Cosmological models with variable parameters

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Abstract. The present study deals with the investigation of the Friedmann-Lemaitre-Robertson-Walker models (often FLRW-models) with time varying G and Λ in the framework of General theory of Relativity. The Einstein field equations have been solved by considering the time-varying deceleration parameter $q(t)$ and Scale factor $a(t) = e^{\beta t} + (\sinh \beta t)^{\frac{1}{m}}$, where m and β are arbitrary constants. We have analysed the value of m, which will generate a transition for universe from early decelerating phase to recent acceleration phase. The physical and graphic behaviour have also been planned to study in this communication.

Keywords: Cosmological models, deceleration parameter, scale factor, Einstein field equations.

1. Introduction

The value of cosmological parameters has always been a central query in the area of cosmology. However, in recent years, this subject has become more important, and the natural attention of the researcher has increased day by day. As widely accepted, a cosmological model is a mathematical explanation of the entire cosmos. The study of various categories of cosmological models tries to explain the reasons for their current aspects along with the nature of evolution over time.

Homogeneity and Isotropy are the structural consequences of the cosmological principle that may be evaluated the most. Both homogeneity and isotropy imply that the cosmos appears the same at every point in space and that the universe appears the same in all directions, respectively. One of the most recent and notable advancement in contemporary cosmology is the finding of the cosmic acceleration. Understanding the physical cause of cosmic acceleration is still a challenging endeavour.

Our universe is expanding, as we all know, and according to Einstein's General Theory of Relativity (GTR), if the cosmos's primary effects are matter and radiation, the universe's expansion will slow down due to the presence of gravity. Here, two options present themselves, either of which could have significant repercussions on how we truly comprehend the universe and the fundamental rules of physics. The first is that around 75% of the universe's energy density exists as a smooth component with a significant negative pressure, and according to GTR, a fluid with static may accelerate the expansion. The second idea is that GTR collapses at cosmological scales, and cosmic acceleration results from fundamental gravitational physics, perhaps by assuming extra dimensions. These days, dark energy is the term used to describe the fictional fluids with static density mentioned above under the first possibility. It has been noted during the literature review that there are numerous hypotheses for dark energy, but the most elementary and most likely candidate is the positive cosmological constant Λ [1,2].

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According to research findings, the universe is currently in the "dark energy paramount era," in which an as-yet-unidentified dark energy has been accountable for the observable acceleration of the cosmos' expansion and prevails over gravitation. Initially, it was believed that the model of the universe was unchanging and remained in its original state. Thus, to justify this static nature of the universe and to provide a mathematical framework, Einstein 1917 [3] added the cosmological term $\Lambda g_{\mu\nu}$ to his field equations. But later on, in 1929, Edwin Hubble provided the first evidence for the universe having finite age [4]. He found that the galaxies are relocating away from us. This implies that the universe is naturally expanding equally in all directions. More so, many cosmologists now think that the cosmological constant might be positive given the knowledge that the universe's expansion is speeding [5,6]. Thus, it can be certified that Λ is not a consistent quantity but a varying quantity that varies with respect to time. The same can be said for the gravitational constant G [7-9].Thus, to include G and Λ in the field equations, Berman has proposed his theory on the bases of conservation law by assuming $c = 1$ as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\Pi G T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{1}$$

And the conservation law is defined as:

$$T^{\mu\nu}_{;\mu} = 0, \tag{2}$$

Using (1) and (2), we will get the following equation

$$8\pi G_{,\mu} T^{\mu}_{\nu} + \Lambda_{,\mu} g^{\mu}_{\nu} = 0, \tag{3}$$

where G and Λ varies with time.

2. Equations governing cosmological models

As we know, the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, serves as a mathematical representation that characterizes the geometry of space-time based on the cosmological principle. This principle asserts that, at substantial cosmic distances, our Universe exhibits both homogeneity and isotropy. Therefore, when crafting the cosmological model, we define the FLRW metric expression as follows:

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right], \tag{4}$$

where K denotes the curvature. For various values of K, i.e., $K = -1, 0, 1$, we can define an open, flat and closed universe. Here, the coordinates (r, θ, ϕ) are spherical, t is the cosmic time and $a(t)$ denotes the scale factor. Furthermore, employing the convention $c = 1$, we can use this framework to describe the stress-energy tensor for a perfect fluid characterized by isotropic pressure p, the force exerted by dust particles on their surroundings, and the energy density, ρ which measures the quantity of energy enclosed within a particular region of space per unit volume.

$$T^{\mu\nu} = -p g^{\mu\nu} + (p + \rho) u^{\mu} u^{\nu}, \tag{5}$$

where u^{μ} is the four-velocity vector with $u^{\mu}u^{\nu} = 1$, when $\mu \neq \nu$ and is 0 when $\mu = \nu$. Since in comoving coordinate system, the particle is in rest, we have

$$u^{\mu} = (0,0,0,1). \tag{6}$$

Now by substituting the values from equations (4), (5) and (6), the field equation (1) takes the form

$$\frac{-2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} = 8\Pi Gp - \Lambda, \tag{7}$$

and

$$3\,\frac{\dot{a}^2}{a^2} + \frac{3K}{a^2} = 8\Pi G\rho + \Lambda. \tag{8}$$

On addition, we get

$$\frac{-2\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2K}{a^2} = 8\Pi G\,(p + \rho), \tag{9}$$

Differentiating equation (8) w.r.t 't' and on simplification by using equation (9), we have

$$\dot{G}\rho + G\dot{\rho} + 3\frac{\dot{a}}{a}G\,(p + \rho) + \frac{\dot{\Lambda}}{8\pi} = 0. \tag{10}$$

Multiplying equation (10) by a^3, we get

$$a^3\,(\dot{G}\rho + \frac{\dot{\Lambda}}{8\pi}) + G\,(\frac{d}{dt}(\rho a^3) + 3pa^2\dot{a}) = 0. \tag{11}$$

On simplification, we get

$$\frac{d}{dt}(\rho a^3) = -3pa^2\dot{a}, \tag{12}$$

$$8\pi\dot{G}\rho + \dot{\Lambda} = 0. \tag{13}$$

By using equations (9) and (10), one gets

$$3\frac{\ddot{a}}{a} = -4\pi G\,\left(3p + \rho - \frac{\Lambda}{4\pi G}\right). \tag{14}$$

Further for the construction of the homogeneous and isotropic model, we assume the dependence of ρ and p as

$$p = \gamma\rho, \tag{15}$$

where γ is not necessarily a constant. If γ becomes zero, then we will get the pressure less model with $p = 0$ and if γ acquires the non-negative value $\gamma = \frac{1}{3}$, then model becomes the radiation dominated model. As suggested by Mishra et al., [9], Berman [7] and Rahman [10], we consider the dependence of G and Λ as

$$G \propto \frac{\Lambda}{\rho}. \tag{16}$$

For any physically relevant model, there exist two important observational quantities defined by Hubble's parameter $H(t)$ and deceleration parameter $q(t)$. While the deceleration parameter $q(t)$ quantifies how the expansion rate varies over time, $H(t)$ indicates the time dependence of the universe's expansion, or how quickly the cosmos is expanding. For further investigation, we may define the Hubble and deceleration parameter as

$$H(t) = \frac{\dot{a}}{a}, \tag{17}$$

and

$$q(t) = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{18}$$

3. Variation of H, q, Λ, G, ρ

To get the solution of field equation, we have already taken the suitable assumption under the equations (15) and (16) as suggested by Mishra *et al.* [11]. Now we define the deceleration parameter's expression as

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\left(\frac{\dot{H}+H^2}{H^2}\right). \tag{19}$$

Given that the universe is currently expanding faster than previously thought, as seen in several studies of SNe Ia [12-16], and CMB anisotropies [17-19]. Thus, for further calculation and construction of the cosmological model, we have an additional requirement. So here we have taken scale factor $a(t)$ as

$$a(t) = e^{\beta t} + (\sinh \beta t)^{\frac{1}{m}}, \tag{20}$$

where β and m are arbitrary constants with $m > 1$. So, after simplifications we will obtain the following results

$$\dot{a}(t) = \beta e^{\beta t} + \frac{\beta}{m}(\sinh \beta t)^{\frac{1}{m}-1}(\cosh \beta t), \tag{21}$$

$$\ddot{a}(t) = \beta^2 e^{\beta t} + \frac{\beta^2}{m^2}(\sinh \beta t)^{\frac{1}{m}-2}\left((\cosh \beta t)^2 - m\right). \tag{22}$$

By using equations (12) and (15) along with scale factor $a(t) = e^{\beta t} + (\sinh \beta t)^{\frac{1}{m}}$, we will get following set of equations

$$\frac{\dot{\rho}}{\rho} = -3\,\frac{\beta e^{\beta t} + \frac{\beta}{m}(\sinh \beta t)^{\frac{1}{m}-1}\cosh \beta t}{e^{\beta t} + (\sinh \beta t)^{\frac{1}{m}}}(1+\gamma). \tag{23}$$

From equations (13), (16) and (23), we get

$$\frac{\dot{G}}{G} = 3\,\frac{\beta e^{\beta t} + \frac{\beta}{m}(\sinh \beta t)^{\frac{1}{m}-1}\cosh \beta t}{e^{\beta t} + (\sinh \beta t)^{\frac{1}{m}}}\left(\frac{1+\gamma}{1+\alpha'}\right), \tag{24}$$

and

$$\frac{\dot{\Lambda}}{\Lambda} = -3\alpha'\,\frac{\beta e^{\beta t} + \frac{\beta}{m}(\sinh \beta t)^{\frac{1}{m}-1}\cosh \beta t}{e^{\beta t} + (\sinh \beta t)^{\frac{1}{m}}}\left(\frac{1+\gamma}{1+\alpha'}\right). \tag{25}$$

Now after integrating equations (23), (24) and (25) we will get the values of G, ρ, Λ as a function of cosmic time t as follows

$$\rho = A\left[e^{\beta t} + (\sinh \beta t)^{\frac{1}{m}}\right]^{-3(1+\gamma)}, \tag{26}$$

$$G = B\left[e^{\beta t} + (\sinh\beta t)^{\frac{1}{m}}\right]^{\frac{3(1+\gamma)}{1+\alpha'}}, \tag{27}$$

and

$$\Lambda = C\left[e^{\beta t} + (\sinh\beta t)^{\frac{1}{m}}\right]^{\frac{-3\alpha'(1+\gamma)}{(1+\alpha')}}, \tag{28}$$

here A, B and C are arbitrary constants. Further after few more computational steps we will get Hubble parameter $H(t)$ and Deceleration parameter $q(t)$ with the help of equations (17), (18) and (20) as

$$H = \frac{\beta e^{\beta t} + \frac{\beta}{m}(\sinh\beta t)^{\frac{1}{m}-1}(\cosh\beta t)}{e^{\beta t} + (\sinh\beta t)^{\frac{1}{m}}}, \tag{29}$$

$$q = \frac{-\left[\beta^2 m^2 e^{\beta t}(\sinh\beta t)^2\left((\sinh\beta t)^{\frac{1}{m}} + e^{\beta t}\right) + \beta^2(\sinh\beta t)^{\frac{1}{m}}\left((\cosh\beta t)^2 - m\right)\left((\sinh\beta t)^{\frac{1}{m}} + e^{\beta t}\right)\right]}{\beta^2 m^2 e^{2\beta t}(\sinh\beta t)^2 + \beta^2(\sinh\beta t)^{\frac{2}{m}}(\cosh\beta t)^2 + 2\beta^2 m e^{\beta t}(\sinh\beta t)^{\frac{1}{m}+1}\cosh\beta t}. \tag{30}$$

If $\sinh\beta t = y$, the above equation can be rewritten as

$$q = \frac{\left[\beta^2 m^2 e^{\beta t} y^2\left((y)^{\frac{1}{m}} + e^{\beta t}\right) + \beta^2 (y)^{\frac{1}{m}}(1 + y^2 - m)\left((y)^{\frac{1}{m}} + e^{\beta t}\right)\right]}{\beta^2 m^2 e^{2\beta t} y^2 + \beta^2 (y)^{\frac{2}{m}}(1 + y^2) + 2\beta^2 m e^{\beta t}(y)^{\frac{1}{m}+1}\cosh\beta t}. \tag{31}$$

We propose that as cosmic time increases, the density ρ and gravitational constant G drops, i.e., $\dot{\rho} < 0$ and $\dot{G} < 0$ occurs during the advancement of the universe. Thus equations (26) and (27) implies that

$$1 + \gamma > 0 \text{ and } 1 + \alpha' < 0. \tag{32}$$

4. The physical and geometrical properties

Considering the equation of state, i.e., $p = \gamma\rho$, where $0 \leq \gamma \leq \frac{1}{3}$, we will now explore the model's physical and geometrical characteristics by referring to equations (28), (29) and (31). The universe model is categorised as a matter-dominated universe when $\gamma = 0$. The radiation-dominated universe when $\gamma = \frac{1}{3}$ is obtained in the meantime and Zel' dovich fluid or shift fluid universe model may be discussed at $\gamma = 1$ in further study. For better understanding of solution, we have taken the pictorial representation of parameters in Fig 1-6.

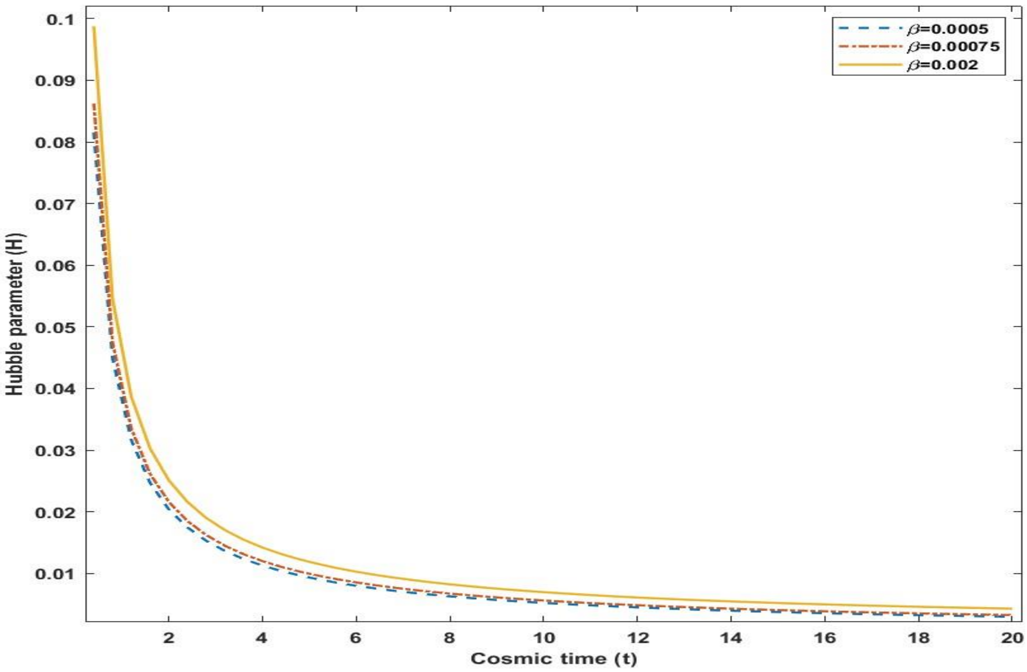

Fig.1 Plot of Hubble's parameter (H) in relation with cosmic time (t) with $m = 6$

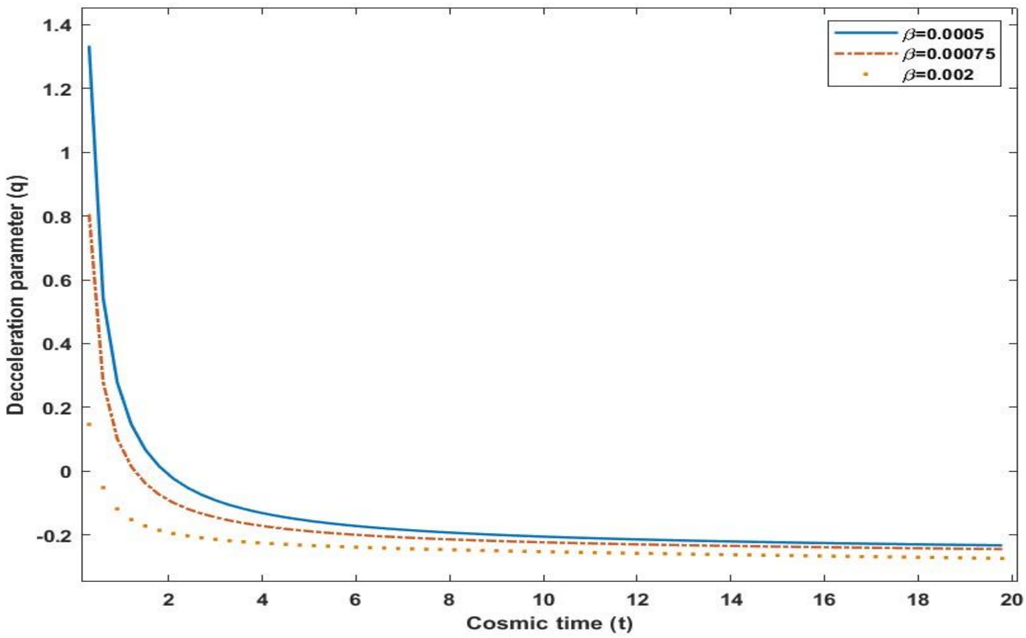

Fig.2 Plot of Deceleration parameter (q) in relation with cosmic time (t)

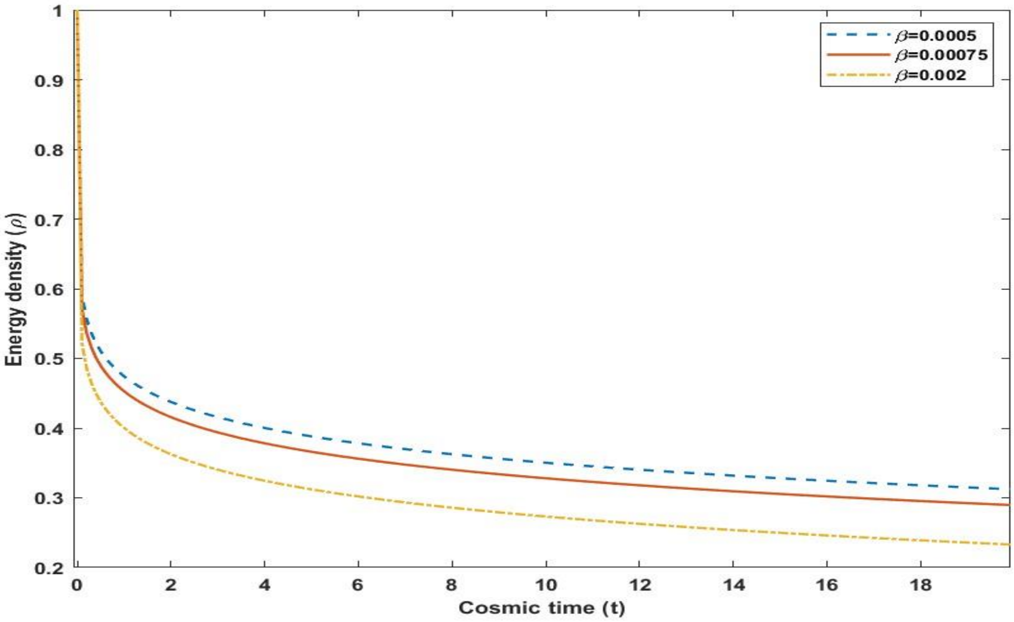

Fig. 3 Plot of Energy density in relation with cosmic time (t) by considering $m = 6, \;\; A = 1\; and\; \gamma = 0$

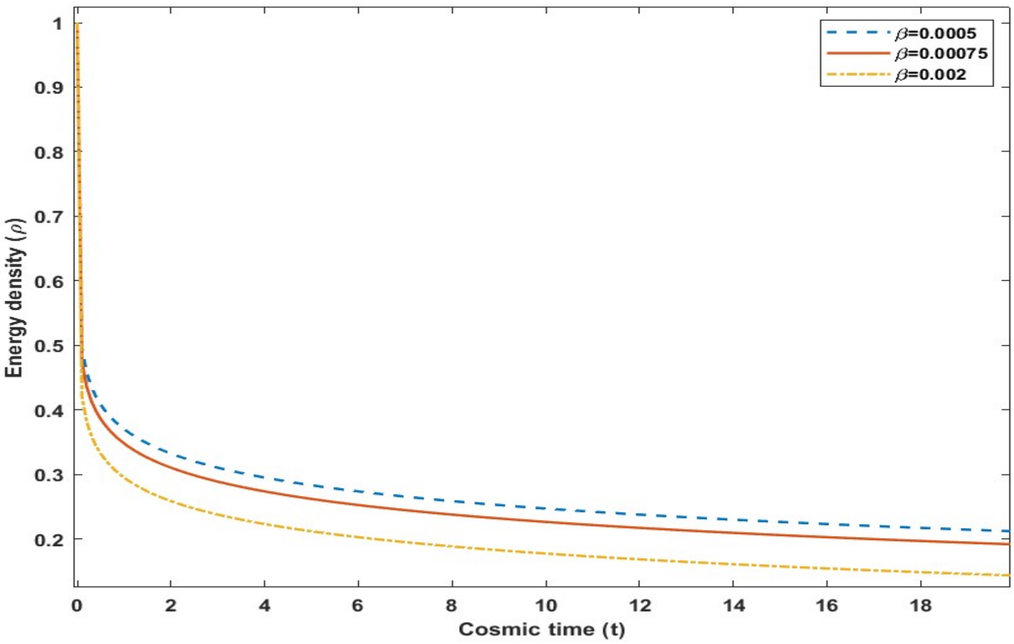

Fig. 4 Plot of Energy density in relation with cosmic time (t) by considering $m = 6, A = 1\; and\; \gamma = \frac{1}{3}$

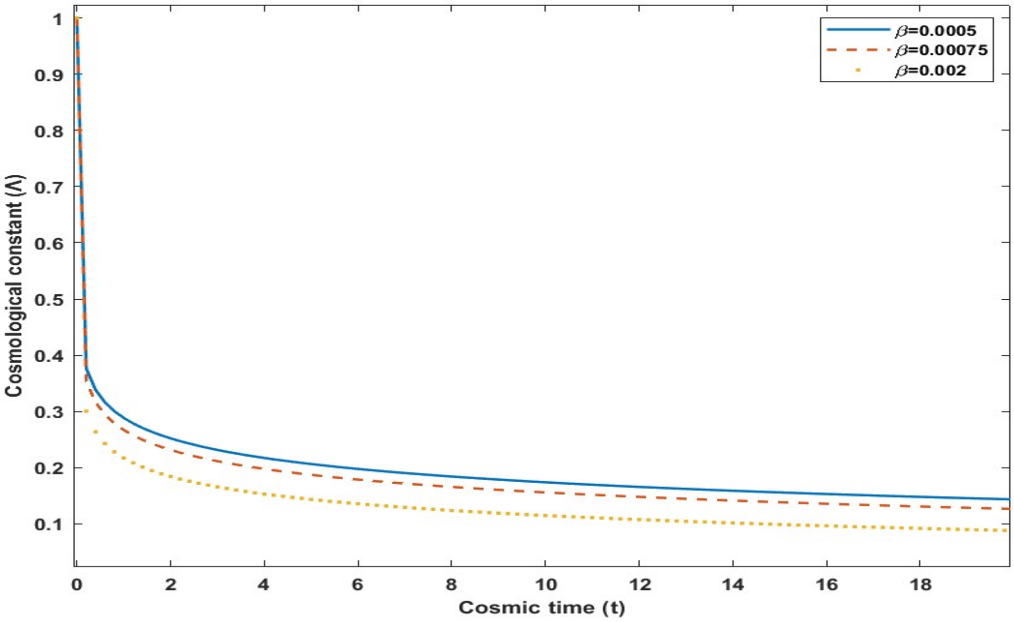

Fig. 5 Plot of Cosmological constant (Λ) in relation with Cosmic time (t) by considering $m = 6,\ \gamma = 0,\ C = 1$ and $\alpha' = -2.5$

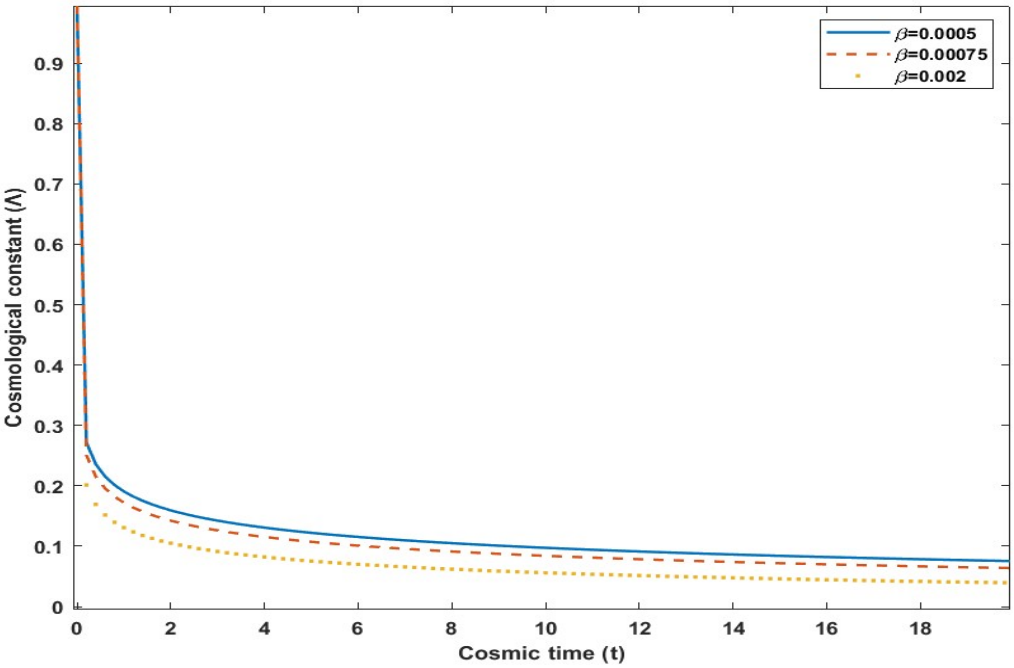

Fig. 6 Plot of Cosmological constant (Λ) in relation with cosmic time (t) by considering $m = 6,\ \gamma = \frac{1}{3},\ C = 1$ and $\alpha' = -2.5$

5. Results and discussions

From the pictorial representation of above cited figures, following conclusions are made:

- Fig. 1 predicts the variation of Hubble parameter (H) in relation with cosmic time t for numerous choices of β and m as cited in the figure. The graph of H shows the steep decline behaviour and approaches to negligible value after crossing early inflationary stage of universe.
- It is evident from Fig. 2 that as cosmic time increases, the graph of the deceleration parameter displays the universe's transitional phase, which is the period of early time deceleration to the present time speeding. One can also observe that for positive β with $m > 1$ and $0< t < 3.8$, q is always positive, i.e., $q > 0$ which means the universe's expansion pace is slowing down with time and for $t > 3.8$, q lies in some interval $(-1,0)$ which means the universe shows the exponential expansion where entropy of universe is directly proportional to the expansion of the universe.

  According to the discussion above, the cosmos was decelerating in the beginning and speeding at the moment.
- The alteration in energy density parameter (ρ) versus cosmic time (t) is shown in Fig. 3 and Fig. 4 for different choices of β and m as cited in the figure assuming two values of $\gamma = 0, \frac{1}{3}$. We noticed that $\rho > 0$ and $\dot{\rho} < 0$, i.e., the energy density decreases positively as time increases.
- In Fig 5 and Fig. 6, we have shown the change of cosmological constant (Λ) in relation with cosmic time (t) for the different choices of β and m as cited in the figure. Clearly in both the cases $\gamma = 0, \frac{1}{3}$, the graph comes out to be a positive function of time which drops as time increases and at present moment converges to the small value.

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