CARTAN CALCULUS ON QUANTUM LIE ALGEBRAS

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Abstract. A generalization of the differential geometry of forms and vector fields to the case of quantum Lie algebras is given. In an abstract formulation that incorporates many existing examples of differential geometry on quantum spaces we combine an exterior derivative, inner derivations, Lie derivatives, forms and functions all into one big algebra, the "Cartan Calculus".

Key words: Quantum Groups — Differential Geometry — Lie Algebras

1. Introduction

The central idea behind Connes' Universal Calculus (Connes, 1985) in the context of non-commutative geometry was to retain the classical differential geometric properties of $\mathbf{d}$, *i.e.* nilpotency and the undeformed Leibniz rule: $\mathbf{d}\alpha = \mathbf{d}(\alpha) + (-1)^p \alpha \mathbf{d}$ for any p-form α.

We use parentheses to delimit operations like $\mathbf{d}$, $\boldsymbol{i}_x$ and $\mathcal{L}_x$, *e.g.* $\mathbf{d}a = \mathbf{d}(a) + a\mathbf{d}$. However, if the limit of the operation is clear from the context, we will suppress the parentheses, *e.g.* $\mathbf{d}(\boldsymbol{i}_x \mathbf{d}a) \equiv \mathbf{d}(\boldsymbol{i}_x(\mathbf{d}(a)))$.

Here we want to base the construction of a differential calculus on quantum groups on two additional classical formulas: to extend the definition of a Lie derivative from functions and vector fields to forms we postulate

$$\mathcal{L} \circ \mathbf{d} = \mathbf{d} \circ \mathcal{L}; \tag{1}$$

this is essential for a geometrical interpretation of vector fields. The second formula that we can — somewhat surprisingly — keep undeformed in the quantum case is

$$\mathcal{L}_{\chi_i} = \boldsymbol{i}_{\chi_i}\mathbf{d} + \mathbf{d}\boldsymbol{i}_{\chi_i}, \qquad \textit{(Cartan Identity)} \tag{2}$$

where $\{\chi_i\}$ are the generators of some quantum Lie algebra.

2. Quantum Lie Algebras

A quantum Lie algebra is a Hopf algebra $\mathcal{U}$ with a finite-dimensional biinvariant subvector space $\mathcal{T}_q$ spanned by generators $\{\chi_i\}$ with coproduct

$$\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j. \tag{3}$$

More precisely we will call this a quantum Lie algebra of **type II.** Let $\{\omega^j \in \mathcal{T}_q{}^*\}$ be a dual basis of 1-forms corresponding to a set of functions $b^j \in \mathcal{A}$ via $\omega^j \equiv Sb^j_{(1)}\mathrm{d}b^j_{(2)}$; *i.e.*

$$\begin{aligned}
{}_{\mathcal{A}}\Delta(\chi_i) &= 1 \otimes \chi_i, & (4)\\
\Delta_{\mathcal{A}}(\chi_i) &= \chi_j \otimes T^j{}_i, \quad T^j{}_i \in \mathrm{Fun}(G_q), & (5)\\
i_{\chi_i}(\omega^j) &= -<\chi_i, Sb^j> = \delta^j_i, & (6)\\
{}_{\mathcal{A}}\Delta(\omega^i) &= 1 \otimes \omega^i, & (7)\\
\Delta_{\mathcal{A}}(\omega^i) &= \omega^j \otimes S^{-1}T^i{}_j. & (8)
\end{aligned}$$

If the functions b^i also close under adjoint coaction $\Delta^{Ad}(b^i) = b^j \otimes S^{-1}T^i{}_j$, we will call the corresponding quantum Lie algebra one of **type I.**

We can derive two alternate expressions for the exterior derivative of a function from the Cartan identity (2) in terms of these bases

$$\mathbf{d}(f) = \omega^j \pounds_{\chi_j}(f) = -\pounds_{S\chi_j}(f)\omega^j. \tag{9}$$

Combining the two expressions for $\mathbf{d}$ one easily derives the well-known $f-\omega$ commutation relations

$$f\omega^i = \omega^j \pounds_{O_j{}^i}(f). \tag{10}$$

The classical limit is given by $O_j{}^i \to 1\delta^i_j$, so that forms commute with functions.

3. Generators, Metrics and the Pure Braid Group

How does one go about finding the basis of generators $\{\chi_i\}$ and the set of functions $\{b^i\}$ that define the basis of 1-forms $\{\omega^i\}$? Here we would like to present a method that utilizes pure braid group elements as introduced in (Schupp *et al.*, 1992).

Let us recall that a pure braid element Υ is an element of $\mathcal{U}\hat{\otimes}\mathcal{U}$ that commutes with all coproducts of elements of $\mathcal{U}$, *i.e.*

$$\Upsilon\Delta(y) = \Delta(y)\Upsilon, \qquad \forall y \in \mathcal{U}. \tag{11}$$

Υ maps elements of $\mathcal{A}$ to elements of $\mathcal{U}$ with special transformation properties under the right coaction:

$$\begin{aligned}
&\Upsilon : \mathcal{A} \to \mathcal{U} : \quad b \mapsto \Upsilon_b \equiv <\Upsilon, b \otimes \mathrm{id}>;\\
&\Delta_{\mathcal{A}}(\Upsilon_b) = \Upsilon_{b_{(2)}} \otimes S(b_{(1)})b_{(3)} = <\Upsilon \otimes \mathrm{id}, \tau^{23}(\Delta^{Ad}(b) \otimes \mathrm{id})>.
\end{aligned} \tag{12}$$

An element Υ of the pure braid group defines furthermore a bilinear quadratic form on $\mathcal{A}$

$$(\, , \,) : \mathcal{A} \otimes \mathcal{A} \to k : \quad a \otimes b \mapsto (a,b) = - < \Upsilon, a \otimes S(b) > \in k, \tag{13}$$

with respect to which we can construct orthonormal $(b_i, b^j) = \delta_i^j$ bases $\{b_i\}$ and $\{b^j\}$ of functions that in turn will give generators $\chi_i := \Upsilon_{b_i}$ and 1-forms $\omega^j := S(b^j_{(1)})\mathrm{d}b^j_{(2)}$. Typically, one can choose $\mathrm{span}\{b_i\} = \mathrm{span}\{b^j\}$; then one starts by constructing one set, say $\{b_i\}$, of functions that close under adjoint coaction

$$\Delta^{Ad} b_i = b_j \otimes T^j{}_i. \tag{14}$$

If the numerical matrix

$$\eta_{ij} := - < \Upsilon, b_i \otimes S b_j > \qquad \textit{(metric)} \tag{15}$$

is invertible, *i.e.* $\det(\eta) \neq 0$, then we can use its inverse $\eta^{ij} := (\eta^{-1})_{ij}$ to raise indices

$$b^i = b_j \eta^{ji}. \tag{16}$$

This metric is invariant — or T is orthogonal — in the sense

$$\eta_{ji} = \eta_{kl} T^k{}_j T^l{}_i. \tag{17}$$

Once we have obtained a metric η, we can truncate the pure braid element Υ and work instead with:

$$\Upsilon \to \Upsilon_{trunc} = -S(\chi_i) \otimes \chi^i = -S(\chi_i) \otimes \chi_j \eta^{ji}, \tag{18}$$

which also commutes with all coproducts. Casimir operators can also be constructed from elements of the pure braid group. The truncated pure braid element gives for instance the quadratic casimir:

$$[\cdot \circ \tau \circ (S^{-1} \otimes \mathrm{id})](\Upsilon_{trunc}) = \eta^{ji} \chi_j \chi_i. \qquad \textit{(casimir)} \tag{19}$$

Now we would like to show that we have actually obtained a quantum Lie algebra of type I:

$$- < \chi_i, S b^j > = - < \Upsilon, b_i \otimes S b_k > \eta^{kj} = \eta_{ik} \eta^{kj} = \delta_i^j, \tag{20}$$

$$\Delta_{\mathcal{A}}(\chi_i) = \Upsilon_{b_{i(2)}} \otimes S(b_{i(1)}) b_{i(3)} = \Upsilon_{b_j} \otimes T^j{}_i = \chi_j \otimes T^j{}_i \tag{21}$$

and

$$\Delta^{Ad}(b^i) = b_k \otimes T^k{}_j \eta^{ji} = b_k \otimes \eta^{kl} \eta_{ln} T^n{}_j \eta^{ji} = b^k \otimes S^{-1} T^i{}_k. \tag{22}$$

Note, that Υ has to be carefully chosen to insure the correct number of generators. Furthermore, we still have to check the coproduct of the generators. If they are not of the form $\Delta \chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j$ then we might still consider a calculus with deformed Leibniz rule.

3.1. EXAMPLES

3.1.1. *The R-matrix approach*

Often one can take $b_i \in \text{span}\{t^n{}_m\}$, where $t^n{}_m$ is a quantum matrix in the defining representation of the quantum group under consideration. If we are dealing with a quasitriangular Hopf algebra with universal $\mathcal{R} \equiv \alpha_i \otimes \beta^i$, a natural choice for the pure braid element is

$$\Upsilon_R = \frac{1}{\lambda}\left(1 \otimes 1 - \mathcal{R}^{21}\mathcal{R}^{12}\right), \tag{23}$$

where the term $\mathcal{R}^{21}\mathcal{R}^{12}$ has been introduced and extensively studied by Reshetikhin & Semenov-Tian-Shansky (1990) and later by Jurčo (1991), Majid (1993) and Schupp, Watts & Zumino (1992). These choices of b_i's and Υ lead to the R-matrix approach to differential geometry on quantum groups. The metric is

$$\eta = - < X_1, St_2 >= \frac{1}{\lambda}\left(1 - \left[\left(R_{21}{}^{-1}\right)^{t_2}\left(R_{12}{}^{t_2}\right)^{-1}\right]^{t_2}\right), \tag{24}$$

where $X_1 =< \Upsilon_R, t_1 \otimes \text{id} >$ and $R_{12} =< \mathcal{R}, t_1 \otimes t_2 >$. In the case of $\text{GL}_q(2)$ we find.

$$\eta_{\text{GL}_q(2)} = \begin{pmatrix} q^{-3} & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-3} & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \tag{25}$$

In its reduced form, this matrix agrees with the metric obtained from quantum traces (see next section) except in the casimir sector $X^1{}_1 + q^{-2}X^2{}_2$. The formulation in terms of the pure braid element has the advantage that it does not require the existence of an element like u that implements the square of the antipode.

Using this metric we recover — as expected — the well-known (Zumino, 1992 and Schupp *et al.*(2), 1992) expression of the exterior derivative $\mathbf{d}$ on functions in terms of the quantum trace over X and the Cartan-Maurer form $\Omega = t^{-1}\mathrm{d}t$:

$$\mathbf{d} = \omega^i \chi_i = \text{tr}_q(\Omega \cdot X) \qquad \textit{(on functions)}. \tag{26}$$

(This follows essentially from $D_2^{-1}\eta_{12} = P_{12}$, where $D =< u, t >$ with $u = S(\beta^i)\alpha_i$ and P is the permutation matrix.)

3.1.2. *Trace formula for the metric*

Again, in the case where $\mathcal{U}$ is a quasitriangular Hopf algebra, there exists an alternate way of defining a Killing form; let $\rho : \mathcal{U} \to M_n(k)$ be an $n \times n$

matrix representation of $\mathcal{U}$ with entries in k. Define the map $\eta^{(\rho)} : \mathcal{U}\otimes\mathcal{U} \to k$ as

$$\eta^{(\rho)}(x \otimes y) := \mathrm{tr}_\rho(uxy), \tag{27}$$

where $x, y \in \mathcal{U}$, tr_ρ is the trace over the given representation, and u (see above) implements the square of the antipode. The map $\eta^{(\rho)}$ has the following properties:

$$\eta^{(\rho)}(y \otimes x) = \eta^{(\rho)}(x \otimes S^2(y)), \tag{28}$$

$$\eta^{(\rho)}((z_{(1)} \overset{\mathrm{ad}}{\triangleright} x) \otimes (z_{(2)} \overset{\mathrm{ad}}{\triangleright} y)) = \eta^{(\rho)}(x \otimes y)\epsilon(z), \tag{29}$$

for all $x, y, z \in \mathcal{U}$. Respectively, these express the symmetry of $\eta^{(\rho)}$ and its invariance under the adjoint action. In the case when $\mathcal{U}$ is a quantum Lie algebra with generators $\{\chi_i\}$, we can define the Killing metric for the representation ρ as

$$\eta_{ij}^{(\rho)} := \eta^{(\rho)}(\chi_i \otimes \chi_j). \tag{30}$$

3.1.3. The 2-dim quantum euclidean group

This is an example of a quantum Lie algebra that seems to have no universal $\mathcal{R}$ and where the set of functions $\{b_i\}$ does not arise from the matrix elements of some quantum matrix. In (Schupp *et al.*, 1992) we constructed such a set of functions b_0, b_+, b_-, b_1 and a pure braid element $\Upsilon_e = \frac{1}{\lambda}(\Delta c - c \otimes 1)$ from the casimir $c := P_+P_-$ of $\mathrm{e}_q(2)$. Now we can put the new machinery to work and calculate the (invertible) metric

$$\eta_{\mathrm{E}_q(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -q^{-2} & 0 \end{pmatrix}, \tag{31}$$

which immediately gives an expression for $\mathbf{d}$ on functions:

$$\mathbf{d} = \omega_0\chi_1 + \omega_1\chi_0 - q^2\omega_+\chi_- - \omega_-\chi_+. \tag{32}$$

4. Calculus of Functions, Vector Fields and Forms

Here we will generalize the Cartan calculus of ordinary *commutative* differential geometry to the case of quantum Lie algebras.

As in the classical case, the Lie derivative of a function is given by the action of the corresponding vector field, *i.e.*

$$\begin{aligned} \pounds_x(a) &= x \triangleright a = a_{(1)} < x, a_{(2)} >, \\ \pounds_x a &= a_{(1)} < x_{(1)}, a_{(2)} > \pounds_{x_{(2)}}. \end{aligned} \tag{33}$$

The action on products is given through the coproduct of x:

$$x \triangleright ab = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b). \tag{34}$$

The Lie derivative along x of an element $y \in \mathcal{U}$ is given by the adjoint action in $\mathcal{U}$:

$$\mathcal{L}_x(y) = x \overset{\text{ad}}{\triangleright} y = x_{(1)} y S(x_{(2)}). \tag{35}$$

To find the action of $\boldsymbol{i}_{\chi_i}$ we can now attempt to use the Cartan identity (2):

$$\chi_i \triangleright a = \mathcal{L}_{\chi_i}(a) = \boldsymbol{i}_{\chi_i}(\mathrm{d}a) + \mathrm{d}(\boldsymbol{i}_{\chi_i} a). \tag{36}$$

The idea is to use this identity as long as it is consistent and modify it if needed.

As the inner derivation $\boldsymbol{i}_{\chi_i}$ contracts 1-forms and is zero on 0-forms like a, we find

$$\boldsymbol{i}_{\chi_i}(\mathrm{d}a) = \chi_i \triangleright a = a_{(1)} < \chi_i, a_{(2)} > . \tag{37}$$

Next consider that for any form α,

$$\mathcal{L}_{\chi_i}(\mathrm{d}\alpha) = \mathrm{d}(\boldsymbol{i}_{\chi_i}\mathrm{d}\alpha) + \boldsymbol{i}_{\chi_i}(\mathrm{d}\mathrm{d}\alpha) = \mathrm{d}(\mathcal{L}_{\chi_i}\alpha) + 0, \tag{38}$$

which shows that Lie derivatives commute with the exterior derivative; $\mathcal{L}_{\chi_i}\mathbf{d} = \mathbf{d}\mathcal{L}_{\chi_i}$. We will later need to extend this equation to all elements of $\mathcal{U}$: $\mathcal{L}_x\mathbf{d} = \mathbf{d}\mathcal{L}_x$. From this and (33) we find

$$\mathcal{L}_x\mathbf{d}(a) = \mathbf{d}(a_{(1)}) < x_{(1)}, a_{(2)} > \mathcal{L}_{x_{(2)}}. \tag{39}$$

To find the complete commutation relations of $\boldsymbol{i}_{\chi_i}$ with functions and forms rather than just its action on them, we next compute the action of $\mathcal{L}_{\chi_i}$ on a product of functions a, $b \in \mathcal{A}$, *i.e.*

$$\mathcal{L}_{\chi_i}(ab) = \boldsymbol{i}_{\chi_i}\mathbf{d}(ab) = \boldsymbol{i}_{\chi_i}(\mathbf{d}(a)b + a\mathbf{d}(b)), \tag{40}$$

and compare with equation (34). Recalling that the χ_i have coproducts of the form $\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j$, $O_i{}^j \in \mathcal{U}$, we obtain

$$\boldsymbol{i}_{\chi_i} a = (O_i{}^j \triangleright a)\, \boldsymbol{i}_{\chi_j} = \mathcal{L}_{O_i{}^j}(a)\, \boldsymbol{i}_{\chi_j} \tag{41}$$

if we assume that the commutation relation of $\boldsymbol{i}_{\chi_i}$ with $\mathbf{d}(a)$ is of the general form

$$\boldsymbol{i}_{\chi_i}\mathbf{d}(a) = \underbrace{\boldsymbol{i}_{\chi_i}(\mathrm{d}a)}_{\in\mathcal{A}} + \text{“braiding term”} \cdot \boldsymbol{i}_{\chi_i} . \tag{42}$$

A calculation of $\mathcal{L}_{\chi_i}(\mathbf{d}(a)\mathbf{d}(b))$ along similar lines gives in fact

$$\begin{aligned} \boldsymbol{i}_{\chi_i}\mathbf{d}(a) &= (\chi_i \triangleright a) - \mathbf{d}(O_i{}^j \triangleright a)\, \boldsymbol{i}_{\chi_j} \\ &= \boldsymbol{i}_{\chi_i}(\mathrm{d}a) - \mathcal{L}_{O_i{}^j}(\mathrm{d}a)\, \boldsymbol{i}_{\chi_j}, \end{aligned} \tag{43}$$

and we propose for any p-form α:

$$i_{\chi_i}\alpha = i_{\chi_i}(\alpha) + (-1)^p \pounds_{O_i{}^j}(\alpha)\, i_{\chi_j}. \tag{44}$$

Using the Cartan identity we can derive commutation relations for (Lie) derivatives and functions from equation (41), which can be written in Hopf algebra language as

$$\chi a = a_{(1)} < \chi_{(1)}, a_{(2)} > \chi_{(2)}. \tag{45}$$

This actually defines the product in the cross-product algebra $\mathcal{A}\rtimes\mathcal{U}$ of general vector fields that one obtains by combining the Hopf algebras $\mathcal{A}$ and $\mathcal{U}$; see *e.g.* (Schupp *et al.*, 1992).

4.1. Maurer-Cartan Forms

The most general left-invariant 1-form can be written (Woronowicz 1989)

$$\omega_b := S(b_{(1)})\mathrm{d}(b_{(2)}) = -\mathrm{d}(Sb_{(1)})b_{(2)}, \tag{46}$$

$$\textit{left-invariance:}\quad {}_{\mathcal{A}}\Delta(\omega_b) = S(b_{(2)})b_{(3)} \otimes S(b_{(1)})\mathrm{d}(b_{(4)}) = 1 \otimes \omega_b, \tag{47}$$

corresponding to a function $b \in \mathcal{A}$. If this function happens to be $t^i{}_k$, where $t \in M_m(\mathcal{A})$ is an $m \times m$ matrix representation of $\mathcal{U}$ with $\Delta(t^i{}_k) = t^i{}_j \otimes t^j{}_k$ and $S(t) = t^{-1}$, we obtain the well-known Cartan-Maurer form $\omega_t = t^{-1}\mathrm{d}(t)$. Here is a nice formula for the exterior derivative of ω_b:

$$\mathrm{d}(\omega_b) = -\omega_{b_{(1)}}\omega_{b_{(2)}}. \tag{48}$$

The Lie derivative is

$$\pounds_\chi(\omega_b) = \omega_{b_{(2)}} < \chi, S(b_{(1)})b_{(3)} > . \tag{49}$$

The contraction of left-invariant forms with i_χ is

$$i_\chi(\omega_b) = - < \chi, S(b) > \in k. \tag{50}$$

4.2. Tensor Product Realization of the Wedge

From (49) and (50) we find commutation relations for i_{χ_i} with ω^j,

$$\begin{aligned} i_{\chi_i}\omega^j &= \delta^j_i - \pounds_{O_i{}^k}(\omega^j) i_{\chi_k} \\ &= \delta^j_i - \omega^m < O_i{}^k, S^{-1}(T^j{}_m) > i_{\chi_k}, \end{aligned} \tag{51}$$

which can be used to define the wedge product $\wedge$ of forms as some kind of antisymmetrized tensor product. So far we have suppressed the $\wedge$-symbol; to avoid confusion we will reinsert it in this paragraph. As in the classical case we make an ansatz for the product of two forms in terms of tensor products

$$\omega^i \wedge \omega^j = \omega^i \otimes \omega^j - \hat{\sigma}^{ij}{}_{mn}\omega^m \otimes \omega^n, \tag{52}$$

with as yet unknown numerical constants $\hat{\sigma}^{ij}{}_{mn} \in k$, and define $\boldsymbol{i}_{\chi_i}$ to act on this product by contracting in the first tensor product space, *i.e.*

$$\boldsymbol{i}_{\chi_i}(\omega^j \wedge \omega^k) = \delta^j_i \omega^k - \hat{\sigma}^{jk}{}_{mn}\delta^m_i \omega^n. \tag{53}$$

But from (51) we already know how to compute this, and we find

$$\hat{\sigma}^{ij}{}_{mn} = < O_m{}^j, S^{-1}(T^i{}_n) >, \tag{54}$$

or

$$\begin{aligned} \omega^i \wedge \omega^j &= (I - \hat{\sigma})^{ij}{}_{mn}\omega^m \otimes \omega^n \\ &= \omega^i \otimes \omega^j - \omega^k \otimes \pounds_{O_k{}^j}(\omega^i). \end{aligned} \tag{55}$$

These equations give implicit (anti)commutation relations between the ω^is. Note that $(1 - \hat{\sigma})$ has a sensible classical limit — it becomes $(1 - P)$ where P is the permutation matrix. Using the same method as for ω we can also obtain a tensor product decomposition of products of inner derivations.

Example: Maurer-Cartan Equation

$$\begin{aligned} \mathbf{d}\omega^j &= \mathbf{d}\omega_{b^j} = -\omega_{b^j_{(1)}} \wedge \omega_{b^j_{(2)}} \\ &= -\omega_{S^{-1}(Sb^j_{(1)} b^j_{(3)})} \otimes \omega_{b^j_{(2)}} \\ &= -\omega^k \otimes \omega^l < -S\chi_k, S^{-1}(Sb^j_{(1)} b^j_{(3)}) >< -S\chi_l, b^j_{(2)} > \\ &= -\omega^k \otimes \omega^l < \underbrace{(S^{-1}\chi_k)_{(1)}\chi_l S(S^{-1}\chi_k)_{(2)}}_{S^{-1}\chi_k \overset{\text{ad}}{\triangleright} \chi_l}, Sb^j > \\ &= -f'_k{}^j{}_l \omega^k \otimes \omega^l. \end{aligned} \tag{56}$$

In the previous equation we have introduced the adjoint action of a left-invariant vector field on another vector field. A short calculation gives

$$S^{-1}\chi_k \overset{\text{ad}}{\triangleright} \chi_l = \chi_b \chi_c (\delta^c_k \delta^b_l - \hat{\sigma}^{cb}{}_{kl}) = \chi_a < S^{-1}\chi_k, T^a{}_l >= \chi_a f'_k{}^a{}_l \tag{57}$$

as compared to

$$\chi_k \overset{\text{ad}}{\triangleright} \chi_l \equiv \pounds_{\chi_k}(\chi_l) = \chi_b \chi_c (\delta^c_k \delta^b_l - \hat{R}^{cb}{}_{kl}) = \chi_a f_k{}^a{}_l, \tag{58}$$

with $\hat{R}^{cb}{}_{kl} =< O_k{}^b, T^c{}_l >$. The two sets of structure constants are related by $< \chi_k, T^a{}_l >= f_k{}^a{}_l = -f'_i{}^a{}_l R^{ij}{}_{kl}$. See (Castellani *et al.* 1993) for a detailed discussion of such structure constants.

4.2.1. The "Anti-Wedge" Operator.

There is actually an operator W that recursively translates wedge products into the tensor product representation:

$$\begin{aligned} &W : \Lambda_q^p \to \mathcal{T}_q^* \otimes \Lambda_q^{p-1}, \quad p \geq 1, \\ &W(\alpha) = \omega^n \otimes i_{\chi_n}(\alpha), \end{aligned} \tag{59}$$

for any p-form α. For example,

$$\begin{aligned} \omega^j \wedge \omega^k &= \omega^n \otimes i_{\chi_n}(\omega^j \wedge \omega^k) \\ &= \omega^n \otimes (\delta_n^j \omega^k - \mathcal{L}_{O_n{}^m}(\omega^j)\delta_m^k). \end{aligned} \tag{60}$$

4.3. Summary of Relations in the Cartan Calculus

Commutation Relations

For any p-form α:

$$\mathbf{d}\alpha = \mathbf{d}(\alpha) + (-1)^p \alpha \mathbf{d} \tag{61}$$

$$i_{\chi_i}\alpha = i_{\chi_i}(\alpha) + (-1)^p \mathcal{L}_{O_i{}^j}(\alpha) i_{\chi_j} \tag{62}$$

$$\mathcal{L}_{\chi_i}\alpha = \mathcal{L}_{\chi_i}(\alpha) + \mathcal{L}_{O_i{}^j}(\alpha)\mathcal{L}_{\chi_j} \tag{63}$$

Actions

For any function $f \in \mathcal{A}$, 1-form $\omega_f \equiv Sf_{(1)}\mathbf{d}f_{(2)}$ and vector field $\phi \in \mathcal{A} \rtimes \mathcal{U}$:

$$i_{\chi_i}(f) = 0 \tag{64}$$

$$i_{\chi_i}(\mathbf{d}f) = \mathbf{d}f_{(1)} < \chi_i, f_{(2)} > \tag{65}$$

$$i_{\chi_i}(\omega_f) = - < \chi_i, Sf > \tag{66}$$

$$\mathcal{L}_\chi(f) = \chi(f) = f_{(1)} < \chi, f_{(2)} > \tag{67}$$

$$\mathcal{L}_\chi(\omega_f) = \omega_{f_{(2)}} < \chi, S(f_{(1)})f_{(3)} > \tag{68}$$

$$\mathcal{L}_\chi(\phi) = \chi_{(1)}\phi S(\chi_{(2)}) \tag{69}$$

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$$\mathbf{d}\mathbf{d} = 0 \tag{70}$$

$$\mathbf{d}\mathcal{L}_\chi = \mathcal{L}_\chi \mathbf{d} \tag{71}$$

$$\mathcal{L}_{\chi_i} = \mathbf{d} i_{\chi_i} + i_{\chi_i}\mathbf{d} \tag{72}$$

$$[\mathcal{L}_{\chi_i}, \mathcal{L}_{\chi_k}]_q = \mathcal{L}_{\chi_l} f_i{}^l{}_k \tag{73}$$

$$[\mathcal{L}_{\chi_i}, i_{\chi_k}]_q = i_{\chi_l} f_i{}^l{}_k \tag{74}$$

The quantum commutator $[\, ,\,]_q$ is here defined as follows:

$$[\mathcal{L}_{\chi_i}, \square]_q := \mathcal{L}_{\chi_i}\square - \mathcal{L}_{O_i{}^j(\square)}\mathcal{L}_{\chi_j}. \tag{75}$$

This quantum Lie algebra becomes infinite-dimensional as soon as we introduce derivatives along general vector fields.

4.4. Lie Derivatives Along General Vector Fields

So far we have focused on Lie derivatives and inner derivations along *left-invariant* vector fields, *i.e.* along elements of $\mathcal{T}_q$. The classical theory allows functional coefficients, *i.e.* the vector fields need not be left-invariant. Here we may introduce derivatives along elements in the $\mathcal{A} \rtimes \mathcal{T}_q$ plane by the following set of equations valid on forms: (note: $\epsilon(\chi) = 0$ for $\chi \in \mathcal{T}_q$)

$$i_{f\chi} = f i_\chi, \tag{76}$$

$$\mathcal{L}_{f\chi} = \mathrm{d} i_{f\chi} + i_{f\chi}\mathrm{d}, \tag{77}$$

$$\mathcal{L}_{f\chi} = f\mathcal{L}_\chi + \mathrm{d}(f) i_\chi, \tag{78}$$

$$\mathcal{L}_{f\chi}\mathrm{d} = \mathrm{d}\mathcal{L}_{f\chi}. \tag{79}$$

Equation (78) can be used to define Lie derivatives recursively on any form.

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