be used to build all conformal transformations.

After studying some examples of conformal inversion we are left with the somewhat surprising result that a Hopf knot turns into a different, but simply related Hopf knot under this transformation. However, it remains difficult to attach meaning to the transformed fields in general. For the time being, perhaps the best we can do is to subscribe the conclusion of Parrott [59], "This [conformal invariance] rather remarkable mathematical fact seems to have no obvious physical interpretation.", and consider the nice transformation formula for the Riemann-Silberstein vector only as a way to generate new solutions from old ones.

Appendix A method of Lie

Starting point is the linear differential equation $\hat{L}(x^\mu, \partial_\mu)\psi(x^\mu) = 0$, in which $\hat{L}$ is a linear operator and ψ stands for $(\psi_1, ..., \psi_r)$. First order differential operators, $\hat{Q}_A = \xi^\mu_A(x,\psi)\partial_\mu + \eta^k_A(x,\psi)\frac{\partial}{\partial\psi_k}$, are sought that satisfy the invariance condition $\hat{L}\hat{Q}_A\psi(x) = 0$ and form a Lie algebra $[\hat{Q}_A, \hat{Q}_B] = C_{ABC}\hat{Q}_C$. The solution leads to the invariance algebra for the differential equation, with help of which the symmetry group can be constructed [44].
This construction clearly shows that related to a symmetry, new solutions can be obtained from old ones. It is also clear that there exists many more symmetries not obtainable by this method.

Appendix B frequently used equalities

(1) $\frac{\partial f}{\partial x_\alpha} = \frac{\partial f}{\partial x^\beta}\frac{\partial x^\beta}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}(\eta^{\beta\sigma}x_\sigma)\frac{\partial f}{\partial x^\beta} = \eta^{\beta\alpha}\frac{\partial f}{\partial x^\beta}$

This relation would not be valid if raising and lowering is done by g.

(2) $|\frac{\partial(\tilde{x})}{\partial(x)}| = |\frac{1}{(x.x)^4}| = |\frac{\partial(x)}{\partial(\tilde{x})}|^{-1}$

This is a long but straightforward calculation.
The Jacobian of the transformation equals $\frac{-1}{(x.x)^4}$. We use the convention $|\frac{\partial(\tilde{x})}{\partial(x)}| =$ absolute value of the Jacobian.

(3) $\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu}\eta_{\alpha\beta} = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\eta_{\mu\nu}$ and $\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}\eta_{\alpha\beta} = |\frac{\partial(x)}{\partial(\tilde{x})}|^{\frac{1}{2}}\eta_{\mu\nu}$

The second follows from the first by noting the functional similarity between f and f^{-1}.

(4) $\frac{\partial\tilde{x}_\mu}{\partial x_\nu} = \eta_{\mu\lambda}\frac{\partial\tilde{x}^\lambda}{\partial x^\rho}\eta^{\rho\nu} = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\lambda}\eta_{\alpha\beta}\frac{\partial\tilde{x}^\lambda}{\partial x^\rho}\eta^{\rho\nu} = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\delta^\beta_\rho\eta_{\alpha\beta}\eta^{\rho\nu} = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\delta^\nu_\alpha = |\frac{\partial(\tilde{x})}{\partial(x)}|^{\frac{1}{2}}\frac{\partial x^\nu}{\partial\tilde{x}^\mu}$

Use has been made of (1) and (3).

(5) For conformal inversions: $\frac{\partial\tilde{x}_\mu}{\partial x_\nu} = \frac{\partial\tilde{x}^\nu}{\partial x^\mu}$

Appendix C is F a Faraday tensor?

We need

(1) $\partial_\mu \frac{\partial x^\mu}{\partial \tilde{x}^\lambda} = \partial_\mu(\frac{\delta^\mu_\lambda}{(\tilde{x}.\tilde{x})} - 2\frac{\tilde{x}^\mu \tilde{x}_\lambda}{(\tilde{x}.\tilde{x})^2}) = \partial_\mu(\frac{\delta^\mu_\lambda}{(\tilde{x}.\tilde{x})} - 2x^\mu \eta_{\lambda\alpha} x^\alpha) = \partial_\lambda \frac{1}{\tilde{x}.\tilde{x}} - 8\eta_{\lambda\alpha}x^\alpha - 2x^\mu \eta_{\lambda\alpha}\delta^\alpha_\mu = \partial_\lambda(x.x) - 10\eta_{\lambda\alpha}x^\alpha = 2x_\lambda - 10\eta_{\lambda\alpha}x^\alpha = -8\eta_{\lambda\alpha}x^\alpha$

(2) $\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}(\partial_\mu \frac{\partial x^\nu}{\partial \tilde{x}^\rho})\tilde{F}^{\lambda\rho} = \frac{\partial^2 x^\nu}{\partial \tilde{x}^\lambda \partial \tilde{x}^\rho}\tilde{F}^{\lambda\rho} = 0$

(3) $\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\partial_\mu \tilde{F}^{\lambda\rho} = \partial_{\tilde{\lambda}}\tilde{F}^{\lambda\rho} = 0$

Thus:

$\partial_\mu(|\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\tilde{F}^{\lambda\rho}(\tilde{x}(x))) = (\partial_\mu|\frac{\partial(\tilde{x})}{\partial(x)}|)\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\tilde{F}^{\lambda\rho}(\tilde{x}(x)) + |\frac{\partial(\tilde{x})}{\partial(x)}|(-8\eta_{\lambda\alpha}x^\alpha)\frac{\partial x^\nu}{\partial \tilde{x}^\rho}\tilde{F}^{\lambda\rho}(\tilde{x}(x)) + 0 + 0$

The first part of the first term is:

$(\partial_\mu|\frac{\partial(\tilde{x})}{\partial(x)}|)\frac{\partial x^\mu}{\partial \tilde{x}^\lambda} = (\partial_\mu(x.x)^{-4})\frac{\partial x^\mu}{\partial \tilde{x}^\lambda} = \frac{-8x_\mu}{(x.x)^5}\frac{\partial x^\mu}{\partial \tilde{x}^\lambda} = -8(x.x)^{-5}\eta_{\mu\alpha}x^\alpha(\frac{\delta^\mu_\lambda}{(\tilde{x}.\tilde{x})} - 2\frac{\tilde{x}^\mu \tilde{x}_\lambda}{(\tilde{x}.\tilde{x})^2}) = -8(x.x)^{-4}(\eta_{\lambda\alpha}x^\alpha - 2x^\mu \tilde{x}_\lambda x^\alpha \eta_{\mu\alpha})$

The last term in brackets is $-2x^\mu \tilde{x}_\lambda x^\alpha \eta_{\mu\alpha} = -2\eta_{\lambda\alpha}x^\alpha$, and this makes the first term (in the complete expression) cancel the second in the complete expression.

Appendix D transformation of self-dual form

We have: $F'^{(sd)}_{\mu\nu} = (f'_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\lambda\rho}f'^{\lambda\rho}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)f_{\alpha\beta}(\tilde{x}) - \frac{\partial \tilde{x}^\alpha}{\partial x^\kappa}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\sigma}(x)\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}f^{\kappa\sigma}(\tilde{x})$.

Now $\frac{\partial \tilde{x}^\alpha}{\partial x^\kappa}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\sigma}(x)\epsilon_{\alpha\beta\mu\nu} = -\epsilon_{\kappa\sigma\alpha\beta}|\frac{\partial(\tilde{x})}{\partial(x)}|\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} = -\epsilon_{\kappa\sigma\alpha\beta}\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x)\frac{\partial \tilde{x}^\beta}{\partial x^\nu}(x)$.

The second equality follows from appendix **B**(4)-(5) and the first comes from the following property of the ϵ symbol:

$M^\alpha_a M^\beta_b M^\gamma_c M^\delta_d \epsilon_{\alpha\beta\gamma\delta} = \epsilon_{abcd}(det(M))$

and thus $M^\alpha_a M^\beta_b \epsilon_{\alpha\beta\lambda\rho} = \epsilon_{abcd}(M^{-1})^c_\lambda (M^{-1})^d_\rho (det(M))$.

The required equality follows from taking $M^\alpha_\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}$ (so $det(M) = \frac{-1}{(x.x)^4} = -|\frac{\partial(\tilde{x})}{\partial(x)}|$).

Appendix E conformal inversion of electromagnetic Hopf knot

Here we will use the transformation formula for the (anti-) self-dual form of the electromagnetic field tensor.
We have:

$$(F^{(asd)}_{\mu\nu}(x)) = \begin{pmatrix} 0 & E_x + iB_x & E_y + iB_y & E_z + iB_z \\ -E_x - iB_x & 0 & -B_z + iE_z & B_y - iE_y \\ -E_y - iB_y & B_z - iE_z & 0 & -B_x + iE_x \\ -E_z - iB_z & -B_y + iE_y & B_x - iE_x & 0 \end{pmatrix}^* =$$

$$\frac{1}{((t+i)^2 - r^2)^3} \begin{pmatrix} 0 & a^2 - d^2 & -ia^2 - id^2 & -2ad \\ -a^2 + d^2 & 0 & 2iad & a^2 + d^2 \\ ia^2 + id^2 & -2iad & 0 & -ia^2 + id^2 \\ 2ad & -a^2 - d^2 & ia^2 - id^2 & 0 \end{pmatrix}$$

in which $a \equiv x + iy$ and $d \equiv t - z + i$.

Thus: $$(F^{(asd)}_{\mu\nu}(\tfrac{x}{t^2 - r^2})) = \frac{-(t^2 - r^2)}{((t-i)^2 - r^2)^3} \begin{pmatrix} 0 & a^2 - b^2 & -ia^2 - ib^2 & -2ab \\ -a^2 + b^2 & 0 & 2iab & a^2 + b^2 \\ ia^2 + ib^2 & -2iab & 0 & -ia^2 + ib^2 \\ 2ab & -a^2 - b^2 & ia^2 - ib^2 & 0 \end{pmatrix}$$

in which $a \equiv x + iy$ and $b \equiv t - z + i(t^2 - r^2)$.

Using the transformation formula, we now find:

$F'^{(sd)}_{0i}(x) = \frac{F^{(asd)}_{0i}(\frac{x}{t^2-r^2})}{(t^2-r^2)^2} + 2\frac{x^i x^j}{(t^2-r^2)^3} F^{(asd)}_{0j}(\frac{x}{t^2-r^2}) - 2\frac{t x^\alpha}{(t^2-r^2)^3} F^{(asd)}_{\alpha i}(\frac{x}{t^2-r^2})$
$= \frac{1}{(t^2-r^2)^3}\{(t^2 - r^2) - 2t^2\} F^{(asd)}_{0i}(\frac{x}{t^2-r^2}) + 2\frac{x^i x^j}{(t^2-r^2)^3} F^{(asd)}_{0j}(\frac{x}{t^2-r^2}) - 2\frac{t x^j}{(t^2-r^2)^3} F^{(asd)}_{ji}(\frac{x}{t^2-r^2})$
$= \frac{-1}{(t^2-r^2)^3}\{t^2 + r^2 - 2(x^i)^2\} F^{(asd)}_{0i}(\frac{x}{t^2-r^2}) + \{2\frac{x^i x^j}{(t^2-r^2)^3} F^{(asd)}_{0j}(\frac{x}{t^2-r^2}) -$

$2\frac{tx^j}{(t^2-r^2)^3}F_{ji}^{(asd)}(\frac{x}{t^2-r^2})\}_{j\neq i}$

Substituting from above we find for the x component of the RS vector:

$F_{01}^{\prime(sd)}(x) = \frac{1}{(t^2-r^2)^3}(t^2-x^2+y^2+z^2)\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{(x+iy)^2-(t-z+i(t^2-r^2))^2\} - 2\frac{xy}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-i(x+iy)^2-i(t-z+i(t^2-r^2))^2\} - 2\frac{xz}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-2(x+iy)(t-z+i(t^2-r^2))\} + 2\frac{ty}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-2i(x+iy)(t-z+i(t^2-r^2))\} + 2\frac{tz}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-(x+iy)^2-(t-z+i(t^2-r^2))^2\}$

$\equiv \frac{A}{(t^2-r^2)^2((t-i)^2-r^2)^3}$

in which:

$A = (t^2-x^2+y^2+z^2)\{(x+iy)^2-(t-z+i(t^2-r^2))^2\} + 2ixy\{(x+iy)^2+(t-z+i(t^2-r^2))^2\} + 4xz\{(x+iy)(t-z+i(t^2-r^2))\} - 4ity\{(x+iy)(t-z+i(t^2-r^2))\} - 2tz\{(x+iy)^2+(t-z+i(t^2-r^2))^2\}$

$= (-(t+z)^2+(x+iy)^2)\{t-z+i(t^2-r^2)\}^2 + ((t-z)^2-(x-iy)^2)(x+iy)^2 + 4(xz-ity)(x+iy)(t-z+i(t^2-r^2))$

(Now use $(xz-ity) = \frac{1}{2}\{(x-iy)(t+z)-(x+iy)(t-z)\}$)

$= (-(t+z)^2+(x+iy)^2)\{t-z+i(t^2-r^2)\}^2 + ((t-z)^2-(x-iy)^2)(x+iy)^2 + 2\{(x-iy)(t+z)-(x+iy)(t-z)\}(x+iy)(t-z+i(t^2-r^2))$

$= \{-(t+z)^2+(x+iy)^2\}\{(t-z)+i(t^2-r^2)\}^2 + \{(t-z)^2-(x-iy)^2\}(x+iy)^2 + 2(x-iy)(t+z)(x+iy)\{(t-z)+i(t^2-r^2)\} - 2(x+iy)^2(t-z)\{(t-z)+i(t^2-r^2)\}$

$= \{-(t+z)^2+(x+iy)^2\}\{(t-z)+i(t^2-r^2)\}^2 + \{-(t-z)^2-(x-iy)^2\}(x+iy)^2 + 2(x-iy)(x+iy)(t+z)\{(t-z)+i(t^2-r^2)\} - 2(x+iy)^2(t-z)i(t^2-r^2)$

$= -(t+z)^2\{(t-z)+i(t^2-r^2)\}^2 + (x+iy)^2\{(t-z)^2-(t^2-r^2)^2\} + \{-(t-z)^2-(x-iy)^2\}(x+iy)^2 + 2(x-iy)(x+iy)(t+z)\{(t-z)+i(t^2-r^2)\}$

$= -(t+z)^2\{(t-z)+i(t^2-r^2)\}^2 - (x+iy)^2(t^2-r^2)^2 - (x-iy)^2(x+iy)^2 + 2(x-iy)(x+iy)(t+z)\{(t-z)+i(t^2-r^2)\}$

$= -(t+z)^2\{(t-z)^2-(t^2-r^2)^2+2i(t-z)(t^2-r^2)\} - (x+iy)^2\{(t^2-r^2)^2+(x-iy)^2\} + 2(x-iy)(x+iy)(t+z)(t-z) + 2i(x-iy)(x+iy)(t+z)(t^2-r^2)$

$= -(t+z)^2\{(t-z)^2-(t^2-r^2)^2\} - (x+iy)^2\{(t^2-r^2)^2+(x-iy)^2\} + 2(x-iy)(x+iy)(t+z)(t-z) - 2i(t+z)(t^2-r^2)^2$

$= (t^2 - r^2)^2\{(t+z)^2 - (x+iy)^2 - 2i(t+z)\} - (t-z)^2(t+z)^2 - (x+iy)^2(x-iy)^2 + 2(x^2+y^2)(t^2-z^2)$

$= (t^2 - r^2)^2\{(t+z)^2 - (x+iy)^2 - 2i(t+z)\} - (t^2-z^2)^2 - (x^2+y^2)^2 + 2(x^2+y^2)(t^2-z^2)$

$= (t^2 - r^2)^2\{(t+z)^2 - (x+iy)^2 - 2i(t+z)\} - \{(t^2-z^2) - (x^2+y^2)\}^2$

$= (t^2 - r^2)^2\{-1 + (t+z)^2 - (x+iy)^2 - 2i(t+z)\}$

And thus:

$(\mathbf{F}^{RS\prime})_x(x) = \frac{1}{((t-i)^2-r^2)^3}\{-1 + (t+z)^2 - x^2 + y^2 - 2i(t+z+xy)\}$

$= \frac{-1}{((t-i)^2-r^2)^3}\{(x+iy)^2 - (t-i+z)^2\}$

$= -(\mathbf{F}^{RS}(t \to -t))^*_x$

The y component is rather simple now:

$F'^{(sd)}_{02}(x) = \frac{1}{(t^2-r^2)^3}(t^2-y^2+x^2+z^2)\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-i(x+iy)^2 - i(t-z+i(t^2-r^2))^2\} - 2\frac{xy}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{(x+iy)^2 - (t-z+i(t^2-r^2))^2\} - 2\frac{yz}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{-2(x+iy)(t-z+i(t^2-r^2))\} + 2\frac{tx}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{2i(x+iy)(t-z+i(t^2-r^2))\} + 2\frac{tz}{(t^2-r^2)^3}\frac{t^2-r^2}{((t-i)^2-r^2)^3}\{i(x+iy)^2 - i(t-z+i(t^2-r^2))^2\}$

Notice: $F'^{(sd)}_{02}(x) = iF'^{(sd)}_{01}(x \to -y, y \to x)$ as can be seen from the expression for $F'^{(sd)}_{01}$ above.

So: $F'^{(sd)}_{02}(x) = \frac{-i}{((t-i)^2-r^2)^3}\{(-y+ix)^2 - (t-i+z)^2\} = \frac{-1}{((t-i)^2-r^2)^3}\{-i(x+iy)^2 - i(t-i+z)^2\}$

$= -(\mathbf{F}^{RS}(t \to -t))^*_y.$

What remains is the z component (in less detail than before):

With the details of $F'^{(sd)}_{01}$ fresh in mind it is not difficult to see the first step:

$F'^{(sd)}_{03}(x) = \frac{B}{(t^2-r^2)^2((t-i)^2-r^2)^3}$

in which:

$B = (t^2 + x^2 + y^2 - z^2)\{-2(x + iy)(t - z + i(t^2 - r^2))\} - 2xz\{(x + iy)^2 - (t - z + i(t^2 - r^2)^2\} + 2iyz\{(x + iy)^2 + (t - z + i(t^2 - r^2))^2\} + 2tx\{(x + iy)^2 + (t - z + i(t^2 - r^2))^2\} + 2ity\{-(x + iy)^2 + (t - z + i(t^2 - r^2))^2\}$

$= \{t - z + i(t^2 - r^2)\}^2 2(x + iy)(t + z) + (x + iy)^2 2(x - iy)(t - z) - 2(x + iy)(t - z + i(t^2 - r^2))(t^2 + x^2 + y^2 - z^2)$

$= \{(t - z)^2 - (t^2 - r^2)^2 + 2i(t - z)(t^2 - r^2)\}2(x + iy)(t + z) + (x + iy)^2 2(x - iy)(t - z) - 2(x + iy)(t - z)(t^2 - z^2 + x^2 + y^2) - 2(x + iy)i(t^2 - r^2)(t^2 - z^2 + x^2 + y^2)$

$= 2(x + iy)\{-(t^2 - r^2)^2(t + z) + 2i(t - z)(t + z)(t^2 - r^2) - i(t^2 - r^2)(t^2 - z^2 + x^2 + y^2)\}$

$= 2(x + iy)\{-(t^2 - r^2)^2(t + z) + i(t^2 - r^2)\{t^2 - z^2 - x^2 - y^2\}\}$

$= -2(x + iy)(t^2 - r^2)^2(t - i + z)$

Thus: $F_{03}^{\prime(sd)}(x) = \frac{-1}{((t-i)^2 - r^2)^3} 2(x + iy)(t - i + z)$ and this agrees with the previous results. Combining the components:

$\mathbf{F}^{RS\prime}(x) = -\{\mathbf{F}^{RS}(t \to -t)\}^*$

Q.E.D.

6

Linked field configurations and the Penrose transform

This chapter marks the transition from electrodynamics to general relativity. Here we 'transform' an electromagnetic Hopf knot into a gravitational Hopf knot. For this we need the twistor functions leading via the Penrose transform to respective knots. In a sense these functions are to be considered the same and were, in present context, first considered in [11]. An article on the main results of this chapter, J. Swearngin, A. Thompson, A.Wickes, J.W. Dalhuisen, and D. Bouwmeester, Linked Gravitational Radiation, arXiv preprint arXiv:1302.1431 (2013) has been submitted for publication.

6.1 introduction

In chapter 4 we used Robinson's theorem to show that there is a correspondence between a non-null twistor and an electromagnetic Hopf knot. Robinson's theorem associates a null electromagnetic field to every geodesic shear free null congruence. Here we will consider the Penrose transform, a contour integral formula that associates solutions to the zero rest mass free field equations with twistor functions that satisfy certain conditions. Since vacuum Maxwell's equations form a particular instance of a zero rest mass equation, it must be possible to arrive at an electromagnetic Hopf knot from a specific twistor function. Having found this twistor function, it can be used in a modified form to generate solutions to linearized Einstein's equations, which forms another example of a zero rest mass equation. Although this strategy could have been used with respect to a generalized Robinson's theorem [60], its solutions depend on solving a set of difficult partial differential equations. When using the contour integral formula the solution comes 'automatically'.

From the discussion of the Kerr theorem in chapter 1 we know that every analytic geodesic shear-free null congruence is related to a holomorphic twistor function that is homogeneous of some degree (1.16). Combined with Robinson's theorem we thus have a relation between homogeneous holomorphic twistor functions and solutions to the zero rest mass equation. The precise relation is encoded in the following contour integral, known as the Penrose transform [27]:

$$\bar{\phi}_{X'...Y'}(x^{\mu}) = \frac{1}{2\pi i}\oint \bar{\pi}_{X'}...\bar{\pi}_{Y'}\rho_x f(\mathcal{Z}^{\alpha})\bar{\pi}_{Z'}d\bar{\pi}^{Z'} \tag{6.1}$$

or

$$\psi_{A...B}(x^{\mu}) = \frac{1}{2\pi i}\oint \rho_x \frac{\partial}{\partial\omega^A}...\frac{\partial}{\partial\omega^B} f(\mathcal{Z}^{\alpha})\bar{\pi}_{Z'}d\bar{\pi}^{Z'} \tag{6.2}$$

where the symbol ρ_x means that the twistors are restricted to correspond to the line through $x = (x^{\mu})$:

$(\mathcal{Z}^{\alpha}) = (\omega^A, \bar{\pi}_{X'})$, $\omega^A = iX^{AX'}\bar{\pi}_{X'}$ and $X^{AX'}$ corresponds to x (1.13).

The integrals in equations (6.1) and (6.2) make sense only when the twistor function has the correct degree of homogeneity (see also the remark concerning projective twistor space in chapter 1). In equation (6.1) this means -n-2, with n the number of indices in $X'...Y'$ and in (6.2) this means n-2, now with n the number of indices in $A...B$.
That the fields $\phi_{X'...Y'}$ and $\psi_{A...B}$ satisfy the zero rest mass free field equation follows by noting that $\nabla_A{}^{X'} = \epsilon^{X'Y'}\frac{\partial}{\partial X^{AY'}}$ and $\frac{\partial}{\partial X^{AY'}}\rho_x f(\mathcal{Z}^{\alpha}) = i\bar{\pi}_{Y'}\rho_x\frac{\partial f}{\partial\omega^A}$: the resulting expression is a product of an anti-symmetric quantity in two indices and a symmetric quantity with respect to the same indices. The n=0 case leads to the wave equation $\Box\phi = 0$.
From equations (6.1) and (6.2) it follows that there is not a single twistor function that corresponds to a particular solution to the zero rest mass equation, but a whole class of twistor functions. This touches upon the subject of sheaf cohomology [61], that will not be needed for our purposes.

6.2 examples

Fields that result from the Penrose transform of twistor functions that have the form $\frac{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)^p (\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)^q}{(\bar{\mathcal{C}}_\alpha \mathcal{Z}^\alpha)^r (\bar{\mathcal{D}}_\beta \mathcal{Z}^\beta)^s}$ $(p, q, r, s \in \mathbb{N} \cup \{0\})$, are called elementary states.
We will consider three examples, n=0, n=1 and n=2, in which in the first example we use the simplest possible twistor function in (6.1) leading to a non-trivial elementary state. In the two following examples we then alter the twistor function minimally, by raising one of the powers in the denominator, in order to satisfy the homogeneity requirements. In a quantum context this could be seen as a spin or helicity raising operation. In a sense the resulting fields can be considered as generalizations of each other.

6.2.1 wave equation

(n=0) and we take $f(\mathcal{Z}^\alpha) = \frac{1}{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)(\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)}$

$\bar{\mathcal{A}}_\alpha = (A_A, \bar{C}^{X'})$ and $\bar{\mathcal{B}}_\alpha = (B_A, \bar{D}^{X'})$ are fixed dual twistors.

This leads to the integral $\bar{\phi}(x^\mu) = \frac{1}{2\pi i} \oint \frac{1}{(\bar{\alpha}^{X'} \bar{\pi}_{X'})(\bar{\beta}^{Y'} \bar{\pi}_{Y'})} \bar{\pi}_{Z'} d\bar{\pi}^{Z'}$,

where $\bar{\alpha}^{X'} = iA_A X^{AX'} + \bar{C}^{X'}$ and $\bar{\beta}^{Y'} = iB_A X^{AY'} + \bar{D}^{Y'}$.

Suppose that $\bar{\mathcal{A}}_\alpha$ and $\bar{\mathcal{B}}_\alpha$ are such that $\bar{\alpha}^{X'}$ and $\bar{\beta}^{X'}$ provide a basis in (conjugate) spin-space. The poles in the integral are then different and we can write $\bar{\pi}_{Z'} = \bar{\alpha}_{Z'} + z\bar{\beta}_{Z'}$ and $d\bar{\pi}^{Z'} = \bar{\beta}^{Z'} dz$ for some complex z. Choosing the contour to enclose only the pole with respect to $\bar{\mathcal{A}}_\alpha$ we now have

$\bar{\phi}(x^\mu) = \frac{1}{2\pi i} \oint \frac{dz}{(\bar{\alpha}^{X'} \bar{\beta}_{X'}) z} = \frac{1}{\bar{\alpha}^{X'} \bar{\beta}_{X'}} = \frac{2}{A_A B^A (x^\mu - y^\mu)(x_\mu - y_\mu)}$.

Where the last equality follows if we assume in addition that there exists a $y \in M^4_{\mathbb{C}}$ corresponding to $Y^{AX'}$ such that

$\bar{C}^{X'} = -iY^{AX'} A_A$ and $\bar{D}^{X'} = -iY^{AX'} B_A$.

(Compare the discussion of the incidence relation (1.12); note that above equation implies $C^A = i\bar{Y}^{AX'} \bar{A}_{X'}$.)
This assumption is equivalent to $\mathcal{A}^\alpha \bar{\mathcal{B}}_\alpha = iB_A(\bar{Y}^{AX'} - Y^{AX'})\bar{A}_{X'}$ and in case the two twistors are null, this equals 0 and then it implies that the corresponding geodesics have one point in M^4 in common.
We can solve the two equations to obtain an expression for $Y^{AX'}$:

$Y^{AX'} = i\frac{A^A\bar{D}^{X'} - B^A\bar{C}^{X'}}{A^A B_A}$

In the calculation of the last expression of $\bar{\phi}$ the following identity is useful:

$A_B B^A - A^A B_B = \delta_B{}^A A_C B^C.$

6.2.2 source-free electrodynamics

(n=1) and we take $f(\mathcal{Z}^\alpha) = \frac{1}{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)(\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)^3}$

We now have to compute

$\bar{\phi}_{X'Y'}(x^\mu) = \frac{1}{2\pi i}\oint \bar{\pi}_{X'}\bar{\pi}_{Y'}\rho_x \frac{1}{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)(\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)^3}\bar{\pi}_{Z'}d\bar{\pi}^{Z'}$,

and with the same assumptions for $\mathcal{A}^\alpha$ and $\mathcal{B}^\alpha$ as in the previous example this leads to

$\bar{\phi}_{X'Y'}(x^\mu) = \frac{1}{2\pi i}\oint(\bar{\alpha}_{X'} + \bar{\beta}_{X'}z)(\bar{\alpha}_{Y'} + \bar{\beta}_{Y'}z)\frac{1}{z(\bar{\alpha}^{Z'}\bar{\beta}_{Z'})^3}dz = (\frac{2}{A_A B^A(x^\mu - y^\mu)(x_\mu - y_\mu)})^3\bar{\alpha}_{X'}\bar{\alpha}_{Y'}$,

or, for comparison with equation (4.7):

$\phi_{AB}(x^\mu) = (\frac{2}{\bar{A}_{X'}\bar{B}^{X'}(x^\mu - \bar{y}^\mu)(x_\mu - \bar{y}_\mu)})^3\alpha_A\alpha_B.$

If we now take $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$ we get $(\alpha_A) = \frac{-i}{\sqrt{2}}(-t + z - i, x + iy)$. Comparison with equation (4.6) shows that the solution belongs to the same class of electromagnetic Hopf knots as the one given in equation (4.1), which will be reproduced exactly for $(\mathcal{B}^\alpha) = (\frac{-2}{\sqrt{2}}, \frac{b}{\sqrt{2}}, -2, b)$ and arbitrary b. With present choices of $(\mathcal{A}^\alpha)$ and $(\mathcal{B}^\alpha)$ we have $(y^\mu) = (i, 0, 0, 0)$, $\bar{A}_{X'}\bar{B}^{X'} = 2$ and $\bar{\alpha}_{X'}\bar{\beta}^{X'}(x^\mu) \neq 0$, so that indeed $\bar{\alpha}^{X'}$ and $\bar{\beta}^{X'}$ do form a basis for all (x^μ).
A possible choice for b would be -2 and we know from chapter 4 that in this case $\mathcal{B}$ corresponds via Robinson's theorem with the class of which a rotated version of equation (4.1) is a representative.

Starting with the twistor $(\mathcal{A}^\alpha) = (0, \frac{1}{\sqrt{2}}, 0, 1)$, in chapter 4 we had to solve partial differential equations for $g(x^\mu)$ in order to arrive at a solution of Maxwell's equations. Here we start with the same twistor $\mathcal{A}$ and have to find another twistor $\mathcal{B}$ algebraically. In chapter 4 we could solve the problem easily because we knew what to look for, otherwise it would have been a hard exercise. The Penrose transform makes this exercise much simpler.

In the following example we will use $(\mathcal{A}^\alpha) = (i, 1, i\sqrt{2}, \sqrt{2})$ and therefore we note here that a representative of the class that Robinson's theorem associates with this twistor is given by a rotated version of the field (4.1) (rotation around x-axis such that the z-axis turns into the y-axis):

$$\mathbf{F}^{RS}(t, \mathbf{r}) = \frac{1}{((t-i)^2 - r^2)^3} \begin{pmatrix} (x-iz)^2 - (t-i+y)^2 \\ 2(x-iz)(t-i+y) \\ i(x-iz)^2 + i(t-i+y)^2 \end{pmatrix}. \tag{6.3}$$

When combined with $(\mathcal{B}^\alpha) = (\frac{1}{2})^{\frac{2}{3}}(1, i, \sqrt{2}, i\sqrt{2})$ the Penrose transform also leads to this field. The interchange of $\mathcal{A}$ and $\mathcal{B}$ would lead to a 180^0 rotation around the z-axis of the field (6.3).

Present discussion shows that an electromagnetic Hopf knot is to be considered as what is known in the twistor literature as an elementary state.

6.2.3 linearized Einstein equation

(n=2) and we take $f(\mathcal{Z}^\alpha) = \frac{1}{(\bar{\mathcal{A}}_\alpha \mathcal{Z}^\alpha)(\bar{\mathcal{B}}_\beta \mathcal{Z}^\beta)^5}$

In chapter 3 we showed how the spinor form of Maxwell's equations can be derived. A similar but much more complicated analysis applied to Einstein's equations lead to the following result [62]. The spinor form of the Riemann tensor is (a short mathematical introduction to the tensors used here is given in section 7.2)

$R_{AW'BX'CY'DZ'} =$

$\Phi_{ABCD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \bar{\Phi}_{W'X'Y'Z'}\epsilon_{AB}\epsilon_{CD} +$
$\Psi_{ABY'Z'}\epsilon_{W'X'}\epsilon_{CD} + \bar{\Psi}_{CDW'X'}\epsilon_{AB}\epsilon_{Y'Z'} +$
$2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{W'Z'}\epsilon_{X'Y'})$,

in which

$\Phi_{ABCD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \bar{\Phi}_{W'X'Y'Z'}\epsilon_{AB}\epsilon_{CD}$ corresponds to the Weyl tensor $C_{\kappa\lambda\mu\nu}$,

$-2\Psi_{ABW'X'}$ corresponds to the trace free Ricci tensor $R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}$ and 24Λ corresponds to the Ricci scalar R.

From this we see that in vacuum the Einstein field equations are simply $\Psi_{ABW'X'} = 0 = \Lambda$. The Bianchi identity in vacuum gives $\nabla^{AX'}\Phi_{ABCD} = 0$. It is important to note that in a curved space-time the form (1.4) for the Infeld van der Waerden symbols do not solve the requirement (1.3). We first need the metric before we can find a suitable representation for the symbols. Therefore we cannot solve the vacuum Bianchi identity for Φ_{ABCD} without further assumptions. But, in the weak field limit (or linearized Einstein theory) we have $\nabla^{AX'}\phi_{ABCD} = 0$ for the linearized Weyl spinor ϕ_{ABCD} and the usual Minkowski space form of the differential operators. Conversely, in vacuum, a solution to this zero rest mass field equation defines a linearized Weyl tensor [63].

In electrodynamics, the sources are represented by the four-current (j^{μ}) and the field degrees of freedom by the Faraday tensor $(F^{\mu\nu})$. The analogues in general relativity are the energy-momentum tensor $(T^{\mu\nu})$ for the sources and the Weyl tensor (C^{abcd}) for the field. The Weyl tensor can be regarded as the Riemann tensor with the influences of the sources "taken out of it". So, when classifying gravitational fields, it is natural to look at the Weyl tensor, not the Riemann tensor. The Weyl spinor, the spinor equivalent of the Weyl tensor, is completely symmetric. Its principal spinors define four principal null directions that are used to classify fields.

There exists a striking algebraic similarity between the electric $(E_i \equiv F_{0i})$ and magnetic $(B_i \equiv *F_{0i})$ part of a Faraday tensor in Maxwell theory and the electric part $(E_{ij} \equiv C_{i0j0})$ and magnetic part $(B_{ij} \equiv - * C_{i0j0})$ of the Weyl tensor in general relativity [64,65] .
Since $E_{ij}\xi^j$ measures the relative (tidal) acceleration between to points separated by a small vector $\boldsymbol{\xi}$, we can interpret the electric part of the Weyl tensor as a tidal field. The matrix (E_{ij}) is traceless and symmetric, and can therefore be characterized by its eigenvalues and eigenvectors, the integral curves of which are called tendex lines [66] and can be considered the gravitational analogue of electric field lines. An extended object placed in the field E_{ij} will be stretched in the direction of the lines corresponding to positive eigenvalues and compressed along the curves corresponding to negative eigenvalues. The strength of this effect is related to the eigenvalue.

The interpretation of the magnetic part of the Weyl tensor is as a frame-drag field (possibly a misnomer, see [67,68]). A gyroscope at the tip of the vector $\boldsymbol{\xi}$ will precess with angular velocity $B_{ij}\xi^j$ relative to inertial frames at the tail. The matrix (B_{ij}) is also traceless and symmetric and the integral curves of the eigenvectors are now called vortex lines [66].

Continuing now the n=2 example, with the help of the previous examples

it can be seen immediately that $\phi_{ABCD}(x^\mu) \propto \alpha_A\alpha_B\alpha_C\alpha_D$. There is only one fourfold degenerate principal null direction. In the Petrov classification of gravitational fields this corresponds to a type N field (see section **7.6**).
For $(\mathcal{A}^\alpha) = (i, 1, i\sqrt{2}, \sqrt{2})$ and $(\mathcal{B}^\alpha) = (\frac{1}{2})^{\frac{2}{3}}(1, i, \sqrt{2}, i\sqrt{2})$ the calculation leads to the eigenvalues $-\mathsf{a}, 0$ and a for both the electric part and the magnetic part of the Weyl tensor. The corresponding (eigen) vector fields will be denoted by $\mathbf{E}_-, \mathbf{E}_0, \mathbf{E}_+$ and $\mathbf{B}_-, \mathbf{B}_0, \mathbf{B}_+$ respectively, and we will use the Riemann-Silberstein like combination $\mathbf{F}_E \equiv \mathbf{E}_- + i\mathbf{E}_+$ and $\mathbf{F}_B \equiv \mathbf{B}_- + i\mathbf{B}_+$.

A detailed calculation shows:

$$\mathsf{a}(x^\mu) = \frac{(1+x^2+(y+t)^2+z^2)^2}{2^{\frac{2}{3}}((t^2-r^2)^2+1+2(t^2+r^2))^{\frac{5}{2}}},$$

$$\mathbf{E}_0(x^\mu) = \mathbf{B}_0(x^\mu) \propto \begin{pmatrix} 2(x(t+y)+z) \\ 1+(t+y)^2-x^2-z^2 \\ 2(z(t+y)-x) \end{pmatrix}$$

and

$$\mathbf{F}_E(x^\mu) = e^{i\frac{\pi}{4}}\mathbf{F}_B(x^\mu) \propto e^{i\theta}\mathbf{F}^{RS}_{(6.3)}(x^\mu)$$

Here, $\mathbf{F}^{RS}_{(6.3)}$ is given by (6.3) and $\theta(x^\mu) = Arg(\sqrt{-(t-i)^2+r^2})$. Both proportionality factors are real constants, not functions and can simultaneously be made equal to unity by multiplying the twistor $(\mathcal{B}^\alpha)$ with a suitable factor. This then changes the eigenvalue field accordingly.

We thus have a solution to linearized Einstein's equations in which five Hopf knots appear. Two perpendicular tendex (or tidal) knots and two perpendicular vortex (or frame drag) knots. These two sets are rotated 45° with respect to each other. Perpendicular to these sets we have another Hopf structure, at every point defining the direction where no stretching, no compression and no precession occurs. Only this last structure moves without distortion.
At $t = 0$ the two tendex Hopf knots (one indicating compression, the other stretching) are exactly aligned with the electric and magnetic fields of the electromagnetic Hopf knot (6.3). At other times the difference, apart from the proportionality constant, is a local duality transformation between the corresponding Riemann-Silberstein vectors. Both elementary states were obtained from the Penrose transform using the same twistors $(\mathcal{A}^\alpha)$ and $(\mathcal{B}^\alpha)$ in a slightly different function, to meet the requirements of homogeneity. Note that the fields $\mathbf{E}_0$ and $\mathbf{B}_0$ are proportional to the Poynting vector corresponding to the field (6.3). In analogy with electrodynamics, in [64] a super Poynting vector is defined that in present context is aligned with $\mathbf{E}_0(x^\mu)$ and $\mathbf{B}_0(x^\mu)$. It is thus possible to

describe this gravitational Hopf knot or "linked gravitational radiation" in terms of the electromagnetic Hopf knot [10].

Note that $\theta(x^\mu) = \arctan\{\frac{1}{2t}(t^2 - r^2 - 1 + \sqrt{(t^2-r^2)^2+1+2(t^2+r^2)})\}$ and does not satisfy the generalized wave equation (4.12) and (4.13): as expected $\mathbf{F}_E$ is not a solution of Maxwell's equations.

Although in general relativity the interpretation of quantities that appear in equations has to be done very carefully due to diffeomorphism invariance of the theory [58, 69], it is important to realize that according to a valid interpretation of *linearized* general relativity the physical fields can be considered as Lorentz tensors on Minkowski (flat) space-time [68]. A nice feature, since every physicists has gained some intuition in this respect. This changes dramatically when considering the full nonlinear equations of Einstein's theory, and some of the problems that this gives rise to will be discussed in later chapters.
Since in linearized general relativity the fields can be considered as fields in Minkowski space, it is still possible to construct the Riemann-Silberstein like structures appearing in present section, in the obvious way. In the full theory however one should be more careful. Now one needs for its definition a time-like congruence, or, what amounts to the same, two real null congruences.

As a final remark, we mention that besides the aforementioned similarity between the Faraday tensor in electrodynamics and the Weyl tensor in general relativity, there is another electromagnetic analogy in *linearized* general relativity [70]. Under certain restrictions this analogy can for instance be used to predict the motion of test particles in a gravitational field, based on the knowledge of the corresponding problem in electrodynamics. With the help of this analogy it is however not possible to simply transform a solution of Maxwell's equations into a solution of linearized Einstein equations. It is therefore of no help in our endeavour to 'generalize' an ectromagnetic Hopf knot to a gravitational one.
A lucid account of this and the previous analogy, including a discussion of their limitations, is to be found in [71,72].

7

Review of tetrad formalism in general relativity

7.1 introduction

Einstein's equations, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$, relate the energy-momentum tensor of sources with the geometry of space-time. It tells us how matter curves space-time and contains the equations of motion of the matter via Bianchi's identity. It looks deceptively simple: when realizing that we need hundreds of pages to write them out in terms of components of the metric tensor and its derivatives, it comes as no surprise that only very few exact solutions are known that have physical, astronomical or cosmological relevance. Karl Schwarzschild found a spherical symmetric static solution shortly after the birth of general relativity. His solution describes the space-time of non-rotating spherically symmetric matter distributions, for example a non-rotating star or black hole. More relevant to astronomy is the solution that includes the effect of rotation, the Kerr solution, valid for rotating black holes or as an approximation to the exterior solution outside an axially symmetric rotating mass distribution [73]. This was found by Roy Kerr (and independently by Ezra Newman) in 1963: it took almost 50 years just to include angular momentum! This solution is more mysterious than Schwarzschild's in that still, after many years of unsuccessful search, no acceptable interior solution has been found that matches the exterior solution [74] (however, for a different view see [75]). According to Kerr this interior solution will probably never be found [76]. In addition, the maximal analytic extension contains closed timelike curves [77], which form a problem for causality. An exact solution that is important in cosmology is the Friedmann-Lemaitre-Robertson-Walker solution describing a spatially homogeneous and isotropic universe.

Obtaining exact solutions of Einstein's equations has become a field of research in its own. In general relativity, a theory where experiments are extremely difficult, where it is not immediately clear what coordinates mean and where you have to think hard how to define the most basic physical quantities, it is important to think about solutions that have no counterpart in the world around us, the category that contains almost all known exact solutions.
Confronted with the complexity of the field equations of general relativity, there are essentially three options to proceed. We could consider weak fields only and linearize the theory, resulting in a theory that has some similarities with electrodynamics. We could turn to numerics and, not surprisingly in a computer dominated era, numerical relativity has become an important branch of Einstein's theory in the last decade. Finally, we can try to solve the equations analytically with the help of assumptions, like symmetry, that reduces the complexity.

In chapter 8 and chapter 10 we will consider exact solutions. Here we review the mathematical notions that are needed in those chapters. Reference [12] comes closest to our preferences with respect to notation and signature choices and can therefore be best consulted for additional information. In geometrized units with $4\pi G \equiv 1$, Einstein's equations read $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu}$. Here, as elsewhere, $c \equiv 1$.

7.2 differential geometric notations

In chapter 1 we considered null tetrads in Minkowski space. In general relativity Minkowski space figures as tangent space to the curved space-time. It is one of our goals in this mathematical intermezzo to introduce the Newman-Penrose formalism, a specific example of the tetrad formalism in curved space-time.

A basis of the tangent space defined by local coordinates (x) of the space-time manifold is $\{\mathbf{e}_\mu\} \equiv \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the dual basis is the collection of one-forms (or linear functionals or covariant vectors) $\{\mathbf{e}^\mu\}$:

$$\mathbf{e}^\mu(\mathbf{e}_\nu) = \mathbf{e}^\mu \cdot \mathbf{e}_\nu = \delta^\mu_\nu$$

The following notation is often used: $\mathbf{e}_\mu = \partial_\mu$ and $\mathbf{e}^\mu = dx^\mu$.

Any other n linearly independent tangent vectors $\{\mathbf{e}_\alpha\}$ define a different basis related to the coordinate basis in the following manner:
$\mathbf{e}_\alpha = M_\alpha^{\ \mu}\partial_\mu$.

The dual basis to this is $\{\mathbf{e}^\alpha\}$, $\mathbf{e}^\alpha = M^\alpha_{\ \mu} dx^\mu$.

The matrices $(M_\alpha^{\ \mu})$ and $(M^\alpha_{\ \mu})$ are inverses of each other, the upper index labeling different elements in one row (the column index). Whenever $\{\mathbf{e}_\alpha\}$ comes from a different basis $(\tilde{x})$, the matrix elements are

$$M_\alpha^{\ \mu} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}.$$

Of course, the aforementioned pointwise definitions are valid throughout the manifold, thus defining vector fields.

7.2.1 tensors

Let T_p denote the tangent space at point p and T_p^* the cotangent space at p. Define

$$\Pi^s_{\ r} = T_p^* \times ... \times T_p^* \times T_p \times ... \times T_p,$$

in which the cotangent space appears r times and the tangent space s times. A tensor of type (r, s) is a multilinear mapping from $\Pi^s_{\ r}$ to $\mathbb{R}$: $\mathbf{T}(\omega^1, ..., \omega^r, \mathbf{V}_1, ..., \mathbf{V}_s) \in \mathbb{R}$. The space of these tensors is denoted

$$T^r_{\ s}(p) = T_p \otimes ... \otimes T_p \otimes T_p^* \otimes ... \otimes T_p^*,$$

in which the tangent space appears r times and the cotangent space s times (note the interchange of r and s). Clearly, the collection $\{\mathbf{e}_{\alpha_1} \otimes ... \otimes \mathbf{e}_{\alpha_r} \otimes \mathbf{e}^{\beta_1} \otimes ... \otimes \mathbf{e}^{\beta_s}\}$ is a basis for $T^r_{\ s}$.

7.2.2 forms

An important subclass of T^r_s consists of antisymmetric tensors. Combined with functions and dual vectors they constitute the class of forms, for which a calculus can be developed that serves as a convenient instrument for calculations in general relativity.

A zero-form, f, is just a function from the manifold into the reals. One-forms are linear combinations of dual basis vectors. A s-form is a totally antisymmetric tensor of type (0, s). We can construct a s-form from any tensor of type (0, s) with aid of the alternating operator $\hat{\mathcal{A}}$:

$$\hat{\mathcal{A}}\mathbf{T}(\mathbf{V}_1, ..., \mathbf{V}_s) = \frac{1}{s!} \sum_{j_1,...,j_s} sgn(j_1, ..., j_s) \mathbf{T}(\mathbf{V}_{j_1}, ..., \mathbf{V}_{j_s}).$$

The $\frac{n!}{s!(n-s)!}$ dimensional vector space of s-forms is denoted $\Lambda^s T_p^*$. A basis for this vector space can be obtained from a basis of T^0_s:

$$\{\hat{A}(\mathbf{e}^{\alpha_1} \otimes ... \otimes \mathbf{e}^{\alpha_s})\} \equiv \{\mathbf{e}^{\alpha_1} \wedge ... \wedge \mathbf{e}^{\alpha_s} \mid \alpha_1 > ... > \alpha_s\}.$$

This defines the exterior or wedge product of one-forms.
The exterior differential operator, d, is defined for forms. For a zero-form f, df is a one-form:

$$df(\mathbf{V}) = df \cdot \mathbf{V} = \mathbf{V}(f).$$

In a local coordinate basis: $df = f_{,\mu} dx^\mu$.

The abstract definition for the other forms comes down to, in a local coordinate basis:

$$d\mathbf{A} = d(A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}) = A_{\mu_1 \ldots \mu_p, \nu} dx^\nu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}.$$

This rule is consistent with an important fact that is not a result of the foregoing, but part of the abstract definition of exterior differentiation: $d(d\mathbf{A}) = 0$.

7.2.3 covariant differentiation

We define an affine (or Koszul) connection ∇ in such a way that for any vector field $\mathbf{V}$, $\nabla_\mathbf{V}$ maps another vector field $\mathbf{W}$ into the vector field $\nabla_\mathbf{V}\mathbf{W}$. We require the following conditions (with $\mathbf{U}$ an arbitrary vector field):

$$\nabla_{(f\mathbf{V}+g\mathbf{W})}\mathbf{U} = f\nabla_\mathbf{V}\mathbf{U} + g\nabla_\mathbf{W}\mathbf{U}$$

$$\nabla_\mathbf{V}(\mathbf{W}+\mathbf{U}) = \nabla_\mathbf{V}\mathbf{W} + \nabla_\mathbf{V}\mathbf{U}$$

$$\nabla_\mathbf{V} f = \mathbf{V}(f)$$

$$\nabla_\mathbf{V}(f\mathbf{W}) = (\nabla_\mathbf{V} f)\mathbf{W} + f\nabla_\mathbf{V}\mathbf{W}$$

and define $\nabla\mathbf{W}$ to be the following tensor field of type (1,1): $\nabla\mathbf{W}(\mathbf{V}) = \nabla_\mathbf{V}\mathbf{W}$.

For higher order tensors we demand the Leibniz rule to hold.

In terms of an arbitrary dual basis $\{\mathbf{e}_\alpha\}$ and $\{\mathbf{e}^\alpha\}$, since $\nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha$ is a vector field:

$$\nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha = \omega^\gamma_{\ \alpha}(\mathbf{e}_\beta)\mathbf{e}_\gamma \equiv \omega^\gamma_{\ \alpha\beta}\mathbf{e}_\gamma,$$

in which $\omega^{\gamma}_{\ \alpha}$ are one-forms. Thus a connection is specified by 4^2 one-forms $\omega^{\gamma}_{\ \alpha}$ or by 4^3 scalar fields $\omega^{\gamma}_{\ \alpha\beta}$. We write $\Gamma^{\lambda}_{\ \mu\nu}$ for $\omega^{\lambda}_{\ \mu\nu}$ and $W^{\lambda}_{\ ;\mu}$ for $(\nabla_{\partial_\mu}\mathbf{W})^{\lambda}$ in case a local coordinate basis is used. We thus arrive, for example, at the familiar

$$W^{\lambda}_{\ ;\mu} = W^{\lambda}_{\ ,\mu} + \Gamma^{\lambda}_{\ \nu\mu} W^{\nu}. \tag{7.1}$$

We will be concerned with connections that are derived from a metric,

$\Gamma^{\lambda}_{\ \mu\nu} = \frac{1}{2} g^{\lambda\kappa} (\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}})$

and for these the torsion tensor vanishes: $T^{\lambda}_{\ \mu\nu} = -(\Gamma^{\lambda}_{\ \mu\nu} - \Gamma^{\lambda}_{\ \nu\mu}) = 0$.

The connection coefficients are then called the Christoffel symbols and a vanishing torsion means a symmetric connection.

7.2.4 parallel propagation and geodesics

A vector $\mathbf{V}$ is parallely propagated along a curve c if

$V^{\mu}_{\ ;\nu} \frac{dx^{\nu}(\mathsf{c}(t))}{dt} \delta t = 0$.

A curve is a geodesic when the parallel propagation of its tangent vector is a multiple of itself, the proportionality possibly changing from point to point:

$\frac{d^2 x^{\mu}}{dt^2} + \Gamma^{\mu}_{\ \lambda\kappa} \frac{dx^{\lambda}}{dt} \frac{dx^{\kappa}}{dt} = f(t) \frac{dx^{\mu}}{dt}$.

If we reparametrize the geodesic according to $s = \int^t dt'' e^{\int^{t''} dt' f(t')}$ this becomes

$\frac{d^2 x^{\mu}}{ds^2} + \Gamma^{\mu}_{\ \lambda\kappa} \frac{dx^{\lambda}}{ds} \frac{dx^{\kappa}}{ds} = 0$. In this case s is called an affine parameter.

Written in terms of the 'four velocity' $u^{\mu} = \frac{dx^{\mu}}{ds}$ we do get

$u^{\mu}_{\ ;\lambda} u^{\lambda} = 0$.

7.2.5 Riemann tensor, Ricci tensor, Bianchi identity and more

In general the Riemann tensor, $\mathbf{R}$, is defined by

$$\mathbf{R}(\mathbf{V},\mathbf{W}) = \nabla_{\mathbf{V}}\nabla_{\mathbf{W}} - \nabla_{\mathbf{W}}\nabla_{\mathbf{V}} - \nabla_{[\mathbf{V},\mathbf{W}]}.$$

We will use a basis, and for a coordinate basis this definition boils down to:

$$R^{\kappa}{}_{\lambda\mu\nu} = \Gamma^{\kappa}{}_{\lambda\nu,\mu} - \Gamma^{\kappa}{}_{\lambda\mu,\nu} + \Gamma^{\kappa}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\lambda\nu} - \Gamma^{\kappa}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\lambda\mu}.$$

The Ricci tensor is defined by $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$, and the scalar curvature is defined as $R = R^{\mu}{}_{\mu}$.

Some important formula that can be derived with the help of the definitions stated thus far are:

$V^{\mu}{}_{;\kappa;\lambda} - V^{\mu}{}_{;\lambda;\kappa} = -R^{\mu}{}_{\nu\kappa\lambda}V^{\nu}$ (*Ricci identity*) and obvious generalizations

$g_{\mu\nu;\lambda} = 0$

$R^{\nu}{}_{\lambda\kappa\mu} + R^{\nu}{}_{\kappa\mu\lambda} + R^{\nu}{}_{\mu\lambda\kappa} = 0$ (*cyclic identity*)

$R^{\nu}{}_{\lambda\kappa\mu;\sigma} + R^{\nu}{}_{\lambda\mu\sigma;\kappa} + R^{\nu}{}_{\lambda\sigma\kappa;\mu} = 0$ (or $R_{\nu\lambda[\kappa\mu;\sigma]} = 0$) (*Bianchi identity*)

$R_{\mu\nu\kappa\lambda} + R_{\nu\mu\kappa\lambda} = 0,\ R_{\mu\nu\kappa\lambda} + R_{\mu\nu\lambda\kappa} = 0,\ R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu},\ R_{\mu\nu} = R_{\nu\mu}$

From the symmetries of the Riemann tensor we conclude that it contains 20 independent components, whereas the Ricci tensor contains 10 independent components.

7.2.6 Einstein tensor and Weyl tensor

The Einstein tensor $\mathbf{G}$ is defined as:

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R$$

and from the Bianchi identity it follows that

$$G^{\mu}{}_{\nu;\mu} = 0,$$

an identity with important implications for the structure of general relativity. The Weyl tensor $\mathbf{C}$ is defined as follows

$$C_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} - \tfrac{1}{2}(g_{\mu\kappa}R_{\nu\lambda} + g_{\nu\lambda}R_{\mu\kappa} - g_{\nu\kappa}R_{\mu\lambda} - g_{\mu\lambda}R_{\nu\kappa}) + \tfrac{1}{6}(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa})R.$$

It shares all the symmetries of the Riemann tensor, and in addition satisfies $g^{\nu\lambda}C_{\mu\nu\kappa\lambda} = 0$.

As can be seen from the symmetries it satisfies, the number of independent components is 10. The most important property of this tensor is the invariance under conformal transformations of $C^{\mu}_{\ \nu\kappa\lambda}$. In vacuum it coincides with the Riemann tensor.

7.3 tetrad formalism

The tetrad formalism can be used as a mathematical tool by which vector fields with complex functional dependence become the basis vector fields in which all other tensor fields are expressed.
We consider four contravariant vector fields that in every point of the manifold form a basis, called a vierbein or a tetrad:

$$\{\mathbf{e}_{(k)}\} \equiv \{\mathbf{e}_{(1)}, \mathbf{e}_{(2)}, \mathbf{e}_{(3)}, \mathbf{e}_{(4)}\}$$

with $\mathbf{e}_{(k)}^{\ \ \mu}$ the components along the (coordinate) basis $\{\mathbf{e}_\mu\}$:

$$\mathbf{e}_{(k)} = \mathbf{e}_{(k)}^{\ \ \mu}\mathbf{e}_\mu.$$

For tetrad and tetrad components we will use round brackets around indices. The discussion somewhat parallels the introduction of tetrads in chapter 1. There is indeed a very elegant spinor alternative to all that follows. Here we opt for the more conventional route, that, except perhaps for the expert, is more intuitive.

Components of the corresponding covariant vector fields are

$$\mathbf{e}_{(k)\mu} = g_{\mu\nu}\mathbf{e}_{(k)}^{\ \ \nu}.$$

We consider tetrads for which

$\mathbf{g}(\mathbf{e}_{(k)}, \mathbf{e}_{(l)}) = \mathbf{e}_{(k)}^{\ \ \mu}\mathbf{e}_{(l)\mu} \equiv \eta_{(k)(l)}$ form a constant (symmetric) matrix with inverse $[\eta^{(k)(l)}]$.

The components of the tetrad do form a matrix $[\mathbf{e}_{(k)}^{\ \ \mu}]$. We define the inverse to be $[\mathbf{e}^{(k)}_{\ \ \mu}]$:

${\bf e}_{(k)}{}^{\mu}{\bf e}^{(l)}{}_{\mu} = \delta^{(l)}{}_{(k)}$ and ${\bf e}_{(k)}{}^{\mu}{\bf e}^{(k)}{}_{\nu} = \delta^{\mu}_{\nu}$.

Thus the upper index labels different elements in a row (the column index). It is easy to check the following rules:

$\eta_{(k)(l)}{\bf e}^{(k)}{}_{\mu} = {\bf e}_{(l)\mu}$

$\eta^{(k)(l)}{\bf e}_{(k)\mu} = {\bf e}^{(l)}{}_{\mu}$

${\bf e}_{(k)\mu}{\bf e}^{(k)}{}_{\nu} = g_{\mu\nu}$

$A_{(k)} = {\bf e}_{(k)\mu}A^{\mu} = {\bf e}_{(k)}{}^{\mu}A_{\mu}$

$A^{(k)} = \eta^{(k)(l)}A_{(l)} = {\bf e}^{(k)}{}_{\mu}A^{\mu} = {\bf e}^{(k)\mu}A_{\mu}$ are the tetrad components of the vector field $\mathbf{A}$: $\mathbf{A} = A^{(k)}{\bf e}_{(k)}$

$A^{\mu} = {\bf e}_{(k)}{}^{\mu}A^{(k)} = {\bf e}^{(k)\mu}A_{(k)}$

$T_{(k)(l)} = {\bf e}_{(k)}{}^{\mu}{\bf e}_{(l)}{}^{\nu}T_{\mu\nu} = {\bf e}_{(k)}{}^{\mu}T_{\mu(l)}$

etc.

So, for tetrad components η *plays the role of the metric tensor*, and we can change from tetrad indices to tensor indices and vice versa with the help of the matrices formed by the components of the tetrad along the original basis. These matrices now play the role of (M) in the beginning of section 7.2. Tetrads are defined independent of coordinates. Tetrad components of tensors therefore do not change when a coordinate transformation is applied: tetrad components of tensors are scalar fields. Especially when one learned about tensors from the older literature, often still used by physicists, in which a tensor is defined by its transformation properties, this fact has to be remembered carefully.

When the basis $\{{\bf e}_{\mu}\}$ is a coordinate basis it is also clear that

${\bf e}_{(k)} = e_{(k)}{}^{\mu}\partial_{\mu}$

$A_{(k),(l)} = {\bf e}_{(l)}{}^{\mu}\partial_{\mu}A_{(k)} = ... = A_{(k)|(l)} + \gamma_{(m)(k)(l)}A^{(m)}$ in which

$\gamma_{(m)(k)(l)} = -\gamma_{(k)(m)(l)} \equiv {\bf e}_{(m)}{}^{\mu}{\bf e}_{(l)}{}^{\nu}{\bf e}_{(k)\mu;\nu}$ are called the Ricci rotation coefficients

$A_{(k)|(l)} \equiv {\bf e}_{(k)}{}^{\mu}{\bf e}_{(l)}{}^{\nu}A_{\mu;\nu}$ is called the intrinsic derivative.

Thus, from comparison with (7.1): *the intrinsic derivative takes over the role* *of covariant derivative*, and

$\gamma^{(m)}{}_{(k)(l)}$ *comes instead of* $\Gamma^{\lambda}_{\mu\nu}$.

Calculation of the Ricci rotation coefficients do not require knowledge of the Christoffel symbols, because of:

$\gamma_{(k)(l)(m)} = \frac{1}{2}[\lambda_{(k)(l)(m)} + \lambda_{(m)(k)(l)} - \lambda_{(l)(m)(k)}]$

with $\lambda_{(k)(l)(m)} \equiv [\mathbf{e}_{(l)\mu,\nu} - \mathbf{e}_{(l)\nu,\mu}]\mathbf{e}_{(k)}{}^{\mu}\mathbf{e}_{(m)}{}^{\nu}$ in which only partial, not covariant, derivatives appear.

7.4 Newman-Penrose formalism

The Newman-Penrose formalism [78] results from a particular choice for the matrix $(\eta_{(k)(l)})$, namely the form (1.1). We take the tetrad $\{\mathbf{e}_{(k)}\}$ to be $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$ with the first pair being real null vectors and the last pair complex conjugate null vectors. We require the following relations to hold:

$\mathbf{l}\cdot\mathbf{l} = \mathbf{n}\cdot\mathbf{n} = \mathbf{m}\cdot\mathbf{m} = \bar{\mathbf{m}}\cdot\bar{\mathbf{m}} = 0$ *null condition*

$\mathbf{l}\cdot\mathbf{m} = \mathbf{l}\cdot\bar{\mathbf{m}} = \mathbf{n}\cdot\mathbf{m} = \mathbf{n}\cdot\bar{\mathbf{m}} = 0$ *orthogonality condition*

$\mathbf{l}\cdot\mathbf{n} = 1 = -\mathbf{m}\cdot\bar{\mathbf{m}}$ *normalization condition*

Then the dual tetrad basis $\{\mathbf{e}^{(k)}\}$ is $\{\mathbf{n}, \mathbf{l}, -\bar{\mathbf{m}}, -\mathbf{m}\}$, and

$$[\eta_{(k)(l)}] = [\eta^{(k)(l)}] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{7.2}$$

When considering directional derivatives, we write $\{D, \Delta, \delta, \delta^*\}$ for $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$.

The Ricci rotation coefficients are now called *spin coefficients*:

$\kappa = \gamma_{(3)(1)(1)}$
$\sigma = \gamma_{(3)(1)(3)}$
$\lambda = \gamma_{(2)(4)(4)}$
$\nu = \gamma_{(2)(4)(2)}$

$\rho = \gamma_{(3)(1)(4)}$
$\mu = \gamma_{(2)(4)(3)}$
$\tau = \gamma_{(3)(1)(2)}$
$\pi = \gamma_{(2)(4)(1)}$
$\epsilon = \frac{1}{2}(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)})$
$\gamma = \frac{1}{2}(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)})$
$\alpha = \frac{1}{2}(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)})$
$\beta = \frac{1}{2}(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)})$

and the complex conjugates will be arrived at when interchanging the labels (3) and (4).

The equations of the Newman-Penrose formalism consist of the commutation relations for the tetrad seen as derivatives, the Ricci identities, the so called eliminant relations and the Bianchi identities. These equations, some real and some complex, stand in the place of Einstein's equations and will be given in chapter 10, where we try to solve them simultaneously in case the curvature of space-time is due to an electromagnetic Hopf knot.

7.4.1 Weyl tensor

It is a time consuming but otherwise simple exercise to show that the ten independent components of the Weyl tensor in the present formalism can be represented by the complex scalar fields $\Psi_0, ..., \Psi_4$:

$\Psi_0 \equiv -C_{(1)(3)(1)(3)} = -C_{\kappa\lambda\mu\nu} l^\kappa m^\lambda l^\mu m^\nu$
$\Psi_1 \equiv -C_{(1)(2)(1)(3)} = -C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda l^\mu m^\nu$
$\Psi_2 \equiv -C_{(1)(3)(4)(2)} = -C_{\kappa\lambda\mu\nu} l^\kappa m^\lambda \bar{m}^\mu n^\nu$
$\Psi_3 \equiv -C_{(1)(2)(4)(2)} = -C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda \bar{m}^\mu n^l \nu$
$\Psi_4 \equiv -C_{(2)(4)(2)(4)} = -C_{\kappa\lambda\mu\nu} n^\kappa \bar{m}^\lambda n^\mu \bar{m}^\nu$

Note that there is no need for parentheses around indices of the Ψ's since, besides the fact that they are only defined for the tetrad formalism, the indices do not refer to a decomposition with respect to basis vectors. The explicit form of the Weyl tensor in a coordinate basis in terms of these complex scalar fields is the daunting expression:

$C_{\kappa\lambda\mu\nu} = -(\Psi_2 + \Psi_2^*)[\{l_\kappa n_\lambda l_\mu n_\nu\} + \{m_\kappa \bar{m}_\lambda m_\mu \bar{m}_\nu\}] + (\Psi_2 - \Psi_2^*)\{l_\kappa n_\lambda m_\mu \bar{m}_\nu\} +$
$2Re\{-\Psi_0\{n_\kappa \bar{m}_\lambda n_\mu \bar{m}_\nu\} - \Psi_4\{l_\kappa m_\lambda l_\mu m_\nu\} + \Psi_2\{l_\kappa m_\lambda n_\mu \bar{m}_\nu\} - \Psi_1[\{l_\kappa n_\lambda n_\mu \bar{m}_\nu\} +$
$\{n_\kappa \kappa \bar{m}_\lambda \bar{m}_\mu m_\nu\}] + \Psi_3[\{l_\kappa n_\lambda l_\mu m_\nu\} - \{l_\kappa m_\lambda m_\mu \bar{m}_\nu\}]\}$

in which $\{l_\kappa n_\lambda m_\mu \bar{m}_\nu\}$ stands for

$l_\kappa n_\lambda m_\mu \bar{m}_\nu - l_\kappa n_\lambda \bar{m}_\mu m_\nu - n_\kappa l_\lambda m_\mu \bar{m}_\nu + n_\kappa l_\lambda \bar{m}_\mu m_\nu + m_\kappa \bar{m}_\lambda l_\mu n_\nu - m_\kappa \bar{m}_\lambda n_\mu l_\nu - \bar{m}_\kappa m_\lambda l_\mu n_\nu + \bar{m}_\kappa m_\lambda n_\mu l_\nu$.

This time, simply by looking at the expression above, one convinces oneself of the following form for all the distinct tetrad components of the Weyl tensor in terms of the Ψ's:

$$C_{(1)(3)(1)(4)} = C_{(2)(3)(2)(4)} = C_{(1)(3)(3)(2)} = C_{(1)(4)(4)(2)} = 0$$
$$C_{(1)(3)(3)(4)} = \Psi_1$$
$$C_{(2)(4)(4)(3)} = \Psi_3$$
$$C_{(1)(2)(1)(2)} = C_{(3)(4)(3)(4)} = -(\Psi_2 + \Psi_2^*)$$
$$C_{(1)(2)(3)(4)} = (\Psi_2 - \Psi_2^*)$$

supplemented with the defining equations for the Ψ's, and the complex conjugates of all these (interchanging labels (3) and (4)). The equations that follow for the complex scalar fields, as well as those of the following subsection, as a result of the equations of general relativity will be given in chapter 10.

7.4.2 Ricci tensor

We have seen that the Ricci tensor contains ten independent components. In the Newman-Penrose formalism the tetrad components of this tensor will be represented in terms of four real and three complex scalar fields as follows:

$$\Lambda = \tfrac{1}{24}R = \tfrac{1}{12}(R_{(1)(2)} - R_{(3)(4)})$$
$$\Phi_{00} = -\tfrac{1}{2}R_{(1)(1)}$$
$$\Phi_{11} = -\tfrac{1}{4}(R_{(1)(2)} + R_{(3)(4)})$$
$$\Phi_{22} = -\tfrac{1}{2}R_{(2)(2)}$$
$$\Phi_{01} = -\tfrac{1}{2}R_{(1)(3)} \quad (\text{and } \Phi_{10} = -\tfrac{1}{2}R_{(1)(4)})$$
$$\Phi_{02} = -\tfrac{1}{2}R_{(3)(3)} \quad (\text{and } \Phi_{20} = -\tfrac{1}{2}R_{(4)(4)})$$
$$\Phi_{12} = -\tfrac{1}{2}R_{(2)(3)} \quad (\text{and } \Phi_{21} = -\tfrac{1}{2}R_{(2)(4)}).$$

7.4.3 electrodynamical quantities and equations

Einstein's (strong) equivalence principle implies that the equations of physics in the presence of a gravitational field, when written in a freely falling (and non-rotating) coordinate system, have the same form (locally) as in special relativity. This means that we do get the manifest covariant tensor equations in the presence of gravity from the special relativistic form by substituting semicolons for commas. See in this respect equation (7.1). The source-free Maxwell equations in the absence of gravity can, according to (3.11) and (3.12), be written as ${F^{\mu}}_{\nu,\mu} = 0$ and $F_{[\mu\nu,\lambda]} = 0$. Maxwell's equations without source in a curved space-time become therefore ${F^{\mu}}_{\nu;\mu} = 0$ and $F_{[\mu\nu;\lambda]} = 0$.

The six independent components of the electromagnetic field tensor **F** will be represented by three complex fields:

$$\phi_0 \equiv F_{(1)(3)} = F_{\mu\nu} l^{\mu} m^{\nu}$$
$$\phi_1 \equiv \tfrac{1}{2}(F_{(1)(2)} + F_{(4)(3)}) = \tfrac{1}{2} F_{\mu\nu}(l^{\mu} n^{\nu} + \bar{m}^{\mu} m^{\nu})$$
$$\phi_2 \equiv F_{(4)(2)} = F_{\mu\nu} \bar{m}^{\mu} n^{\nu}$$

in terms of which Maxwell's equations in curved space-time

$F_{[\mu\nu;\lambda]} = 0,\ \ {F^{\mu}}_{\nu;\mu} = 0$ (in a coordinate basis) or

$F_{[(k)(l)|(m)]} = 0,\ \ {F^{(m)}}_{(k)|(m)} = 0$ (in a tetrad basis) reads

$$D\phi_1 - \delta^* \phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2$$

$$D\phi_2 - \delta^* \phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2$$

$$\delta\phi_1 - \Delta\phi_0 = (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2$$

$$\delta\phi_2 - \Delta\phi_1 = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2$$

and in terms of which the nine independent components of the electromagnetic energy-momentum tensor $\mathbf{T}^{e.m.}$ (3.13) will be represented as

$$T_{(1)(1)} = -2\phi_0\phi_0^*$$
$$T_{(1)(2)} + T_{(3)(4)} = -4\phi_1\phi_1^*$$
$$T_{(2)(2)} = -2\phi_2\phi_2^*$$
$$T_{(1)(3)} = -2\phi_0\phi_1^*$$
$$T_{(2)(3)} = -2\phi_1\phi_2^*$$
$$T_{(3)(3)} = -2\phi_0\phi_2^*$$

Whenever the electromagnetic field is the only source for gravity, $\mathbf{T} = \mathbf{T}^{e.m.}$, we find, using the fact that $\mathbf{T}^{e.m.}$ is traceless (see (3.13)), from Einstein's equations, $R_{(k)(l)} = 2T_{(k)(l)}$, that

$$\Phi_{nm} = 2\phi_n\phi_m^* \ \text{ and } \ \Lambda = 0 \tag{7.3}$$

This relates the Ricci tensor with the electromagnetic field that acts as its sole source.

7.4.4 tetrad transformations

We will now consider the effect of changing the tetrad. By considering the relation between the tetrad and the basis vectors coresponding to local geodesic coordinates at a point p, for which $g_{\mu\nu}(p) = \eta_{\mu\nu}(p)$, the Minkowski metric, it is clear that we have as much freedom to change the tetrad as we have in changing these basis vectors such that they remain related to local geodesic coordinates at p. This, of course, means a Lorentz transformation and correspondingly we have six degrees of freedom. If we extend this transformation continuously throughout the manifold, we do get a tetrad transformation depending on six function that does not alter $(\eta_{(k)(l)})$. Direct calculation of inner products will show that these tetrad transformations can be represented as follows.

With a and b arbitrary complex functions, A and θ arbitrary real functions:

$$\text{type 1)} \quad \boxed{\begin{array}{l} \mathbf{l} \to \mathbf{l} \\ \mathbf{n} \to \mathbf{n} + a^*\mathbf{m} + a\bar{\mathbf{m}} + aa^*\mathbf{l} \\ \mathbf{m} \to \mathbf{m} + a\mathbf{l} \\ \bar{\mathbf{m}} \to \bar{\mathbf{m}} + a^*\mathbf{l} \end{array}} \tag{7.4}$$

$$\text{type 2)} \quad \boxed{\begin{array}{l} \mathbf{l} \to \mathbf{l} + b^*\mathbf{m} + b\bar{\mathbf{m}} + bb^*\mathbf{n} \\ \mathbf{n} \to \mathbf{n} \\ \mathbf{m} \to \mathbf{m} + b\mathbf{n} \\ \bar{\mathbf{m}} \to \bar{\mathbf{m}} + b^*\mathbf{n} \end{array}} \tag{7.5}$$

$$\text{type 3)} \quad \boxed{\begin{array}{l} \mathbf{l} \to A^{-1}\mathbf{l} \\ \mathbf{n} \to A\mathbf{n} \\ \mathbf{m} \to e^{i\theta}\mathbf{m} \\ \bar{\mathbf{m}} \to e^{-i\theta}\bar{\mathbf{m}} \end{array}} \tag{7.6}$$

As straightforward calculations show, these tetrad transformations do have an effect on the previous defined quantities in the following way.

type 1)

$\Psi_0 \rightarrow \Psi_0$
$\Psi_1 \rightarrow \Psi_1 + a^*\Psi_0$
$\Psi_2 \rightarrow \Psi_2 + 2a^*\Psi_1 + (a^*)^2\Psi_0$
$\Psi_3 \rightarrow \Psi_3 + 3a^*\Psi_2 + 3(a^*)^2\Psi_1 + (a^*)^3\Psi_0$
$\Psi_4 \rightarrow \Psi_4 + 4a^*\Psi_3 + 6(a^*)^2\Psi_2 + 4(a^*)^3\Psi_1 + (a^*)^4\Psi_4$

$\kappa \rightarrow \kappa$
$\sigma \rightarrow \sigma + a\kappa$
$\lambda \rightarrow \lambda + a^*(2\alpha + \pi) + (a^*)^2(\rho + 2\epsilon) + (a^*)^3\kappa + \delta^*a^* + a^*Da^*$
$\rho \rightarrow \rho + a^*\kappa$
$\mu \rightarrow \mu + a\pi + 2a^*\beta + 2aa^*\epsilon + (a^*)^2\sigma + a(a^*)^2\kappa + \delta a^* + aDa^*$
$\tau \rightarrow \tau + a\rho + a^*\sigma + aa^*\kappa$
$\pi \rightarrow \pi + 2a^*\epsilon + (a^*)^2\kappa + Da^*$
$\epsilon \rightarrow \epsilon + a^*\kappa$
$\gamma \rightarrow \gamma + a\alpha + a^*(\beta + \tau) + aa^*(\rho + \epsilon) + (a^*)^2\sigma + a(a^*)^2\kappa$
$\alpha \rightarrow \alpha + a^*(\rho + \epsilon) + (a^*)^2\kappa$
$\beta \rightarrow \beta + a\epsilon + a^*\sigma + aa^*\kappa$
$\nu \rightarrow \nu + a\lambda + a^*(\mu + 2\gamma) + (a^*)^2(\tau + 2\beta) + (a^*)^3\sigma + aa^*(\pi + 2\alpha) + a(a^*)^2(\rho + 2\epsilon) + a(a^*)^3\kappa + (\Delta + a^*\delta + a\delta^* + aa^*D)a^*$

$\phi_0 \rightarrow \phi_0$
$\phi_1 \rightarrow \phi_1 + a^*\phi_0$
$\phi_2 \rightarrow \phi_2 + 2a^*\phi_1 + (a^*)^2\phi_0$

type 2)

With respect to type 1) **l** and **n** have been interchanged, resulting in the transformation

$\Psi_0 \leftrightarrows \Psi_4^*$
$\Psi_1 \leftrightarrows \Psi_3^*$
$\Psi_2 \leftrightarrows \Psi_2^*$
$\phi_0 \leftrightarrows -\phi_2^*$
$\phi_1 \leftrightarrows -\phi_1^*$
$\kappa \leftrightarrows -\nu^*$
$\rho \leftrightarrows -\mu^*$
$\sigma \leftrightarrows -\lambda^*$
$\alpha \leftrightarrows -\beta^*$
$\epsilon \leftrightarrows -\gamma^*$
$\pi \leftrightarrows -\tau^*$

Thus, for example: $\Psi_1 \to \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4$, etc.

type 3)

$\Psi_0 \to A^{-2}e^{2i\theta}\Psi_0$
$\Psi_1 \to A^{-1}e^{i\theta}\Psi_1$
$\Psi_2 \to \Psi_2$
$\Psi_3 \to Ae^{-i\theta}\Psi_3$
$\Psi_4 \to A^2e^{-2i\theta}\Psi_4$

$\kappa \to A^{-2}e^{i\theta}\kappa$
$\sigma \to A^{-1}e^{2i\theta}\sigma$
$\rho \to A^{-1}\rho$
$\tau \to e^{i\theta}\tau$
$\pi \to e^{-i\theta}\pi$
$\lambda \to Ae^{-2i\theta}\lambda$
$\mu \to A\mu$
$\nu \to A^2e^{-i\theta}\nu$
$\gamma \to A\gamma - \frac{1}{2}\Delta A + \frac{i}{2}A\Delta\theta$
$\epsilon \to A^{-1}\epsilon - \frac{1}{2}A^{-2}DA + \frac{i}{2}A^{-1}D\theta$
$\alpha \to e^{-i\theta}\alpha + \frac{i}{2}e^{-i\theta}\delta^*\theta - \frac{1}{2}A^{-1}e^{-i\theta}\delta^*A$
$\beta \to e^{i\theta}\beta + \frac{i}{2}e^{i\theta}\delta\theta - \frac{1}{2}A^{-1}e^{i\theta}\delta A$

$\phi_0 \to A^{-1}e^{i\theta}\phi_0$
$\phi_1 \to \phi_1$
$\phi_2 \to Ae^{-i\theta}\phi_2$

7.5 the optical scalars

In section **1.2.1** we introduced the optical scalars rotation, expansion and shear for a null congruence in Minkowski space. Here we review a more general mathematical treatment, not confined to flat space-time. In doing so, we will learn about the geometrical interpretation of at least some of the spin coefficients. In the Newman-Penrose formalism Einstein's equations have been replaced by 43 equations, each of which does not have a physical interpretation individually. Often in solving equations of physics, (physical) intuition or (physical) analogies are helpful. Here, in the present formalism, we have to solve many equations containing symbols of a highly mathematical character, the physics of which is hidden far behind the surface. Ultimately the expert will possibly acquire some intuition ("when doing this or that with this parameter such or so will happen"), but, no doubt, in cases like this it is desirable to have powerful general theorems (see next section) or the relief of having an interpretation for at least some of the symbols.

The change in the basis vector $\mathbf{e}_{(k)}$ between two infinitesimally ($\delta\mathbf{s}-$) separated points is $\delta\mathbf{e}_{(k)\mu} = \mathbf{e}_{(k)\mu;\nu}\delta s^{\nu} = \mathbf{e}^{(l)}{}_{\mu}\gamma_{(l)(k)(m)}\mathbf{e}^{(m)}{}_{\nu}\delta s^{\nu} = -\gamma_{(k)(l)(m)}\mathbf{e}^{(l)}{}_{\mu}\delta s^{(m)}$. This leads to $\delta\mathbf{e}_{(k)}(m) = -\gamma_{(k)(l)(m)}\mathbf{e}^{(l)}$ for the change in $\mathbf{e}_{(k)}$ per unit displacement along the direction (m) . Applied to the vectors from the Newman-Penrose tetrad, the change in $\mathbf{l}$ per unit displacement along $\mathbf{l}$ is:

$$\delta\mathbf{l}(1) = -\gamma_{(1)(k)(1)}\mathbf{e}^{(k)} = -\gamma_{(1)(2)(1)}\mathbf{l}+\gamma_{(1)(3)(1)}\bar{\mathbf{m}}+\gamma_{(1)(4)(1)}\mathbf{m} = (\epsilon+\epsilon^*)\mathbf{l}-\kappa\bar{\mathbf{m}}-\kappa^*\mathbf{m}$$

or: $l_{\mu;\nu}l^{\nu} = (\epsilon+\epsilon^*)l_{\mu} - \kappa\bar{m}_{\mu} - \kappa^* m_{\mu}$.

In the same way we find for $\mathbf{n}$ and $\mathbf{m}$:

$\delta\mathbf{n}(1) = -(\epsilon+\epsilon^*)\mathbf{n} + \pi\mathbf{m} + \pi^*\bar{\mathbf{m}}$, $\delta\mathbf{n}(2) = -(\gamma+\gamma^*)\mathbf{n} + \nu\mathbf{m} + \nu^*\bar{\mathbf{m}}$,

$\delta\mathbf{m}(1) = (\epsilon+\epsilon^*)\mathbf{m} + \pi^*\mathbf{l} - \kappa\mathbf{n}$, etc.

From this and section **7.2.4** we see that the vector field $\mathbf{l}$ is a congruence of null geodesics if and only if $\kappa = 0$ and that these are affinely parametrized when in addition $Re(\epsilon) = 0$. We can apply a tetrad type 3) transformation to achieve $\epsilon = 0$, whenever this is not already the case and this will not change the direction of $\mathbf{l}$ or the fact that $\kappa = 0$.
With $\kappa = \epsilon = 0$, writing out $l_{\mu;\nu}$ in terms of spin coefficients and antisymmetrizing the result, we arrive at

$$l_{[\mu;\nu]} = -(\alpha^* + \beta - \tau)l_{[\mu}\bar{m}_{\nu]} - (\alpha + \beta^* - \tau^*)l_{[\mu}m_{\nu]} + (\rho - \rho^*)\bar{m}_{[\mu}m_{\nu]}$$

and thus

$$l_{[\mu;\nu}l_{\kappa]} = (\rho - \rho^*)\bar{m}_{[\mu}m_{\nu}l_{\kappa]}.$$

These two equations tell us that whenever $\rho = \rho^*$, $\mathbf{l}$ will be *proportional* to the gradient of a scalar field (we say: the congruence of the null geodesics is hyper-surface orthogonal) and that when in addition $\alpha^* + \beta = \tau$, $\mathbf{l}$ will be *equal* to the gradient of this scalar field. We also find (again, $\kappa = \epsilon = 0$)

$$\tfrac{1}{2}l^{\mu}{}_{;\mu} = -\tfrac{1}{2}(\rho + \rho^*) \equiv \Theta$$

$$\tfrac{1}{2}l_{[\mu;\nu]}l^{\mu;\nu} = -\tfrac{1}{4}(\rho - \rho^*)^2 \equiv \omega^2$$

$\frac{1}{2}l_{(\mu;\nu)}l^{\mu;\nu} = \Theta^2 + |\sigma|^2.$

Θ, ω and σ are called the optical scalars (expansion, rotation and shear; these names indicate what happens with bundles of light-rays as they travel through a gravitational field for example). The significance of $\Theta = -Re(\rho), \omega = Im(\rho)$ and σ can also be grasped from the change in $\mathbf{l}$ in the orthogonal direction $\mathbf{m}$:

$\delta\mathbf{l}(3) = (\alpha^* + \beta)\mathbf{l} - \rho^*\mathbf{m} - \sigma\bar{\mathbf{m}}.$

When $\kappa = 0 = \epsilon$ we do find the following propagation equations (variation along $\mathbf{l}$) from the form the Riemann tensor takes in the present formalism (see also section **10.3**):

$D\sigma = \sigma(\rho + \rho^*) + \Psi_0 = -2\Theta\sigma + \Psi_0$

$D\rho = (\rho^2 + |\sigma|^2) + \Phi_{00}$ (or $D\omega = -2\Theta\omega$ and $D\Theta = \omega^2 - \Theta^2 - |\sigma|^2 - \Phi_{00}$).

7.6 Petrov classification and Goldberg-Sachs theorem

In chapter 6 it was already mentioned that the spinor equivalent of the Weyl tensor is completely symmetric and from chapter 1 we know that as a result the Weyl spinor defines four principal null directions via its principal spinors. These null directions are used to distinguish gravitational fields according to the Petrov classification. In contrast with electrodynamics, where an analoguous classification is possible since the Maxwell spinor is symmetric (see section **3.4**), the invariant classification of the Weyl tensor is very important. Many powerful theorems exist that relate certain properties of the field with its Petrov type, one example of which will be given below: the Goldberg-Sachs theorem.
Many exact solutions have been rediscovered again and again. Perhaps, even likely, our chapter 10, if successful is an example of this. The problem of proving or disproving that two solutions are the same can be very hard, but the complexity is reduced by the invariant classification of fields, combined with the powerful theorems available.

The Petrov classification is as follows (PND = principal null directions):

Type I: four distinct PND
Type II: two PND coincide
Type D: two degenerate PND
Type III: three PND coincide
Type N: four PND coincide

When at least two principal null directions coincide, the space-time is called algebraically special. By considering the tetrad transformations from section **7.4.4** it is not difficult to show that there exists a tetrad such that:

Type I: only $\Psi_0 = \Psi_4 = 0$
Type II: only Ψ_2 and $\Psi_3 \neq 0$
Type D: only $\Psi_2 \neq 0$
Type III: only $\Psi_3 \neq 0$
Type N: only $\Psi_4 \neq 0$

There are many forms of the theorem that goes under the name Goldberg-Sachs [79]. Even the lemmas or corollaries are sometimes presented as the Goldberg-Sachs theorem. Here we list indiscriminately two of these, the first of which will be used in chapter 10:

If a gravitational field contains a shear-free geodesic null congruence $\mathbf{l}$ (so, $\kappa = 0 = \sigma$) and $\Phi_{00} = \Phi_{01} = \Phi_{02} = 0$, then the field is algebraically special (so, $\Psi_0 = 0 = \Psi_1$) with $\mathbf{l}$ a degenerate principal null direction.

A vacuum metric is algebraically special if and only if it contains a shear-free geodesic null congruence.

Similar in spirit is the Mariot-Robinson theorem [65]:

An arbitrary space-time admits a geodesic shear-free null congruence if and only if it admits an electromagnetic null field satisfying Maxwell's equations in it

and its corollary:

The Weyl tensor of Einstein-Maxwell fields with an electromagnetic null field is algebraically special.

8

The Robinson congruence in general relativity

8.1 introduction

In previous chapters the Robinson congruence or its projection on a time-slice, the Hopf fibration, was seen to correspond to a non-null twistor, an exact solution to source-free Maxwell equations and a solution to linearized Einstein equations. The purpose of this chapter is to show that it is also related to an exact solution in general relativity. To this end we summarize and, to suit our purposes, supplement an article by Debney, Kerr and Schild [13] in which a formalism was established that can be considered as a solution generating technique "avant-la-lettre".

Consider a metric that can be cast in so called Kerr-Schild form:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h\mathbf{e}_\mu \mathbf{e}_\nu,$$

in which η is the metric of Minkowski space (M^4 with usual coordinates {t,x,y,z} that will also be used (with different interpretation of course) as coordinates in the curved manifold), $h = h(x^\mu)$ and $\mathbf{e}$ is a null vector:

$$g^{\mu\nu}\mathbf{e}_\nu \mathbf{e}_\mu = \mathbf{e}^\mu \mathbf{e}_\mu = 0.$$

The contravariant form of the metric tensor can easily be seen to be

$g^{\mu\nu} = \eta^{\mu\nu} - 2h\mathbf{e}^{\mu}\mathbf{e}^{\nu}.$

It follows that $\eta^{\mu\nu}\mathbf{e}_{\mu}\mathbf{e}_{\nu} = 0$: $\mathbf{e}$ is also a null vector with respect to "auxiliary" Minkowski space. Raising and lowering of indices of vectors that are orthogonal to $\mathbf{e}$, including $\mathbf{e}$ itself, can be done with η.

In terms of the null coordinates $\{u, v, w, \bar{w}\}$:

$u \equiv \frac{1}{\sqrt{2}}(t+z)$

$v \equiv \frac{1}{\sqrt{2}}(t-z)$

$w \equiv \frac{1}{\sqrt{2}}(x+iy)$

$\bar{w} = \frac{1}{\sqrt{2}}(x-iy)$

we can write a general (covariant) vector with unit coefficient in front of du as

$\mathbf{e} = du + Adv + \bar{Y}dw + Yd\bar{w}$

(the case of vanishing coefficient in front of du can be treated by a suitable limiting process, but will be of no concern here)
and the Minkowski metric tensor as

$$(\eta_{\mu\nu}) = (\eta^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The condition for $\mathbf{e}$ to be null with respect to η now gives $A = Y\bar{Y}$:

$\mathbf{e} = du + Y\bar{Y}dv + \bar{Y}dw + Yd\bar{w}, \quad \text{or}$

$$(\mathbf{e}_{\mu}) = \begin{pmatrix} 1 \\ Y\bar{Y} \\ \bar{Y} \\ Y \end{pmatrix} \quad \text{and} \quad (\mathbf{e}^{\mu}) = (\eta^{\mu\nu}\mathbf{e}_{\nu}) = \begin{pmatrix} Y\bar{Y} \\ 1 \\ -Y \\ -\bar{Y} \end{pmatrix}.$$

The line element in null coordinates is $ds^2 = 2dudv - 2dwd\bar{w} + 2h(du + Y\bar{Y}dv + \bar{Y}dw + Yd\bar{w})^2$, and therefore

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} + 2h \begin{pmatrix} 1 & Y\bar{Y} & \bar{Y} & Y \\ Y\bar{Y} & (Y\bar{Y})^2 & Y\bar{Y}^2 & Y^2\bar{Y} \\ \bar{Y} & Y\bar{Y}^2 & \bar{Y}^2 & Y\bar{Y} \\ Y & Y^2\bar{Y} & Y\bar{Y} & Y^2 \end{pmatrix}.$$

The inverse can be calculated most directly from $g^{\mu\nu} = \eta^{\mu\nu} - 2h\mathbf{e}^\mu \mathbf{e}^\nu$:

$$(g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} - 2h \begin{pmatrix} (Y\bar{Y})^2 & Y\bar{Y} & -Y^2\bar{Y} & -Y\bar{Y}^2 \\ Y\bar{Y} & 1 & -Y & -\bar{Y} \\ -Y^2\bar{Y} & -Y & Y^2 & Y\bar{Y} \\ -Y\bar{Y}^2 & -\bar{Y} & Y\bar{Y} & \bar{Y}^2 \end{pmatrix}.$$

Using the null vector $\mathbf{e}$, we can form a Newman-Penrose tetrad $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$, $\mathbf{l} = (l^\mu) \equiv (\mathbf{e}^\mu)$, $\mathbf{n} = (n^\mu)$, $\mathbf{m} = (m^\mu)$, $\bar{\mathbf{m}} = (\bar{m}^\mu)$, with, in null coordinates

$$(l^\mu) = \begin{pmatrix} Y\bar{Y} \\ 1 \\ -Y \\ -\bar{Y} \end{pmatrix}, \quad (n^\mu) \equiv \begin{pmatrix} 1 - hY\bar{Y} \\ -h \\ hY \\ h\bar{Y} \end{pmatrix}, \quad (m^\mu) \equiv \begin{pmatrix} -Y \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (\bar{m}^\mu) \equiv \begin{pmatrix} -\bar{Y} \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Note that in these coordinates $\bar{m}^\mu$ is not simply the complex conjugate of m^μ, as would be the case in real coordinates. In terms of the more familiar $\{t, x, y, z\}$ these definitions would be more complicated. We will need only $\mathbf{e}$ in these coordinates:

$$(\mathbf{e}^\mu) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + Y\bar{Y} \\ -(Y + \bar{Y}) \\ i(Y - \bar{Y}) \\ -1 + Y\bar{Y} \end{pmatrix} \tag{8.1}$$

When the spinor field $(\Pi^A) = (\Pi^1, \Pi^2)$ corresponds to the direction of $\mathbf{e}$, considered as a vector in auxiliary Minkowski space, M^4, we find from $\mathbf{e}^\mu \propto \sigma^\mu{}_{AX'} \Pi^A \bar{\Pi}^{X'}$ and equations (8.1), (1.4) that

$$Y = -\frac{\Pi^1}{\Pi^2} = \frac{\Pi_2}{\Pi_1}. \tag{8.2}$$

8.2 Debney, Kerr and Schild

The result of the article by Debney, Kerr and Schild that is of interest here can be summarized as follows.

For arbitrary (complex) analytic functions Φ and Ψ, constant $q \in \mathbb{C}$ and constants $p, c, m \in \mathbb{R}$ we generate a solution of Einstein-Maxwell equations in terms of null coordinates $\{u, v, w, \bar{w}\}$:

$$F \equiv \Phi(Y) + (qY + c)(w + Yv) - (pY + \bar{q})(u + Y\bar{w})$$

generates, via $F \equiv 0$:

<u>e.m. field:</u>

$$\phi_0 = 0, \quad \phi_1 = \tfrac{1}{2}\Psi(Y)P^{-2}\rho^2, \quad \phi_2 = -\tfrac{1}{2}(\partial_Y \Psi)(\partial_Y F)^{-2} + \tfrac{1}{2}\Psi(Y)(\partial_Y F)^{-3}\partial_Y \partial_Y F$$

<u>metric:</u>

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h\mathbf{e}_\mu \mathbf{e}_\nu$$

where

$$P \equiv pY\bar{Y} + qY + \bar{q}\bar{Y} + c$$

$$\rho \equiv -P(\partial_Y F)^{-1}$$

$$h \equiv \tfrac{1}{2}mP^{-3}(\rho + \bar{\rho}) - \tfrac{1}{2}\Psi\bar{\Psi}P^{-4}\rho\bar{\rho}$$

$\mathbf{e} \equiv du + Y\bar{Y}dv + Yd\bar{w} + \bar{Y}dw$ is null geodesic and shear-free

ϕ_0, ϕ_1, ϕ_2 and ρ are defined as in the Newman-Penrose formalism with $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$ as in section **8.1**.

Comparison of this function F with equations (8.2) and (1.10), which is also valid in curved spaces of Kerr-Schild form, explains one part of the working of this prescription to generate solutions to the combined system of Maxwell and Einstein equations, the fact that $\mathbf{e}$ is geodesic and shear-free.

The defining equations of ϕ_0, ϕ_1 and ϕ_2 in terms of the tetrad components of the Faraday tensor in section **7.4.3** can be used to obtain the inverse relations, expressing the tetrad components in terms of the ϕ's: $F_{(1)(3)} = -F_{(3)(1)} = F^*_{(1)(4)} = \phi_o$, $F_{(1)(2)} = -F_{(2)(1)} = \phi_1 + \phi_1^*$, etc. Equation (7.2) can now be used to obtain the $F^{(k)(l)}$'s. From these, the calculation of $F^{(k)(l)}F_{(k)(l)}$ leads to $4(\phi_0\phi_2 + \phi_0^*\phi_2^*) - 4(\phi_1^2 + \phi_1^{*2})$. This shows that the formalism of Debney, Kerr and Schild does not lead to null electromagnetic fields, since, from section **3.3**, $2(\mathbf{E}^2 - \mathbf{B}^2) = F^{\mu\nu}F_{\mu\nu} = F^{(k)(l)}F_{(k)(l)} = 0$ only for $\Psi(Y) = 0$: when an electromagnetic field is present, at least one of the requirements for this field to be null is not satisfied.

8.3 examples

In [13] the choice $\Phi = iaY,\ q = 0$ and $p = \frac{1}{\sqrt{2}} = c$ was shown to lead to the now famous Kerr metric for $\Psi = 0$ and to its electrically charged version, the Kerr-Newman metric, for $\Psi = e$. Since the angular momentum of these solutions is proportional to a it follows that we arrive at the Schwarzschild and the charged Schwarzschild or Reissner-Nordström solutions by putting $a = 0$ in the choices from above. Here we are interested in other choices of the parameters. However, for comparison we first give the results of the present procedure for the Schwarzschild and Kerr solutions.

Schwarzschild solution, $\Phi = 0 = \Psi,\ q = 0,\ p = \frac{1}{\sqrt{2}} = c:$

$$\rho = \sqrt{2}\frac{z+r}{x^2+y^2},\ h = \frac{m}{2}\frac{(x^2+y^2)^2}{(r+z)^2r^3},\ r^2 \equiv x^2+y^2+z^2$$

$$(\mathbf{e}^\mu) = \frac{\sqrt{2}(r+z)}{x^2+y^2}\begin{pmatrix} r \\ x \\ y \\ z \end{pmatrix}$$

Note that the combination $h\mathbf{e}_\mu\mathbf{e}_\nu$ that appears in the metric does not contain the common factor in $(\mathbf{e}^\mu)$, in agreement with the sperical symmetry of the solution. The form of the expansion, -$Re(\rho) = \rho$ here, seems to contradict this symmetry. However, the spin coefficients depend on the chosen tetrad. A type 3) transformation with A=$\sqrt{2}\frac{z+r}{r(x^2+y^2)}$ leads to a manifest spherical symmetric expansion.

Kerr solution, $\Phi = iaY,\ \Psi = 0,\ q = 0,\ p = \frac{1}{\sqrt{2}} = c:$

$$\rho = \sqrt{2}\frac{z+\tilde{r}}{x^2+y^2}\frac{\tilde{r}^2}{\tilde{r}^2+iaz},\ h = \frac{m}{2}\frac{(x^2+y^2)^2}{(\tilde{r}+z)^2}\frac{\tilde{r}}{\tilde{r}^4+a^2z^2}\ , \tilde{r}^4 + (a^2 - x^2 - y^2 - z^2)\tilde{r}^2 - a^2z^2 = 0$$

$$(\mathbf{e}^\mu) = \frac{\sqrt{2}(\tilde{r}+z)}{x^2+y^2}\begin{pmatrix} \tilde{r} \\ \tilde{r}\frac{x\tilde{r}+ay}{\tilde{r}^2+a^2} \\ \tilde{r}\frac{y\tilde{r}-ax}{\tilde{r}^2+a^2} \\ z \end{pmatrix}$$

Note the difference between $\tilde{r}$ (an ellipsoidal radial coordinate) and r. In the limit $a \to 0$ we have $\tilde{r} \to r$, resulting in the Schwarzschild solution. Apart

from the flat space-time part, the metric now contains the factor $\frac{2m\tilde{r}^3}{\tilde{r}^4+a^2z^2}$, shared by all coefficients. It it clear that the angular momentum is directed in the z-direction.
The coordinate transformation $u = t + \tilde{r},\ e^{i\phi}\sin\theta = (x + iy)(\tilde{r} - ia)^{-1},\ z = \tilde{r}\cos\theta$ leads to the form of the Kerr metric as it appeared for the first time in the literature [80]:

$$ds^2(= g_{\mu\nu}dx^\mu dx^\nu) = [1 - \frac{2m\tilde{r}}{\tilde{r}^2+a^2\cos^2\theta}](du + a\sin^2\theta d\phi)^2 - 2(du + a\sin^2\theta d\phi)(d\tilde{r} + a\sin^2\theta d\phi) - (\tilde{r}^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2).$$

The limit $a \to 0$ of this expression gives the Schwarzschild metric in the familiar advanced Eddington-Finkelstein coordinates. For $\tilde{r} = 0$ and $\theta = \frac{\pi}{2}$ there is a curvature singularity (not just a coordinate singularity) that has the geometry of a ring *in auxiliary Minkowski space.*

8.3.1 Robinson congruence

$\Phi = \frac{i}{\sqrt{2}}Y,\ \ \Psi = 0,\ \ p = 0 = q,\ \ c = 1$

These choices lead to $P = 1,\ \ F = (\frac{i}{\sqrt{2}} + v)Y + w,$ thus $F \equiv 0 \iff Y = \frac{-\sqrt{2}w}{i+\sqrt{2}v}$.

In terms of the auxiliary coordinates $\{t, x, y, z\}$ we have:

$Y = \frac{x+iy}{-t+z-i},\quad \rho = \frac{\sqrt{2}}{1+(t-z)^2}(-t + z + i),\quad h = m\sqrt{2}(\frac{-t+z}{1+(t-z)^2})$ and

$$(\mathrm{e}^\mu) = \frac{1}{\sqrt{2}(1 + (t - z)^2)}\begin{pmatrix} 1 + x^2 + y^2 + (t - z)^2 \\ 2(x(t - z) + y) \\ 2(y(t - z) - x) \\ x^2 + y^2 - 1 - (t - z)^2 \end{pmatrix}. \tag{8.3}$$

We recognise the Robinson congruence or Hopf fibration in this form, see equations (4.2), (4.9) and (4.10). Alternatively, the corresponding spinor field (Π_1, Π_2) with $\frac{\Pi_2}{\Pi_1} = Y$ is $(\Pi_1, \Pi_2) = f(x^\mu)(-t + z - i, x + iy)$ which is exactly equation (4.6), which led to the Robinson congruence.

In all the previous appearances of the Hopf fibration we were able to give a physical interpretation of the integral curves of the vector field: as electric or magnetic field lines, lines related to energy flow ((super-)Poynting vector),

tendex or vortex lines. But now the vector field appears twice in the metric without obvious interpretation. In addition, we do not have a clear and global distinction between time and space here, as was the case in past examples, where fields could be considered to be defined on flat space-time. The vector field in (8.3) is still to be considered as a Robinson congruence, but a projection on a time-slice in order to arrive at the structure of a Hopf fibration cannot be done, except in auxiliary space.
Of course, the congruence is built from null geodesics and these are possible photon paths or light rays. But in every point there are possible photon paths in all directions.

If we change Ψ from zero to constant e, the difference appears only in the expressions for h and the electromagnetic field. There is an extra term in h due to the contribution of the electromagnetic energy density to the curvature of space-time: $\frac{-e^2}{1+(z-t)^2}$. The electromagnetic field is represented as $\phi_0 = 0 = \phi_2$, $\phi_1 = e\frac{(z-t+i)^2}{(1+(z-t)^2)^2}$. This expression is not very illuminating, but again a Robinson congruence pops up, since the corresponding electromagnetic field can also be obtained from a one form α: $F_{\mu\nu}dx^\mu \wedge dx^\nu = d\alpha$, in which

$\alpha = e\frac{2\sqrt{2}(z-t)}{1+(z-t)^2}\mathbf{e}$.

8.3.2 degenerate Robinson congruence

$\Phi = 0 = \Psi, \;\; p = 0 = q, \;\; c = 1$

This leads to $Y = \frac{x+iy}{z-t}$, $\rho = \frac{\sqrt{2}}{z-t}$, $h = \frac{m\sqrt{2}}{z-t}$ and

$$(\mathbf{e}^\mu) = \frac{1}{\sqrt{2}(t-z)^2}\begin{pmatrix} x^2+y^2+(t-z)^2 \\ 2x(t-z) \\ 2y(t-z) \\ x^2+y^2-(t-z)^2 \end{pmatrix},$$

in which we recognise a degenerate Robinson congruence, see (4.22).

Putting $\Psi = e$, a constant, leads to an additional term in h: $-\frac{e^2}{(z-t)^2}$, and an electromagnetic field that can be represented as in example (**8.3.1**), with $\alpha = e\frac{2\sqrt{2}}{z-t}\mathbf{e}$.

8.4 conclusion and final remarks

In this chapter we used a method developed by Debney, Kerr and Schild to show that the Robinson congruence also appears in exact solutions of the full Einstein equations. However, within the confines of classical general relativity it is fair to say that in contrast with previous cases we do not know whether a physical interpretation is possible. An attempt at an interpretation along the lines of [35,36,81,82] could be worthwhile.
An interesting possibility suggests itself when comparing the solution of section **8.3.1** with the solution in chapter 6. Could the latter be the linearized version of the former? If yes, can the interpretation of the curves of the Robinson congruence in the solution of the linearized theory in some sense be taken over to the exact solution in the full theory?

It has not yet been investigated whether the solution in section **8.3.1** belongs to the class of solutions for which no general solution is known [83] or perhaps to the class of which only a few solutions are known [84], or neither of these possibilities.

It may further be noted that from [85] we may conclude the Petrov type-D character of at least the vacuum solutions in sections **8.3.1** and **8.3.2**, as well as the fact that these solutions must contain singularities not confined to a bounded region. This makes it unlikely that they will bring any changes to the observation in [86]: 'But the hope of finding metrics amongst the solutions [. . .] which describe the radiation field of a physically meaningful matter distribution has not been realized.'

As in chapter 4, there is a complex shift related to the transition degenerate Robinson congruence → Robinson congruence. The relation between the two shifts has not been investigated. Again, there is a possible connection with work done by E. Newman [31,33].

Finally, it is of interest to note that in the case of rotation-free Kerr-Schild metrics there is a geometrical interpretation for (e^{μ}) and other quantities appearing in the present procedure [87] (also described in [88]). For this interpretation to work for the degenerate Robinson congruence (rotation-free, since ρ is a real function) we need the unphysical assumption of a massive particle travelling along the z-axis with the speed of light in order to have the correct retarded distance [88]. In addition, other quantities do not fit into the scheme presented in said references.

9

Electromagnetic Hopf knot in tetrad formalism

In this chapter we search for a tetrad adapted to the field of an electromagnetic Hopf knot, such that as many of the spin coefficients as possible are vanishing or otherwise do have a simple expression. It mainly consists of calculations. Doing these by hand, one soon acquires an intuition of how to change the tetrad in order to achieve a desired expression for one of the coefficients. The choice presented here is only one of the nicer possibilities. The results of this chapter are needed in the next.

9.1 introduction

Among the many exact solutions of Einstein's equations with a source, almost all sources are unphysical. Only very few known exact solutions correspond to a realistic source, such as the Schwarzschild or Kerr solution. In the rest of this thesis we will be concerned with finding an exact solution due to a source that might be of physical relevance.

Parallel light beams have been considered as a source to Einstein's equations, leading to plane gravitational waves [89, 90]. Plane waves in general relativity played an important role in the acceptance of the existence of gravitational radiation.
Experimental physicists are currently thinking about the production in the laboratory of electromagnetic knots. Although the extend of a Hopf knot is infinite, and in this respect an experimental realization can only approximate

the full Hopf knot, in contrast to a plane electromagnetic wave the energy of an electromagnetic Hopf knot is finite and very much concentrated [1].

In chapter 10 we will try to find an exact solution of the combined system of Einstein and Maxwell equations with as sole source an electromagnetic Hopf knot. Starting with a source, a fluid, an electromagnetic field or anything else, considered in flat space-time, thus neglecting gravity, that is fully understood and specified in terms of familiar Minkowski coordinates $\{t, x, y, z\}$ and using this as input for the energy-momentum tensor in Einstein's equations in order to calculate the metric leads to a problem that is peculiar to general relativity: in order to calculate the metric we need the source, but to describe the source correctly, we must know the metric. In the problem at hand this peculiarity leads to a comfortable freedom in the definition of an electromagnetic Hopf knot in the context of curved space-time. In chapter 10 it will become clear how we can trim our sails according to the current wind.
Due to the complicated t, x, y, z dependence of the Robinson congruence and the fact that this congruence is null, geodesic and shear free, the Newman-Penrose formalism seems most suited for solving our problem. We therefore first need to know what a Hopf knot in flat space-time looks like in this formalism.

9.2 Hopf knot in flat space-time in Newman-Penrose formalism

A possible tetrad adapted to the congruence of Robinson consists of the following basis.

$$\boxed{\begin{aligned} \mathbf{l} &= \tfrac{1}{A}(S, \mathbf{S}) \\ \mathbf{n} &= A(\tfrac{1}{2S}, -\tfrac{\mathbf{S}}{2S^2}) \\ \mathbf{m} &= \tfrac{e^{i\theta}}{\sqrt{2S}}(0, \mathbf{F}) \\ \bar{\mathbf{m}} &= \tfrac{e^{-i\theta}}{\sqrt{2S}}(0, \mathbf{F}^*) \end{aligned}} \tag{9.1}$$

in which A and θ are unspecified real functions, and

$S = |\mathbf{S}| = |\mathbf{E} \times \mathbf{B}| = E^2 = \epsilon$ (energy density, for $\mathbf{E} \cdot \mathbf{B} = 0, |\mathbf{E}| = |\mathbf{B}|$)

$E = |\mathbf{E}|$

$\mathbf{F} = \mathbf{E} + i\mathbf{B}$

A nice property of this choice of tetrad is the direct relation between any of the vector fields contained in it with the electric, magnetic or Poynting vector of the electromagnetic field under consideration. We take

$$\mathbf{F} = \mathbf{F}^{RS}(t, \mathbf{r}) = \frac{1}{((t-i)^2 - r^2)^3} \begin{pmatrix} (x-iy)^2 - (t-i-z)^2 \\ i(x-iy)^2 + i(t-i-z)^2 \\ -2(x-iy)(t-i-z) \end{pmatrix} \tag{9.2}$$

which defines an electromagnetic Hopf knot (equation (4.1)). The first vector field from the tetrad therefore defines a Robinson congruence: we know from chapter 4 ((4.9), (4.10)) that $\mathbf{l}$ can be written as $\mathbf{l} = 2e^{\psi}e^{\psi^c}k^0(k^\alpha)$, with $(k^\alpha) = (k^0, \mathbf{k})$, the Robinson congruence associated with (4.2), and $|k^0| = k^0 = |\mathbf{k}|$.

Since we use the Minkowski metric, the following relations hold. These can be seen almost without effort from what we know from electromagnetic null fields.

$m^\mu m_\mu = 0 = \bar{m}^\mu \bar{m}_\mu$
$m^\mu \bar{m}_\mu = -1$
$l^\mu m_\mu = 0 = l^\mu \bar{m}_\mu$
$l^\mu l_\mu = 0 = n^\mu n_\mu$
$l^\mu n_\mu = 1$
$m^\mu n_\mu = 0 = \bar{m}^\mu n_\mu$

The tetrad thus qualifies as a Newman-Penrose tetrad (section **7.4**).

9.2.1 the spin coefficients

We now calculate the spin coefficients (section **7.4**), including the optical scalars, and use the fact that we are working in Minkowski space, so the Christoffel symbols vanish. First we derive general expressions for κ, σ, ρ with respect to (9.1), the boxed equations below. They can be formulated straightforwardly in terms of properties of the electromagnetic field considered. For example, κ is proportional to the projection along the Poynting vector of the directional derivative in the direction of the Poynting vector of the Rieman-Silberstein vector. The fields A and θ could be adjusted in such a way as to make this proportionality factor equal to 1. But we will need a different choice, the one that makes $\epsilon = 0$. As soon as we start using (9.2) the spin coefficients will receive the label Rob.

$$\kappa = \gamma_{(3)(1)(1)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(1)}^{\ \nu}\mathbf{e}_{(1)\mu;\nu} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(1)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu}$$

and thus (for the tetrad (9.1) defined above)

$$\kappa = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot(-\frac{\mathbf{S}}{A})_{,\nu}\,\frac{S^{\nu}}{A}$$

It follows that (remember $D = l^{\mu}\partial_{\mu}$)

$$\boxed{\kappa = \frac{-e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot D\frac{\mathbf{S}}{A} = \frac{e^{i\theta}}{\sqrt{2S}A}\mathbf{S}\cdot D\mathbf{F}} \text{ since } \mathbf{F}\cdot\mathbf{S} = 0$$

$\sigma = \gamma_{(3)(1)(3)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(3)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\delta\frac{\mathbf{S}}{A}$, or

$$\boxed{\sigma = \frac{e^{i\theta}}{\sqrt{2S}A}\mathbf{S}\cdot\delta\mathbf{F}}$$

$$\rho = \gamma_{(3)(1)(4)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(4)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu}$$

The difference with respect to σ is that now we have a directional derivative $\bar{m}^{\mu}\partial_{\mu}$, thus

$$\boxed{\rho = \frac{e^{i\theta}}{\sqrt{2S}A}\mathbf{S}\cdot\delta^{*}\mathbf{F}}$$

$$\epsilon = \tfrac{1}{2}(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}) = \tfrac{1}{2}(\mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(1)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} + \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(1)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu}) =$$

$$\tfrac{1}{2}\{\tfrac{1}{2S}DS + \tfrac{A}{2}D\tfrac{1}{A} + \tfrac{-A\mathbf{S}}{2S^2}\cdot D\tfrac{-\mathbf{S}}{A} + \tfrac{-1}{2S}\mathbf{F}\cdot D\mathbf{F}^{*} - \tfrac{e^{i\theta}}{2E}\mathbf{F}\cdot\mathbf{F}^{*}D\tfrac{e^{-i\theta}}{E}\} =$$
$$\tfrac{1}{2}\{\tfrac{1}{2S}DS - \tfrac{1}{2A}DA + \tfrac{1}{2S}DS - \tfrac{1}{2A}DA - \tfrac{1}{2S}\mathbf{F}\cdot D\mathbf{F}^{*} + \tfrac{1}{E}DE + iD\theta\} =$$
$$\tfrac{1}{2}\{\tfrac{1}{S}DS - \tfrac{1}{A}DA - \tfrac{1}{2S}\mathbf{F}\cdot D\mathbf{F}^{*} + \tfrac{1}{2S}DS + iD\theta$$

Thus $\epsilon = \frac{1}{2}\{\frac{3}{2S}DS - \frac{1}{A}DA - \frac{1}{2S}\mathbf{F}\cdot D\mathbf{F}^{*} + iD\theta\}$

Although we did not calculate κ yet, we do know that the first tetrad vector field is aligned with a geodesic Robinson congruence. Therefore we expect κ to be zero (see section **7.5** for the interpretation of some of the spin coefficients). Further, it is always possible to use an affine parameter along the geodesics (section **7.2.4**), and then $Re(\epsilon)$=0 also. Note that here we actually are considering an affine parameter *field*. We will try to find A and θ such that $\epsilon = 0$.

To compute the above expressions concretely, we first need the derivatives of the Riemann-Silberstein vector

$$\mathbf{F} = \frac{1}{((t-i)^2-r^2)^3}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix}.$$

$$\partial_t\mathbf{F} = \frac{-6(t-i)}{((t-i)^2-r^2)^4}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix} + \frac{-2}{((t-i)^2-r^2)^3}\begin{pmatrix}(t-i-z)\\ -i(t-i-z)\\ (x-iy)\end{pmatrix}$$

$$\partial_x\mathbf{F} = \frac{6x}{((t-i)^2-r^2)^4}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix} + \frac{2}{((t-i)^2-r^2)^3}\begin{pmatrix}(x-iy)\\ i(x-iy)\\ -(t-i-z)\end{pmatrix}$$

$$\partial_y\mathbf{F} = \frac{6y}{((t-i)^2-r^2)^4}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix} + \frac{2}{((t-i)^2-r^2)^3}\begin{pmatrix}-i(x-iy)\\ (x-iy)\\ i(t-i-z)\end{pmatrix}$$

$$\partial_z\mathbf{F} = \frac{6z}{((t-i)^2-r^2)^4}\begin{pmatrix}(x-iy)^2-(t-i-z)^2\\ i(x-iy)^2+i(t-i-z)^2\\ -2(x-iy)(t-i-z)\end{pmatrix} + \frac{2}{((t-i)^2-r^2)^3}\begin{pmatrix}(t-i-z)\\ -i(t-i-z)\\ (x-iy)\end{pmatrix}$$

The first term in these four derivatives is proportional to $\mathbf{F}$ and is therefore not needed in the calculation of the first three spin coefficients, κ, σ, ρ, because $\mathbf{S}\cdot\mathbf{F}=0$. Therefore:

$$\mathbf{S}\cdot\partial_t\mathbf{F} = \frac{-2}{((t-i)^2-r^2)^3}\frac{(1+x^2+y^2+(t-z)^2)}{((r^2-(t^2+1))^2+4r^2)^3}\begin{pmatrix}2(x(t-z)+y)\\ 2(y(t-z)-x)\\ x^2+y^2-(t-z)^2-1\end{pmatrix}\cdot\begin{pmatrix}(t-i-z)\\ -i(t-i-z)\\ (x-iy)\end{pmatrix} =$$
$$\frac{-2}{((t-i)^2-r^2)^3}\frac{(1+x^2+y^2+(t-z)^2)^2}{((r^2-(t^2+1))^2+4r^2)^3}(x-iy) = \frac{-2(x-iy)}{((t-i)^2-r^2)^3}S$$

Similarly:

$$\mathbf{S}\cdot\partial_x\mathbf{F} = \frac{2(t-z-i)}{((t-i)^2-r^2)^3}S$$

$$\mathbf{S}\cdot\partial_y\mathbf{F} = \frac{-2i(t-z-i)}{((t-i)^2-r^2)^3}S$$

$$\mathbf{S}\cdot\partial_z\mathbf{F} = \frac{2(x-iy)}{((t-i)^2-r^2)^3}S$$

We now calculate the spin coefficients explicitly.

$\kappa^{Rob.} \propto l^\mu \mathbf{S}\cdot\partial_\mu\mathbf{F} \propto (1+x^2+y^2+(t-z)^2)\mathbf{S}\cdot\partial_t\mathbf{F} + 2(x(t-z)+y)\mathbf{S}\cdot\partial_x\mathbf{F} + 2(y(t-z)-x)\mathbf{S}\cdot\partial_y\mathbf{F} + (x^2+y^2-(t-z)^2-1)\mathbf{S}\cdot\partial_z\mathbf{F} = 0$

This expresses the known fact that the integral curves of the vector field $\mathbf{l}$ are geodesics.

$\sigma^{Rob.} \propto m^\mu \mathbf{S}\cdot\partial_\mu\mathbf{F} \propto ((x-iy)^2-(t-i-z)^2)\mathbf{S}\cdot\partial_x\mathbf{F} + (i(x-iy)^2+i(t-i-z)^2)\mathbf{S}\cdot\partial_y\mathbf{F} - 2(x-iy)(t-i-z)\mathbf{S}\cdot\partial_z\mathbf{F} = 0$

As expected, the Robinson congruence is shear-free.

Returning now to the expression for $\epsilon^{Rob.}$, the real part vanishes for A such that $\frac{3}{2S}DS - \frac{1}{A}DA - \frac{1}{2S}\mathbf{F}\cdot D\mathbf{F}^*$ is imaginary. From $DS = \frac{1}{2}(\mathbf{F}\cdot D\mathbf{F}^* + \mathbf{F}^*\cdot D\mathbf{F})$ we infer that $A = S$ is the desired choice. If in addition we want $Im(\epsilon) = 0$, we have to solve the following equation for θ:

$D\theta = \frac{i}{4S}(\mathbf{F}^*\cdot D\mathbf{F} - \mathbf{F}\cdot D\mathbf{F}^*) \quad \Leftrightarrow \quad (\partial_t + \frac{\mathbf{S}}{S}\cdot\nabla)\theta = \frac{2}{1+x^2+y^2+(t-z)^2}.$

The right hand side of this equation, combined with our knowledge of the electromagnetic Hopf knot justify the "ansatz" $\theta(t,x,y,z) = \theta(x^2+y^2, t-z)$. We write $\rho \equiv x^2+y^2, \xi \equiv t-z$ to obtain:

$(1+\xi^2)\partial_\xi\theta + 2\rho\xi\partial_\rho\theta = 1$

For $\theta \neq \theta(\rho)$ the equation is simple to solve:

$\partial_\xi\theta = \frac{1}{1+\xi^2}$, with solution $\tan\theta = \xi$ or $\theta = \tan^{-1}(t-z)$.

With the choices $\boxed{A = S}$ and $\boxed{\theta = \tan^{-1}(t-z)}$ we now also have $\epsilon^{Rob.} = 0$.

Note that this choice leads to $e^{i\theta} = \frac{1+i(t-z)}{\sqrt{1+(t-z)^2}}$.

Obtaining explicit expressions for the other coefficients requires lengthy but simple calculations, that will be presented in compressed form:

$\rho^{Rob.} = \frac{e^{i\theta}}{\sqrt{2S}S}\bar{m}^\mu\mathbf{S}\cdot\partial_\mu\mathbf{F} =$
$\frac{1}{2S^2}\frac{1}{((t+i)^2-r^2)^3}\{((x+iy)^2-(t+i-z)^2)\mathbf{S}\cdot\partial_x\mathbf{F} + (-i(x+iy)^2 - i(t+i-z)^2)\mathbf{S}\cdot\partial_y\mathbf{F} + (-2(x+iy)(t+i-z))\mathbf{S}\cdot\partial_z\mathbf{F}\} =$

$\frac{1}{2S^2}\frac{1}{((t+i)^2-r^2)^3}\frac{2}{((t-i)^2-r^2)^3}S\{((x+iy)^2-(t+i-z)^2)(t-z-i) + (-i(x+iy)^2 - i(t+i-z)^2)(-i)(t-z-i) + (-2(x+iy)(t+i-z))(x-iy)\} =$

$\frac{1}{S}\frac{1}{((r^2-(t^2+1))^2+4r^2)^3}\{((x+iy)^2-(t+i-z)^2)(t-z-i)+(-i(x+iy)^2-i(t+i-z)^2)(-i)(t-z-i)+(-2(x+iy)(t+i-z))(x-iy)\}=$

$\frac{1}{S}\frac{-2(1+x^2+y^2+(t-z)^2)}{((r^2-(t^2+1))^2+4r^2)^3}(t-z+i)=-2\frac{t-z+i}{1+x^2+y^2+(t-z)^2}$

So that

$$\boxed{\Theta^{Rob.}=\frac{2(t-z)}{1+x^2+y^2+(t-z)^2}}$$

and

$$\boxed{\omega^{Rob.}=\frac{-2}{1+x^2+y^2+(t-z)^2}}$$

From this and the fact that there is no freedom left for a suitable tetrad transformation to achieve $\rho=\rho^*$ (see section **7.4.4**), it follows that the Robinson congruence is not hyper-surface orthogonal (i.e. not proportional to the gradient of a scalar function).

For the propagation equations, expressing the change along the geodesics of the congruence, we find

$D\omega^{Rob.}=\frac{8(t-z)}{(1+x^2+y^2+(t-z)^2)^2}$ and $D\Theta^{Rob.}=\frac{4(1-(t-z)^2)}{(1+x^2+y^2+(t-z)^2)^2}$

From equations like this ($D(...)=f(x^\mu)$), of which one more will be given later in this chapter, we could take advantage in the next chapter, were, depending on some choices we make, we have to solve these kind of equations for (...).

We continue with the remaining spin coefficients.

$\pi^{Rob.}=\gamma_{(2)(4)(1)}=\mathbf{e}_{(2)}{}^{\mu}\mathbf{e}_{(1)}{}^{\nu}\mathbf{e}_{(4)\mu,\nu}=-\frac{\mathbf{S}}{2S}\cdot D(\frac{-e^{-i\theta}}{\sqrt{2}S}\mathbf{F}^*)$

This expression contains a term proportional to $\mathbf{S}\cdot D\mathbf{F}^*=(\mathbf{S}\cdot D\mathbf{F})^*\propto(\kappa^{Rob.})^*=0$, and a term proportional to $\mathbf{S}\cdot\mathbf{F}^*=0$. Therefore $\pi^{Rob.}=0$.

$\lambda^{Rob.}=\gamma_{(2)(4)(4)}=\mathbf{e}_{(2)}{}^{\mu}\mathbf{e}_{(4)}{}^{\nu}\mathbf{e}_{(4)\mu,\nu}=-\frac{\mathbf{S}}{2S}\cdot\delta^*(\frac{-e^{-i\theta}}{\sqrt{2}S}\mathbf{F}^*)$

This expression contains a term proportional to $\mathbf{S}\cdot\delta^*\mathbf{F}^*=(\mathbf{S}\cdot\delta\mathbf{F})^*\propto(\sigma^{Rob.})^*=0$, and a term proportional to $\mathbf{S}\cdot\mathbf{F}^*=0$. Therefore $\lambda^{Rob.}=0$.

$\mu^{Rob.}=\gamma_{(2)(4)(3)}=\mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(3)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu}=$

$-\frac{\mathbf{S}}{2S}\cdot\delta(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*)=\frac{e^{-i\theta}}{\sqrt{2S}}\frac{1}{2S}\mathbf{S}\cdot\delta\mathbf{F}^*=\frac{1}{2}(\frac{e^{i\theta}}{S\sqrt{2S}}\mathbf{S}\cdot\delta^*\mathbf{F})^*=\frac{1}{2}(\rho^{Rob.})^*=\frac{-(t-z)+i}{1+x^2+y^2+(t-z)^2}$

$\nu^{Rob.}=\gamma_{(2)(4)(2)}=\mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(2)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu}=-\frac{\mathbf{S}}{2S}\cdot\Delta(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*)=\frac{e^{-i\theta}}{\sqrt{2S}}\frac{1}{2S}\mathbf{S}\cdot\Delta\mathbf{F}^*$

Note that $\Delta=\partial_t-\frac{1}{2}D$ and $\mathbf{S}\cdot D\mathbf{F}^*\propto\kappa^*=0$, thus

$\nu^{Rob.}=\frac{e^{-i\theta}}{\sqrt{2S}}\frac{1}{2S}\mathbf{S}\cdot\partial_t\mathbf{F}^*=\frac{-e^{-i\theta}(x+iy)}{\sqrt{2S}((t+i)^2-r^2)^3}$

$\tau^{Rob.}=\gamma_{(3)(1)(2)}=\mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(2)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu}=\frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\Delta(-\frac{\mathbf{S}}{S})=\frac{e^{i\theta}}{\sqrt{2S}}\frac{1}{S}\mathbf{S}\cdot\Delta\mathbf{F}=2(\nu^{Rob.})^*$

$\alpha^{Rob.}=\frac{1}{2}(\gamma_{(2)(1)(4)}+\gamma_{(3)(4)(4)})$ and

$\gamma_{(2)(1)(4)}=\mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(4)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu}=-\frac{\mathbf{S}}{2S}\cdot\delta^*(-\frac{\mathbf{S}}{S})=\frac{1}{4}\delta^*\frac{\mathbf{S}\cdot\mathbf{S}}{S^2}=0,$

$\gamma_{(3)(4)(4)}=\mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(4)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu}=\frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\delta^*(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*)=-\frac{1}{2S}\mathbf{F}\cdot\delta^*\mathbf{F}^*-e^{i\theta}\sqrt{2S}\delta^*\frac{e^{-i\theta}}{\sqrt{2S}}=$

$-\frac{1}{2S}\mathbf{F}\cdot\delta^*\mathbf{F}^*+i\delta^*\theta+\frac{1}{2S}\delta^*S=\frac{1}{4S}(\mathbf{F}^*\cdot\delta^*\mathbf{F}-\mathbf{F}\cdot\delta^*\mathbf{F}^*)+i\delta^*\theta$

We have $\delta\theta=\frac{e^{i\theta}}{\sqrt{2S}}\frac{-2}{((t-i)^2-r^2)^3}(x-iy)(t-i-z)\partial_z\theta=-\tau^{Rob.}(t-i-z)\partial_{t-z}\theta=\frac{-\tau^{Rob.}(t-i-z)}{1+(t-z)^2}$,

therefore $i\delta^*\theta=\frac{-2i\nu^{Rob.}(t-z+i)}{1+(t-z)^2}$

The other terms turn out to be

$\frac{-1}{4S}\mathbf{F}\cdot\delta^*\mathbf{F}^*=\frac{e^{-i\theta}(x+iy)}{\sqrt{2S}((t+i)^2-r^2)^3}=-\nu^{Rob.}$ and $\frac{1}{4S}\mathbf{F}^*\cdot\delta^*\mathbf{F}=3\frac{(t+i)^2-r^2}{(t-i)^2-r^2}\nu^{Rob.}$

These terms sum to $\alpha^{Rob.}=\frac{1}{2}[3\frac{(t+i)^2-r^2}{(t-i)^2-r^2}-\frac{(t-z)+i}{(t-z)-i}]\nu^{Rob.}$

$\beta^{Rob.}=\frac{1}{2}(\gamma_{(2)(1)(3)}+\gamma_{(3)(4)(3)})$ and

$\gamma_{(2)(1)(3)} = \mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(3)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} = -\frac{\mathbf{S}}{2S}\cdot\delta(-\frac{\mathbf{S}}{S}) = \frac{1}{4}\delta\frac{\mathbf{S}\cdot\mathbf{S}}{S^2} = 0,$

$\gamma_{(3)(4)(3)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(3)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu} = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\delta(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*) = -\frac{1}{2S}\mathbf{F}\cdot\delta\mathbf{F}^* - e^{i\theta}\sqrt{2S}\delta\frac{e^{-i\theta}}{\sqrt{2S}} =$

$\frac{1}{4S}(\mathbf{F}^*\cdot\delta\mathbf{F} - \mathbf{F}\cdot\delta\mathbf{F}^*) + i\delta\theta = -2(\alpha^{Rob.})^*$ and this implies:

$\beta^{Rob.} = -(\alpha^{Rob.})^*$

$\gamma^{Rob.} = \frac{1}{2}(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)})$ and

$\gamma_{(2)(1)(2)} = \mathbf{e}_{(2)}^{\ \mu}\mathbf{e}_{(2)}^{\ \nu}\mathbf{e}_{(1)\mu,\nu} = -\frac{\mathbf{S}}{2S}\cdot\Delta(-\frac{\mathbf{S}}{S}) = \frac{1}{4}\Delta\frac{\mathbf{S}\cdot\mathbf{S}}{S^2} = 0,$

$\gamma_{(3)(4)(2)} = \mathbf{e}_{(3)}^{\ \mu}\mathbf{e}_{(2)}^{\ \nu}\mathbf{e}_{(4)\mu,\nu} = \frac{e^{i\theta}}{\sqrt{2S}}\mathbf{F}\cdot\Delta(\frac{-e^{-i\theta}}{\sqrt{2S}}\mathbf{F}^*) = -\frac{1}{2S}\mathbf{F}\cdot\Delta\mathbf{F}^* - e^{i\theta}\sqrt{2S}\Delta\frac{e^{-i\theta}}{\sqrt{2S}} =$

$\frac{1}{4S}(\mathbf{F}^*\cdot\Delta\mathbf{F} - \mathbf{F}\cdot\Delta\mathbf{F}^*) + i\Delta\theta$

Now use $\Delta = \partial_t - \frac{1}{2}D$ and $D\theta = \frac{i}{4S}(\mathbf{F}^*\cdot D\mathbf{F} - \mathbf{F}\cdot D\mathbf{F}^*)$ to get

$\gamma^{Rob.} = \frac{1}{8S}(\mathbf{F}^*\cdot\partial_t\mathbf{F} - \mathbf{F}\cdot\partial_t\mathbf{F}^*) + \frac{i}{2}\partial_t\theta = \frac{i}{1+x^2+y^2+(t-z)^2} - \frac{3i(t^2+r^2+1)}{(t^2-r^2)^2+1+2(t^2+r^2)} + \frac{1}{2}\frac{i}{1+(t-z)^2}$

We could try to express γ in terms of the other spin coefficients with the help of relations like

$\nu\tau = \frac{x^2+y^2}{(1+x^2+y^2+(t-z)^2)^2}, \quad \mu\mu^* = \frac{1+(t-z)^2}{(1+x^2+y^2+(t-z)^2)^2}$, or

$1+(t-z)^2 = \frac{\mu\mu^*}{(\nu\tau+\mu\mu^*)^2}, \quad x^2+y^2 = \frac{\nu\tau}{(\nu\tau+\mu\mu^*)^2},$

supplemented with formulas that contain α and β, but it is more important for the needs of the following chapter to note that $\gamma^{Rob.} \in \mathbb{I}$. Chapter 10 starts with a summary of the results obtained here.

9.2.2 the electromagnetic field

We now calculate the tetrad representation for the electromagnetic field tensor that corresponds to this particular electromagnetic Hopf knot. For this we need the following relations

$$(F^{(asd)}_{\mu\nu}) = \begin{pmatrix} 0 & F_x & F_y & F_z \\ -F_x & 0 & iF_z & -iF_y \\ -F_y & -iF_z & 0 & iF_x \\ -F_z & iF_y & -iF_x & 0 \end{pmatrix}$$

$$(F^{(sd)}_{\mu\nu}) = \begin{pmatrix} 0 & F^*_x & F^*_y & F^*_z \\ -F^*_x & 0 & -iF^*_z & iF^*_y \\ -F^*_y & iF^*_z & 0 & -iF^*_x \\ -F^*_z & -iF^*_y & iF^*_x & 0 \end{pmatrix}$$

$$(F_{\mu\nu}) = \tfrac{1}{2}(F^{(asd)}_{\mu\nu} + F^{(sd)}_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \text{ thus}$$

$$F_{ij} = \tfrac{i}{2}\epsilon_{ijk}(F^k - F^{*k}), \quad (F^k) = (F_x, F_y, F_z) = (-F_k).$$

Using $\mathbf{F}\cdot\mathbf{F} = 0 = \mathbf{F}\cdot\mathbf{S}$, $\mathbf{F}\cdot\mathbf{F}^* = 2S$, we find for the Faraday tensor:

$$\phi_0 = F_{(1)(3)} = F_{\mu\nu}l^\mu m^\nu =$$

$$F_{0j}\tfrac{e^{i\theta}}{\sqrt{2S}}F^j + F_{ij}\tfrac{S^i}{S}\tfrac{e^{i\theta}}{\sqrt{2S}}F^j = -\tfrac{1}{2}(F_j + F^*_j)F^j\tfrac{e^{i\theta}}{\sqrt{2S}} + \tfrac{i}{2}\epsilon_{ijk}(F^k - F^{*k})\tfrac{S^i}{S}\tfrac{e^{i\theta}}{\sqrt{2S}}F^j =$$

$$\tfrac{1}{2}(\mathbf{F}\cdot\mathbf{F} + \mathbf{F}^*\cdot\mathbf{F})\tfrac{e^{i\theta}}{\sqrt{2S}} + \tfrac{ie^{i\theta}}{2S\sqrt{2S}}\epsilon_{ijk}(F^k - F^{*k})S^iF^j = \tfrac{e^{i\theta}\sqrt{S}}{\sqrt{2}} + \tfrac{ie^{i\theta}}{2S\sqrt{2S}}(\mathbf{S}\cdot(\mathbf{F}\times\mathbf{F}) + \mathbf{S}\cdot(\mathbf{F}^*\times\mathbf{F})) =$$

$$\tfrac{e^{i\theta}\sqrt{S}}{\sqrt{2}} - \tfrac{e^{i\theta}}{S\sqrt{2S}}\mathbf{S}\cdot\mathbf{S} = 0.$$

$$\phi_1 = \tfrac{1}{2}(F_{(1)(2)} + F_{(4)(3)}) = \tfrac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu) =$$

$$-\tfrac{1}{4}F_{0j}\tfrac{S^j}{S} + \tfrac{1}{4}F_{i0}\tfrac{S^i}{S} - \tfrac{1}{2}F_{ij}\tfrac{S^i}{S}\tfrac{S^j}{S} + \tfrac{1}{2}F_{ij}\tfrac{1}{2S}F^{*i}F^j = -\tfrac{1}{2}F_{0j}\tfrac{S^j}{S} + \tfrac{i}{4}\tfrac{1}{2S}\epsilon_{ijk}(F^k - F^{*k})F^{*i}F^j = 0.$$

$$\phi_2 = F_{(4)(2)} = F_{\mu\nu}\bar{m}^\mu n^\nu =$$

$$\tfrac{e^{-i\theta}}{2\sqrt{2S}}F_{i0}F^{*i} - \tfrac{e^{-i\theta}}{2S\sqrt{2S}}F_{ij}F^{*i}S^j = -\tfrac{e^{-i\theta}}{4\sqrt{2S}}\mathbf{F}\cdot\mathbf{F}^* - \tfrac{ie^{-i\theta}}{4S\sqrt{2S}}\epsilon_{ijk}F^kF^{*i}S^j =$$

$$-\tfrac{e^{-i\theta}\sqrt{S}}{2\sqrt{2}} - \tfrac{e^{-i\theta}}{2S\sqrt{2S}}\mathbf{S}\cdot\mathbf{S} = -e^{-i\theta}\sqrt{\tfrac{S}{2}},$$

so:

$$\boxed{\phi_2 = -\frac{e^{-i\theta}}{\sqrt{2}} \frac{1+x^2+y^2+(t-z)^2}{((t^2-r^2)^2+1+2(t^2+r^2))^{\frac{3}{2}}}} \text{ and } \boxed{\phi_0 = 0 = \phi_1}.$$

This immediately leads to the energy-momentum tensor for the electromagnetic field in tetrad terms:

the only nonzero component is

$T_{(2)(2)} = -2\phi_2\phi_2^* = -S = -\frac{(1+x^2+y^2+(t-z)^2)^2}{((t^2-r^2)^2+1+2(t^2+r^2)^2)^3}$.

9.2.3 miscellanea

If this tensor acts as the only source for Einstein's equations we arrive at a space-time in which all Ricci coefficients except one vanish:

$R_{(2)(2)} = -2S$, or $\quad \Phi_{22} = S = \frac{(1+x^2+y^2+(t-z)^2)^2}{((t^2-r^2)^2+1+2(t^2+r^2)^2)^3}$

Looking back at the Maxwell equations in Newman-Penrose formalism (section **7.4.3**), we see that for the present electromagnetic field they read:

$\kappa\phi_2 = 0$, which is satisfied trivially because $\kappa = 0$
$D\phi_2 = \rho\phi_2$
$\sigma\phi_2 = 0$, which is satisfied because $\sigma = 0$
$\delta\phi_2 = (\tau - 2\beta)\phi_2$.

From the second equation we infer that $DS = -4S\frac{t-z}{1+x^2+y^2+(t-z)^2}$, as can be checked by direct calculation. This, like the explicit propagation equations for the expansion and rotation given above, could be of use when solving the combined system of Maxwell and Einstein equations with an electromagnetic Hopf knot as source. It is not difficult to add to this list more equations of the same form or with δ substituted for D, etc. However, since in the example given in the next chapter we do not use any of these, they will not be presented here. Other choices than those of chapter 10 have been considered in which this list is of importance.

9.2.4 another choice for the tetrad

We had good physical reasons to consider the tetrad in (9.1). This choice led to reasonable expressions for the spin coefficients, five of which turned out to vanish. However, for the sake of the problem posed in chapter 10, it can be advantageous to have an alternative at hand. Therefore, here we consider an alternative choice for the tetrad that will have an additional vanishing spin coefficient.

We perform a type 1 tetrad transformation (section **7.4.4**) to the tetrad (9.1) such that $\tau \to 0$. Since, with $\kappa = 0 = \sigma$, for type 1 transformations $\tau \to \tau + a\rho$, this will be achieved for

$$a = -\frac{\tau}{\rho} = -\frac{i}{\sqrt{2}} \frac{x-iy}{\sqrt{1+(t-z)^2}} \frac{(t-z-i)((t+i)^2-r^2)^{\frac{3}{2}}}{(t-z+i)((t-i)^2-r^2)^{\frac{3}{2}}},$$

thus $aa^* = \frac{1}{2}\frac{x^2+y^2}{1+(t-z)^2}$.

From equation (7.4) we find for the new tetrad:

$$\boxed{\begin{aligned} &\mathbf{l} = (1, \tfrac{\mathbf{S}}{S}) \\ &\mathbf{n} = \tfrac{1}{2}\tfrac{1+x^2+y^2+(t-z)^2}{1+(t-z)^2}(1,0,0,1) \\ &\mathbf{m} = a(1,0,0,1) + a\tfrac{t-z-i}{x-iy}(0,1,-i,0) \\ &\bar{\mathbf{m}} = a^*(1,0,0,1) + a^*\tfrac{t-z+i}{x+iy}(0,1,i,0) \end{aligned}} \tag{9.3}$$

Note that $\mathbf{n} = (\frac{1}{2} + aa^*)(1,0,0,1)$.

The spin coefficients can be calculated as before, or, much simpler, from the previous ones by the transformation formula in section **7.4.4**. For the latter strategy, use can be made of equations that are given in section **10.3**. This leads to the following expressions for the spin coefficients:

$$\sigma = \kappa = \tau = \epsilon = \pi = \lambda = 0,$$

$$\gamma = \frac{3i}{1+(t-z)^2} - \frac{6i(t^2+r^2+1)}{(t^2-r^2)^2+1+2(t^2+r^2)} + \frac{6i(x^2+y^2)}{1+(t-z)^2}\frac{t^2-r^2-1}{(t^2-r^2)^2+1+2(t^2+r^2)},$$

$$\rho = -2\frac{t-z+i}{1+x^2+y^2+(t-z)^2},$$

$$\alpha = \frac{6ia^*}{(t-i)^2-r^2},$$

$\beta = \frac{6ia}{(t+i)^2-r^2} - \frac{2a(t-z-i)}{1+x^2+y^2+(t-z)^2}$,

$\mu = \frac{t-z-i}{1+(t-z)^2}(((t-i)^2-r^2)^3-1)$,

$\nu = \frac{2ia^*}{1+x^2+y^2+(t-z)^2} + \frac{ia^*}{1+(t-z)^2} - \frac{6ia^*(t^2+r^2+1)}{(t^2-r^2)^2+1+2(t^2+r^2)} + \frac{a^*(t-z-i)\{((t-i)^2-r^2)^3-1\}}{1+(t-z)^2}$

Part of the last expression comes from $\Delta a^* = \partial_t a^*$, which is valid since $Da^* = 0$. Note that Δ here refers to the original tetrad (9.1), and not (9.3). Again we have a simple relation between α and β: $\beta = -\alpha^* + a\rho^*$.

A type 1 tetrad transformation does not change the representation of an electromagnetic field for which $\phi_0 = 0 = \phi_1$, so with tetrad (9.3) we still have

$\phi_0 = 0 = \phi_1,\ \phi_2 = -\frac{1-i(t-z)}{\sqrt{2(1+(t-z)^2)}} \frac{1+x^2+y^2+(t-z)^2}{((t^2-r^2)^2+1+2(t^2+r^2))^{\frac{3}{2}}}$.

10

Hopf knot curves space-time

The basic issue is this: one can extend the notion of electric and magnetic fields to quite arbitrary coordinates in arbitrary space times, and one can extend the notion of the Poynting vector with it, but one does not understand what these things mean physically. In other words, the extension is purely formal. This is a very general problem in G[eneral] R[elativity], . . . [91].

In the final chapter of this thesis we set ourselves the ambitious task of solving the full nonlinear Einstein equations with an electromagnetic Hopf knot as its source. In doing so we audaciously neglect what can be considered as a warning or request, not to come up with new exact solutions but instead trying to interpret the ones that have already been found. This warning, that can be sensed for example in [92], expresses the fact that it can be very hard to give an interpretation for exact solutions. However, when our attempt will be successful, we do have an understanding of the source.
We will use the Newman-Penrose formalism for which the preparations appear in the previous chapter, that will be summarized first.
It should be noted that in the process of solving the equations every now and then a choice has to be made. The examples presented here are just two of the few options that have been considered, some of which, like the one presented first, lead to a contradiction, whereas other lead to a set of equations that have not been solved. The list of equations of the form $D(...) = f(x^{\mu})$ partly presented in chapter 9 could be of help in solving said equations. However, the complexity of the equations to be solved is daunting. The purpose of present example is to convey the method used, for which these choices are best suited.

10.1 introduction (summary of chapter 9)

In the previous chapter we have seen that an electromagnetic Hopf knot can be represented in the Newman-Penrose formalism in the following way.

In M^4 we use the tetrad

$$\mathbf{l} = (1, \frac{\mathbf{S}}{S}), \quad \mathbf{n} = \frac{1}{2}(1, -\frac{\mathbf{S}}{S}), \quad \mathbf{m} = \frac{1+i(t-z)}{\sqrt{2S(1+(t-z)^2)}}(0, \mathbf{F}), \quad \bar{\mathbf{m}} = \frac{1-i(t-z)}{\sqrt{2S(1+(t-z)^2)}}(0, \mathbf{F}^*),$$

with $\mathbf{F}$ and $\mathbf{S}$ as in chapter 4, equations (4.1) and (4.2).

The spin coefficients that correspond with this tetrad are:

$$\kappa = \sigma = \epsilon = \pi = \lambda = 0, \quad \tau = 2\nu^*, \quad \beta = -\alpha^*, \quad \mu = \frac{1}{2}\rho^*,$$

$$\rho = -2\frac{t-z+i}{1+x^2+y^2+(t-z)^2},$$

$$\nu = \frac{-(1-i(t-z))(x+iy)}{\sqrt{2S(1+(t-z)^2)((t+i)^2-r^2)^3}},$$

$$\alpha = [3\frac{(t+i)^2-r^2}{(t-i)^2-r^2} - \frac{t-z+i}{t-z-i}]\nu,$$

$$\gamma = \frac{2i}{1+x^2+y^2+(t-z)^2} - \frac{6i(t^2+r^2+1)}{(t^2-r^2)^2+1+2(t^2+r^2)} + \frac{i}{1+(t-z)^2}$$

in which $r^2 = x^2 + y^2 + z^2$.

The electromagnetic Hopf knot is given by:

$$\phi_0 = \phi_1 = 0, \quad \phi_2 = -(1 - i(t-z))\sqrt{\frac{S}{2(1+(t-z)^2)}}$$

which leads to the only nonzero component of the energy-momentum tensor: $T_{(2)(2)} = -S$.

10.2 how does an electromagnetic Hopf knot curve space-time?

Associated with an electromagnetic field is an energy-momentum distribution. When considering Einstein's equations, electromagnetic fields should therefore be included as a source. We now attempt to solve the combined Einstein-Maxwell equations in case the energy-momentum tensor of the electromagnetic Hopf knot is the only source for curvature. Due to the complicated functional relationship of the various physical quantities in terms of Minkowski coordinates, it seems appropriate to use the Newman-Penrose formalism in which we can take advantage of the incorporation of these complicated structures in the tetrad vector fields, thereby simplifying the equations to be solved.

The combined system of Einstein and Maxwell equations in this formalism consists of 64 (real) equations [12] that contain the tetrad as derivative operators, $D, \Delta, \delta, \delta^*$, the spin coefficients, $\kappa, \sigma, ...$ and the tetrad representatives of the Weyl tensor, $\Psi_0, ..., \Psi_4$, the Ricci tensor, $\Phi_{00}, ..., \Phi_{22}, \Lambda$, and the Faraday tensor, ϕ_1, ϕ_2, ϕ_3.

If we would use the quantities that define an electromagnetic Hopf knot in M^4, given in the introduction, in the 64 equations, we would get back flat space-time. Of course, this is no surprise, since it has already been build in. This relates to the fact that, in general relativity, in order to describe the sources in detail we do need the metric first. There is a circularity, since the metric depends on the details of the sources. Apart from linearizing the theory, few circumventions to this circularity exists:
1. We could try to find an exact solution mathematically, without consideration to the physical sources. When the solution has been found we try to interpret the sources.
2. We could give up a detailed description of the source, and only demand for example axial symmetry. The more symmetry we demand, the less likely we find a solution.
3. In the present context, we should "declare" what really is essential to the source, and only demand those properties for the source. This then *defines* a generalization of a "Minkowski electromagnetic Hopf knot" into the realm of general relativity.

It can (and will) be argued that $\kappa = \sigma = 0$ is essential to a Hopf knot. We surely want this to be true when commencing our task of solving the 64 equations simultaneously. But what about the fact that $\gamma \in \mathbb{I}$ or $\beta = -\alpha^*$? What do they mean? Notice that in the equations (see below) the combination $\gamma + \gamma^*$ as well as $\alpha + \beta^*$ appears, often in complicated equations. Demanding these combinations to vanish would certainly help to reduce the difficulty of our task. However, it is very well possible that with these extra restrictions there will be no solution, other than the one already found: flat space-time. The more restrictions we impose, the less likely it becomes that we find a new solution. On the other hand, with very few restrictions it becomes almost impossible to solve the equations simultaneously. This might result in a delicate balance between too many and too few requirements in the definition for the knot.

10.2.1 the strategy

We must start with what we consider to be essential, try to solve as many equations as possible, and in the process decide wether we impose additional restrictions in order to relieve the difficulty of the equations encountered. We possibly then find our previous result (flat space-time) or some contradiction. In that case we know that we imposed the wrong extra condition. We return to that point, alter the restrictions and try again. If there is no progress after many repetitions, we come to a point of finally changing the assumptions in the definition of the source and start the whole process over again. There could of course be multiple repetitions of this also. It seems reasonable to assume that a non trivial solution will in the end be found. It is however uncertain how much of what we hoped to include in the definition of an electromagnetic Hopf knot in the context of curved space-time is also possible. It can only be hoped that when the sought after solution has been obtained, enough reason is left for justifying the said knowledge of the source.

We first list the equations to be solved (**10.3**), mention some equations that can be used advantageously (**10.4**), state what is to be considered essential to an electromagnetic Hopf knot (**10.5**) and then show one attempt to solve the equations in detail (**10.6.1**) and a summary of another (**10.6.2**).

10.3 the equations to be solved

From the Ricci identities (36 real or 18 complex equations):

$$D\rho - \delta^*\kappa = (\rho^2 + \sigma\sigma^*) + \rho(\epsilon + \epsilon^*) - \kappa^*\tau - \kappa(3\alpha + \beta^* - \pi) + \Phi_{00} \tag{10.1}$$

$$D\sigma - \delta\kappa = \sigma(\rho + \rho^* + 3\epsilon - \epsilon^*) - \kappa(\tau - \pi^* + \alpha^* + 3\beta) + \Psi_0 \tag{10.2}$$

$$D\tau - \Delta\kappa = \rho(\tau + \pi^*) + \sigma(\tau^* + \pi) + \tau(\epsilon - \epsilon^*) - \kappa(3\gamma + \gamma^*) + \Psi_1 + \Phi_{01} \tag{10.3}$$

$$D\alpha - \delta^*\epsilon = \alpha(\rho + \epsilon^* - 2\epsilon) + \beta\sigma^* - \beta^*\epsilon - \kappa\lambda - \kappa^*\gamma + \pi(\epsilon + \rho) + \Phi_{10} \tag{10.4}$$

$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\rho^* - \epsilon^*) - \kappa(\mu + \gamma) - \epsilon(\alpha^* - \pi^*) + \Psi_1 \tag{10.5}$$

$$D\gamma - \Delta\epsilon = \alpha(\tau + \pi^*) + \beta(\tau * + \pi) - \gamma(\epsilon + \epsilon^*) - \epsilon(\gamma + \gamma^*) + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda \tag{10.6}$$

$$D\lambda - \delta^*\pi = (\rho\lambda + \sigma^*\mu) + \pi(\pi + \alpha - \beta^*) - \nu\kappa^* - \lambda(3\epsilon - \epsilon^*) + \Phi_{20} \tag{10.7}$$

$$D\mu - \delta\pi = (\rho^*\mu + \sigma\lambda) + \pi(\pi^* - \alpha^* + \beta) - \mu(\epsilon + \epsilon^*) - \nu\kappa + \Psi_2 + 2\Lambda \tag{10.8}$$

$$D\nu - \Delta\pi = \mu(\pi + \tau^*) + \lambda(\pi^* + \tau) + \pi(\gamma - \gamma^*) - \nu(3\epsilon + \epsilon^*) + \Psi_3 + \Phi_{21} \tag{10.9}$$

$$\Delta\lambda - \delta^*\nu = -\lambda(\mu + \mu^* + 3\gamma - \gamma^*) + \nu(3\alpha + \beta^* + \pi - \tau^*) - \Psi_4 \tag{10.10}$$

$$\delta\rho - \delta^*\sigma = \rho(\alpha^* + \beta) - \sigma(3\alpha - \beta^*) + \tau(\rho - \rho^*) + \kappa(\mu - \mu^*) - \Psi_1 + \Phi_{01} \tag{10.11}$$

$$\delta\alpha - \delta^*\beta = (\mu\rho - \lambda\sigma) + \alpha\alpha^* + \beta\beta^* - 2\alpha\beta + \gamma(\rho - \rho^*) + \epsilon(\mu - \mu^*) - \Psi_2 + \Phi_{11} + \Lambda \tag{10.12}$$

$$\delta\lambda - \delta^*\mu = \nu(\rho - \rho^*) + \pi(\mu - \mu^*) + \mu(\alpha + \beta^*) + \lambda(\alpha^* - 3\beta) - \Psi_3 + \Phi_{21} \tag{10.13}$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\lambda^*) + \mu(\gamma + \gamma^*) - \nu^*\pi + \nu(\tau - 3\beta - \alpha^*) + \Phi_{22} \tag{10.14}$$

$$\delta\gamma - \Delta\beta = \gamma(\tau - \alpha^* - \beta) + \mu\tau - \sigma\nu - \epsilon\nu^* - \beta(\gamma - \gamma^* - \mu) + \alpha\lambda^* + \Phi_{12} \tag{10.15}$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \lambda^*\rho) + \tau(\tau + \beta - \alpha^*) - \sigma(3\gamma - \gamma^*) - \kappa\nu^* + \Phi_{02} \tag{10.16}$$

$$\Delta\rho - \delta^*\tau = -(\rho\mu^* + \sigma\lambda) + \tau(\beta^* - \alpha - \tau^*) + \rho(\gamma + \gamma^*) + \nu\kappa - \Psi_2 - 2\Lambda \tag{10.17}$$

$$\Delta\alpha - \delta^*\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\gamma^* - \mu^*) + \gamma(\beta^* - \tau^*) - \Psi_3 \tag{10.18}$$

From the Bianchi identities (16 real or 8 complex equations):

$$-\delta^*\Psi_0 + D\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 + R_1 = 0 \tag{10.19}$$

in which
$R_1 = -D\Phi_{01} + \delta\Phi_{00} + 2(\epsilon + \rho^*)\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \kappa^*\Phi_{02} + (\pi^* - 2\alpha^* - 2\beta)\Phi_{00}$

$$\delta^*\Psi_1 - D\Psi_2 - \lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 + R_2 = 0 \tag{10.20}$$

in which
$R_2 = \delta^*\Phi_{01} - \Delta\Phi_{00} - 2(\alpha + \tau^*)\Phi_{01} + 2\rho\Phi_{11} + \sigma^*\Phi_{02} - (\mu^* - 2\gamma - 2\gamma^*)\Phi_{00} - 2\tau\Phi_{10} - 2D\Lambda$

$$-\delta^*\Psi_2 + D\Psi_3 + 2\lambda\Psi_1 - 3\pi\Psi_2 + 2(\epsilon - \rho)\Psi_3 + \kappa\Psi_4 + R_3 = 0 \tag{10.21}$$

in which
$R_3 = -D\Phi_{21} + \delta\Phi_{20} + 2(\rho^* - \epsilon)\Phi_{21} - 2\mu\Phi_{10} + 2\pi\Phi_{11} - \kappa^*\Phi_{22} - (2\alpha^* - 2\beta - \pi^*)\Phi_{20} - 2\delta^*\Lambda$

$$\delta^*\Psi_3 - D\Psi_4 - 3\lambda\Psi_2 + 2(2\pi + \alpha)\Psi_3 - (4\epsilon - \rho)\Psi_4 + R_4 = 0 \tag{10.22}$$

in which
$R_4 = -\Delta\Phi_{20} + \delta^*\Phi_{21} + 2(\alpha - \tau^*)\Phi_{21} + 2\nu\Phi_{10} + \sigma^*\Phi_{22} - 2\lambda\Phi_{11} - (\mu^* + 2\gamma - 2\gamma^*)\Phi_{20}$

$$-\Delta\Psi_0 + \delta\Psi_1 + (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 + R_5 = 0 \tag{10.23}$$

in which
$R_5 = -D\Phi_{02} + \delta\Phi_{01} + 2(\pi^* - \beta)\Phi_{01} - 2\kappa\Phi_{12} - \lambda^*\Phi_{00} + 2\sigma\Phi_{11} + (\rho^* + 2\epsilon - 2\epsilon^*)\Phi_{02}$

$$-\Delta\Psi_1 + \delta\Psi_2 + \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 + R_6 = 0 \tag{10.24}$$

in which
$R_6 = \Delta\Phi_{01} - \delta^*\Phi_{02} + 2(\mu^* - \gamma)\Phi_{01} - 2\rho\Phi_{12} - \nu^*\Phi_{00} + 2\tau\Phi_{11} + (\tau^* - 2\beta^* + 2\alpha)\Phi_{02} + 2\delta\Lambda$

$$-\Delta\Psi_2 + \delta\Psi_3 + 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4 + R_7 = 0 \tag{10.25}$$

in which
$R_7 = -D\Phi_{22} + \delta\Phi_{21} + 2(\pi^* + \beta)\Phi_{21} - 2\mu\Phi_{11} - \lambda^*\Phi_{20} + 2\pi\Phi_{12} + (\rho^* - 2\epsilon - 2\epsilon^*)\Phi_{22} - 2\Delta\Lambda$

$$-\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - (\tau - 4\beta)\Psi_4 + R_8 = 0 \tag{10.26}$$

in which
$R_8 = \Delta\Phi_{21} - \delta^*\Phi_{22} + 2(\mu^* + \gamma)\Phi_{21} - 2\nu\Phi_{11} - \nu^*\Phi_{20} + 2\lambda\Phi_{12} + (\tau^* - 2\alpha - 2\beta^*)\Phi_{22}$

and further, also from the Bianchi identities (2 real and 1 complex equations):

$\delta^*\Phi_{01} + \delta\Phi_{10} - D(\Phi_{11} + 3\Lambda) - \Delta\Phi_{00} =$
$\kappa^*\Phi_{12} + \kappa\Phi_{21} + (2\alpha + 2\tau^* - \pi)\Phi_{01} + (2\alpha^* + 2\tau - \pi^*)\Phi_{10} - 2(\rho + \rho^*)\Phi_{11} - \sigma^*\Phi_{02} -$
$\sigma\Phi_{20} + (\mu + \mu^* - 2(\gamma + \gamma^*))\Phi_{00}$

$\delta^*\Phi_{12} + \delta\Phi_{21} - \Delta(\Phi_{11} + 3\Lambda) - D\Phi_{22} =$
$-\nu\Phi_{01} - \nu^*\Phi_{10} + (\tau^* - 2\beta^* - 2\pi)\Phi_{12} + (\tau - 2\beta - 2\pi^*)\Phi_{21} + 2(\mu + \mu^*)\Phi_{11} - (\rho +$
$\rho^* - 2\epsilon - 2\epsilon^*)\Phi_{22} + \lambda\Phi_{02} + \lambda^*\Phi_{20}$

$\delta(\Phi_{11} - 3\Lambda) - D\Phi_{12} - \Delta\Phi_{01} + \delta^*\Phi_{02} =$
$\kappa\Phi_{22} - \nu^*\Phi_{00} + (\tau^* - \pi + 2\alpha - 2\beta^*)\Phi_{02} - \sigma\Phi_{21} + \lambda^*\Phi_{10} + 2(\tau - \pi^*)\Phi_{11} - (2\rho +$
$\rho^* - 2\epsilon^*)\Phi_{12} + (2\mu^* + \mu - 2\gamma)\Phi_{01}$

And finally, Maxwell's equations:

$$D\phi_1 - \delta^*\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2 \tag{10.27}$$

$$D\phi_2 - \delta^*\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2 \tag{10.28}$$

$$\delta\phi_1 - \Delta\phi_0 = (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2 \tag{10.29}$$

$$\delta\phi_2 - \Delta\phi_1 = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2 \tag{10.30}$$

10.4 useful equations

When solving the combined system of equations, use can be made of the commutation relations, which are easy to obtain from the definitions of the spin coefficients:

$$\Delta D - D\Delta = (\gamma + \gamma^*)D + (\epsilon + \epsilon^*)\Delta - (\tau^* + \pi)\delta - (\tau + \pi^*)\delta^* \tag{10.31}$$

$$\delta D - D\delta = (\alpha^* + \beta - \pi^*)D + \kappa\Delta - (\rho^* + \epsilon - \epsilon^*)\delta - \sigma\delta^* \tag{10.32}$$

$$\delta\Delta - \Delta\delta = -\nu^* D + (\tau - \alpha^* - \beta)\Delta + (\mu - \gamma + \gamma^*)\delta + \lambda^*\delta^* \tag{10.33}$$

$$\delta^*\delta - \delta\delta^* = (\mu^* - \mu)D + (\rho^* - \rho)\Delta + (\alpha - \beta^*)\delta + (\beta - \alpha^*)\delta^* \tag{10.34}$$

The following set of so called eliminant equations can be useful too. They can be derived from combinations of (10.1) - (10.18).

$D(\rho-\rho^*)+\delta\kappa^*-\delta^*\kappa = (\rho-\rho^*)(\rho+\rho^*+\epsilon+\epsilon^*)+\kappa(\tau^*+\pi-3\alpha-\beta^*)-\kappa^*(\tau+\pi^*-3\alpha^*-\beta)$

$D(\mu - \mu^*) + \delta(\alpha + \beta^* - \pi) - \delta^*(\alpha^* + \beta - \pi^*) = (\gamma + \gamma^*)(\rho - \rho^*) + \alpha(\pi^* - 2\beta) - \alpha^*(\pi - 2\beta^*) + \kappa^*\nu^* - \kappa\nu + \beta\pi - \beta^*\pi^* + (\rho + \rho^*)(\mu - \mu^*)$

$D(\mu - \mu^* - \gamma + \gamma^*) + \Delta(\epsilon - \epsilon^*) - \delta\pi + \delta^*\pi^* = (\epsilon + \epsilon^*)(\mu^* - \mu) + \tau^*(\alpha^* + \pi^* - \beta) - \tau(\alpha + \pi - \beta^*) + \lambda\sigma - \lambda^*\sigma^* + \rho^*\mu - \rho\mu^* + 2(\epsilon\gamma - \epsilon^*\gamma^*)$

$\Delta(\mu^* - \mu) + \delta\nu - \delta^*\nu^* = (\mu - \mu^*)(\mu + \mu^* + \gamma + \gamma^*) + \nu(\tau - 3\beta - \alpha^* + \pi^*) - \nu^*(\tau^* + \pi - 3\beta^* - \alpha)$

$D(\tau - \alpha^* - \beta) - \Delta\kappa + \delta(\epsilon + \epsilon^*) = \rho(\tau + \pi^*) + \kappa^*\lambda^* + \sigma(\tau^* - \alpha - \beta^*) + \epsilon(\tau - \pi^*) - \rho^*(\beta + \alpha^* + \pi^*) + \epsilon^*(2\alpha^* + 2\beta - \tau - \pi^*) + \kappa(\mu - 2\gamma)$

$\delta(\rho - \epsilon + \epsilon^*) - \delta^*\sigma + D(\beta - \alpha^*) = \rho(\alpha^* + \beta + \tau) - \rho^*(\tau - \beta + \alpha^* + \pi^*) + (\epsilon^* - \epsilon)(2\alpha^* - \pi^*) + \sigma(\pi - 2\alpha) + \kappa(\gamma^* - \gamma - \mu^*) + \kappa^*\lambda^*$

$D\lambda + \Delta\sigma^* - \delta^*(\tau^* + \pi) = \sigma^*(3\gamma^* - \gamma + \mu - \mu^*) + (\pi + \tau^*)(\pi - \tau^* + \alpha) + \lambda(\rho - \rho^* - 3\epsilon + \epsilon^*) - \beta\pi - \tau^*\beta^*$

$D\nu + \Delta(\alpha + \beta^* - \pi) - \delta^*(\gamma + \gamma^*) = \nu(\rho - 2\epsilon) + \lambda(\pi^* - \alpha^* - \beta) + \mu(\pi + \tau^*) - \mu^*(\alpha + \beta^* + \tau^*) + \gamma(\pi - \tau^*) + \gamma^*(2\alpha + 2\beta^* - \pi - \tau^*) + \sigma^*\nu^*$

$\Delta(\beta^* - \alpha) + \delta\lambda + \delta^*(\gamma - \gamma^* - \mu) = \nu(\epsilon^* - \epsilon - \rho^*) + \lambda(\tau - 2\beta) + \alpha(\mu + \mu^*) - \mu^*(\pi + \tau^* + \beta^*) + \mu(\pi + \beta^*) + (\gamma - \gamma^*)(\tau^* - 2\beta^*) + \sigma^*\nu^*$

$D\mu + \Delta\rho - \delta\pi - \delta^*\tau = \rho^*\mu - \rho\mu^* + \pi(\pi^* - \alpha^* + \beta) + \tau(\beta^* - \alpha - \tau^*) + \rho(\gamma + \gamma^*) - \mu(\epsilon + \epsilon^*)$

10.5 essentials of a Hopf knot

We must define the source now. For every constant real a the vector (compare (4.1))

$$\mathbf{F}(t, \mathbf{r}) = \frac{a}{((t-i)^2 - r^2)^3} \begin{pmatrix} (x - iy)^2 - (t - i - z)^2 \\ i(x - iy)^2 + i(t - i - z)^2 \\ -2(x - iy)(t - i - z) \end{pmatrix}$$

defines the Riemann-Silberstein vector of an electromagnetic knot. In the limit of very small a, the energy(density) is accordingly small and the curvature of space-time due to this field can be safely neglected. This is the situation described in the introduction, except that the constant a should now appear in the representation of the Faraday tensor. In this case there is a direct relation between the first tetrad vector field, **l**, and the electromagnetic field. We expect this relation to hold also in the case of stronger fields, so that stating properties for the first of tetrads is tantamount to having corresponding properties for the electromagnetic field.

An important property of an electromagnetic Hopf knot is that the field is null. From section **3.4** we know that the principal null directions of the field coincide: in spinor-language we have $\phi_{AB} \propto \vartheta_A \vartheta_B$ for some spinor ϑ. In the notation of section **1.2.1**, let $\mathbf{l} \leftrightarrow \vartheta\bar{\vartheta}$ and $\mathbf{n} \leftrightarrow \iota\bar{\iota}$ for some suitable but unspecified ι. The spinor form of the defining equations for the tetrad representation of the faraday tensor (section **7.4.3**) can be seen to be $\phi_0 = \phi_{AB}\vartheta^A\vartheta^B, \phi_1 = \phi_{AB}\vartheta^A\iota^B$ and $\phi_2 = \phi_{AB}\iota^A\iota^B$. With present choice for **l** we therefore have ϕ_2 as the only nonzero component. From section **3.4** we also know that Maxwell equations in flat space-time implies that $\mathbf{l}$ defines a geodesic and shear-free null congruence.

This remains true in a general curved space-time (see below), and we therefore proceed as follows.

We choose $\mathbf{l}$ to be the eigenvector field of the Faraday tensor. This implies that the Faraday tensor is represented by ϕ_2 alone, the other components vanish. Maxwell's equations (10.27) and (10.29) now give $\kappa = 0 = \sigma$.

In M^4 the Robinson congruence is not proportional to the gradient of a scalar field. Having in mind the converse of the limiting procedure just described, it seems implausible that at one stage in the process we suddenly do have this proportionality. We therefore assume $\rho \neq \rho^*$, in particular $\rho \neq 0$. Only the latter property will be of importance in the sequel.

The choices that we will make next represent only one possible set. Others have been considered, some of which could afterwards be rejected. With one particular set of choices we finally obtained few difficult equations that still have to be solved. However, we would like to present an example that has been brought to completion. Besides, the choices discussed in the following section allow for a better survey of the method used.
We would also like to mention that changes in the choice of the form of $\mathbf{l}$ (see below) have been considered, with no prospect of obtaining a solution quickly.

10.6 an attempt to solve the equations

10.6.1 first attempt

We assume that there exists coordinates $\{t, x, y, z\}$ such that

$\mathbf{l} = \mathcal{A}(1, \frac{\mathbf{S}}{bS})$

$T_{(2)(2)} = -bS$, other components vanishing

$bS = |\mathbf{S}|$, $\mathbf{S} = \frac{(1+x^2+y^2+(t-z)^2)}{((t^2-r^2)^2+1+2(t^2+r^2))^3} \begin{pmatrix} 2(x(t-z)+y) \\ 2(y(t-z)-x) \\ x^2+y^2-(t-z)^2-1 \end{pmatrix}$,

$\mathcal{A} = \mathcal{A}(t, x, y, z)$ and $b = b(t, x, y, z)$ undetermined

and $\phi_0 = 0 = \phi_1$; $\kappa = 0 = \sigma$, $\rho \neq 0$.

The space-time under consideration contains a geodesic shear-free null congruence, **l**. In addition, since $\phi_0 = 0 = \phi_1$, we have $\Phi_{00} = \Phi_{01} = \Phi_{02} = 0$. All the requirements for the Goldberg-Sachs theorem (section **7.6**) to be valid are therefore satisfied. We conclude that the space-time is algebraically special and:

$\Psi_0 = 0 = \Psi_1$.

Now apply a type 3 transformation to the tetrad (see section **7.4.4**) such that $\frac{\epsilon}{A} - \frac{DA}{2A^2} + \frac{iD\theta}{2A} = 0$. The function A will be absorbed in $\mathcal{A}$. From now we also have:

$\epsilon = 0$.

Next, consider a type 1 transformation with $a = -\frac{\tau}{\rho}$. From the Ricci identities, (10.1)-(10.18), and what we know so far, it can be inferred that $Da = -\pi^*$ and $\delta a = -\lambda^* + 2\alpha^* \frac{\tau}{\rho} - \rho^* \frac{\tau^2}{\rho^2}$. These relations for a, Da and δa imply that we now have in addition:

$\tau = \pi = \lambda = 0$.

Plugging the known variables into the equations listed above and skipping the ones that have become trivial, we see that, only due to $\phi_0 = \phi_1 = \kappa = \sigma = 0, \quad \rho \neq 0$ and suitable tetrad transformations, a great simplification has occurred:

$D\rho = \rho^2$
$D\alpha = \alpha\rho$
$D\beta = \beta\rho^*$
$D\gamma = \Psi_2$
$D\mu = \rho^*\mu + \Psi_2$
$D\nu = \Psi_3$
$-\delta^*\nu = \nu(3\alpha + \beta^*) - \Psi_4$
$\delta\rho = \rho(\alpha^* + \beta)$
$\delta\alpha - \delta^*\beta = \mu\rho + \alpha\alpha^* + \beta\beta^* - 2\alpha\beta + \gamma(\rho - \rho^*) - \Psi_2$
$-\delta^*\mu = \nu(\rho - \rho^*) + \mu(\alpha + \beta^*) - \Psi_3$
$\delta\nu - \Delta\mu = \mu^2 + \mu(\gamma + \gamma^*) - \nu(3\beta + \alpha^*) + \Phi_{22}$
$\delta\gamma - \Delta\beta = \gamma(-\alpha^* - \beta) - \beta(\gamma - \gamma^* - \mu)$
$\Delta\rho = -\rho\mu^* + \rho(\gamma + \gamma^*) - \Psi_2$
$\Delta\alpha - \delta^*\gamma = \nu\rho + \alpha(\gamma^* - \mu^*) + \gamma\beta^* - \Psi_3$

$D\Psi_2 = 3\rho\Psi_2$
$-\delta^*\Psi_2 + D\Psi_3 - 2\rho\Psi_3 = 0$
$\delta^*\Psi_3 - D\Psi_4 + 2\alpha\Psi_3 + \rho\Psi_4 = 0$
$\delta\Psi_2 = 0$

$$-\Delta\Psi_2 + \delta\Psi_3 - 3\mu\Psi_2 + 2\beta\Psi_3 - D\Phi_{22} + \rho^*\Phi_{22} = 0$$
$$-\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + 4\beta\Psi_4 - \delta^*\Phi_{22} - 2(\alpha + \beta^*)\Phi_{22} = 0$$

$$D\Phi_{22} = (\rho + \rho^*)\Phi_{22}$$

$$D\phi_2 = \rho\phi_2$$
$$\delta\phi_2 = -2\beta\phi_2$$

Since $\Phi_{22} \propto \phi_2\phi_2^*$, $D\phi_2 = \rho\phi_2$ implies $D\Phi_{22} = (\rho + \rho^*)\Phi_{22}$, which can therefore be omitted from the list.
From $T_{(2)(2)} = -bS$ it follows that $\phi_2 = e^{i\Omega}\sqrt{\frac{bS}{2}}$ for some real function Ω. $D\phi_2 = \rho\phi_2$ then results in

$$\rho = iD\Omega - \mathcal{A}\frac{2(t-z)}{1+x^2+y^2+(t-z)^2} + \frac{1}{2b}Db.$$

Since $\rho, \alpha, \beta^*, \phi_2$ and $(\Psi_2)^{\frac{1}{3}}$ all satisfy the same equation $D(...) = \rho(...)$, it is tempting to express all these variables in terms of one of them:

$\alpha = f^\alpha\rho$, $\beta = f^\beta\rho^*$, $\phi_2 = f^\phi\rho$, $\Psi_2 = f^\Psi\rho^3$ in which all the $f^{\cdots}$ satisfy $Df^{\cdots} = 0$.
In the same spirit $D\gamma = \Psi_2$ then implies $\gamma = \frac{1}{2}f^\Psi\rho^2 + f^\gamma$, and $D\mu = \rho^*\mu + \Psi_2$ suggests the form $\mu = f_1\rho^* + f_2\rho + f_3\rho^2$. Substitution of this expression into the equation leads to $f_3 = \frac{1}{2}f^\Psi$ and $f_2 = f^\Psi\frac{\rho\rho^*}{2(\rho-\rho^*)}$. Note that indeed $Df_2 = 0$. To summarize:

$$\alpha = f^\alpha\rho$$
$$\beta = f^\beta\rho^*$$
$$\phi_2 = f^\phi\rho$$
$$\Psi_2 = f^\Psi\rho^3$$
$$\gamma = \frac{1}{2}f^\Psi\rho^2 + f^\gamma$$
$$\mu = f^\mu\rho^* + f^\Psi\frac{\rho^3}{2(\rho-\rho^*)}$$
$$\Phi_{22} = 2f^\phi(f^\phi)^*\rho\rho^*$$

$$Df^{\cdots} = 0.$$

$$D\rho = \rho^2$$
$$D\nu = \Psi_3$$
$$-\delta^*\nu = \nu(3\alpha + \beta^*) - \Psi_4$$
$$\delta\rho = \rho(\alpha^* + \beta)$$

$$\delta\alpha - \delta^*\beta = \mu\rho + \alpha\alpha^* + \beta\beta^* - 2\alpha\beta + \gamma(\rho - \rho^*) - \Psi_2$$
$$-\delta^*\mu = \nu(\rho - \rho^*) + \mu(\alpha + \beta^*) - \Psi_3$$
$$\delta\nu - \Delta\mu = \mu^2 + \mu(\gamma + \gamma^*) - \nu(3\beta + \alpha^*) + \Phi_{22}$$
$$\delta\gamma - \Delta\beta = \gamma(-\alpha^* - \beta) - \beta(\gamma - \gamma^* - \mu)$$
$$\Delta\rho = -\rho\mu^* + \rho(\gamma + \gamma^*) - \Psi_2$$
$$\Delta\alpha - \delta^*\gamma = \nu\rho + \alpha(\gamma^* - \mu^*) + \gamma\beta^* - \Psi_3$$

$$-\delta^*\Psi_2 + D\Psi_3 - 2\rho\Psi_3 = 0$$
$$\delta^*\Psi_3 - D\Psi_4 + 2\alpha\Psi_3 + \rho\Psi_4 = 0$$
$$\delta\Psi_2 = 0$$
$$-\Delta\Psi_2 + \delta\Psi_3 - 3\mu\Psi_2 + 2\beta\Psi_3 - \rho\Phi_{22} = 0$$
$$-\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + 4\beta\Psi_4 - \delta^*\Phi_{22} - 2(\alpha + \beta^*)\Phi_{22} = 0$$

$$\delta\phi_2 = -2\beta\phi_2$$

From the equations for $\delta\rho, \delta\Psi_2$ and $\delta\phi_2$ we deduce:

$$\delta f^\Psi = -3f^\Psi(f^\beta + f^{\alpha*})\rho^*, \quad \delta f^\phi = -f^\phi(3f^\beta + f^{\alpha*})\rho^*, \quad \delta\rho = \rho\rho^*(f^\beta + f^{\alpha*}).$$

This is a good point for an extra assumption that will lead to a welcome further simplification.
If we assume $\beta = -\alpha^*$, valid in the flat space-time case, we get:

$$D\rho = \rho^2$$
$$D\nu = \Psi_3$$
$$-\delta^*\nu = 2\nu\alpha - \Psi_4$$
$$\delta\rho = 0$$
$$\delta\alpha + \delta^*\alpha^* = \mu\rho + 4\alpha\alpha^* + \gamma(\rho - \rho^*) - \Psi_2$$
$$-\delta^*\mu = \nu(\rho - \rho^*) - \Psi_3$$
$$\delta\nu - \Delta\mu = \mu^2 + \mu(\gamma + \gamma^*) + 2\nu\alpha^* + \Phi_{22}$$
$$\delta\gamma + \Delta\alpha^* = \alpha^*(\gamma - \gamma^* - \mu)$$
$$\Delta\rho = -\rho\mu^* + \rho(\gamma + \gamma^*) - \Psi_2$$
$$\Delta\alpha - \delta^*\gamma = \nu\rho + \alpha(\gamma^* - \mu^*) - \gamma\alpha - \Psi_3$$

$$-\delta^*\Psi_2 + D\Psi_3 - 2\rho\Psi_3 = 0$$
$$\delta^*\Psi_3 - D\Psi_4 + 2\alpha\Psi_3 + \rho\Psi_4 = 0$$
$$-\Delta\Psi_2 + \delta\Psi_3 - 3\mu\Psi_2 - 2\alpha^*\Psi_3 - \rho\Phi_{22} = 0$$
$$-\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - 4\alpha^*\Psi_4 - \delta^*\Phi_{22} = 0$$

$$\delta f^\Psi = 0$$
$$\delta f^\phi = 2f^\phi f^{\alpha*}\rho^*$$

Except for ν, we have expressed all spin coefficients in terms of ρ. From $D\nu = \Psi_3$ and $-\delta^*\Psi_2 + D\Psi_3 - 2\rho\Psi_3 = 0$, we see that if $\delta^*\Psi_2 = 0$, we can express Ψ_3 and ν in terms of ρ.
Alternatively, and less restrictive, we can use the commutation relations, which now read:

$$\begin{aligned}
&\Delta D - D\Delta = (\gamma + \gamma^*)D \\
&\delta D - D\delta = -\rho^*\delta \\
&\delta\Delta - \Delta\delta = -\nu^* D + (\mu - \gamma + \gamma^*)\delta \\
&\delta^*\delta - \delta\delta^* = (\mu^* - \mu)D + (\rho^* - \rho)\Delta + 2\alpha\delta - 2\alpha^*\delta^*,
\end{aligned}$$

the second of which gives $(\delta^* D - D\delta^*)\rho = -\rho\delta^*\rho$, or $D(\delta^*\rho) = 3\rho(\delta^*\rho)$. We therefore put

$$\delta^*\rho = f^\rho\rho^3, \quad Df^\rho = 0.$$

Also, $(\delta^* D - D\delta^*)f^\Psi = -\rho\delta^* f^\Psi$, thus $\delta^* f^\Psi = f^{ps}\rho, \quad Df^{ps} = 0$.

The same commutation relation also gives rise to $D(\delta^*\Psi_2) = 4\rho(\delta^*\Psi_2) + 3f^\Psi f^\rho\rho^6$, which suggests $\delta^*\Psi_2 = f_4\rho^4 + f_5\rho^5$. Substitution of this expression leads to $\delta^*\Psi_2 = f^p\rho^4 + 3f^\rho f^\Psi\rho^6$ and $Df^p = 0$. The equation that contains $D\Psi_3$ now becomes:

$$D\Psi_3 = 2\rho\Psi_3 + f^p\rho^4 + 3f^\rho f^\Psi\rho^6.$$

This again leads to a suggestion for Ψ_3 that, when substituted, results in:

$$\Psi_3 = f^p\rho^3 + f^\rho f^\Psi\rho^5.$$

Alternatively, use $\Delta\alpha - \delta^*\gamma = \nu\rho + \alpha(\gamma^* - \mu^*) - \gamma\alpha - \Psi_3$ and $\delta\gamma + \Delta\alpha^* = \alpha^*(\gamma - \gamma^* - \mu)$ to get $\Psi_3 = \nu\rho + \delta^*(\gamma^* + \gamma)$. Substitute γ in terms of ρ and use $\delta D - D\delta = -\rho^*\delta$ several times in order to get:

$$\Psi_3 = (\nu + f^g)\rho + \tfrac{1}{2}f^{ps}\rho^3 + f^\rho f^\Psi\rho^4, \quad Df^g = 0.$$

Comparing these results gives an expression for ν in terms of ρ:

$$\nu = -f^g + (f^p - \tfrac{1}{2}f^{ps})\rho^2 - f^\rho f^\Psi(\rho^3 - \rho^4).$$

$D\nu = \Psi_3$ can now be used to arrive at $f^p = f^{ps}$ and $f^\rho = 0$ or $f^\Psi = 0$.

Time to pause and recapitulate what the result is up till now.

$\alpha = f^{\alpha}\rho = -\beta^*$
$\gamma = \frac{1}{2} f^{\Psi}\rho^2 + f^{\gamma}$
$\mu = f^{\mu}\rho^* + f^{\Psi}\frac{\rho^3}{2(\rho-\rho^*)}$
$\nu = -f^g + \frac{1}{2} f^p \rho^2$

$\phi_2 = f^{\phi}\rho$
$\Phi_{22} = 2f^{\phi}(f^{\phi})^*\rho\rho^*$

$\Psi_2 = f^{\Psi}\rho^3$
$\Psi_3 = f^p\rho^3$
$\Psi_4 = -(f^h + 2f^g f^{\alpha})\rho + (f^p f^{\alpha} + \frac{1}{2} f^q)\rho^3 + f^p f^{\rho}\rho^4$

$D\rho = \rho^2$
$\delta\rho = 0$
$\delta^*\rho = f^{\rho}\rho^3$
$\Delta\rho = (f^{\gamma} + f^{\gamma *})\rho - f^{\mu *}\rho^2 - \frac{1}{2} f^{\Psi}\rho^3 + \frac{1}{2} f^{\Psi *}\rho(\rho^*)^2 + f^{\Psi *}\frac{\rho(\rho^*)^3}{2(\rho-\rho^*)}$

$\delta f^{\Psi} = 0$
$\delta f^{\phi} = 2f^{\phi} f^{\alpha *}\rho^*$
in general $\delta f^a = f^b\rho^*$

$\delta^* f^{\Psi} = f^p\rho$
in general $\delta^* f^a = f^b\rho: \quad \delta^* f^g = f^h\rho, \quad \delta^* f^p = f^q\rho, \quad \delta^* f^{\mu} = f^m\rho$

$\delta^*\Psi_2 = f^p\rho^4$

$Df^{\cdots} = 0$
$f^{\rho} = 0$ or $f^{\Psi} = 0$

$\delta\alpha + \delta^*\alpha^* = \mu\rho + 4\alpha\alpha^* + \gamma(\rho - \rho^*) - \Psi_2$
$-\delta^*\mu = \nu(\rho - \rho^*) - \Psi_3$
$\delta\nu - \Delta\mu = \mu^2 + \mu(\gamma + \gamma^*) + 2\nu\alpha^* + \Phi_{22}$
$\delta\gamma + \Delta\alpha^* = \alpha^*(\gamma - \gamma^* - \mu)$
$\Delta\alpha - \delta^*\gamma = \nu\rho + \alpha(\gamma^* - \mu^*) - \gamma\alpha - \Psi_3$

$\delta^*\Psi_3 - D\Psi_4 + 2\alpha\Psi_3 + \rho\Psi_4 = 0$
$-\Delta\Psi_2 + \delta\Psi_3 - 3\mu\Psi_2 - 2\alpha^*\Psi_3 - \rho\Phi_{22} = 0$
$-\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - 4\alpha^*\Psi_4 - \delta^*\Phi_{22} = 0$

The equation $-\delta^*\mu = \nu(\rho - \rho^*) - \Psi_3$ can be used to show that $\frac{1}{2}\frac{1}{\rho\rho^*} f^p[\rho^3 - \frac{\rho^4}{\rho-\rho^*} + \rho^*\rho^2] = f^g\frac{(\rho^*-\rho)}{\rho\rho^*} + f^m$. Since $D(f^g\frac{(\rho^*-\rho)}{\rho\rho^*} + f^m) = 0$ we should also have $D(\frac{1}{\rho\rho^*}[\rho^3 - \frac{\rho^4}{\rho-\rho^*} + \rho^*\rho^2]) = 0$. It is easy to show that this implies:

$\rho = \rho^*$.

Although we did not expect this to happen, in fact we only used $\rho \neq 0$. Again we get a pleasant reduction in the complexity of the remaining equations. The reality of ρ implies $f^\rho = 0$, since $(\delta^*\rho)^* = \delta\rho^*$. It also implies $f^\Psi = 0$, as can be seen from the derivation of the expression for μ in terms of ρ. This in turn implies $f^p = 0$, and therefore $f^q = 0$ also. Now, the only nonzero Ψ is Ψ_4. The equation $-\Delta\Psi_2 + \delta\Psi_3 - 3\mu\Psi_2 - 2\alpha^*\Psi_3 - \rho\Phi_{22} = 0$ however tells us that we ran into a contradiction, since $\Phi_{22} \neq 0$.

10.6.2 second attempt

In the previous attempt we imposed $\beta = -\alpha^*$, that was also valid in the flat space-time in case we use the tetrad (9.1). However, in section **9.2.1** we did not fix $\mathbf{n}$ in such a way as to obtain $\tau = 0$, whereas in **10.6.1** we did have $\tau = 0$. This does not mean that the attempt in the previous section *must* fail, but we might have better prospects when we use relations valid in the flat space-time case related to a tetrad that is more akin to the one in the curved space-time. Therefore, here we present a summary of an attempt to solve the equations, in which we assume a relation that is valid in the flat space-time case when we use the tetrad (9.3): $\beta = -\alpha^* + a\rho^*$. This exercise has not been brought to completion, but is presented here to show to what kind of task it leads.

Like before, we start with $\phi_0 = 0 = \phi_1$, $\kappa = 0 = \sigma$ and $\rho \neq 0$. Again, this implies $\Psi_0 = 0 = \Psi_1$. We now choose a such that $\tau + a\rho = 0$, and apply a tetrad type 1 transformation with this a. This gives $\tau = 0$, which in combination with (10.3) and (10.16) leads to $\pi = 0$ and $\lambda = 0$ in addition. All this leads to equations that still appear difficult. We therefore use the freedom of a type 3 transformation to get $\epsilon = 0$ like before. (We could go a different direction here, since this is just a choice, motivated only by the reduction of the complexity. Other choices also reduce the complexity. It is difficult to come up with a physically preferred choice. At least, present choice does admit an interpretation: the use of an affine parameter.) It is clear that we now have exactly the same set of equations as the first set in the previous subsection. But now, as a second choice, we make a different one from before: $\beta = -\alpha^* + a\rho^*$, and combine this with the *assumption* that ρ is real. We should therefore refrain from using the previous expression for μ in terms of ρ, and have instead $\mu = f^\mu\rho + f^\Psi\rho^2$. It is easy to find $Da = 0$. From the equations for $\delta\rho, \delta\Psi_2$ and $\delta\phi_2$ we deduce $\delta f^\Psi = -3af^\Psi\rho$ and $\delta f^\phi = -f^\phi(3f^\beta + (f^\alpha)^*)\rho$. Applying the commutation relation for δ^* and D (now different from before) to Ψ_2 leads to

$\delta^*\Psi_2 = f^p\rho^4$, which in turn leads to $\Psi_3 = f^\rho\rho^2 + f^p\rho^3$. $D\nu = \Psi_3$ then gives $\nu = f^\rho\rho + \frac{1}{2}f^p\rho^2 + f^\nu$. Continuing in this spirit, it is perhaps not too difficult to find few more expressions for the tetrad representatives in terms of ρ and functions $f^{\cdot\cdot}$ that satisfy $Df^{\cdot\cdot} = 0 = D\Delta f^{\cdot\cdot}, \delta f^{\cdot\cdot} = f^{'\cdot\cdot}\rho$ and $\delta^* f^{\cdot\cdot} = f^{''\cdot\cdot}\rho$. We end up with only a few of the original equations that are not identically satisfied, and many unknown functions $f^{\cdot\cdot}$ that all satisfy the same relatively simple equations. What is left from the original set can be used to relate some of the $f^{\cdot\cdot}$'s. Additional assumptions lead to simpler equations, but could also lead to contradictions.

10.7 final remarks

The contradiction in the first attempt is the result of the collection of assumptions made, at least one of which should not have been made. It seems reasonable to assume that it is possible, changing assumptions every time when needed, to bring this exercise to a satisfactory end. In practise however this turns out to be very laborious. The Geroch-Held-Penrose [93] formalism is an alternative for the Newman-Penrose formalism that might be better suited to our problem.

Suppose we succeeded in our task and would have an expression for all tetrad components of the Weyl tensor in terms of t, x, y, z. We would then have a formidable new task of showing that this solution really is a new one or that it is a known one in disguise. Diffeomorphism invariance in general relativity leads to an egalitarian principle among all coordinate systems. The same solution expressed in weird but equally valid coordinates could look very different from the solution in the usual coordinates. The problem of proving or disproving that two solutions are the same, expressed in different coordinate systems, is called the equivalence problem and is notorious. If it would lead to a solution already known, the present context could possibly clarify questions about its source.

Bibliography

[1] W. T. Irvine and D. Bouwmeester, *Linked and knotted beams of light*, Nature Physics **4**, 716 (2008).

[2] A. F. Rañada, *A topological theory of the electromagnetic field*, letters in mathematical physics **18**, 97 (1989).

[3] R. Penrose and W. Rindler, *Spinors and Space-time: Spinor and twistor methods in space-time geometry. volume 2*, Cambridge University Press, 1986 .

[4] J. Dalhuisen and D. Bouwmeester, *Twistors and electromagnetic knots*, Journal of Physics A: Mathematical and Theoretical **45**, 135201 (2012).

[5] W. T. Irvine, *Linked and knotted beams of light, conservation of helicity and the flow of null electromagnetic fields*, Journal of Physics A: Mathematical and Theoretical **43**, 385203 (2010).

[6] W. Irvine, *Single photons in nonlinear photonic crystals and studies on electromagnetic knots*, thesis University Santa Barbara (2006).

[7] I. Bialynicki-Birula, *Electromagnetic vortex lines riding atop null solutions of the Maxwell equations*, Journal of Optics A: Pure and Applied Optics **6**, S181 (2004).

[8] I.Bialinicky-Birula, *private communication*, Warsaw (2010).

[9] E. T. Newman, *Maxwell's equations and complex Minkowski space*, Journal of Mathematical Physics **14**, 102 (1973).

[10] J. Swearngin, A. Thompson, A. Wickes, J. W. Dalhuisen, and D. Bouwmeester, *Linked Gravitational Radiation*, arXiv preprint arXiv:1302.1431 (2013).

[11] J. Swearngin, *Linked Gravitational Radiation*, unpublished thesis University of Leiden (2012).

[12] S. Chandrasekhar, *The mathematical theory of black holes*, Oxford University Press, 1998 .

[13] G. Debney, R. Kerr, and A. Schild, *Solutions of the Einstein and Einstein-Maxwell Equations*, Journal of Mathematical Physics **10**, 1842 (1969).

[14] M. Carmeli, *Group theory and general relativity: representations of the Lorentz group and their applications to the gravitational field*, Imperial College Press, 2000 .

[15] R. Penrose, *Twistor algebra*, Journal of Mathematical physics **8**, 345 (1967).

[16] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, Communications in Mathematical Physics **252**, 189 (2004).

[17] T. Adamo, M. Bullimore, L. Mason, and D. Skinner, *Scattering amplitudes and Wilson loops in twistor space*, Journal of Physics A: Mathematical and Theoretical **44**, 454008 (2011).

[18] A. Zee, *Einstein Gravity in a Nutshell*, Princeton University Press, 2013 .

[19] H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Mathematische Annalen **104**, 637 (1931).

[20] P. J. Hilton, *An introduction to homotopy theory*, Cambridge University Press, 1953 .

[21] H. Urbantke, *The Hopf fibration - seven times in physics*, Journal of Geometry and Physics **46**, 125 (2003).

[22] A. Hirsch, *Extension of the 'Villarceau-Section' to Surfaces of Revolution with a Generating Conic*, Journal for Geometry and Graphics **6**, 121 (2002).

[23] H. Stachel, *Remarks on A. Hirsch's Paper concerning Villarceau Sections*, J. Geometry Graphics **6**, 133 (2002).

[24] K.-J. Kim and M.-S. Kim, in *Proc. of Israel-Korea Bi-National Conference on New Themes in Computerized Geometrical Modeling, Tel-Aviv, Israel*, Citeseer (Postech, Israel, 1998), pp. 11–20.

[25] D. Rolfsen, *Knots and links*, AMS Chelsea Pub., 2003 .

[26] W. Fulton, *Algebraic topology: a first course*, Springer, 1995 .

[27] S. A. Huggett and K. P. Tod, *An introduction to twistor theory*, Cambridge University Press, 1994 .

[28] I. Robinson, *Null electromagnetic fields*, Journal of Mathematical Physics **2**, 290 (1961).

[29] J. D. Jackson, *Classical Electrodynamics*, 3rd Edition 1999. Wiley-VCH .

[30] E. T. Newman, *Heaven and its properties*, General Relativity and Gravitation **7**, 107 (1976).

[31] E. Newman, *Complex coordinate transformations and the Schwarzschild-Kerr metrics*, Journal of Mathematical Physics **14**, 774 (1973).

[32] E. T. Newman and J. Winicour, *A curiosity concerning angular momentum*, Journal of Mathematical Physics **15**, 1113 (1974).

[33] E. T. Newman, *Classical, geometric origin of magnetic moments, spin-angular momentum, and the Dirac gyromagnetic ratio*, Physical Review D **65**, 104005 (2002).

[34] R. W. Lind and E. T. Newman, *Complexification of the algebraically special gravitational fields*, Journal of Mathematical Physics **15**, 1103 (1974).

[35] A. Burinskii, *The Dirac-Kerr-Newman electron*, Gravitation and Cosmology **14**, 109 (2008).

[36] A. Burinskii and G. Magli, *Kerr-Schild approach to the boosted Kerr solution*, Physical Review D **61**, 044017 (2000).

[37] B. Trishin, *unknown title, only available in Russian*, 2012 .

[38] H. Kedia, I. Bialynicki-Birula, D. Peralta-Salas, and W. T. Irvine, *Tying Knots in Light Fields*, Physical Review Letters, 111, 150404, 2013 .

[39] H. Bateman, *The conformal transformations of a space of four dimensions and their applications to geometrical optics*, Proceedings of the London Mathematical Society **2**, 70 (1909).

[40] H. Bateman, *The transformation of the electrodynamical equations*, Proceedings of the London Mathematical Society **2**, 223 (1910).

[41] H. Bateman, *The transformations of coordinates which can be used to transform one physical problem into another*, Proceedings of the London Mathematical Society **2**, 469 (1910).

[42] E. Cunningham, *The principle of relativity in electrodynamics and an extension thereof*, Proceedings of the London Mathematical Society **2**, 77 (1910).

[43] H. A. Kastrup, *On the advancements of conformal transformations and their associated symmetries in geometry and theoretical physics*, Annalen der Physik **17**, 631 (2008).

[44] V. I. Fouchtchitch and A. G. Nikitin, *Symmetries of Maxwell's equations*, Springer, 1987 .

[45] J. V. Narlikar, *Introduction to cosmology*, Cambridge University Press, 2nd edition, 1998 .

[46] S. Lyle, *Self-Force and Inertia: Old Light on New Ideas*, Springer, Berlin, 2010, pp. 65-71 .

[47] S. M. Carroll, *Spacetime and geometry. An introduction to general relativity*, Pearson, Addison Wesley, 2004 .

[48] M. Tsamparlis, *Special relativity*, Springer, 2009 .

[49] E. Gourgoulhon, *Special Relativity in General Frames*, Springer, 2013 .

[50] A. Barut, *Electrodynamics and classical theory of fields & particles*, Dover Publications (New York), 1980 .

[51] H. Stephani, *Exact Solutions of Eisntein's Field Equations*, Cambridge University Press, 2003, pp. 155-156 .

[52] N. Van den Bergh, *Conformally Ricci flat Einstein-Maxwell solutions with a null electromagnetic field*, General relativity and gravitation **18**, 1105 (1986).

[53] H. Bondi and T. Gold, *The field of a uniformly accelerated charge, with special reference to the problem of gravitational acceleration*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **229**, 416 (1955).

[54] T. Fulton and F. Rohrlich, *Classical radiation from a uniformly accelerated charge*, Annals of Physics **9**, 499 (1960).

[55] B. S. DeWitt and R. W. Brehme, *Radiation damping in a gravitational field*, Annals of Physics **9**, 220 (1960).

[56] D. G. Boulware, *Radiation from a uniformly accelerated charge*, Annals of Physics **124**, 169 (1980).

[57] S. Parrott, *Radiation from a uniformly accelerated charge and the equivalence principle*, Foundations of Physics **32**, 407 (2002).

[58] S. Lyle, *Uniformly accelerating charged particles: a threat to equivalence principle*, Springer, 2008 .

[59] S. Parrott, *Relativistic electrodynamics and differential geometry.*, Springer-Verlag, New York, 1987 .

[60] L. P. Hughston and L. J. Mason, *A generalised Kerr-Robinson theorem*, Classical and Quantum Gravity **5**, 275 (1988).

[61] D. Huybrechts, *Complex geometry-an introduction*, Springer, 2005 .

[62] M. Carmeli and S. Malin, *Theory of spinors: an introduction*, World Scientific, 2000 .

[63] R. Penrose and W. Rindler, *Spinors and Space-time: Spinor and twistor methods in space-time geometry. volume 1*, Cambridge University Press, 1984 .

[64] R. Maartens and B. A. Bassett, *Gravito-electromagnetism*, Classical and Quantum Gravity **15**, 705 (1998).

[65] H. Stephani, *Exact Solutions of Eisntein's Field Equations*, Cambridge University Press, 2003 .

[66] D. A. Nichols, R. Owen, F. Zhang, A. Zimmerman, J. Brink, Y. Chen, J. D. Kaplan, G. Lovelace, K. D. Matthews, M. A. Scheel, *et al.*, *Visualizing spacetime curvature via frame-drag vortexes and tidal tendexes: General theory and weak-gravity applications*, Physical Review D **84**, 124014 (2011).

[67] W. Rindler, *The case against space dragging*, Physics Letters A **233**, 25 (1997).

[68] W. Rindler, *Relativity: special, general, and cosmological*, Oxford University Press, 2nd edition, 2006 .

[69] F. De Felice and D. Bini, *Classical Measurements in Curved Space-Times*, Cambridge University Press, 2010 .

[70] W. Rindler, *Relativity: special, general, and cosmological*, Oxford University Press, 2nd edition, 2006, pp. 335-341 .

[71] L. F. O. Costa and C. A. Herdeiro, *Gravitoelectromagnetic analogy based on tidal tensors*, Physical Review D **78**, 024021 (2008).

[72] L. F. Costa and C. A. Herdeiro, *A gravito-electromagnetic analogy based on tidal tensors*, arXiv preprint gr-qc/0612140 (2006).

[73] D. L. Wiltshire, M. Visser, and S. Scott, *The Kerr spacetime: Rotating black holes in general relativity*, Cambridge University Press Cambridge, 2009, p. 6 .

[74] D. L. Wiltshire, M. Visser, and S. Scott, *The Kerr spacetime: Rotating black holes in general relativity*, Cambridge University Press Cambridge, 2009, p. 95 and p. 116 .

[75] A. Burinskii, *Regular superconducting source of the Kerr-Newman solution*, arXiv preprint arXiv:1003.2928 (2010).

[76] Kerr, *talk*, Marcell Grossmann meeting, Stockholm (2012).

[77] B. O'Neill, *The geometry of Kerr black holes*, Wellesley, Mass. AK Peters, 1995 .

[78] E. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, Journal of Mathematical Physics **3**, 566 (1962).

[79] J. Goldberg and R. Sachs, *Republication of: A theorem on Petrov types*, General Relativity and Gravitation **41**, 433 (2009).

[80] R. P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, Phys. Rev. Letters **11**, (1963).

[81] A. Burinskii, E. Elizalde, S. R. Hildebrandt, and G. Magli, *Regular sources of the Kerr-Schild class for rotating and nonrotating black hole solutions*, Physical Review D **65**, 064039 (2002).

[82] A. Burinskii, *Kerr-Schild photonlike metric solutions*, arXiv preprint arXiv:0704.3184 (2007).

[83] H. Stephani, *Exact Solutions of Eisntein's Field Equations*, Cambridge University Press, 2003, pp. 493-494 .

[84] H. Stephani, *Exact Solutions of Eisntein's Field Equations*, Cambridge University Press, 2003, pp. 497-498 .

[85] H. Stephani, *Exact Solutions of Eisntein's Field Equations*, Cambridge University Press, 2003, pp. 492-493 .

[86] H. Stephani, *Relativity: An introduction to special and general relativity*, Cambridge university press, 2004, pp. 294-295 .

[87] W. Bonnor and P. Vaidya, *General Relativity, papers in honor of JL Synge, Edited by L*, O'Raifeartaigh (Dublin Institute for Advanced Studies) p **119**, (1972).

[88] H. Stephani, *Exact Solutions of Eisntein's Field Equations*, Cambridge University Press, 2003, pp. 487-489 .

[89] A. Peres, *Null electromagnetic fields in general relativity theory*, Physical Review **118**, 1105 (1960).

[90] W. Bonnor, *The gravitational field of light*, Communications in Mathematical Physics **13**, 163 (1969).

[91] S. Lyle, *Uniformly accelerating charged particles: a threat to equivalence principle*, Springer, 2008, p. 256 .

[92] J. B. Griffiths and J. Podolskỳ, *Exact space-times in Einstein's general relativity*, Cambridge University Press, 2009 .

[93] R. Geroch, A. Held, and R. Penrose, *A space-time calculus based on pairs of null directions*, Journal of Mathematical Physics **14**, 874 (1973).

Samenvatting

Dit proefschrift, "De Robinson Congruentie in Elektrodynamica en in de Algemene Relativiteitstheorie", heeft als uitgangspunt een recent gevonden oplossing van de Maxwell vergelijkingen, een zogenaamde elektromagnetische Hopf knoop. Deze kenmerkt zich door een bijzondere structuur van de elektrische en magnetische veldlijnen: elke elektrische (magnetische) veldlijn is gesloten en twee willekeurige elektrische (magnetische) veldlijnen zijn aan elkaar gekoppeld zoals twee naburige olympische ringen. Deze twee eigenschappen gelden voor alle tijdstippen. Er is een uniek moment waarop elke veldlijn een cirkel is. De ruimtevullende verzameling van alle elektrische veldlijnen tezamen vormt op dit unieke tijdstip een Hopf fibratie. De magnetische veldlijnen vormen op dit tijdstip ook een Hopf fibratie, negentig graden gedraaid ten opzichte van de elektrische Hopf fibratie.
In hoofdstuk twee wordt de Hopf fibratie elementair en uitvoerig beschreven.

Deze structuur speelt ook een rol in twistortheorie: een nultwistor wordt voorgesteld als een Hopf fibratie die met de lichtsnelheid langs een centrale as beweegt. Na wiskundige voorbereidingen in de hoofdstukken één en drie, laat hoofdstuk vier zien dat de overeenkomst tussen een elektromagnetische Hopf knoop en een nultwistor geen toeval is. Het Poynting vectorveld van een elektromagnetische knoop heeft op alle tijdstippen de structuur van een Hopf fibratie en deze beweegt met de lichtsnelheid langs de centrale as, net als het beeld van een nultwistor. Een stelling van Robinson wordt gebruikt voor de wiskundige relatie tussen beide.

De bekende Lorentz transformaties maken deel uit van de groep van conforme transformaties. Een conforme inversie is een ander voorbeeld hiervan. De vergelijkingen van Maxwell in vacuum zijn invariant onder alle conforme transformaties. Hoofdstuk vijf ontwikkelt een methode om middels conforme inversie van een oplossing van Maxwell vergelijkingen een andere oplossing te verkrijgen. Deze methode wordt toegepast op een elektromagnetische Hopf knoop met als verrassend resultaat een andere, maar eenvoudig gerelateerde, Hopf knoop.

De tot de driedimensionale Euclidische ruimte behorende raakvectoren van de Hopf fibratie kunnen met een tijdcomponent worden aangevuld tot een nulvectorveld in Minkowski ruimte, een vlakke ruimtetijd. Dit leidt tot een zogenaamde Robinson congruentie, welke in een vlakke ruimtetijd onlosmakelijk met de Hopf fibratie is verbonden. In een gekromde ruimtetijd is dit verband er niet meer. Omdat in het lineaire regime van Einsteins veldvergelijkingen een oplossing gedacht kan worden als een veld in de Minkowski ruimte, heeft het zin om, in dit regime, te zoeken naar oplossingen met een Hopf structuur. Hoofdstuk zes laat zien dat met behulp van de Penrose transformatie uit de twistortheorie een elektromagnetische Hopf knoop 'gegeneraliseerd' kan worden tot een 'gravitationele Hopf knoop': een oplossing van de gelineariseerde Einstein vergelijkingen waarin vijf Hopf fibraties een rol spelen.

Hoofdstuk acht laat zien dat de Robinson congruentie ook een rol speelt bij oplossingen van de volledige (niet-lineaire) Einstein vergelijkingen. Hoewel de relatie met een Hopf fibratie er nu niet is, kan in de hier beschouwde specifieke klasse van gekromde ruimtetijd toch *in zekere zin* gesproken worden van een Hopf fibratie.

Omdat met elektromagnetische velden een energie-impulsdichtheid geassocieerd wordt, moet bij het oplossen van Einsteins vergelijkingen ook dit veld in de bronterm worden verwerkt. Hoofdstuk tien beschrijft een poging om de gekromde ruimtetijd te vinden met een elektromagnetische Hopf knoop als enige bron. Een bij dit vraagstuk behorende moeilijkheid betreft de definitie van de bron in de context van een gekromde ruimtetijd: om de bron te beschrijven moeten we de ruimtetijd kennen en om de ruimtetijd te kennen moeten we Einsteins vergelijkingen oplossen met bronterm. Om dit probleem op te lossen kiezen we voor enige vrijheid in de definitie van onze bron.
Het formalisme dat gebruikt wordt is het in hoofdstuk zeven geintroduceerde Newman-Penrose formalisme. In hoofdstuk negen wordt de elektromagnetische Hopf knoop in dit formalisme *in een vlakke ruimtetijd* behandeld. Dit geeft mogelijke aanknopingspunten voor de definitie van de bron in de context van een gekromde ruimtetijd.

Curriculum Vitae

Jan Willem Dalhuisen

5 July 1966	Born in Elst (Gld), The Netherlands
1986	Astronomy, Leiden University freshman year completed with distinction
1987-1992	B.Sc. and M.Sc. Physics, Leiden University completed cum laude Thesis: Equations of Motion in General Relativity Advisor: prof. dr. R.H. Terwiel
2009-2013	Ph.D. research, Leiden University Supervisor: prof. dr. D. Bouwmeester