Non-Abelian T-duality in Superspace

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Scientific abstract

In this thesis we take a superspace perspective on T-duality, focusing on sigma models defined on background geometries that are constructed in terms of Lie supergroups. We briefly review Abelian bosonic and fermionic T-duality and the derivation of Buscher's rules, moving then to the dualisation of principal chiral models on group manifolds. Extension of the latter to the case of supergroup manifolds represents the starting point of our analysis, which features an extended discussion about the explicit dualisation of the supergroup $\mathsf{OSp}(1|2)$. While the initial model represents an appropriate three-dimensional supergravity background, the T-dual one hints in the opposite direction, as the ansatz adopted to construct the dual veilbeine fails to satisfy the supergravity torsion constraints. Such result, together with the complexity of the ansatz-based approach, suggests that a more abstract and general point of view should be taken on the dualisation procedure. This represents the next step of our analysis and allows a simpler dualisation of principal chiral models and a clearer argument that the above T-dual model falls outside the class of three-dimensional supergravity backgrounds. Extension of the dualisation procedure to symmetric and semi-symmetric coset space sigma models based on Lie supergroups G/H is also favored by the more abstract perspective, which allows to recover the well-known exchange of equations of motion and Maurer-Cartan equations typically observed in purely bosonic settings, hence leading to the construction of a dual Lax connection and ensuring preservation of classical integrability. While dualisation of principal chiral models can be performed in full generality, for coset models the procedure might be affected by impediments appearing in the process of integrating out the gauge fields in favor of the dual variables, and thus requires a case by case analysis. We proceed by solving those gauge fields equations of motion that allow for a general solution, thus confining the potential obstruction to a single equation, whose solvability depends on the invertibility of two linear operators. We conclude by discussing two explicit examples in which dualisation goes through, the first based on the symmetric space $\mathsf{S}^3 \simeq \mathsf{SO}(4)/\mathsf{SO}(3)$, well-known for its dualisability, the second on the semi-symmetric space $\mathsf{OSp}(1|2)/\mathsf{SO}(1,1)$, already approached in the literature from the point of view of holography and representing a Green-Schwarz-like sigma model satisfying the supergravity torsion constraints.

Statement of Originality Declaration

This thesis and the work to which it refers [1, 2] are the results of my own efforts. Any ideas, data, images or text resulting from the work of others (whether published or unpublished) are fully identified as such within the work and attributed to their originator in the text, bibliography or in footnotes. This thesis has been submitted to the University of Surrey and the University of Milano - Bicocca in agreement with the established dual doctorate program. I agree that the University has the right to submit my work to the plagiarism detection service TurnitinUK for originality checks. Whether or not drafts have been so-assessed, the University reserves the right to require an electronic version of the final document (as submitted) for assessment as above.

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Contents

1 Introduction

In this chapter we provide a brief introduction to T-duality, highlighting some of its features and providing a non-exhaustive list of references on the topic. We conclude with a short description of the remaining chapters and their structure.

1.1. A landscape of T-dualties

The main focus of this thesis lies in an intriguing property arising in the context of sigma models, known under the name of *Target Space Duality*, or more simply *T-duality*. Many examples of *dualities* have been found, conjectured and studied in the last few decades in the physics literature and such word originates from the possibility of providing two different descriptions of a common underlying physical system. From this point of view, the name *target space duality* hints toward the possibility of establishing a relation between different target space geometries, equally perceived from the sigma model perspective and giving rise to a single physical picture. For this reason T-duality has played a major role in the context of *string theory*, where sigma models are commonly exploited to describe the motion of strings in curved backgrounds. The origin of target space duality is in fact rooted in the string theory framework as it was first discovered and understood, in the context of string compactifications, as the invariance of the string spectrum and the full worldsheet conformal field theory under the exchange of a geometry having a compact direction of radius R with another one compactified on a circle of radius $\frac{\alpha'}{R}$ [3–5]. This phenomenon represents a characteristic feature of strings and originates from the possibility, enjoyed by one-dimensional extended objects as opposed to point-like particles, to wrap around compact directions.

In [6, 7] Buscher further extended the above picture by showing the invariance, under target space duality, of a generic string sigma model on an arbitrary curved manifold enjoying an Abelian isometry, while Roček and Verlinde showed in [8] that couples of conformal sigma models on curved backgrounds related by T-duality of a single Abelian isometry represent equivalent conformal field theories. This was soon extended to the case of d commuting isometries [9]. Importantly, the approaches adopted by Buscher and Roček-Verlinde presented a slight conceptual difference

- Buscher's results were based on the idea of rewriting a sigma model action, in which a certain background coordinate x_0 only appears in terms of worldsheet derivatives ∂x_0 and $\bar{\partial} x_0$, in the so-called first order form: substituting such terms with a new field $A, \bar{A}$ and adding to the action the extra term $\Lambda(\partial \bar{A} - \bar{\partial} A)$ one may recover the initial model integrating out Λ, while a new model is obtained upon integrating out $A, \bar{A}$.

- Roček and Verlinde realised how the above rewriting is actually equivalent to the gauging of a global shift symmetry along the direction x_0. Such shift symmetry can indeed be made local by covariantising the derivatives with a gauge field transforming appropriately $\partial x_0 \to (\partial + A) x_0$ $\bar{\partial} x_0 \to (\bar{\partial} + \bar{A}) x_0$ and the extra term $\Lambda(\partial \bar{A} - \bar{\partial} A)$ can be regarded as enforcing the flatness of the gauge field by means of a Lagrange multiplier.

The second approach represented an important change of viewpoint, as the gauging procedure could be straightforwardly generalised to settings with non-Abelian isometries and this was indeed soon recognised by de la Ossa and Quevedo, who introduced non-Abelian T-duality in [10]. The idea of performing duality transformations for non-Abelian isometries had actually already been considered earlier in Buscher's PhD thesis, which remains unpublished, and several years before in [11–13]. It was immediately realised, already in [10], that the newly introduced non-Abelian dualisation carried some peculiar features as compared to the Abelian one, and in paricular a systematic loss of isometries in the T-dual model. Indeed, while dualising with respect to a set of commuting isometries one finds that these are preserved in the T-dual model, so that the gauging procedure could be performed again leading back to the original model, when dualising with respect to a non-Abelian group G, this is generically broken and the dual model could even enjoy no isometry at all, implying that one may not be able to perform again the gauging and reach back the original model. It was indeed argued by Giveon and Roček [14] that, contrarily to the Abelian case, non-Abelian T-duality is not a symmetry of a single underlying worldsheed conformal field theory, but rather a relation between inequivalent theories. They also described how the exchange of equations of motion with Maurer-Cartan equations, which had already been observed for Abelian duality, is reproduced in the non-Abelian setting. Despite its peculiarities, this new duality attracted a lot of attention and also started to be used as a solution generating technique in supergravity [15–19]. In the search for an approach allowing to recover the initial model starting from the T-dual one, novel descriptions of the dualisation procedure started to emerge. A *canonical* perspective [20–22] turned out to be effective for generalisation, at least in the Abelian case, to the study of higher genus worldsheets, while the *Poisson-Lie* approach [23, 24], motivated by the idea that isometries could have not been the truly relevant structure to consider in duality transformations, initiated a novel research line in which T-duality is understood in terms of Lie bi-algebras $(\mathfrak{g}, \tilde{\mathfrak{g}})$ and regarded as a symmetry which exchanges the roles of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ (see [25, 26] for reviews of recent developments). About at the same time, people also started studying the possibility of extending T-duality to more complicated theories involving not only metric, B-field and dilaton, but also Ramond-Ramond and fermionic fields. This extension of the formalism was first achieved, in the Abelian case, from a target space perspective [27, 28] and successively generalised to the Green-Schwarz [29, 30] and pure-spinor [31] formalisms.

More recently, another important development took place when Berkovits and Maldacena proposed in [32] an extension of the Abelian bosonic dualisation procedure to the case of anticommuting fermionic isometries and showed how this new type of duality could be used to prove self-duality of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string background. This achievement allowed to clarify the relation between Wilson loops and scattering amplitudes which had already emerged from the field theory point of view in the planar limit of N= 4 super Yang-Mills theory [33]. Self-duality of $\mathrm{AdS}_5 \times \mathrm{S}^5$ was also studied from the integrability perspective in [34] (see also [35] for previous work and [36] for a review), and it was shown how the combination of bosonic and fermionic Abelian T-dualities sending the background to itself could be understood as a precise mapping between the conserved charges of the two models. Under the latter, some of the local charges in the initial model become delocalised in the dual model and, vice versa, some non-local charges become local. The combination of bosonic and fermionic Abelian T-dualities was later also used to show self-duality of other string backgrounds [37–40] and particular attention was given to the case of $\mathrm{AdS}_4 \times \mathbb{CP}^3$, for which self-duality was expected to hold as a consequence of the relation between Wilson loops and scattering amplitudes observed in Aharony– Bergman–Jafferis–Maldacena theory [41–44], in analogy with N= 4 super Yang-Mills theory for $\mathrm{AdS}_5 \times \mathrm{S}^5$, and of the integrability of both the string theory [45–48] and gauge theory sides [49, 50]. The appearance of singularities under combinations of Abelian dualities was however argued to prevent self-duality [51–54]. The idea of a fermionic duality was soon also considered from the Poisson-Lie perspective [55–59] and in the context of purely fermionic coset models [60], which highlighted the presence of obstructions to the dualisation originating from both the fermionic and coset nature of the models and called for the use of Becchi-Rouet-Stora-Tyutin techniques.

The idea of using T-duality as a solution generating technique in supergravity was re-discovered and generalised to the bosonic non-Abelian setting in [61, 62], where dualisation of principal chiral models and coset models in combination with fields from the Ramond-Ramond sector was introduced. For example, novel supergravity solutions were found starting from the notable backgrounds $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ and $\mathrm{AdS}_5 \times \mathrm{S}^5$ by performing dualisation along the $SO(4)$ and $SO(6)$ isometries of S^3 and S^5, first regarded as group manifolds and successively as symmetric cosets. These results also initiated the search for possible holographic interpretations of the newly obtained T-dual supergravity solutions [63–67], hence giving birth to an intriguing interplay between T-duality and holography. Other interesting features of non-Abelian T-duality emerged as a result of the semi-symmetric coset realisation for the $AdS_5 \times S^5$ background [68] (later also extended to $AdS_4 \times \mathbb{CP}^3$ [45]) and the discovery of its integrability [69]: these initiated the search for integrable deformations of such model, which led to the discovery of η-deformations [70] and λ-deformations [71, 72], respectively generalising the constructions in [73, 74]. A relation between these two was then pointed out in [75]. It was soon realised that while the η-deformed $AdS_5 \times S^5$ does not represent a Type II background [76], by performing Abelian T-duality on this model one obtains a proper supergravity solution exhibiting a novel feature, namely the presence of a dilaton depending on the directions along which T-duality has been performed, hence making the reverse dualisation not straightforward. This puzzle was then clarified by the discovery that the η-deformed background enjoys scale, but not Weyl invariance [77] and for this reason satisfies a generalised version of the supergravity equations of motion. The full set of such equations was then found in [78] by studying the constraints imposed by classical κ-symmetry on the target superspace geometry, which turned out to be not equivalent to 2d Weyl invariance (i.e. Type II equations of motion), but to scale invariance.

In [79] the background superfields associated to η and λ deformations were constructed and it was discovered that for unimodular R-matrices they lead to proper Type II backgrounds. In [80] it was further shown that deformations with Abelian R-matrices are equivalent to T-duality-shift-T-duality transformations performed on the undeformed background. For certain bosonic models, deformations with non-Abelian R-matrices turned out to be equivalent to non-Abelian T-duality of the undeformed model [81]. The latter finding was then proved in [82–84], where η deformations of principal chiral models, semi-symmetric cosets and Green-Schwarz sigma models were shown to be equivalent to a deformation of their T-duals by means of an invertible two-cocycle.

Dualisation of non-Abelian fermionic directions has furthermore been discussed in [82–85] and, in connection with λ-deformed models, in [86]. Quite recently, non-Abelian fermionic T-duality has also been approached from the double field theory perspective [87, 88] and a new supergravity solution has been constructed by performing bosonic dualisation of the non-compact isometry group $SL(2, \mathbb{R})$ acting on the AdS_3 factor contained in the $AdS_3 \times S^3 \times CY_2$ background [89].

1.2. Structure of the thesis

In chapter 2 we provide a quick review of some known results, first deriving the Buscher's rules for Abelian bosonic and fermionic duality of a single isometry and subsequently moving to the dualisation of principal chiral models on group manifolds. This prepares the ground for the case of supergroup manifolds G considered in chapter 3: we first extend to superspace the results obtained in the purely bosonic setting by dualising the left sector G_L of the full isometry group $G_L \times G_R$ and subsequently consider the example of $\mathsf{G} = \mathsf{OSp}(1|2)$. We explicitly construct the initial model, and its Killing vectors, showing it satisfies the supergravity torsion constraints. T-dualisation is then considered, first with respect to the maximal bosonic subgroup $\mathsf{SL}(2,\mathbb{R})_L$ and next with respect to the full $\mathsf{OSp}(1|2)_L$ supergroup. Residual isometries for both T-dual models are constructed and an argument is presented, hinting toward the fact that both models break the supergravity torsion constraints. The argument is based on a choice of ansatz for the vielbeine of the T-dual models and the need for a more general choice, together with the complexity of the approach, leads us to take a more abstract perspective on the whole procedure. Chapter 4 is based on this point of view, which allows to perform T-duality of principal chiral models in great generality, automatically providing a choice of T-dual vielbeine and allowing a simple construction of the 3-form $H_3 = \mathrm{d}B_2$. Exploiting these results we revise and improve our analysis of the principal chiral model on $\mathsf{OSp}(1|2)$, showing that both the dual vielbeine and H_3 break the superspace supergravity requirements. The new point of view also allows for simpler generalisation of the dualisation procedure to the case of coset models G/H based on Lie supegroups and we thus analyse the case of symmetric and semi-symmetric spaces. In such classes of models, the requirement for local invariance under the right action of H leads to potential obstructions to dualisation, hence forcing a case by case analysis. We proceed however integrating out all the gauge fields, except for the ones requiring a choice of explicit model, thus obtaining a hybrid T-dual action. This way we are able to recover, as for the case of principal chiral models, the exchange in role of Maurer-Cartan equations and equations of motion typically exhibited by bosonic T-duality. For all three classes of models, we also include in the discussion a topological deformation term, based on two-cocycles and recently introduced in the literature. In the final section we recast solvability of the potentially obstructing equation as the invertibility condition for two linear operators and consider two explicit examples in which these turn out to be invertible, hence allowing completion of the dualisation procedure.

2

Short Review Of Some Known Results

We revise some known results about T-duality, so as to introduce conventions and notation.

2.1. Abelian T-duality

In this section we shall partially take inspiration from [90] to briefly revise the concepts of bosonic and fermionic Abelian T-dualities as respectively introduced in [6–8] and [32]. The starting point of the discussion is the sigma model action

$$S = \frac{1}{2}\int_\Sigma \mathrm{d}\tau\mathrm{d}\sigma\big(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\nu\partial_\alpha X^\mu g_{\mu\nu} + \epsilon^{\alpha\beta}\partial_\beta X^\nu\partial_\alpha X^\mu B_{\mu\nu}\big)\ , \tag{2.1}$$

where, aiming at a purely classical treatment, we discarded the Dilaton term which might possibly be included. In this model the fields $X^\mu = X^\mu(\sigma,\tau)$, coordinates on the background space (superspace in the fermionic case), are regarded as maps $X^\mu : \Sigma \to \mathcal{M} \quad \forall\mu = 1,\ldots,dim(\mathcal{M})$ from a two-dimensional Lorentzian worldsheet $(\Sigma, h_{\alpha\beta})$[1] to some curved space $(\mathcal{M}, g_{\mu\nu})$ (superspace). Before proceeding, we shall first rewrite the action in a convenient compact form. To do this we first exploit worldsheet reparametrisation and Weyl invariance of the action to write the worldsheet metric in the conformal gauge, i.e. $h_{\alpha\beta} = \eta_{\alpha\beta} = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ with $h = det(h_{\alpha\beta}) = -1$. Then, choosing $\epsilon^{\alpha\beta} = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and introducing lightcone coordinates

$$\begin{cases} z = \frac{1}{2}(\tau+\sigma) \\ \bar{z} = \frac{1}{2}(\tau-\sigma)\end{cases} \Rightarrow \begin{cases} \tau = z+\bar{z} \\ \sigma = z-\bar{z}\end{cases} \Rightarrow \begin{cases} \partial_\tau = \frac{1}{2}(\partial+\bar{\partial}) \\ \partial_\sigma = \frac{1}{2}(\partial-\bar{\partial})\end{cases} \tag{2.2}$$
$$\text{and} \qquad \mathrm{d}\tau\mathrm{d}\sigma \quad\to\quad \mathrm{d}z\mathrm{d}\bar{z}\,det(\frac{\partial\xi}{\partial\xi'}) = -2\mathrm{d}z\mathrm{d}\bar{z}\ ,$$

[1] From now on Σ will be chosen to have sphere-like topology, to avoid dealing with non-trivial holonomies of the gauge fields along non-contractible loops. While generalisation to other topologies turned out to be manageable via the canonical picture [20–22] in the Abelian setting, this is still an open problem in the non-Abelian case.

the action takes the following form

$$S = \int_\Sigma \mathrm{d}^2 z\, \partial X^\nu \bar{\partial} X^\mu E_{\mu\nu} \qquad \text{with} \qquad E_{\mu\nu} := g_{\mu\nu} + B_{\mu\nu} \ . \tag{2.3}$$

Where the metric and B-field have been packed together as the symmetric and antisymmetric components of E.

2.1.1. Bosonic

The bosonic T-duality procedure relies on the assumption that the background metric $g_{\mu\nu}$ enjoys at least one continuous isometry $X^\mu \to X^\mu + K^\mu \epsilon$ generated by the Killing vector $K = K^\mu \partial_\mu$. Gauging the symmetry and adding to the action a term enforcing the flatness of the gauge fields by means of Lagrange multipliers, one can recover the initial model upon integrating out the multipliers, while another model is obtained by removal of the gauge fields. Together with the requirement $\mathscr{L}_K g = 0$, for the intial action (2.3) to be dualised one further needs that $\mathscr{L}_K B = 0$. With such an assumption one has a global symmetry of the full action

$$\delta_\epsilon S = \int_\Sigma \mathrm{d}^2 z\, \partial X^\nu \bar{\partial} X^\mu (\mathscr{L}_K E)_{\mu\nu} = 0 \ . \tag{2.4}$$

To proceed with the dualisation it is convenient to make use of coordinates $\{X^\mu\} = \{X^1, X^m\}$ adapted to the Killing vector. This way one has $K = \partial_1$, i.e. the Killing vector acts as a simple translation along X^1, and the metric and B-field are independent of such coordinate, which only appears in the action via ∂X^1 and $\bar{\partial} X^1$.

The symmetry can now be made local by gauging

$$\begin{cases} \partial X^m \to DX^m := \partial X^m \\ \partial X^1 \to DX^1 := \partial X^1 + A \end{cases} \qquad \begin{cases} \bar{\partial} X^m \to \bar{D}X^m := \bar{\partial} X^m \\ \bar{\partial} X^1 \to \bar{D}X^1 := \bar{\partial} X^1 + \bar{A} \end{cases} \tag{2.5}$$

and to recover the starting model one needs to include the term $\tilde{X}^1 F$, with $\tilde{X}^1$ a Lagrange multiplier and $F = \partial \bar{A} - \bar{\partial} A$ the Abelian field strength. The minimally coupled gauged action hence reads

$$S_{MCG} = \int_\Sigma \mathrm{d}^2 z \left(DX^\nu \bar{D} X^\mu E_{\mu\nu} + \tilde{X}^1 F \right) \ , \tag{2.6}$$

and is invariant under local transformations with $\epsilon = \epsilon(z, \bar{z})$

$$\delta_\epsilon X^m = 0 \qquad \delta_\epsilon X^1 = \epsilon \qquad \delta_\epsilon \tilde{X}^1 = 0 \qquad \delta_\epsilon A = -\partial\epsilon \qquad \delta_\epsilon \bar{A} = -\bar{\partial}\epsilon \ . \tag{2.7}$$

As mentioned above, the addition of the field strength term to the action is fundamental in recovering the original model (2.3), as the equations of motion for the Lagrange multiplier $\tilde{X}^1$ forces the vanishing of the field strength and thus the gauge field to be pure gauge $A = \partial\chi$, $\bar{A} = \bar{\partial}\chi$. The original model is recovered for the convenient choice of χ =const which makes $A = \bar{A} = 0$.

On the other hand, one can obtain a new model by proceeding the other way around, i.e. first integrating out the gauge field and then gauge fixing. The equations of motion for A and $\bar{A}$ read

$$A = \frac{1}{E_{11}}(\partial\tilde{X}^1 - \partial X^1 E_{11} - \partial X^i E_{1i}) \qquad \bar{A} = \frac{1}{E_{11}}(\bar{\partial}\tilde{X}^1 - \bar{\partial}X^1 E_{11} - \bar{\partial}X^i E_{i1}) \ . \tag{2.8}$$

Substituting them into the action, fixing $X^1 = 0$ and rearranging the remaining terms, one obtains a model which formally has the same structure as the original one

$$\tilde{S} = \int_\Sigma \mathrm{d}^2 z\, \partial\tilde{X}^\nu \bar{\partial}\tilde{X}^\mu \tilde{E}_{\mu\nu} \ , \tag{2.9}$$

with new coordinates $\{\tilde{X}^\mu\} = \{\tilde{X}^1, X^m\}$ and the following relations

$$\begin{aligned} \tilde{E}_{11} &= \tilde{g}_{11} = \frac{1}{g_{11}} & \tilde{E}_{mn} &= E_{mn} - \frac{1}{g_{11}} E_{m1}E_{1n} \\ \tilde{E}_{1n} &= -\frac{E_{1n}}{g_{11}} & \tilde{E}_{m1} &= \frac{E_{m1}}{g_{11}} \ . \end{aligned} \tag{2.10}$$

Using $\tilde{g}_{\mu\nu} = \frac{1}{2}(\tilde{E}_{\mu\nu} + \tilde{E}_{\mu\nu})$ and $\tilde{B}_{\mu\nu} = \frac{1}{2}(\tilde{E}_{\mu\nu} - \tilde{E}_{\mu\nu})$, leads to the famous Buscher's rules

$$\begin{aligned} &\tilde{g}_{11} = \frac{1}{g_{11}} \qquad \tilde{g}_{m1} = \frac{B_{m1}}{g_{11}} \qquad \tilde{g}_{mn} = g_{mn} - \frac{1}{g_{11}}(g_{m1}g_{1n} + B_{m1}B_{1n}) \\ &\tilde{B}_{m1} = \frac{g_{m1}}{g_{11}} \qquad \tilde{B}_{mn} = B_{mn} - \frac{1}{g_{11}}(g_{m1}B_{1n} + B_{m1}g_{1n}) \ . \end{aligned} \tag{2.11}$$

It is important to notice that the T-Dual model exhibits a shift isometry along the $\tilde{X}^1$ direction, as the metric and B-field are independent of it. This allows to re-apply the dualisation procedure to the dual model and go back to the original one.

2.1.2. Fermionic

The fermionic duality introduced in [32] is performed in the same spirit as the bosonic one and thus follows similar steps. The starting point is again the action

$$S = \int_\Sigma \mathrm{d}^2 z\, \partial X^N \bar{\partial} X^M E_{MN} \ , \qquad \text{with} \qquad E_{MN} = g_{MN} + B_{MN} \ , \tag{2.12}$$

the main difference being that now the target space is a superspace $\mathcal{M}$, with bosonic and fermionic coordinates $\{X^M\} = \{X^m, \theta^\mu\}$ such that $X^M : \Sigma \to \mathcal{M}$. In this setting g_{MN} and B_{MN} are respectively graded-symmetric and graded-antisymmetric and thus satisfy $g_{MN} = (-1)^{|M||N|} g_{NM}$ and $B_{MN} = -(-1)^{|M||N|} B_{NM}$, where we introduced the Grassmann parity $|M| = 0$ and $|M| = 1$ for respectively bosonic and fermionic directions. The total parity of E_{MN} is then simply given by $|M| + |N|$. Similarly to the bosonic case, the action is assumed to be written in coordinates such that the metric and B-field are independent of the fermionic θ^1, which only appears in terms of $\partial\theta^1$ and $\bar{\partial}\theta^1$. The action is thus invariant under a constant fermionic shift of θ^1 by ρ

$$X^{\hat{M}} \to X^{\hat{M}} \qquad\qquad \theta^1 \to \theta^1 + \rho \,, \tag{2.13}$$

where we introduced $X^{\hat{M}} = \{X^m, \theta^{\hat{\mu}}\}$ with $\hat{\mu}$ running over all fermionic coordinates except for θ^1. In this setting, the dualisation procedure can be performed pretty much like for the bosonic case. The symmetry is made local by gauging

$$\begin{cases} \partial X^{\hat{M}} \to DX^{\hat{M}} = \partial X^{\hat{M}} \\ \partial\theta^1 \to D\theta^1 = \partial\theta^1 + A \end{cases} \qquad \begin{cases} \bar{\partial} X^{\hat{M}} \to \bar{D}X^{\hat{M}} = \bar{\partial} X^{\hat{M}} \\ \bar{\partial}\theta^1 \to \bar{D}\theta^1 = \bar{\partial}\theta^1 + \bar{A} \end{cases} \tag{2.14}$$

and adding a term $\tilde{\theta}^1 F$, with $\tilde{\theta}^1$ a fermionic Lagrange multiplier and $F = \partial\bar{A} - \bar{\partial}A$ the fermionic Abelian gauge field strength. We thus get the minimally coupled gauged action

$$S_{MCG} = \int_\Sigma \mathrm{d}^2 z \left(DX^N \bar{D}X^M E_{MN} + \tilde{\theta}^1 F \right) \,, \tag{2.15}$$

which is invariant under local transformations with $\rho = \rho(z, \bar{z})$

$$\delta_\rho X^{\hat{M}} = 0 \qquad \delta_\rho \theta^1 = \rho \qquad \delta_\rho \hat{\theta}^1 = 0 \qquad \delta_\rho A = -\partial\rho \qquad \delta_\rho \bar{A} = -\bar{\partial}\rho \,. \tag{2.16}$$

As already mentioned, the main difference with respect to the bosonic case is that the gauge field, the Lagrange multiplier and the cross terms E_{1m}, $E_{\tilde{\mu}m}$ are fermionic quantities, which means they anticommute. This can be handled with some extra attention when reshuffling the various terms and as a result of such property the dual fields will be different from the ones found in the bosonic case. Once again, integrating out the Lagrange multipliers forces the vanishing of the field strength, hence setting the gauge field to be pure gauge and allowing to recover the initial action. On the other hand, computing the equations of motion for the gauge field one finds

$$A = \frac{1}{E_{11}} (\partial\tilde{\theta}^1 - \partial\theta^1 E_{11} + (-1)^{\hat{N}} \partial X^{\hat{N}} E_{1\hat{N}}) \qquad \bar{A} = \frac{1}{E_{11}} (\bar{\partial}\tilde{\theta}^1 - \bar{\partial}\theta^1 E_{11} - \bar{\partial} X^{\hat{M}} E_{\hat{M}1}) \,, \tag{2.17}$$

so that substituting back into the action, fixing $\theta^1 = 0$ and rearranging, a model with the same structure as the initial one is obtained

$$\tilde{S} = \int_\Sigma \mathrm{d}^2 z\, \partial \tilde{X}^N \bar{\partial} \tilde{X}^M \tilde{E}_{MN} \;, \tag{2.18}$$

with coordinates $\{\tilde{X}^M\} = \{X^{\hat{M}}, \tilde{\theta}^1\}$ and the following components

$$\begin{aligned} \tilde{E}_{11} &= \tilde{B}_{11} = -\frac{1}{B_{11}} & \qquad \tilde{E}_{\hat{M}\hat{N}} &= E_{\hat{M}\hat{N}} - \frac{1}{B_{11}} E_{\hat{N}1} E_{1\hat{M}} \\ \tilde{E}_{1\hat{M}} &= \frac{E_{1\hat{M}}}{B_{11}} & \tilde{E}_{\hat{M}1} &= \frac{E_{\hat{M}1}}{B_{11}} \;. \end{aligned} \tag{2.19}$$

Paying attention to the exchange of fermionic quantities and respecting the graded symmetry and antisymmetry of the metric and B-field via $\hat{g}_{MN} = \frac{1}{2}(\hat{E}_{MN} + (-1)^{|M||N|}\hat{E}_{NM})$ and $\hat{B}_{MN} = \frac{1}{2}(\hat{E}_{MN} - (-1)^{|M||N|}\hat{E}_{NM})$, we end up with the fermionic analogue of the Buscher's rules, again relating the two sigma models

$$\begin{aligned} \tilde{g}_{11} &= 0 \quad \tilde{g}_{1\hat{M}} = \frac{g_{1\hat{M}}}{B_{11}} & \qquad \tilde{B}_{11} &= -\frac{1}{B_{11}} \quad \tilde{B}_{1\hat{M}} = \frac{B_{1\hat{M}}}{B_{11}} \\ \tilde{g}_{\hat{M}\hat{N}} &= g_{\hat{M}\hat{N}} - \frac{1}{B_{11}}(g_{\hat{M}1}B_{1\hat{N}} + B_{\hat{M}1}g_{1\hat{N}}) & \tilde{B}_{\hat{M}\hat{N}} &= B_{\hat{M}\hat{N}} - \frac{1}{B_{11}}(g_{\hat{M}1}g_{1\hat{N}} + B_{\hat{M}1}B_{1\hat{N}}) \;. \end{aligned} \tag{2.20}$$

Notice that also in this case the dual model exhibits a shift isometry along the $\tilde{\theta}^1$ direction, as the metric and B-field are independent of it, and the procedure might thus be repeated.

2.2. Non-Abelian T-duality

Non-Abelian T-duality was first introduced in [10] and conceptually represents the natural extension of the procedure described in the previous section, to backgrounds with a set of non commuting isometries. Despite the name, a key feature of such generalisation is that it does not represent a true duality [14], as it is generally not possible to recover the initial model starting from the T-dual one and in this sense it is not invertble. Indeed, in contrast to the Abelian case, the dual model does not usually have the same amount of isometries as the initial model and even gauging the residual ones, the original setup is generally not recovered. The systematic loss of isometries is due to the fact that only those isometries which commute with the gauged ones survive the dualisation procedure [90, 91]. As discussed in the introduction, despite this drawback non-Abelian T-duality has been largely studied and has played an important role in various research directions. In the rest of this chapter we shall focus on reviewing a class of sigma models, known as *principal chiral models* (PCM), defined on group manifolds, as they will be of primary importance for later purposes.

2.2.1. Principal chiral models

Principal chiral models are sigma models in which the background is a group manifold G. The basic constituent of this class of models it to the so-called *principal chiral field* $\mathrm{g} : \Sigma \to \mathrm{G}$, where Σ is a two-dimensional Lorentzian worldsheet and G is a Lie group with Lie algebra $\mathfrak{g}$. Starting from g one can construct the Lie algebra valued 1-form current $j = \mathrm{g}^{-1}\mathrm{dg} \in \Omega^1(\Sigma, \mathfrak{g})$ satisfying, by construction, the Maurer-Cartan equation[2]

$$\mathrm{d}j = \mathrm{d}(\mathrm{g}^{-1}) \wedge \mathrm{dg} = -\mathrm{g}^{-1}(\mathrm{dg})\mathrm{g}^{-1} \wedge \mathrm{dg} = -j \wedge j = -\tfrac{1}{2}[j, j] \quad \Rightarrow \quad \mathrm{d}j + \tfrac{1}{2}[j, j] = 0 \ . \tag{2.21}$$

The action is then constructed out of the current j as

$$S_{PCM} = \frac{1}{2}\int_\Sigma Tr\big[(\mathrm{g}^{-1}\mathrm{dg}) \wedge \star(\mathrm{g}^{-1}\mathrm{dg})\big] \ , \tag{2.22}$$

where Tr represents the Ad-invariant inner product on the Lie algebra generators and $\star$ the Hodge operator with respect to the worldsheet metric. Switching to lightcone coordinates and choosing the inner product such that $Tr[T_a T_b] = \delta_{ab}$, the action can be further recast into

$$S_{PCM} = \int_\Sigma \mathrm{d}^2z\, Tr[(\mathrm{g}^{-1}\partial\mathrm{g})(\mathrm{g}^{-1}\bar\partial\mathrm{g})] = \int_\Sigma \mathrm{d}^2z\, (\mathrm{g}^{-1}\partial\mathrm{g})^b(\mathrm{g}^{-1}\bar\partial\mathrm{g})^a\delta_{ab} \ . \tag{2.23}$$

Principal chiral models have a large set of isometries $\mathrm{G}_L \times \mathrm{G}_R$, as they are invariant under the global left $\mathrm{g} \to g_L^{-1}\mathrm{g}$ and global right $\mathrm{g} \to \mathrm{g}g_R$ action of $g_L, g_R \in \mathrm{G}$.

- Invariance under G_L is a direct consequence of the fact that j itself is left unchanged

$$\begin{aligned} S'_{PCM} &= \int_\Sigma \mathrm{d}^2z\ Tr\big[\big((g_L^{-1}\mathrm{g})^{-1}\partial(g_L^{-1}\mathrm{g})\big)\big((g_L^{-1}\mathrm{g})^{-1}\bar\partial(g_L^{-1}\mathrm{g})\big)\big] = \\ &= \int_\Sigma \mathrm{d}^2z\ Tr[(\mathrm{g}^{-1}g_L g_L^{-1}\partial\mathrm{g})(\mathrm{g}^{-1}g_L g_L^{-1}\bar\partial\mathrm{g})] = S_{PCM} \ . \end{aligned} \tag{2.24}$$

- For G_R one needs to exploit Ad-invariance of the inner product, namely ciclicity of the trace

$$\begin{aligned} S'_{PCM} &= \int_\Sigma \mathrm{d}^2z\ Tr\big[\big((\mathrm{g}g_R)^{-1}\partial(\mathrm{g}g_R)\big)\big((\mathrm{g}g_R)^{-1}\bar\partial(\mathrm{g}g_R)\big)\big] = \\ &= \int_\Sigma \mathrm{d}^2z\ Tr[g_R^{-1}\mathrm{g}^{-1}(\partial\mathrm{g})g_R g_R^{-1}\mathrm{g}^{-1}(\bar\partial\mathrm{g})g_R] = S_{PCM} \ . \end{aligned} \tag{2.25}$$

Given such set of isometries, T-duality can be performed by following the same key steps highlighted in the past section. For these models, the procedure can be carried out in such a way that a quite

[2] we consider $\mathrm{g} \in \mathcal{C}^\infty(\Sigma, \mathrm{G})$, the set of smooth maps from the worldsheet to G, and we denote by d the exterior derivative on the worldsheet and by $\Omega^1(\Sigma, \mathfrak{g})$ the set of 1-forms on Σ taking values in $\mathfrak{g}$

general expression for the dual action is achieved. We shall proceed, as done for example in [61], by gauging the invariance under the G_L action. These global symmetries can be made local by introducing non-Abelian gauge fields via minimal coupling

$$D\mathrm{g} := \partial\mathrm{g} + A \qquad\qquad \bar{D}\mathrm{g} := \bar{\partial}\mathrm{g} + \bar{A}\ . \tag{2.26}$$

Indeed, given the local transformation $\mathrm{g} \to h^{-1}\mathrm{g}$ with $h^{-1} = h^{-1}(z, \bar{z}) \in G_L$ and

$$A \to h^{-1}Ah - (\partial h^{-1})h \qquad\qquad \bar{A} \to h^{-1}\bar{A}h - (\bar{\partial}h^{-1})h\ , \tag{2.27}$$

the covariant derivatives transform as $D\mathrm{g} \to h^{-1}D\mathrm{g}$ and $\bar{D}\mathrm{g} \to h^{-1}\bar{D}\mathrm{g}$, so that the covariantised action enjoys local invariance. At this stage one further needs to enforce the flatness of the gauge fields by introducing the appropriate Lagrange multiplier term. The full minimally coupled gauged action then reads

$$S^{MCG}_{PCM} = \int_\Sigma \mathrm{d}^2z\ Tr[(\mathrm{g}^{-1}D\mathrm{g})(\mathrm{g}^{-1}\bar{D}\mathrm{g})] + Tr[\Lambda F]\ , \tag{2.28}$$

where $F = \partial\bar{A} - \bar{\partial}A + [A, \bar{A}]$ and Λ are Lie algebra valued Lagrange multipliers. Given the local transformations (2.27), the field strength transforms as $F \to h^{-1}Fh$ and consecutively the newly introduced piece is invariant provided the Lagrange multipliers transform as $\Lambda \to h^{-1}\Lambda h$. As in the Abelian cases, variations of the Lagrange multipliers Λ imposes the vanishing of the field strength $F = \partial\bar{A} - \bar{\partial}A + [A, \bar{A}] = 0$, which in turn implies $A = h\partial h^{-1}$ and $\bar{A} = h\bar{\partial}h^{-1}$, i.e. the gauge fields to be pure gauge. The original model (2.23) is thus recovered by choosing the gauge $h = \mathbb{1}$.

The dual model can once again be obtained by integrating out the gauge fields rather than the multipliers. Writing (2.28) in components, the equations of motion for the gauge fields read

$$A^a = -\big([(\partial\mathrm{g})\mathrm{g}^{-1}]^b - \partial\Lambda^b\big)(M^{-1})_b{}^a \qquad \bar{A}^a = -\big([(\bar{\partial}\mathrm{g})\mathrm{g}^{-1}]^b + \bar{\partial}\Lambda^b\big)(N^{-1})_b{}^a\ . \tag{2.29}$$

Where we defined matrices $M_a{}^b := \delta_a{}^b + \Lambda^c f_{ca}{}^b$ and $N_a{}^b := \delta_a{}^b - \Lambda^c f_{ca}{}^b$, with $f_{ab}{}^c$ the structure constant of the Lie algebra $\mathfrak{g}$ appearing explicitly in the component form of the field strength $F^a = \partial\bar{A}^a - \bar{\partial}A^a - A^b\bar{A}^c f_{cb}{}^a$. Substituting the latter equations back into the action and exploiting the gauge freedom to choose $\mathrm{g} = \mathbb{1}$, the action becomes

$$\begin{aligned} S^{MCG}_{PCM} = \int_\Sigma d^2z\Big[& -(\partial\Lambda^d)(M^{-1})_c{}^a\bar{\partial}\Lambda^d(N^{-1})_d{}^b\delta_{ba} + \partial\Lambda^a\bar{\partial}\Lambda^c(N^{-1})_c{}^b\delta_{ba} + \\ & + \bar{\partial}\Lambda^a\partial\Lambda^c(M^{-1})_c{}^b\delta_{ba} + \Lambda^a\partial\Lambda^c(M^{-1})_c{}^d\bar{\partial}\Lambda^e(N^{-1})_e{}^p f_{pd}{}^b\delta_{ba}\Big]\ . \end{aligned} \tag{2.30}$$

At this point, one can use the definition of $N_a{}^b$ to find that

$$(N^{-1})_a{}^c N_c{}^b = \delta_a{}^b \qquad \Rightarrow \qquad (N^{-1})_a{}^b = \delta_a{}^b + (N^{-1})_a{}^c \Lambda^d f_{dc}{}^b \ , \tag{2.31}$$

which substituted into the first of the above four terms, cancels the last two and thus leaves

$$\tilde{S}_{PCM} = \int_\Sigma d^2z\, \partial\Lambda^a \bar{\partial}\Lambda^c (N^{-1})_c{}^b \delta_{ba} \ . \tag{2.32}$$

From the latter expression one can then extract the dual metric and B-field as the symmetric and antisymmetric components of $(N^{-1})_{ab} := (N^{-1})_a{}^c \delta_{cb}$

$$\tilde{g}_{ab} = \frac{1}{2}[(N^{-1})_{ab} + (N^{-1})_{ba}] \qquad\qquad \tilde{B}_{ab} = \frac{1}{2}[(N^{-1})_{ab} - (N^{-1})_{ba}] \ . \tag{2.33}$$

Given a choice of group manifold G and the above result, one may in principle straightforwardly compute the T-dual fields by explicitly writing down the matrix N and inverting it using computational techniques. This might however not always be the best way to proceed, especially in cases where one would like to retain the underlying index structure. In the next chapter we shall generalise the above result to the case of supergroup manifolds and consider an explicit example in which the dualisation can be performed in full details without the need for computational tools.

3

An Explicit Approach To Super Non-Abelian T-Duality

In light of the great versatility of the gauging procedure introduced in [8], which allowed to extend T-duality in bosonic backgrounds from the simplest case of a single isometry [6, 7] to the case of a non-Abelian group of isometries [10], it is natural to consider the possiblity of extending the Abelian fermionic procedure introduced in [32] to the more complicated case of multiple bosonic and fermionic isometries with non trivial commutation properties. The rich structure and successful exploitation of bosonic non-Abelian T-duality in various contexts represents a strong motivation to wonder about potential extensions and applications of a super non-Abelian setting. Some steps have already been taken in this direction [60, 82–86], nevertheless a clear picture on the topic is still missing and the construction of explicit examples (currently almost absent in the literature) might prove useful in shading some light. With such motivation, we begin addressing the problem by extending the discussion of principal chiral models from the previous chapter to the case of supergroup manifolds, which, to the best of our knowledge, has not been discussed with this flavour in the literature. We then analyse in detail, for the first time from the supergroup manifold perspective, dualisation of $\mathsf{OSp}(1|2)$, which had been considered in [60] in the context of purely fermionic cosets via Becchi-Rouet-Stora-Tyutin techniques. The material of this chapter hence already introduces some novelties, but is not taken from [1, 2] and rather serves as a motivation for the next chapter, which is on the other hand based on the latter publications.

3.1. Principal chiral models on supergroup manifolds

The starting point and setup of our discussion goes along the same lines as for principal chiral models defined on group manifolds. We consider a background which is a Lie supergroup G, with associated Lie superalgebra $\mathfrak{g}$, and define the principal chiral field $\mathrm{g} : \Sigma \to \mathsf{G}$ together with the Lie algebra valued 1-form principal chiral current $j = \mathrm{g}^{-1}\mathrm{dg} \in \Omega^1(\Sigma, \mathfrak{g})$, which again satisfies by construction the Maurer-Cartan equation (2.21). In lightcone coordinates on the worldsheet the action takes the form (2.23) with the replacement of the trace by the supertrace

$$S_{PCM} = \int_\Sigma \mathrm{d}^2 z\, STr[(\mathrm{g}^{-1}\partial \mathrm{g})(\mathrm{g}^{-1}\bar{\partial}\mathrm{g})]\ . \tag{3.1}$$

The symmetry properties of the model are the same as the ones discussed in section 2.2.1., i.e. the action enjoys $\mathrm{G}_L \times \mathrm{G}_R$ invariance under the global left and right actions $\mathrm{g} \to g_L^{-1}\mathrm{g}$ and $\mathrm{g} \to \mathrm{g}g_R$ for $g_L, g_R \in \mathrm{G}$. We shall proceed again with the gauging of the left action, which can be made local by introducing non-Abelian Lie algebra valued gauge fields with minimal coupling $D\mathrm{g} = \partial\mathrm{g} + A\mathrm{g}$ and $\bar{D}\mathrm{g} = \bar{\partial}\mathrm{g} + \bar{A}\mathrm{g}$. These enjoy the transformation properties $D\mathrm{g} \to h^{-1}D\mathrm{g}$ and $\bar{D}\mathrm{g} \to h^{-1}\bar{D}\mathrm{g}$, for $h^{-1} = h^{-1}(z,\bar{z}) \in \mathrm{G}$, provided that the gauge fields transform as in (2.27), i.e. $A \to h^{-1}Ah - (\partial h^{-1})h$ and $\bar{A} \to h^{-1}\bar{A}h - (\bar{\partial}h^{-1})h$. These modifications make the action (3.1) invariant under the local G_L action, but to be able to get back the original model we must include the extra term $STr[\Lambda F]$, with $F = \partial\bar{A} - \bar{\partial}A + [A, \bar{A}]$. The field strength transforms as $F \to h^{-1}Fh$, so that the minimally coupled gauged action

$$S_{PCM}^{MCG} = \int_\Sigma \mathrm{d}^2 z\, STr[(\mathrm{g}^{-1}D\mathrm{g})(\mathrm{g}^{-1}\bar{D}\mathrm{g})] + STr[\Lambda F] \tag{3.2}$$

is fully invariant under the local G_L action provided that $\Lambda \to h^{-1}\Lambda h$. Once again, variations of the Lagrange multipliers Λ impose the vanishing of the field strength $F = \partial\bar{A} - \bar{\partial}A + [A, \bar{A}] = 0$, which in turn requires the gauge fields to be pure gauge $A = h\partial h^{-1}$ and $\bar{A} = h\bar{\partial}h^{-1}$. The original model (3.1) is thus recovered by conveniently choosing $h = \mathbb{1}$, while the dual model can be obtained by integrating out the gauge fields rather than the multipliers.

At this level the whole setup and procedure is thus morally the same as in the purely bosonic case and the main difference, so far still hidden by having considered fully contracted Lie algebra valued quantities, lies in the fact that we also have fermionic gauge fields and multipliers. This implies that particular attention should be paid due to the different Graßmann nature of the various objects appearing in the action. In order to proceed let us first introduce some notation. We label the Lie superalgebra generators as $\{T_A\}$ with $A \in \{m, \mu\}$: lower case latin indices label bosonic generators, while lower case greek indices label fermionic generators. The Graßmann parity of a generator is thus given by its index following standard conventions

$$|A| \equiv \begin{cases} 0 & \text{if} \quad A = m \\ 1 & \text{if} \quad A = \mu \end{cases}\ , \tag{3.3}$$

and in turn allows to identify the parity of all fields in terms of the parity of their indices. To simplify the notation we shall from now on remove the absolute value. This notation allows us to define the graded commutator of generators, the graded antisymmetry of the structure constants and the action of the supertrace on generators as[1]

$$\begin{aligned} &[T_A, T_B] := T_A T_B - (-1)^{AB} T_B T_A = f_{AB}{}^C T_C \qquad \text{with} \qquad f_{AB}{}^C = -(-1)^{AB} f_{BA}{}^C \\ &\delta_{AB} := STr[T_A T_B] = (-1)^{AB} STr[T_B T_A] = (-1)^{AB} \delta_{BA} \ . \end{aligned} \tag{3.4}$$

We furthermore exploit δ_{AB} to construct structure constants with lowered indices $f_{ABC} := f_{AB}{}^D \delta_{DC}$, which are graded antisymmetric in the exchange of any two indices

$$f_{ABC} = -(-1)^{BC} f_{ACB} = (-1)^{C(A+B)} f_{CAB} = -(-1)^{C(A+B)+BA} f_{CBA} \ . \tag{3.5}$$

This is a direct result of the supertrace identity holding for any triplet of generators

$$\begin{aligned} STr[T_A[T_B, T_C]] &= STr[T_A T_B T_C] - (-1)^{BC} STr[T_A T_C T_B] = \\ &= STr[T_A T_B T_C] - (-1)^{BC+B(A+C)} STr[T_B T_A T_C] = \\ &= STr[[T_A, T_B], T_C] \ . \end{aligned} \tag{3.6}$$

The component form of the action (3.2) reads

$$\begin{aligned} S^{MCG}_{PCM} = \int_\Sigma d^2 z \Big[&+ (g^{-1}\partial g)^A (g^{-1}\bar\partial g)^B + [(\partial g)g^{-1}]^A \bar A^B + [(\bar\partial g) g^{-1}]^A A^B + A^A \bar A^B + \\ &+ \Lambda^A \partial \bar A^B - \Lambda^A \bar\partial A^B - \Lambda^A A^P \bar A^Q f_{QP}{}^B \Big] \delta_{BA} \ , \end{aligned} \tag{3.7}$$

and exploiting the above property (3.5), while being careful with the exchange of Graßmann odd quantities, it is possible to find the following equations of motion for the gauge fields

$$A^A = -\big\{[(\partial g)g^{-1}]^C - \partial\Lambda^C\big\}(M^{-1})_C{}^A \qquad\qquad \bar A^A = -\big\{[(\bar\partial g)g^{-1}]^C + \bar\partial\Lambda^C\big\}(N^{-1})_C{}^A \ . \tag{3.8}$$

Where we defined

$$\begin{cases} M_A{}^B := \delta_A{}^B + \Lambda^C f_{CA}{}^B \\ N_A{}^B := \delta_A{}^B - \Lambda^C f_{CA}{}^B \end{cases} \qquad \text{such that} \qquad \begin{cases} (M^{-1})_A{}^C M_C{}^B = \delta_A{}^B \\ (N^{-1})_A{}^C N_C{}^B = \delta_A{}^B \end{cases} \ . \tag{3.9}$$

[1] To properly define PCMs, and later on coset models, the (super)trace among generators of the Lie (super)algebra should be non-degenerate, graded-symmetric and Ad-invariant. The Cartan-Killing form, i.e. the (super)trace of the generators in the adjoint representation, is a common choice for such an inner product, but is not the only one available. Indeed, as discussed in footnote 24 at page 18-19 of [92] and papers therein, (super)trace of any other representation of generators would provide an equivalently good form. This remark is particularly useful in those cases where the Cartan-Killing form, proportional to the dual-Coxeter number, vanishes identically, such as the series $\mathfrak{sl}(n|n)$ and $\mathfrak{osp}(2n+2|2n)$. These have indeed vanishing Cartan-Killing form, but still fall in the class of *basic* classical Lie superalgebras, always admitting a non-degenerate, graded-symmetric and Ad-invariant bilinear form [93].

At this stage it is important to notice that the supermatrices $M_A{}^B$ and $N_A{}^B$ happen to be of even nature, i.e. they have bosonic fields in the diagonal blocks and fermionic fields in the off-diagonal ones. This feature can be recognised by analysing the index structure of the terms proportional to the Lagrange multipliers: a generic superalgebra is characterised by the fact that the commutator (anticommutator) of two bosonic (fermionic) generators is again bosonic, while the commutator of a bosonic and a fermionic element is fermionic. This property shows that the diagonal blocks $M_a{}^b$,$N_a{}^b$ and $M_\alpha{}^\beta$, $N_\alpha{}^\beta$ only contain bosonic multipliers Λ^c, as the structure constants $f_{\gamma a}{}^b$ and $f_{\gamma\alpha}{}^\beta$ vanish. On the other hand, the off-diagonal blocks $M_a{}^\beta$,$N_a{}^\beta$ and $M_\alpha{}^b$, $N_\alpha{}^B$ only contain fermionic multipliers Λ^γ as the structure constants $f_{ca}{}^\beta$ and $f_{c\alpha}{}^b$ vanish. This property is important in the dualisation procedure, since it immediately allows to recognise the partity of $M_A{}^B$,$N_A{}^B$ as being given by $A+B$. At the same time, since the inverse of an even supermatrix is again even the Graßmann parity of M^{-1},N^{-1} exhibit exactly the same structure. Additionally, it should be noted that invertibility of M and N is ensured by the presence of the identity matrix, which makes the determinant of their bosonic part non-vanishing. Substituting the equations of motion (3.8) back into the action (3.2) and exploiting the gauge freedom to set $\mathrm{g} = \mathbb{1}$ we obtain the dual model

$$\tilde{S}_{PCM} = \int_\Sigma \mathrm{d}^2 z\ \tilde{S}^{(1)} + \tilde{S}^{(2)} + \tilde{S}^{(3)} + \tilde{S}^{(4)}\ , \tag{3.10}$$

where

$$\begin{cases} \tilde{S}^{(1)} = -\partial\Lambda^C (M^{-1})_C{}^A \bar{\partial}\Lambda^D (N^{-1})_D{}^B \delta_{BA} \\ \tilde{S}^{(2)} = \partial\Lambda^A \bar{\partial}\Lambda^C (N^{-1})_C{}^B \delta_{BA} \\ \tilde{S}^{(3)} = \bar{\partial}\Lambda^A \partial\Lambda^C (M^{-1})_C{}^B \delta_{BA} \\ \tilde{S}^{(4)} = \Lambda^A \partial\Lambda^C (M^{-1})_C{}^D \bar{\partial}\Lambda^E (N^{-1})_E{}^F f_{FD}{}^B \delta_{BA} \end{cases} . \tag{3.11}$$

As a final step, exploiting the relation

$$\begin{aligned} (N^{-1})_A{}^C N_C{}^B = (N^{-1})_A{}^C (\delta_C{}^B - \Lambda^D f_{DC}{}^B) \equiv \delta_A{}^B \\ \Rightarrow \qquad (N^{-1})_A{}^B = \delta_A{}^B + (N^{-1})_A{}^C \Lambda^D f_{DC}{}^B \end{aligned} \tag{3.12}$$

on the term $\tilde{S}^{(1)}$ one can cancel again the terms $\tilde{S}^{(3)}$, $\tilde{S}^{(4)}$, thus obtaining the dual action

$$\tilde{S}_{PCM} = \int_\Sigma \mathrm{d}^2 z\, \partial\Lambda^A \bar{\partial}\Lambda^B (N^{-1})_B{}^C \delta_{CA} = \int_\Sigma \mathrm{d}^2 z\, \partial\Lambda^A \bar{\partial}\Lambda^B \tilde{g}_{BA} + \partial\Lambda^A \bar{\partial}\Lambda^B \tilde{B}_{BA}\ . \tag{3.13}$$

Where we have identified the dual metric and B-field with

$$\tilde{g}_{MN} = \tfrac{1}{2}[(N^{-1})_{MN} + (-1)^{MN}(N^{-1})_{NM}] \qquad \tilde{B}_{MN} = \tfrac{1}{2}[(N^{-1})_{MN} - (-1)^{MN}(N^{-1})_{NM}]\ , \tag{3.14}$$

and defined $(N^{-1})_{AB} := (N^{-1})_A{}^C\delta_{CB}$. The latter result looks, as desired and expected, like the generalised version of (2.32), which is indeed recovered by setting to zero the fermionic generators, i.e. by going back to a bosonic Lie algebra. We stress again that given the definition (3.9) of N, it would in principle be possible to write down the supermatrix explicitly and make use of computational techniques to find its inverse N^{-1}, immediately leading to the T-dual fields (3.14) upon extraction of the graded symmetric and antisymmetric components of the latter. Such type of inversion would however be affected by the loss of index structure characterising the model and underlying algebra, thus making harder the analysis of the T-dual model. Additionally, in the supersymmetric case, an explicit inversion of N would also be made harder by the presence of fermionic multipliers, and would thus require the use of specific computational tools[2].

3.2. Principal chiral model on OSp(1|2)

In this section we take into account the principal chiral model on OSp(1|2) and perform explicitly its T-dualisation. There are multiple reasons to start tackling concrete examples of super non-Abelian T-duality from such a model: its low dimensionality (3|2) certainly provides a relatively simple starting point and, even more importantly, it also enjoys very interesting physical properties which make of it an intriguing and rich playground. The supergroup manifold OSp(1|2) can indeed be interpreted as the supersymmetric extension of an AdS_3 background with the minimum amount of supersymmetry and satisfying the supergravity torsion constraints discussed in A.6.. This feature renders the model a proper three dimensional supergravity background, relevant in light of the successful exploitation of bosonic non-Abelian T-duality as a solution generating technique in supergravity. The possibility of describing super AdS_3 as a simple supergroup manifold represents a peculiarity of three dimensions, where the purely bosonic AdS_3 space might either be realised as the group manifold $SL(2,\mathbb{R})$ or the coset $SO(2,2)/SO(2,1)$. A similar feature has also been exploited in the case of the purely bosonic sphere S^3, which has been T-dualised from both the group manifold and coset perspective in [61] and [62]. The coset description of AdS_3 has been largely exploited in the supergravity literature, where a vast class of AdS superspaces with various amounts of supersymmetries has been realised as $AdS_{(3|p,q)} \simeq \frac{OSp(p|2)\times OSp(q|2)}{SL(2,\mathbb{R})\times SO(p)\times SO(q)}$ [94–97]. Another very interesting feature of the principal chiral model on OSp(1|2) is that the T-dual model to its purely bosonic part, namely the principal chiral model on $SL(2,\mathbb{R})$, can be interpreted as describing a three dimensional black hole [15]. This immediately rises the question of whether a possible supersymmetric extension of such interpretation might be attempted, i.e. whether or not the T-dual model to OSp(1|2) might be interpreted as a supersymmetric black hole directly realised in superspace. Last but not least, we shall see in the next chapter that the $\mathfrak{osp}(1|2)$ algebra admits

[2]For example, one may consider using the Mathematica package Grassmann, by Matthew Headrick.

a $\mathbb{Z}_4$ grading which allows the construction of another very interesting type of model, namely the semi-symmetric space $\mathsf{OSp}(1|2)/\mathsf{SO}(1,1)$. Before dualising $\mathsf{OSp}(1|2)$, we remark once again that one could in principle carry out the procedure by using the result of the previous section and exploiting computer algebra techniques to invert the supermatrix N in equation (3.9), thus obtaining the dual fields (3.14). The complete loss of the index structure resulting from such operation would however make it much harder to study the dual model, especially in this supersymmetric setting where the physical interpretation of the background in terms of a supergravity theory requires the study of geometric requirements such as the torsion constraints. For this reason we shall perform dualisation by analytically inverting the operators involved in the process.

3.2.1. Initial sigma model

Metric and vielbeine. To begin, we consider a convenient form of the $\mathfrak{osp}(1|2)$ algebra and define inner products among generators

$$\begin{gathered}\{Q_\alpha, Q_\beta\} = -i(\gamma^a)_{\alpha\beta} L_a \qquad [L_a, Q_\alpha] = -\tfrac{1}{2}(\gamma_a)_\alpha{}^\beta Q_\beta \qquad [L_a, L_b] = \varepsilon_{ab}{}^c L_c \\ STr[L_a L_b] = -\tfrac{1}{2}\eta_{ab} \qquad STr[Q_\alpha Q_\beta] = i\epsilon_{\alpha\beta}\ .\end{gathered} \tag{3.15}$$

Details on how to obtain this form of the algebra and about the conventions for the raising and lowering of indices can be found in B.1.. We can now start constructing explicitly the initial sigma model action (3.1) by taking the following parametrisation of $\mathsf{OSp}(1|2)$

$$g = e^{x^a L_a} e^{-\theta^\alpha Q_\alpha} = g_{bos} g_{fer} \qquad g^{-1} = e^{\theta^\alpha Q_\alpha} e^{-x^a L_a} = g_{fer}^{-1} g_{bos}^{-1}\ . \tag{3.16}$$

From this we can construct the current $j = g^{-1}dg$ as

$$\begin{aligned} j &= g_{fer}^{-1} g_{bos}^{-1} d(g_{bos} g_{fer}) = g_{fer}^{-1} g_{bos}^{-1} d(g_{bos}) g_{fer} + g_{fer}^{-1} dg_{fer} = \\ &= g_{fer}^{-1} j_{bos} g_{fer} + g_{fer}^{-1} dg_{fer} =: j_{bos} + j_{fer}\ .\end{aligned} \tag{3.17}$$

The above two contributions to j are derived in C.1. and read

$$\begin{cases} j_{fer} = \frac{i}{4}\theta^2 \lambda^a L_a - (1 - \frac{i}{8}\theta^2) d\theta^\alpha Q_\alpha + \frac{1}{2}\lambda_a \theta^\alpha (\gamma^a)_\alpha{}^\beta Q_\beta - \frac{i}{2}\theta^\alpha d\theta^\beta (\gamma^a)_{\alpha\beta} L_a \\ j_{bos} = dx^m \lambda_m{}^a L_a \end{cases}, \tag{3.18}$$

where λ^a are the Maurer-Cartan forms of $\mathsf{Sp}(2,\mathbb{R}) \simeq \mathsf{SL}(2,\mathbb{R})$, derived in C.2. and reported below. Rearranging $j = j_{bos} + j_{fer} =: j^a L_a + j^\alpha Q_\alpha$ we find the following expansion on the generators

$$j^a = (1 + \frac{i}{4}\theta^2)\big[\lambda^a - \frac{i}{2}\theta^\alpha d\theta^\beta (\gamma^a)_{\alpha\beta}\big] \qquad j^\alpha = -(1 - \frac{i}{8}\theta^2)\big[d\theta^\alpha - \frac{1}{2}\lambda^a \theta^\beta (\gamma_a)_\beta{}^\alpha\big]\ . \tag{3.19}$$

Substituting into the action and exploiting the inner products (3.15) we end up with

$$\begin{aligned} S_{PCM} &= \frac{1}{2}\int_\Sigma STr[j\wedge\star j] = \frac{1}{2}\int_\Sigma STr[j^A T_A\wedge\star j^B T_B] = \\ &= \frac{1}{2}\int_\Sigma (-1)^{AB} j^A\wedge\star j^B STr[T_A T_B] = \frac{1}{2}\int_\Sigma j^A\wedge\star j^B STr[T_B T_A] = \\ &= \frac{1}{2}\int_\Sigma \Big[-\frac{1}{2}\lambda^a\wedge\star\lambda_a - \frac{i}{4}\theta^\alpha(\gamma_a)_{\alpha\beta}\lambda^a\wedge\star d\theta^\beta + \\ &\qquad -\frac{i}{4}\theta^\alpha(\gamma_a)_{\alpha\beta}d\theta^\beta\wedge\star\lambda^a - i(1-\frac{i}{16}\theta^2)d\theta^\alpha\wedge\star d\theta_\alpha\Big] \\ &= \frac{1}{2}\int_\Sigma \mathrm{d}x^q\wedge\star\mathrm{d}x^p g_{pq} + \mathrm{d}x^p\wedge\star\mathrm{d}\theta^\sigma g_{\sigma p} + \mathrm{d}\theta^\sigma\wedge\star\mathrm{d}x^p g_{p\sigma} + \mathrm{d}\theta^\sigma\wedge\star\mathrm{d}\theta^\rho g_{\rho\sigma}\ , \end{aligned} \tag{3.20}$$

where, after introducing $R^2 := x^q x^p \eta_{pq}$ and some manipulations, we defined the metric components

$$\begin{cases} g_{pq} &= g_1\eta_{pq} + g_2 x_p x_q = g_{qp} \\ g_{p\sigma} &= \frac{i}{4}\theta^\alpha(\gamma_a)_{\alpha\sigma}\lambda^a_p = g_{\sigma p} \\ g_{\rho\sigma} &= i\epsilon_{\rho\sigma}(1-\frac{i\theta^2}{16}) = -g_{\sigma\rho} \end{cases} \qquad \text{with} \qquad \begin{cases} g_1 = \frac{1-\cosh(R)}{R^2} \\ g_2 = \frac{2[\cosh(R)-1]-R^2}{2R^4} \end{cases} . \tag{3.21}$$

The current j defines a set of orthonormal frames on the supermanifold

$$\begin{aligned} &S_{PCM} = \frac{1}{2}\int_\Sigma \mathrm{d}x^N\wedge\star\mathrm{d}x^M g_{MN} = \frac{1}{2}\int_\Sigma j^A\wedge\star j^B\delta_{BA} \\ &\text{with}\qquad \delta_{BA} = \begin{pmatrix} -\frac{1}{2}\eta_{ba} & 0 \\ 0 & i\epsilon_{\beta\alpha}\end{pmatrix} \qquad j = \mathrm{d}x^M {j_M}^A T_A\ , \end{aligned} \tag{3.22}$$

and for later use one can write down their components and those of the inverse frame fields

$$ {j_M}^A {j_A}^N = {\delta_M}^N \qquad {j_A}^M {j_M}^B = {\delta_A}^B \qquad \text{with} \tag{3.23}$$

$$\begin{aligned} &{j_m}^a = (1+\tfrac{i\theta^2}{4}){\lambda_m}^a && {j_\mu}^a = \tfrac{i}{2}\theta^\nu(\gamma^a)_{\nu\mu} && {j_m}^\alpha = \tfrac{1}{2}{\lambda_m}^a\theta^\nu{(\gamma_a)_\nu}^\alpha && {j_\mu}^\alpha = -(1-\tfrac{i\theta^2}{8}){\delta_\mu}^\alpha \\ &{j_a}^m = {(\lambda^{-1})_a}^m && {j_a}^\mu = \tfrac{1}{2}\theta^\nu{(\gamma_a)_\nu}^\mu && {j_\alpha}^m = \tfrac{i}{2}\theta^\nu(\gamma^a)_{\nu\alpha}{(\lambda^{-1})_a}^m && {j_\alpha}^\mu = -(1-\tfrac{i\theta^2}{4}){\delta_\alpha}^\mu \end{aligned}$$

Where the SL(2, ℝ) frame fields and their inverse read

$$ {\lambda_m}^a = l_1{\delta_m}^a + l_2 x_m x^a + l_3 x^c {\varepsilon_{cm}}^a \qquad {(\lambda^{-1})_a}^m = l_4{\delta_a}^m + l_5 x_a x^m + l_6 x^c{\varepsilon_{ca}}^m\ . \tag{3.24}$$

They satisfy ${\lambda_m}^a{(\lambda^{-1})_a}^n = {\delta_m}^n$ and ${(\lambda^{-1})_a}^m{\lambda_m}^b = {\delta_a}^b$ with coefficients

$$\begin{aligned} l_1 &= \frac{\sinh R}{R} & l_2 &= \frac{R-\sinh R}{R^3} & l_3 &= -\frac{2\sinh^2(R/2)}{R^2} \\ l_4 &= \frac{R}{2}\coth(R/2) & l_5 &= \frac{2-R\coth(R/2)}{2R^2} & l_6 &= \frac{1}{2}\ . \end{aligned} \tag{3.25}$$

Isometries and their realisation. In this paragraph we construct explicitly the Noether currents and Killing vectors associated to the $\mathrm{G}_L \times \mathrm{G}_R$ isometry group of the principal chiral model on OSp(1|2). In C.3. we discuss Noether currents for generic principal models, while in C.4. and C.5. we focus on OSp(1|2), first deriving the left Noether current and successively the full set of Killing vectors. From the action (3.20) one can extract the Noether currents associated to the left and right group action

$$\begin{array}{lllll} \mathrm{G}_L: & \mathrm{g} \to g_L^{-1}\,\mathrm{g} & \Rightarrow & \mathrm{L}_N = -\mathrm{g}\, j\, \mathrm{g}^{-1} = -(\mathrm{dg})\mathrm{g}^{-1} \\ \mathrm{G}_R: & \mathrm{g} \to \mathrm{g}\, g_R & \Rightarrow & \mathrm{R}_N = j = \mathrm{g}^{-1}\mathrm{dg} \end{array} . \tag{3.26}$$

One can then generalise to the superspace setting the result of [35], relating the Noether currents for the principal chiral model to Killing vectors. Under an infinitesimal transformation

$$\delta_\epsilon x^M = \epsilon^V \xi_V{}^M \ , \tag{3.27}$$

generated by a Killing vector $\xi_V = \xi_V{}^M \partial_M$ associated to the generator V of the isometry algebra, the variations of the action read

$$\delta_\epsilon S_{PCM} = \int_\Sigma \mathrm{d}\epsilon^V \wedge \star \xi_V{}^N \mathrm{d}x^M g_{MN} = -\int_\Sigma \epsilon^V \wedge \mathrm{d}\Big(\star \xi_V{}^N \mathrm{d}x^M g_{MN} \Big) \ , \tag{3.28}$$

so that the Noether current associated to the generator V takes the form

$$J_V := \langle J, V \rangle = \xi_V{}^N \mathrm{d}x^M g_{MN} \ . \tag{3.29}$$

Computing explicitly the Noether currents one can then extract the components of the Killing vectors by inverting the latter formula. From (3.26) and (3.19) we immediately find the current associated to the right sector of the isometry group, namely $\mathrm{OSp}(1|2)_R$

$$\mathrm{R}_N = \mathrm{R}_N{}^a L_a + \mathrm{R}_N{}^\alpha Q_\alpha \qquad \text{with} \qquad \begin{cases} \mathrm{R}_N{}^a = (1 + \frac{i}{4}\theta^2)\big[\lambda^a - \frac{i}{2}\theta^\alpha \mathrm{d}\theta^\beta (\gamma^a)_{\alpha\beta}\big] \\ \mathrm{R}_N{}^\alpha = -(1 - \frac{i}{8}\theta^2)\big[\mathrm{d}\theta^\alpha - \frac{1}{2}\lambda^a \theta^\rho (\gamma_a)_\rho{}^\alpha\big] \end{cases} . \tag{3.30}$$

For the current associated to the left sector $\mathrm{OSp}(1|2)_L$ we have

$$\mathrm{L}_N = -(\mathrm{dg})\mathrm{g}^{-1} = \mathrm{g}\mathrm{dg}^{-1} = \mathrm{g}_{bos}\mathrm{dg}^{-1}_{bos} + \mathrm{g}_{bos}(\mathrm{g}_{fer}\mathrm{dg}^{-1}_{fer})\mathrm{g}^{1}_{bos} = (\mathrm{L}_N)_{bos} + (\mathrm{L}_N)_{fer} \tag{3.31}$$

and the bosonic piece can be simply extracted by letting $x \to -x$ in λ^a with components (3.24)

$$(\mathrm{L}_N)_{bos} = \mathrm{g}_{bos}(\mathrm{dg}^{-1}_{bos}) = \lambda^a[x \to -x] L_a = \mathrm{d}x^m[-l_1 \delta_m{}^a - l_2 x_m x^a + l_3 x^c \varepsilon_{cm}{}^a] L_a \ . \tag{3.32}$$

On the other hand, to get the fermionic piece one first needs to compute

$$\mathrm{g}_{fer}\mathrm{d}\mathrm{g}_{fer}^{-1} = \frac{i}{2}\mathrm{d}\theta^\beta\theta^\alpha(\gamma^a)_{\alpha\beta}L_a + (1-\frac{i}{8}\theta^2)\mathrm{d}\theta^\alpha Q_\alpha \ , \tag{3.33}$$

and can then proceed by exploiting the relation $e^A B e^{-A} = \sum_{k=0}^{\infty}\frac{1}{k!}\mathrm{ad}_A^k(B)$ to find

$$\begin{aligned} \mathrm{g}_{bos}Q_\alpha\mathrm{g}_{bos}^{-1} &= \big[\cosh(R/2)\delta_\alpha{}^\beta - \frac{\sinh(R/2)}{R}x^a(\gamma_a)_\alpha{}^\beta\big]Q_\beta \\ \mathrm{g}_{bos}L_a\mathrm{g}_{bos}^{-1} &= \big[\cosh(R)\delta_a^b + \frac{1-\cosh R}{R^2}x_a x^b + \frac{\sinh R}{R}x^c\varepsilon_{ca}{}^b\big]L_b \end{aligned} \ . \tag{3.34}$$

Putting the above pieces together one finally obtains $\mathrm{L}_N = \mathrm{L}_N{}^a L_a + \mathrm{L}_N{}^\alpha Q_\alpha$ with

$$\begin{cases} \mathrm{L}_N{}^a = \lambda^a[x\to -x] + \frac{i}{2}\mathrm{d}\theta^\beta\theta^\alpha(\gamma^b)_{\alpha\beta}[\cosh R\delta_b{}^a + \frac{1-\cosh R}{R^2}x_b x^a + \frac{\sinh R}{R}x^c\varepsilon_{cb}{}^a] \\ \mathrm{L}_N{}^\alpha = \mathrm{d}\theta^\beta(1-\frac{i}{8}\theta^2)[\cosh(R/2)\delta_\beta{}^\alpha - \frac{\sinh(R/2)}{R}x^a(\gamma_a)_\beta{}^\alpha] \end{cases} \ . \tag{3.35}$$

At this stage one can apply the relation (3.29) to both the left and right Noether currents to extract the two sets of Killing vectors

$$\begin{cases} \xi^R_{L_a} = (A\delta_a{}^c + Bx_a x^c + \frac{1}{2}x^d\varepsilon_{da}{}^c)\partial_c + \frac{1}{2}\theta^\rho(\gamma_a)_\rho{}^\lambda\partial_\lambda \\ \xi^R_{Q_\alpha} = \frac{i}{2}\theta^\rho(\gamma^a)_{\rho\alpha}(A\delta_a{}^c + Bx_a x^c + \frac{1}{2}x^d\varepsilon_{da}{}^c)\partial_c - (1-\frac{i}{4}\theta^2)\partial_\alpha \end{cases}$$

$$\begin{cases} \xi^L_{L_a} = -(A\delta_a{}^c + Bx_a x^c - \frac{1}{2}x^d\varepsilon_{da}{}^c)\partial_c \\ \xi^L_{Q_\alpha} = \big[\frac{i}{2}N\theta_\alpha x^c - \frac{i}{4}\theta^\lambda(\gamma^b)_{\lambda\alpha}(U\delta_b{}^c + Vx_b x^c)\big]\partial_c - (1+\frac{i}{8}\theta^2)(Nx^a(\gamma_a)_\alpha{}^\sigma - M\delta_\alpha{}^\sigma)\partial_\sigma \end{cases} \tag{3.36}$$

where we defined coefficients

$$\begin{aligned} &A = \frac{R}{2}\coth(R/2) &&B = \frac{2-R\coth(R/2)}{2R^2} &&M = \cosh(R/2) \\ &N = -\frac{\sinh(R/2)}{R} &&U = -R\,\mathrm{csch}(R/2) &&V = \frac{R-\sinh R}{R^2}\mathrm{csch}(R/2) \ . \end{aligned} \tag{3.37}$$

It is possible to check that the latter two sets of vector fields correctly satisfy the $\mathfrak{osp}(1|2)$ algebra

$$\begin{aligned} &[\xi^R_{L_a},\xi^R_{L_b}] = \varepsilon_{ab}{}^c\xi^R_{L_c} &&[\xi^L_{L_a},\xi^L_{L_b}] = \varepsilon_{ab}{}^c\xi^L_{L_c} \\ &[\xi^R_{L_a},\xi^R_{Q_\alpha}] = -\frac{1}{2}(\gamma_a)_\alpha{}^\beta\xi^R_{Q_\beta} &&[\xi^L_{L_a},\xi^L_{Q_\alpha}] = -\frac{1}{2}(\gamma_a)_\alpha{}^\beta\xi^L_{Q_\beta} \\ &[\xi^R_{Q_\alpha},\xi^R_{Q_\beta}] = -i(\gamma^a)_{\alpha\beta}\xi^R_{L_a} &&[\xi^L_{Q_\alpha},\xi^L_{Q_\beta}] = -i(\gamma^a)_{\alpha\beta}\xi^L_{L_a} \end{aligned} \tag{3.38}$$

and commute as required for the $G_L \times G_R$ isometry to be correctly realised

$$[\xi^R_{L_a}, \xi^L_{L_b}] = 0 \qquad [\xi^R_{L_a}, \xi^L_{Q_\alpha}] = 0 = [\xi^L_{L_a}, \xi^R_{Q_\alpha}] \qquad \{\xi^R_{Q_\alpha}, \xi^L_{Q_\beta}\} = 0 \,. \tag{3.39}$$

Finally, both sets of vectors satisfy the graded Killing equation derived in A.2.

$$(\mathscr{L}_{\xi_V} g)_{AB} = (-1)^{V(A+B)}\xi^C_V(\partial_C g_{AB}) + (-1)^{VB}(\partial_A \xi^C_V)g_{CB} + (-1)^{A(B+C+V)}(\partial_B \xi^C_V)g_{AC} \,. \tag{3.40}$$

Supergravity constraints. These represent a set of conditions that any supergeometry should satisfy to be understood as an appropriate supergravity background. Originally introduced to connect the superspace approach to supergravity with the component one [98–101], they were successively re-derived in various occasions and shown to be implied by κ-symmetry of the Green-Schwarz action, describing the motion of a superstring on a supergravity background [102, 103, 78]. Here we shall show that, in spite of its simplicity, the principal chiral model on OSp(1|2) satisfies the supergravity torsion constraints described in A.6., hence representing an appropriate supergravity background. In the next section and next chapter we shall then study whether the T-dual model satisfies as well the supegravity constraints, i.e. whether or not dualisation preserves them. While κ-symmetry will not be considered in the context of principal chiral models, it will be discussed in the next chapter for semi-symmetric space sigma models. To see how the OSp(1|2) principal chiral model satisfies the supergravity torsion constraints in A.6. it is sufficient to recall that on a group manifold the structure functions $\mathcal{F}_{AB}{}^C$ are constant and for the case at hand take the form (B.11)

$$f_{ab}{}^c = \varepsilon_{ab}{}^c \qquad f_{a\beta}{}^\gamma = -\tfrac{1}{2}(\gamma_a)_\beta{}^\gamma \qquad f_{\beta a}{}^\gamma = \tfrac{1}{2}(\gamma_a)_\beta{}^\gamma \qquad f_{\alpha\beta}{}^c = -i(\gamma^c)_{\alpha\beta} \,. \tag{3.41}$$

The constraints on the structure functions (A.38) are thus automatically satisfied with $f_{\alpha b}{}^c = 0 = f_{\alpha\beta}{}^\gamma$ and $k_1 = -1$ and the non-vanishing components of connection and torsion are

$$\begin{aligned}
\Omega_{ab}{}^c &= \tfrac{1}{2}\big[f_{ab}{}^c + \delta^{cd}(f_{dab} + f_{dba})\big] = \tfrac{1}{2}\varepsilon_{ab}{}^c \\
\Omega_{a\beta}{}^\gamma &= \Omega_{abc}[\gamma^b, \gamma^c]_\beta{}^\gamma = (\gamma_a)_\beta{}^\gamma \\
T_{\alpha\beta}{}^c &= k_1 f_{\alpha\beta}{}^c = i(\gamma^c)_{\alpha\beta} \\
T_{a\beta}{}^\gamma &= -f_{a\beta}{}^\gamma + \Omega_{a\beta}{}^\gamma = \tfrac{3}{2}(\gamma_a)_\beta{}^\gamma \,,
\end{aligned} \tag{3.42}$$

where we exploited the identities for 3d gamma matrices reported in B.1., together with the relation $(-1)^C \delta_{AC}\delta^{CB} = \delta_A{}^B$ and

$$\delta_{AC} = \begin{pmatrix} -\frac{1}{2}\eta_{ac} & 0 \\ 0 & i\epsilon_{\alpha\gamma} \end{pmatrix} \qquad \delta^{CB} = \begin{pmatrix} -2\eta^{cb} & 0 \\ 0 & i\epsilon^{\gamma\beta} \end{pmatrix} \,. \tag{3.43}$$

3.2.2. T-dual model

We perform explicitly T-dualisation of the principal chiral model on OSp(1|2) with respect to the G_L part of the isometry group. We first consider dualisation of the maximal bosonic subgroup $SL(2,\mathbb{R})_L$ and successively of the full $OSp(1|2)_L$, showing that in both cases the $OSp(1|2)_R$ part of the initial isometry group is preserved and explicitly constructing its realisation. We then try to solve the torsion constraints by proposing an ansatz for the vielbeine which encompasses the three models under consideration, namely the principal chiral model, the dual with respect to $SL(2,\mathbb{R})_L$ and the dual with respect to $OSp(1|2)_L$. This allows to study the torsion constraints for the dual models while using the principal chiral model as a sanity check of the procedure and leads to the conclusion that the constraints cannot be satisfied within this framework. Such result, together with the complexity of the argument, hints toward the need for a more general treatment of the dualisation procedure, that may allow to take into account more complete ansatze and provide a clearer picture. This is the goal of the next chapter.

Bosonic dualisation. In this paragraph we perform dualisation with respect to the bosonic subgroup $SL(2,\mathbb{R})_L \subset OSp(1|2)_L$. We shall only highlight the procedure, as a more thorough treatment of it will be given in the next paragraph, dealing with dualisation of the full $OSp(1|2)_L$. The starting point is again the gauged action (3.2)

$$S^{MCG}_{PCM} = \int_\Sigma \mathrm{d}^2 z\, STr[(\mathrm{g}^{-1}D\mathrm{g})(\mathrm{g}^{-1}\bar{D}\mathrm{g})] + STr[\Lambda F]\ . \tag{3.44}$$

Recalling the useful choice of parametrisation for the group element $\mathrm{g} = \mathrm{g}_{bos}\mathrm{g}_{fer}$, the left action of $SL(2,\mathbb{R})$ reads $\mathrm{g} \to h^{-1}_{bos}\mathrm{g}$, hence the gauging goes through as in the previous section with purely bosonic gauge fields and Lagrange multipliers transforming as $A_{bos} \to h^{-1}_{bos}A_{bos}h_{bos} - (\partial h^{-1}_{bos})h_{bos}$ and $\Lambda_{bos} \to h^{-1}_{bos}\Lambda_{bos}h_{bos}$. Upon fixing the gauge $\mathrm{g}_{bos} = \mathbb{1}$ one is left with $\mathrm{g} = \mathrm{g}_{fer}$ and the action can be rearranged as

$$\begin{aligned} S^{MCG}_{PCM} = \int_\Sigma \mathrm{d}^2 z\, STr[&(\mathrm{g}^{-1}_{fer}\partial\mathrm{g}_{fer})(\mathrm{g}^{-1}_{fer}\bar{\partial}\mathrm{g}_{fer}) + (\partial\mathrm{g}_{fer})\mathrm{g}^{-1}_{fer}\bar{A}_{bos} + (\bar{\partial}\mathrm{g}_{fer})\mathrm{g}^{-1}_{fer}A_{bos} + A_{bos}\bar{A}_{bos}]+ \\ &+ STr[-\partial\Lambda_{bos}\bar{A}_{bos} + \bar{\partial}\Lambda_{bos}A_{bos} + \Lambda_{bos}A^a_{bos}\bar{A}^b_{bos}f_{ab}{}^c L_c]\ , \end{aligned} \tag{3.45}$$

so that upon using (3.15) and expanding the multipliers as $\Lambda_{bos} := \tilde{x}^a L_a$, the equations of motion for the gauge fields are easily computed

$$\begin{cases} \bar{A}^a_{bos} = -[(\bar{\partial}\mathrm{g}_{fer}\mathrm{g}^{-1}_{fer})^b + (\bar{\partial}\tilde{x}^b)](M^{-1})_b{}^a \\ A^a_{bos} = -[(\partial\mathrm{g}_{fer}\mathrm{g}^{-1}_{fer})^b - (\partial\tilde{x}^b)](N^{-1})_b{}^a \end{cases}, \tag{3.46}$$

with matrices M, N and their inverses defined as in (D.3)

$$\begin{cases} M_c{}^a := \delta_c{}^a - \varepsilon_c{}^{ab}\tilde{x}_b \quad \Rightarrow \quad (M^{-1})_a{}^k = \frac{1}{1-r^2}(\delta_a{}^k - \tilde{x}_a\tilde{x}^k + \varepsilon_a{}^{kn}\tilde{x}_n) \\ N_c{}^a := \delta_c{}^a + \varepsilon_c{}^{ab}\tilde{x}_b \quad \Rightarrow \quad (N^{-1})_a{}^k = \frac{1}{1-r^2}(\delta_a{}^k - \tilde{x}_a\tilde{x}^k - \varepsilon_a{}^{kn}\tilde{x}_n) \end{cases} \tag{3.47}$$
$$\text{with} \qquad r^2 := \tilde{x}^b\tilde{x}^a\eta_{ab} \; .$$

Substituting the equations back into the action and using the result (3.33)

$$\begin{cases} \mathrm{g}_{fer}^{-1}\mathrm{d}\mathrm{g}_{fer} = \frac{i}{2}\mathrm{d}\theta^\beta\theta^\alpha(\gamma^a)_{\alpha\beta}L_a - (1 - \frac{i\theta^2}{8})\mathrm{d}\theta^\alpha Q_\alpha \\ (\mathrm{d}\mathrm{g}_{fer})\mathrm{g}_{fer}^{-1} = -\frac{i}{2}\mathrm{d}\theta^\beta\theta^\alpha(\gamma^a)_{\alpha\beta}L_a - (1 - \frac{i\theta^2}{8})\mathrm{d}\theta^\alpha Q_\alpha \end{cases} , \tag{3.48}$$

together with the inner products (3.15) and the identities (D.10) (D.22), the dual action reads

$$\tilde{S} = \frac{1}{2}\int_\Sigma \mathrm{d}\Lambda^N \wedge \star\mathrm{d}\Lambda^M\tilde{g}_{MN} + \mathrm{d}\Lambda^N \wedge \mathrm{d}\Lambda^M\tilde{B}_{MN} \qquad \text{with} \qquad \begin{cases} \Lambda^n \equiv \tilde{x}^n \\ \Lambda^\nu \equiv \theta^\nu \end{cases} . \tag{3.49}$$

The dual metric and B-field take the explicit form

$$\begin{cases} \tilde{g}_{mn} = L_1[\eta_{mn} - \tilde{x}_m\tilde{x}_n] \\ \tilde{g}_{m\nu} = L_3\varepsilon_{mdk}(\gamma^d)_{\nu\lambda}\tilde{x}^k\theta^\lambda \\ \tilde{g}_{\mu\nu} = i\epsilon_{\mu\nu}(1 + i\theta^2L_2) \end{cases} \qquad \begin{cases} \tilde{B}_{mn} = L_1\varepsilon_{mnk}\tilde{x}^k \\ \tilde{B}_{m\nu} = L_3[\eta_{mk} - \tilde{x}_m\tilde{x}_k](\gamma^k)_{\nu\lambda}\theta^\lambda \\ \tilde{B}_{\mu\nu} = -\frac{1}{4}L_1\tilde{x}_k(\gamma^k)_{\mu\nu}\theta^2 \end{cases} , \tag{3.50}$$

with coefficients

$$L_1 := \frac{1}{2(r^2-1)} \qquad L_2 := -\frac{1}{8}\frac{(r^2-2)}{(r^2-1)} \qquad L_3 := \frac{i}{4(r^2-1)} \; . \tag{3.51}$$

Full dualisation. We move to the dualisation of the full $\mathrm{OSp}(1|2)_L$ isometry subgroup starting from the usual minimally coupled gauged action (3.2), which now also involves fermionic gauge fields and Lagrange multipliers. We give here the main steps and refer to D.1. for more details. Expanding the multipliers as $\Lambda := \tilde{x}^a L_a + \tilde{\theta}^\alpha Q_\alpha$ and using (3.15) the Lagrangian takes the following explicit form

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}[\mathrm{g}^{-1}\partial\mathrm{g}]_a[\mathrm{g}^{-1}\bar{\partial}\mathrm{g}]^a - \frac{1}{2}[(\partial\mathrm{g})\mathrm{g}^{-1}]_a\bar{A}^a - \frac{1}{2}[(\bar{\partial}\mathrm{g})\mathrm{g}^{-1}]_aA^a - \frac{1}{2}A_a\bar{A}^a + \\ &+ \frac{1}{2}(\partial\tilde{x}_a)\bar{A}^a - \frac{1}{2}(\bar{\partial}\tilde{x}_a)A^a - \frac{i}{2}\tilde{x}_aA^\alpha\bar{A}^\beta(\gamma^a)_{\beta\alpha} + \frac{1}{2}\tilde{x}_aA^c\bar{A}^d\varepsilon_{dc}{}^a + \\ &+ i[\mathrm{g}^{-1}\partial\mathrm{g}]_\beta[\mathrm{g}^{-1}\bar{\partial}\mathrm{g}]^\beta + i[(\partial\mathrm{g})\mathrm{g}^{-1}]_\beta\bar{A}^\beta + i[(\bar{\partial}\mathrm{g})\mathrm{g}^{-1}]_\beta A^\beta + iA_\beta\bar{A}^\beta + \\ &- i(\partial\tilde{\theta}_\beta)\bar{A}^\beta + i(\bar{\partial}\tilde{\theta}_\beta)A^\beta + \frac{i}{2}\tilde{\theta}_\beta A^\alpha\bar{A}^a(\gamma_a)_\alpha^\beta - \frac{i}{2}\tilde{\theta}_\beta A^a\bar{A}^\alpha(\gamma_a)_\alpha{}^\beta \; . \end{aligned} \tag{3.52}$$

One can thus proceed by calculating the equations of motion for the gauge fields

$$\bar{A}^\alpha = \left[\frac{1}{2}\big([(\bar\partial \mathrm{g})\mathrm{g}^{-1}]^b + \bar\partial\tilde{x}^b\big)(M^{-1})_b{}^a(\gamma_a)^{\rho\sigma}\tilde\theta_\sigma - [(\bar\partial \mathrm{g})\mathrm{g}^{-1}]^\rho - \bar\partial\tilde\theta^\rho\right](W_1^{-1})_\rho{}^\alpha$$
$$A^\alpha = \left[\frac{1}{2}\big(-[(\partial \mathrm{g})\mathrm{g}^{-1}]^b + \partial\tilde{x}^b\big)(N^{-1})_b{}^a(\gamma_a)^{\rho\sigma}\tilde\theta_\sigma - [(\partial \mathrm{g})\mathrm{g}^{-1}]^\rho + \partial\tilde\theta^\rho\right](W_2^{-1})_\rho{}^\alpha \tag{3.53}$$
$$\bar{A}^a = \Bigg[-[(\bar\partial \mathrm{g})\mathrm{g}^{-1}]^b - \bar\partial\tilde{x}^b - i\Big([(\bar\partial \mathrm{g})\mathrm{g}^{-1}]^\rho + \bar\partial\tilde\theta^\rho\Big)(W_1^{-1})_\rho{}^\alpha(\gamma^b)_\alpha{}^\beta\tilde\theta_\beta +$$
$$+\frac{i}{2}\Big([(\bar\partial \mathrm{g})\mathrm{g}^{-1}]^c + \bar\partial\tilde{x}^c\Big)(M^{-1})_c{}^d(\gamma_d)^{\rho\sigma}\tilde\theta_\sigma(W_1^{-1})_\rho{}^\alpha(\gamma^b)_\alpha^\beta\tilde\theta_\beta\Bigg](M^{-1})_b{}^a$$
$$A^a = \Bigg[-[(\partial \mathrm{g})\mathrm{g}^{-1}]^b + \partial\tilde{x}^b + i\Big([(\partial \mathrm{g})\mathrm{g}^{-1}]^\rho - \partial\tilde\theta^\rho\Big)(W_2^{-1})_\rho{}^\alpha(\gamma^b)_\alpha{}^\beta\tilde\theta_\beta +$$
$$+\frac{i}{2}\Big([(\partial \mathrm{g})\mathrm{g}^{-1}]^c - \partial\tilde{x}^c\Big)(N^{-1})_c{}^d(\gamma_d)^{\rho\sigma}\tilde\theta_\sigma(W_2^{-1})_\rho{}^\alpha(\gamma^b)_\alpha{}^\beta\tilde\theta_\beta\Bigg](N^{-1})_b{}^a\ .$$

To obtain the latter we defined and inverted matrices M, N, as for the dualisation with respect to $\mathsf{SL}(2,\mathbb{R})_L$ in (3.47), and W_1, W_2, which also involve fermionic multipliers. These read explicitly

$$\begin{cases}(W_1)_\beta{}^\alpha := \delta_\beta{}^\alpha(1 + A\tilde\theta^2) + \frac{1}{2}\tilde{x}_a(\gamma^a)_\beta{}^\alpha(1 + B\tilde\theta^2)\\ (W_2)_\beta{}^\alpha := \delta_\beta{}^\alpha(1 + A\tilde\theta^2) - \frac{1}{2}\tilde{x}_a(\gamma^a)_\beta{}^\alpha(1 + B\tilde\theta^2)\end{cases}\quad,\quad \text{with}\quad \begin{cases}A := \frac{i}{4}\frac{(r^2-3)}{(r^2-1)}\\ B := \frac{i}{r^2-1}\\ \tilde\theta^2 := \tilde\theta^\alpha\tilde\theta_\alpha\end{cases}\quad, \tag{3.54}$$

and using the identities in B.1. can be inverted as

$$\begin{cases}(W_1^{-1})_\alpha{}^\rho = \frac{4}{(4-r^2)}\big[\delta_\alpha{}^\rho(1 + C\tilde\theta^2) - \frac{1}{2}\tilde{x}_c(\gamma^c)_\alpha{}^\rho(1 + D\tilde\theta^2)\big]\\ (W_2^{-1})_\alpha{}^\rho = \frac{4}{(4-r^2)}\big[\delta_\alpha{}^\rho(1 + C\tilde\theta^2) + \frac{1}{2}\tilde{x}_c(\gamma^c)_\alpha{}^\rho(1 + D\tilde\theta^2)\big]\end{cases}\ . \tag{3.55}$$

Where we defined coefficients

$$C := \frac{i}{4}\frac{(-12 - 7r^2 + r^4)}{(r^2-1)(r^2-4)}\qquad\qquad D := i\frac{(-10 + r^2)}{(r^2-1)(r^2-4)}\ . \tag{3.56}$$

Substituting the equations of motion (3.53) back into (3.52), choosing the gauge $\mathrm{g} = \mathbb{1}$ and performing some manipulations, which mainly involve the identities (B.8) and (B.7), one lands on the following dual action

$$\tilde{S}_{PCM} = \int_\Sigma \mathrm{d}^2 z\ (\partial\tilde{x}^q)(\bar\partial\tilde{x}^p)\tilde{L}_{pq} + (\partial\tilde{x}^q)(\bar\partial\tilde\theta^\rho)\tilde{L}_{\rho q} + (\partial\tilde\theta^\sigma)(\bar\partial\tilde{x}^p)\tilde{L}_{p\sigma} + (\partial\tilde\theta^\sigma)(\bar\partial\tilde\theta^\rho)\tilde{L}_{\rho\sigma}\ , \tag{3.57}$$

with explicit components

$$\begin{aligned}
\tilde{L}_{pq} &= L_1(r^2)\Big[\eta_{pq}\big(1+ih(r^2)\tilde{\theta}^2\big) - \tilde{x}_p\tilde{x}_q\big(1+im(r^2)\tilde{\theta}^2\big) + \varepsilon_{pqk}\tilde{x}^k\big(1+in(r^2)\tilde{\theta}^2\big)\Big] \\
\tilde{L}_{\rho\sigma} &= L_2(r^2)\Big[-\epsilon_{\rho\sigma}\big(1+if(r^2)\tilde{\theta}^2\big) - \frac{1}{2}\tilde{x}_a(\gamma^a)_{\rho\sigma}\big(1+ig(r)\tilde{\theta}^2\big)\Big] \\
\tilde{L}_{p\sigma} &= L_3(r^2)\Big[(2+r^2)(\gamma_p)_{\sigma\alpha} - 3\tilde{x}_a(\gamma^a)_{\sigma\alpha}\tilde{x}_p - (1-r^2)\tilde{x}_p\epsilon_{\sigma\alpha} + 3\varepsilon_{pab}(\gamma^a)_{\sigma\alpha}\tilde{x}^b\Big]\tilde{\theta}^\alpha \\
\tilde{L}_{\rho q} &= L_3(r^2)\Big[-(2+r^2)(\gamma_q)_{\rho\alpha} + 3\tilde{x}_a(\gamma^a)_{\rho\alpha}\tilde{x}_q - (1-r^2)\tilde{x}_q\epsilon_{\rho\alpha} + 3\varepsilon_{qab}(\gamma^a)_{\rho\alpha}\tilde{x}^b\Big]\tilde{\theta}^\alpha \ ,
\end{aligned} \tag{3.58}$$

involving the following quantities

$$\begin{aligned}
&L_1(r^2) := \frac{1}{2(r^2-1)} && L_2(r^2) := \frac{4i}{(r^2-4)} && L_3(r^2) := \frac{-i}{(r^2-1)(r^2-4)} \\
&h(r^2) := \frac{2(1+2r^2)}{(r^2-1)(r^2-4)} && m(r^2) := \frac{-2(r^2-4)}{(r^2-1)(r^2-4)} && n(r^2) := \frac{5+r^2}{(r^2-1)(r^2-4)} \\
&f(r^2) := \frac{-12-7r^2+r^4}{4(r^2-1)(r^2-4)} && g(r^2) := \frac{-10+r^2}{(r^2-1)(r^2-4)} \ .
\end{aligned} \tag{3.59}$$

We can finally extract the dual metric and B-field as the graded symmetric and antisymmetric components of the above expressions $\tilde{g}_{MN} = \frac{1}{2}(\tilde{L}_{MN} + (-1)^{MN}\tilde{L}_{NM})$ and $\tilde{B}_{MN} = \frac{1}{2}(\tilde{L}_{MN} - (-1)^{MN}\tilde{L}_{MN})$. This leads to the following metric

$$\begin{cases}
\tilde{g}_{pq} = L_1(r^2)\big[\eta_{pq}\big(1+ih(r^2)\tilde{\theta}^2\big) - \tilde{x}_p\tilde{x}_q\big(1+im(r^2)\tilde{\theta}^2\big)\big] = \tilde{g}_{qp} \\
\tilde{g}_{p\sigma} = L_3(r^2)\big[-(1-r^2)\tilde{x}_p\epsilon_{\sigma\alpha} + 3\varepsilon_{pab}(\gamma^a)_{\sigma\alpha}\tilde{x}^b\big]\tilde{\theta}^\alpha = \tilde{g}_{\sigma p} \\
\tilde{g}_{\rho\sigma} = -L_2(r^2)\epsilon_{\rho\sigma}\big(1+if(r^2)\tilde{\theta}^2\big) = -\tilde{g}_{\sigma\rho}
\end{cases} \tag{3.60}$$

and B-field

$$\begin{cases}
\tilde{B}_{pq} = L_1(r^2)\varepsilon_{pqk}\tilde{x}^k\big(1+in(r^2)\tilde{\theta}^2\big) = -\tilde{B}_{qp} \\
\tilde{B}_{p\sigma} = L_3(r^2)\big[(2+r^2)(\gamma_p)_{\sigma\alpha} - 3\tilde{x}_a(\gamma^a)_{\sigma\alpha}\tilde{x}_p\big]\tilde{\theta}^\alpha = -\tilde{B}_{\sigma p} \\
\tilde{B}_{\rho\sigma} = -\frac{1}{2}L_2(r^2)\tilde{x}_a(\gamma^a)_{\rho\sigma}\big(1+ig(r)\tilde{\theta}^2\big) = \tilde{B}_{\sigma\rho}
\end{cases} \ . \tag{3.61}$$

Residual isometries. In general, only the isometries which commute with the gauged ones survive dualisation [91, 90]. Hence, given the $\mathrm{OSp}(1|2)_L \times \mathrm{OSp}(1|2)_R$ isometry group of the starting principal chiral model and the gauging of the left sector for the T-dual models derived in the previous paragraphs, we should expect the $\mathrm{OSp}(1|2)_R$ sector to be still intact in the dual models (3.60) and (3.50). We show that this is indeed the case and find how such residual isometries are explicitly realised by constructing an ansatz for the Killing vectors and studying the constraints that the graded Killing equation (A.13) imposes on it. The main results are reported here, while a complete list of the constraints and a description of their resolution can be found in D.2..

For the model obtained by dualisation of $\mathrm{OSp}(1|2)_L$ one finds the following set of Killing vectors

$$\xi_{L_a} = \tilde{x}^d \varepsilon_{da}{}^c \partial_c + \tfrac{1}{2}\tilde{\theta}^\lambda (\gamma_a)_\lambda{}^\nu \partial_\nu \qquad \xi_{Q_\alpha} = -i\tilde{\theta}^\lambda (\gamma^c)_{\lambda\alpha}\partial_c + \tilde{x}_c (\gamma^c)_\alpha{}^\nu \partial_\nu \ , \tag{3.62}$$

while for the T-dual model obtained by dualisation of $\mathrm{SL}(2,\mathbb{R})_L$ one finds

$$\xi_{L_a} = \tilde{x}^d \varepsilon_{da}{}^c \partial_c + \tfrac{1}{2}\tilde{\theta}^\lambda (\gamma_a)_\lambda{}^\nu \partial_\nu \qquad \xi_{Q_\alpha} = \tfrac{i}{2}\tilde{\theta}^\lambda (\gamma^b)_{\lambda\alpha}\tilde{x}^d \varepsilon_{db}{}^c \partial_c - [1 - \tfrac{i}{4}\tilde{\theta}^2]\partial_\alpha \ . \tag{3.63}$$

It should be noted that the bosonic isometries turn out to be realised exactly in the same way for both T-dual models. This was reasonably expected, as the model (3.50) is effectively obtained by dualising a subsector of the full $\mathrm{OSp}(1|2)_L$ isometry group. As needed, both sets of vectors correctly satisfy the $\mathfrak{osp}(1|2)$ algebra (B.11)

$$[\xi_{L_a}, \xi_{L_b}] = \varepsilon_{ab}{}^c \xi_{L_c} \qquad [\xi_{L_a}, \xi_{Q_\beta}] = -\tfrac{1}{2}(\gamma_a)_\beta{}^\alpha \xi_{Q_\alpha} \qquad [\xi_{Q_\alpha}, \xi_{Q_\beta}] = -i(\gamma^a)_{\alpha\beta}\xi_{L_a} \ . \tag{3.64}$$

T-dual supergravity constraints. In this paragraph we go back to the supergravity torsion constraints, focusing on the T-dual models constructed above. Ideally, one would like to show that such models satisfy again the constraints, so as to be able to interpret them as appropriate supergravity backgrounds. In particular, this might extend the 3d black hole interpretation given in [15] for the T-dual model to the PCM on $\mathrm{SL}(2,\mathbb{R})$, in terms of a supersymmetric black hole directly realised in superspace. Studying this problem for the dual models is much harder than for the principal chiral model, as after dualisation one loses the supergroup manifold picture and the structure constants of the initial model are turned into complicated structure functions. The torsion constraints impose on the latter three non trivial conditions (A.38), which for the models under consideration read

$$\begin{array}{lc} \text{(A)} & \mathcal{F}_{\alpha\beta}{}^c \equiv i k_1 (\gamma^c)_{\alpha\beta} \\ \text{(B)} & \mathcal{F}_{\alpha(bc)} \equiv 0 \\ \text{(C)} & \tfrac{1}{2}[\mathcal{F}_{\alpha\beta}{}^\gamma + i\epsilon^{\gamma\delta}(\mathcal{F}_{\delta\alpha\beta} - \mathcal{F}_{\delta\beta\alpha})] \equiv -k_2 \mathcal{F}_{\alpha bc}\varepsilon^{bca}(\gamma_a)_\beta{}^\gamma \end{array} \quad . \tag{3.65}$$

In the spirit of the present chapter we would now like to study the above three constraints explicitly. This requires an explicit expression for the structure functions of the T-dual models, which in turn depend on the choice of vielbeine and inverse vielbeine

$$\mathcal{F}_{AB}{}^C = (-1)^{MB} e_A{}^M e_B{}^N [\partial_N e_M{}^C - (-1)^{MN}\partial_M e_N{}^C] \ . \tag{3.66}$$

The latter relation can be extracted from the Maurer-Cartan equation (A.15) by writing the one forms and exterior derivative in the coordinate basis $e^A = \mathrm{d}\Lambda^M e_M{}^A$, $\mathrm{d} = \mathrm{d}\Lambda^M \partial_M$ and exploiting the defining relation for the inverse vielbeine $e_A{}^M e_M{}^B = \delta_A{}^B$ or $e_M{}^A e_A{}^N = \delta_M{}^N$. Given the T-dual metrics (3.60) and (3.50) it is not complicated to make a choice of vielbeine, invert it and consequently derive the structure functions. However, one should keep in mind that the final aim is finding an appropriate choice of vielbeine which also allows to satisfy the supergravity torsion constraints reported above, and for this reason it is certainly not a good idea to fix them straight away, as these would not be guaranteed to solve the constraints. Due to this problem one should rather proceed by leaving the vielbeine unspecified and letting the torsion constraints determine the most appropriate set (if any). To continue in full generality one should expand the vielbeine in powers of the fermionic coordinates with arbitrary coefficient functions. However, to make the analysis more concrete, we shall restrict ourselves to a quite general, even though possibly not exhaustive, ansatz. The conclusion will be that no choice of vielbeine with the chosen structure manages to satisfy the torsion constraints while reproducing the T-dual metrics and this will in turn motivate the use of a more general formalism to deal with the whole dualisation procedure and supergravity constraints, that will be discussed in the next chapter.

To introduce and motivate our ansatz for the vielbeine we notice that the three models so far considered, namely the principal chiral model on OSp(1|2), the T-dual model with respect to $SL(2,\mathbb{R})_L$ and the T-dual model with respect to $OSp(1|2)_L$, can be described in terms of a single enlarged metric. For simplicity the three models will be from now on referred to as *Initial Model*, *Bosonic Model* and *Fermionic Model*. Taking coordinates $\{x^m, \theta^\mu\}$ with contractions $r^2 \equiv x^n x^m \eta_{mn}$ and $\theta^2 \equiv \theta^\mu \theta^\nu \epsilon_{\mu\nu}$ we can write the following metric

$$
\begin{aligned}
g_{pq} &= (\lambda_1)_p{}^m \eta_{mq} + \theta^2 (\lambda_2)_p{}^m \eta_{mq} \\
g_{p\sigma} &= \theta^\lambda [(\lambda_3)_p{}^q (\gamma_q)_{\lambda\sigma} + g_8 \epsilon_{\sigma\lambda} x_p] \quad \text{with} \begin{cases} (\lambda_1)_p{}^m = g_1 \delta_p{}^m + g_3 x_p x^m + g_{11} x^c \varepsilon_{cp}{}^m \\ (\lambda_2)_p{}^m = g_2 \delta_p{}^m + g_4 x_p x^m + g_{12} x^c \varepsilon_{cp}{}^m \\ (\lambda_3)_p{}^m = g_5 \delta_p{}^m + g_6 x_p x^m + g_7 x^c \varepsilon_{cp}{}^m \\ \text{for} \quad g_{11} = g_{12} = 0 \end{cases} , \\
g_{\mu\nu} &= \epsilon_{\mu\nu}[g_9 + g_{10}\theta^2]
\end{aligned} \tag{3.67}
$$

which reduces to each of the three models above by appropriately renaming the coordinates.

- Initial model - unchanged name of coordinates $\{x^m, \theta^\mu\}$

$$
\begin{array}{llll}
g_1 = \frac{1-\cosh r}{r^2} & g_3 = -\frac{1+2g_1}{2r^2} & g_5 = \frac{i \sinh r}{4r} & g_6 = \frac{i-4g_5}{4r^2} \\
g_7 = \frac{i g_1}{4} & g_9 = i & g_{10} = \frac{1}{16} \; . &
\end{array} \tag{3.68}
$$

- Bosonic model - rename coordinates as $\{x^m \equiv \tilde{x}^m, \theta^\mu \equiv \theta^\mu\}$

$$g_1 = \frac{1}{2(r^2-1)} \qquad g_3 = -g_1 \qquad g_7 = \frac{ig_1}{2} \qquad g_9 = i \qquad g_{10} = \frac{1-2g_1}{8} \ . \tag{3.69}$$

- Fermionic model - rename coordinates as $\{x^m \equiv \tilde{x}^m, \theta^\mu \equiv \tilde{\theta}^\mu\}$

$$\begin{aligned} &g_1 = \frac{1}{2(r^2-1)} \qquad g_2 = -(1+2r^2)g_1^2 g_9 \qquad g_3 = -g_1 \qquad g_4 = 4ig_1^2 \qquad g_7 = \frac{3g_1g_9}{2} \\ &g_8 = \frac{g_9}{4} \qquad g_9 = \frac{-4i}{r^2-4} \qquad g_{10} = (12+7r^2-r^4)\frac{g_1g_9^2}{8} \ . \end{aligned} \tag{3.70}$$

The enlarged metric has inverse metric of the form

$$\begin{aligned} &g^{pq} = \eta^{pn}(\lambda_1^{inv})_n{}^q + \theta^2\eta^{pn}(\lambda_2^{inv})_n{}^q \\ &g^{p\sigma} = \theta^\lambda[(\gamma^q)_\lambda{}^\sigma(\lambda_3^{inv})_q{}^p + g_8^{inv}\delta_\lambda{}^\sigma x^p] \quad \text{with} \begin{cases} (\lambda_1^{inv})_p{}^m = g_1^{inv}\delta_p{}^m + g_3^{inv}x_px^m + g_{11}^{inv}x^c\varepsilon_{cp}{}^m \\ (\lambda_2^{inv})_p{}^m = g_2^{inv}\delta_p{}^m + g_4^{inv}x_px^m + g_{12}^{inv}x^c\varepsilon_{cp}{}^m \\ (\lambda_3^{inv})_p{}^m = g_5^{inv}\delta_p{}^m + g_6^{inv}x_px^m + g_7^{inv}x^c\varepsilon_{cp}{}^m \\ \text{for} \quad g_{11}^{inv} = g_{12}^{inv} = 0 \end{cases} \\ &g^{\mu\nu} = \epsilon^{\mu\nu}[g_9^{inv} + g_{10}^{inv}\theta^2] \end{aligned} \tag{3.71}$$

and its defining relation, $(-1)^P g_{MP}g^{PN} = \delta_M{}^N$, allows to express the above coefficients as functions of those of the metric as in (D.46), leading to

- Initial model

$$\begin{aligned} &g_1^{inv} = \frac{r^2}{1-\cosh r} \quad g_2^{inv} = \frac{-ir^2}{8(1-\cosh r)} \quad g_3^{inv} = \frac{-r^2-2(1-\cosh r)}{r^2(1-\cosh r)} \quad g_4^{inv} = \frac{i(4-r^2\,\mathrm{csch}\,(r/2)^2)}{16r^2} \\ &g_5^{inv} = -\frac{r\coth(r/2)}{4} \quad g_6^{inv} = \frac{-2+r\coth(r/2)}{4r^2} \quad g_7^{inv} = -\frac{1}{4} \quad g_9^{inv} = i \quad g_{10}^{inv} = -\frac{1}{4} \ . \end{aligned} \tag{3.72}$$

- Bosonic model

$$\begin{aligned} &g_1^{inv} = 2(r^2-1) \qquad g_2^{inv} = -\frac{ir^2}{4} \qquad g_3^{inv} = -2 \qquad g_4^{inv} = \frac{i}{4} \\ &g_7^{inv} = -\frac{1}{2} \qquad g_9^{inv} = i \qquad g_{10}^{inv} = -\frac{1}{4} \ . \end{aligned} \tag{3.73}$$

- Fermionic model

$$\begin{aligned} &g_1^{inv} = 2(r^2-1) \qquad g_2^{inv} = i \qquad g_3^{inv} = -2 \qquad g_7^{inv} = -\frac{3}{2} \\ &g_8^{inv} = -\frac{1}{2} \qquad g_9^{inv} = -\frac{i(r^2-4)}{4} \qquad g_{10}^{inv} = -\frac{3}{4} \ . \end{aligned} \tag{3.74}$$

We can finally write down the ansatz for the vielbeine that will allow us to explicitly study the torsion constraints. This will also need to be supplemented by inverse vielbeine, as these also appear in the structure functions. We choose vielbeine

$$\begin{cases} e_m{}^a = (\omega_1)_m{}^a + \theta^2(\omega_2)_m{}^a \\ \\ e_\mu{}^a = \theta^\lambda[(\gamma^q)_{\mu\lambda}(\omega_3)_q{}^a + B_3\epsilon_{\mu\lambda}x^a] \\ \\ e_m{}^\alpha = \theta^\lambda[(\omega_4)_m{}^q(\gamma_q)_\lambda{}^\alpha + D_3\delta_\lambda{}^\alpha x_m] \\ \\ e_\mu{}^\alpha = (\kappa_1)_\mu{}^\alpha + \theta^2(\kappa_2)_\mu{}^\alpha \end{cases} \quad \text{with} \quad \begin{cases} (\omega_1)^a_m = A_1\delta_m{}^a + A_3x_mx^a + A_5x^c\varepsilon_{cm}{}^a \\ (\omega_2)_m{}^a = A_2\delta_m{}^a + A_4x_mx^a + A_6x^c\varepsilon_{cm}{}^a \\ (\omega_3)_m{}^a = B_1\delta_m{}^a + B_4x_mx^a + B_2x^c\varepsilon_{cm}{}^a \\ (\omega_4)_m{}^a = D_1\delta_m{}^a + D_4x_mx^a + D_2x^c\varepsilon_{cm}{}^a \\ (\kappa_1)_\mu{}^\alpha = E_1\delta_\mu{}^\alpha + E_3x_p(\gamma^p)_\mu{}^\alpha \\ (\kappa_2)_\mu{}^\alpha = E_2\delta_\mu{}^\alpha + E_4x_p(\gamma^p)_\mu{}^\alpha \end{cases} \tag{3.75}$$

and similarly looking inverse vielbeine

$$\begin{cases} e_a{}^m = (\omega_1^{inv})_a{}^m + \theta^2(\omega_2^{inv})_a{}^m \\ \\ e_\alpha{}^n = \theta^\lambda[(\gamma^q)_{\alpha\lambda}(\omega_3^{inv})_q{}^n + N_3\epsilon_{\alpha\lambda}x^n] \\ \\ e_a{}^\nu = \theta^\lambda[(\omega_4^{inv})_a{}^q(\gamma_q)_\lambda{}^\nu + R_3\delta_\lambda{}^\nu x_a] \\ \\ e_\alpha{}^\nu = (\kappa_1^{inv})_\alpha{}^\nu + \theta^2(\kappa_2^{inv})_\alpha{}^\nu \end{cases} \quad \text{with} \quad \begin{cases} (\omega_1^{inv})_a{}^n = M_1\delta_a{}^n + M_3x_ax^n + M_5x^c\varepsilon_{ca}{}^n \\ (\omega_2^{inv})_a{}^n = M_2\delta_a{}^n + M_4x_ax^n + M_6x^x\varepsilon_{ca}{}^n \\ (\omega_3^{inv})_a{}^n = N_1\delta_a{}^n + N_4x_ax^n + N_2x^c\varepsilon_{ca}{}^n \\ (\omega_4^{inv})_a{}^n = R_1\delta_a{}^n + R_4x_ax^n + R_2x^c\varepsilon_{ca}{}^n \\ (\kappa_1^{inv})_\alpha{}^\nu = S_1\delta_\alpha{}^\nu + S_3x_p(\gamma^p)_\alpha{}^\nu \\ (\kappa_2^{inv})_\alpha{}^\nu = S_2\delta_\alpha{}^\nu + S_4x_p(\gamma^p)_\alpha{}^\nu \end{cases} \quad . \tag{3.76}$$

The coefficients $A_1, ..., E_4$ of the vielbeine, and those of their inverses, are arbitrary functions of r and pretty much like for the study of Killing vectors, we would now like to fix them by substituting the ansatze into the torsion constraints. One should however keep in mind that all such degrees of freedom are actually not completely free, as the coefficients of the vielbeine should also satisfy a set of equations ensuring that they correctly reproduce the metric, while the coefficients of the inverse vielbeine should satisfy another set of equations ensuring that they effectively represent an appropriate inverse matrix. A reasonable way to proceed would thus be that of determining the latter set of equations, coming from the defining relation $e_A{}^M e_M{}^B = \delta_A{}^B$, and solving them for the coefficients of the inverse vielbeine as functions of those of the vielbeine. This approach however leads to quite complicated expressions, naturally affected by highly non-linear dependencies and makes it really difficult to study the torsion constraints. For this reason, a much more efficient way to proceed is exploiting knowledge of the inverse metric to write down the coefficients of the inverse vielbeine as linear functions of those of the vielbeine with insertions of the known coefficients of the inverse metric. This can be achieved by writing down the metric and its inverse in terms of vielbeine and inverse vielbeine

$$g_{MN} = (-1)^{M(N+A)} e_N{}^A e_M{}^B \delta_{BA} \qquad\qquad g^{MN} = (-1)^{N(M+B)} \delta^{BA} e_A{}^N e_B{}^M \ , \tag{3.77}$$

such that $(-1)^P g_{MP} g^{PN} = \delta_M{}^N$. Using then $e_A{}^M e_M{}^B = \delta_A{}^B$, the second relation allows to extract

$$e_A{}^M = (-1)^{AM} g^{MN} e_N{}^B \delta_{BA} \ , \tag{3.78}$$

which leads to the linear expressions (D.47) for the coefficients of the inverse vielbeine as functions of those of the vielbeine and fixes the problem of having too many degrees of freedom to deal with. One could now substitute the vielbeine and their inverses into the torsion constraints, leading to a set of quite involved, but still tractable, set of differential equations for the coefficients of the vielbeine only. What still remains to be addressed is the other requirement the vielbeine should satisfy, namely they should reproduce the metric. There is no way around this, and for this reason we shall simply spell out the whole set of equations and treat them as an additional constraint, to be added to the ones on torsion. Writing the metric in terms of the vielbeine as in (3.77) leads to the following 10 conditions, which we shall from now on refer to as *metric equations*

$$\begin{aligned}
&① \qquad g_1 \equiv -\frac{1}{2}(A_1^2 - r^2 A_5^2)\\
&② \qquad g_2 \equiv i[D_1^2 - r^2 D_2^2 + iA_1A_2 - ir^2 A_5 A_6]\\
&③ \qquad g_3 \equiv -A_1A_3 - \frac{1}{2}r^2 A_3^2 - \frac{1}{2}A_5^2\\
&④ \qquad g_4 \equiv -i[D_3^2 - 2D_1D_4 - D_2^2 - r^2 D_4^2 - iA_4(A_1 + r^2 A_3) - iA_2A_3 - iA_5A_6]\\
&⑤ \qquad g_5 \equiv -\frac{1}{2}A_1B_1 + \frac{1}{2}r^2 A_5 B_2 - iD_1E_1 - ir^2 D_2 E_3\\
&⑥ \qquad g_6 \equiv -\frac{1}{2}B_4(A_1 + r^2 A_3) - \frac{1}{2}A_3B_1 - \frac{1}{2}A_5B_2 - iD_4E_1 + iE_3(D_3 + D_2)\\
&⑦ \qquad g_7 \equiv \frac{1}{2}A_1B_2 - \frac{1}{2}A_5B_1 - iD_1E_3 - iD_2E_1\\
&⑧ \qquad g_8 \equiv -\frac{1}{2}B_3(A_1 + r^2 A_3) + iD_1E_3 + ir^2 D_4E_3 - iD_3E_1\\
&⑨ \qquad g_9 \equiv i(E_1^2 - r^2 E_3^2)\\
&⑩ \qquad g_{10} \equiv i[2E_1E_2 - 2r^2 E_3E_4 - \frac{3i}{4}B_1^2 + \frac{i}{2}r^2 B_2^2 + \frac{i}{4}r^2 B_3^2 - \frac{i}{4}r^4 B_4^2 - \frac{i}{2}r^2 B_1B_4] \ .
\end{aligned} \tag{3.79}$$

At this point we are almost ready to proceed with our plan: substitute the ansatze for vielbeine and inverse vielbeine into the explicit expression (3.66) for the structure functions; determine from these the explicit set of differential equations imposed on the coefficients by the torsion constraints (3.65); substitute the expressions for the coefficients of the inverse vielbeine as linear functions of those of the vielbeine and try to solve the conditions resulting from the torsion constraints together with the metric equations reported above. We anticipate that, at the end of the story, we shall be dealing with a system of 26 equations, including the 10 (algebraic) metric equation and the 16 (differential) equations resulting from the torsion constraints, and we shall argue that the system admits no solution for either the bosonic or the fermionic T-dual model. As a consistency and sanity check for our procedure, we shall also treat the initial principal chiral model on the same footing as the T-dual models, that is as if we didn't know about its super group manifold structure and we had only been given its metric. This reverse engineering treatment of the principal chiral model highlights how the components of its metric satisfy complicated sets of differential equations which the T-dual models are not able to satisfy. This effect, brought in by the dualisation, might be interpreted as the result of switching hyperbolic functions of r with polynomials in the same variable. In a sense, dualisation truncates the infinite series expansions encoded in the hyperbolic functions to some finite polynomials so that, while differentiating the former one generates terms that might be recombined exploiting trigonometric identities, the same is not generally true for the latter, which give rise to new independent structures upon differentiation. Before writing down the full set of equations resulting from the torsion constraints, there are a few comments related to the metric equations (3.79) that we should make, which will be useful in the subsequent analysis.

- In solving (3.79) together with the constraints, it will be very convenient not to solve eqs ① and ⑨ explicitly, as these would bring unwanted square roots into the game. Such equations shall rather be imposed on the other constrains to symplify them and will only be solved at the very end of the calculation.

- Solving ③ as a quadratic equation for A_3 and using ① as a constraint, one can rewrite the former as the linear equation (3s) $A_1 + r^2 A_3 = 1$. This simplification is due to the fact that all the three models satisfy the relation $g_1 + r^2 g_3 = -\frac{1}{2}$. The newly obtained equation can then be conveniently solved for A_3. This simplifies the above system (and successively the constraints) and leads to naturally solve ④ for the coefficient A_4. Together with the just mentioned relation between g_1 and g_3, it will also be useful to exploit the following relations, holding among the coefficients of all the three models above

$$\begin{gathered} g_1 + r^2 g_3 = -\frac{1}{2} \qquad g_2 + r^2 g_4 + g_8 = 0 \qquad g_8[1 - 2(r^2 - 1)g_1] = 0 \\ g_5 g_2 = g_5 g_4 = g_5 g_8 = 0 = g_6 g_2 = g_6 g_4 = g_6 g_8 \\ 2r g_6 + r^2 g_6' + g_5' = 0 \qquad\qquad g_8' - \frac{1}{4} g_9' = 0 \ . \end{gathered} \tag{3.80}$$

- Equations ⑤ to ⑧ can be easily solved for the coefficients $B_1, ..., B_4$ or $D_1, ..., D_4$: we shall later choose the second option, as some of the constraints will be naturally solved in terms of the B-coefficients.

- Writing down explicitly the invertibility condition for the vielbeine, $e_A{}^M e_M{}^B = \delta_A{}^B$, one obtains a linear system of the form $Wx = v$, where x represents the unknown coefficients of the inverse veilbeine $\{M_1, ..., S_4\}$ and the matrix W contains the coefficients $\{A_1, ..., E_4\}$ of the vielbeine. The condition for the invertibility of W, and thus of the vielbeine, then reads

$$det(W) = (A_1 + r^2 A_3)^4 (A_1^2 - r^2 A_5^2)^3 (E_1^2 - r^2 E_3^2)^4 \propto g_1^3 g_9^4 \neq 0 \ . \tag{3.81}$$

- The two couples of eqs ① − ② and ⑨ − ⑩ present a similar structure: both of them involve the off-diagonal coefficients of the vielbeine in quadratic form (i.e. $D's$ and $B's$), while the coefficients on the diagonal blocks (i.e. $A's$ and $E's$) appear multiplied by each other. Assuming one of the couples $\{A_1, E_1\}$, $\{A_1, E_3\}$, $\{A_5, E_1\}$, $\{A_5, E_3\}$ to be non-vanishing, which is needed to ensure invertibility of the vielbeine as from the above condition, one could easily solve ② and ⑩ for the couple of variables $\{A_2, E_2\}$, $\{A_2, E_4\}$, $\{A_6, E_2\}$, $\{A_6, E_4\}$.

- The choice of non-vanishing couple also allows to extract another information: by differentiating ① and ⑨ one obtains

$$g_1' = -(A_1 A_1' - r A_5^2 - r^2 A_5 A_5') \qquad g_9' = 2i(E_1 E_1' - r E_3^2 - r^2 E_3 E_3') \ , \tag{3.82}$$

  which depending on the choice of non-vanishing coefficients made above, might respectively be used to get an expression for A_1' or A_5' and E_1' or E_3'. This helps in exchanging differential for algebraic equations, effectively disentangling some of the constraints.

We can now move on to the explicit construction of the structure functions, for which more details are provided in D.3.. Recalling the definition (3.66) one finds the following expressions

$$\mathcal{F}_{\alpha\beta}{}^{c} = (\gamma^b)_{\alpha\beta}[(F_1 \delta_b{}^c + F_2 x_b x^c + F_3 x^d \varepsilon_{db}{}^c) + \theta^2 (F_4 \delta_b{}^c + F_5 x_b x^c + F_6 x^d \varepsilon_{db}{}^c)] \ , \tag{3.83}$$

$$\begin{aligned} \mathcal{F}_{\alpha b}{}^{d} = \theta^\lambda \big[& \epsilon_{\alpha\lambda}(F_7 \delta_b{}^d + F_8 x_b \Lambda^d + F_9 x^c \varepsilon_{cb}{}^d) + \\ & + (\gamma^q)_{\alpha\lambda}(F_{10} x^c \varepsilon_{cqb} x^d + F_{11} x^c \varepsilon_{cq}{}^d x_b + F_{12} x^c \varepsilon_{cb}{}^d x_q + \\ & \qquad + F_{13} \eta_{qb} x^d + F_{14} \delta_b{}^d x_q + F_{15} \delta_q{}^d x_b + F_{16} x_q x_b x^d) \big] \ , \end{aligned} \tag{3.84}$$

$$\begin{aligned} \mathcal{F}_{\alpha\beta}{}^{\gamma} = \theta^\lambda \{ & F_{17}[\delta_\alpha{}^\gamma \epsilon_{\beta\lambda} + \delta_\beta{}^\gamma \epsilon_{\alpha\lambda}] + \\ & + F_{18} x_p [(\gamma^p)_\alpha{}^\gamma \epsilon_{\beta\lambda} + (\gamma^p)_\beta{}^\gamma \epsilon_{\alpha\lambda}] + F_{19} x_p [\delta_\alpha{}^\gamma (\gamma^p)_{\beta\lambda} + \delta_\beta{}^\gamma (\gamma^p)_{\alpha\lambda}] + \\ & + [(\gamma^p)_\alpha{}^\gamma (\gamma_q)_{\beta\lambda} + (\gamma^p)_\beta{}^\gamma (\gamma_q)_{\alpha\lambda}](F_{20} \delta_q{}^p + F_{21} x_q x^p + F_{22} x^c \varepsilon_{cp}{}^q) \} \ , \end{aligned} \tag{3.85}$$

with the coefficients reported in (D.56),(D.57),(D.58). From the latter, it is not hard to extract the conditions imposed by the torsion constraints (3.65) on the coefficients:

- From (3.83) one immediately recognises that constraint Ⓐ imposes six conditions

$$\begin{cases} (11) & F_1 \equiv ik_1 \\ (12) & F_2 \equiv 0 \\ (13) & F_3 \equiv 0 \end{cases} \qquad \begin{cases} (14) & F_4 \equiv 0 \\ (15) & F_5 \equiv 0 \\ (16) & F_6 \equiv 0 \end{cases} \quad . \tag{3.86}$$

- Similarly, from (3.84) one can recognise the following six conditions imposed by Ⓑ

$$\begin{cases} (17) & F_7 \equiv 0 \\ (18) & F_8 \equiv 0 \\ (19) & F_{10} + F_{11} \equiv 0 \end{cases} \qquad \begin{cases} (20) & F_{13} + F_{15} \equiv 0 \\ (21) & F_{14} \equiv 0 \\ (22) & F_{16} \equiv 0 \end{cases} \quad . \tag{3.87}$$

- The remaining four conditions imposed by Ⓒ can be conveniently obtained by slightly rearranging the constraint. Contracting both sides with $\varepsilon_{pq}{}^k(\gamma_k)_\gamma{}^\beta$ and exploiting the identities in B.1. this can be rewritten as

$$\varepsilon_{pq}{}^k[2\mathcal{F}_{\alpha\beta}{}^\gamma(\gamma_k)_\gamma{}^\beta + \mathcal{F}_{\delta\beta}{}^\lambda\epsilon_{\alpha\lambda}(\gamma_k)^{\delta\beta}] \equiv 4k_2(\mathcal{F}_{\alpha p}{}^d\eta_{dq} - \mathcal{F}_{\alpha q}{}^d\eta_{dp}) \ . \tag{3.88}$$

Notice that the latter operation can be inverted by contracting again with $\varepsilon^{pq}{}_n(\gamma^n)_\mu{}^\nu$ as

$$\varepsilon_{pq}{}^k(\gamma_k)_\gamma{}^\beta\varepsilon^{pq}{}_n(\gamma^n)_\mu{}^\nu = -4\delta_\mu{}^\beta\delta_\gamma{}^\nu + 2\delta_\mu{}^\nu\delta_\gamma{}^\beta \ , \tag{3.89}$$

and the second term vanishes on both sides of Ⓒ as they are traceless. After a short calculation, the above rearranged constraint leads to the following conditions

$$\begin{cases} (23)\ F_{18} - F_{19} \equiv 2k_2F_9 \\ (24)\ F_{20} - F_{17} \equiv k_2r^2(F_{10} - F_{11}) \end{cases} \qquad \begin{cases} (25)\ F_{19} - F_{22} \equiv k_2(F_{13} - F_{15}) \\ (26)\ F_{20} - F_{17} + r^2F_{21} \equiv 2k_2r^2F_{12} \end{cases} \quad . \tag{3.90}$$

At this point one can proceed with the aid of Mathematica, so as to more efficiently manipulate and solve the 10 metric equations and the 16 constraints just introduced. At first we shall proceed by solving as many equations as possible without specifying any model, i.e. solving for the coefficients of the vielbeine in terms of generic metric coefficients. This allows to solve 8 metric equations and 6 constraints, fixing 14 out of the 18 available coefficients of the vielbeine. At this stage the remaining equations become quite involved and in order to proceed further in the analysis one needs to specify the model. While the equations for the initial model greatly simplify and allow for a solution which recovers the vielbeine introduced at the beginning of the chapter, the other two models exhibit a complicated structure and the set of equations turns out to be inconsistent.

As a first step in the above argument, we shall assume coefficients A_1 and E_1 of the vielbeine to be non-vanishing. This ensures their invertibility, as discussed around (3.81). The metric equations ① and ⑨ might in principle be easily solved, but this would bring into the game square roots, which would make the rest of the equations more complicated and harder to simplify. For this reason we shall proceed by solving only linear equations and imposing the above two, together with the conditions (3.80), as constraints on the resolution of the remaining equations. This can be achieved in Mathematica by using the command *Assume*, which allows to simplify expressions under some given assumptions. One can then notice that constraints ⑪ − ⑬ are algebraic and can thus be solved together with some of the metric equations: conditions ② − ⑧, ⑩ and ⑪ − ⑬ are easily solved in terms of the coefficients $A_2, A_3, A_4, D_1, D_2, D_3, D_4, E_2, B_1, B_2, B_4$. Notice that despite the appearance, all these equations can be solved linearly and only ② and ⑩ actually require dividing by A_1 and E_1.

The second step consists in solving the conditions (3.82) for A_1' and E_1' and substituting the result in all the remaining constraints. Simplifying all of them imposing once again the vanishing of ① and ⑨, together with (3.80), one finds that constraints ⑰ and ㉑ have become algebraic and can be easily solved for A_5 and B_3, while ⑯ can easily be solved for A_6. The latter equation is not an algebraic one, but can be safely solved for A_6 as this only appears in ⑯, ㉓, ㉖ and never differentiated.

At this point one is left with free coefficients A_1, E_1, E_3, E_4 and unsolved equations ①, ⑨, ⑭, ⑮, ⑱ − ⑳, ㉒ − ㉖, of which only the first two are algebraic. It is hence the moment to specify the model under consideration

- **Initial model.** After introducing explicitly the coefficients of the metric and solving ① and ⑨ for A_1 and E_1 all the remaining equations undergo a great simplification and one can notice that most of them vanish provided that $k_1^2 \equiv 1$. This is however not sufficient to solve all the constraints, as ㉔, ㉕ are only satisfied provided that $k_1 = -1$ and $E_3 = 0$. At this stage only equation ㉓ is left unsolved and one can easily put remedy to this by setting $E_4 = 0$. At the end of these steps all the constraints and metric equations are satisfied for the metric coefficients of the initial model and one recovers the coefficients of the vielbeine that we introduced in the previous section, together with the requirement that $k_1 = -1$.

- **Bosonic model.** Also for this model one can proceed by solving ① and ⑨ for A_1 and E_1. However, contrarily to the previous case, one immediately finds problems upon inspecting

(14), as this turns out to be only dependent on k_1 and r^2, thus only allowing for solutions with non-constant k_1.

- **Fermionic model.** For this model things are slightly more involved, as proceeding like for the other two models brings into the game more complicated square roots. One can however proceed by leaving (1) and (9) unsolved and checking consistency of the remaining set of differential equations, which are coupled and all dependent on A'_1, E'_1, E'_3. Indeed, after isolating A'_1 from equation (14) and substituting into all the others, one finds that (15), (19), (22), (24), (25) become algebraic in the variables A_1, E_1, E_3. In particular, (15) and (25) can be linearly solved for A_1 and E_3 and this causes (19) to only allow for the solution $E_1 = 0$, hence leading to an inconsistency.

From this analysis, the T-dual models constructed in the previous paragraphs seem not to satisfy the supergravity torsion constraints. However, we restricted ourselves to an ansatz for the choice of vielbeine and our argument might be improved by considering a more general one. One could have proceeded by considering an expansion of the vielbeine in the fermionic coordinates, but the complexity of the approach strongly suggests one should really resort to other techniques. In the next chapter we shall thus introduce a more abstract point of view on the dualisation: this will allow to argue much more quickly that the T-dual models break the torsion constraints and will also be of easier extension to other types of geometries, in particular coset sigma models of the symmetric and semi-symmetric kind.

4

A More Abstract Perspective On Super Non-Abelian T-Duality

Motivated by the results of the previous chapter and the difficulties encountered in analysing T-dual models of principal chiral models from an explicit ansatz-based approach, in this chapter we shall take a slightly more abstract perspective on T-duality, relying on the possibility of solving the equations of motion for the gauge fields in a general model-independent manner. This will have the advantage of leading to a T-dual action directly written down in terms of a set of dual vielbeine, which can be more efficiently exploited to study the supergravity requirements of the model. Additionally, this will allow to recover the exchange of Maurer-Cartan equations and equations of motion notoriously resulting from the bosonic dualisation. This approach will also have the advantage of being more easily generalisable to other types of geometries and in particular we shall focus on symmetric and semi-symmetric spaces, re-deriving the exchange of Maurer-Cartan equations and equations of motion found for principal chiral models.

4.1. Principal chiral models

We start from principal chiral models, performing gauging and dualisation with respect to the left sector of the isometry group, then re-considering the explicit example of $\mathsf{OSp}(1|2)$.

4.1.1. Setup

Let us consider again a generic Lie (super)group manifold G with associated Lie (super)algebra $\mathfrak{g}$, equipped with a non-degenerate, (graded-)symmetric, Ad-invariant bilinear form $\langle -, - \rangle$. Principal chiral models are defined in terms of smooth maps $\mathrm{g} \in \mathcal{C}^{\infty}(\Sigma, \mathsf{G})$ from the two dimensional Lorentzian worldsheet Σ to G, constructing the pull-back to Σ of the Lie algebra valued Maurer-Cartan 1-form $j := \mathrm{g}^{-1}\mathrm{d}\mathrm{g} \in \Omega^1(\Sigma, \mathfrak{g})$. This satisfies by construction the Maurer-Cartan flatness condition $\mathrm{d}j + \frac{1}{2}[j, j] = 0$, where d represents the exterior derivative on the worldsheet, and is invariant under the global left action of the group on itself $\mathrm{g} \to g_L^{-1}\mathrm{g}$, while transforms adjointly $j \to g_R^{-1} j g_R$ under the global right action $g \to g g_R$, for $g_L, g_R \in G$. The principal chiral model action

$$S_{PCM} := \int_\Sigma \langle j, \star j \rangle \ , \tag{4.1}$$

then enjoys $G_L \times G_R$ global invariance, with associated Noether currents $L_N := -g j g$ and $R_N := j$. The equations of motion read $d(\star j) = 0$ and, together with the Maurer-Cartan equation, can be understood as resulting from the flatness condition of Lax connection $J(z)$. This depends on a complex spectral parameter z and its existence ensures classical integrability of the model [104–106]. Parametrising the Lax connection as $J = aj + b \star j$ one finds

$$dJ + \frac{1}{2}[J, J] = 0 \qquad \Rightarrow \qquad \begin{cases} d(\star j) = 0 \\ dj + \frac{1}{2}[j, j] = 0 \end{cases} \qquad \text{iff} \qquad a^2 - b^2 \equiv a \ . \tag{4.2}$$

This condition can be solved as $a := -\frac{1}{4}(z - z^{-1})^2$ and $b := \frac{1}{4}(z^2 - z^{-2})$, so that $J \equiv J(z)$. We stress that, strictly speaking, the existence of a Lax connection is not enough to guarantee full integrability of a system, as this ensures the presence of an infinite set of conserved charges, which might however be not in involution. For this reason the Lax connection is said to imply *weak* integrabiliy, while *strong* integrability, namely the involution property of the charges, is ensured when the components of the Lax connection satisfy a specific Poisson structure [107].

Topological deformations. We can introduce one more ingredient in the definition of our starting model, namely a deformation first introduced in [82–84] and referred to as *topological*, as it does not alter the equations of motion of the model nor its global symmetries, thus leaving untouched the construction of the Lax connection. This class of deformations relies on the existence of two-cocylces Ω[1] on the Lie (super)algebra $\mathfrak{g}$, or possibly on a subalgebra of it. Given $\Omega \in H^2(\mathfrak{g})$ satisfying the two-cocycle condition

$$\Omega(X, [Y, Z]) + (-1)^{X(Y+Z)}\Omega(Y, [Z, X]) + (-1)^{Z(X+Y)}\Omega(Z, [X, Y]) = 0 \qquad \forall X, Y, Z \in \mathfrak{g} \ , \tag{4.3}$$

[1] As discussed in [82], deformation by a two-cocycle becomes trivial when the latter is a couboundary. This happens when $\Omega(X, Y) := f([X, Y])$, with $X, Y \in \mathfrak{g}$ and $f : \mathfrak{g} \to \mathbb{F}$ some linear map, called 1-cochain, from the algebra to $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. In such cases the deformation can be removed by a field redefinition and for this reason non-trivial deformations are in correspondence with two-cocycles modulo coboundaries. This relates non-trivial deformations to elements of the second cohomology group $H^2(\mathfrak{g})$, which also correspond to non-trivial central extensions of $\mathfrak{g}$. For finite-dimensional semisimple Lie algebras, the Whitehead lemma ensures that $H^2(\mathfrak{g}) = 0$ (see for example chapter 4 in [108] and references therein) and a similar result has been proven in chapter 18 of the collection [109, 110] for finite-dimensional simple Lie algebras. Non-trivial deformations of Lie algebras can thus be considered when $\mathfrak{g}$ is neither simple nor semisimple, as for the case of $\mathfrak{u}(2)$ discussed in [82]. For Lie superalgebras these results do not hold in general and one needs to perform a case by case analysis. See for example [111–113] for results on the cohomology of Lie superalgebras, also in relation to string theory and supergravity.

by the Riesz representation theorem one can associate to this a unique Lie algebra endomorphism $D : \mathfrak{g} \to \mathfrak{g}$ relating Ω to the inner product as

$$\Omega(X, Y) = \langle D(X), Y \rangle \qquad \forall X, Y \in \mathfrak{g} \; . \tag{4.4}$$

The graded antisymmetry of Ω and the two-cocylce condition (4.3) respectively imply the following properties of D, which are derived in E.1.

$$\langle X, D(Y) \rangle = -\langle D(X), Y \rangle \qquad D[X, Y] = [D(X), Y] + [X, D(Y)] \qquad \forall X, Y \in \mathfrak{g} \; . \tag{4.5}$$

Hence, D is antisymmetric with respect to the inner product and acts as a derivation on the Lie bracket. One can then show that D can be extended to a left-invariant vector field on G

$$D(gYg^{-1}) = g\big(D(Y) + [g^{-1}D(g), Y]\big)g^{-1} \qquad \forall Y \in \mathfrak{g} \tag{4.6}$$

upon defining

$$g^{-1}D(g) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \mathrm{ad}_X^k(D(X)) = -D(g^{-1})g \qquad \text{for} \qquad g := e^X \; . \tag{4.7}$$

Notice that $g^{-1}D(g) = -D(g^{-1})g$ is a result of the fact that $g^{-1}D(g)$ is defined via the same series expansion as the Maurer-Cartan forms $j := \mathrm{g}^{-1}\mathrm{dg}$ in (C.12). This makes D inherit the same properties as the exterior derivative d. The above property allows to show, together with Ad-invariance of the inner product, that

$$\Omega(g^{-1}Xg, g^{-1}Yg) = \Omega(X, Y) + \langle g^{-1}D(g), [X, Y] \rangle \; . \tag{4.8}$$

Extending the above definitions to p-forms $\Omega^p(\Sigma, \mathfrak{g})$ one can then define the deformed action

$$S_{PCM}^{\Omega} := \tfrac{1}{2} \int_{\Sigma} \langle j, \star j \rangle + \zeta \langle D(j), j \rangle \; , \tag{4.9}$$

with $\zeta \in \mathbb{R}$ an arbitrary deformation parameter, from now on set to one without loss of generality.

Gauging. We proceed, as in the previous chapter, with the gauging of a subgroup $\mathrm{K}_L \subseteq \mathrm{G}_L$ in the left sector of the isometry group of the deformed principal chiral model. Denoting by $\mathfrak{k}_L$ the Lie algebra of K_L, we introduce $\mathfrak{k}_L$-valued gauge fields $\omega \in \Omega^1(\Sigma, \mathfrak{k}_L)$ and modify the current j as

$$j_\omega := \mathrm{g}^{-1}\mathrm{dg} + \mathrm{g}^{-1}\omega\mathrm{g} \qquad \text{with} \qquad \mathrm{g} \in \mathcal{C}^\infty(\Sigma, \mathrm{G}) \; . \tag{4.10}$$

The latter is now invariant under the local left K_L action $g \to k^{-1}g$ with $k \in \mathsf{K}_L$ provided that

$$\omega \quad \to \quad k^{-1}\omega k + k^{-1}\mathrm{d}k \ . \tag{4.11}$$

The action must then be modified by means of the Lagrange multiplier term, enforcing the flatness of the gauge fields. Introducing $\Lambda \in \mathcal{C}^\infty(\Sigma, \mathfrak{k}_L)$ we thus have the following master action

$$S_\omega := \tfrac{1}{2}\int_\Sigma \langle j_\omega, \star j_\omega\rangle + \langle D(j_\omega), j_\omega\rangle + \int_\Sigma \langle \Lambda + D(g)g^{-1}, F_\omega\rangle \quad \text{with} \quad F_\omega := d\omega + \tfrac{1}{2}[\omega,\omega] \ , \tag{4.12}$$

which given the property $F_\omega \to k^{-1}F_\omega k$, under local K_L transformations, is invariant provided that

$$\Lambda \quad \to \quad k^{-1}\Lambda k + k^{-1}D(k) \ . \tag{4.13}$$

Notice that the contribution $\langle D(g)g^{-1}, F_\omega\rangle$ must be included to retain invariance under the global G_R action in the presence of the deformation. Indeed, $g \to g g_R$ does not affect the multipliers but (4.8) implies $\langle D(j_\omega), j_\omega\rangle \to \langle D(j_\omega), j_\omega\rangle + \langle g_R^{-1}D(g_R), [j_\omega, j_\omega]\rangle$, thus leading to the need for a compensating term. More details are given in E.1.. Finally, we can simplify the master action as

$$S_\omega = \tfrac{1}{2}\int_\Sigma \langle j_\omega, \star j_\omega\rangle + \langle D(j_\omega), j_\omega\rangle + \int_\Sigma \langle \tilde\Lambda, F_{j_\omega}\rangle \qquad \text{with} \qquad \tilde\Lambda := g^{-1}\Lambda g + g^{-1}D(g) \ , \tag{4.14}$$

where we exploited that $g^{-1}F_\omega g = F_{j_\omega}$. The latter is manifestly K_L invariant and can now be T-dualised. Integrating out the Lagrange multipliers one enforces the flatness of the gauge fields, which can be removed by an appropriate choice of K_L gauge, while integrating out the gauge fields one obtains the T-dual model, where the Lagrange multipliers play the role of dual coordinates. K_L gauge invariance can then be exploited to remove $dim(\mathsf{K}_L)$ coordinates of the initial model, thus restoring the correct number of degrees of freedom. When $\mathsf{K}_L = \mathsf{G}_L$ one can choose gauge $g = \mathbb{1}$, so that all information from the initial model is lost.

4.1.2. T-Dual model

One can now proceed with T-dualisation by integrating out the gauge fields from the master action (4.14), which varied with respect to ω leads to

$$\star j_\omega + \mathrm{d}\tilde\Lambda - D_{\tilde\Lambda}(j_\omega) = 0 \qquad \text{with} \qquad D_{\tilde\Lambda} := D + \mathrm{ad}_{\tilde\Lambda} \ . \tag{4.15}$$

The latter equation can be solved as

$$j_\omega = -\frac{1}{1 - D_{\tilde\Lambda}}P_+(\mathrm{d}\tilde\Lambda) + \frac{1}{1 + D_{\tilde\Lambda}}P_-(\mathrm{d}\tilde\Lambda) \qquad \text{with} \qquad D_{\tilde\Lambda} := D + \mathrm{ad}_{\tilde\Lambda} \ , \tag{4.16}$$

where we introduced projectors $P_\pm : \Omega^1(\Sigma, \mathfrak{g}) \to \Omega^1_\pm(\Sigma, \mathfrak{g})$

$$P_\pm := \tfrac{1}{2}(1 \pm \star) \qquad \text{with} \qquad P_\pm P_\pm = P_\pm \qquad P_\pm P_\mp = 0 \ , \tag{4.17}$$

defining an orthogonal decomposition of worldsheet 1-forms $\Omega^1(\Sigma, \mathfrak{g}) \simeq \Omega^1_+(\Sigma, \mathfrak{g}) \oplus \Omega^1_-(\Sigma, \mathfrak{g})$[2].

Substituting the latter back into the master action (4.14) and exploiting the relation

$$\langle X, \frac{1}{1 \pm D_Y} Z \rangle = \langle \frac{1}{1 \mp D_Y} X, Z \rangle \qquad \forall X, Y, Z \in \mathfrak{g} \ , \tag{4.18}$$

which results from $\langle D_X(Y), Z \rangle = -\langle Y, D_X(Z) \rangle$, one obtains the T-dual action

$$\tilde{S} = \int_\Sigma \langle \mathrm{d}\tilde{\Lambda}, \frac{1}{1 - D_{\tilde{\Lambda}}} P_+(\mathrm{d}\tilde{\Lambda}) \rangle \ . \tag{4.19}$$

Equations of motion and Lax connection. One can now verify the exchange of equations of motion and Maurer-Cartan equations by computing the T-dual equations of motion as in E.2.. Varying the dual action with respect to $\tilde{\Lambda}$ and exploiting $\delta D_{\tilde{\Lambda}} = \mathrm{ad}_{\delta\tilde{\Lambda}}$ together with the relation

$$\frac{1}{1 \pm (D_{\tilde{\Lambda}} + \delta D_{\tilde{\Lambda}})} = \frac{1}{1 \pm D_{\tilde{\Lambda}}} \mp \frac{1}{1 \pm D_{\tilde{\Lambda}}} \circ \mathrm{ad}_{\delta\tilde{\Lambda}} \circ \frac{1}{1 \pm D_{\tilde{\Lambda}}} + \dots \ , \tag{4.20}$$

where $\circ$ denotes composition, one indeed finds Maurer-Cartan-like equations of motion

$$\mathrm{d}\tilde{j} + \tfrac{1}{2}[\tilde{j}, \tilde{j}] = 0 \qquad \text{with} \qquad \tilde{j} := j_\omega \ . \tag{4.21}$$

Combining the latter with (4.16), and exploiting Jacobi identity, one then recovers the conservation equation for $\tilde{j}$

$$\mathrm{d}(\star\tilde{j}) = 0 \ . \tag{4.22}$$

This exchange allows to construct the dual Lax connection as the one of the initial model

$$\tilde{J}(\tilde{z}) := -\tfrac{1}{4}(\tilde{z} - \tilde{z}^{-1})^2 \tilde{j} + \tfrac{1}{4}(\tilde{z}^2 - \tilde{z}^{-2}) \star \tilde{j} \ , \tag{4.23}$$

with $\tilde{z}$ a new spectral parameter, thus ensuring classical integrability of the dual model.

Residual isometries. As discussed in the previous chapter, since T-duality preserves the isometries commuting with the ones that have been gauged [90,91], for principal chiral models one expects

[2] Equation (4.15) can be easily solved by rearranging as $j_\omega = -\star T + S(\star j_\omega)$, with $S := D_{\tilde{\Lambda}}$ and $T := \mathrm{d}\tilde{\Lambda}$, and recursively substituting j_ω into itself. Exploiting $\star^2 = 1$ one finds $j_\omega = -\sum_{k=0}^\infty S^{2k} \star T - \sum_{k=0}^\infty S^{2k+1} T = -\frac{1}{1-S} P_+(T) + \frac{1}{1+S} P_-(T)$, using in the last step $\sum_{k=0}^\infty S^{2k} = \frac{1}{2}\left(\frac{1}{1-S} + \frac{1}{1+S}\right)$ and $\sum_{k=0}^\infty S^{2k+1} = \frac{1}{2}\left(\frac{1}{1-S} - \frac{1}{1+S}\right)$.

the T-dual model to still retain the global G_R invariance. Indeed under $g \to g g_R$ one has

$$\tilde{\Lambda} \quad \to \quad g_R^{-1}\tilde{\Lambda} g_R + g_R^{-1} D(g_R) \tag{4.24}$$

and the dual action is manifestly invariant. This comes from g_R, D being constant on the world-sheet, as g_R is a global G_R transformation and D a Lie algebra endomorphism $D : \mathfrak{g} \to \mathfrak{g}$, and the relation

$$D_{g_R^{-1}\tilde{\Lambda} g_R + g_R^{-1} D(g_R)}(g_R^{-1} \mathrm{d}\tilde{\Lambda} g_R) = g_R^{-1} D_{\tilde{\Lambda}}(\mathrm{d}\tilde{\Lambda}) g_R \ , \tag{4.25}$$

implied by (4.6). Ad-invariance of the inner product then ensures invariance of the action at all orders. One can then also compute the associated Noether current

$$\tilde{J}_N = \star D_{\tilde{\Lambda}} \tilde{j} \ . \tag{4.26}$$

The latter can be combined with the result (C.21) to extract the Killing vectors generating the residual isometries, for which more details are provided in E.2.. Given generators T_A of the Lie algebra $\mathfrak{g}$, with commutators $[T_A, T_B] = f_{AB}{}^C T_C$, the Killing vectors read

$$\xi_{T_A} = (D_A{}^B + \tilde{\Lambda}^M f_{MA}{}^B)\partial_B \qquad \text{with} \qquad \partial_B := \frac{\partial}{\partial \tilde{\Lambda}^B} \ , \tag{4.27}$$

where we defined components $D(T_A) = D_A{}^B T_B$ of the Lie algebra endomorphism. From Jacobi identity and two-cocycle condition one can then verify they satisfy the commutation relations

$$[\xi_{T_A}, \xi_{T_B}] = f_{AB}{}^C \xi_{T_C} \ . \tag{4.28}$$

T-dual vielbeine. The T-dual action (4.19) can be alternatively rewritten as

$$\tilde{S} = \frac{1}{2}\int_\Sigma \langle \tilde{e}, \star\tilde{e}\rangle + \langle \tilde{e}, D_{\tilde{\Lambda}}\tilde{e}\rangle \qquad \text{with} \qquad \tilde{e} := -\frac{1}{1 - D_{\tilde{\Lambda}}}\mathrm{d}\tilde{\Lambda} \ , \tag{4.29}$$

where we introduced T-dual vielbeine $\tilde{e}$. Notice that the choice of vielbeine is not unique. One may indeed define $\tilde{e}_- := -\frac{1}{1-D_{\tilde{\Lambda}}}\mathrm{d}\tilde{\Lambda}$ or $\tilde{e}_+ := -\frac{1}{1+D_{\tilde{\Lambda}}}\mathrm{d}\tilde{\Lambda}$ and the dual action (4.19) would still be rewritten as (4.29), respectively in terms of $\tilde{e}_-$ or $\tilde{e}_+$[3]. From the above definition one can find the following modified Maurer-Cartan equation (see the last paragraph in E.2. for more details)

$$\mathrm{d}\tilde{e} + \frac{1}{2}[\tilde{e}, \tilde{e}] = -\frac{1}{2}\frac{1}{1 - D_{\tilde{\Lambda}}}[\tilde{e}, \tilde{e}] \ . \tag{4.30}$$

[3]The two choices are related by $\tilde{e}_- = \frac{1+D_{\tilde{\Lambda}}}{1-D_{\tilde{\Lambda}}}\tilde{e}_+$ and the operator $\frac{1+D_{\tilde{\Lambda}}}{1-D_{\tilde{\Lambda}}}$ acts as a local vielbeine rotation leaving unchanged the structure of the action. Additionally, the action is not affected by overall changes of sign in the vielbeine definition: choosing $\tilde{e}_- := \pm\frac{1}{1-D_{\tilde{\Lambda}}}\mathrm{d}\tilde{\Lambda}$ or $\tilde{e}_+ := \pm\frac{1}{1+D_{\tilde{\Lambda}}}\mathrm{d}\tilde{\Lambda}$ $\tilde{e}_+$ simply changes the relative sign between terms in the left member of (4.30).

Furthermore, recognising the presence of a T-dual B-field

$$\tilde{B}_2 := \frac{1}{2}\langle \tilde{e}, D_{\tilde{\Lambda}}\tilde{e}\rangle \ , \tag{4.31}$$

one can determine E.2., from the point of view of the background geometry, its field strength

$$H_3 := \mathrm{d}\tilde{B}_2 = \frac{1}{2}\langle \tilde{e}, \frac{1}{1-D_{\tilde{\Lambda}}}[\tilde{e},\tilde{e}]\rangle \ . \tag{4.32}$$

The latter relation is always true, at least locally, due to Poincaré lemma, as the field strength H_3 should be a closed 3-form. From the global point of view, H_3 might be closed but not exact, hence being a representative of the third cohomology class.

4.1.3. OSp$(1|2)$ revised and improved

We are now in the position to reconsider the principal chiral model on OSp$(1|2)$ and study its properties by using the formalism described in the previous section.

Lie algebra and initial setup. Throughout this section we shall change a bit our notation as compared to the previous chapter, and use the OSp$(1|2)$ algebra in spinorial form. More details about it are provided in B.1., and we report here the non-trivial commutators for clarity

$$\begin{gathered}[L_{\alpha\beta}, L_{\gamma\delta}] = -\,i\big(\epsilon_{\gamma(\alpha}L_{\beta)\delta} + \epsilon_{\delta(\alpha}L_{\beta)\gamma}\big)\\ [Q_\alpha, Q_\beta] = L_{\alpha\beta} \qquad\qquad [L_{\alpha\beta}, Q_\gamma] = -i\epsilon_{\gamma(\alpha}Q_{\beta)} \ .\end{gathered} \tag{4.33}$$

With this notation, the inner products take the form

$$\langle L_{\alpha\beta}, L_{\gamma\delta}\rangle = \epsilon_{\alpha(\gamma}\epsilon_{\delta)\beta} \qquad \langle Q_\alpha, Q_\beta\rangle = i\epsilon_{\alpha\beta} \qquad \langle L_{\alpha\beta}, Q_\gamma\rangle = 0 \ . \tag{4.34}$$

Since $H^2(\mathfrak{osp}(1|2)) = 0$ [111–113], we proceed with the undeformed principal chiral model (4.1), i.e. setting $D = 0$ in all formulae from the previous section. We shall not discuss, as in chapter 3, the explicit structure of the dual fields, but rather concentrate on the supergravity requirements.

Initial model - torsion constraints. Given the 3d supergravity torsion constraints (E.34), it is not hard to recognise that the initial model satisfies them. All one needs to do is expanding the generic Maurer-Cartan form on the Lie algebra generators $j = j^{\alpha\beta}L_{\alpha\beta} + j^\alpha Q_\alpha$, exploiting the commutators (4.33) to write down the Maurer-Cartan equation in components

$$\mathrm{d}j^{\alpha\beta} + ij^{\gamma\alpha}\wedge j_\gamma{}^\beta = \frac{1}{2}j^\alpha\wedge j^\beta \qquad \mathrm{d}j^\alpha + ij^\beta\wedge j_\beta{}^\alpha = 0 \ . \tag{4.35}$$

The torsion constraints (E.34) are then satisfied with the identifications

$$e^{\alpha\beta} = -j^{\alpha\beta} \qquad e^{\alpha} = -j^{\alpha} \qquad \Omega_{\alpha}{}^{\beta} = -\tfrac{i}{2} j_{\alpha}{}^{\beta} \qquad T_{\alpha\beta\gamma}{}^{\delta} = -\epsilon_{\gamma(\alpha}\delta_{\beta)}{}^{\delta} \, , \tag{4.36}$$

and conditions (A.33) for the connection are respected due to the symmetry of $j_{\alpha\beta}$. Even though the initial sigma model does not contain a B-field, one could include in the 3d action an H_3 contribution that would dynamically generate the cosmological constant via the equations of motion, thus ensuring the existence of a super-AdS$_3$ solution. In this case the three-form would read

$$H_3 = e_{\alpha} \wedge e^{\alpha\beta} \wedge e_{\beta} + \tfrac{4i}{3R} e_{\alpha}{}^{\beta} \wedge e_{\beta}{}^{\gamma} \wedge e_{\gamma}{}^{\alpha} \, , \tag{4.37}$$

which is indeed of the form (E.36), with R proportional to the AdS$_3$ radius. From the sigma model perspective, this three-form might be included via a Wess-Zumino term

$$S_{WZ} = \tfrac{1}{3!} \int_M \langle j, [j, j] \rangle \qquad \text{with} \qquad \partial M = \Sigma \, . \tag{4.38}$$

T-dual model - torsion constraints. In order to study the torsion constraints for the dual model, we need to compute the exterior derivative of the dual vielbeine (4.29). Using the modified Maurer-Cartan equation (4.30) we can extract $\mathrm{d}\tilde{e}$ by computing

$$\mathrm{d}\tilde{e} = -\frac{1}{2}[\tilde{e}, \tilde{e}] - \frac{1}{2}\frac{1}{1 - ad_{\tilde{\Lambda}}}[\tilde{e}, \tilde{e}] \, . \tag{4.39}$$

This would allow us to substitute $\mathrm{d}\tilde{e}^{\alpha\beta}$ and $\mathrm{d}\tilde{e}^{\alpha}$ into the torsion constraints (E.34) and check explicitly whether these can be satisfied for some appropriate choice of connection. To find the exterior derivative of the vielbeine one needs, from the above equation, to compute the action of the operator $\frac{1}{1-ad_{\tilde{\Lambda}}}$ on $[\tilde{e}, \tilde{e}]$ and to this aim we now make the assumption of having dualised with respect to the full $\mathrm{OSp}(1|2)_L$ sector of the isometry group, so as to be able to choose the gauge $\mathrm{g} = \mathbb{1}$ in which no trace is left of the initial model and $\tilde{\Lambda} := \mathrm{g}^{-1}\Lambda\mathrm{g} + \mathrm{g}^{-1}D(\mathrm{g}) \to \Lambda \in \mathfrak{osp}(1|2)$. Then, expanding the vielbeine and Lagrange multipliers on the $\mathrm{OSp}(1|2)$ generators as $\tilde{e} = \tilde{e}^{\alpha\beta}L_{\alpha\beta} + \tilde{e}^{\alpha}Q_{\alpha}$ and $\Lambda := \tilde{x}^{\alpha\beta}L_{\alpha\beta} + \tilde{\theta}^{\alpha}Q_{\alpha}$ we can start computing (see the last paragraph in E.3. for more details)

$$[\tilde{e}, \tilde{e}] = \left(2i\tilde{e}^{\gamma(\alpha} \wedge \tilde{e}_{\gamma}{}^{\beta)} - \tilde{e}^{\alpha} \wedge \tilde{e}^{\beta}\right)L_{\alpha\beta} + \left(2i\tilde{e}^{\alpha\beta} \wedge \tilde{e}_{\beta}\right)Q_{\alpha} =: \upsilon^{\alpha\beta}L_{\alpha\beta} + \upsilon^{\alpha}Q_{\alpha} \, . \tag{4.40}$$

One can then exploit the results (E.49) and (E.50) with the choice $U := \Lambda$ and $V := [\tilde{e}, \tilde{e}]$ to find

$$\begin{aligned} \frac{1}{1 - ad_{\Lambda}}[\tilde{e}, \tilde{e}] = \frac{1}{1 - 2\tilde{x}^2}\big[(1 - \tilde{x}^2)Z^{\alpha\beta} - 2\tilde{x}_{\gamma}{}^{\alpha}\tilde{x}_{\delta}{}^{\beta}Z^{\gamma\delta} - 2i\tilde{x}_{\gamma}{}^{(\alpha}Z^{\beta)\gamma}\big]L_{\alpha\beta} + \\ + \frac{2}{2 - \tilde{x}^2}\zeta^{\alpha}[\delta_{\alpha}{}^{\beta} - i\tilde{x}_{\alpha}{}^{\beta}]Q_{\beta} \, , \end{aligned} \tag{4.41}$$

with

$$
\begin{aligned}
Z^{\alpha\beta} &= \upsilon^{\alpha\beta} + \frac{2}{2-\tilde{x}^2}\big[\upsilon^{(\alpha} - i\upsilon^{\gamma}\tilde{x}_\gamma{}^{(\alpha}\big]\tilde{\theta}^{\beta)} + \\
&\quad + \frac{i}{(2-\tilde{x}^2)(1-2\tilde{x}^2)}\big[(1-\tfrac{1}{2}\tilde{x}^2)\upsilon^{\alpha\beta} - 3\tilde{x}_\gamma{}^\alpha\tilde{x}_\delta{}^\beta\upsilon^{\gamma\delta} - 3i\tilde{x}_\gamma{}^{(\alpha}\upsilon^{\beta)\gamma}\big]\tilde{\theta}^2 \\
\zeta^\alpha &= \upsilon^\alpha - \frac{i}{(1-2\tilde{x}^2)}\big[(1-\tilde{x}^2)\upsilon^{\alpha\beta} - 2\tilde{x}_\gamma{}^\alpha\tilde{x}_\delta{}^\beta\upsilon^{\gamma\delta} - 2i\tilde{x}_\gamma{}^{(\alpha}\upsilon^{\beta)\gamma}\big]\tilde{\theta}_\beta + \\
&\quad - \frac{i}{(2-\tilde{x}^2)(1-2\tilde{x}^2)}\big[\tfrac{3}{2}\upsilon^\alpha - i(\tfrac{7}{2}-\tilde{x}^2)\upsilon^\beta\tilde{x}_\beta{}^\alpha\big]\tilde{\theta}^2 \ .
\end{aligned}
\tag{4.42}
$$

The exterior derivatives of the components of the vielbeine are then easily extracted

$$
\begin{aligned}
\mathrm{d}\tilde{e}^{\alpha\beta} &= -\frac{1}{2}\upsilon^{\alpha\beta} - \frac{1}{2(1-2\tilde{x}^2)}\big[(1-\tilde{x}^2)Z^{\alpha\beta} - 2\tilde{x}_\gamma{}^\alpha\tilde{x}_\delta{}^\beta Z^{\gamma\delta} - 2i\tilde{x}_\gamma{}^{(\alpha}Z^{\beta)\gamma}\big] \\
\mathrm{d}\tilde{e}^{\alpha} &= -\frac{1}{2}\upsilon^\alpha - \frac{2}{2(2-\tilde{x}^2)}\zeta^\beta[\delta_\beta{}^\alpha - i\tilde{x}_\beta{}^\alpha]
\end{aligned}
\tag{4.43}
$$

and we are ready to start looking at the torsion constraints (E.34), starting from the first one

$$
\mathrm{d}\tilde{e}^{\alpha\beta} - 2\tilde{e}^{\gamma(\alpha}\wedge\Omega_\gamma{}^{\beta)} = -\tfrac{1}{2}\tilde{e}^\alpha\wedge\tilde{e}^\beta \ . \tag{4.44}
$$

This is the most restrictive constraint, as it contains no free component of torsion and the only non-vanishing one has to be a constant. Looking at the left hand side of the constraint, we recognise that even after expanding the connection one form on the vielbeine $\Omega_\alpha{}^\beta = \tilde{e}^{\gamma\delta}\Omega_{\gamma\delta\,\alpha}{}^\beta + \tilde{e}^\gamma\Omega_{\gamma\,\alpha}{}^\beta$ it cannot contribute with a term proportional to $\tilde{e}^\alpha\wedge\tilde{e}^\beta$ that could compensate for the one on the right hand side of the equation. Hence, for the constraint to be satisfied one strictly needs that terms proportional to $\tilde{e}^\alpha\wedge\tilde{e}^\beta$ coming from $\mathrm{d}\tilde{e}^{\alpha\beta}$ should match the constant term on the right hand side of the equation. To study this requirement, it is sufficient to look at those terms in (4.43) which are proportional to $\upsilon^{\alpha\beta}$, as from (4.40) these are the only ones containing $\tilde{e}^\alpha\wedge\tilde{e}^\beta$. Hence, concentrating on terms involving the latter contribution we find

$$
\begin{aligned}
\mathrm{d}\tilde{e}^{\alpha\beta}|_{\upsilon^{\alpha\beta}} &= -\frac{1}{2}\upsilon^{\alpha\beta} - \frac{1}{2(1-2\tilde{x}^2)}\big[(1-\tilde{x}^2)\upsilon^{\alpha\beta} - 2\tilde{x}_\gamma{}^\alpha\tilde{x}_\delta{}^\beta\upsilon^{\gamma\delta} - 2i\tilde{x}_\gamma{}^{(\alpha}\upsilon^{\beta)\gamma}\big] + \\
&\quad - \frac{i}{2(2-\tilde{x}^2)(1-2\tilde{x}^2)^2}\big[(1+2\tilde{x}^4)\upsilon^{\alpha\beta} - 4(2-\tilde{x}^2)\tilde{x}_\gamma{}^\alpha\tilde{x}_\delta{}^\beta\upsilon^{\gamma\delta} - i(5+2\tilde{x}^2)\tilde{x}_\gamma{}^{(\alpha}\upsilon^{\beta)\gamma}\big]\tilde{\theta}^2.
\end{aligned}
\tag{4.45}
$$

From the latter expression and the definition $\upsilon^{\alpha\beta} := 2i\tilde{e}^{\gamma(\alpha}\wedge\tilde{e}_\gamma{}^{\beta)} - \tilde{e}^\alpha\wedge\tilde{e}^\beta$, one can then recognise that despite a first constant term proportional to $\tilde{e}^\alpha\wedge\tilde{e}^\beta$, all the remaining ones lead to non-vanishing and non-constant contributions which cannot be compensated by the right hand side of the above torsion constraint. For this reason we conclude that the torsion constraint is broken by the T-dual model. Exploiting the result (4.32) it is then also possible to get an additional confirmation of the incompatibility of the T-dual model with the supergravity requirements. Indeed combining (4.41) and the inner products (4.34) it is not hard to notice that the T-dual three-form contains components that should not be present in order to satisfy the requirement (E.36). In particular, one finds terms proportional to $\tilde{e}^\alpha \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma$ by looking at

$$H_3|_{\tilde{e}^\alpha\tilde{e}^\beta\tilde{e}^\gamma} = \frac{1}{2}\langle \tilde{e}^\alpha Q_\alpha, \frac{2}{2-\tilde{x}^2}\zeta^\beta[\delta_\beta{}^\delta - i\tilde{x}_\beta{}^\delta]Q_\delta\rangle \tag{4.46}$$

and considering once again the terms proportional to $\upsilon^{\alpha\beta}$ contained in ζ^α. This leads to

$$H_3|_{\tilde{e}^\alpha\tilde{e}^\beta\tilde{e}^\gamma} = \tilde{e}^\alpha \wedge \tilde{e}^\beta \wedge \tilde{e}^\gamma \frac{2}{1-2\tilde{x}^2}\left[i\tilde{x}_{(\alpha\beta}\tilde{\theta}_{\gamma)} - \frac{3}{2-\tilde{x}^2}\tilde{x}_{(\alpha\beta}\tilde{x}_{\gamma)\delta}\tilde{\theta}^\delta\right] . \tag{4.47}$$

Before surrendering to the impossibility of satisfying the supergravity constraints with the T-dual model, we shall consider the possibility of performing a local $\mathbf{OSp}(1|2)$ rotation on the vielbeine $\tilde{e} \to \tilde{e}^g := g^{-1}\tilde{e}g$ for $g \in \mathcal{C}^\infty(\Sigma, \mathbf{OSp}(1|2))$. While leaving the metric term in (4.29) unchanged, due to the Ad-invariance of the inner product, this rotation introduces new contributions to the B-field and one may thus hope to be able to choose g so as to satisfy both the requirements on torsion and H_3. To get the modified structure equation for the vielbeine $\tilde{e}^g$ we start noting that

$$\mathrm{d}\tilde{e}^g = g^{-1}(\mathrm{d}\tilde{e})g - [A, \tilde{e}^g] \qquad \text{with} \qquad A := g^{-1}\mathrm{d}g . \tag{4.48}$$

Hence, substituting $\mathrm{d}\tilde{e}$ with (4.30) one finds immediately

$$\nabla_A\tilde{e}^g = -\frac{1}{2}[\tilde{e}^g, \tilde{e}^g] - \frac{1}{2}\frac{1}{1-ad_{\Lambda^g}}[\tilde{e}^g, \tilde{e}^g] \qquad \text{with} \qquad \Lambda^g := g^{-1}\Lambda g \qquad \nabla_A := \mathrm{d} + [A, -]. \tag{4.49}$$

We can thus proceed as above, and start by expanding

$$\Lambda^g := y^{\alpha\beta}L_{\alpha\beta} + \lambda^\alpha Q_\alpha \qquad \tilde{e}^g := \tilde{e}^{\alpha\beta}L_{\alpha\beta} + \tilde{e}^\alpha Q_\alpha \qquad A := A^{\alpha\beta}L_{\alpha\beta} + A^\alpha Q_\alpha . \tag{4.50}$$

From (4.49) it is clear that new contributions to $\mathrm{d}\tilde{e}^{\alpha\beta}$ come from $[A, \tilde{e}^g]$, while the rest is left unchanged. For this reason we start looking for a possible choice of A that could cancel the unwanted terms proportional to $\tilde{e}^\alpha \wedge \tilde{e}^\beta$ encountered in (4.45), in which one should simply make the replacements $\tilde{x}^{\alpha\beta} \to y^{\alpha\beta}$ and $\tilde{\theta}^\alpha \to \lambda^\alpha$. One can then see that

$$[A, \tilde{e}^g]|_{\tilde{e}^\alpha\tilde{e}^\beta} = -\tilde{e}^\gamma \wedge \tilde{e}^\delta A_{(\gamma}{}^{(\alpha}\delta_{\delta)}{}^{\beta)} \tag{4.51}$$

and while this might be exploited to cancel terms in (4.45) which are linear in $y_\gamma{}^\alpha$, terms of the form $\tilde{e}^\gamma \wedge \tilde{e}^\delta y_{(\gamma}{}^{(\alpha}y_{\delta)}{}^{\beta)}$ could not be cancelled and hence $\mathrm{d}\tilde{e}^{\alpha\beta}$ still exhibits some non-constant contributions which spoil the torsion constraint.

In light of the above analysis it is now also possible to recognise that even upon dualising the $SL(2,\mathbb{R})_L$ subgroup of the left-isometry sector one would still not be able to satisfy the torsion constraints. Since $H^2(\mathfrak{sl}(2,\mathbb{R})) = 0$ also in this case we proceed by setting $D = 0$ in all the expressions derived in the previous section, and parametrising the group element for the initial model as $g = g_{bos}g_{fer}$ with $g_{bos} := e^{x^{\alpha\beta}L_{\alpha\beta}}$ and $g_{fer} := e^{-\theta^\alpha Q_\alpha}$, one can choose gauge such that $g_{bos} = \mathbb{1}$. Expanding the multipliers as $\Lambda := \tilde{x}^{\alpha\beta}L_{\alpha\beta}$ the dual model is then written in terms of

$$\tilde{\Lambda} := g_{fer}^{-1}\Lambda g_{fer} = (1 + \tfrac{i}{4}\theta^2)\tilde{x}^{\alpha\beta}L_{\alpha\beta} + i\theta^\gamma \tilde{x}_\gamma{}^\alpha Q_\alpha =: y^{\alpha\beta}L_{\alpha\beta} + \lambda^\alpha Q_\alpha \ . \tag{4.52}$$

The analysis carried out for the dualisation of the full $OSp(1|2)_L$ can thus be re-used by replacing $\tilde{x}^{\alpha\beta} \to y^{\alpha\beta}$ and $\tilde{\theta}^\alpha \to \lambda^\alpha$ in (4.45). This leads once again to the conclusion that the torsion constraint is broken by the dualisation.

We should mention that while restoring the fermionic components of connection could help solving the torsion constraints, this would be in contradiction with the fundamental assumption of Lorentzian structure group. This would correspond to studying super Riemannian geometry [114], known to have a complicated connection to supergravity [115, 116], via a limiting procedure in superspace [117]. As a final remark, while we considered general local $OSp(1|2)$ rotation of vielbeine, to try solving the torsion constraints, there might in principle still exist some choice of shift and/or field redefinition improving the situation, possibly at the cost of modifying the canonical torsion constraints A.6.. We thought about this possibility, but could not find any of them.

4.2. Symmetric and semi-symmetric spaces

We shall now discuss extension of the T-duality procedure introduced in the previous section to two other types of geometries, namely symmetric and semi-symmetric spaces.

4.2.1. Setup

These two classes of models belong to the family of *coset sigma models* on spaces G/H with G a Lie (super)group and H a Lie (super)subgroup, and are characterised by the fact that H arises as the invariant subset under the action of an automorphism σ of G. This has the property that $\sigma^k = 1$ with $k = 2$ for symmetric and $k = 4$ for semi-symmetric spaces [118, 119] [4] and induces an orthogonal decomposition of the Lie algebra $\mathfrak{g}$ which in turn exhibits restricted commutation relations. Like group manifolds, coset spaces G/H are characterised by the transitive left action G_L

[4] See also chapter 5 in [105] for a nice introduction and [120] for coset spaces admitting $\mathbb{Z}_m$-grading.

of the group on itself, but the right transitive action G_R is lost in favor of the equivalence of any two group elements differing by the right action of any $h \in H$. Physical sigma models on G/H will thus have to exhibit a global G_L invariance as well as a local $H_R : g \to gh$ gauge symmetry, and the description in terms of the Maurer-Cartan form $j := g^{-1}dg \in \Omega^1(\Sigma, \mathfrak{g})$, which is by construction invariant under the global left action $G_L : g \to g_L^{-1} g$ and transforms as $j \to h^{-1}jh + h^{-1}dh$ under local $H_R : g \to gh$, will exhibit some differences with respect to the principal chiral model case.

Symmetric spaces. For this class of models the automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ is an involution and the Lie (super)subalgebra $\mathfrak{h}$ has the property $\sigma(\mathfrak{h}) = \mathfrak{h}$. The rest of the (super)algebra, denoted by $\mathfrak{m}$, is then characterised by $\sigma(\mathfrak{m}) = -\mathfrak{m}$ and the following orthogonal decomposition takes place

$$\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{m} \qquad \text{with} \qquad \mathfrak{h} := \underbrace{\tfrac{1}{2}(1+\sigma)}_{=:P_{\mathfrak{h}}}(\mathfrak{g}) \qquad \mathfrak{m} := \underbrace{\tfrac{1}{2}(1-\sigma)}_{=:P_{\mathfrak{m}}}(\mathfrak{g}) \ . \tag{4.53}$$

The automorphism property $\sigma[X, Y] = [\sigma(X), \sigma(Y)] \quad \forall X, Y \in \mathfrak{g}$ then leads to commutators

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \qquad [\mathfrak{m}, \mathfrak{h}] \subseteq \mathfrak{m} \qquad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h} \ . \tag{4.54}$$

The non-degenerate, (graded-)symmetric, Ad-invariant bilinear form $\langle -, - \rangle$ on the Lie (super)algebra is taken to be compatible with the above decomposition and only non-vanishing for couples of elements in the same subspace. Elements in $\mathfrak{h}$ are said to be of homogeneity $|\mathfrak{h}| = 0$, while elements in $\mathfrak{m}$ of homogeneity $|\mathfrak{m}| = 2$, hence one has that $\langle X, Y \rangle \neq 0$ only for $|X| + |Y| = 0 \bmod 4$.

The above algebra decomposition also reflects on the Maurer-Cartan form $j := g^{-1}dg \in \Omega^1(\Sigma, \mathfrak{g})$

$$j = A + m \qquad \text{with} \qquad A := P_{\mathfrak{h}}(j) \ \in \ \Omega^1(\Sigma, \mathfrak{h}) \qquad m := P_{\mathfrak{m}}(j) \ \in \ \Omega^1(\Sigma, \mathfrak{m}) \ . \tag{4.55}$$

While invariance of j under global $G_L : g \to g_L^{-1} g$ implies the invariance of A and m, the transformation $j \to h^{-1}jh + h^{-1}dh$ under local $H_R : g \to gh$ implies that $A \to A + h^{-1}dh$ and $m \to h^{-1}mh$. For this reason, symmetric-space sigma model actions can be constructed as

$$S := \frac{1}{2} \int_\Sigma \langle m, \star m \rangle \tag{4.56}$$

and exhibit global G_L invariance, with associated Noether current $L_N := gmg^{-1}$, as well as local H_R invariance. The equations of motion of the above action read

$$\nabla_A \star m = 0 \tag{4.57}$$

while the Maurer-Cartan equation $dj + \frac{1}{2}[j, j] = 0$ now decomposes on the two subspaces as

$$\mathfrak{h}: \quad F_A + \tfrac{1}{2}[m, m] = 0 \qquad\qquad \mathfrak{m}: \quad \nabla_A m = 0 \,. \tag{4.58}$$

Like for principal chiral models, the Maurer-Cartan equations and equations of motion for symmetric spaces can be encoded in the flatness condition of a Lax connection, which ensures classical integrability of the models. Parametrising $J = aA + b \star A + cm + e \star m$, with a, b, c, e coefficients to be determined, the flatness condition $\mathrm{d}J + \frac{1}{2}[J, J] = 0$ implies the above equations of motion and Maurer-Cartan equations provided that $b = 0$, $a = 1$ and $c^2 - e^2 = 1$. These can be solved, for example, as $c := \frac{1}{4}(z^2 + z^{-2})$ and $e := \frac{1}{4}(z^2 - z^{-2})$, so that $J \equiv J(z)$.

Semi-symmetric spaces. For these models, the automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ satisfies $\sigma^4 = 1$ and the Lie subalgebra $\mathfrak{h} \equiv \mathfrak{g}_0$ enjoys the property $\sigma(\mathfrak{g}_0) = \mathfrak{g}_0$. The remaining elements of the superalgebra are characterised by $\sigma(\mathfrak{g}_1) = i\mathfrak{g}_1$, $\sigma(\mathfrak{g}_2) = -\mathfrak{g}_2$ or $\sigma(\mathfrak{g}_3) = -i\mathfrak{g}_3$, so that one could identify the action of σ on four subspaces $\sigma(\mathfrak{g}_k) = i^k \mathfrak{g}_k$, with $k = 0, 1, 2, 3$, leading to the decomposition

$$\mathfrak{g} \simeq \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \quad \text{with} \quad \mathfrak{g}_k := \tfrac{1}{4}(1 + i^{3k}\sigma + i^{2k}\sigma^2 + i^k\sigma^3) =: P_k(\mathfrak{g}) \ , \ \ P_k P_l = \delta_{kl} P_l . \tag{4.59}$$

Exploiting again the automorphism property $\sigma[\mathfrak{g}_i, \mathfrak{g}_j] = [\sigma(\mathfrak{g}_i), \sigma(\mathfrak{g}_j)]$ one finds commutators

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{(i+j) mod 4} \tag{4.60}$$

From $\sigma(\mathfrak{g}_k) = i^k \mathfrak{g}_k$ one also recognises that $\sigma^2(\mathfrak{g}_k) = (-1)^k \mathfrak{g}_k$, i.e. σ^2 acts as $(-1)^F$ and for this reason $\mathfrak{g}_0, \mathfrak{g}_2$ and $\mathfrak{g}_1, \mathfrak{g}_3$ are respectively purely bosonic and purely fermionic subspaces. Elements in $\mathfrak{g}_k$ are said to be of homogeneity k and also in this case the non-degenerate, (graded-)symmetric, Ad-invariant bilinear form $\langle -, - \rangle$ is taken to be compatible with the above decomposition, such that $\langle X, Y \rangle \neq 0$ for $|X| + |Y| = 0$ mod 4. To simplify the notation in subsequent paragraphs, and more easily make contact with symmetric spaces, we rename the four subspaces as

$$\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{p} \oplus \mathfrak{m} \oplus \mathfrak{q} \ , \tag{4.61}$$

so that non-vanishing commutation relations read

$$\begin{gathered}
[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \qquad [\mathfrak{p}, \mathfrak{h}] \subseteq \mathfrak{p} \qquad [\mathfrak{m}, \mathfrak{h}] \subseteq \mathfrak{m} \qquad [\mathfrak{q}, \mathfrak{h}] \subseteq \mathfrak{q} \\
[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{m} \qquad [\mathfrak{m}, \mathfrak{p}] \subseteq \mathfrak{q} \qquad [\mathfrak{q}, \mathfrak{p}] \subseteq \mathfrak{h} \\
[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h} \qquad [\mathfrak{q}, \mathfrak{m}] \subseteq \mathfrak{p} \\
[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{m} \ .
\end{gathered} \tag{4.62}$$

Also in this case the above algebra decomposition reflects on the Maurer-Cartan form j

$$\begin{aligned} & j = A + p + m + q \qquad \text{with} \\ A := P_{\mathfrak{h}}(j) \in \Omega^1(\Sigma, \mathfrak{h}) \qquad & p := P_{\mathfrak{p}}(j) \in \Omega^1(\Sigma, \mathfrak{p}) \\ m := P_{\mathfrak{m}}(j) \in \Omega^1(\Sigma, \mathfrak{m}) \qquad & q := P_{\mathfrak{q}}(j) \in \Omega^1(\Sigma, \mathfrak{q}) \ . \end{aligned} \tag{4.63}$$

While invariance of j under global $\mathsf{G}_L : \mathrm{g} \to g_L^{-1}\mathrm{g}$ implies invariance of A, p, m, q, the transformation $j \to h^{-1}jh+h^{-1}\mathrm{d}h$ under local $\mathsf{H}_R : \mathrm{g} \to \mathrm{g}h$ implies $A \to A+h^{-1}\mathrm{d}h$ and $\{p, m, q\} \to h^{-1}\{p, m, q\}h$, so that semi-symmetric space sigma models can be constructed as

$$S := \frac{1}{2}\int_\Sigma \langle m, \star m\rangle + \kappa \langle p, q\rangle \ . \tag{4.64}$$

These exhibit global G_L invariance, with associated Noether current $L_N := \mathrm{g}\big(m - \frac{\kappa}{2} \star (p-q)\big)\mathrm{g}^{-1}$, as well as local H_R invariance. The equations of motion of the above action read F.1.

$$\begin{aligned} \nabla_A \star m - \frac{\kappa}{2}[p, p] + \frac{\kappa}{2}[q, q] = 0 \\ \frac{\kappa}{2}\nabla_A q + [p, \star m - \frac{\kappa}{2}m] = 0 \\ \frac{\kappa}{2}\nabla_A p - [q, \star m + \frac{\kappa}{2}m] = 0 \ , \end{aligned} \tag{4.65}$$

while the Maurer-Cartan equation $dj + \frac{1}{2}[j, j] = 0$ decomposes on the four subspaces as

$$\begin{aligned} \mathfrak{h} : \qquad & F_A + \frac{1}{2}[m, m] + [p, q] = 0 \\ \mathfrak{p} : \qquad & \nabla_A p + [m, q] = 0 \\ \mathfrak{m} : \qquad & \nabla_A m + \frac{1}{2}[p, p] + \frac{1}{2}[q, q] = 0 \\ \mathfrak{q} : \qquad & \nabla_A q + [m, p] = 0 \ . \end{aligned} \tag{4.66}$$

Also for this class of models, the equations of motion and Maurer-Cartan equations can be encoded in the flatness of a Lax connection, which now takes the form of

$$J(z) := A + zp + \frac{1}{2}(z^2 + z^{-2})m + z^{-1}q - \frac{1}{2\kappa}(z^2 - z^{-2}) \star m \ . \tag{4.67}$$

It should furthermore be noted that the action (4.64) is of the Green-Schwarz form and the above Lax connection can only be constructed provided that the relative coefficient κ between the metric and B-field term is either $+1$ or -1. See F.1. for more on this. Importantly, this condition is the same one required to have κ-symmetry of the action [121, 45], as we shall discuss soon, and hence supercoset sigma models of the form (4.64) with $\kappa = \pm 1$ are both integrable and κ-

symmetric [103], thus representing very special instances of superstring sigma models.

From the above discussion one can recognise that semi-symmetric spaces are special cases of symmetric ones and indeed all relevant expressions for the latter can be obtained by formally setting to zero any element in $\mathfrak{p}, \mathfrak{q}$ from the former. For this reason we shall from now on present the analysis only in terms of semi-symmetric models.

Topological deformations. Symmetric and semi-symmetric space sigma models can be deformed in analogy with principal chiral models, hence leading to the following action

$$S^{\Omega}_{SS} := \frac{1}{2}\int_{\Sigma} \langle m, \star m\rangle + \kappa\langle p, q\rangle + \Omega(j, j)\ . \tag{4.68}$$

Exactly like for principal chiral models, this does not alter the equations of motion nor the Noether current, thus leaving unaffected the construction of the Lax connection. The only difference encountered for the above two classes of coset models, lies in the requirement of H_R invariance of the action under local transformations $\mathrm{g} \to \mathrm{g}h$ for $h \in \mathsf{H}$, which makes the Maurer-Cartan current transform as $j \to h^{-1}jh + h^{-1}\mathrm{d}h$. In turn, exploiting the relation (4.6) and $D(h^{-1}\mathrm{d}h) = h^{-1}\mathrm{d}\big(D(h)h^{-1}\big)h$, the latter implies that

$$\Omega(j, j) \quad \to \quad \Omega(j, j) + \Omega(h^{-1}\mathrm{d}h, h^{-1}\mathrm{d}h)\ , \tag{4.69}$$

so that H_R invariance is retained upon restricting $\Omega|_{\mathfrak{h}} = 0$. For symmetric spaces, this requirement leads to the condition $D : \mathfrak{h} \to \mathfrak{m}$, as the inner product ensures that $\Omega(H_1, H_2) = \langle D(H_1), H_2\rangle = \langle M_1, H_2\rangle = 0$ for $H_1, H_2 \in \mathfrak{h}$ and $M_1 \in \mathfrak{m}$. From the derivation property of D it then follows that for the commutation relations to be preserved one also needs $D : \mathfrak{m} \to \mathfrak{h}$. In semi-symmetric spaces, these conditions are further supplemented by the requirements $D : \mathfrak{p} \to \mathfrak{q}$ and $D : \mathfrak{q} \to \mathfrak{p}$.

Gauging. Given the deformed (semi)-symmetric coset action (4.68) one can gauge a subgroup of the isometry group $\mathsf{K}_L \subseteq \mathsf{G}_L$ as done for the principal chiral models around equation (4.10), so that upon introducing Lagrange multipliers $\Lambda \in \mathcal{C}^{\infty}(\Sigma, \mathfrak{k}_L)$ the following master action is obtained

$$S_{\omega} := \frac{1}{2}\int_{\Sigma} \langle m_{\omega}, \star m_{\omega}\rangle + \kappa\langle p_{\omega}, q_{\omega}\rangle + \langle D(j_{\omega}), j_{\omega}\rangle + \int_{\Sigma}\langle \tilde{\Lambda}, F_{j_{\omega}}\rangle\ , \tag{4.70}$$

where $\tilde{\Lambda} = \mathrm{g}^{-1}\Lambda\mathrm{g} + \mathrm{g}^{-1}D(\mathrm{g})$ and $A_{\omega}, p_{\omega}, m_{\omega}, q_{\omega}$ are the projections of j_{ω} on the four subspaces. The above action is again manifestly K_L gauge invariant due to the invariance of j_{ω} and of $\tilde{\Lambda}$, which follows from the transformation law $\Lambda \to k^{-1}\Lambda k + k^{-1}D(k)$ with $k \in \mathsf{K}_L$ under $\mathrm{g} \to k^{-1}\mathrm{g}$. At the same time, it also retains H_R gauge invariance as, by construction, the first two terms are left unchanged by $\mathrm{g} \to \mathrm{g}h$ due to the transformation laws $\{p_{\omega}, m_{\omega}, q_{\omega}\} \to h^{-1}\{p_{\omega}, m_{\omega}, q_{\omega}\}h$, while the last two terms generate compensating contributions. Indeed using $\Omega|_{\mathfrak{h}} = 0$, $D(h^{-1}\mathrm{d}h) = h^{-1}\mathrm{d}\big(D(h)h^{-1}\big)h$ and (4.6), one finds that $\langle D(j_\omega), j_\omega\rangle \to \langle D(j_\omega), j_\omega\rangle - 2\langle D(h)h^{-1}, F_{j_\omega}\rangle$, while $\langle\tilde{\Lambda}, F_{j_\omega}\rangle \to \langle\tilde{\Lambda}, F_{j_\omega}\rangle + \langle D(h)h^{-1}, F_{j_\omega}\rangle$. Cancellation of the two contributions thus fixes the relative coefficient between the deformation and Lagrange multiplier terms in the action. T-duality can now be performed by integrating out the gauge fields rather than the Lagrange multipliers, which would otherwise lead back to the initial action after an appropriate choice of gauge. At the end of the dualisation one can finally proceed with the gauge fixing of the extra coordinates, so that the correct number of degrees of freedom is recovered. For coset models G/H this procedure is slightly different from the case of principal chiral models, as one generally needs to gauge fix some of the Lagrange multipliers together with the initial coordinates. Consider for example the case in which one gauges the full group of isometry $\mathrm{K}_L = \mathrm{G}_L$

- Fixing an H_R gauge in the initial model, one begins with $dim(\mathrm{G}) - dim(\mathrm{H})$ initial coordinates x. At the end of the dualisation one also has $dim(\mathrm{G})$ multipliers $\tilde{x}$, but having already exploited the H_R gauge symmetry to fix some of the initial coordinates, only the G_L gauge symmetry remains available. This can be exploited to remove $dim(\mathrm{G})$ degrees of freedom, but since the initial coordinates left are only $dim(\mathrm{G}) - dim(\mathrm{H})$, one will necessarily have to fix some of the multipliers as well.

- Performing dualisation before exploiting the H_R gauge symmetry one obtains a T-dual model containing $dim(\mathrm{G})$ initial coordinates x and $dim(\mathrm{G})$ Lagrange multipliers $\tilde{x}$. Having gauged the full group of isometry one can now exploit the G_L gauge symmetry to completely get rid of the initial coordinates, i.e. choose gauge $\mathrm{g} = \mathbb{1}$. To correctly recover $dim(\mathrm{G}) - dim(\mathrm{H})$ degrees of freedom it is then necessary to exploit the residual H_R gauge symmetry, which can however only remove Lagrange multipliers.

κ-symmetry of the master action. We now describe the conditions under which the generic master action (4.70) of a semi-symmetric space sigma model enjoys κ-symmetry. Interestingly, these will formally look the same as the ones required for κ-symmetry of the initial action (4.64), derived in [103] (from which we shall borrow some notation) and argued to imply the supergravity torsion constraints. While the requirements for κ-symmetry of the T-dual models will not be discussed here, these should be similar to those of the master action, as the gauge fields integrated out along the dualisation are inert under such fermionic transformations. This should in turn imply that the T-dual model satifies the supergravity torsion constraints and indeed it has been argued in [83]. To begin, we recall the McArthur interpretation of κ-symmetry in terms of the right action of a purely fermionic local and infinitesimal group element g_f on the group element g [122]

$$\mathrm{g} \quad\rightarrow\quad \mathrm{g}g_f \qquad \text{with} \qquad g_f = e^{\epsilon} \simeq 1 + \epsilon \qquad \text{and} \qquad \epsilon := \epsilon_{\mathfrak{p}} + \epsilon_{\mathfrak{q}} \ . \tag{4.71}$$

Having, for coset models, no canonical definition for the right action of a generic group element, the idea of McArthur is that of determining the fermionic parameters $\epsilon_{\mathfrak{p}}$ and $\epsilon_{\mathfrak{q}}$ by requiring the transformation to leave the action invariant. Since the gauge fields ω are supposed to be integrated out in the dualisation procedure, discussed in the next section, κ-symmetry of the master action should not rely on their transformation properties under (4.71), as they do not appear in the T-dual model. On the other hand, Lagrange multipliers may in principle transform under the above fermionic action, as they play the role of coordinates in the dual model. We shall indeed see that they need to transform for the master action to be invariant. From the definition $j_\omega := j + g^{-1}\omega g$ one finds that

$$\mathrm{g} \quad \to \quad \mathrm{g}g_f \quad \Rightarrow \quad j_\omega \to j_\omega + \nabla_{j_\omega}\epsilon \quad \Rightarrow \begin{cases} \delta_\kappa A_\omega = [q_\omega, \epsilon_{\mathfrak{p}}] + [p_\omega, \epsilon_{\mathfrak{q}}] \\ \delta_\kappa p_\omega = \mathrm{d}\epsilon_{\mathfrak{p}} + [A_\omega, \epsilon_{\mathfrak{p}}] + [m_\omega, \epsilon_{\mathfrak{q}}] \\ \delta_\kappa m_\omega = [p_\omega, \epsilon_{\mathfrak{p}}] + [q_\omega, \epsilon_{\mathfrak{q}}] \\ \delta_\kappa q_\omega = \mathrm{d}\epsilon_{\mathfrak{q}} + [A_\omega, \epsilon_{\mathfrak{q}}] + [m_\omega, \epsilon_{\mathfrak{p}}] \end{cases} \tag{4.72}$$

and additionally

$$\mathrm{g} \quad \to \quad \mathrm{g}g_f \quad \Rightarrow \quad D(\mathrm{g})\mathrm{g}^{-1} \quad \to \quad D(\mathrm{g})\mathrm{g}^{-1} + \mathrm{g}D(\epsilon)\mathrm{g}^{-1} \ . \tag{4.73}$$

One can then compute the variations of the master action (4.70) under $\mathrm{g} \to \mathrm{g}g_f$ and substitute the above relations. Since j_ω is not flat one cannot use the Maurer-Cartan equations to simplify the variations, as it could on the other hand be done for κ-symmetry of the initial model. Nevertheless, recalling that $F_{j_\omega} = \mathrm{g}^{-1}F_\omega \mathrm{g}$ one can rearrange the variations as

$$\delta_\kappa S_\omega = \int_\Sigma \langle \epsilon_{\mathfrak{p}}, [p_\omega, (\kappa - \star)m_\omega]\rangle - \langle \epsilon_{\mathfrak{q}}, [q_\omega, (\kappa + \star)m_\omega]\rangle + \delta_\kappa\gamma + \langle \delta_\kappa\Lambda - \tfrac{\kappa}{2}\mathrm{g}[(P_{\mathfrak{p}} - P_{\mathfrak{q}})\epsilon]\mathrm{g}^{-1}, F_\omega\rangle \tag{4.74}$$

where the term $\delta_\kappa\gamma$ encodes the variations of the worldsheet metric, which will be specified soon. When the last term in the above variations vanishes, one formally recovers the same structure that would have been obtained by varying, with respect to $\mathrm{g} \to \mathrm{g}g_f$, the initial action (4.64), with the simple replacements $\{p_\omega, m_\omega, q_\omega\} \to \{p, m, q\}$. Hence we learn the Lagrange multipliers should transform under κ-symmetry so as to cancel the last term. To analyse it in more details, we first notice that $F_\omega \in \mathfrak{k}_L$ and $\delta_\kappa\Lambda \in \mathfrak{k}_L$, while $\frac{\kappa}{2}\mathrm{g}[(P_{\mathfrak{p}} - P_{\mathfrak{q}})\epsilon]\mathrm{g}^{-1} \in \mathfrak{g}$ and thus proceed by decomposing

$$X := \tfrac{\kappa}{2}\mathrm{g}[(P_{\mathfrak{p}} - P_{\mathfrak{q}})\epsilon]\mathrm{g}^{-1} = \mathcal{K}(X) + \mathcal{K}^\perp(X) \ , \tag{4.75}$$

where $\mathcal{K}$ and $\mathcal{K}^\perp$ respectively represent projectors on the gauged subalgebra $\mathfrak{k}_L$ and its orthogonal complement $\mathfrak{k}_L^\perp$, on which $\mathfrak{g}$ decomposes as $\mathfrak{g} \simeq \mathfrak{k}_L \oplus \mathfrak{k}_L^\perp$ with respect to the inner product. This means that one can rewrite the extra term as

$$\langle \delta_\kappa \Lambda - X, F_\omega \rangle = \langle \delta_\kappa \Lambda - \mathcal{K}(X) - \mathcal{K}^\perp(X), F_\omega \rangle = \langle \delta_\kappa \Lambda - \mathcal{K}(X), F_\omega \rangle \ , \tag{4.76}$$

using in the last step that, by definition of orthogonal complement, $\langle \mathfrak{k}_L^\perp, \mathfrak{k}_L \rangle = 0$. Consequently, the unwanted term vanishes provided that the Lagrange multipliers enjoy the following κ-symmetry transformations

$$\delta_\kappa \Lambda = \mathcal{K}(\tfrac{1}{2} g[(P^{(1)} - P^{(3)})\epsilon] g^{-1}) \ . \tag{4.77}$$

Under this assumption one is left with

$$\delta_\kappa S_\omega = \int_\Sigma \langle \epsilon_\mathfrak{p}, [p_\omega, (\kappa - \star) m_\omega] \rangle - \langle \epsilon_\mathfrak{q}, [q_\omega, (\kappa + \star) m_\omega] \rangle + \delta_\kappa \gamma \ . \tag{4.78}$$

As anticipated, this has the same structure as the variations one would have obtained for the initial action (4.64). In the latter expression one can define worldsheet operators $P_\pm^\kappa := \frac{1}{2}(\kappa \pm \star)$ and notice they satisfy relations

$$P_\pm^\kappa P_\pm^\kappa = \tfrac{1}{4}[(\kappa^2 + 1) \pm 2\kappa\star] \qquad\qquad P_\pm^\kappa P_\mp^\kappa = \tfrac{1}{4}(\kappa^2 - 1) \ . \tag{4.79}$$

Hence, for $\kappa = \pm 1$ one recovers the worldsheet projectors $P_\pm := \frac{1}{2}(1 \pm \star)$

$$\kappa = 1 : \qquad P_\pm^\kappa \equiv P_\pm \qquad\qquad \kappa = -1 : \quad P_\pm^\kappa \equiv -P_\mp \ . \tag{4.80}$$

The projection property of the above operators will soon prove important in showing κ-symmetry of the master action - and the same would be true for the initial action (4.64) - for this reason we shall from now on assume that $\kappa = +1$. Before proceeding further we also need to introduce some extra notation, for both the Lie algebra $\mathfrak{g}$ and the component form of the above projectors. For an algebra $\mathfrak{g}$ enjoying the decomposition (4.61) we explicitly denote the generators from the various subspaces as

$$\mathfrak{h} = span\{H_{\hat{a}}\} \qquad \mathfrak{p} = span\{Q_\alpha\} \qquad \mathfrak{m} = span\{M_a\} \qquad \mathfrak{q} = span\{S_{\hat{\alpha}}\} \ . \tag{4.81}$$

The projectors enjoy the following properties

$$\begin{gathered} P_\pm P_\pm = P_\pm \qquad P_\pm P_\mp = 0 \qquad 1 = P_+ + P_- \qquad \star = P_+ - P_- \\ \langle \alpha, P_\pm \beta \rangle = \langle P_\mp \alpha, \beta \rangle \quad \Rightarrow \quad \langle P_\pm \alpha, P_\pm \beta \rangle = 0 \qquad \forall \alpha, \beta \in \Omega^1(\Sigma, \mathfrak{g}) \ , \end{gathered} \tag{4.82}$$

but in checking κ-symmetry we shall need their component form $P_{\pm}^{ij}$. To find this we start by recalling that given 1-forms α, β one has

$$\langle \alpha, \beta \rangle = \epsilon^{ij} \langle \alpha_i, \beta_j \rangle d^2\sigma \qquad \langle \alpha, \star\beta \rangle = \gamma^{ij} \langle \alpha_i, \beta_j \rangle d^2\sigma \ , \tag{4.83}$$

with $\epsilon^{ij} = -\epsilon^{ji}$ and $\gamma^{ij} := \sqrt{-h} h^{ij}$, for $h := det(h_{ij})$ and h_{ij} the worldsheet metric. In the above notation α_i, β_j are still Lie algebra valued but lose the 1-form property and for this reason the inner product is now symmetric, i.e. while $\langle \alpha, \beta \rangle = -\langle \beta, \alpha \rangle$ we have $\langle \alpha_i, \beta_j \rangle = \langle \beta_j, \alpha_i \rangle$. The component form of the projectors can then be obtained by defining

$$\alpha_i := \gamma_{ij} \alpha^j \qquad \alpha_{\pm}^i := P_{\pm}^{ij} \alpha_j \tag{4.84}$$

and looking at the following two expressions

$$\begin{aligned}
\langle \alpha, P_{\pm}\beta \rangle &= \epsilon^{ij} \langle \alpha_i, (\beta^{\pm})_j \rangle d^2\sigma = \epsilon^{ij} \gamma_{jk} \langle \alpha_i, (\beta_{\pm})^k \rangle = \epsilon^{ij} \gamma_{jk} P_{\pm}^{kl} \langle \alpha_i, \beta_l \rangle d^2\sigma \\
\langle \alpha, P_{\pm}\beta \rangle &= \tfrac{1}{2} \langle \alpha, \beta \rangle \pm \tfrac{1}{2} \langle \alpha, \star\beta \rangle = \tfrac{1}{2} (\epsilon^{ij} \pm \gamma^{ij}) \langle \alpha_i, \beta_j \rangle d^2\sigma \ .
\end{aligned} \tag{4.85}$$

Comparison leads to a condition and the required component form

$$\epsilon^{il} \gamma_{lk} P_{\pm}^{kj} \equiv \tfrac{1}{2} (\epsilon^{ij} \pm \gamma^{ij}) \qquad \Rightarrow \qquad P_{\pm}^{ij} = \tfrac{1}{2} (\gamma^{ij} \pm \epsilon^{ij}) = P_{\mp}^{ji} \ . \tag{4.86}$$

In components, the projectors satisfy the following important properties

$$\begin{aligned}
P_{\pm}^{ij} \gamma_{jk} P_{\pm}^{kl} &= P_{\pm}^{il} & \qquad P_{\pm}^{ij} \gamma_{jk} P_{\mp}^{kl} &= 0 \\
\gamma^{ij} &= P_{+}^{ij} + P_{-}^{ij} & \qquad \epsilon^{ij} &= P_{+}^{ij} - P_{-}^{ij} \\
P_{\pm}^{ij} P_{\pm}^{kl} &= P_{\pm}^{kj} P_{\pm}^{il} & \qquad P_{\pm}^{ij} P_{\pm}^{kl} &= P_{\pm}^{il} P_{\pm}^{kj} \ .
\end{aligned} \tag{4.87}$$

All of them are simple to verify and, in particular, the last one is a direct consequence of the fact that $det(\gamma^{ij}) = -1$. Indeed one can explicitly write down the relation $det(\gamma^{ij}) = \gamma^{11}\gamma^{22} - (\gamma^{12})^2 = -1$ and use it to check that $P_{\pm}^{ij} P_{\pm}^{kl} = P_{\pm}^{kj} P_{\pm}^{il}$ holds true for all possible values of $i, j, k, l \in \{1, 2\}$. We are now ready to rewrite the variations of the master action as

$$\delta_\kappa S_\omega = \int_\Sigma 2\langle \epsilon_{\mathfrak{p}}, [p_\omega, m_\omega^-] \rangle - 2\langle \epsilon_{\mathfrak{q}}, [q_\omega, m_\omega^+] \rangle + \tfrac{1}{2} \delta_\kappa \gamma^{ij} \langle (m_\omega)_i, (m_\omega)_j \rangle \mathrm{d}^2\sigma \ , \tag{4.88}$$

where we explicitly wrote the term

$$\delta_\kappa \gamma := +\tfrac{1}{2} \delta_\kappa \gamma^{ij} \langle (m_\omega)_i, (m_\omega)_j \rangle \mathrm{d}^2\sigma \ . \tag{4.89}$$

To proceed further we need to make a choice of fermionic parameters which allows the two unwanted terms to be cancelled by the variations of the worldsheet metric. Introducing $\kappa_{\mathfrak{p}} \in \mathfrak{p}$ and $\kappa_{\mathfrak{q}} \in \mathfrak{q}$ we thus make the ansatz

$$\epsilon_{\mathfrak{p}} := [(m_\omega)_i, \kappa^i_{\mathfrak{q}}] \quad \in \mathfrak{p} \qquad\qquad \epsilon_{\mathfrak{q}} := [(m_\omega)_i, \kappa^i_{\mathfrak{p}}] \quad \in \mathfrak{q} \ , \tag{4.90}$$

and proceed by introducing explicit worldsheet indices in all terms

$$\begin{aligned}\delta_\kappa S_\omega &= \int_\Sigma 2\epsilon^{kl}\langle\epsilon_{\mathfrak{p}}, [(p^+_\omega)_k, (m_\omega)_l]\rangle d^2\sigma - 2\epsilon^{kl}\langle\epsilon_{\mathfrak{q}}, [(q^-_\omega)_k, (m_\omega)_l]\rangle d^2\sigma + \delta_\kappa\gamma = \\ &= \int_\Sigma 2\epsilon^{kl}\gamma_{kn}\big(P^{nj}_+\langle\epsilon_{\mathfrak{p}}, [(p_\omega)_j, (m_\omega)_l]\rangle - P^{nj}_-\langle\epsilon_{\mathfrak{q}}, [(q_\omega)_j, (m_\omega)_l]\rangle\big) d^2\sigma + \delta_\kappa\gamma = \\ &= \int_\Sigma \mathrm{d}^2\sigma \tfrac{\delta_\kappa\gamma^{ij}}{2}\langle(m_\omega)_i, (m_\omega)_j\rangle - 2\langle[(m_\omega)_i, \kappa^i_{\mathfrak{q}}], [(p^+_\omega)^j, (m_\omega)_j]\rangle - 2\langle[(m_\omega)_i, \kappa^i_{\mathfrak{p}}], [(q^-_\omega)^j, (m_\omega)_j]\rangle\end{aligned} \tag{4.91}$$

where we used the relation $\epsilon^{kl}\gamma_{kn}P^{nj}_\pm = \mp P^{lj}_\pm$ to get to the third line. At this point we exploit the notation introduced above for the Lie algebra generators to rewrite

$$\begin{cases}[(m_\omega)_i, \kappa^i_{\mathfrak{q}}] = (m_\omega)^a_i \kappa^{i\hat\alpha}_{\mathfrak{q}} f_{a\hat\alpha}{}^\beta Q_\beta \\ [(m_\omega)_i, \kappa^i_{\mathfrak{p}}] = (m_\omega)^a_i \kappa^{i\alpha}_{\mathfrak{p}} f_{a\alpha}{}^{\hat\beta} S_{\hat\beta}\end{cases} \qquad \begin{cases}[(p^+_\omega)^j, (m_\omega)_j] = (p^+_\omega)^{j\alpha}(m_\omega)^b_j f_{\alpha b}{}^{\hat\beta} S_{\hat\beta} \\ [(q^-_\omega)^j, (m_\omega)_j] = (q^-_\omega)^{j\hat\alpha}(m_\omega)^b_j f_{\hat\alpha b}{}^{\beta} Q_\beta\end{cases} \ , \tag{4.92}$$

so that the variations can be rearranged as

$$\delta_\kappa S_\omega = \int_\Sigma \mathrm{d}^2\sigma\big[\tfrac{1}{2}(\delta_\kappa\gamma^{ij})\delta_{ab} + 2(p^+_\omega)^{j\alpha}\kappa^{i\hat\alpha}_{\mathfrak{q}}\delta_{\hat\beta\beta} f_{\alpha a}{}^{\hat\beta} f_{b\hat\alpha}{}^\beta - 2(q^-_\omega)^{j\hat\alpha}\kappa^{i\alpha}_{\mathfrak{p}}\delta_{\beta\hat\beta} f_{a\hat\alpha}{}^\beta f_{b\alpha}{}^{\hat\beta}\big](m_\omega)^a_j(m_\omega)^b_i \ . \tag{4.93}$$

To manipulate the above variations we add zero in the form

$$\begin{aligned}&\big[(p^+_\omega)^{j\alpha}\kappa^{i\hat\alpha}_{\mathfrak{q}}\delta_{\hat\beta\beta} f_{\alpha b}{}^{\hat\beta} f_{a\hat\alpha}{}^\beta - (p^+_\omega)^{j\alpha}\kappa^{i\hat\alpha}_{\mathfrak{q}}\delta_{\hat\beta\beta} f_{\alpha b}{}^{\hat\beta} f_{a\hat\alpha}{}^\beta\big](m_\omega)^a_j(m_\omega)^b_i = 0 \\ &\big[(q^-_\omega)^{j\hat\alpha}\kappa^{i\alpha}_{\mathfrak{p}}\delta_{\beta\hat\beta} f_{b\hat\alpha}{}^{\beta} f_{a\alpha}{}^{\hat\beta} - (q^-_\omega)^{j\hat\alpha}\kappa^{i\alpha}_{\mathfrak{p}}\delta_{\beta\hat\beta} f_{b\hat\alpha}{}^{\beta} f_{a\alpha}{}^{\hat\beta}\big](m_\omega)^a_j(m_\omega)^b_i = 0 \ ,\end{aligned} \tag{4.94}$$

so that using $\delta_{\hat\beta\beta} = -\delta_{\beta\hat\beta}$ and $f_{\alpha a}{}^{\hat\beta} = -f_{a\alpha}{}^{\hat\beta}$ the terms can be rearranges as

$$\begin{aligned}\delta_\kappa S_\omega = \int_\Sigma \mathrm{d}^2\sigma\Big(&\tfrac{1}{2}(\delta_\kappa\gamma^{ij})\delta_{ab} + [(p^+_\omega)^{j\alpha}\kappa^{i\hat\alpha}_{\mathfrak{q}} - (q^-_\omega)^{j\hat\alpha}\kappa^{i\alpha}_{\mathfrak{p}}]\delta_{\beta\hat\beta}[f_{a\hat\alpha}{}^\beta f_{b\alpha}{}^{\hat\beta} + f_{b\hat\alpha}{}^\beta f_{a\alpha}{}^{\hat\beta}] + \\ &- [(p^+_\omega)^{j\alpha}\kappa^{i\hat\alpha}_{\mathfrak{q}} + (q^-_\omega)^{j\hat\alpha}\kappa^{i\alpha}_{\mathfrak{p}}]\delta_{\beta\hat\beta}[f_{a\hat\alpha}{}^\beta f_{b\alpha}{}^{\hat\beta} - f_{b\hat\alpha}{}^\beta f_{a\alpha}{}^{\hat\beta}]\Big)(m_\omega)^a_j(m_\omega)^b_i \ .\end{aligned} \tag{4.95}$$

We now recognise that the extra terms on the first line can interact with the variations of the worldsheet metric if the following condition is satisfied, as found in [103] for the undeformed initial model (4.64)

$$\delta_{\beta\hat{\beta}}(f_{a\hat{\alpha}}{}^{\beta} f_{b\alpha}{}^{\hat{\beta}} + f_{b\hat{\alpha}}{}^{\beta} f_{a\alpha}{}^{\hat{\beta}}) \equiv c_{\alpha\hat{\alpha}}\delta_{ab} \ , \tag{4.96}$$

for $c_{\alpha\hat{\alpha}}$ some matrix to be determined. Under this assumption the variations become

$$\begin{aligned}\delta_\kappa S_\omega = \int_\Sigma \mathrm{d}^2\sigma \Big(\tfrac{1}{2}(\delta_\kappa\gamma^{ij}) + c_{\alpha\hat{\alpha}}[(p^+_\omega)^{j\alpha}\kappa^{i\hat{\alpha}}_{\mathfrak{q}} - (q^-_\omega)^{j\hat{\alpha}}\kappa^{i\alpha}_{\mathfrak{p}}]\Big)\delta_{ab}(m_\omega)^a_j(m_\omega)^b_i + \\ - [(p^+_\omega)^{j\alpha}\kappa^{i\hat{\alpha}}_{\mathfrak{q}} + (q^-_\omega)^{j\hat{\alpha}}\kappa^{i\alpha}_{\mathfrak{p}}]\delta_{\beta\hat{\beta}}[f_{a\hat{\alpha}}{}^{\beta} f_{b\alpha}{}^{\hat{\beta}} - f_{b\hat{\alpha}}{}^{\beta} f_{a\alpha}{}^{\hat{\beta}}](m_\omega)^a_j(m_\omega)^b_i \ .\end{aligned} \tag{4.97}$$

To conclude the reasoning we then need to recall two further properties that variation of the worldsheet metric should have: obviously it should be symmetric in the exchange of i and j and additionally it should satisfy the condition $\gamma_{ij}\delta_\kappa\gamma^{ij} = 0$. The latter requirement is a direct consequence of the definition $\gamma^{ij} := \sqrt{-h}h^{ij}$, as under the general variations

$$\delta(\sqrt{-h}) = \tfrac{1}{2}\sqrt{-h}h^{-1}\delta h \qquad \text{with} \qquad \delta h = -hh_{ij}\delta h^{ij} \tag{4.98}$$

one finds that

$$\gamma_{ij}\delta_\kappa\gamma^{ij} = \sqrt{-h}h_{ij}(\delta_\kappa\sqrt{-h})h^{ij} - hh_{ij}\delta_\kappa h^{ij} = -\delta h - hh_{ij}\delta_\kappa h^{ij} = 0 \ . \tag{4.99}$$

In order to ensure that $\delta_\kappa\gamma^{ij}$ enjoys the latter two properties, it is sufficient to notice that the term we would like to cancel from the first line in (4.97) reads

$$(p^+_\omega)^{j\alpha}\kappa^{i\hat{\alpha}}_{\mathfrak{q}} - (q^-_\omega)^{j\hat{\alpha}}\kappa^{i\alpha}_{\mathfrak{p}} = P^{jk}_+(p_\omega)^\alpha_k\kappa^{i\hat{\alpha}}_{\mathfrak{q}} - P^{jk}_-(q_\omega)^{\hat{\alpha}}_k\kappa^{i\alpha}_{\mathfrak{p}} \ , \tag{4.100}$$

so that by requiring

$$\kappa^{i\hat{\alpha}}_{\mathfrak{q}} := P^{il}_+\kappa^{\hat{\alpha}}_{\mathfrak{q}l} \qquad\qquad \kappa^{i\alpha}_{\mathfrak{p}} := P^{il}_-\kappa^{\alpha}_{\mathfrak{p}l} \tag{4.101}$$

symmetry in the exchange of i and j is ensured by the property $P^{jk}_\pm P^{il}_\pm = P^{ik}_\pm P^{jl}_\pm$. It should also be noticed how the requirement (4.101) guarantees the vanishing of the second line in (4.97), which is antisymmetric in the exchange of i and j due to antisymmetry in a and b. Hence, choosing

$$\delta_\kappa\gamma^{ij} = 2c_{\alpha\hat{\alpha}}\big[(q^-_\omega)^{j\hat{\alpha}}\kappa^{i\alpha}_{\mathfrak{p}} - (p^+_\omega)^{j\alpha}\kappa^{i\hat{\alpha}}_{\mathfrak{q}}\big] \tag{4.102}$$

and requiring (4.101) is sufficient to set $\delta_\kappa S_\omega = 0$. Finally, the condition $\gamma_{ij}\delta_\kappa\gamma^{ij} = 0$ is ensured by the relation $\gamma_{ij} = P^+_{ij} + P^-_{ij}$ and the properties $P^{jl}_\pm P^{in}_\pm = P^{il}_\pm P^{jn}_\pm$, $P^{ij}_\pm = P^{ji}_\mp$, $P^\pm_{ij}P^{jk}_\mp = 0$ of the projectors. Notice at this stage how the properties enjoyed by the projectors played a crucial role in finding the cancellation of κ-symmetry variations, hence justifying the restriction $\kappa = \pm 1$.

To summarise, the master action (4.70) of a semi-symmetric deformed coset model enjoys κ-symmetry provided the structure constants and inner products of the underlying algebra satisfy

$$\delta_{\beta\hat{\beta}}(f_{a\hat{\alpha}}{}^{\beta} f_{b\alpha}{}^{\hat{\beta}} + f_{b\hat{\alpha}}{}^{\beta} f_{a\alpha}{}^{\hat{\beta}}) \equiv c_{\alpha\hat{\alpha}}\delta_{ab} \ , \tag{4.103}$$

for some matrix $c_{\alpha\hat{\alpha}}$, and one chooses worldsheet metric variations and fermionic parameters

$$\delta_\kappa \gamma^{ij} = 2c_{\alpha\hat{\alpha}}\big[(q_\omega^-)^{j\hat{\alpha}}\kappa_{\mathfrak{p}}^{i\alpha} - (p_\omega^+)^{j\alpha}\kappa_{\mathfrak{q}}^{i\hat{\alpha}}\big] \quad ; \quad \begin{cases} \epsilon_{\mathfrak{p}} := [(m_\omega)_i, \kappa_{\mathfrak{q}}^i] & \text{with} \quad \kappa_{\mathfrak{q}}^i := P_+^{il}\kappa_{\mathfrak{q}l} \\ \epsilon_{\mathfrak{q}} := [(m_\omega)_i, \kappa_{\mathfrak{p}}^i] & \text{with} \quad \kappa_{\mathfrak{p}}^i := P_-^{il}\kappa_{\mathfrak{p}l} \end{cases} . \tag{4.104}$$

The condition (4.103) seems to represent a restriction on the structure constants of the underlying algebra and it would be interesting to understand under which circumstances this is satisfied. To the best of our knowledge, this problem has not yet been addressed in the literature.

4.2.2. T-dual model

One can now proceed in the dualisation of the master action (4.70) by integrating out the gauge fields. Variation with respect to ω gives the following equations of motion

$$\star m_\omega - \tfrac{1}{2}p_\omega + \tfrac{1}{2}q_\omega + \nabla_{j_\omega}\tilde{\Lambda} - D(j_\omega) = 0 \qquad \text{with} \qquad \nabla_{j_\omega} := \mathrm{d} + [j_\omega, -] \ , \tag{4.105}$$

which after projecting on the four subspaces read

$$\begin{aligned}
\mathfrak{h} : \quad & [\tilde{\Lambda}_{\mathfrak{q}}, p_\omega] + D_{\tilde{\Lambda}_{\mathfrak{m}}}(m_\omega) + [\tilde{\Lambda}_{\mathfrak{p}}, q_\omega] = \nabla_{A_\omega}\tilde{\Lambda}_{\mathfrak{h}} \\
\mathfrak{p} : \quad & [\tilde{\Lambda}_{\mathfrak{h}}, p_\omega] + [\tilde{\Lambda}_{\mathfrak{q}}, m_\omega] + D_{\tilde{\Lambda}_{\mathfrak{m}}}(q_\omega) + \tfrac{1}{2}p_\omega = \nabla_{A_\omega}\tilde{\Lambda}_{\mathfrak{p}} \\
\mathfrak{m} : \quad & -\star m_\omega + [\tilde{\Lambda}_{\mathfrak{p}}, p_\omega] + [\tilde{\Lambda}_{\mathfrak{h}}, m_\omega] + [\tilde{\Lambda}_{\mathfrak{q}}, q_\omega] = \mathrm{d}\tilde{\Lambda}_{\mathfrak{m}} - D_{\Lambda_{\mathfrak{m}}}(A_\omega) \\
\mathfrak{q} : \quad & D_{\tilde{\Lambda}_{\mathfrak{m}}}(p_\omega) + [\tilde{\Lambda}_{\mathfrak{p}}, m_\omega] + [\tilde{\Lambda}_{\mathfrak{h}}, q_\omega] - \tfrac{1}{2}q_\omega = \nabla_{A_\omega}\tilde{\Lambda}_{\mathfrak{q}} \ .
\end{aligned} \tag{4.106}$$

The first important feature one can notice about the latter system is that the first equation, unlike the remaining three, does not exhibit a linear term in A_ω. This is a direct result of the need for H_R gauge invariance of the initial model, which forced us to exclude the $\mathfrak{h}$-projection of j from the action. In turn this implies that while the last three equations can be solved in general for $p_\omega, m_\omega, q_\omega$, the first equation may potentially represent an obstruction to dualisation, as the underlying algebra will determine whether or not this can effectively be solved. For this reason, completion of the T-duality procedure requires a case by case analysis. We shall now proceed by solving the last three equations, landing on a hybrid T-dual action which allows to recover the exchange of equations of motion and Maurer-Cartan equations found for principal chiral models. In the next section we shall describe two examples in which the first equation can be solved as well.

The equations in the subspaces $\mathfrak{p}$ and $\mathfrak{q}$ in (4.105) can be solved, see F.2. for more details, as

$$\begin{aligned}\begin{bmatrix} p_\omega \\ q_\omega \end{bmatrix} &= R \begin{bmatrix} -\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{p} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}}(m_\omega) \\ -\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{q} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}(m_\omega) \end{bmatrix} = \\ &= \begin{bmatrix} -R_{11}(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{p}) - R_{12}(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{q}) + (R_{11}\circ \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}} + R_{12}\circ \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}})(m_\omega) \\ -R_{21}(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{p}) - R_{22}(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{q}) + (R_{21}\circ \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}} + R_{22}\circ \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}})(m_\omega) \end{bmatrix} ,\end{aligned} \tag{4.107}$$

where we defined

$$R := \begin{bmatrix} -\frac{1}{1+c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}}\circ c_+ & -\frac{1}{1+c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}}\circ c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_- \\ -\frac{1}{1+c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_+ & \frac{1}{1+c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}}\circ c_- \end{bmatrix} \tag{4.108}$$

and

$$c_\pm := \frac{2}{1 \pm 2\mathrm{ad}_{\tilde{\Lambda}_\mathfrak{h}}} . \tag{4.109}$$

After substituting p_ω and q_ω back into the equation of motion in $\mathfrak{m}$, this can be rewritten in a form similar to the equation of motion (4.15) encountered for principal chiral models, namely $m_\omega = -\star T + S(\star m_\omega)$, with the definitions

$$\begin{aligned} S &:= \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{h}} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}\circ(R_{11}\circ\mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}} + R_{12}\circ\mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}) + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}}\circ(R_{21}\circ\mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}} + R_{22}\circ\mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}) \\ T &:= \mathrm{d}\tilde{\Lambda}_\mathfrak{m} - D_{\tilde{\Lambda}_\mathfrak{m}}(A_\omega) + (\mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}\circ R_{11} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}}\circ R_{21})(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{p}) + (\mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}\circ R_{12} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}}\circ R_{22})(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{q}) \end{aligned} \tag{4.110}$$

and can thus be solved in a similar way

$$m_\omega = -\frac{1}{1-S}\circ P_+(T) + \frac{1}{1+S}\circ P_-(T) . \tag{4.111}$$

Substituting $p_\omega, m_\omega, q_\omega$ back into (4.70) and rearranging, one obtains the hybrid T-dual action

$$\begin{aligned}\tilde{S} &= \int_\Sigma \langle T, \frac{1}{1-S}P_+(T)\rangle + \langle\tilde{\Lambda}_\mathfrak{h}, F_{A_\omega}\rangle + \\ &+ \tfrac{1}{2}\int_\Sigma \langle\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{p}, R_{21}(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{p}) + R_{22}(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{q})\rangle + \langle\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{q}, R_{11}(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{p}) + R_{12}(\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{q})\rangle ,\end{aligned} \tag{4.112}$$

which is manifestly invariant under $\mathrm{K}_L : \mathrm{g} \to k^{-1}\mathrm{g}$ gauge transformations due to the invariance of $\tilde{\Lambda} := \mathrm{g}^{-1}\Lambda\mathrm{g} + \mathrm{g}^{-1}D(\mathrm{g})$ and A_ω. See the second paragraph in F.2. for more details on the rearrangement. As needed, the above action is also unchanged under $\mathrm{H}_R : \mathrm{g} \to \mathrm{g}h$ gauge transformations: the law $A_\omega \to h^{-1}A_\omega h + h^{-1}\mathrm{d}h$ makes $\nabla_{A_\omega}\tilde{\Lambda}_x \to h^{-1}\nabla_{A_\omega}\tilde{\Lambda}_x h$ and consequently the law $\tilde{\Lambda} \to h^{-1}\tilde{\Lambda}h + h^{-1}D(h)$ ensures that $R_{ij}(\nabla_{A_\omega}\tilde{\Lambda}_x) \to h^{-1}R_{ij}(\nabla_{A_\omega}\tilde{\Lambda}_x)h$, for $x = \{\mathfrak{p}, \mathfrak{m}, \mathfrak{q}\}$ and $i, j \in \{1, 2\}$, so that $T \to h^{-1}Th$ and $S(T) \to h^{-1}S(T)h$.

Equations of motion and Lax connection. Varying the action (4.112) with respect to the multipliers $\tilde{\Lambda}_{\mathfrak{h}}, \tilde{\Lambda}_{\mathfrak{p}}, \tilde{\Lambda}_{\mathfrak{m}}, \tilde{\Lambda}_{\mathfrak{q}}$ (see the last paragraph in F.2. for details) one finds equations of motion

$$\begin{aligned} \tilde{\Lambda}_{\mathfrak{h}} : &\qquad F_{\tilde{A}} + \tfrac{1}{2}[\tilde{m}, \tilde{m}] + [\tilde{p}, \tilde{q}] = 0 \\ \tilde{\Lambda}_{\mathfrak{p}} : &\qquad \nabla_{\tilde{A}}\tilde{p} + [\tilde{m}, \tilde{q}] = 0 \\ \tilde{\Lambda}_{\mathfrak{m}} : &\qquad \nabla_{\tilde{A}}\tilde{m} + \frac{1}{2}[\tilde{p}, \tilde{p}] + \frac{1}{2}[\tilde{q}, \tilde{q}] = 0 \\ \tilde{\Lambda}_{\mathfrak{q}} : &\qquad \nabla_{\tilde{A}}\tilde{q} + [\tilde{m}, \tilde{p}] = 0 \ , \end{aligned} \tag{4.113}$$

which have the form of Maurer-Cartan equations (4.66) after defining $\tilde{A} := A_\omega$, $\tilde{p} := p_\omega$, $\tilde{m} := m_\omega$ and $\tilde{q} := q_\omega$, with $p_\omega, m_\omega, q_\omega$ as given in (4.107) and (4.111). Variations with respect to $\tilde{A}$ gives back the unsolved equation in (4.105), namely

$$[\tilde{\Lambda}_{\mathfrak{p}}, \tilde{p}] + D_{\tilde{\Lambda}_{\mathfrak{m}}}(\tilde{m}) + [\tilde{\Lambda}_{\mathfrak{p}}, \tilde{q}] = \nabla_{\tilde{A}}\tilde{\Lambda}_{\mathfrak{h}} \ . \tag{4.114}$$

Using the above equations of motion together with Jacobi identity and the fact that $\tilde{p}, \tilde{m}, \tilde{q}$ satisfy (4.105) one then also recovers, having chosen $\kappa = 1$, the equations of motion of the initial model

$$\begin{aligned} \nabla_{\tilde{A}} \star \tilde{m} - \frac{1}{2}[\tilde{p}, \tilde{p}] + \frac{1}{2}[\tilde{q}, \tilde{q}] = 0 \\ \tfrac{1}{2}\nabla_{\tilde{A}}\tilde{q} + [\tilde{p}, \star\tilde{m} - \tfrac{1}{2}\tilde{m}] = 0 \\ \tfrac{1}{2}\nabla_{\tilde{A}}\tilde{p} - [\tilde{q}, \star\tilde{m} + \tfrac{1}{2}\tilde{m}] = 0 \ , \end{aligned} \tag{4.115}$$

which confirms the exchange of equations of motion and Maurer-Cartan equations observed in principal chiral models. In turn, this allows to write down the T-dual Lax connection as for the initial model (4.67)

$$\tilde{J}(\tilde{z}) := \tilde{A} + \tilde{z}\tilde{p} + \tfrac{1}{2}(\tilde{z}^2 - \tilde{z}^{-2})\tilde{m} + \tilde{z}^{-1}\tilde{q} - \tfrac{1}{2}(\tilde{z}^2 - \tilde{z}^{-2}) \star \tilde{m} \ , \tag{4.116}$$

with $\tilde{z}$ a new spectral parameter. Once again this ensures classical integrability of the T-dual model.

4.2.3. Potential obstructions and examples

In this section we focus on the unsolved equation (4.114). We first rewrite it in a convenient manner, making it clear that the possibility to solve it depends on the underlying algebra and thus highlighting the presence of potential obstructions in the dualisation procedure. Subsequently we discuss two simple examples, one of a symmetric and one of a semi-symmetric space, in which a solution can be explicitly found.

Substituting the solution (4.107) for p_ω, q_ω into the unsolved equation (4.114), one obtains

$$\begin{aligned}\nabla_{\tilde{A}}\tilde{\Lambda}_{\mathfrak{h}} = &-\mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} \circ R_{11}(\nabla_{\tilde{A}}\tilde{\Lambda}_{\mathfrak{p}}) - \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} \circ R_{12}(\nabla_{\tilde{A}}\tilde{\Lambda}_{\mathfrak{q}}) + \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} \circ (R_{11} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} + R_{12} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}})(\tilde{m}) + \\ &- \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \circ R_{21}(\nabla_{\tilde{A}}\tilde{\Lambda}_{\mathfrak{p}}) - \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \circ R_{22}(\nabla_{\tilde{A}}\tilde{\Lambda}_{\mathfrak{q}}) + \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \circ (R_{21} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} + R_{22} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}})(\tilde{m}) + \\ &+ D_{\tilde{\Lambda}_{\mathfrak{m}}}(\tilde{m}) \ . \end{aligned} \tag{4.117}$$

It is now convenient to define the following set of operators

$$\begin{cases} \mathcal{O}_1 := R_{11} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} + R_{12} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \\ \mathcal{O}_2 := R_{21} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} + R_{22} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \\ \mathcal{O}_3 := R_{12} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} + R_{11} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \\ \mathcal{O}_4 := R_{22} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} + R_{21} \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \end{cases} \qquad \begin{cases} \mathcal{O}_1^\dagger := \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} \circ R_{22} + \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \circ R_{12} \\ \mathcal{O}_2^\dagger := \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} \circ R_{21} + \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \circ R_{11} \\ \mathcal{O}_3^\dagger := \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} \circ R_{12} + \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \circ R_{22} \\ \mathcal{O}_4^\dagger := \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} \circ R_{11} + \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \circ R_{21} \end{cases} \tag{4.118}$$

where conjugates $\mathcal{O}^\dagger$ are defined via $\langle \mathcal{O}^\dagger(X), Y\rangle = \langle X, \mathcal{O}(Y)\rangle \ \ \forall X, Y \in \mathfrak{g}$ exploiting (F.18). Using the latter definitions and substituting also the solution (4.111) for m_ω one then finds

$$\begin{aligned}\nabla_{\tilde{A}}\tilde{\Lambda}_{\mathfrak{h}} = &-\mathcal{O}_3^\dagger(\nabla_{\tilde{A}}\tilde{\Lambda}_{\mathfrak{q}}) - \mathcal{O}_4^\dagger(\nabla_{\tilde{A}}\tilde{\Lambda}_{\mathfrak{p}}) + \\ &- \{D_{\tilde{\Lambda}_{\mathfrak{m}}} + \mathcal{O}_3^\dagger \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} + \mathcal{O}_4^\dagger \circ \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}}\} \circ \{\sum_{k=0}^{\infty} S^{2k}(\star T) + \sum_{k=0}^{\infty} S^{2k+1}(T)\} \ . \end{aligned} \tag{4.119}$$

The definition (4.110) of the 1-form T can also be rewritten, using the operators (4.118), as

$$T = \mathrm{d}\tilde{\Lambda}_{\mathfrak{m}} + \mathcal{O}_1^\dagger(\mathrm{d}\tilde{\Lambda}_{\mathfrak{q}}) + \mathcal{O}_2^\dagger(\mathrm{d}\tilde{\Lambda}_{\mathfrak{q}}) - (D_{\tilde{\Lambda}_{\mathfrak{m}}} + \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{p}}} \circ \mathcal{O}_3 + \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{q}}} \circ \mathcal{O}_4)(\tilde{A}) \ , \tag{4.120}$$

so that upon substituting in the above equation and rearranging terms in such a way that $\tilde{A} \equiv A_\omega$ appears on the left hand side, one finally obtains the desired form of the unsolved equation (4.114)

$$W(A_\omega) + Z(\star A_\omega) = \zeta \ , \tag{4.121}$$

where we introduced operators $W, Z : \mathfrak{h} \to \mathfrak{h}$ and the 1-form $\zeta \in \Omega^1(\Sigma, \mathfrak{h})$ as

$$\begin{aligned} W &:= \mathrm{ad}_{\tilde{\Lambda}_{\mathfrak{h}}} + N + (D_{\tilde{\Lambda}_{\mathfrak{m}}} - M^\dagger) \circ \sum_{k=0}^{\infty} S^{2k+1} \circ (D_{\tilde{\Lambda}_{\mathfrak{m}}} + M) \\ Z &:= (D_{\tilde{\Lambda}_{\mathfrak{m}}} - M^\dagger) \circ \sum_{k=0}^{\infty} S^{2k} \circ (D_{\tilde{\Lambda}_{\mathfrak{m}}} + M) \\ \zeta &:= \mathrm{d}\tilde{\Lambda}_{\mathfrak{h}} + \xi + (D_{\tilde{\Lambda}_{\mathfrak{m}}} - M^\dagger) \circ (\star + S) \circ \sum_{k=0}^{\infty} S^{2k} \circ (\mathrm{d}\tilde{\Lambda}_{\mathfrak{m}} + \chi) \ . \end{aligned} \tag{4.122}$$

In the above expressions, we isolated terms from the $\mathfrak{p},\mathfrak{q}$ subspaces by defining $S := \mathrm{ad}_{\Lambda_\mathfrak{h}} + L$ with

$$M := \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}} \circ \mathcal{O}_3 + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}} \circ \mathcal{O}_4 \qquad L := \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}} \circ \mathcal{O}_1 + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}} \circ \mathcal{O}_2 \qquad N := \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}} \circ \mathcal{O}_3 + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}} \circ \mathcal{O}_4$$
$$\xi := \mathcal{O}_3^\dagger(\mathrm{d}\tilde{\Lambda}_\mathfrak{q}) + \mathcal{O}_4^\dagger(\mathrm{d}\tilde{\Lambda}_\mathfrak{p}) \qquad \chi := \mathcal{O}_1^\dagger(\mathrm{d}\tilde{\Lambda}_\mathfrak{q}) + \mathcal{O}_2^\dagger(\mathrm{d}\tilde{\Lambda}_\mathfrak{p}) \ , \tag{4.123}$$

so that one can recover the expressions for symmetric-spaces by setting such contributions to zero.

The rewriting (4.121) of the unsolved equation of motion (4.114) makes it clear that it cannot be generally solved for A_ω, as this possibility depends on the invertibility of the operators W and Z and in turn on the structure of the underlying algebra. In fact, using the projectors $P_\pm$ introduced in (4.17) one can write $1 = P_+ + P_-$ and $\star = P_+ - P_-$ and use this to separately solve for the components $A_\omega^\pm := P_\pm A_\omega \in \Omega^1_\pm(\Sigma,\mathfrak{h})$ of the gauge field. This leads to

$$(W+Z)A_\omega^+ = \zeta^+ \qquad\qquad (W-Z)A_\omega^- = \zeta^- \ . \tag{4.124}$$

Consecutively, one can solve the equation for A_ω provided the two operators $W \pm Z$ are invertible. From a first inspection, one may recognise three situations: in the first two W and $1 \pm ZW^{-1}$ or Z and $1 \pm WZ^{-1}$ are inverible, with respectively Z and W not invertible, while in the third case both W and Z are not invertible, but their sum and difference are

$$A_\omega = \frac{1}{2}(\zeta + \star\zeta)B_+ + \frac{1}{2}(\zeta - \star\zeta)B_- \qquad \text{with} \qquad \begin{cases} B_\pm := W^{-1}[(1 \pm ZW^{-1})^{-1}] \\ B_\pm := \pm Z^{-1}[(1 \pm WZ^{-1})^{-1}] \\ B_\pm := (W \pm Z)^{-1} \end{cases} . \tag{4.125}$$

Even though the above result requires a case-by-case analysis and does not allow for a general solution that could be substituted back into the hybrid action (4.112), one could make one further step to formally integrate out A_ω and obtain the full T-dual action, separating the metric from the B-field. From the definition (4.122) of ζ one can recognise that

$$P_\pm\zeta = P_\pm\zeta_\pm \qquad \text{with} \qquad \zeta_\pm := \mathrm{d}\tilde{\Lambda}_\mathfrak{h} + \xi \pm (D_{\tilde{\Lambda}} - M^\dagger) \circ \sum_{k=0}^{\infty} (\pm S)^k \circ (\mathrm{d}\tilde{\Lambda}_\mathfrak{m} + \chi) \tag{4.126}$$

and the gauge field A_ω can thus be split into terms with and without Hodge-star operator

$$A_\omega := \star\alpha + \beta \qquad \text{with} \qquad \alpha := \frac{1}{2}(\zeta_+ B_+ - \zeta_- B_-) \qquad \beta := \frac{1}{2}(\zeta_+ B_+ + \zeta_- B_-) \ . \tag{4.127}$$

This can now be substituted back into the hybrid action (4.112), obtainting $\tilde{S} = \int_\Sigma \tilde{g} + \tilde{B}$ with

$$\tilde{g} := \frac{1}{2}\langle \lambda_-, \frac{1}{1-S} \star \lambda_+ \rangle - \langle \nabla_\beta \tilde{\Lambda}_\mathfrak{h} + \mathcal{O}_3^\dagger(\nabla_\beta \tilde{\Lambda}_\mathfrak{q}) + \mathcal{O}_4^\dagger(\nabla_\beta \tilde{\Lambda}_\mathfrak{p}), \star\alpha \rangle \ , \tag{4.128}$$

$$\tilde{B} := \frac{1}{2}\langle \lambda_-, \frac{1}{1-S}\lambda_+\rangle + \langle \tilde{\Lambda}_{\mathfrak{h}}, F_\beta - \frac{1}{2}[\alpha,\alpha]\rangle + \frac{1}{2}\langle \alpha, N(\alpha)\rangle + \\ + \frac{1}{2}\langle \nabla_\beta \tilde{\Lambda}_{\mathfrak{p}}, R_{21}(\nabla_\beta \tilde{\Lambda}_{\mathfrak{p}}) + R_{22}(\nabla_\beta \tilde{\Lambda}_{\mathfrak{q}})\rangle + \frac{1}{2}\langle \nabla_\beta \tilde{\Lambda}_{\mathfrak{p}}, R_{21}(\nabla_\beta \tilde{\Lambda}_{\mathfrak{p}}) + R_{22}(\nabla_\beta \tilde{\Lambda}_{\mathfrak{q}})\rangle \; . \tag{4.129}$$

Where we further defined the 1-forms $\lambda_\pm := \nabla_{\beta\pm\alpha}\tilde{\Lambda}_{\mathfrak{m}} + \mathcal{O}_1^\dagger(\nabla_{\beta\pm\alpha}\tilde{\Lambda}_{\mathfrak{q}}) + \mathcal{O}_2^\dagger(\nabla_{\beta\pm\alpha}\tilde{\Lambda}_{\mathfrak{p}}) - D(\beta\pm\alpha)$.

While the above action is formally T-dual to (4.68), as all gauge fields have been integrated out assuming invertibility of $W \pm Z$, we stress again that the $\mathfrak{h}$-valued 1-forms α and β encode all such information and cannot be made more explicit without performing a choice of model. We shall now discuss two examples in which the equation (4.121) can effectively be solved.

$S^3 \simeq SO(4)/SO(3)$. The symmetric space S^3 has already been studied in the literature [62] and it's known to be T-dualisable. Having explicitly solved the equations of motion (4.106) in the subspaces $\mathfrak{p}, \mathfrak{m}, \mathfrak{q}$ and exploited the solution to rearrange the action and the $\mathfrak{h}$-projection, we shall recover dualisability in a slightly different fashion, as solving the equation (4.121) will require the inversion of two 3x3 operators, as opposed to the 6x6 one studied in [62]. These exhibit the nice property of being analytically invertible, thus allowing to avoid computational methods and to retain the underlying index structure, which would otherwise be lost. We shall assume to have dualised the full $G_L = SO(4)$ group of isometry, so as to choose gauge $g = \mathbb{1}$. Further assuming vanishing deformation $D = 0$, we end up with $\tilde{\Lambda} := g^{-1}\Lambda g + g^{-1}D(g) \to \Lambda \in \mathfrak{so}(4)$. We thus start by considering the $\mathfrak{so}(4)$ subalgebra contained in $\mathfrak{osp}(M|2)$, with $M = 4$, reported in B.1.

$$[R_{IJ}, R_{KL}] = -\tfrac{i}{2}(\delta_{IK}R_{JL} - \delta_{JK}R_{IL} - \delta_{IL}R_{JK} + \delta_{JL}R_{IK}) \; , \tag{4.130}$$

with indices $I, J = \{1,2,3,4\}$ raised and lowered using the Euclidean metric δ_{IJ}. The subalgebra $\mathfrak{h} = \mathfrak{so}(3) = \{H_i := -\frac{1}{2}\varepsilon_i{}^{jk}R_{jk}\}$ can then be separated from the rest $\mathfrak{m} := \{M_i := R_{i4}\}$ by using indices $i, j = \{1,2,3\}$ and exploiting the above relation one can find

$$[H_i, H_j] = \frac{i}{2}\varepsilon_{ij}{}^k H_k \qquad [M_i, H_j] = \frac{i}{2}\varepsilon_{ij}{}^k M_k \qquad [M_i, M_j] = \frac{i}{2}\varepsilon_{ij}{}^k H_k \; . \tag{4.131}$$

Expanding the gauge fields and the multipliers as

$$A_\omega = A^i_\omega H_i \qquad \Lambda = \Lambda_{\mathfrak{h}} + \Lambda_{\mathfrak{m}} = y^i H_i + x^i M_i \; , \tag{4.132}$$

one can compute, see F.3. for more details, the explicit form of the operators W and Z

$$W(A_\omega) = v^i \varepsilon_{ij}{}^k A^j_\omega H_k \qquad \text{with} \qquad v^i := \tfrac{i}{2}[y^i + \tfrac{x\cdot y}{(4-y^2)}x^i] \\ Z(\star A_\omega) = \tfrac{1}{4(y^2-4)}\{[(y^2-4)x^2 - (y\cdot x)^2]\delta_j{}^k + \\ + [(y\cdot x)y_j - (y^2-4)x_j]x^k + [(y\cdot x)x_j - x^2 y_j]y^k\} \star A^j_\omega H_k \; . \tag{4.133}$$

Looking at the above expressions one realises that v^j and x^j respectively lie in the kernels of $W_j{}^k$ and $Z_j{}^k$, which are thus not invertible. However it is also not too complicated to recognise that $W \pm Z$ are both invertible and in particular that one can write the two inverses as

$$[(W \pm Z)^{-1}]_k{}^l = a_1^{\pm}\delta_k{}^l + x_k(a_2^{\pm}x^l + a_3^{\pm}y^l) + y_k(a_4^{\pm}x^l + a_5^{\pm}y^l) + \varepsilon_{ak}{}^l(a_6^{\pm}x^a + a_7^{\pm}y^a) + \\ + x^a y^b \varepsilon_{ab}{}^l(a_8^{\pm}x^k + a_9^{\pm}y^k) + x^a y^b \varepsilon_{abk}(a_{10}^{\pm}x^l + a_{11}^{\pm}y^l) + a_{12}^{\pm}x^a y^b \varepsilon_{abk} x^c y^d \varepsilon_{cd}{}^l \,, \tag{4.134}$$

with coefficients $a_1^{\pm}, \dots, a_{12}^{\pm}$ complicated functions of $x^2, y^2, (y \cdot x)$ which we report in F.3..

$\mathrm{OSp}(1|2)/\mathrm{SO}(1,1)$. This semi-symmetric coset has been studied in the literature in the context of holography [123] and we shall here consider it as one of the simplest explicit examples of super non-Abelian dualisation, showing that the equations of motion (4.121) can indeed be solved. The interest in such a model comes from its structure, which is that of a 2d Green-Schwarz string sigma model satisfying the supergravity torsion constraints. Furthermore, dualisation of such a coset naturally generalises that of the principal chiral model on $\mathrm{OSp}(1|2)$, considered in the previous sections, and it would thus be interesting to study in full details. The T-dual model may also in principle be approached from the point of view of holography, as it has already been done for certain classes of T-dual models [63–67]. Performing dualisation with respect to the full isometry group $\mathrm{G}_L = \mathrm{OSp}(1|2)$ one can choose gauge $\mathrm{g} = \mathbb{1}$ and upon setting $D = 0$ the multipliers become $\tilde{\Lambda} := \mathrm{g}^{-1}\Lambda\mathrm{g} + \mathrm{g}^{-1}D(\mathrm{g}) \to \Lambda \in \mathfrak{osp}(1|2)$. The starting point for this model is the $\mathfrak{osp}(1|2)$ algebra (4.33), which we rewrite in lightcone notation using $\alpha, \beta = \{+, -\}$ and $\epsilon_{+-} = -1$

$$\begin{aligned} &\{Q_\pm, Q_\pm\} = L_{\pm\pm} && \{Q_+, Q_-\} = L_{+-} && [L_{\pm\pm}, Q_\mp] = \mp i Q_\pm \\ &[L_{+-}, L_{\pm\pm}] = \pm i L_{\pm\pm} && [L_{++}, L_{--}] = -2i L_{+-} && [L_{+-}, Q_\pm] = \pm \tfrac{i}{2} Q_\pm \,. \end{aligned} \tag{4.135}$$

Upon identifying the four subspaces as

$$\mathfrak{h} = \{L_{+-}\} \qquad \mathfrak{p} = \{Q_+\} \qquad \mathfrak{m} = \{L_{++}, L_{--}\} \qquad \mathfrak{q} = \{Q_-\} \,, \tag{4.136}$$

one can see that the commutator structure (4.62) is indeed satisfied. We thus proceed expanding the gauge field and multipliers as

$$A_\omega = A L_{+-} \qquad \Lambda = \Lambda_{\mathfrak{h}} + \Lambda_{\mathfrak{p}} + \Lambda_{\mathfrak{m}} + \Lambda_{\mathfrak{q}} = y L_{+-} + \theta^+ Q_+ + x^{++} L_{++} + x^{--} L_{--} + \theta^- Q_- \tag{4.137}$$

and after some computations, see F.3. for more details, we find that

$$W(A_\omega) = 0 \qquad Z(\star A_\omega) = \frac{4x^{++}x^{--}}{1+y^2}\left[1 + \frac{4i\theta^+\theta^-}{(1-iy)[4x^{++}x^{--} + (1+iy)^2]}\right] \star A L_{+-} \,. \tag{4.138}$$

Hence equation (4.121) can immediately be solved exploiting nilpotency of the fermionic multipliers

$$Z^{-1} = \frac{1+y^2}{4x^{++}x^{--}}\left[1 - \frac{4i\theta^+\theta^-}{(1-iy)[4x^{++}x^{--}+(1+iy)^2]}\right] . \qquad (4.139)$$

H-gauge fixing. In both examples considered above we have performed dualisation by gauging the full group G of isometry of the initial model, successively exploiting the K = G gauge freedom to get rid of the initial coordinates. As mentioned below equation (4.70), the dual models are then written in terms of dim(G) Lagrange multipliers, which need to be reduced by exploiting the inherited H-gauge symmetry in order to recover the right number dim(G/H) of degrees of freedom. To deal with such requirement one can then proceed as in [62], that is by using the multipliers to construct precisely dim(G/H) invariant quantities under the local H action and using the gauge transformations to get rid of those multipliers whose disappearance could not lead to the vanishing of the invariant quantities. This way one is effectively establishing a one-to-one relation between the invariant quantities and the remaining multipliers, so that the invariants themselves could be used to describe the T-dual model. This reasoning had already been exploited in [124, 125], where the authors argued about the possibilty of constructing precisely dim(G/H) H-invariant quantities out of dim(G) group parameters. We thus conclude with a brief discussion about the construction of invariant quantities for the two models examined above. To this aim we start by recalling that under a local H_R transformation $\mathrm{g} \to \mathrm{g}h$ the multipliers $\tilde{\Lambda} := \mathrm{g}^{-1}\Lambda\mathrm{g} + \mathrm{g}^{-1}D(\mathrm{g})$ transform as

$$\tilde{\Lambda} \quad \rightarrow \quad h^{-1}\tilde{\Lambda}h + h^{-1}D(h) \qquad \text{with} \qquad h \in \mathsf{H} , \qquad (4.140)$$

so that upon gauging the full group of isometry, choosing $\mathrm{g} = \mathbb{1}$ and setting $D = 0$ one finds

$$\Lambda \quad \rightarrow \quad h^{-1}\Lambda h = \mathrm{Ad}_h^{-1}\Lambda = \sum_{k=0}^{\infty}\frac{1}{k!}\mathrm{ad}_C^k(\Lambda) \qquad \text{with} \qquad h := e^{-C} \qquad C := C^i H_i . \qquad (4.141)$$

Where we respectively denoted by H_i and C^i the generators of the Lie algebra $\mathfrak{h}$ and the local parameters of the transformation. The latter relation can then be used to determine the explicit transformation law of the Lagrange multipliers and construct invariant quantities.

For the case of the symmetric space SO(4)/SO(3) it is not hard to exploit the commutation relations (4.131) and the expansion (4.132) of the multipliers to find that

$$z^i \quad \rightarrow \quad z^j(\mathrm{Ad}_h^{-1})_j{}^i := z^j\left[\cosh(C/2)\delta_j{}^i + \frac{1-\cosh(C/2)}{2}C_jC^i + \frac{i\sinh(C/2)}{C}C^k\varepsilon_{kj}{}^i\right] , \qquad (4.142)$$

for $z^i = \{x^i, y^i\}$ and $C^2 := C^iC_i$. Noting that $(\mathrm{Ad}_h^{-1})_k{}^i(\mathrm{Ad}_h^{-1})_l{}^j\delta_{ij} = \delta_{kl}$ one can then immediately construct three invariant quantities

$$y^2 := y^i y_i \qquad x^2 := x^i x_i \qquad y \cdot x := y^i x_i \ , \tag{4.143}$$

so that $dim(\mathsf{H}) = 3$ out of the $dim(\mathsf{G}) = 6$ multipliers can be gauge fixed, recovering $dim(\mathsf{G/H}) = 3$ dual coordinates as for the initial model.

For the case of the semi-symmetric space $\mathsf{OSp}(1|2)/\mathsf{SO}(1,1)$ the above computation is even simpler, as $\mathfrak{h}$ is Abelian and one has $h = e^{CL_{+-}}$. From the relations (4.135) and the multipliers expansion (4.137) one then obtains that y is itself invariant and

$$\theta^+ \;\to\; e^{iC/2}\theta^+ \qquad x^{++} \;\to\; e^{iC}x^{++} \qquad x^{--} \;\to\; e^{-iC}x^{--} \qquad \theta^- \;\to\; e^{-iC/2}\theta^- \ , \tag{4.144}$$

from which one can immediately construct the bosonic invariant $x^{++}x^{--}$. Notice that in this case the gauge transformations act as a phase on the multipliers and hence do not allow to set to zero any of them. This is in agreement with the fact that the inverse operator Z^{-1} in equation (4.139) would otherwise not exist. The best one could achieve in this situation is hence setting either x^{++} or x^{--} to a non-vanishing constant, effectively getting rid of it and recovering the number $dim(\mathsf{G/H}) = 2$ of bosonic coordinates of the initial model.

5

Conclusions and Outlook

In this thesis we have taken a superspace approach to non-Abelian T-duality, aiming for a better understanding of examples of dualisation procedure in which the background under investigation enjoys a set of superisometries closing on a certain superalgebra. The well-known process of gauging bosonic isometries, enforcing the flatness of the gauge fields and integrating them out, can be nicely extended to the supersymmetric setting and we concentrated on three main families of models constructed in terms of Lie supergroups, namely principal chiral and coset models on symmetric and semi-symmetric spaces.

After briefly reviewing the derivation of Buscher's rules for the case of a single bosonic or fermionic isometry, we highlighted the dualisation procedure for bosonic principal chiral models, successively extending it to the case of supergroup manifolds and concentrating on the simple, but physically relevant, concrete example of $\mathsf{OSp}(1|2)$. Even for such a relatively simple model, an explicit and ansatz-based approach to dualisation proved to be quite involved due to the enriched amount of technical complications deriving from the need for a physical understanding of supergeometries, which have to satisfy a non-trivial set of geometric requirements to be granted the status of appropriate supergravity backgrounds. The study of such requirements turns out to be much clearer from a more abstract and algebraic perspective, which allows to perform dualisation of principal chiral models in great generality and to draw conclusions about the above specific model with a faster and more linear argument: while the initial model represents an appropriate supergravity background, its T-dual falls outside this class of geometries. Beyond this, the more abstract point of view also allows an intuitive extension of T-duality to coset models based on Lie supergroups and to explicitly verifiy the exchange in role of the equations of motion and Maurer-Cartan equations, expected to hold as for the purely bosonic setting, for all the three classes of models investigated.

Dualisation of symmetric and semi-symmetric spaces exhibits more subtleties, as compared to the case of principal chiral models, since the process of integrating out the gauge fields cannot be performed in general and obstructions may arise due to the local H-invariance characterising cosets G/H. The potential impediment can however be confined to the study of a single equation, involving quantities taking values in the Lie algebra of the subgroup H, in which invertibility of two linear operators determines on a case-by-case basis the possibility of completing the procedure. This allows, as a first step, to recover the known dualisability of the symmetric space $S^3 \simeq SO(4)/SO(3)$ and to introduce a prime concrete example of semi-symmetric dualisation in terms of the $OSp(1|2)/SO(1,1)$ coset, which exhibits the structure of a 2d Green-Schwarz-like superstring satisfying the supergravity torsion constraints.

The above results and considerations are far from providing a complete picture on super non-Abelian T-duality, but certainly suggest some possible directions to take into account to shed more light on the topic. In first place, it would certainly be interesting to investigate further the breaking of the supergravity torsion constraints upon dualisation of the principal chiral model on $OSp(1|2)$. A possible explanation for this phenomenon might lie in the simplicity of the model, which despite its interesting properties does not describe the dynamics of a string on a supergravity background. This would require the use a Green-Schwarz action [102], argued to preserve the supergravity interpretation of the backgrounds [83,84], and hence the removal of the fermionic vielbeine contribution to the metric and the inclusion of a Wess-Zumino term. The gauging procedure used in this work would not be applicable in such case [126], but this would certainly deserve further investigations. Another interesting possibility is that the breaking of three-dimensional supergravity requirements might be the result of a more general breaking pattern $ds^2_{S^3} \rightarrow dr^2 + f(r)ds^2_{S^2}$ already observed in [15] for the dualisation of the bosonic compact space S^3, and more recently also in the non compact case of AdS_3 [89]. Understanding whether this pattern might exhibit a supersymmetric extension could be used to argue in favour of a lower dimensional supergravity in the T-dual model, which would represent a novel peculiar feature brought into the game by the dualisation of non-Abelian fermionic isometries.

Other fascinating directions to pursue are certainly related to a deeper understanding of semi-symmetric spaces, which often play an important role in the AdS/CFT context. To begin, in light of the resolution of the potential obstruction for the semi-symmetric coset $OSp(1|2)/SO(1,1)$ it would be interesting to complete in full details the dualisation procedure of such model, as this might serve as a base reference for the study of more complicated examples. Additionally, in light of the holography perspective from which the above model was initially studied [123], it would be very fascinating to consider again this direction by looking for a possible holographic description of the T-dual model. The connection between bosonic T-duality and holography has already been explored in the literature for certain classes of models [63–67], for which a holographic interpretation of the T-dual backgrounds has been achieved. It thus seems reasonable that novel examples of such relation should appear in a super non-Abelian T-duality setting and, beyond the simple semi-symmetric spaced mentioned above, it would naturally be worth tackling this type of analysis for the various examples of semi-symmetric spaces of physical relevance appeaing in the AdS/CFT literature. This line of research would also benefit from a deeper understanding of the structure of the operators W and Z, defined in (4.122), involved in the potentially obstructing equation for coset models. Indeed, criteria for T-dualisability of a background might be extracted by understanding the invertibility conditions of such operators. These conditions might lead to some restrictions on the form of the underlying algebra and knowing the latter would immediately allow for a more systematic study of the dualisation procedure.

Last but not least, another interesting scenario in which the ideas and techniques developed in the gauging approach to T-duality might potentially find application, is represented by non-relativistic string theory. In this context, the so-called *Lie algebra expansion* [127] has indeed recently allowed the construction of coset-like sigma models [128] thus opening up the possibility of re-using the T-duality toolkit to attempt gaining further insight on the topic.

Appendices

$\mathcal{A}$

Superspace Conventions

A.1. Setup

Consider a $(p, q|2n)$-dimensional manifold $\mathcal{M}$ with metric g of Graßmann degree zero. This locally resembles $\mathbb{R}^{p,q|2n}$, naturally equipped with a block-diagonal graded-symmetric bilinear form $\delta_{AB} = (-1)^{|A||B|}\delta_{BA}$[1], built out of the flat metric of signature (p, q) in the bosonic block and the standard symplectic form in the fermionic one, that is

$$\delta_{AB} := \begin{bmatrix} \eta_{ab} & 0 \\ 0 & J_{\alpha\beta} \end{bmatrix} \qquad \text{with} \qquad \eta_{ab} := \begin{bmatrix} -\mathbb{1}_{p\times p} & \mathbb{0}_{p\times q} \\ \mathbb{0}_{q\times p} & \mathbb{1}_{q\times q} \end{bmatrix} \quad , \quad J_{\alpha\beta} := \begin{bmatrix} \mathbb{0}_{n\times n} & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & \mathbb{0}_{n\times n} \end{bmatrix} . \tag{A.1}$$

The supergroup $\mathsf{OSp}(p, q|2n) := \{A \in \mathsf{GL}(p, q|2n) : A^{ST}\eta A = \eta\}$, where A^{ST} denotes super transposition of A, leaves invariant the above bilinear form and hence represents the structure group of $\mathcal{M}$. This supergroup has $\frac{1}{2}(p+q)(p+q-1)+n(2n+1)$ bosonic generators and $2(p+q)n$ fermionic generators. Upper case letters from the beginning of the latin alphabet $A, B, C, D, \ldots$ will refer to tangent space indices, while upper case letters from the middle of the latin alphabet $M, N, P, Q, \ldots$ to curved indices. Coordinates on $\mathcal{M}$ will be collectively denoted by z^M.

We shall adopt the northwest-southeast (NW-SE) summation convention, commonly adopted in superspace, and in a given set of coordinates the metric tensor will be written as

$$g = \frac{1}{2}\mathrm{d}z^M \odot \mathrm{d}z^N g_{NM} \ , \tag{A.2}$$

with $g_{NM} = (-1)^{MN}g_{MN}$ and $\odot$ denoting the graded symmetric tensor product $\mathrm{d}z^M \odot \mathrm{d}z^N := \mathrm{d}z^M \otimes \mathrm{d}z^N + (-1)^{MN}\mathrm{d}z^N \otimes \mathrm{d}z^M$. The symbol $\wedge$ denotes graded antisymmetric tensor product

[1] The symbol $|\cdot|$ refers to the Graßmann degree and we shall generally simply replace $|A|$ with A.

$\mathrm{d}z^M \wedge \mathrm{d}z^N := \mathrm{d}z^M \otimes \mathrm{d}z^N - (-1)^{MN}\mathrm{d}z^N \otimes \mathrm{d}z^M$ and k-forms are expanded as

$$\omega_k = \frac{1}{k!}\mathrm{d}z^{M_k} \wedge \ldots \wedge \mathrm{d}z^{M_1}\omega_{M_1\ldots M_k} \ . \tag{A.3}$$

Under a change of coordinates $z^M \to \tilde{z}^M(z)$ we have the following transformation properties

$$\mathrm{d}z^M \quad \to \quad \mathrm{d}\tilde{z}^M = \mathrm{d}z^N \frac{\partial \tilde{z}^M}{\partial z^N} \qquad\qquad \frac{\partial}{\partial z^M} \quad \to \quad \frac{\partial}{\partial \tilde{z}^M} = \frac{\partial z^N}{\partial \tilde{z}^M}\frac{\partial}{\partial z^N} \ , \tag{A.4}$$

leaving the exterior derivative $\mathrm{d} = \mathrm{d}z^M \partial_M$ invariant and transforming the metric components as

$$\tilde{g}_{MN}(\tilde{z}) = (-1)^{(Q+M)N} \frac{\partial z^Q}{\partial \tilde{z}^N}\frac{\partial z^P}{\partial \tilde{z}^M} g_{PQ}(z) \ . \tag{A.5}$$

One can then introduce frame fields, or vielbeine, e^A such that

$$g = \frac{1}{2}e^B \odot e^A \eta_{AB} \qquad \text{with} \qquad e^A = \mathrm{d}z^M e_M{}^A \qquad g_{MN} = (-1)^{NB} e_M{}^B e_N{}^A \delta_{AB} \tag{A.6}$$

and a set of inverse vielbeine e_A, satisfying the relation $e_A \lrcorner\, e^B = \delta_A{}^B$ and defining the structure functions on $\mathcal{M}$

$$[e_A, e_B] = \mathcal{F}_{AB}{}^C e_C \ . \tag{A.7}$$

In the above we used $\lrcorner$ to denote the interior product, while $\delta_A{}^B$ is the Kronecker symbol and $[-,-]$ the graded Lie bracket. By setting $\delta^B{}_A = (-1)^{AB}\delta_A{}^B$ we can introduce δ^{AB} and g^{MN}, respectively the inverses of δ_{AB} and g_{MN}, such that

$$O^{AC}O_{CB} = \delta^A{}_B \quad \Longleftrightarrow \quad (-1)^C O_{BC}O^{CA} = \delta_B{}^A \qquad \text{with} \qquad O = \delta, g \ . \tag{A.8}$$

A generic (p,q)-tensor S can be expanded on a basis as

$$S = \mathrm{d}z^{M_q} \otimes \ldots \otimes \mathrm{d}z^{M_1} \ S_{M_1\ldots M_q}{}^{N_p\ldots N_1} \ \partial_{N_1} \otimes \ldots \otimes \partial_{N_p} \tag{A.9}$$

and its components transform as

$$\begin{aligned}\tilde{S}_{M_1\ldots M_q}{}^{N_p\ldots N_1} =& (-1)^{\sum_{n=2}^{q}(M_n+Q_n)\sum_{m=1}^{n-1}M_m}(-1)^{\sum_{k=2}^{p}(N_k+P_k)\sum_{l=1}^{k-1}N_l} \cdot \\ & \cdot \frac{\partial z^{Q_q}}{\partial \tilde{z}^{M_q}}\ldots\ldots\frac{\partial z^{Q_1}}{\partial \tilde{z}^{M_1}} \ S_{Q_1\ldots Q_q}{}^{P_p\ldots P_1} \ \frac{\partial \tilde{z}^{N_1}}{\partial z^{P_1}}\ldots\ldots\frac{\partial \tilde{z}^{N_p}}{\partial z^{P_p}} \ . \end{aligned} \tag{A.10}$$

A.2. Graded Lie derivative and Killing equation

The Lie derivative along a vector field X on $\mathcal{M}$ will be denoted by $\mathscr{L}_X$. This acts on functions f and vector fields Y as

$$\mathscr{L}_X(f) = X(f) \qquad\qquad \mathscr{L}_X(Y) = [X, Y] \ . \tag{A.11}$$

By requiring $\mathscr{L}_X$ to act as a graded derivation on tensor product and commute with interior product

$$\begin{aligned} \mathscr{L}_X(Y \otimes \omega) &= \mathscr{L}_X(Y) \otimes \omega + (-1)^{XY} Y \otimes \mathscr{L}_X(\omega) \\ \mathscr{L}_X(Y \lrcorner\, \omega) &= \mathscr{L}_X(Y) \lrcorner\, \omega + (-1)^{XY} Y \lrcorner\, \mathscr{L}_X(\omega) \ , \end{aligned} \tag{A.12}$$

one can determine the graded Lie derivative of the metric and B-field as

$$(\mathscr{L}_{K_V} T)_{MN} = (-1)^{V(M+N)} K_V^P \partial_P T_{MN} + (-1)^{VN} (\partial_M K_V^P) T_{PN} + (-1)^{M(N+P+V)} (\partial_N K_V^P) T_{MP} \ , \tag{A.13}$$

for $T = \{g, B\}$. The latter can be obtained by considering

$$\mathscr{L}_{K_V}(Y \lrcorner\, X \lrcorner\, T) = \mathscr{L}_{K_V}(Y) \lrcorner\, X \lrcorner\, T + (-1)^{VY} Y \lrcorner\, \mathscr{L}_{K_V}(X) \lrcorner\, T + (-1)^{V(X+Y)} Y \lrcorner\, X \lrcorner\, \mathscr{L}_{K_V}(T)$$

and choosing $Y = \partial_M$, $X = \partial_N$, $K_V = {K_V}^P \partial_P$, so as to extract the components as

$$\partial_M \lrcorner\, \partial_N \lrcorner\, T = T_{MN} \qquad \partial_M \lrcorner\, \partial_N \lrcorner\, (\mathscr{L}_K T) = (\mathscr{L}_K T)_{MN} \qquad [\partial_M, K_V^P \partial_P] = (\partial_M K_V^P) \partial_P \ . \tag{A.14}$$

Notice that $|\partial_M| = M$ and $|K_V^P| = V + P$, with V the grading of the isometry generated by K_V.

A.3. Maurer-Cartan equations

The vielbeine e^A satisfy the Maurer-Cartan equation $de + \frac{1}{2}[e, e] = 0$. In components this reads

$$de^C - \frac{1}{2} e^A \wedge e^B \mathcal{F}_{BA}{}^C = 0 \ . \tag{A.15}$$

To show this one uses Cartan's formula and that Lie derivative commutes with the interior product

$$\mathscr{L}_X(Y \lrcorner\, \omega) = \mathscr{L}_X(Y) \lrcorner\, \omega + (-1)^{XY} Y \lrcorner\, \mathscr{L}_X(\omega) \qquad \mathscr{L}_X \omega = d(X \lrcorner\, \omega) + X \lrcorner\, (d\omega) \ . \tag{A.16}$$

Indeed, taking $Y \lrcorner\, \mathscr{L}_X \omega$ and choosing $X = e_A$, $Y = e_B$, $\omega = e^C$ these two equations lead to

$$\begin{cases} e_B \lrcorner\, \mathscr{L}_{e_A}(e^C) = \mathcal{F}_{BA}{}^C \\ e_B \lrcorner\, \mathscr{L}_{e_A}(e^C) = e_B \lrcorner\, e_A \lrcorner\, de^C \end{cases} \quad \Rightarrow \quad e_B \lrcorner\, e_A \lrcorner\, de^C = \mathcal{F}_{BA}{}^C \ . \tag{A.17}$$

A.4. Cartan structure equations

The torsion and curvature 2-forms

$$T^C = \frac{1}{2} e^B \wedge e^A T_{AB}{}^C \qquad\qquad R_C{}^D = \frac{1}{2} e^B \wedge e^A R_{ABC}{}^D \tag{A.18}$$

satisfy Cartan's structure equations

$$T^C = -de^C + e^B \wedge \Omega_B{}^C \qquad\qquad R_C{}^D = -d\Omega_C{}^D + \Omega_C{}^E \wedge \Omega_E{}^D \ , \tag{A.19}$$

with Bianchi identities

$$dT^C + T^B \wedge \Omega_B{}^C = e^B \wedge R_B{}^C \qquad\qquad dR_C{}^D + R_C{}^E \wedge \Omega_E{}^D = \Omega_C{}^E \wedge R_E{}^D . \tag{A.20}$$

$\Omega_B{}^C$ denotes the connection 1-form with coefficients $\Omega_{AB}{}^C$ defined by the covariant derivative of the inverse vielbeine $\nabla_{e_A} e_B = \Omega_{AB}{}^C e_C$. The Ricci tensor and scalar curvature are then defined as

$$R_{AB} := (-1)^{C(1+A+B)} R_{CAB}{}^C \qquad\qquad R := \delta^{BA} R_{AB} \ . \tag{A.21}$$

To obtain (A.19) one starts from the definition of $(1,2)$-torsion and $(1,3)$-curvature tensors

$$\begin{aligned} T(X,Y) &:= \nabla_X Y - (-1)^{XY} \nabla_Y X - [X,Y] \\ R(X,Y)Z &:= \nabla_X \nabla_Y Z - (-1)^{XY} \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ , \end{aligned} \tag{A.22}$$

which choosing $X = e_A$, $Y = e_B$, $Z = e_C$ lead to

$$\begin{aligned} T_{AB}{}^C e_C &= \left[\Omega_{AB}{}^C - (-1)^{AB} \Omega_{BA}{}^C - \mathcal{F}_{AB}{}^C\right] e_C \\ R_{ABC}{}^D e_D &= \left[e_A \Omega_{BC}{}^D - (-1)^{AB} e_B \Omega_{AC}{}^D + (-1)^{A(B+C+E)} \Omega_{BC}{}^E \Omega_{AE}{}^D + \right. \\ &\quad \left. - (-1)^{B(C+E)} \Omega_{AC}{}^E \Omega_{BE}{}^D - \mathcal{F}_{AB}{}^E \Omega_{EC}{}^D\right] e_D \ . \end{aligned} \tag{A.23}$$

A.5. Connection from metric compatibility

Metric compatibility leads to the condition $\Omega_{A(BC)} = 0$ for $\Omega_{ABC} := \Omega_{AB}{}^D \delta_{DC}$, and can be combined with (A.19) to rewrite the connection components as

$$\Omega_{AB}{}^C = \frac{1}{2}\left[F_{AB}{}^C + F^C{}_{AB} + (-1)^{AB} F^C{}_{BA}\right] \ , \tag{A.24}$$

where we defined $F_{AB}{}^{C} := \mathcal{F}_{AB}{}^{C} + T_{AB}{}^{C}$ and $F^{C}{}_{AB} := (-1)^{C(A+B)}\delta^{CD}F_{DA}{}^{E}\delta_{EB}$.

To begin, one can obtain the relation $\Omega_{C(AB)} := \Omega_{CAB} + (-1)^{AB}\Omega_{CBA} = 0$ by looking at

$$\nabla_Z(X \lrcorner Y \lrcorner g) = (\nabla_Z X) \lrcorner Y \lrcorner g + (-1)^{XZ} X \lrcorner (\nabla_Z Y) \lrcorner g + (-1)^{Z(X+Y)} X \lrcorner Y \lrcorner \nabla_Z g \ , \tag{A.25}$$

imposing that $\nabla_Z g = 0$ and choosing $X = e_A$, $Y = e_B$, $Z = e_c$. In order to determine the connection in terms of structure functions and torsion one can then lower the index in the Cartan structure equation

$$T_{ABC} = \Omega_{ABC} - (-1)^{AB}\Omega_{BAC} - \mathcal{F}_{ABC} \qquad \text{with} \qquad \begin{cases} T_{ABC} := T_{AB}{}^{D}\delta_{DC} \\ \mathcal{F}_{ABC} := f_{AB}{}^{D}\delta_{DC} \end{cases} , \tag{A.26}$$

and take the symmetric part of the equation with respect to the indices $B - C$. Exploiting the condition $\Omega_{A(BC)} = 0$ this leads to

$$\Omega_{B)A(C} = -(-1)^{AB}\big[\mathcal{F}_{A(BC)} + T_{A(BC)}\big] \ . \tag{A.27}$$

Looking at the symmetric part in $B - C$ of the condition $\Omega_{B(CA)} = 0$ one then finds

$$\Omega_{(BC)A} + (-1)^{AC}\Omega_{B)A(C} = 0 \qquad \Rightarrow \qquad \Omega_{B)A(C} = -(-1)^{AC}\Omega_{(BC)A} \ , \tag{A.28}$$

which can be substituted in the former equation leading to

$$\Omega_{(BC)A} = (-1)^{A(B+C)}\big[\mathcal{F}_{A(BC)} + T_{A(BC)}\big] \ . \tag{A.29}$$

One can then simply write $\Omega_{BCA} = \Omega_{[BC]A} + \Omega_{(BC)A}$ and use that the first antisymmetric term is precisely contained in the Cartan structure equation, hence obtaining

$$\Omega_{BCA} = \frac{1}{2}\big[F_{BCA} + (-1)^{A(B+C)}(F_{ABC} + (-1)^{BC}F_{ACB})\big] \ , \tag{A.30}$$

with $F_{BCA} := \mathcal{F}_{BCA} + T_{BCA}$. Finally, recalling that $\Omega_{BCA} = \Omega_{BC}{}^{D}\delta_{DA}$ and $(-1)^{A}\delta_{DA}\delta^{AE} = \delta_{D}{}^{E}$ one can contract both sides of the above equation with $(-1)^{A}\delta^{AE}$ to raise back the index.

A.6. Supergravity constraints

In this section we introduce a set of requirements that any superspace should satisfy in order to be given the physical interpretation of appropriate supergravity background [129, 130]. The fundamental postulate of any supergravity theory, introduced in direct analogy with general relativity, is that the superspace structure group should be a Lorentz group, relating equivalent vielbeine via a reducible combination of vector and spinor representations. This assumption imposes a strong restriction on the generic structure supergroup and, practically speaking, imposes connection and curvature to take values in the Lorentz algebra in such a way that they result to be diagonal

$$\Omega_{AB}{}^{C} = \begin{bmatrix} \Omega_{Ab}{}^{c} & 0 \\ 0 & \Omega_{A\beta}{}^{\gamma} \end{bmatrix} \qquad R_{ABC}{}^{D} = \begin{bmatrix} R_{ABc}{}^{d} & 0 \\ 0 & R_{AB\gamma}{}^{\delta} \ , \end{bmatrix} \tag{A.31}$$

with vector and spinor components related by the commutator of gamma matrices associated to the Lorentz group

$$\Omega_{A\beta}{}^{\gamma} \propto \Omega_{Abc}[\Gamma^{b}, \Gamma^{c}]_{\beta}{}^{\gamma} \qquad R_{AB\gamma}{}^{\delta} \propto R_{ABcd}[\Gamma^{c}, \Gamma^{d}]_{\gamma}{}^{\delta} \ . \tag{A.32}$$

In particular we notice that such constraints lead to the following restrictions on the connection

$$\Omega_{Ab}{}^{\gamma} = 0 = \Omega_{A\beta}{}^{c} \qquad \Omega_{A(BC)} = \Omega_{ABC} + (-1)^{BC}\Omega_{ACB} = 0 \ , \tag{A.33}$$

and similarly for the curvature.

This postulate implies another important feature of supergravities, known as *Dragon's theorem* [101, 129, 131]: a non-trivial relation between supertorsion and supercurvature can be established and the latter can be completely expressed in terms of the former. Due to this result, supertorsion represents the truly fundamental quantity in superspace, while supercurvature is effectively redundant. This implies that proceeding further in direct analogy with general relativity and imposing the vanishing of torsion to find a Levi-Civita connection is not a good choice in superspace, as this would also lead to vanishing curvature, and hence to a trivial geometry. Additionally, such a choice would be in contradiction with the global supersymmetry perspective, which exhibits a single non-vanishing component of torsion: imposing all components to vanish would make the simplest flat superspace fall outside the class of geometries encompassed by our description, thus immediately breaking the physical interpretation. Despite the restrictions imposed by the above postulate, superspaces still feature a quite large number of fields and it thus makes sense to look for a set of requirements which could reduce such set of degrees of freedom to a consistent minimal one. This motivates the following set of torsion constraints

$$T_{\alpha\beta}{}^{c} = k_1 f_{\alpha\beta}{}^{c} \qquad T_{ab}{}^{c} = T_{a\beta}{}^{c} = T_{\alpha\beta}{}^{\gamma} = 0 \ , \tag{A.34}$$

introduced to establish a connection with the component formulation of supergravity [98–100], while ensuring consistency with flat superspace. In the above equation k_1 represents a proportionality constant, while $f_{\alpha\beta}{}^{c}$ are the structure constants of the underlying super Poincaré algebra, and their exact form may vary depending on the dimension. The above set of "canonical" torsion constraints represent the basis of our analysis for the PCM on $\mathsf{OSp}(1|2)$ and its T-dual model, but it should be kept in mind that not all of them are essentials and one may possibly consider variations. In fact, the truly fundamental constraint is the first one $T_{\alpha\beta}{}^{c} = k_1 f_{\alpha\beta}{}^{c}$, which ensures consistency with flat superspace and also comes out as a requirement when imposing κ-symmetry of the Green-Schwarz action [102,78]. The remaining constraints are often said to be conventional, as they are generically chosen so as to fix components of the connection or the vielbeine, but might for example be modified as a result of shifts and field re-definitions [132].

We can now spell out explicitly the equations contained in (A.24) imposing (A.33), (A.34)

$$\begin{aligned}
&\textcircled{1} \quad \Omega_{ab}{}^{c} &&= \tfrac{1}{2}\big[\mathcal{F}_{ab}{}^{c} + \delta^{cd}(\mathcal{F}_{dab} + \mathcal{F}_{dba})\big]\\
&\textcircled{2} \quad \Omega_{\alpha b}{}^{c} &&= \tfrac{1}{2}\big[\mathcal{F}_{\alpha b}{}^{c} + \delta^{cd}(\mathcal{F}_{d\alpha b} + \mathcal{F}_{db\alpha} + T_{db\alpha})\big]\\
&\textcircled{3} \quad \Omega_{a\beta}{}^{\gamma} &&= \tfrac{1}{2}\big[\mathcal{F}_{a\beta}{}^{\gamma} + T_{a\beta}{}^{\gamma} - \delta^{\gamma\delta}(\mathcal{F}_{\delta a\beta} + T_{\delta a\beta} + \mathcal{F}_{\delta\beta a} + k_1 f_{\delta\beta a})\big]\\
&\textcircled{4} \quad \Omega_{\alpha\beta}{}^{\gamma} &&= \tfrac{1}{2}\big[\mathcal{F}_{\alpha\beta}{}^{\gamma} + \delta^{\gamma\delta}(\mathcal{F}_{\delta\alpha\beta} - \mathcal{F}_{\delta\beta\alpha})\big]\\
&\textcircled{5} \quad 0 &&= \tfrac{1}{2}\big[\mathcal{F}_{a\beta}{}^{c} + \delta^{cd}(\mathcal{F}_{da\beta} + T_{da\beta} + \mathcal{F}_{d\beta a})\big]\\
&\textcircled{6} \quad 0 &&= \tfrac{1}{2}\big[\mathcal{F}_{ab}{}^{\gamma} + T_{ab}{}^{\gamma} + \delta^{\gamma\delta}(\mathcal{F}_{\delta ab} + \mathcal{F}_{\delta ba})\big]\\
&\textcircled{7} \quad 0 &&= \tfrac{1}{2}\big[\mathcal{F}_{\alpha\beta}{}^{c} + k_1 f_{\alpha\beta}{}^{c} + \delta^{cd}(\mathcal{F}_{d\alpha\beta} + T_{d\alpha\beta} - \mathcal{F}_{d\beta\alpha} - T_{d\beta\alpha})\big]\\
&\textcircled{8} \quad 0 &&= \tfrac{1}{2}\big[\mathcal{F}_{\alpha b}{}^{\gamma} + T_{\alpha b}{}^{\gamma} - \delta^{\gamma\delta}(\mathcal{F}_{\delta\alpha b} + k_1 f_{\delta\alpha b} - \mathcal{F}_{\delta b\alpha} - T_{\delta b\alpha})\big]\ .
\end{aligned} \tag{A.35}$$

The first thing one can notice is that equations $\textcircled{1}$ and $\textcircled{4}$ define two unconstrained components of the connection in terms of the structure functions of the geometry. By construction, these components satisfy the requirement $\Omega_{A(BC)} = 0$. The second thing to notice is that equations $\textcircled{5}$–$\textcircled{8}$ do not represent four independent constraints on the structure functions and components of torsion. Indeed, given the graded antisymmetry of $\mathcal{F}_{AB}{}^{C}$ and $T_{AB}{}^{C}$ in the exchange of the first two indices, it is not difficult to notice that exploiting $\delta_{cb}\delta^{cd} = \delta_b{}^{d}$ and $\delta_{\gamma\beta}\delta^{\gamma\delta} = \delta_\beta{}^{\delta}$ one finds

$$\textcircled{5}\delta_{cb} = -\textcircled{6}\delta_{\gamma\beta} \qquad\qquad \textcircled{7}\delta_{cb} = -\textcircled{8}\delta_{\gamma\beta}\ . \tag{A.36}$$

It is thus sufficient to solve equations $\textcircled{6}$ and $\textcircled{7}$ by noting that the symmetric and antisymmetric parts in the lower indices should vanish separately.

$$\textcircled{6}\begin{cases} T_{ab}{}^{\gamma} := -\mathcal{F}_{ab}{}^{\gamma} & \textcircled{6A} \\ \mathcal{F}_{\delta ab} + \mathcal{F}_{\delta ba} \equiv 0 & \textcircled{6S} \end{cases} \qquad \textcircled{7}\begin{cases} (\mathcal{F}_{d\alpha\beta} + T_{d\alpha\beta}) - (\mathcal{F}_{d\beta\alpha} + T_{d\beta\alpha}) \equiv 0 & \textcircled{7A} \\ \mathcal{F}_{\alpha\beta}{}^{c} \equiv -k_1 f_{\alpha\beta}{}^{c} & \textcircled{7S} \end{cases} \tag{A.37}$$

Hence one is left with solving equations $\textcircled{2}$ and $\textcircled{3}$, which can however be first simplified by using the constraints imposed by (6) and (7).

- Exploiting (6A) and (6S), equation (2) leads to the condition $\Omega_{\alpha b}{}^{c} = \mathcal{F}_{\alpha b}{}^{c}$. This fixes one further component of the connection and (6S) ensures that it correctly satisfies $\Omega_{\alpha(bc)} = 0$. Condition (6S) will hence represent one of the actual constraints imposed by the supergravity requirements. This will finally have to be supplemented by (A.32).

- Exploiting (7A) and (7S), equation (3) leads to the condition $T_{a\beta}{}^{\gamma} := -\mathcal{F}_{a\beta}{}^{\gamma} + \Omega_{a\beta}{}^{\gamma}$. This just fixes one of the unconstrained components of torsion, leaving $\Omega_{a\beta}{}^{\gamma}$ still free. Substituting back into (7A) one finds it is satisfied provided that $\Omega_{d(\alpha\beta)} = 0$. This will be ensured by the requirement (A.32), which will completely fix the latter component of connection.

To summarise, the eight equations on the previous page have been solved for the components of connection $\Omega_{ab}{}^{c}, \Omega_{\alpha\beta}{}^{\gamma}, \Omega_{\alpha b}{}^{c}$ and torsion $T_{ab}{}^{\gamma}, T_{a\beta}{}^{\gamma}$. This requires to constrain the structure functions $\mathcal{F}_{\alpha\beta}{}^{c}$ to be constant and $\mathcal{F}_{\alpha ab}$ to be antisymmetric in the last two indices. The connection component $\Omega_{a\beta}{}^{\gamma}$ is left unconstrained except for the requirement $\Omega_{a(\alpha\beta)} = 0$. Hence, at this stage, the actual constraints on the structure functions resulting from imposing the supergravity requirements are (6S) and (7S). One can now finally impose the requirement (A.32), which puts an additional condition on the structure functions $\mathcal{F}_{\alpha ab}$ and completely fixes the connection component $\Omega_{a\alpha\beta}$, also ensuring its graded antisymmetry in the last two indices. Altogether, the resulting constraints on the structure functions read

$$\begin{aligned} \mathcal{F}_{\alpha\beta}{}^{c} &\equiv -k_1 f_{\alpha\beta}{}^{c} \\ \mathcal{F}_{\alpha(bc)} &\equiv 0 \\ \tfrac{1}{2}[\mathcal{F}_{\alpha\beta}{}^{\gamma} + \delta^{\gamma\delta}(\mathcal{F}_{\delta\alpha\beta} - \mathcal{F}_{\delta\beta\alpha})] &\equiv k_2 \mathcal{F}_{\alpha bc}[\Gamma^b, \Gamma^c]_{\beta}{}^{\gamma} , \end{aligned} \tag{A.38}$$

where k_1, k_2 are proportionality constants. And the non-vanishing components of connection and torsion read

$$\begin{aligned} \Omega_{ab}{}^{c} &= \tfrac{1}{2}\big[\mathcal{F}_{ab}{}^{c} + \delta^{cd}(\mathcal{F}_{dab} + \mathcal{F}_{dba})\big] \\ \Omega_{a\beta}{}^{\gamma} &= \Omega_{abc}[\Gamma^b, \Gamma^c]_{\beta}{}^{\gamma} \\ \Omega_{\alpha b}{}^{c} &= \mathcal{F}_{\alpha b}{}^{c} \\ \Omega_{\alpha\beta}{}^{\gamma} &= \tfrac{1}{2}\big[\mathcal{F}_{\alpha\beta}{}^{\gamma} + \delta^{\gamma\delta}(\mathcal{F}_{\delta\alpha\beta} - \mathcal{F}_{\delta\beta\alpha})\big] = k_2 \mathcal{F}_{\alpha bc}[\Gamma^b, \Gamma^c]_{\beta}{}^{\gamma} \\ T_{ab}{}^{\gamma} &= -\mathcal{F}_{ab}{}^{\gamma} \\ T_{\alpha\beta}{}^{c} &= k_1 f_{\alpha\beta}{}^{c} \\ T_{a\beta}{}^{\gamma} &= -\mathcal{F}_{a\beta}{}^{\gamma} + \Omega_{a\beta}{}^{\gamma} \\ T_{\beta a}{}^{\gamma} &= -\mathcal{F}_{\beta a}{}^{\gamma} - \Omega_{a\beta}{}^{\gamma} . \end{aligned} \tag{A.39}$$

B

Orthosymplectic Algebra

B.1. The $\mathfrak{osp}(1|2)$ algebra

To derive the algebra (3.15) we start from the more general $\mathfrak{osp}(M|2)$ algebra [1]

$$\begin{aligned} \{Q_{I\alpha}, Q_{J\beta}\} &= \delta_{IJ} L_{\alpha\beta} + \epsilon_{\alpha\beta} R_{IJ} \\ [R_{IJ}, Q_{K\alpha}] &= -\mathrm{i}\delta_{K[I} Q_{J]\alpha} \qquad [L_{\alpha\beta}, Q_{I\gamma}] = -\mathrm{i}\epsilon_{\gamma(\alpha} Q_{I\beta)} \\ [L_{\alpha\beta}, L^{\gamma\delta}] &= -2\mathrm{i}\delta_{(\alpha}{}^{(\gamma} L_{\beta)}{}^{\delta)} \qquad [R_{IJ}, R^{KL}] = -2\mathrm{i}\delta_{[I}{}^{[K} R_{J]}{}^{L]} \ , \end{aligned} \tag{B.1}$$

where $Q_{I\alpha}$ are Graßmann-odd generators carrying the index $I = 1, \dots, M$ of the vector representation of $\mathfrak{so}(M)$, generated by $R_{IJ} = \delta_{IK}\delta_{JL}R^{KL} = -R_{JI}$, and the spinor index $\alpha = 1, 2$ of $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R})$, generated by $L_{\alpha\beta} = \epsilon_{\alpha\gamma}L^{\gamma\delta}\epsilon_{\beta\delta} = L_{\beta\alpha}$ with $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ and inverse $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ such that $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \epsilon_{\beta\gamma}\epsilon^{\gamma\alpha} = \delta_\beta{}^\alpha$.

The above algebra reduces to the following one when considering the $\mathfrak{osp}(1|2)$ case

$$\{Q_\alpha, Q_\beta\} = L_{\alpha\beta} \qquad [L_{\alpha\beta}, Q_\gamma] = -i\epsilon_{\gamma(\alpha}Q_{\beta)} \qquad [L_{\alpha\beta}, L^{\gamma\delta}] = -2\mathrm{i}\delta_{(\alpha}{}^{(\gamma} L_{\beta)}{}^{\delta)} \ . \tag{B.2}$$

It is here convenient to introduce generators

$$L_a = -\frac{i}{2}(\gamma_a)^{\alpha\beta} L_{\alpha\beta} \ , \tag{B.3}$$

defined in terms of the 3d gamma matrices $(\gamma^a)_\alpha{}^\beta$, generators of $\mathfrak{sl}(2, \mathbb{R})$ in the fundamental

[1] For a complete discussion of which we refer to [93, 133]. Here we use convention according to which parentheses and square brackets respectively denote normalised symmetrisation and normalised anti-symmetrisation of the enclosed indices.

representation

$$\{\gamma^a,\gamma^b\}_\alpha{}^\beta = 2\delta_\alpha{}^\beta\eta^{ab} \qquad [\gamma^a,\gamma^b]_\alpha{}^\beta = 2\varepsilon^{abc}(\gamma_c)_\alpha{}^\beta$$
$$\text{with} \qquad \eta^{ab} = \mathrm{diag}(-1,+1,+1) = \eta_{ab} \quad \text{s.t.} \quad \eta^{ab}\eta_{bc} = \eta_{cb}\eta^{ba} = \delta_c{}^a \ . \tag{B.4}$$

Vctor indices $a = 0,1,2$ are raised and lowered by η

$$\gamma^a = \eta^{ab}\gamma_b \qquad \varepsilon^{abc}\gamma_c = \varepsilon^{abc}\eta_{cd}\gamma^d = \varepsilon^{ab}{}_c\gamma^c \qquad \gamma^a L_a = \gamma_b\eta^{ba}L_a = \gamma_a L^a \ , \tag{B.5}$$

while spinor indices are raised and lowered by ϵ

$$(\gamma^a)_{\alpha\beta} = \epsilon_{\alpha\delta}(\gamma^a)^\delta{}_\beta = (\gamma^a)_\alpha{}^\delta\epsilon_{\beta\delta} \qquad (\gamma^a)^{\alpha\beta} = \epsilon^{\alpha\delta}(\gamma^a)_\delta{}^\beta = (\gamma^a)^\alpha{}_\delta\epsilon^{\beta\delta} \ . \tag{B.6}$$

We also have the following identities

$$\varepsilon_{abc}\varepsilon^{adk} = \delta_b{}^k\delta_c{}^d - \delta_b{}^d\delta_c{}^k \qquad \varepsilon_{abc}\varepsilon^{abd} = -2\delta_c{}^d \ , \tag{B.7}$$

and combining the anticommutator and commutator of gamma matrices (B.4) one can obtain

$$(\gamma^a)_\alpha{}^\delta(\gamma^b)_\delta{}^\beta = \delta_\alpha{}^\beta\eta^{ab} + \varepsilon^{abc}(\gamma_c)_\alpha{}^\beta \quad \Rightarrow \quad (\gamma^a)_{\alpha\delta}(\gamma^b)^{\delta\beta} = -\delta_\alpha{}^\beta\eta^{ab} - \varepsilon^{abc}(\gamma_c)_\alpha{}^\beta$$
$$(\gamma^a)_\alpha{}^\delta(\gamma^b)_\delta{}^\alpha = 2\eta^{ab} \quad \Rightarrow \quad (\gamma^a)_{\alpha\delta}(\gamma^b)^{\delta\alpha} = -2\eta^{ab} \ . \tag{B.8}$$

Moreover, the 3d gamma matrices enjoy the following properties

$$(\gamma^a)_{\alpha\beta} = (\gamma^a)_{\beta\alpha} \qquad (\gamma^a)^{\alpha\beta} = (\gamma^a)^{\beta\alpha} \qquad (\gamma^a)_\alpha{}^\alpha = 0$$
$$(\gamma^a)_{\alpha\beta}(\gamma_a)_{\rho\sigma} = \epsilon_{\rho\alpha}\epsilon_{\beta\sigma} + \epsilon_{\rho\beta}\epsilon_{\alpha\sigma} \ . \tag{B.9}$$

These can be understood as descending from those of the 4d sigma matrices upon dimensional reduction, as noted in Appendix A of [134]. Using the above properties one can verify that[2]

$$L_a = -\frac{i}{2}(\gamma_a)^{\alpha\beta}L_{\alpha\beta} \qquad \Rightarrow \qquad L_{\alpha\beta} = -i(\gamma^b)_{\alpha\beta}L_b \tag{B.10}$$

and, exploiting the relations (B.8), one can recover the $\mathfrak{osp}(1|2)$ algebra in (3.15)

$$\{Q_\alpha, Q_\beta\} = -i(\gamma^b)_{\alpha\beta}L_b \qquad [L_a, Q_\gamma] = -\frac{1}{2}(\gamma_a)_\gamma{}^\beta Q_\beta \qquad [L_a, L_b] = \varepsilon_{ab}{}^c L_c \ . \tag{B.11}$$

[2] We do this by checking consistency of the two definitions using (B.8)

$$L_a = -\frac{i}{2}(\gamma_a)^{\alpha\beta}L_{\alpha\beta} = -\frac{1}{2}(\gamma_a)^{\alpha\beta}(\gamma^b)_{\alpha\beta}L_b = -\frac{1}{2}(-2)\delta_a{}^b L_b = L_a \ .$$

Notice also that the definition of inner products provided in (3.15) has been fixed by setting

$$STr[L_a L_b] = A\eta_{ab} \qquad STr[Q_\alpha Q_\beta] = iB\epsilon_{\alpha\beta} \tag{B.12}$$

and checking consistency with the following supertrace identity

$$\begin{aligned} STr[T_A[T_B, T_C]] &= STr[T_A T_B T_C] - (-1)^{BC} STr[T_A T_C T_B] = \\ &= STr[T_A T_B T_C] - (-1)^{BC+B(C+A)} STr[T_B T_A T_C] = \\ &= STr[[T_A, T_B] T_C] \ , \end{aligned} \tag{B.13}$$

which should be satisfied by all triplets of generators (T_A, T_B, T_C). The triplet (L_a, Q_α, Q_β) forces $A = -B/2$, so that choosing $B = 1$ leads to the desired inner products (3.15).

C

Initial Action On $\mathsf{OSp}(1|2)$

C.1. Derivation of the action

We recall the chosen parametrisation for $\mathsf{OSp}(1|2)$

$$\mathrm{g} = e^{x^a L_a} e^{-\theta^\alpha Q_\alpha} = \mathrm{g}_{bos}\mathrm{g}_{fer} \qquad \mathrm{g}^{-1} = e^{\theta^\alpha Q_\alpha} e^{-x^a L_a} = \mathrm{g}_{fer}^{-1}\mathrm{g}_{bos}^{-1} \tag{C.1}$$

and the structure of the current $j = \mathrm{g}^{-1}\mathrm{d}\mathrm{g}$

$$j = \mathrm{g}_{fer}^{-1} j_{bos} \mathrm{g}_{fer} + \mathrm{g}_{fer}^{-1}\mathrm{d}\mathrm{g}_{fer} =: j_{bos} + j_{fer} \ . \tag{C.2}$$

The bosonic contribution can be written as $j_{bos} = \lambda^a L_a$, where λ^a are the Maurer-Cartan forms of $\mathsf{Sp}(2,\mathbb{R}) \simeq \mathsf{SL}(2,\mathbb{R})$, the bosonic part of $\mathsf{OSp}(1|2)$. To calculate the fermionic current one can define $F := \theta^\alpha Q_\alpha$ and proceed by using the following formula, which first appeared in [135, 136]

$$j_{fer} := -\frac{\sinh(\mathcal{M})}{\mathcal{M}} \nabla_{j_{bos}} F - 2\left[F, \frac{\sinh^2(\mathcal{M}/2)}{\mathcal{M}^2} \nabla_{j_{bos}} F\right] , \tag{C.3}$$

where we introduced notation

$$\nabla_A := \mathrm{d} + [A, -] \qquad \text{and} \qquad \mathcal{M} = \mathrm{ad}_F := [F, -] \tag{C.4}$$

and the hyperbolic functions are understood as formal Taylor series. In the case under consideration, with only two fermionic coordinates θ^α, one finds

$$j_{fer} = -\nabla_{j_{bos}} F - \frac{1}{2}[F, \nabla_{j_{bos}} F] - \frac{1}{6}[F, [F, \mathrm{d}F]] \ . \tag{C.5}$$

We can thus calculate the various contributions

- $\nabla_{j_{bos}} F = \mathrm{d}F + [j_{bos}, F] = \mathrm{d}\theta^\alpha Q_\alpha - \frac{1}{2}\lambda_a \theta^\alpha (\gamma^a)_\alpha^{\ \beta} Q_\beta$

- $[F, \nabla_{j_{bos}} F] = i\Big(\theta^\alpha \mathrm{d}\theta^\beta - \frac{1}{2}\theta^\alpha \lambda_a \theta^\rho (\gamma_a)_\rho{}^\beta\Big)(\gamma^b)_{\alpha\beta} L_b$
- $[F, \mathrm{d}F] = i\theta^\alpha \mathrm{d}\theta^\beta (\gamma^a)_{\alpha\beta} L_a$
- $[F, [F, \mathrm{d}F]] = \frac{i}{2}\theta^\rho \theta^\alpha \mathrm{d}\theta^\beta (\gamma^a)_{\alpha\beta} (\gamma_a)_\rho{}^\sigma Q_\sigma$

and using the identity (D.8) we can combine such terms into the fermionic current

$$j_{fer} = -(1 - \frac{i}{8}\theta^2)\mathrm{d}\theta^\alpha Q_\alpha + \frac{i}{4}\theta^2 \lambda^a L_a + \frac{1}{2}\lambda_a \theta^\alpha (\gamma^a)_\alpha{}^\beta Q_\beta - \frac{i}{2}\theta^\alpha \mathrm{d}\theta^\beta (\gamma^a)_{\alpha\beta} L_a \; , \tag{C.6}$$

so that equation (3.19) is recovered.

Exploiting the inner products (3.15) the action reads

$$S = \frac{1}{2}\int_\Sigma STr[j \wedge \star j] = \frac{1}{2}\int_\Sigma j^b \wedge \star j^a (-\tfrac{1}{2}\eta_{ab}) + j^\beta \wedge \star j^\alpha (i\epsilon_{\alpha\beta}) \tag{C.7}$$

and using that for any two 1-forms α and β one has $\alpha \wedge \star\beta = -\star\alpha \wedge \beta$ and $\alpha \wedge \beta = -(-1)^{|\alpha||\beta|}\beta \wedge \alpha$, where $|\alpha|$ and $|\beta|$ represent their grading, the two terms in the action become

$$\begin{aligned} -\frac{1}{2} j^a \wedge \star j_a &= -\frac{1}{2}(1 + \frac{i}{2}\theta^2)\big(\lambda^a \wedge \star\lambda_a - i(\gamma_a)_{\alpha\beta}\lambda^a \wedge \star\theta^\alpha \mathrm{d}\theta^\beta - \frac{3}{8}\theta^2 \mathrm{d}\theta^\alpha \wedge \star \mathrm{d}\theta_\alpha\big) \\ -i\, j^\alpha \wedge \star j_\alpha &= -i(1 - \frac{i}{4}\theta^2)\big(\mathrm{d}\theta^\alpha \wedge \star \mathrm{d}\theta_\alpha + (\gamma_a)_{\alpha\beta}\lambda^a \wedge \star\theta^\alpha \mathrm{d}\theta^\beta - \frac{1}{4}\theta^2 \lambda^a \wedge \star\lambda_a\big) \; , \end{aligned} \tag{C.8}$$

so that they can be combined into (3.20).

C.2. Maurer-Cartan forms of $\mathbf{SL}(2,\mathbb{R})$

To explicitly construct the Maurer-Cartan forms λ^a we start by defining $L \equiv x^a L_a$ and considering the following parametric current

$$j(t) = e^{-tL}\mathrm{d}e^{tL} \qquad \text{such that} \qquad j(t=1) = j_{bos} \qquad j(t=0) = 0 \; . \tag{C.9}$$

One can thus notice that

$$\partial_t j(t) = -Lj(t) + e^{-tL}d(Le^{tL}) = -Lj(t) + e^{-tL}\mathrm{d}(e^{tL}L) = \mathrm{d}L + [j(t), L] \; , \tag{C.10}$$

so that integrating both sides one obtains

$$j(t) = t\mathrm{d}L - \left[L, \int_0^t \mathrm{d}u\, j(u)\right] \; . \tag{C.11}$$

Recursively substituting the current into itself one then gets

$$j(t) = t\mathrm{d}L - \left[L, \int_0^t \mathrm{d}u\, u\mathrm{d}L\right] + \left[L, \int_0^t \mathrm{d}u\left[L, \int_0^u \mathrm{d}v v\mathrm{d}L\right]\right] - \left[L, \int_0^t \mathrm{d}u\left[L, \int_0^u \mathrm{d}v\left[L, \int_0^v \mathrm{d}w w\mathrm{d}L\right]\right]\right] + \dots$$

and after integration finally arrives to

$$\begin{aligned} j(t) &= t\mathrm{d}L - \tfrac{t^2}{2!}[L, \mathrm{d}L] + \tfrac{t^3}{3!}[L,[L,dL]] - \frac{t^4}{4!}[L,[L,[L,\mathrm{d}L]]] + \frac{t^5}{5!}[L,[L,[L,[L,\mathrm{d}L]]]] + \dots = \\ &= t\sum_{k=0}^{\infty}\frac{(-t)^k}{(k+1)!}\mathrm{ad}_L^k(\mathrm{d}L) \qquad \text{with} \qquad \mathrm{ad}_L := [L, -] \quad ; \quad \mathrm{ad}_L^0 := 1\ . \end{aligned} \tag{C.12}$$

Exploiting the $\mathfrak{sl}(2,\mathbb{R})$ algebra $[L_a, L_b] = \varepsilon_{ab}{}^c L_c$ and the identity (B.7) one then finds the pattern

$$\begin{cases} \mathrm{ad}_L^{2k}(\mathrm{d}L) = R^{2k-2} J^a L_a & \text{for} \quad k \geqslant 1 \\ \mathrm{ad}_L^{2k+1}(\mathrm{d}L) = R^{2k} dx^b x^a \varepsilon_{ab}{}^c L_c & \text{for} \quad k \geqslant 0\ , \end{cases} \tag{C.13}$$

where we defined $R^2 := x^q x^p \eta_{pq}$ and $J^a = R^2(\mathrm{d}x^a) - (x_i \mathrm{d}x^i)x^a$. The series is then resummed as

$$\begin{aligned} j(t) &= t\sum_{k=0}^{\infty}\frac{(-t)^k}{(k+1)!}\mathrm{ad}_L^k(\mathrm{d}L) = t\sum_{k=0}^{\infty}\frac{(-t)^{2k}}{(2k+1)!}\mathrm{ad}_L^{2k}(\mathrm{d}L) + t\sum_{k=1}^{\infty}\frac{(-t)^{2k+1}}{(2k+2)!}\mathrm{ad}_L^{2k+1}(\mathrm{d}L) = \\ &= t\mathrm{d}L + R^{-3}\sum_{k=1}^{\infty}\frac{(tR)^{2k+1}}{(2k+1)!}J^a L_a - R^{-2}\sum_{k=0}^{\infty}\frac{(tR)^{2k+2}}{(2k+2)!}\mathrm{d}x^b x^c \varepsilon_{cb}{}^a L_a = \\ &= t\mathrm{d}L + R^{-3}[\sinh(tR) - tR]J^a L_a - R^{-2}[\cosh(tR) - 1]\mathrm{d}x^b x^c \varepsilon_{cb}{}^a L_a = \\ &= \mathrm{d}x^m\left[\frac{\sinh(tR)}{R}\delta_m{}^a + \frac{tR - \sinh(tR)}{R^3}x_m x^a - 2\frac{\sinh^2(tR/2)}{R^2}x^c \varepsilon_{cm}{}^a\right]L_a\ , \end{aligned} \tag{C.14}$$

which reduces to (3.24) for $t = 1$.

C.3. Noether currents for principal chiral models

In this section we work out the Noether currents for principal chiral models and relate them to the Killing vectors of the model, so as to recover equation (3.29).

Group action perspective. Given the invariance under global left and global right action

$$\mathrm{G}_L : \mathrm{g} \quad \to \quad g_L^{-1}\mathrm{g} \qquad\qquad \mathrm{G}_R : \mathrm{g} \quad \to \quad \mathrm{g}\, g_R\ , \tag{C.15}$$

one can extract the Noether currents by using the standard trick of taking local infinitesimal transformations $g_R = 1 + \epsilon_R$, with $\epsilon_R = \epsilon_R^A T_A$.

Under the above actions, the current j respectively transforms as

$$\mathrm{G}_L : \quad j \quad \rightarrow \quad j - \mathrm{g}^{-1}(\mathrm{d}\epsilon_L)\mathrm{g} \qquad\qquad \mathrm{G}_R : \quad j \quad \rightarrow \quad j + \nabla_j \epsilon_R \ , \tag{C.16}$$

with ∇_j defined in (C.4). Integrating by parts and exploiting $[j, \star j] = 0$, the variations then read

$$\delta_L S_{PCM} = -\int_\Sigma \langle \epsilon_L, \mathrm{d}(\star L_N)\rangle \qquad \text{and} \qquad \delta_R S_{PCM} = -\int_\Sigma \langle \epsilon_R, \mathrm{d}(\star R_L)\rangle \ , \tag{C.17}$$

with the Noether currents $\mathrm{L}_N := -\mathrm{g} j \mathrm{g}^{-1}$ and $\mathrm{R}_N := j$ as in (3.26).

Coordinate transformation perspective. We consider for the moment a sigma model including metric and B-field terms $S = S_g + S_B$ with

$$S_g = \frac{1}{2}\int_\Sigma \mathrm{d}x^N \wedge \star \mathrm{d}x^M\, g_{MN} \qquad \text{and} \qquad S_B = \frac{1}{2}\int_\Sigma \mathrm{d}x^N \wedge \mathrm{d}x^M\, B_{MN} \ , \tag{C.18}$$

such that $g_{MN} = (-1)^{MN} g_{NM}$ and $B_{MN} = -(-1)^{MN} B_{NM}$. Then, considering a local infinitesimal coordinate transformation $\delta x^A = \epsilon^V \xi_V^A$ with gradings $\left|\delta x^A\right| = A$, $\left|\epsilon^V\right| = V$, $\left|\xi_V^A\right| = V + A$, the variations of the metric term in the action read

$$\begin{aligned}
\delta S_g = \frac{1}{2}\int_\Sigma \big[&(\mathrm{d}\epsilon^V)\xi_V^N \wedge \star \mathrm{d}x^M g_{MN} + \epsilon^V(\mathrm{d}\xi_V^N) \wedge \star \mathrm{d}x^M g_{MN} + \\
&+ \mathrm{d}x^N \wedge \star(\mathrm{d}\epsilon^V)\xi_V^M g_{MN} + \mathrm{d}x^N \wedge \star\epsilon^V(\mathrm{d}\xi_V^M) g_{MN} + \mathrm{d}x^N \wedge \star \mathrm{d}x^M \epsilon^V \xi_V^P \partial_P g_{MN}\big] \\
= \frac{1}{2}\int_\Sigma \big[&2\mathrm{d}\epsilon^V \wedge \xi_V^N \star \mathrm{d}x^M g_{MN} + \\
+ \epsilon^V\big(&\mathrm{d}x^N \partial_N \xi_V^P \wedge \star \mathrm{d}x^M g_{MP} + (-1)^{NV} \mathrm{d}x^N \wedge \star \mathrm{d}x^M \partial^M \xi_V^P g_{PN} + (-1)^{V(M+N)} \mathrm{d}x^N \wedge \star \mathrm{d}x^M \xi_V^P \partial_P g_{MN}\big)\big] = \\
= \int_\Sigma &\mathrm{d}\epsilon^V \wedge \xi_V^N \star \mathrm{d}x^M g_{MN} + \frac{1}{2}\epsilon^V \mathrm{d}x^N \wedge \star \mathrm{d}x^M (\mathscr{L}_{\xi_V} g)_{MN} \ ,
\end{aligned} \tag{C.19}$$

and similarly for the B-field

$$\delta S_B = \int_\Sigma \mathrm{d}\epsilon^V \wedge \xi_V^N \mathrm{d}x^M B_{MN} + \frac{1}{2}\epsilon^V \mathrm{d}x^N \wedge \mathrm{d}x^M (\mathscr{L}_{\xi_V} B)_{MN} \ , \tag{C.20}$$

with the graded Lie derivative given in (A.13). At this stage, requiring the coordinate transformation to represent an isometry of the full action, the Lie derivative terms vanish and after integrating by parts one is left with the Noether current

$$\delta S = -\int_\Sigma \epsilon^V \mathrm{d}(\star J_V) \qquad \text{with} \qquad J_V := \xi_V^N \mathrm{d}x^M g_{MN} + \xi_V^N \star \mathrm{d}x^M B_{MN} \ . \tag{C.21}$$

For principal chiral models, which have no B-field contribution, this expression reduces to (3.29).

C.4. Left Noether current on OSp(1|2)

In this section we construct the Noether currents associated to the G_L invariance of the principal chiral model on OSp(1|2). Let us recall the left Noether current (3.26)

$$L_N = g_{bos} dg_{bos}^{-1} + g_{bos}(g_{fer} dg_{fer}^{-1})g_{bos}^{-1} \qquad \text{with} \qquad g_{bos} := e^{x^a L_a} \qquad g_{fer} := e^{-\theta^\alpha Q_\alpha} \ . \tag{C.22}$$

The bosonic contribution has been constructed in (3.32) and to compute the remaining terms we start by expanding the fermionic group element and exploiting the algebra (B.11) to find

$$\begin{aligned} g_{fer} dg_{fer}^{-1} &= (1 - Q + \tfrac{1}{2}Q^2) d(Q + \tfrac{1}{2!}Q^2 + \tfrac{1}{3!}Q^3) = dQ - \tfrac{1}{2}[Q, dQ] + \tfrac{1}{6}[Q,[Q,dQ]] = \\ &= \frac{i}{2} d\theta^\beta \theta^\alpha (\gamma^a)_{\alpha\beta} L_a + (1 - \frac{i}{8}\theta^2) d\theta^\alpha Q_\alpha \ . \end{aligned} \tag{C.23}$$

We then use the relation $e^A B e^{-A} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}_A^k(B)$ to include the effect of the bosonic rotation. For the L_a generators, a few commutators are sufficient to find a pattern and re-sum the series as

$$\begin{aligned} g_{bos} L_a g_{bos}^{-1} &= L_a + \sum_{k=0}^{\infty} \frac{R^{2k-2}}{(2k)!}(R^2 \delta_a{}^b - x_a x^b) L_b + \sum_{k=0}^{\infty} \frac{R^{2k}}{(2k+1)!} x^c \varepsilon_{ca}{}^b L_b = \\ &= L_a + R^{-1} \sinh(R) x^c \varepsilon_{ca}{}^b L_b + R^{-2}(\cosh(R) - 1)(R^2 \delta_a{}^b - x_a x^b) L_b = \\ &= \Big[\cosh(R) \delta_a{}^b + \frac{1 - \cosh R}{R^2} x_a x^b + \frac{\sinh R}{R} x^c \varepsilon_{ca}{}^b \Big] L_b \ , \end{aligned} \tag{C.24}$$

and similarly for the Q_α generators

$$\begin{aligned} g_{bos} Q_\alpha g_{bos}^{-1} &= \sum_{k=0}^{\infty} \frac{(R/2)^{2k}}{(2k)!} Q_\alpha - \frac{1}{2} \sum_{k=0}^{\infty} \frac{(R/2)^{2k}}{(2k+1)!} x_a (\gamma^a)_\alpha{}^\beta Q_\beta = \\ &= \Big[\cosh(R/2) \delta_\alpha{}^\beta - \frac{\sinh(R/2)}{R} x_a (\gamma^a)_\alpha{}^\beta \Big] Q_\beta \ . \end{aligned} \tag{C.25}$$

Putting together the above ingredients one then finds $L_N = L_N^a L_a + L_N^\alpha Q_\alpha$ as in (3.35).

C.5. Killing vectors of the initial model

In this section we construct explicitly the Killing vectors for the principal chiral model on OSp(1|2). Before specialising to such model, we consider again a more general sigma model including a B-field term, for which we derived in (C.21) the Noether current associated to the isometries

$$J_V = \langle J, V \rangle = \xi_V^N dx^M g_{MN} + \xi_V^N \star dx^M B_{MN} \ , \tag{C.26}$$

with V denoting either one of the bosonic generators $\{B_a\}$ or one of the fermionic generators $\{F_\alpha\}$. The inner product is taken to be non-degenerate and block diagonal with $\langle B_a, B_b\rangle = \delta_{ab}$ and $\langle F_\alpha, F_\beta\rangle = \delta_{\alpha\beta}$. Our aim here is extracting the components of the Killing vectors from those of the Noether current and for this reason we proceed assuming to have an explicit expression for J, which can be expanded as $J = \mathrm{d}x^N J^{(1)}_N + \star\mathrm{d}x^N J^{(2)}_N$. Comparing the two sides of the equation above for $V = B_b$ and $V = F_\beta$ one then finds

$$\begin{aligned} V = B_b: &\qquad J^{(1)a}_N\delta_{ab} = \xi^M_{B_b} g_{MN} &\qquad J^{(2)a}_N\delta_{ab} &= -\xi^M_{B_b} B_{MN} \\ V = F_\beta: &\qquad J^{(1)\alpha}_N\delta_{\alpha\beta} = (-1)^N \xi^M_{F_\beta} g_{MN} &\qquad J^{(2)\alpha}_N\delta_{\alpha\beta} &= -(-1)^N \xi^M_{F_\beta} B_{MN}\ . \end{aligned} \tag{C.27}$$

The two equations involving the metric will now be exploited to extract the components of the Killing vectors. If the model also has a B-field which does not break the isometries, the equations involving such field will then be automatically respected. The principal chiral model will not have problems from this point of view, but later on we shall deal with this in the T-dual model.

Setting $V = B_b$ the equations involving the metric read

$$\textcircled{1}\quad J^{(1)a}_n\delta_{ab} = \xi^m_{B_b} g_{mn} + \xi^\mu_{B_b} g_{\mu n} \qquad\qquad \textcircled{2}\quad J^{(1)a}_\nu\delta_{ab} = \xi^m_{B_b} g_{m\nu} + \xi^\mu_{B_b} g_{\mu\nu}\ . \tag{C.28}$$

For the full metric to be invertible $(-1)^P g_{MP} g^{PN} = \delta_M{}^N$, the two block diagonal terms g_{mn} and $g_{\mu\nu}$ should both be separately invertible and we define their inverse matrices via $g_{mp}\hat{g}^{pn} = \delta_m{}^n$ and $g_{\mu\rho}\hat{g}^{\rho\nu} = \delta_\mu{}^\nu$. We put a hat on the inverses to highlight that those are not the components of the full inverse metric g^{MB}. This fact can be exploited to solve the second equation for $\xi^\mu_{B_b}$, substitute back into the first equation and solve it for $\xi^m_{B_b}$. In this second step one further needs to invert the symmetric operator $G_{mn} := g_{mn} + \hat{g}^{\mu\nu} g_{m\nu} g_{n\mu}$. Its invertibility is ensured by the invertibility of g_{mn} and we write its inverse as $G_{mp}G^{pn} = \delta_m{}^n$. In the end one finds Killing vector components

$$\begin{cases} \xi^m_{B_b} = G^{mn}(J^{(1)a}_n\delta_{ab} + \hat{g}^{\mu\nu} J^{(1)a}_\nu \delta_{ab} g_{\mu n}) \\ \xi^\mu_{B_b} = \hat{g}^{\mu\nu}[G^{mn}(J^{(1)a}_n\delta_{ab} + \hat{g}^{\rho\sigma} J^{(1)a}_\sigma \delta_{ab} g_{\rho n}) g_{m\nu} - J^{(1)a}_\nu \delta_{ab}] \end{cases} . \tag{C.29}$$

One can then proceed in a very similar way for the case $V = F_\beta$. The equations read

$$\textcircled{4}\quad J^{(1)\alpha}_n\delta_{\alpha\beta} = \xi^m_{Q_\beta} g_{mn} + \xi^\mu_{Q_\beta} g_{\mu n} \qquad\qquad \textcircled{4}\quad J^{(1)\alpha}_\nu\delta_{\alpha\beta} = -\xi^m_{Q_\beta} g_{m\nu} - \xi^\mu_{Q_\beta} g_{\mu\nu} \tag{C.30}$$

and the resulting Killing vector components are

$$\begin{cases} \xi^m_{F_\beta} = G^{mn}[J^{(1)\alpha}_n\delta_{\alpha\beta} - \hat{g}^{\mu\nu} J^{(1)\alpha}_\nu \delta_{\alpha\beta} g_{\mu n}] \\ \xi^\mu_{F_\beta} = \hat{g}^{\mu\nu}[G^{mn}(J^{(1)\alpha}_n\delta_{\alpha\beta} - \hat{g}^{\rho\sigma} J^{(1)\alpha}_\sigma \delta_{\alpha\beta} g_{n\rho}) g_{m\nu} + J^{(1)\alpha}_\nu \delta_{\alpha\beta}] \end{cases} . \tag{C.31}$$

We now specialise to OSp(1|2) with metric (3.21) and inner products (3.15). From these one finds

$$\begin{aligned} g_{\mu\rho} &= i\epsilon_{\mu\rho}(1-\frac{i}{16}\theta^2) \quad \Rightarrow \quad \hat{g}^{\rho\nu} = -i\epsilon^{\lambda\rho}(1+\frac{i}{16}\theta^2) \quad \text{such that} \quad g_{\mu\rho}\hat{g}^{\rho\nu} = \delta_\mu{}^\nu \\ g_{mp} &= g_1\eta_{mp} + g_2 x_m x_p \quad \Rightarrow \quad \hat{g}^{pn} = g_3\eta^{pn} + g_4 x^p x^n \quad \text{such that} \quad g_{mp}\hat{g}^{pn} = \delta_m{}^n \end{aligned} \tag{C.32}$$

with coefficients g_1, g_2 given in (3.21) and $g_3 = \frac{1}{g_1}$, $g_4 = \frac{g_2}{2g_1}$. Additionally the operator G_{ab} reads

$$G_{mp} := g_{mp} + g^{\mu\nu} g_{m\nu} g_{p\mu} = (1+\frac{i}{8}\theta^2) g_{mp} \quad \Rightarrow \quad G^{pn} = (1-\frac{i}{8}\theta^2)\hat{g}^{pn} \ . \tag{C.33}$$

The right sector. To obtain the components of the right set of Killing vectors it is sufficient to recall the right Noether current (3.30) to extract

$$(\mathrm{R}_N)^a_n = (1+\frac{i}{4}\theta^2)\lambda_n{}^a \quad (\mathrm{R}_N)^a_\nu = \frac{i}{2}\theta^\rho(\gamma^a)_{\rho\nu} \quad (\mathrm{R}_N)^\alpha_n = \frac{1}{2}\lambda_n{}^a\theta^\rho(\gamma^a)_\rho{}^\alpha \quad (\mathrm{R}_N)^\alpha_\nu = -(1-\frac{i}{8}\theta^2)\delta_\nu{}^\alpha .$$

Using then the inner products (3.15) and the expressions derived above one obtains components

$$\begin{aligned} (\xi^R_{L_k})^c &= A\delta_k{}^c + Bx_k x^c + \frac{1}{2}x^d\varepsilon_{dk}{}^c \qquad & (\xi^R_{L_k})^\rho &= \frac{1}{2}\theta^\nu(\gamma_k)_\nu{}^\rho \\ (\xi^R_{Q_\beta})^c &= \frac{i}{2}\theta^\nu(\gamma^a)_{\nu\beta}(A\delta_a{}^c + Bx_a x^c + \frac{1}{2}x^d\varepsilon_{da}{}^c) \qquad & (\xi^R_{Q_\beta})^\lambda &= -(1-\frac{i}{4}\theta^2)\delta_\beta{}^\lambda \ . \end{aligned} \tag{C.34}$$

The left sector. Similarly, for the left sector, recalling the Noether current (3.35) one extracts

$$\begin{aligned} (\mathrm{L}_N)^a_m &= -\frac{\sinh R}{R}\delta_m{}^a - \frac{R-\sinh R}{R^3}x_m x^a - \frac{2\sinh^2(R/2)}{R^2}x^c\varepsilon_{cm}{}^a \\ (\mathrm{L}_N)^a_\nu &= \tfrac{i}{2}\theta^\rho(\gamma^b)_{\rho\nu}[\cosh(R)\delta_b{}^a + \frac{1-\cosh R}{R^2}x_b x^a + \frac{\sinh R}{R}x^c\varepsilon_{cb}{}^a] \\ (\mathrm{L}_N)^\alpha_\nu &= (1-\frac{i}{8}\theta^2)[\cosh(R/2)\delta_\nu{}^\alpha - \frac{\sinh(R/2)}{R}x_a(\gamma^a)_\nu{}^\alpha] \qquad (\mathrm{L}_N)^a_\nu = 0 \ . \end{aligned} \tag{C.35}$$

Using again the inner products and the expressions above one then finds

$$\begin{aligned} (\xi^L_{L_k})^c &= -(A\delta_k{}^c + Bx_k x^x - \frac{1}{2}x^d\varepsilon_{dk}{}^c) \qquad & (\xi^L_{L_k})^\rho &= 0 \\ (\xi^L_{Q_\beta})^c &= \frac{i}{2}N\theta_\beta x^c - \frac{i}{4}\theta^\lambda(\gamma^b)_{\lambda\beta}(U\delta_b{}^c + Vx_b x^x) \qquad & (\xi^L_{Q_\beta})^\rho &= -(1+\frac{i}{8}\theta^2)(Nx_a(\gamma^a)_\beta{}^\rho - M\delta_\beta{}^\rho) \ , \end{aligned} \tag{C.36}$$

with the coefficients A, B, U, V, M, N introduced in equation (3.37).

D

T-Dual Action On OSp(1|2)

D.1. Details of the procedure

In this section we provide details relative to the dualisation of the principal chiral model on OSp(1|2) with respect to the $\mathrm{OSp}(1|2)_L$ isometries. Starting from the explicit Lagrangian (3.52), which we report here for clarity

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}[g^{-1}\partial g]_a[g^{-1}\bar{\partial}g]^a - \frac{1}{2}[(\partial g)g^{-1}]_a\bar{A}^a - \frac{1}{2}[(\bar{\partial}g)g^{-1}]_aA^a - \frac{1}{2}A_a\bar{A}^a +\\ &+\frac{1}{2}(\partial\tilde{x}_a)\bar{A}^a - \frac{1}{2}(\bar{\partial}\tilde{x}_a)A^a - \frac{i}{2}\tilde{x}_aA^\alpha\bar{A}^\beta(\gamma^a)_{\beta\alpha} + \frac{1}{2}\tilde{x}_aA^c\bar{A}^d\varepsilon_{dc}{}^a +\\ &+ i[g^{-1}\partial g]_\beta[g^{-1}\bar{\partial}g]^\beta + i[(\partial g)g^{-1}]_\beta\bar{A}^\beta + i[(\bar{\partial}g)g^{-1}]_\beta A^\beta + iA_\beta\bar{A}^\beta +\\ &- i(\partial\tilde{\theta}_\beta)\bar{A}^\beta + i(\bar{\partial}\tilde{\theta}_\beta)A^\beta + \frac{i}{2}\tilde{\theta}_\beta A^\alpha\bar{A}^a(\gamma_a)_\alpha{}^\beta - \frac{i}{2}\tilde{\theta}_\beta A^a\bar{A}^\alpha(\gamma_a)_\alpha{}^\beta \ .\end{aligned} \tag{D.1}$$

Variations of the gauge fields lead to the equations

$$\begin{aligned}\bar{A}^a - \bar{A}^c\varepsilon_c{}^{ab}\tilde{x}_b &= -[(\bar{\partial}g)g^{-1}]^a - \bar{\partial}\tilde{x}^a + i\bar{A}^\alpha(\gamma^a)_\alpha{}^\beta\tilde{\theta}_\beta\\ A^a + A^c\varepsilon_c{}^{ab}\tilde{x}_b &= -[(\partial g)g^{-1}]^a + \partial\tilde{x}^a - iA^\alpha(\gamma^a)_\alpha{}^\beta\tilde{\theta}_\beta\\ \bar{A}^\alpha + \frac{1}{2}\bar{A}^\beta(\gamma^a)_\beta{}^\alpha\tilde{x}_a + \frac{1}{2}\bar{A}^a(\gamma_a)^{\alpha\beta}\tilde{\theta}_\beta &= -[(\bar{\partial}g)g^{-1}]^\alpha - \bar{\partial}\tilde{\theta}^\alpha\\ A^\alpha - \frac{1}{2}A^\beta(\gamma^a)_\beta{}^\alpha\tilde{x}_a - \frac{1}{2}A^a(\gamma_a)^{\alpha\beta}\tilde{\theta}_\beta &= -[(\partial g)g^{-1}]^\alpha + \partial\tilde{\theta}^\alpha \ ,\end{aligned} \tag{D.2}$$

and to solve the latter system for the gauge fields we start from the first couple of equations. These allow us to find $\bar{A}^a$ and A^a in terms of the fermionic fields by defining the matrices M and

N, that can be easily inverted exploiting the identities in B.1.

$$\begin{cases} M_c{}^a \equiv \delta_c{}^a - \varepsilon_c{}^{ab}\tilde{x}_b & \Rightarrow \quad (M^{-1})_a{}^k = \frac{1}{1-r^2}(\delta_a{}^k - \tilde{x}_a\tilde{x}^k + \varepsilon_a{}^{kn}\tilde{x}_n) \\ N_c{}^a \equiv \delta_c{}^a + \varepsilon_c{}^{ab}\tilde{x}_b & \Rightarrow \quad (N^{-1})_a{}^k = \frac{1}{1-r^2}(\delta_a{}^k - \tilde{x}_a\tilde{x}^k - \varepsilon_a{}^{kn}\tilde{x}_n) \end{cases} \tag{D.3}$$
$$\text{with} \qquad r^2 \equiv \tilde{x}^a\tilde{x}_a = \tilde{x}_a\tilde{x}^a = \tilde{x}^b\tilde{x}^a\eta_{ab} \ .$$

This leads to rewriting the first two equations as

$$\begin{cases} \bar{A}^a & = \left[-[(\bar{\partial}\mathrm{g})\mathrm{g}^{-1}]^b - \bar{\partial}\tilde{x}^b + i\bar{A}^\alpha(\gamma^b)_\alpha{}^\beta\tilde{\theta}_\beta\right](M^{-1})_b{}^a \\ A^a & = \left[-[(\partial\mathrm{g})\mathrm{g}^{-1}]^b + \partial\tilde{x}^b - iA^\alpha(\gamma^b)_\alpha{}^\beta\tilde{\theta}_\beta\right](N^{-1})_b{}^a \end{cases} , \tag{D.4}$$

so that substituting the latter into the second pair of equations and rearranging one obtains

$$\begin{cases} \bar{A}^\beta(W_1)_\beta{}^\alpha & = \frac{1}{2}\big([(\bar{\partial}\mathrm{g})\mathrm{g}^{-1}]^b + \bar{\partial}\tilde{x}^b\big)(M^{-1})_b{}^a(\gamma_a)^{\alpha\sigma}\tilde{\theta}_\sigma - [(\bar{\partial}\mathrm{g})\mathrm{g}^{-1}]^\alpha - \bar{\partial}\tilde{\theta}^\alpha \\ A^\beta(W_2)_\beta{}^\alpha & = \frac{1}{2}\big(-[(\partial\mathrm{g})\mathrm{g}^{-1}]^b + \partial\tilde{x}^b\big)(N^{-1})_b{}^a(\gamma_a)^{\alpha\sigma}\tilde{\theta}_\sigma - [(\partial\mathrm{g})\mathrm{g}^{-1}]^\alpha + \partial\tilde{\theta}^\alpha \end{cases} , \tag{D.5}$$

with the definitions

$$\begin{cases} (W_1)_\beta{}^\alpha \equiv \delta_\beta{}^\alpha + \frac{1}{2}\tilde{x}_a(\gamma^a)_\beta{}^\alpha + \frac{i}{2}(\gamma^a)_\beta{}^\rho(M^{-1})_a{}^b(\gamma_b)^{\alpha\sigma}\tilde{\theta}_\rho\tilde{\theta}_\sigma \\ (W_2)_\beta{}^\alpha \equiv \delta_\beta{}^\alpha - \frac{1}{2}\tilde{x}_a(\gamma^a)_\beta{}^\alpha + \frac{i}{2}(\gamma^a)_\beta{}^\rho(N^{-1})_a{}^b(\gamma_b)^{\alpha\sigma}\tilde{\theta}_\rho\tilde{\theta}_\sigma \end{cases} . \tag{D.6}$$

In order to recast W_1 and W_2 in the form (3.54) and to invert them, we need to make some simplifications. The first step is exploiting the identity

$$\begin{aligned} \tilde{\theta}_\rho\tilde{\theta}_\sigma &= \frac{1}{2}(\tilde{\theta}_\rho\tilde{\theta}_\sigma - \tilde{\theta}_\sigma\tilde{\theta}_\rho) = \frac{1}{2}(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)\tilde{\theta}_\alpha\tilde{\theta}_\beta = \\ &= \frac{1}{2}\epsilon_{\rho\sigma}\epsilon^{\beta\alpha}\tilde{\theta}_\alpha\tilde{\theta}_\beta = \frac{1}{2}\epsilon_{\rho\sigma}(\tilde{\theta}^\beta\tilde{\theta}_\beta) = \frac{1}{2}\epsilon_{\rho\sigma}\tilde{\theta}^2 \ , \end{aligned} \tag{D.7}$$

which also has the raised-index analogue

$$\tilde{\theta}^\rho\tilde{\theta}^\sigma = -\frac{1}{2}\epsilon^{\rho\sigma}\tilde{\theta}^2 \ . \tag{D.8}$$

By using (D.7) in the above definitions of W_1 and W_2 we obtain

$$\begin{cases} \frac{i}{2}(\gamma^a)_\beta{}^\rho(M^{-1})_a{}^b(\gamma_b)^{\alpha\sigma}\tilde{\theta}_\rho\tilde{\theta}_\sigma = \frac{i}{4}\tilde{\theta}^2\Big(\delta_\beta{}^\alpha(M^{-1})_a{}^a + (\gamma^c)_\beta{}^\alpha(M^{-1})_a{}^b\varepsilon^a{}_{bc}\Big) \\ \frac{i}{2}(\gamma^a)_\beta{}^\rho(N^{-1})_a{}^b(\gamma_b)^{\alpha\sigma}\tilde{\theta}_\rho\tilde{\theta}_\sigma = \frac{i}{4}\tilde{\theta}^2\Big(\delta_\beta{}^\alpha(N^{-1})_a{}^a + (\gamma^c)_\beta{}^\alpha(N^{-1})_a{}^b\varepsilon^a{}_{bc}\Big) \end{cases} , \tag{D.9}$$

so that from the explicit form of M^{-1} and N^{-1} one can find

$$(M^{-1})_a{}^a = (N^{-1})_a{}^a = \frac{r^2-3}{r^2-1} \qquad (M^{-1})_a{}^b\varepsilon^a{}_{bc} = -(N^{-1})_a{}^b\varepsilon^a{}_{bc} = \frac{2\tilde{x}_c}{r^2-1} \tag{D.10}$$

and rearrange (D.6) to obtain (3.54). The two matrices are then easily inverted using the identities for the γ-matrices in B.1., leading to (3.55). This allows to solve for the fermionic gauge fields in (D.5) and to substitute the result back into (D.4) to obtain the equations of motion (3.53).

Substituting the equations of motion back into the Lagrangian (3.52) and choosing gauge $g = \mathbb{1}$, we thus get

$$\begin{aligned}
\bar{A}^\alpha &= -(\bar\partial\tilde\theta^\beta)(W_1^{-1})_\beta{}^\alpha + \frac{1}{2}(\bar\partial\tilde{x}^b)(M^{-1})_b{}^a(\gamma_a)^{\beta\lambda}\tilde\theta_\lambda(W_1^{-1})_\beta{}^\alpha \\
A^\alpha &= +(\partial\tilde\theta^\beta)(W_2^{-1})_\beta{}^\alpha + \frac{1}{2}(\partial\tilde{x}^b)(N^{-1})_b{}^a(\gamma_a)^{\beta\lambda}\tilde\theta_\lambda(W_2^{-1})_\beta{}^\alpha \\
\bar{A}^a &= -(\bar\partial\tilde{x}^b)(M^{-1})_b{}^a - i(\bar\partial\tilde\theta^\beta)(W_1^{-1})_\beta{}^\alpha(\gamma^b)_\alpha{}^\rho\tilde\theta_\rho(M^{-1})_b{}^a + \\
&\quad + \frac{i}{2}(\bar\partial\tilde{x}^c)(M^{-1})_c{}^d(\gamma_d)^{\beta\lambda}\tilde\theta_\lambda(W_1^{-1})_\beta{}^\alpha(\gamma^b)_\alpha{}^\rho\tilde\theta_\rho(M^{-1})_b{}^a \\
A^a &= +(\partial\tilde{x}^b)(N^{-1})_b{}^a - i(\partial\tilde\theta^\beta)(W_2^{-1})_\beta{}^\alpha(\gamma^b)_\alpha{}^\rho\tilde\theta_\rho(N^{-1})_b{}^a + \\
&\quad - \frac{i}{2}(\partial\tilde{x}^c)(N^{-1})_c{}^d(\gamma_d)^{\beta\lambda}\tilde\theta_\lambda(W_2^{-1})_\beta{}^\alpha(\gamma^b)_\alpha{}^\rho\tilde\theta_\rho(N^{-1})_b{}^a
\end{aligned} \tag{D.11}$$

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}(\eta_{ab} + \varepsilon_{abc}\tilde{x}^c)A^a\bar{A}^b - i(\epsilon_{\alpha\beta} + \frac{1}{2}\tilde{x}_c(\gamma^c)_{\alpha\beta})A^\alpha\bar{A}^\beta + \\
&+ \frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde\theta_\beta A^\alpha\bar{A}^a - \frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde\theta_\beta A^a\bar{A}^\alpha + \\
&+ \frac{1}{2}(\partial\tilde{x}_a)\bar{A}^a - \frac{1}{2}(\bar\partial\tilde{x}_a)A^a - i(\partial\tilde\theta_\alpha)\bar{A}^\alpha + i(\bar\partial\tilde\theta_\alpha)A^\alpha \ .
\end{aligned} \tag{D.12}$$

Using the explicit form of $M^{-1}, N^{-1}, W_1^{-1}, W_2^{-1}$ and exploiting the identities (B.8) and (B.7) it is finally possible to rewrite the gauge fields more explicitly. These are reported below broken down into terms leading to different contributions to the dual action.

$$\bar{A}^a := \bar{A}^a_1 + \bar{A}^a_2 + \bar{A}^a_3 \qquad \text{with:} \tag{D.13}$$

$$\begin{aligned}
\bar{A}^a_1 &:= -(\bar\partial\tilde{x}^c)(M^{-1})_c{}^a \\
\bar{A}^a_2 &:= \frac{i\tilde\theta_\beta(\bar\partial\tilde\theta^\rho)}{(4-r^2)(1-r^2)}\left[2(2+r^2)(\gamma^a)_\rho{}^\beta - 6\tilde{x}_c(\gamma^c)_\rho{}^\beta\tilde{x}^a - 2(1-r^2)\delta_\rho{}^\beta\tilde{x}^a - 6\varepsilon^a{}_{ck}(\gamma^c)_\rho{}^\beta\tilde{x}^k\right] \\
\bar{A}^a_3 &:= \frac{\tilde\theta^2(\bar\partial\tilde{x}^c)}{(4-r^2)(1-r^2)^2}\left[2\tilde{x}_c\tilde{x}^a(4-r^2) - 2\delta_c{}^a(1+2r^2) - \varepsilon_c{}^{ad}\tilde{x}_d(5+r^2)\right]
\end{aligned}$$

$$A^a := A_1^a + A_2^a + A_3^a \qquad \text{with:} \tag{D.14}$$

$$
\begin{aligned}
A_1^a &:= (\partial\tilde{x}^c)(N^{-1})_c{}^a \\
A_2^a &:= \frac{i\tilde{\theta}_\beta(\partial\tilde{\theta}^\rho)}{(4-r^2)(1-r^2)}\Big[2(2+r^2)(\gamma^a)_\rho{}^\beta - 6\tilde{x}_c(\gamma^c)_\rho{}^\beta\tilde{x}^a + 2(1-r^2)\delta_\rho{}^\beta\tilde{x}^a + 6\varepsilon^a{}_{ck}(\gamma^c)_\rho{}^\beta\tilde{x}^k\Big] \\
A_3^a &:= \frac{i\tilde{\theta}^2(\partial\tilde{x}^c)}{(4-r^2)(1-r^2)^2}\Big[-2\tilde{x}_c\tilde{x}^a(4-r^2) + 2\delta_c{}^a(1+2r^2) - \varepsilon_c{}^{ad}\tilde{x}_d(5+r^2)\Big]
\end{aligned}
$$

$$\bar{A}^\alpha := \bar{A}_1^\alpha + \bar{A}_2^\alpha \qquad \text{with:} \tag{D.15}$$

$$
\begin{aligned}
\bar{A}_1^\alpha &:= -(\bar{\partial}\tilde{\theta}^\rho)(W_1^{-1})_\rho{}^\alpha \\
\bar{A}_2^\alpha &:= \frac{\tilde{\theta}_\beta(\bar{\partial}\tilde{x}^b)}{(4-r^2)(1-r^2)}\Big[(2+r^2)(\gamma_b)^{\alpha\beta} - 3\tilde{x}_c(\gamma^c)^{\alpha\beta}\tilde{x}_b + (1-r^2)\epsilon^{\alpha\beta}\tilde{x}_b + 3\varepsilon_{bck}(\gamma^c)^{\alpha\beta}\tilde{x}^k\Big]
\end{aligned}
$$

$$A^\alpha := A_1^\alpha + A_2^\alpha \qquad \text{with:} \tag{D.16}$$

$$
\begin{aligned}
A_1^\alpha &:= (\partial\tilde{\theta}^\rho)(W_2^{-1})_\rho{}^\alpha \\
A_2^\alpha &:= \frac{\tilde{\theta}_\beta(\partial\tilde{x}^b)}{(4-r^2)(1-r^2)}\Big[(2+r^2)(\gamma_b)^{\alpha\beta} - 3\tilde{x}_c(\gamma^c)^{\alpha\beta}\tilde{x}_b - (1-r^2)\epsilon^{\alpha\beta}\tilde{x}_b - 3\varepsilon_{bck}(\gamma^c)^{\alpha\beta}\tilde{x}^k\Big]\,.
\end{aligned}
$$

The above terms respectively contribute to the expressions

$$
\begin{aligned}
(\partial\tilde{x}^q)(\bar{\partial}\tilde{x}^p)\tilde{L}_{pq} = &-\frac{1}{2}(\eta_{ab} + \varepsilon_{abc}\tilde{x}^c)(A_1^a\bar{A}_1^b + A_1^a\bar{A}_3^b + A_3^a\bar{A}_1^b) + \\
&- i(\epsilon_{\alpha\beta} + \frac{1}{2}\tilde{x}_c(\gamma^c)_{\alpha\beta})A_2^\alpha\bar{A}_2^\beta + \\
&+ \frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde{\theta}_\beta A_2^\alpha\bar{A}_1^a - \frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde{\theta}_\beta A_1^a\bar{A}_2^\alpha + \\
&+ \frac{1}{2}(\partial\tilde{x}_a)(\bar{A}_1^a + \bar{A}_3^a) - \frac{1}{2}(\bar{\partial}\tilde{x}_a)(A_1^a + A_3^a)
\end{aligned} \tag{D.17}
$$

$$
\begin{aligned}
(\partial\tilde{\theta}^\sigma)(\bar{\partial}\tilde{\theta}^\rho)\tilde{L}_{\rho\sigma} = &-\frac{1}{2}(\eta_{ab} + \varepsilon_{abc}\tilde{x}^c)A_2^a\bar{A}_2^b - i(\epsilon_{\alpha\beta} + \frac{1}{2}\tilde{x}_c(\gamma^c)_{\alpha\beta})A_1^\alpha\bar{A}_1^\beta + \\
&+ \frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde{\theta}_\beta A_1^\alpha\bar{A}_2^a - \frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde{\theta}_\beta A_2^a\bar{A}_1^\alpha + \\
&- i(\partial\tilde{\theta}_\alpha)\bar{A}_1^\alpha + i(\bar{\partial}\tilde{\theta}_\alpha)A_1^\alpha
\end{aligned} \tag{D.18}
$$

$$
\begin{aligned}
(\partial\tilde{x}^q)(\bar{\partial}\tilde{\theta}^\rho)\tilde{L}_{\rho q} = &-\frac{1}{2}(\eta_{ab}+\varepsilon_{abc}\tilde{x}^c)A_1^a\bar{A}_2^b - i(\epsilon_{\alpha\beta}+\frac{1}{2}\tilde{x}_c(\gamma^c)_{\alpha\beta})A_2^\alpha\bar{A}_1^\beta + \\
&-\frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde{\theta}_\beta A_1^a\bar{A}_1^\alpha + \frac{1}{2}(\partial\tilde{x}_a)\bar{A}_2^a + i(\bar{\partial}\tilde{\theta}_\alpha)A_2^\alpha
\end{aligned} \tag{D.19}
$$

$$
\begin{aligned}
(\partial\tilde{\theta}^\sigma)(\bar{\partial}\tilde{x}^p)\tilde{L}_{p\sigma} = &-\frac{1}{2}(\eta_{ab}+\varepsilon_{abc}\tilde{x}^c)A_2^a\bar{A}_1^b - i(\epsilon_{\alpha\beta}+\frac{1}{2}\tilde{x}_c(\gamma^c)_{\alpha\beta})A_1^\alpha\bar{A}_2^\beta + \\
&+\frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde{\theta}_\beta A_1^\alpha\bar{A}_1^a - \frac{1}{2}(\bar{\partial}\tilde{x}_a)A_2^a - i(\partial\tilde{\theta}_\alpha)\bar{A}_2^\alpha \ .
\end{aligned} \tag{D.20}
$$

It actually turns out that many of the above terms cancel among each other or simply vanish due to the nilpotency of the fermionic multipliers, thus leaving us with the following relevant contributions

$$
\begin{aligned}
(\partial\tilde{x}^q)(\bar{\partial}\tilde{x}^p)\tilde{L}_{pq} = &-i(\epsilon_{\alpha\beta}+\frac{1}{2}\tilde{x}_c(\gamma^c)_{\alpha\beta})A_2^\alpha\bar{A}_2^\beta + \frac{1}{2}(\partial\tilde{x}_a)\bar{A}_1^a \\
&+\frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde{\theta}_\beta A_2^\alpha\bar{A}_1^a - \frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde{\theta}_\beta A_1^a\bar{A}_2^\alpha
\end{aligned}
\qquad
(\partial\tilde{x}^q)(\bar{\partial}\tilde{\theta}^\rho)\tilde{L}_{\rho q} = +i(\bar{\partial}\tilde{\theta}_\alpha)A_2^\alpha
$$

$$
\begin{aligned}
(\partial\tilde{\theta}^\sigma)(\bar{\partial}\tilde{\theta}^\rho)\tilde{L}_{\rho\sigma} = &-i(\epsilon_{\alpha\beta}+\frac{1}{2}\tilde{x}_c(\gamma^c)_{\alpha\beta})A_1^\alpha\bar{A}_1^\beta + \frac{i}{2}(\gamma_a)_\alpha{}^\beta\tilde{\theta}_\beta A_1^\alpha\bar{A}_2^a + \\
&-i(\partial\tilde{\theta}_\alpha)\bar{A}_1^\alpha + i(\bar{\partial}\tilde{\theta}_\alpha)A_1^\alpha
\end{aligned}
\qquad
(\partial\tilde{\theta}^\sigma)(\bar{\partial}\tilde{x}^p)\tilde{L}_{p\sigma} = -i(\partial\tilde{\theta}_\alpha)\bar{A}_2^\alpha
$$

Working these out explicitly one finally obtains (3.58).

Some useful identities. We close this section by briefly summarising some additional identities, particularly useful in obtaining the explicit form of the gauge fields, resulting from the expressions of M^{-1}, $N^{-1}, W_1^{-1}, W_2^{-1}$ and (B.8),(B.7). Notice that the identities involving W_1^{-1} and W_2^{-1} are valid for their purely bosonic part, i.e. upon neglecting the terms proportional to $\tilde{\theta}^2$.

$$
(W_1^{-1})_\alpha{}^\alpha = \frac{8}{4-r^2} = (W_2^{-1})_\alpha{}^\alpha \qquad (W_1^{-1})_\alpha{}^\beta(\gamma^a)_\beta{}^\alpha = \frac{-4\tilde{x}^a}{4-r^2} = -(W_2^{-1})_\alpha{}^\beta(\gamma^a)_\beta{}^\alpha \tag{D.21}
$$

$$
\begin{cases}
(M^{-1})_c{}^b(M^{-1})_b{}^a = \frac{1}{(1-r^2)^2}[(1+r^2)\delta_c{}^a + (r^2-3)\tilde{x}_c\tilde{x}^a + 2\tilde{x}^d\varepsilon_{dc}{}^a] \\
(M^{-1})_c{}^d\varepsilon^b{}_{dk}(M^{-1})_b{}^a(W_1^{-1})_\beta{}^\alpha(\gamma^k)_\alpha{}^\beta = \frac{4}{(4-r^2)(1-r^2)^2}[2r^2\delta_c{}^a - 2\tilde{x}_c\tilde{x}^a + (1+r^2)\tilde{x}^d\varepsilon_{dc}{}^a]
\end{cases}
$$

$$
\begin{cases}
(N^{-1})_c{}^b(N^{-1})_b{}^a = \frac{1}{(1-r^2)^2}[(1+r^2)\delta_c{}^a - (r^2-3)\tilde{x}_c\tilde{x}^a - 2\tilde{x}^d\varepsilon_{dc}{}^a] \\
(N^{-1})_c{}^d\varepsilon^b{}_{dk}(N^{-1})_b{}^a(W_2^{-1})_\beta{}^\alpha(\gamma^k)_\alpha{}^\beta = \frac{4}{(4-r^2)(1-r^2)^2}[2r^2\delta_c{}^a - 2\tilde{x}_c\tilde{x}^a + (1+r^2)\tilde{x}^d\varepsilon_{dc}{}^a]
\end{cases}
$$

Additional identities required to obtain the explicit dual action are

$$
\begin{cases}
(M^{-1})_q{}^b(\eta_{ab}+\tilde{x}^c\varepsilon_{cab}) = \eta_{aq} \\
(N^{-1})_p{}^a(\eta_{ab}+\tilde{x}^c\varepsilon_{cab}) = \eta_{pb}
\end{cases}
\qquad
\begin{cases}
(W_1^{-1})_\sigma{}^\beta(\epsilon_{\alpha\beta}+\frac{1}{2}\tilde{x}_c(\gamma^c)_{\alpha\beta}) = -\epsilon_{\sigma\alpha} \\
(W_2^{-1})_\rho{}^\beta(\epsilon_{\alpha\beta}+\frac{1}{2}\tilde{x}_c(\gamma^c)_{\alpha\beta}) = \epsilon_{\rho\beta}
\end{cases} \tag{D.22}
$$

D.2. Isometries of the T-dual models

In this section we study the residual isometries of the T-dual models by solving explicitly the Killing equations (A.13). We consider in detail the dual model with respect to the full $\mathrm{OSp}(1|2)_L$ isometry subgroup and then discuss briefly the dual with respect to $\mathrm{SL}(2,\mathbb{R})_L$. To begin we rewrite the metric (3.60), obtained by dualising $\mathrm{OSp}(1|2)_L$, as follows

$$\begin{cases} \hat{g}_{pq} = L_1\big[\eta_{pq}\big(1+ih(r^2)\tilde{\theta}^2\big) - \tilde{x}_p\tilde{x}_q\big(1+im(r^2)\tilde{\theta}^2\big)\big] \\ \hat{g}_{p\sigma} = L_3\tilde{x}_p\epsilon_{\sigma\alpha}\tilde{\theta}^\alpha + L_4\varepsilon_{pab}(\gamma^a)_{\sigma\alpha}\tilde{\theta}^\alpha\Lambda^b \\ \hat{g}_{\rho\sigma} = L_2\epsilon_{\rho\sigma}\big(1+if(r^2)\tilde{\theta}^2\big) \end{cases} \quad \text{with} \quad \begin{cases} L_1 = \frac{1}{2(r^2-1)} \\ L_2 = \frac{-4i}{(r^2-4)} \\ L_3 = \frac{i(1-r^2)}{(r^2-1)(r^2-4)} \\ L_4 = \frac{-3i}{(r^2-1)(r^2-4)} \end{cases} \tag{D.23}$$

We broke down the coefficient L_3 defined in (3.59) into two new coefficients L_3 and L_4 and absorbed all the prefactors in L_2, L_3, L_4, leaving L_1 untouched. This will allow us to keep track of such coefficients and their derivatives without looking at their explicit form. We notice indeed that all coefficients are functions of $r^2 := \tilde{x}^b\tilde{x}^a\eta_{ab}$, such that $\forall\ u = u(r^2)$ we can easily compute

$$\partial_a u(r^2) = 2\tilde{x}_a u'(r^2)\ . \tag{D.24}$$

Similarly, recalling the definition $\tilde{\theta}^2 := \tilde{\theta}^\gamma\tilde{\theta}_\gamma = \epsilon_{\gamma\delta}\tilde{\theta}^\gamma\tilde{\theta}^\delta$ we can also easily determine that

$$\partial_\alpha\tilde{\theta}^2 = 2\tilde{\theta}_\alpha\ . \tag{D.25}$$

At this point we have all the ingredients to start computing the partial derivatives of the dual metric, which will be needed to write down the Killing equations

$$\begin{aligned} \partial_a\tilde{g}_{pq} &= +2\tilde{x}_a L_1'[\eta_{pq}(1+ih\tilde{\theta}^2) - \tilde{x}_p\tilde{x}_q(1+im\tilde{\theta}^2)] + 2iL_1h'\tilde{x}_a\eta_{pq}\tilde{\theta}^2 + \\ &\quad - L_1(\eta_{pa}\tilde{x}_q + \eta_{qa}\tilde{x}_p)(1+im\tilde{\theta}^2) - 2iL_1m'\tilde{x}_p\tilde{x}_q\tilde{x}_a\tilde{\theta}^2 \\ \partial_\alpha\tilde{g}_{pq} &= 2iL_1h\eta_{pq}\tilde{\theta}_\alpha - 2iL_1m\tilde{x}_p\tilde{x}_q\tilde{\theta}_\alpha \\ \partial_a\tilde{g}_{p\sigma} &= +2L_3'\tilde{x}_a\tilde{x}_p\epsilon_{\sigma\alpha}\tilde{\theta}^\alpha + L_3\eta_{pa}\epsilon_{\sigma\alpha}\tilde{\theta}^\alpha + \\ &\quad + 2L_4'\tilde{x}_a\varepsilon_{pbc}(\gamma^b)_{\sigma\alpha}\tilde{\theta}^\alpha\tilde{x}^c + L_4\varepsilon_{pba}(\gamma^b)_{\sigma\alpha}\tilde{\theta}^\alpha \\ \partial_\alpha\tilde{g}_{p\sigma} &= L_3\tilde{x}_p\epsilon_{\sigma\alpha} + L_4\varepsilon_{pbc}(\gamma^b)_{\sigma\alpha}\tilde{x}^c \\ \partial_a\tilde{g}_{\rho\sigma} &= 2L_2'\tilde{x}_a\epsilon_{\rho\sigma}(1+if\tilde{\theta}^2) + 2iL_2f'\tilde{x}_a\epsilon_{\rho\sigma}\tilde{\theta}^2 \\ \partial_\alpha\tilde{g}_{\rho\sigma} &= 2iL_2f\epsilon_{\rho\sigma}\tilde{\theta}_\alpha\ . \end{aligned} \tag{D.26}$$

We can now proceed by making a general ansatz for the Killing vectors, so as to be able to explicitly write the Killing equations and determine the constraints they imposes on the ansatz.

Bosonic Killing vectors - $|\xi_{L_c}| = 0$**.** These should satisfy the following three equations, obtained by setting the free indices in (A.13) respectively to $(M, N) = \{(p, q), (p, \sigma), (\rho, \sigma)\}$

$$\begin{aligned}
&\xi_c^a \partial_a \tilde{g}_{pq} + \xi_c^\alpha \partial_\alpha \tilde{g}_{pq} + (\partial_p \xi_c^a)\tilde{g}_{aq} + (\partial_p \xi_c^\alpha)\tilde{g}_{\alpha q} + (\partial_q \xi_c^a)\tilde{g}_{pa} + (\partial_q \xi_c^\alpha)\tilde{g}_{p\alpha} \equiv 0 \quad (1)\\
&\xi_c^a \partial_a \tilde{g}_{p\sigma} + \xi_c^\alpha \partial_\alpha \tilde{g}_{p\sigma} + (\partial_p \xi_c^a)\tilde{g}_{a\sigma} + (\partial_p \xi_c^\alpha)\tilde{g}_{\alpha\sigma} + (\partial_\sigma \xi_c^a)\tilde{g}_{pa} + (\partial_\sigma \xi_c^\alpha)\tilde{g}_{p\alpha} \equiv 0 \quad (2)\\
&\xi_c^a \partial_a \tilde{g}_{\rho\sigma} + \xi_c^\alpha \partial_\alpha \tilde{g}_{\rho\sigma} + (\partial_\rho \xi_c^a)\tilde{g}_{a\sigma} + (\partial_\rho \xi_c^\alpha)\tilde{g}_{\alpha\sigma} - (\partial_\sigma \xi_c^a)\tilde{g}_{\rho a} + (\partial_\sigma \xi_c^\alpha)\tilde{g}_{\rho\alpha} \equiv 0 \quad (3)
\end{aligned} \tag{D.27}$$

The following ansatz can then be constructed

$$\begin{aligned}
\xi_{L_c} &= \xi_c^a \partial_a + \xi_c^\alpha \partial_\alpha \qquad \text{with} \qquad (D.28)\\
\xi_c^a &= \delta_c{}^a[A_1(r^2) + A_2(r^2)\tilde{\theta}^2] + \tilde{x}_c \tilde{x}^a [B_1(r^2) + B_2(r^2)\tilde{\theta}^2] + \tilde{x}^d \varepsilon_{dc}{}^a [D_1(r^2) + D_2(r^2)\tilde{\theta}^2]\\
\xi_c^\alpha &= [F(r^2)(\gamma_c)_\beta{}^\alpha + G(r^2)\tilde{x}_c \delta_\beta{}^\alpha + M(r^2)\tilde{x}_c \tilde{x}_k (\gamma^k)_\beta{}^\alpha + U(r^2)\tilde{x}^d \varepsilon_{dcb}(\gamma^b)_\beta{}^\alpha]\tilde{\theta}^\beta \ ,
\end{aligned}$$

and to simplify the substitution into the above equation we also compute its derivatives

$$\begin{aligned}
\partial_p \xi_c^a &= 2\tilde{x}_p \delta_c{}^a [A_1' + A_2'\tilde{\theta}^2] + \eta_{pc}\tilde{x}^a [B_1 + B_2\tilde{\theta}^2] + \tilde{x}_c \delta_p{}^a [B_1 + B_2\tilde{\theta}^2] +\\
&\quad + \tilde{x}_c \tilde{x}_p \tilde{x}^a [B_1' + B_2'\tilde{\theta}^2] + \varepsilon_{pc}{}^a [D_1 + D_2\tilde{\theta}^2] + \tilde{x}_p \tilde{x}^d \varepsilon_{dc}{}^a [D_1' + D_2'\tilde{\theta}^2]\\
\partial_\rho \xi_c^a &= [2\delta_c{}^a A_2 + 2\tilde{x}_c \tilde{x}^a B_2 + 2\tilde{x}^d \varepsilon_{dc}{}^a D_2]\tilde{\theta}_\rho\\
\partial_p \xi_c^\alpha &= [2F'\tilde{x}_p (\gamma_c)_\beta{}^\alpha + 2G'\tilde{x}_p \tilde{x}_c \delta_\beta{}^\alpha + 2G\eta_{pc}\delta_\beta{}^\alpha + 2M'\tilde{x}_p \tilde{x}_c \tilde{x}_b (\gamma^b)_\beta{}^\alpha +\\
&\quad + M\eta_{pc}\tilde{x}_b (\gamma^b)_\beta{}^\alpha + M\tilde{x}_c (\gamma_p)_\beta{}^\alpha 2U'\tilde{x}_p \tilde{x}^d \varepsilon_{dcb}(\gamma^b)_\beta{}^\alpha + U\varepsilon_{pcb}(\gamma^b)_\beta{}^\alpha]\tilde{\theta}^\beta\\
\partial_\rho \xi_c^\alpha &= F(\gamma_c)_\rho{}^\alpha + G\tilde{x}_c \delta_\rho{}^\alpha + M\tilde{x}_c \tilde{x}_b (\gamma^b)_\rho{}^\alpha + U\tilde{x}^d \varepsilon_{dcb}(\gamma^b)_\rho{}^\alpha \ .
\end{aligned} \tag{D.29}$$

At this stage one can proceed by substituting all the above ingredients into the Killing equations and collecting, for each of them, terms which should vanish independently. This leads to a set of constraining equations for the coefficients of the ansatz, that one can try to solve.
Equation (1) leads to 8 constraints

$$\begin{cases}
L_1' A_1 + (L_1 + r^2 L_1')B_1 = 0\\
i(L_1 h)' A_1 + i(L_1 h + r^2 (L_1 h)')B_1 + (L_1 + r^2 L_1')B_2 + L_1' A_2 + iL_1 hG - L_4 U = 0\\
-L_1' A_1 - L_1 A_1' - (2L_1 + r^2 L_1')B_1 + L_1(1 - r^2)B_1' = 0\\
[-i(L_1 m)' A_1 - iL_1 m A_1' - L_1' A_2 - L_1 A_2' - i(2L_1 m + r^2 (L_1 m)')B_1 +\\
\quad -(2L_1 + r^2 L_1')B_2 + iL_1(h - r^2 m)B_1' + L_1(1 - r^2)B_2' - iL_1 mG + 2L_3 G' - 2L_4 U'] = 0\\
-L_1 A_1 + L_1(1 - r^2)B_1 + 2L_1 A_1' = 0\\
[-iL_1 m A_1 - L_1 A_2 + iL_1(h - r^2 m)B_1 + L_1(1 - r^2)B_2 +\\
\quad +2iL_1 h A_1' + 2L_1 A_2' + 2L_3 G + L_4 U + 2r^2 L_4 U'] = 0\\
L_1 D_1' = 0\\
iL_1 h D_1' + L_1 D_2' + 2L_4 F' = 0
\end{cases} \tag{D.30}$$

Equation (2) leads to 10 constraints

$$\begin{cases} 2L_3'A_1 - 2L_1A_2 + 2(L_3 + r^2L_3')B_1 + 2L1(1-r^2)B_2 + 2L_3A_1' + r^2L_3B_1' + 2L_3G - 2L_2G' & = 0 \\ L_3A_1 + 2L_1A_2 + r^2L_3B_1 - 2L_2G & = 0 \\ L_1D_2 & = 0 \\ (2L_4' + \frac{1}{r^2}L_4)A_1 + 2(L_4 + r^2L_4')B_1 + 2L_4G + \frac{1}{r^2}L_2U & = 0 \\ L_4D_1 - L_2M - 2L_4F & = 0 \\ r^2L_4D_1' - 2L_2F' & = 0 \\ 2L_4F + (2r^2L_4 - L_2)M - L_4D_1 & = 0 \\ -2L_4M - 2L_2M' - L_4D_1' & = 0 \\ (2L_4 - \frac{1}{r^2}L_2)U - \frac{1}{r^2}L_4A_1 & = 0 \\ -\frac{1}{r^2}L_4A_1 + 2L_4A_1' - \frac{1}{r^2}L_2U - 2L_2U' & = 0 \end{cases} \tag{D.31}$$

Equation (3) leads to 2 constraints

$$\begin{cases} L_2G + L_2'A_1 + r^2L_2'B_1 & = 0 \\ i(L_2f)'A_1 + (L_3 + L_2')A_2 + ir^2(L_2f)'B_1 + r^2(L_3 + L_2')B_2 + 2iL_2fG & = 0 \end{cases} \tag{D.32}$$

Notice that in deriving the constraints for equation (3) we exploited the fact that objects anti-symmetric in two spinor indices should be proportional to ϵ, which gives

$$\begin{aligned} &(\partial_\rho\xi_c^a)\tilde{g}_{a\sigma} + (\partial_\rho xi_c^\alpha)\tilde{g}_{\alpha\sigma} - (\partial_\sigma\xi_c^a)\tilde{g}_{\rho a} + (\partial_\sigma\xi_c^\alpha)\tilde{g}_{\rho\alpha} = \\ &= [\epsilon^{\gamma\delta}(\partial_\delta\xi_c^a)\tilde{g}_{a\gamma}]\epsilon_{\rho\sigma} + [\epsilon^{\gamma\delta}(\partial_\delta\xi_c^\alpha)\tilde{g}_{\alpha\gamma}]\epsilon_{\rho\sigma} \ . \end{aligned} \tag{D.33}$$

Studying the above set of conditions with Mathematica it is then not too hard to find that

- The first condition from (1) can be solved expressing A_1 in terms of B_1. One can then integrate the third condition to find $B_1 = \frac{B}{2-r^2}$, with B constant. The latter is however set to zero by the fifth condition.
- The seventh condition from (1) sets $D_1 = A$, with A constant. The third condition from (2) sets $D_2 = 0$.
- The ninth condition from (2) sets $U = 0$ while the first condition from (3) sets $G = 0$.
- From (2), the second equation sets $A_2 = 0$ and consecutively the first one then sets $B_2 = 0$. The sixth condition requires that $F = C$, with C another constant.
- At this point only three conditions are left to solve, namely the fifth, seventh and eighth from (2). The fact that A and C are constants only allows the solution with $M = 0$ and $C = \frac{A}{2}$.

Hence, all constraints are satisfied and only two coefficients are non-vanishing, i.e. $D_1 = A$ and $F = \frac{A}{2}$. One is then left with the following Killing vector

$$K_{L_c} = A\big(\tilde{x}^b \varepsilon_{bc}{}^a \partial_a + \frac{1}{2}\tilde{\theta}^\beta (\gamma_c)_\beta{}^\alpha \partial_\alpha\big) \ . \tag{D.34}$$

Fermionic Killing vectors - $\left|\xi_{Q_\lambda}\right| = 1$. These should satisfy the following three equations, obtained from (A.13) by setting the free indices to $(M, N) = \{(p, q), (p, \sigma), (\rho, \sigma)\}$

$$\begin{aligned}
\xi^a_\lambda \partial_a \tilde{g}_{pq} + \xi^\alpha_\lambda \partial_\alpha \tilde{g}_{pq} + (\partial_p \xi^a_\lambda)\tilde{g}_{aq} + (\partial_p \xi^\alpha_\lambda)\tilde{g}_{\alpha q} + (\partial_q \xi^a_\lambda)\tilde{g}_{pa} + (\partial_q \xi^\alpha_\lambda)\tilde{g}_{p\alpha} &\equiv 0 \quad \textcircled{4}\\
-\xi^a_\lambda \partial_a \tilde{g}_{p\sigma} - \xi^\alpha_\lambda \partial_\alpha \tilde{g}_{p\sigma} - (\partial_p \xi^a_\lambda)\tilde{g}_{a\sigma} - (\partial_p \xi^\alpha_\lambda)\tilde{g}_{\alpha\sigma} + (\partial_\sigma \xi^a_\lambda)\tilde{g}_{pa} + (\partial_\sigma \xi^\alpha_\lambda)\tilde{g}_{p\alpha} &\equiv 0 \quad \textcircled{5}\\
\xi^a_\lambda \partial_a \tilde{g}_{\rho\sigma} + \xi^\alpha_\lambda \partial_\alpha \tilde{g}_{\rho\sigma} - (\partial_\rho \xi^a_\lambda)\tilde{g}_{a\sigma} - (\partial_\rho \xi^\alpha_\lambda)\tilde{g}_{\alpha\sigma} + (\partial_\sigma \xi^a_\lambda)\tilde{g}_{\rho a} - (\partial_\sigma \xi^\alpha_\lambda)\tilde{g}_{\rho\alpha} &\equiv 0 \quad \textcircled{6}
\end{aligned} \tag{D.35}$$

An ansatz similar to the one considered in the bosonic case can then be constructed

$$\begin{aligned}
\xi_{Q_\lambda} &= \xi^a_\lambda \partial_a + \xi^\alpha_\lambda \partial_\alpha \qquad\qquad \text{with}\\
\xi^a_\lambda &= [N(r^2)(\gamma^a)_{\lambda\beta} + P(r^2)\tilde{x}^a \epsilon_{\lambda\beta} + T(r^2)\tilde{x}^b \varepsilon_{bc}{}^a (\gamma^c)_{\lambda\beta} + V(r^2)\tilde{x}^a \tilde{x}_b (\gamma^b)_{\lambda\beta}]\tilde{\theta}^\beta\\
\xi^\alpha_\lambda &= \delta_\lambda{}^\alpha [R_1(r^2) + R_2(r^2)\tilde{\theta}^2] + \tilde{x}_a (\gamma^a)_\lambda{}^\alpha [S_1(r^2) + S_2(r^2)\tilde{\theta}^2] \ ,
\end{aligned} \tag{D.36}$$

for which one has the following derivatives

$$\begin{aligned}
\partial_p \xi^a_\lambda &= [2N'\tilde{x}_p(\gamma^a)_{\lambda\beta} + 2P'\tilde{x}_p\tilde{x}^a\epsilon_{\lambda\beta} + P\delta_p{}^a\epsilon_{\lambda\beta} + 2T'\tilde{x}_p\tilde{x}^b\varepsilon_{bc}{}^a(\gamma^c)_{\lambda\beta} + T\varepsilon_{pc}{}^a(\gamma^c)_{\lambda\beta} +\\
&\quad + 2V'\tilde{x}_p\tilde{x}^a\tilde{x}_b(\gamma^b)_{\lambda\beta} + V\delta_p{}^a\tilde{x}_b(\gamma^b)_{\lambda\beta} + V\tilde{x}^a(\gamma_p)_{\lambda\beta}]\tilde{\theta}^\beta\\
\partial_\rho \xi^a_\lambda &= N(\gamma^a)_{\lambda\rho} + P\tilde{x}^a\epsilon_{\lambda\rho} + T\tilde{x}^b\varepsilon_{bc}{}^a(\gamma^c)_{\lambda\rho} + V\tilde{x}^a\tilde{x}_b(\gamma^b)_{\lambda\rho}\\
\partial_p \xi^\alpha_\lambda &= 2\tilde{x}_p\delta_\lambda{}^\alpha[R_1' + R_2'\tilde{\theta}^2] + (\gamma_p)_\lambda{}^\alpha[S_1 + S_2\tilde{\theta}^2] + 2\tilde{x}_p\tilde{x}_a(\gamma^a)_\lambda{}^\alpha[S_1' + S_2'\tilde{\theta}^2]\\
\partial_\rho \xi^\alpha_\lambda &= 2R_2\delta_\lambda{}^\alpha\tilde{\theta}_\rho + 2S_2\tilde{x}_a(\gamma^a)_\lambda{}^\alpha\tilde{\theta}_\rho \ ,
\end{aligned} \tag{D.37}$$

Substituting all the ingredients in the Killing equations leads to a set of conditions for the ansatz. Equation ④ leads to 6 constraints:

$$\begin{cases}
2L_1(P + iR_1h) + 2r^2L_1'P &= 0\\
2L_1'(N + r^2V) - 2iL_1hS_1 + 2L_1V - 2L_4S_1 &= 0\\
-2r^2L_1'P - 2iL_1mR_1 + 4L_1(1 - r^2)P' - 4L_1P + 4L_3R_1' &= 0\\
[-2L_1'(N + r^2V) + 2iL_1mS_1 + 4L_1(1 - r^2)V' - 4L_1N' - 4L_1V - 4(L_3 + L_4)S_1' &= 0\\
-L_1(N - 2N') + L_1(1 - r^2)V - (L_3 - L_4)S_1 + 2r^2L_4S_1' &= 0\\
2L_1T' - 2L_4R_1' &= 0
\end{cases} \tag{D.38}$$

Equation ⑤ leads to 8 constraints:

$$\begin{cases} -L_1(1-r^2)P - L_3R_1 + 2L_2R_1' & = 0 \\ L_4R_1 + L_1T & = 0 \\ -L_1N + L_1(1-r^2)V + 2L_2S_1' + (L_4-L_3)S_1 & = 0 \\ L_1N + (L_2 - r^2L_4)S_1 & = 0 \\ (L_3 + r^2L_3' + ir^2L_1m - iL_1h)P + r^2L_3P' - 2r^2L_4T' + 2L_2R_2' + 2iL_2fR_1' & = 0 \\ -(L_4 + r^2L_4')P - (L_4 - iL_1h)T + 2L_4R_2 & = 0 \\ [(L_4' - L_3' - iL_1m)N + (2L_4 - L_3)N' + (L_4 - L_3 + r^2L_4' - r^2L_3' - ir^2L_1m + iL_1h)V + \\ \qquad -r^2L_3V' + 2L_4S_2 + 2L_2S_2' + 2iL_2fS_1'] & = 0 \\ (iL_1h - L_4 - r^2L_4' - \frac{1}{2}L_3)N - (r^2L_4 + r^4L_4' + \frac{1}{2}r^2L_3)V + (L_2 - 2r^2L_4)S_2 + iL_2fS_1 & = 0 \end{cases} \quad \text{(D.39)}$$

Equation ⑥ leads to 2 constraints:

$$\begin{cases} (L_3 + 2L_4 + 2L_2')N + r^2(L_3 + 2L_2')V + 2L_2S_2 - 2iL_2fS_1 & = 0 \\ r^2(L_3 + 2L_2')P - 2L_2R_2 + 2r^2L_4T + 2iL_2fR_1 & = 0 \end{cases} \quad \text{(D.40)}$$

Once again, in deriving the constraints for equation ⑥ we used that objects antisymmetric in two spinor indices should be proportional to the ϵ-tensor, so as to simplify

$$\begin{aligned} &-(\partial_\rho \xi^a_\lambda)\tilde{g}_{a\sigma} - (\partial_\rho \xi^\alpha_\lambda)\tilde{g}_{\alpha\sigma} + (\partial_\sigma \xi^a_\lambda)\tilde{g}_{\rho a} - (\partial_\sigma \xi^\alpha_\lambda)\tilde{g}_{\rho\alpha} = \\ &= -[\epsilon^{\gamma\delta}(\partial_\delta \xi^a_\lambda)\tilde{g}_{a\gamma}]\epsilon_{\rho\sigma} - [\epsilon^{\gamma\delta}(\partial_\delta \xi^\alpha_\lambda)\tilde{g}_{\alpha\gamma}]\epsilon_{\rho\sigma} \ . \end{aligned} \quad \text{(D.41)}$$

Studying the system of equations with Mathematica one then finds that

- The fourth constraint resulting from ⑤ is solved by expressing N in terms of S_1 as $N = 2iS_1$.
- As a result of the previous step, the second constraint from ④ requires to set $V = 0$.
- In turn, the fourth condition from ④ is solved if $S_1' = 0$, i.e. for constant $S_1 \equiv B$.
- At this point one can also solve the second condition from ⑤ by expressing R_1 in terms of T and successively integrate the sixth condition from ④ to obtain $T = \frac{C}{\sqrt{8-2r^2}}$, with C constant. This in turn gives an expression for R_1 which can be used to solve the first constraint from ④ by setting $P = \frac{C(1+2r^2)}{3\sqrt{8-2r^2}}$.
- Solving then the third condition from ④ and the last condition from ⑤ respectively requires to set $C = 0$ and $R_2 = 0$.
- Finally, the first condition in ⑥ is solved by setting $S_2 = 0$.

At this stage all the constraints are solved and the remaining non-vanishing coefficients are $N = 2iB$ and $S_1 = B$, with B constant. The fermionic Killing vectors thus read

$$\xi_{Q_\lambda} = B\big(2i\tilde{\theta}^\beta(\gamma^a)_{\beta\lambda}\partial_a + \tilde{x}_a(\gamma^a)_\lambda{}^\alpha\partial_\alpha\big) \ . \tag{D.42}$$

The overall constant prefactors A and B appearing in (D.34) and (D.42) can then be fixed to $A = 1$ and $B = -1/2$ by computing the commutators of Killing vectors and matching the result to

$$[\xi_{L_a}, \xi_{L_b}] = \varepsilon_{ab}{}^c\xi_{L_c} \qquad [\xi_{L_a}, \xi_{Q_\beta}] = -\tfrac{1}{2}(\gamma_a)_\beta{}^\alpha\xi_{Q_\alpha} \qquad [\xi_{Q_\alpha}, \xi_{Q_\beta}] = -i(\gamma^a)_{\alpha\beta}\xi_{L_a} \ . \tag{D.43}$$

Dual model with respect to $\mathrm{SL}(2,\mathbb{R})_L$**.** Residual isometries of this model can be studied by exploiting the above results. One can indeed interpret the dual metric (3.50) as the metric (D.23) with different multipliers and coefficients. Bosonic multipliers are the same as in the previous section, while fermions are the ones of the initial model. The coefficients of (3.50) in the form of (D.23) simplify as $L_3 = h = m = 0$ and

$$L_1 \equiv \frac{1}{2(r^2-1)} \qquad L_2 \equiv i \qquad L_4 \equiv \frac{i}{4(r^2-1)} \qquad f \equiv -\frac{1}{8}\frac{(r^2-2)}{(r^2-1)} \ . \tag{D.44}$$

The ansatz for the Killing vectors and all the derivatives are then formally the same, so that one simply has to study again the constraints with the above coefficients. This leads to

$$\xi_{L_c} = A\big(\tilde{x}^b\varepsilon_{bc}{}^a\partial_a + \frac{1}{2}\tilde{\theta}^\beta(\gamma_c)_\beta{}^\alpha\partial_\alpha\big) \qquad \xi_{Q_\lambda} = B\big(\tilde{\theta}^\beta(\gamma^b)_{\beta\lambda}\tilde{x}^d\varepsilon_{db}{}^a\partial_a + 2i[1-\tfrac{i}{4}\tilde{\theta}^2]\partial_\lambda\big) \tag{D.45}$$

and the constants can be fixed to $A = 1$ and $B = \frac{i}{2}$ by comparison with (D.43).

D.3. T-dual torsion constraints

We summarise formulae and expressions derived in trying to solve the torsion constraints for the T-dual models with respect to $\mathbf{SL}(2,\mathbb{R})_L$ and $\mathbf{OSp}(1|2)_L$. The generic metric (3.67), which includes the initial model as well as its T-duals, has generic inverse metric (3.71) with coefficients

$$\begin{aligned}
g_1^{inv} &= \frac{1}{g_1} \qquad g_2^{inv} = \frac{g_5^2 - g_2 g_9 - g_7^2 r^2}{g_1^2 g_9} \qquad g_3^{inv} = \frac{-g_3}{g_1^2 + g_1 g_3 r^2} \\
g_4^{inv} &= \frac{g_1^2(2g_5g_6 + g_7^2 - g_8^2 - g_4g_9 + g_6^2 r^2) + 2g_1g_3(-g_5^2 + 2g_2g_9 + g_7^2 r^2) + g_3^2 r^2(-g_5^2 + g_2 g_9 + g_7^2 r^2)}{g_1^2 g_9 (g_1 + g_3 r^2)^2} \\
g_5^{inv} &= \frac{g_5}{g_1 g_9} \qquad g_6^{inv} = \frac{-g_3 g_5 + g_1 g_6}{g_1^2 g_9 + g_1 g_3 g_9 r^2} \\
g_7^{inv} &= \frac{-g_7}{g_1 g_9} \qquad g_8^{inv} = \frac{g_8}{g_1 g_9 + g_3 g_9 r^2} \qquad g_9^{inv} = -\frac{1}{g_9} \\
g_{10}^{inv} &= \frac{g_1(2g_1g_{10} + 3g_5^2) + (2g_3g_5^2 + g_1(2g_{10}g_3 + 2g_5g_6 - 2g_7^2 - g_8^2))r^2 + (g_1 g_6^2 - 2g_3 g_7^2) r^4}{2g_1 g_9^2 (g_1 + g_3 r^2)} \,.
\end{aligned} \tag{D.46}$$

Knowledge of the above expressions then allows to write down the coefficients of the inverse vielbeine as linear functions of those of the vielbeine via (3.78)

$$\begin{aligned}
M_1 &= -\frac{1}{2} A_1 g_1^{inv} \qquad M_2 = -\frac{1}{2}(A_1 g_2^{inv} + A_2 g_1^{inv} - B_1 g_5^{inv} + r^2 B_2 g_7^{inv}) \\
M_3 &= -\frac{1}{2}[A_3 g_1^{inv} + g_3^{inv}(A_1 + r^2 A_3)] \\
M_4 &= -\frac{1}{2}[A_3 g_2^{inv} + g_4^{inv}(A_1 + r^2 A_3) + A_4 g_1^{inv} + g_3^{inv}(A_2 + r^2 A_4) - B_4 g_5^{inv} + \\
&\qquad - B_2 g_7^{inv} - g_6^{inv}(B_1 + r^2 B_4) + B_3 g_8^{inv}] \\
M_5 &= \frac{1}{2} A_5 g_1^{inv} \qquad M_6 = \frac{1}{2}[A_5 g_2^{inv} + A_6 g_1^{inv} + B_1 g_7^{inv} - B_2 g_5^{inv}]
\end{aligned} \tag{D.47}$$

$$\begin{aligned}
N_1 &= -i(D_1 g_1^{inv} + E_1 g_5^{inv} + r^2 E_3 g_7^{inv}) \qquad N_2 = -i(E_1 g_7^{inv} + E_3 g_5^{inv} - D_2 g_1^{inv}) \\
N_3 &= -i[D_3(g_1^{inv} + r^2 g_3^{inv}) + E_3(g_5^{inv} + r^2 g_6^{inv}) + E_1 g_8^{inv}] \\
N_4 &= -i[D_4 g_1^{inv} + g_3^{inv}(D_1 + r^2 D_4) + E_1 g_6^{inv} - E_3(g_7^{inv} - g_8^{inv})]
\end{aligned} \tag{D.48}$$

$$\begin{aligned}
R_1 &= -\frac{1}{2}(A_1 g_5^{inv} + r^2 A_5 g_7^{inv} + B_1 g_9^{inv}) \qquad R_2 = \frac{1}{2}(A_5 g_5^{inv} + A_1 g_7^{inv} + B_2 g_9^{inv}) \\
R_3 &= -\frac{1}{2}[B_3 g_9^{inv} + g_8^{inv}(A_1 + r^2 A_3)] \qquad R_4 = -\frac{1}{2}[A_3 g_5^{inv} - A_5 g_7^{inv} + g_6^{inv}(A_1 + r^2 A_3) + B_4 g_9^{inv}]
\end{aligned} \tag{D.49}$$

$$\begin{aligned}
S_1 &= -i E_1 g_9^{inv} \qquad S_3 = i E_3 g_9^{inv} \\
S_2 &= -i[E_2 g_9^{inv} + E_1 g_{10}^{inv} + \frac{3}{2}(D_1 g_5^{inv} + r^2 D_2 g_7^{inv}) + \\
&\qquad + \frac{r^2}{2}(D_4 g_5^{inv} - D_2 g_7^{inv} + g_6^{inv}(D_1 + r^2 D_4) - D_3 g_8^{inv})] \\
S_4 &= i[E_4 g_9^{inv} + E_3 g_{10}^{inv} - D_2 g_5^{inv} - D_1 g_7^{inv} + \frac{1}{2} D_3 (g_5^{inv} + r^2 g_6^{inv}) - \frac{1}{2} g_8^{inv}(D_1 + r^2 D_4)] \,.
\end{aligned} \tag{D.50}$$

We turn now to the construction of the structure functions, needed for the torsion constraints. Recalling the expression (3.66) one can write down

$$\mathcal{F}_{AB}{}^{C} = e_A{}^m e_B{}^n[\partial_n e_m{}^C - \partial_m e_n{}^C] + (-1)^B e_A{}^\mu e_B{}^\nu[\partial_\nu e_\mu{}^C + \partial_\mu e_\nu{}^C] + \\ + [e_A{}^m e_B{}^\nu - (-1)^B e_A{}^\nu e_B{}^m][\partial_\nu e_m{}^C - \partial_m e_\nu{}^C] \ , \tag{D.51}$$

and given the need for $\mathcal{F}_{\alpha\beta}{}^d, \mathcal{F}_{\alpha b}{}^d, \mathcal{F}_{\alpha\beta}{}^\gamma$ in the constraints (3.65), one can use the relations $\partial_a u(r) = x_a u'/r$ and $\partial_\alpha \theta^2 = 2\theta_\alpha$ to compute

$$\begin{aligned}
&[\partial_n e_m{}^d - \partial_m e_n{}^d] = \varepsilon_{nm}{}^k[(\tilde{\omega}_1)_k{}^d + \theta^2(\tilde{\omega}_2)_k{}^d] \\
&[\partial_\nu e_\mu{}^d + \partial_\mu e_\nu{}^d] = 2(\gamma^q)_{\mu\nu}(\omega_3)_q{}^d \\
&[\partial_\nu e_m{}^d - \partial_m e_\nu{}^d] = \theta^\lambda\{\epsilon_{\nu\lambda}(\tilde{\omega}_3)_m{}^d + (\gamma^q)_{\nu\lambda}[x_m(\tilde{\tilde{\omega}}_3)_q{}^d + (x^d\eta_{mq} + x_q\delta^d_m)\mu_{13} + \varepsilon_{mq}{}^d\mu_{14}]\} \\
&[\partial_n e_m{}^\gamma - \partial_m e_n{}^\gamma] = \theta^\lambda(\gamma_q)_\lambda{}^\gamma\varepsilon_{nm}{}^k(\tilde{\omega}_4)_k{}^q \\
&[\partial_\nu e_\mu{}^\gamma + \partial_\mu e_\nu{}^\gamma] = \theta^\lambda\{2E_2[\delta_\nu{}^\gamma\epsilon_{\mu\lambda} + \delta_\mu{}^\gamma\epsilon_{\nu\lambda}] + 2E_4 x_p[(\gamma^p)^\gamma_\nu\epsilon_{\mu\lambda} + (\gamma^p)_\mu{}^\gamma\epsilon_{\nu\lambda}]\} \\
&[\partial_\nu e_m{}^\gamma - \partial_m e_\nu{}^\gamma] = [(\tilde{\tilde{\omega}}_4)_m{}^p(\gamma_p)_\nu{}^\gamma + \rho_7 x_m\delta_\nu{}^\gamma] + \theta^2[(\tilde{\tilde{\tilde{\omega}}}_4)_m{}^p(\gamma_p)_\nu{}^\gamma + \rho_{11}x_m\delta_\nu{}^\gamma] \ ,
\end{aligned} \tag{D.52}$$

where we defined quantities

$$\begin{aligned}
&(\tilde{\omega}_1)_k{}^d = (\mu_1\delta_k{}^d + \mu_2 x_k x^d + \mu_3 x^c\varepsilon_{ck}{}^d) &&(\tilde{\omega}_2)_k^d = (\mu_4\delta_k{}^d + \mu_5 x_k x^d + \mu_6 x^c\varepsilon_{ck}{}^d) \\
&(\tilde{\omega}_3)_m{}^d = (\mu_7\delta_m{}^d + \mu_8 x_m x^d + \mu_9 x^c\varepsilon_{cm}{}^d) &&(\tilde{\tilde{\omega}}_3)_q{}^d = (\mu_{10}\delta_q{}^d + \mu_{11}x_q x^d + \mu_{12}x^c\varepsilon_{cq}{}^d) \\
&(\tilde{\omega}_4)_k{}^q = (\rho_1\delta_k{}^q + \rho_2 x_k x^q + \rho_3 x^c\varepsilon_{ck}{}^q) &&(\tilde{\tilde{\omega}}_4)_m{}^p = (\rho_4\delta_m{}^p + \rho_5 x_m x^p + \rho_6 x^c\varepsilon_{cm}{}^p) \\
&(\tilde{\tilde{\tilde{\omega}}}_4)_m{}^p = (\rho_8\delta_m{}^p + \rho_9 x_m x^p + \rho_{10}x^c\varepsilon_{cm}{}^p)
\end{aligned} \tag{D.53}$$

and, to shorten the notation, new coefficients

$$\begin{aligned}
&\mu_1 = 2A_5 + rA_5' \quad &&\mu_2 = -A_5'/r \quad &&\mu_3 = A_1'/r - A_3 \quad &&\mu_4 = 2A_6 + rA_6' \\
&\mu_5 = -A_6'/r &&\mu_6 = A_2'/r - A_4 &&\mu_7 = 2A_2 - B_3 &&\mu_8 = 2A_4 - B_3'/r \\
&\mu_9 = 2A_6 &&\mu_{10} = -B_1'/r &&\mu_{11} = -B_4'/r &&\mu_{12} = -B_2'/r \\
& &&\mu_{13} = -B_4 &&\mu_{14} = -B_2
\end{aligned} \tag{D.54}$$

$$\begin{aligned}
&\rho_1 = 2D_2 + rD_2' \quad &&\rho_2 = -D_2'/r \quad &&\rho_3 = D_1'/r - D_4 \quad &&\rho_4 = D_1 - E_3 \\
&\rho_5 = D_4 - E_3'/r &&\rho_6 = D_2 &&\rho_7 = D_3 - E_1'/r &&\rho_8 = -E_4 \\
&\rho_9 = -E_4'/r &&\rho_{10} = 0 &&\rho_{11} = -E_2'/r \ .
\end{aligned} \tag{D.55}$$

By then using the ansatz for the inverse vielbeine and the above quantities one finds the expressions (3.83),(3.84),(3.85) for the structure functions, with the following coefficients

$$
\begin{aligned}
F_1 &:= -2B_1(S_1^2 + r^2 S_3^2) + 4r^2 B_2 S_1 S_3 \\
F_2 &:= -2B_4(S_1^2 + r^2 S_3^2) - 4B_2 S_1 S_3 + 4S_3^2(B_1 + r^2 B_4) \\
F_3 &:= -2B_2(S_1^2 + r^2 S_3^2) + 4B_1 S_1 S_3 \\
F_4 &:= 4r^2 B_2(S_1 S_4 + S_2 S_3) - 4B_1(S_1 S_2 + r^2 S_3 S_4) + \mu_1[N_1(N_1 + r^2 N_4) + r^2 N_2 N_3] + \\
&\quad + r^2\mu_3[N_1 N_3 + N_2(N_1 + r^2 N_4)] + \\
&\quad + S_1[N_1(\mu_7 - \mu_{14}) + r^2 N_2(\mu_9 + \mu_{13}) - r^2 N_3 \mu_{10} - (\mu_{14} + r^2\mu_{12})(N_1 + r^2 N_4)] + \\
&\quad + S_3[r^2\mu_{10}(N_1 + r^2 N_4) - r^2 N_1(\mu_9 + \mu_{13}) - r^2 N_2(\mu_7 - \mu_{14}) + r^2 N_3(\mu_{14} + r^2\mu_{12})] \\
F_5 &:= -4B_4(S_1 S_2 + r^2 S_3 S_4) - 4B_2(S_1 S_4 + S_2 S_3) + 8S_3 S_4(B_1 + r^2 B_4) + \\
&\quad + \mu_2[N_1(N_1 + r^2 N_4) + r^2 N_2 N_3] - \mu_3[N_1 N_3 + N_2(N_1 + r^2 N_4)] - (\mu_1 + r^2\mu_2)(N_2^2 + N_2 N_3 + N_1 N_4) + \\
&\quad + S_1[N_1(\mu_8 + \mu_{12}) - N_2(\mu_9 + 3\mu_{13}) - N_3(r^2\mu_{11} + 2\mu_{13}) + N_4(\mu_7 + \mu_{14} + r^2\mu_8 + r^2\mu_{12})] + \\
&\quad + S_3[N_1(\mu_9 - 3\mu_{13}) + N_2(\mu_7 - 3\mu_{14}) - (N_1 + r^2 N_4)(2\mu_{10} + r^2\mu_{11}) - 2r^2 N_4 \mu_{13} + \\
&\qquad\qquad + N_3(\mu_7 - \mu_{14} + r^2\mu_8 - r^2\mu_{12})] \\
F_6 &:= 4B_1(S_1 S_4 + S_2 S_3) - 4B_2(S_1 S_2 + r^2 S_3 S_4) + \\
&\quad + \mu_3[r^2 N_2 N_3 + N_1(N_1 + r^2 N_4)] + \mu_1[N_1 N_3 + N_2(N_1 + r^2 N_4)] + \\
&\quad + S_1[N_1(\mu_9 + \mu_{13}) + N_2(\mu_7 - \mu_{14}) - N_3(\mu_{14} + r^2\mu_{12}) - \mu_{10}(N_1 + r^2 N_4)] + \\
&\quad + S_3[N_1(\mu_{14} - \mu_7) - r^2 N_2(\mu_9 + \mu_{13}) + r^2 N_3 \mu_{10} + (N_1 + r^2 N_4)(\mu_{14} + r^2\mu_{12})]
\end{aligned}
\tag{D.56}
$$

$$
\begin{aligned}
F_7 &:= -r^2N_3(M_1\mu_3 + M_5\mu_1) + S_1[2R_1B_1 + 2r^2R_2B_2 - M_1\mu_7 - r^2M_5\mu_9]+\\
&\qquad + S_3[r^2(M_1\mu_{13} - M_5\mu_{14}) - 2r^2(R_2B_1 + R_1B_2)]\\
F_8 &:= N_3(M_5\mu_1 + M_1\mu_3) + S_1[2R_1B_4 - 2R_2B_2 + 2R_4(B_1 + r^2B_4) - M_1\mu_8 + M_5\mu_9 - M_3(\mu_7 + r^2\mu_8)]+\\
&\quad + S_3[2(R_2 - R_3)(B_1 + r^2B_4) + 2B_2(R_1 + r^2R_4) - 2r^2(R_2B_4 + R_4B_2)+\\
&\qquad + (M_1 + r^2M_3)(\mu_{10} + \mu_{13} + r^2\mu_{11}) + M_5\mu_{14} + r^2M_3\mu_{13}]\\
F_9 &:= -N_3(M_1\mu_1 + r^2M_5\mu_3) + S_1[2R_1B_2 + 2R_2B_1 - M_1\mu_9 - M_5\mu_7]+\\
&\qquad + S_3[-2R_1B_1 - 2r^2R_2B_2 - M_1\mu_{14} + r^2M_5\mu_{13}]\\
F_{10} &:= M_5N_2(\mu_1 + r^2\mu_2) - M_1N_1(\mu_2 + \tfrac{1}{r^2}\mu_1) + S_1[2R_1B_4 + M_5\mu_{13} + \tfrac{1}{r^2}(2R_1B_1 + M_1\mu_{14})]+\\
&\qquad + S_3[2R_2(B_1 + r^2B_4) + M_1\mu_{13} + M_5\mu_{14}]\\
F_{11} &:= M_3N_1\mu_1 + N_2(M_1 + r^2M_3)\mu_3 + \tfrac{1}{r^2}M_1N_1\mu_1 + S_3[2B_2(R_1 + r^2R_4) - 2R_3B_1 + (M_1 + r^2M_3)\mu_{10}]+\\
&\qquad + S_1[2R_3B_2 - 2R_4B_1 - M_3\mu_{14} - (M_1 + r^2M_3)\mu_{12} - \tfrac{1}{r^2}(2R_1B_1 + M_1\mu_{14})]\\
F_{12} &:= -M_1N_4\mu_1 - M_5(N_1 + r^2N_4)\mu_3 - \tfrac{1}{r^2}M_1N_1\mu_1 + S_1[2R_2B_2 - M_5\mu_{13} + \tfrac{1}{r^2}(2R_1B_1 + M_1\mu_{14})]+\\
&\qquad + S_3[-2R_1B_2 - 2R_2B_1 + M_1\mu_9 + M_5\mu_7]\\
F_{13} &:= (M_5N_1 - M_1N_2)(\mu_1 + r^2\mu_2) + S_1[-2r_2(B_1 + r^2B_4) - M_1\mu_{13} - M_5\mu_{14}]+\\
&\qquad + S_3[-2R_1(B_1 + r^2B_4) - M_1\mu_{14} - r^2M_5\mu_{13}]\\
F_{14} &:= -(N_1 + r^2N_4)(M_5\mu_1 + M_1\mu_3) + S_1[2R_2B_1 + 2R_1B_2 - M_1\mu_{13} + M_5\mu_{14}]+\\
&\qquad + S_3[-2R_1B_1 - 2r^2R_2B_2 + M_1\mu_7 + r^2M_5\mu_9]\\
F_{15} &:= (M_1 + r^2M_3)(N_2\mu_1 + N_1\mu_3) + S_1[2R_3B_1 - 2B_2(R_1 + r^2R_4) - (M_1 + r^2M_3)\mu_{10}]+\\
&\qquad + S_3[2B_1(R_1 + r^2R_4) - 2r^2R_3B_2 + (M_1 + r^2M_3)(\mu_{14} + r^2\mu_{12})]\\
F_{16} &:= (M_1N_4 - M_3N_1)\mu_3 + N_2(M_1 + r^2M_3)\mu_2 - M_5(N_1 + r^2N_4)\mu_2 + (M_5N_4 - M_3N_2)(\mu_1 + r^2\mu_2)+\\
&\quad + S_1[2R_4B_2 + 2(R_2 + R_3)B_4 - 2M_3\mu_{13} - (M_1 + r^2M_3)\mu_{11}]+\\
&\quad + S_3[2(R_2 + R_3)B_2 - 2R_1B_4 - 4R_4(B_1 + r^2B_4) + 2B_4(R_1 + r^2R_4)+\\
&\qquad + M_1\mu_8 + M_3(\mu_7 - \mu_{14} + r^2\mu_8) + M_5(\mu_{13} - \mu_9) - (M_1 + r^2M_3)\mu_{12}]
\end{aligned}
\tag{D.57}
$$

$$
\begin{aligned}
F_{17} &:= r^2S_1N_3\rho_7 + r^2S_3N_3(\rho_4 + r^2\rho_5) - 2E_2S_1^2 - 2r^2E_4S_1S_3\\
F_{18} &:= S_1N_3(\rho_4 + r^2\rho_5) + r^2S_3N_3\rho_7 - 2E_4S_1^2 - 2E_2S_1S_3\\
F_{19} &:= (N_1 + r^2N_4)[S_1\rho_7 + S_3(\rho_4 + r^2\rho_5)] + 2r^2E_4S_3^2 + 2E_2S_1S_3\\
F_{20} &:= S_1(N_1\rho_4 + r^2N_2\rho_6) + r^2S_3(N_1\rho_6 + N_2\rho_4)\\
F_{21} &:= S_1[N_1\rho_5 - N_2\rho_6 + N_4(\rho_4 + r^2\rho_5)] + S_3[-N_1\rho_6 - N_2\rho_4 + (N_1 + r^2N_4)\rho_7] + 2E_4S_1S_3 + 2E_2S_3^2\\
F_{22} &:= -S_1(N_1\rho_6 + N_2\rho_4) - S_3(N_1\rho_4 + r^2N_2\rho_6)\ .
\end{aligned}
\tag{D.58}
$$

E T-Duality Of Principal Chiral Models - Abstract Approach

E.1. Properties of topological deformations

Antisymmetry and derivation. Graded antisymmetry of Ω leads to the first property in (4.5)

$$\begin{cases} \Omega(Y,X) = -(-1)^{XY}\Omega(X,Y) = -(-1)^{XY}\langle D(X),Y\rangle \\ \Omega(Y,X) = \langle D(Y),X\rangle = (-1)^{XY}\langle X,D(Y)\rangle \end{cases} \quad \Rightarrow \quad \langle X,D(Y)\rangle = -\langle D(X),Y\rangle \tag{E.1}$$

and combining it with the two-cocycle condition (4.3) one recovers the second relation in (4.5)

$$\begin{aligned} &\Omega(X,[Y,Z]) + (-1)^{X(Y+Z)}\Omega(Y,[Z,X]) + (-1)^{Z(X+Y)}\Omega(Z,[X,Y]) = \\ &= \langle D(X),[Y,Z]\rangle + (-1)^{X(Y+Z)}\langle D(Y),[Z,X]\rangle + (-1)^{Z(X+Y)}\langle D(Z),[X,Y]\rangle = \\ &= -\langle X, D[Y,Z]\rangle + \langle X,[D(Y),Z]\rangle + \langle X,[Y,D(Z)]\rangle \equiv 0 \ . \end{aligned} \tag{E.2}$$

Given a Lie (super)algebra $\mathfrak{g}$ with generators $\{T_A\}$ satisfying $[T_A,T_B] = f_{AB}{}^C T_C$ one can also write D in components $D(T_A) = D_A{}^B T_B$, and define $D_{AB} := D_A{}^C\delta_{CB}$. Then antisymmetry implies that

$$\langle D(T_A),T_B\rangle = -\langle T_A,D(T_B)\rangle \qquad \Rightarrow \qquad D_{AB} = -(-1)^{AB}D_{BA} \tag{E.3}$$

and the two-cocycle condition (4.3) leads to the relation

$$\begin{aligned} &\langle D(T_A),[T_B,T_C]\rangle + (-1)^{A(B+C)}\langle D(T_B),[T_C,T_A]\rangle + (-1)^{C(A+B)}\langle D(T_C),[T_A,T_B]\rangle = \\ &= (-1)^{E(B+C)}D_A{}^E f_{BC}{}^D\delta_{DE} + (-1)^{(A+E)(B+C)}D_B{}^E f_{CA}{}^D\delta_{DE} + (-1)^{(C+E)(A+B)}D_C{}^E f_{AB}{}^D\delta_{DE} = \\ &= D_A{}^E f_{EBC} - (-1)^{AB}D_B{}^E f_{EAC} + (-1)^{E+CE+N} f_{ABE}D_{CN}\delta^{NE} = \\ &= \left(D_A{}^M f_{MB}{}^P - (-1)^{AB}D_B{}^M f_{MA}{}^P - f_{AB}{}^M D_M{}^P\right)\delta_{PC} \equiv 0 \ , \end{aligned} \tag{E.4}$$

where we used the graded antisymmetry of D_{AB} and f_{ABC} with respect to any two indices.

Equations of motion - deformed action. We check that deformations by two-cocycles do not alter equations of motion and Noether symmetries. Recall that under an arbitrary variation of the group element $\delta \mathrm{g}$ the current transforms as $\delta j = \mathrm{d}(\mathrm{g}^{-1}\delta \mathrm{g}) + [j, \mathrm{g}^{-1}\delta \mathrm{g}]$ and consecutively

$$\begin{aligned}\delta S_\Omega = \delta\left(\tfrac{1}{2}\int_\Sigma \Omega(j,j)\right) = \int_\Sigma \Omega(\delta j, j) = \int_\Sigma -\Omega(\mathrm{g}^{-1}\delta\mathrm{g}, \mathrm{d}j) + \Omega([j,\mathrm{g}^{-1}\delta\mathrm{g}], j) = \\ = \int_\Sigma \tfrac{1}{2}\Omega(\mathrm{g}^{-1}\delta\mathrm{g}, [j,j]) + \Omega(j, [j, \mathrm{g}^{-1}\delta\mathrm{g}]) = 0 \ ,\end{aligned} \tag{E.5}$$

where we exploited Maurer-Cartan equation and (4.3) implies cancellation of the two pieces

$$\Omega(\mathrm{g}^{-1}\delta\mathrm{g}, [j,j]) + \Omega(j, [j, \mathrm{g}^{-1}\delta\mathrm{g}]) - \Omega(j, [\mathrm{g}^{-1}\delta\mathrm{g}, j]) = 0 \ . \tag{E.6}$$

G_R invariance - deformed action. Before verifying that the deformed action (4.9) retains invariance under the global G_R action of the group, we shall construct (4.6). Exploiting the relation

$$gYg^{-1} = \mathrm{Ad}_g Y = e^{\mathrm{ad}_X} Y = \sum_{k=0}^{\infty} \tfrac{1}{k!}\mathrm{ad}_X^k(Y) \qquad \text{with} \qquad g = e^X \qquad X, Y \in \mathfrak{g} \tag{E.7}$$

and the derivation property of D on the Lie bracket one finds

$$D(gYg^{-1}) = gD(Y)g^{-1} + \sum_{k,n=0}^{\infty} \frac{1}{(k+n+1)!}\mathrm{ad}_X^k \circ \mathrm{ad}_{D(X)} \circ \mathrm{ad}_X^n(Y) \ . \tag{E.8}$$

Then, noting that $\int_0^1 \mathrm{d}t(1-t)^k t^n = \frac{k!\,n!}{(k+n+1)!}$ the second term can be rewritten as

$$\begin{aligned}\sum_{k,n=0}^{\infty} \frac{1}{(k+n+1)!}\mathrm{ad}_X^k \circ \mathrm{ad}_{D(X)} \circ \mathrm{ad}_X^n(Y) &= \int_0^1 \mathrm{d}t \sum_{k,n=0}^{\infty} \frac{(1-t)^k t^n}{k!\,n!}\mathrm{ad}_X^k \circ \mathrm{ad}_{D(X)} \circ \mathrm{ad}_X^n(Y) = \\ &= \int_0^1 \mathrm{d}t\, e^{(1-t)\mathrm{ad}_X} \circ \mathrm{ad}_{D(X)} \circ e^{t\,\mathrm{ad}_X}(Y) = \\ &= e^{\mathrm{ad}_X}\int_0^1 \mathrm{d}t\, e^{-t\,\mathrm{ad}_X}[D(X), e^{tX}Ye^{-tX}] = \\ &= e^{\mathrm{ad}_X}\int_0^1 \mathrm{d}t[e^{-t\,\mathrm{ad}_X}D(X), Y] = \\ &= e^{\mathrm{ad}_X}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\mathrm{ad}_X^k(D(X)), Y\right] = \\ &= g[g^{-1}D(g), Y]g^{-1} \ ,\end{aligned} \tag{E.9}$$

where in the last line we exploited the definition (4.7) and $e^{\mathrm{ad}_X} = \mathrm{Ad}_g$ for $g = e^X$. Putting together the two pieces above, one recovers (4.6). Then (4.8) is easily verified using the latter and Ad-invariance of the inner product

$$\begin{aligned}\Omega(g^{-1}Xg, g^{-1}Yg) &= \langle D(g^{-1}Xg), g^{-1}Yg\rangle = \\ &= \langle g^{-1}D(X)g, g^{-1}Yg\rangle + \langle g^{-1}[gD(g^{-1}), X]g, g^{-1}Yg\rangle = \\ &= \Omega(X,Y) + \langle gD(g^{-1}), [X,Y]\rangle\ .\end{aligned} \tag{E.10}$$

This allows to show G_R invariance of (4.9). Under $\mathrm{g}\to\mathrm{g}g_R$ one has $j\to g_R^{-1}jg_R$ and hence

$$\begin{aligned}S_\Omega = \tfrac{1}{2}\int_\Sigma \Omega(j,j) \quad \to \quad \tfrac{1}{2}\int_\Sigma \langle D(g_R^{-1}jg_R), g_R^{-1}jg_R\rangle &= S_\Omega + \tfrac{1}{2}\int_\Sigma \langle g_R D(g_R^{-1}), [j,j]\rangle = \\ &= S_\Omega - \int_\Sigma \mathrm{d}\langle g_R D(g_R^{-1}), j\rangle\ .\end{aligned} \tag{E.11}$$

In the last step we used Maurer-Cartan equation and that D, g_R are constant in order to write the extra piece as a boundary term which does not contribute to the action.

A very similar reasoning then shows that the master action (4.12) has the same property: the multipliers do not transform under $\mathrm{g}\to\mathrm{g}g_R$ and it is sufficient to check that

$$\begin{aligned}\tfrac{1}{2}\Omega(j_\omega, j_\omega) + \langle D(\mathrm{g})\mathrm{g}^{-1}, F_\omega\rangle \quad \to \quad & \tfrac{1}{2}\Omega(j_\omega, j_\omega) + \langle D(\mathrm{g})\mathrm{g}^{-1}, F_\omega\rangle + \\ & + \tfrac{1}{2}\langle g_R D(g_R^{-1}), [j_\omega, j_\omega]\rangle - \langle \mathrm{g}g_R D(g_R^{-1})\mathrm{g}^{-1}, F_\omega\rangle = \\ = \, & \tfrac{1}{2}\Omega(j_\omega, j_\omega) + \langle D(\mathrm{g})\mathrm{g}^{-1}, F_\omega\rangle + d\,\langle g_R D(g_R^{-1}), j_\omega\rangle\ ,\end{aligned} \tag{E.12}$$

where in the last step we used $\mathrm{g}^{-1}F_\omega\mathrm{g} = F_{j_\omega}$ and once again that D, g_R are constant.

E.2. T-dual model

Equations of motion. Upon varying the dual action (4.19) with respect to $\tilde{\Lambda}$ one finds

$$\begin{aligned}\delta\tilde{S} &= \int_\Sigma \langle \mathrm{d}\delta\tilde{\Lambda}, \frac{1}{1-D_{\tilde{\Lambda}}}P_+\mathrm{d}\tilde{\Lambda}\rangle + \langle \mathrm{d}\tilde{\Lambda}, \delta\left(\frac{1}{1-D_{\tilde{\Lambda}}}\right)P_+\mathrm{d}\tilde{\Lambda}\rangle + \langle P_-\mathrm{d}\tilde{\Lambda}, \frac{1}{1-D_{\tilde{\Lambda}}}\mathrm{d}\delta\tilde{\Lambda}\rangle = \\ &= \int_\Sigma \langle \delta\tilde{\Lambda}, \mathrm{d}\tilde{j} - \frac{1}{2}\left[\frac{1}{1+D_{\tilde{\Lambda}}}\mathrm{d}\tilde{\Lambda}, \frac{1}{1-D_{\tilde{\Lambda}}}P_+\mathrm{d}\tilde{\Lambda}\right]\rangle\ ,\end{aligned} \tag{E.13}$$

where we used the relation (4.20) and the fact that $\tilde{j} := j_\omega$, with j_ω as in (4.16). This immediately leads to the equations of motion (4.21) after noting that $[\tilde{j},\tilde{j}] = -\big[\frac{1}{1+D_{\tilde{\Lambda}}}\mathrm{d}\tilde{\Lambda}, \frac{1}{1-D_{\tilde{\Lambda}}}P_+\mathrm{d}\tilde{\Lambda}\big]$. Then, $\tilde{j}$ satisfies by construction the relation (4.15) and taking the exterior derivative of it one finds

$$\mathrm{d}(\star\tilde{j}) = -\mathrm{d}[\tilde{j},\tilde{\Lambda}] + D(\mathrm{d}\tilde{j}) = D_{\tilde{\Lambda}}(\mathrm{d}\tilde{j}) + [\tilde{j},\mathrm{d}\tilde{\Lambda}] = D_{\tilde{\Lambda}}(\mathrm{d}\tilde{j} + \tfrac{1}{2}[\tilde{j},\tilde{j}])\ , \tag{E.14}$$

which leads to (4.22) upon using the equations of motion (4.21).

Noether current and Killing vectors. Recalling the transformation law (4.24) of the multipliers under $g \to g g_R$ and taking $g_R \simeq 1 + \epsilon$ one finds that $\delta\tilde{\Lambda} = D_{\tilde{\Lambda}}\epsilon$. Substituting this into the general variations (E.13) of the dual action one finds

$$\delta\tilde{S} = \int_\Sigma \langle D_{\tilde{\Lambda}}\epsilon, \mathrm{d}\tilde{j} + \tfrac{1}{2}[\tilde{j},\tilde{j}]\rangle = -\int_\Sigma \langle \epsilon, D_{\tilde{\Lambda}}(\mathrm{d}\tilde{j}) + [D_{\tilde{\Lambda}}\tilde{j},\tilde{j}]\rangle = -\int_\Sigma \langle \epsilon, \mathrm{d}\big(D_{\tilde{\Lambda}}\tilde{j}\big)\rangle \,, \tag{E.15}$$

where we exploited again the relation (4.15). This immediately leads to the Noether current (4.26). We can now proceed in deriving the explicit form (4.27) of the Killing vectors. To begin we note that inserting the equations of motion (4.15) into the master action (4.14) one finds

$$\tilde{S} = \frac{1}{2}\int_\Sigma \langle \mathrm{d}\tilde{\Lambda}, -\tilde{j}\rangle \qquad \text{with} \qquad \tilde{j} := j_\omega \,. \tag{E.16}$$

Even without explicitly computing $\tilde{j}$, from the expression (4.16) we know we can expand

$$-\tilde{j} = \star\mathrm{d}\tilde{\Lambda}^N X_N{}^A T_A + \mathrm{d}\tilde{\Lambda}^N Y_N{}^A T_A \tag{E.17}$$

for some $X_N{}^A$ and $Y_N{}^A$. Knowing then that the T-dual model contains both a metric and a B-field term, we can also write the dual action as

$$\tilde{S} = \frac{1}{2}\int_\Sigma \mathrm{d}\tilde{\Lambda}^N \wedge \star\mathrm{d}\tilde{\Lambda}^M \tilde{g}_{MN} + \mathrm{d}\tilde{\Lambda}^N \wedge \mathrm{d}\tilde{\Lambda}^M \tilde{B}_{MN} \,, \tag{E.18}$$

so that comparing with the above expression and substituting the expansion of $-\tilde{j}$ one finds

$$\tilde{g}_{MN} = X_M{}^A \delta_{AN} \qquad\qquad B_{MN} = Y_M{}^A \delta_{AN} \,. \tag{E.19}$$

At this point we also substitute the above expansion for $-\tilde{j}$ into the T-dual Noether current (4.26)

$$\tilde{J}_N = \star D_{\tilde{\Lambda}}\tilde{j} = -\mathrm{d}\tilde{\Lambda}^N X_N{}^A (D_A{}^C + \tilde{\Lambda}^P f_{PA}{}^C) T_C - \star\mathrm{d}\tilde{\Lambda}^N Y_N{}^A (D_A{}^C + \tilde{\Lambda}^P f_{PA}{}^C) T_C \tag{E.20}$$

and finally compare to the general expression (C.21)

$$\xi^N_{T_D}\mathrm{d}\tilde{\Lambda}^M \tilde{g}_{MN} + \xi^N_{T_D} \star \mathrm{d}\tilde{\Lambda}^M \tilde{B}_{MN} = -\mathrm{d}\tilde{\Lambda}^N X_N{}^A (D_A{}^C + \tilde{\Lambda}^P f_{PA}{}^C)\delta_{CD} - \star\mathrm{d}\tilde{\Lambda}^N Y_N{}^A (D_A{}^C + \tilde{\Lambda}^P f_{PA}{}^C)\delta_{CD} \,. \tag{E.21}$$

Exploiting the relations (E.19) for the dual metric and B-field and the graded antisymmetry of D_{AB} and f_{ABC} under the exchange of any two indices one can extract the components of the Killing vectors $\xi^M_{T_D} = D_D{}^M + \tilde{\Lambda}^P f_{PD}{}^M$, thus recovering the result (4.27). One can now check the latter correctly satisfy the commutation relations $[\xi_{T_A},\xi_{T_B}] = f_{AB}{}^C\xi_{T_C}$.

$$\begin{aligned}[\xi_{T_A},\xi_{T_B}] &= \big(D_A{}^M f_{MB}{}^P - (-1)^{AB}D_B{}^M f_{MA}{}^P\big)\partial_P + \tilde\Lambda^N\big(f_{NA}{}^M f_{MB}{}^P - (-1)^{AB} f_{NB}{}^M f_{MA}{}^P\big)\partial_P = \\ &= f_{AB}{}^M D_M{}^P\partial_P + f_{AB}{}^M\tilde\Lambda^N f_{NM}{}^P\partial_P = f_{AB}{}^C\xi_{T_C}\ ,\end{aligned} \tag{E.22}$$

where the first two terms are recombined exploiting the two-cocycle condition as in (E.4), while the last two by exploiting Jacobi identity.

Structure equation and H_3. From the definition (4.29) of the dual vielbeine one finds

$$\begin{aligned}\mathrm{d}\tilde e &= \frac{1}{1-D_{\tilde\Lambda}}\mathrm{d}(1-D_{\tilde\Lambda})\frac{1}{1-D_{\tilde\Lambda}}\mathrm{d}\tilde\Lambda = \frac{1}{1-D_{\tilde\Lambda}}\mathrm{ad}_{\mathrm{d}\tilde\Lambda}(\tilde e) = -\frac{1}{1-D_{\tilde\Lambda}}[(1-D_{\tilde\Lambda})\tilde e,\tilde e] = \\ &= -\frac{1-\frac12 D_{\tilde\Lambda}}{1-D_{\tilde\Lambda}}[\tilde e,\tilde e] = -\frac12[\tilde e,\tilde e] - \frac12\frac{1}{1-D_{\tilde\Lambda}}[\tilde e,\tilde e]\ ,\end{aligned} \tag{E.23}$$

which is (4.30). In the first step we used $\mathcal{O}^{-1}\mathcal{O} = 1 \ \Rightarrow\ \mathrm{d}\mathcal{O}^{-1} = -\mathcal{O}^{-1}\mathrm{d}\mathcal{O}\mathcal{O}^{-1}$ for $\mathcal{O} = 1 - D_{\tilde\Lambda}$. Then we used that $\mathrm{d}(D_{\tilde\Lambda}) = \mathrm{ad}_{\mathrm{d}\tilde\Lambda}$ and inverted the definition of the vielbeine as $\mathrm{d}\tilde\Lambda = -(1-D_{\tilde\Lambda})\tilde e$.

To compute H_3 we take the exterior derivative of (4.31) and exploit the above relation

$$\begin{aligned}H_3 := \mathrm{d}\tilde B_2 &= \frac12\langle \mathrm{d}\tilde e, D_{\tilde\Lambda}\tilde e\rangle - \frac12\langle\tilde e, \mathrm{ad}_{\mathrm{d}\tilde\Lambda}\tilde e\rangle - \frac12\langle\tilde e, D_{\tilde\Lambda}\mathrm{d}\tilde e\rangle = \langle\mathrm{d}\tilde e, D_{\tilde\Lambda}\tilde e\rangle + \frac12\langle[\tilde e,\tilde e],(1-D_{\tilde\Lambda})\tilde e\rangle = \\ &= \frac12\langle[\tilde e,\tilde e],\tilde e\rangle - \frac12\langle\frac{1}{1-D_{\tilde\Lambda}}[\tilde e,\tilde e], D_{\tilde\Lambda}\tilde e\rangle - \langle[\tilde e,\tilde e], D_{\tilde\Lambda}\tilde e\rangle = \\ &= \frac12\langle\tilde e,\left(1+\frac{D_{\tilde\Lambda}}{1-D_{\tilde\Lambda}}\right)[\tilde e,\tilde e]\rangle = \frac12\langle\tilde e,\frac{1}{1-D_{\tilde\Lambda}}[\tilde e,\tilde e]\rangle\ .\end{aligned} \tag{E.24}$$

In the first line we inverted the definition of $\tilde e$ for $\mathrm{d}\tilde\Lambda$ and on the second line we used $\langle[\tilde e,\tilde e], D_{\tilde\Lambda}\tilde e\rangle = 0$, which follows from antisymmetry of $D_{\tilde\Lambda}$ and the fact that it acts as a derivation on the Lie bracket.

E.3. OSp(1|2)

Supergravity constraints. To study the requirements imposed by supergravity we proceed now in a slightly different way, that is by taking the constrained structure equations to be satisfied by the vielbeine in 4d theories [100, 134] and reducing them to 3d. We start by writing the first Cartan structure equation in (A.19) using 4d spinorial notation, with indices $B = \{\beta\dot\beta,\beta,\dot\beta\}$

$$\mathrm{d}\hat e^{\alpha\dot\alpha} - \hat e^B\wedge\Omega_B{}^{\alpha\dot\alpha} = -\hat T^{\alpha\dot\alpha} \qquad \mathrm{d}\hat e^{\alpha} - \hat e^B\wedge\Omega_B{}^{\alpha} = -\hat T^{\alpha} \qquad \mathrm{d}\hat e^{\dot\alpha} - \hat e^B\wedge\Omega_B{}^{\dot\alpha} = -\hat T^{\dot\alpha}. \tag{E.25}$$

The connection is then required to be diagonal, with non-vanishing components $\Omega_{\beta\dot\beta}{}^{\alpha\dot\alpha}$, $\Omega_\beta{}^\alpha$, $\Omega_{\dot\beta}{}^{\dot\alpha}$, and the boson-boson part related to the fermion-fermion ones as

$$\Omega_{\beta\dot\beta}{}^{\alpha\dot\alpha} = \Omega_\beta{}^\alpha \delta_{\dot\beta}{}^{\dot\alpha} + \Omega_{\dot\beta}{}^{\dot\alpha}\delta_\beta{}^\alpha \ . \tag{E.26}$$

Imposing then the following constraints on torsion

$$\hat T_{\beta\,\dot\beta}{}^{\alpha\dot\alpha} = \tfrac{1}{2}\delta_\beta{}^\alpha\delta_{\dot\beta}{}^{\dot\alpha} \qquad\qquad \hat T_{\alpha\beta}{}^C = \hat T_{\dot\alpha\dot\beta}{}^C = \hat T_{\alpha\dot\beta}{}^\gamma = \hat T_{\alpha\dot\beta}{}^{\dot\gamma} = \hat T_{A\beta\dot\beta}{}^{\gamma\dot\gamma} = 0 \ , \tag{E.27}$$

the structure equations become

$$\begin{aligned} d\hat e^{\alpha\dot\alpha} - \hat e^{\beta\dot\alpha}\wedge\Omega_\beta{}^\alpha - \hat e^{\alpha\dot\beta}\wedge\Omega_{\dot\beta}{}^{\dot\alpha} &= -\tfrac12 \hat e^\alpha \wedge \hat e^{\dot\alpha} \\ d\hat e^\alpha - \hat e^\beta\wedge\Omega_\beta{}^\alpha &= -\tfrac12 \hat e^{\beta\dot\beta}\wedge\hat e^{\gamma\dot\gamma}\hat T_{\gamma\dot\gamma\beta\dot\beta}{}^\alpha - \hat e^{\beta\dot\beta}\wedge\hat e^\gamma \hat T_{\gamma\beta\dot\beta}{}^\alpha - \hat e^{\beta\dot\beta}\wedge\hat e^{\dot\gamma}\hat T_{\dot\gamma\beta\dot\beta}{}^\alpha \\ d\hat e^{\dot\alpha} - \hat e^{\dot\beta}\wedge\Omega_{\dot\beta}{}^{\dot\alpha} &= -\tfrac12 \hat e^{\beta\dot\beta}\wedge\hat e^{\gamma\dot\gamma}\hat T_{\gamma\dot\gamma\beta\dot\beta}{}^{\dot\alpha} - \hat e^{\beta\dot\beta}\wedge\hat e^\gamma \hat T_{\gamma\beta\dot\beta}{}^{\dot\alpha} - \hat e^{\beta\dot\beta}\wedge\hat e^{\dot\gamma}\hat T_{\dot\gamma\beta\dot\beta}{}^{\dot\alpha} \ . \end{aligned} \tag{E.28}$$

In three-dimensions the distinction between dotted and undotted indices is lost, so one can proceed by identifying the two types of indices, which makes the last two equations the same

$$\begin{aligned} d\hat e^{\alpha\beta} - \hat e^{\gamma\beta}\wedge\Omega_\gamma{}^\alpha - \hat e^{\alpha\gamma}\wedge\Omega_\gamma{}^\beta &= -\hat e^\gamma\wedge\hat e^\delta \hat T_{\delta\,\gamma}{}^{\alpha\beta} \\ d\hat e^\alpha - \hat e^\beta\wedge\Omega_\beta{}^\alpha &= -\tfrac12 \hat e^{\rho\sigma}\wedge\hat e^{\gamma\delta}\hat T_{\gamma\delta\,\rho\sigma}{}^\alpha - 2\hat e^\delta\wedge\hat e^{\beta\gamma}\hat T_{\beta\gamma\,\delta}{}^\alpha \ . \end{aligned} \tag{E.29}$$

Then, using

$$\hat e^{\alpha\beta} := e^{(\alpha\beta)} + S\epsilon^{\alpha\beta} \qquad \Omega^{\alpha\beta} = \Omega^{\beta\alpha} \qquad \hat e^\alpha := e^\alpha \tag{E.30}$$

and choosing

$$\Omega_{[\rho\sigma]\,\gamma}{}^\alpha = 0 \qquad\qquad \hat T_{[\rho\sigma]\,(\gamma\delta)}{}^\alpha = \hat T_{[\rho\sigma]\,[\gamma\delta]}{}^\alpha = \hat T_{\rho\,[\gamma\delta]}{}^\alpha = 0 \ , \tag{E.31}$$

the first equation splits into

$$de^{\alpha\beta} - 2e^{\gamma(\alpha}\wedge\Omega_\gamma{}^{\beta)} = -\tfrac12 e^\alpha\wedge e^\beta \qquad\qquad dS = 0 \ , \tag{E.32}$$

while the second becomes

$$de^\alpha - e^\beta\wedge\Omega_\beta{}^\alpha = -\tfrac12 e^{\rho\sigma}\wedge e^{\gamma\delta}T_{\gamma\delta\,\rho\sigma}{}^\alpha - e^\delta\wedge e^{\beta\gamma}T_{\beta\gamma\,\delta}{}^\alpha \ . \tag{E.33}$$

With the above choice, S completely decouples from the system and the remaining constraints are effectively three-dimensional. As mentioned in Appendix A of [134], in the language of σ-matrices commonly adopted in four dimensions this corresponds to the decoupling of the direction associated with the antisymmetric sigma matrix σ^2, i.e. the second Pauli matrix. Hence we are left with the constrained structure equations

$$\begin{aligned} \mathrm{d}e^{\alpha\beta} - 2e^{\gamma(\alpha} \wedge \Omega_{\gamma}{}^{\beta)} &= -\tfrac{1}{2} e^{\alpha} \wedge e^{\beta} \\ \mathrm{d}e^{\alpha} - e^{\beta} \wedge \Omega_{\beta}{}^{\alpha} &= -\tfrac{1}{2} e^{\delta\epsilon} \wedge e^{\beta\gamma} T_{\beta\gamma\,\delta\epsilon}{}^{\alpha} - e^{\delta} \wedge e^{\beta\gamma} T_{\beta\gamma\,\delta}{}^{\alpha} \ , \end{aligned} \tag{E.34}$$

encoding all the necessary requirements on torsion and connection. The components $T_{\alpha\beta\;\gamma\delta}{}^{\epsilon}$ and $T_{\alpha\beta\,\gamma}{}^{\delta}$ are unconstrained and the diagonal connection components are related by

$$\Omega_{\alpha\beta}{}^{\gamma\delta} = 2\Omega_{(\alpha}{}^{(\gamma}\delta_{\beta)}{}^{\delta)} \ . \tag{E.35}$$

These have expansion on the vielbeine $\Omega_{\alpha}{}^{\beta} = e^{\gamma\delta}\Omega_{\gamma\delta\,\alpha}{}^{\beta} + e^{\gamma}\Omega_{\gamma\,\alpha}{}^{\beta}$ and off-diagonal components vanish as needed by (A.33) and metric compatibility implies $\Omega_{\alpha\beta} = \Omega_{\beta\alpha}$.

There is another requirement, needed for three-dimensional supergravity theories, that thanks to the abstract approach we can now take into account more easily. The components of the three-form H_3 should be restricted and the non-vanishing ones should depend on a scalar superfield L and its derivatives as described in [134]

$$H_3 = e_{\alpha} \wedge e^{\alpha\beta} \wedge e_{\beta}\, L + e_{\beta} \wedge e_{\gamma}{}^{\beta} \wedge e^{\gamma\alpha}\, \mathcal{D}_{\alpha} L - \tfrac{i}{6} e_{\alpha}{}^{\beta} \wedge e_{\beta}{}^{\gamma} \wedge e_{\gamma}{}^{\alpha} (i\mathcal{D}^2 + 8\mathcal{S}) L \ , \tag{E.36}$$

where $\mathcal{D}_{\alpha}$ is the superspace covariant derivative, $\mathcal{D}^2 = \mathcal{D}^{\alpha}\mathcal{D}_{\alpha}$ and $\mathcal{S}$ is another scalar superfield.

Useful identities. We report here a set of identities needed in the study of the OSp(1|2) model. To begin we introduce generic quantities taking values in some Lie superalgebra $\mathfrak{g}$

$$U := U_b + U_f \qquad\qquad V := V_b + V_f \ , \tag{E.37}$$

where we split purely bosonic and purely fermionic contributions. In order to compute the action of $\frac{1}{1-\mathrm{ad}_U}$ on V we exploit the above splitting and obtain the following expansion

$$\begin{aligned} \frac{1}{1 \mp \mathrm{ad}_U} &= \frac{1}{1 \mp \mathrm{ad}_{U_b} \mp \mathrm{ad}_{U_f}} = \frac{1}{[1 \mp \mathrm{ad}_{U_b}][1 \mp (1 \mp \mathrm{ad}_{U_b})^{-1} \circ \mathrm{ad}_{U_f}]} = \\ &= [1 \mp (1 \mp \mathrm{ad}_{U_b})^{-1} \circ \mathrm{ad}_{U_f}]^{-1} [1 \mp \mathrm{ad}_{U_b}]^{-1} = \\ &= \sum_{k=0}^{\infty} \left[\pm \frac{1}{1 \mp \mathrm{ad}_{U_b}} \circ \mathrm{ad}_{U_f} \right]^k \circ \frac{1}{1 \mp \mathrm{ad}_{U_b}} = \\ &= \frac{1}{1 \mp \mathrm{ad}_{U_b}} \pm \frac{1}{1 \mp \mathrm{ad}_{U_b}} \circ \mathrm{ad}_{U_f} \circ \frac{1}{1 \mp \mathrm{ad}_{U_b}} + \dots \end{aligned} \tag{E.38}$$

which due to the nilpotency of fermions terminates at some power $k = N_f$. For OSp(1|2) we have

$$U := u^{\alpha\beta} L_{\alpha\beta} + \chi^\alpha Q_\alpha \qquad\qquad V := v^{\alpha\beta} L_{\alpha\beta} + \eta^\alpha Q_\alpha \tag{E.39}$$

and the above expansion terminates at $k = 2$, leaving us with three contributions to compute. To begin we define $\chi^2 := \chi^\alpha \chi_\alpha = \chi^\alpha \chi^\beta \epsilon_{\alpha\beta}$ and

$$u_\gamma{}^\alpha u_\beta{}^\gamma = -\tfrac{1}{2}\delta_\beta{}^\alpha u_{\gamma\delta} u^{\gamma\delta} =: -\tfrac{1}{2}\delta_\beta{}^\alpha u^2 \qquad \text{with} \qquad u^2 := u_{\alpha\beta} u^{\alpha\beta} \ . \tag{E.40}$$

Using the latter we can start computing the action of the bosonic term in the expansion on V

$$\frac{1}{1 - \mathrm{ad}_{U_b}}(V_b + V_f) = \sum_{k=0}^{\infty} \mathrm{ad}^k_{U_b}(V_u + V_f) \ . \tag{E.41}$$

Exploiting the commutators (4.33) one obtains

$$\begin{aligned} \mathrm{ad}^{2k}_{U_b}(V_b) &= (2u^2)^{k-1}(u^2 v^{\alpha\beta} - 2u_\gamma{}^\alpha u_\delta{}^\beta v^{\gamma\delta}) L_{\alpha\beta} && \text{for} \quad k \geqslant 1 \\ \mathrm{ad}^{2k+1}_{U_b}(V_b) &= -2i(2u^2)^k u_\gamma{}^\alpha v^{\beta\gamma} L_{\alpha\beta} && \text{for} \quad k \geqslant 0 \end{aligned} \tag{E.42}$$

and

$$\begin{aligned} \mathrm{ad}^{2k}_{U_b}(V_f) &= (\tfrac{1}{2}u^2)^k \eta^\alpha Q_\alpha && \text{for} \quad k \geqslant 0 \\ \mathrm{ad}^{2k+1}_{U_b}(V_f) &= -i(\tfrac{1}{2}u^2)^k u_\beta{}^\alpha \eta^\beta Q_\alpha && \text{for} \quad k \geqslant 0 \end{aligned} \tag{E.43}$$

which lead to

$$\begin{aligned} \frac{1}{1 - \mathrm{ad}_{U_b}}(V_b + V_f) &= \frac{1}{1-2u^2}\Big[(1-u^2)v^{\alpha\beta} - 2u_\gamma{}^\alpha u_\delta{}^\beta v^{\gamma\delta} - 2i u_\gamma{}^{(\alpha} v^{\beta)\gamma}\Big] L_{\alpha\beta} + \\ &\quad + \frac{2}{2-u^2}\eta^\beta[\delta_\beta{}^\alpha - i u_\beta{}^\alpha] Q_\alpha = \\ &=: v'^{\alpha\beta} L_{\alpha\beta} + \eta'^\alpha Q_\alpha \end{aligned} \tag{E.44}$$

and consecutively to

$$\mathrm{ad}_{U_f} \circ \frac{1}{1 - \mathrm{ad}_{U_b}}(V) = \eta'^{(\alpha}\chi^{\beta)} L_{\alpha\beta} + i v'^\alpha{}_\beta \chi^\beta Q_\alpha \ . \tag{E.45}$$

One can then compute the terms in the expansion (E.38) recursively, thus obtaining

$$\begin{aligned} \frac{1}{1 - \mathrm{ad}_{U_b}} \circ \mathrm{ad}_{U_f} \circ \frac{1}{1 - \mathrm{ad}_{U_b}}(V) &= \frac{1}{1-2u^2}\Big[(1-u^2)\eta'^{(\alpha}\chi^{\beta)} - 2u_\gamma{}^\alpha u_\delta{}^\beta \eta'^{(\gamma}\chi^{\delta)} - 2i u_\gamma{}^\alpha \eta'^{(\beta}\chi^{\gamma)}\Big] L_{\alpha\beta} + \\ &\quad + \frac{2}{2-u^2} i v'^\alpha{}_\gamma \chi^\gamma [\delta_\beta{}^\alpha - i u_\beta{}^\alpha] Q_\alpha = \\ &=: v''^{\alpha\beta} L_{\alpha\beta} + \eta''^\alpha Q_\alpha \end{aligned} \tag{E.46}$$

and consecutively

$$\mathrm{ad}_{U_f} \circ \frac{1}{1-\mathrm{ad}_{U_b}} \circ \mathrm{ad}_{U_f} \circ \frac{1}{1-\mathrm{ad}_{U_b}}(V) = \eta''^{(\alpha}\chi^{\beta)}L_{\alpha\beta} + iv''^{\alpha}{}_{\beta}\chi^{\beta}Q_{\alpha} \tag{E.47}$$

and the last term

$$\begin{aligned}\frac{1}{1-\mathrm{ad}_{U_b}} \circ \mathrm{ad}_{U_f} \circ \frac{1}{1-\mathrm{ad}_{U_b}} \circ \mathrm{ad}_{U_f} \circ \frac{1}{1-\mathrm{ad}_{U_b}}(V) = \\ = \frac{1}{1-2u^2}\big[(1-u^2)\eta''^{(\alpha}\chi^{\beta)} - 2u_\gamma{}^\alpha u_\delta{}^\beta \eta''^{(\gamma}\chi^{\delta)} - 2iu_\gamma{}^\alpha \eta''^{(\beta}\chi^{\gamma)}\big]L_{\alpha\beta} + \\ + \frac{2}{2-u^2} iv''^{\alpha}{}_{\gamma}\chi^{\gamma}[\delta_\beta{}^\alpha - iu_\beta{}^\alpha]Q_\alpha\ .\end{aligned} \tag{E.48}$$

One can then put together the above pieces, obtaining the following expression

$$\begin{aligned}\frac{1}{1-\mathrm{ad}_U}(V) = \frac{1}{1-2u^2}\big[(1-u^2)Z^{\alpha\beta} - 2u_\gamma{}^\alpha u_\delta{}^\beta Z^{\gamma\delta} - 2iu_\gamma{}^{(\alpha} Z^{\beta)\gamma}\big]L_{\alpha\beta} + \\ + \frac{2}{2-u^2}\zeta^\alpha[\delta_\alpha{}^\beta - iu_\alpha{}^\beta]Q_\beta\ ,\end{aligned} \tag{E.49}$$

with

$$\begin{aligned}Z^{\alpha\beta} &:= v^{\alpha\beta} + (\eta' + \eta'')^{(\alpha}\chi^{\beta)} = \\ &= v^{\alpha\beta} + \frac{2}{2-u^2}\big[\eta^{(\alpha} - i\eta^\gamma u_\gamma{}^{(\alpha}\big]\chi^{\beta)} + \\ &\quad + \frac{i}{(2-u^2)(1-2u^2)}\big[(1-\tfrac{1}{2}u^2)v^{\alpha\beta} - 3u_\gamma{}^\alpha u_\delta{}^\beta v^{\gamma\delta} - 3iu_\gamma{}^{(\alpha}v^{\beta)\gamma}\big]\chi^2 \\ \zeta^\alpha &:= \eta^\alpha - i(v' + v'')^{\alpha\beta}\chi_\beta + \\ &= \eta^\alpha - \frac{i}{1-2u^2}\big[(1-u^2)u^{\alpha\beta} - 2u_\gamma{}^\alpha u_\delta{}^\beta v^{\gamma\delta} - 2iu_\gamma{}^{(\alpha}v^{\beta)\gamma}\big]\chi_\beta + \\ &\quad - \frac{i}{(2-u^2)(1-2u^2)}\big[\tfrac{3}{2}\eta^\alpha - i\big(\tfrac{7}{2} - u^2\big)\eta^\beta u_\beta{}^\alpha\big]\chi^2\ .\end{aligned} \tag{E.50}$$

F

T-Duality Of Symmetric And Semi-Symmetric Spaces

F.1. Initial model

Equations of motion and Noether current. Variations of the action (4.64) read

$$\delta S = \int_\Sigma \langle \delta m, \star m\rangle + \tfrac{\kappa}{2}\langle \delta p, q\rangle - \tfrac{\kappa}{2}\langle \delta q, p\rangle \tag{F.1}$$

and upon recalling that for $\mathrm{g} \to \mathrm{g} + \delta\mathrm{g}$ the current transforms as $j \to j + \delta j$, with $\delta j = \mathrm{d}(\mathrm{g}^{-1}\delta\mathrm{g}) + [j, \mathrm{g}^{-1}\delta\mathrm{g}]$, one can use (4.62) to project on the four subspaces, thus obtaining

$$\begin{aligned}
\delta p &= \mathrm{d}(\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{p}} + [A, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{p}}] + [p, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{h}}] + [m, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{q}}] + [q, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{m}}] \\
\delta m &= \mathrm{d}(\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{m}} + [A, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{m}}] + [p, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{p}}] + [m, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{h}}] + [q, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{q}}] \\
\delta q &= \mathrm{d}(\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{q}} + [A, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{q}}] + [p, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{m}}] + [m, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{p}}] + [q, (\mathrm{g}^{-1}\delta\mathrm{g})|_{\mathfrak{h}}] \ .
\end{aligned} \tag{F.2}$$

Substituting and rearranging one then recovers (4.65). The Noether current associated to the $\mathrm{G}_L : \mathrm{g} \to g_L^{-1}\mathrm{g}$ invariance of the initial action is then obtained by letting $g_L \simeq 1 + \epsilon$ so that $\delta\mathrm{g} = -\mathrm{g}\epsilon$ and $\delta m, \delta p, \delta q$ are simply the projections of $-\mathrm{g}^{-1}\mathrm{d}\epsilon\mathrm{g}$ on the respective subspaces.

LaxcConnection. Recalling Maurer-Cartan equations (4.66) and equations of motion (4.65)

$$\text{MC:} \quad \begin{cases} F_A + \frac{1}{2}[m, m] + [p, q] = 0 \\ \nabla_A p + [m, q] = 0 \\ \nabla_A m + \frac{1}{2}[p, p] + \frac{1}{2}[q, q] = 0 \\ \nabla_A q + [m, p] = 0 \end{cases} \qquad \text{EOM:} \quad \begin{cases} \nabla_A \star m - \frac{\kappa}{2}[p, p] + \frac{\kappa}{2}[q, q] = 0 \\ \nabla_A q + [p, \frac{2}{\kappa} \star m - m] = 0 \\ \nabla_A p - [q, \frac{2}{\kappa} \star m + m] = 0 \end{cases} \tag{F.3}$$

since $\mathrm{d}(\star A), \mathrm{d}(\star p), \mathrm{d}(\star q)$ do not appear in the above equations, one can safely take the following ansatz for the Lax connection, with coefficients a, b, c, w, t to be determined

$$J = aA + bm + c \star m + wp + tq \ . \tag{F.4}$$

One can then construct the curvature $F_J := \mathrm{d}J + \frac{1}{2}[J, J]$ of the connection and rearrange terms in the form of the above equations. In doing so, one should be more careful than with the principal chiral model, as some terms, such as $\mathrm{d}p, \mathrm{d}q$ or $[m, p], [m, q]$ appear both in equations of motion and Maurer-Cartan equations. For this reason one should split such terms and in doing so there is some extra arbitrariness appearing, which could be taken care of by introducing new coefficients to be determined. For example, when splitting the term wdp between the second Maurer-Cartan equation and the third equations of motion one could write $w\mathrm{d}p = \frac{1}{2}(w+\alpha)\mathrm{d}p + \frac{1}{2}(w-\alpha)\mathrm{d}p$, so that each of the two contribution could be used for one of the two equations. Repeating this for all the necessary terms one finally obtains

$$\begin{aligned} F_J = a\Big\{&\mathrm{d}A + \tfrac{a}{2}[A, A] + \tfrac{b^2-c^2}{2a}[m, m] + \tfrac{wt}{a}[p, q]\Big\} + \tfrac{w+\alpha}{2}\Big\{\mathrm{d}p + \tfrac{wa+\beta}{w+\alpha}[A, p] + \tfrac{tb+\gamma}{w+\alpha}[m, q]\Big\}+ \\ &+ b\Big\{\mathrm{d}m + a[A, m] + \tfrac{w^2+\mu}{4b}[p, p] + \tfrac{t^2+\nu}{4b}[q, q]\Big\} + \tfrac{t+\rho}{2}\Big\{\mathrm{d}q + \tfrac{ta+\sigma}{t+\rho}[A, q] + \tfrac{wb+\lambda}{t+\rho}[m, p]\Big\}+ \\ &+ c\Big\{\mathrm{d} \star m + a[A, \star m] + \tfrac{w^2-\mu}{4c}[p, p] + \tfrac{t^2-\nu}{4c}[q, q]\Big\}+ \\ &+ \tfrac{t-\rho}{2}\Big\{\mathrm{d}q + \tfrac{ta-\sigma}{t-\rho}[A, q] + \tfrac{2wc}{t-\rho}[p, \star m] + \tfrac{wb-\lambda}{t-\rho}[p, m]\Big\}+ \\ &+ \tfrac{w-\alpha}{2}\Big\{\mathrm{d}p + \tfrac{wa-\beta}{w-\alpha}[A, p] + \tfrac{2tc}{w-\alpha}[q, \star m] + \tfrac{tb-\gamma}{w-\alpha}[q, m]\Big\} \ , \end{aligned} \tag{F.5}$$

where we introduced $\alpha, \beta, \rho, \sigma, \gamma, \lambda, \mu, \nu$ to be determined. In order for the first four brackets to match the Maurer-Cartan equations and the last three brackets to match the equations of motion one needs that $a = 1$ and

$$\begin{array}{cccc} b^2 - c^2 = 1 & wt = 1 & \frac{w+\beta}{w+\alpha} = 1 & \frac{tb+\gamma}{w+\alpha} = 1 \\ \frac{w^2+\mu}{2b} = 1 & \frac{t^2+\nu}{2b} = 1 & \frac{t+\sigma}{t+\rho} = 1 & \frac{wb+\lambda}{t+\rho} = 1 \\ \frac{w^2-\mu}{2c} = -\kappa & & \frac{t^2-\nu}{2c} = \kappa & \\ \frac{t-\sigma}{t-\rho} = 1 & \frac{wc}{t-\rho} = \frac{1}{\kappa} & \frac{wb-\lambda}{t-\rho} = -1 & \\ \frac{w-\beta}{w-\alpha} = 1 & \frac{tc}{w-\alpha} = -\frac{1}{\kappa} & \frac{tb-\gamma}{w-\alpha} = -1 \ . & \end{array} \tag{F.6}$$

From the above constraints one immediately recognises that $\alpha = \beta$ and $\rho = \sigma$, then

- combining the first two conditions on the second line and the two conditions on the third line one finds that $\nu = w^2$ and $\mu = t^2$
- combining the fourth condition on the first line with the third condition on the fifth line one finds that $\gamma = w$ and $\alpha = tb$

- combining the fourth condition on the second line and the third condition on the fourth line one finds that $\lambda = t$ and $\rho = wb$

At this point all the extra coefficients have been fixed in terms of b, c, w, t and one can find, for example from the first condition on the second line and the second condition on the third line, that $b = \frac{1}{2}(t^2 + w^2)$ and $c = \frac{1}{2\kappa}(t^2 - w^2)$. The second condition on fourth and fifth line are also consistent with this provided one uses $wt = 1$. Now one can impose $b^2 - c^2 = 1$, thus finding

$$b^2 - c^2 = \frac{w^4(\kappa^2 - 1) + 2w^2t^2(\kappa^2 + 1) + t^4(\kappa^2 - 1)}{4\kappa^2} \equiv 1 \qquad \Rightarrow \qquad \kappa^2 \equiv 1 \ . \tag{F.7}$$

Hence the flatness of the lax connection (F.4) can be cast into the Maurer-Cartan equations and equations of motion of semi-symmetric space sigma models provided that $\kappa = \pm 1$. With this restriction, all the above constraints are satisfied and all coefficients are written in terms of w and t, which must be such that $wt = 1$. The simplest choice is to set $w = z$ and $t = z^{-1}$, which finally leads to

$$\begin{aligned} a &= 1 \qquad b = \tfrac{1}{2}(z^2 + z^{-2}) \qquad c = -\tfrac{1}{2\kappa}(z^2 - z^{-2}) \qquad w = z \qquad t = z^{-1} \\ \alpha &= \tfrac{1}{2}(z + z^{-3}) = \beta \qquad \rho = \tfrac{1}{2}(z^3 + z^{-1}) = \sigma \qquad \gamma = z \qquad \lambda = z^{-1} \qquad \mu = z^{-2} \qquad \nu = z^2 \ . \end{aligned} \tag{F.8}$$

F.2. T-dual model

Equations of motion in $\mathfrak{p}, \mathfrak{q}$. To solve (4.105) in the fermionic subspaces we first write them as

$$\begin{aligned} \mathfrak{p}: \qquad &-(\tfrac{1}{2} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{h}})(p_\omega) - D_{\tilde{\Lambda}_\mathfrak{m}}(q_\omega) = -\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{p} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}}(m_\omega) \\ \mathfrak{q}: \qquad &(\tfrac{1}{2} - \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{h}})(q_\omega) - D_{\tilde{\Lambda}_\mathfrak{m}}(p_\omega) = -\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{q} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}(m_\omega) \ , \end{aligned} \tag{F.9}$$

which in matrix form read

$$\underbrace{\begin{bmatrix} -(\frac{1}{2} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{h}}) & -D_{\tilde{\Lambda}_\mathfrak{m}} \\ -D_{\tilde{\Lambda}_\mathfrak{m}} & (\frac{1}{2} - \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{h}}) \end{bmatrix}}_{R^{-1}} \begin{bmatrix} p_\omega \\ q_\omega \end{bmatrix} = \begin{bmatrix} -\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{p} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}}(m_\omega) \\ -\nabla_{A_\omega}\tilde{\Lambda}_\mathfrak{q} + \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}(m_\omega) \end{bmatrix} \ . \tag{F.10}$$

To invert R^{-1} we first define $c_\pm := \frac{2}{1 \pm 2\mathrm{ad}_{\tilde{\Lambda}_\mathfrak{h}}}$ so that it can be rewritten as

$$R^{-1} = \begin{bmatrix} -c_+^{-1} & -D_{\tilde{\Lambda}_\mathfrak{m}} \\ -D_{\tilde{\Lambda}_\mathfrak{m}} & c_-^{-1} \end{bmatrix} = \begin{bmatrix} -c_+^{-1} & 0 \\ 0 & c_-^{-1} \end{bmatrix} (\mathbb{1}_2 - M) \ , \tag{F.11}$$

with

$$\mathbb{1}_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M := \begin{bmatrix} 0 & -c_+ \circ D_{\tilde{\Lambda}_\mathfrak{m}} \\ c_- \circ D_{\tilde{\Lambda}_\mathfrak{m}} & 0 \end{bmatrix} . \tag{F.12}$$

Consequently one finds

$$R = (\mathbb{1}_2 - M)^{-1} \begin{bmatrix} -c_+ & 0 \\ 0 & c_- \end{bmatrix} = \sum_{k=0}^{\infty} M^k \begin{bmatrix} -c_+ & 0 \\ 0 & c_- \end{bmatrix} = \sum_{k=0}^{\infty} M^{2k} (\mathbb{1}_2 + M) \begin{bmatrix} -c_+ & 0 \\ 0 & c_- \end{bmatrix} , \tag{F.13}$$

and (4.108) is recovered after noting that

$$\sum_{k=0}^{\infty} M^{2k} = \sum_{k=0}^{\infty} \begin{bmatrix} -c_+ \circ D_{\tilde{\Lambda}_\mathfrak{m}} \circ c_- \circ D_{\tilde{\Lambda}_\mathfrak{m}} & 0 \\ 0 & -c_- \circ D_{\tilde{\Lambda}_\mathfrak{m}} \circ c_+ \circ D_{\tilde{\Lambda}_\mathfrak{m}} \end{bmatrix}^k = \tag{F.14}$$
$$= \begin{bmatrix} \frac{1}{1 + c_+ \circ D_{\tilde{\Lambda}_\mathfrak{m}} \circ c_- \circ D_{\tilde{\Lambda}_\mathfrak{m}}} & 0 \\ 0 & \frac{1}{1 + c_- \circ D_{\tilde{\Lambda}_\mathfrak{m}} \circ c_+ \circ D_{\tilde{\Lambda}_\mathfrak{m}}} \end{bmatrix} .$$

Hybrid action. Using the properties of the inner product we write the action (4.70) explicitly as

$$\begin{aligned} S_\omega = \int_\Sigma & \langle m_\omega, \star m_\omega \rangle + \frac{1}{2}\langle p_\omega, q_\omega \rangle + \langle \tilde{\Lambda}_\mathfrak{h}, F_{A_\omega} + \frac{1}{2}[m_\omega, m_\omega] + [p_\omega, q_\omega] \rangle + \\ & + \langle \tilde{\Lambda}_\mathfrak{p}, \nabla_{A_\omega} q_\omega + [m_\omega, p_\omega] \rangle + \langle \tilde{\Lambda}_\mathfrak{q}, \nabla_{A_\omega} p_\omega + [m_\omega, q_\omega] \rangle + \\ & + \langle \tilde{\Lambda}_\mathfrak{m}, \nabla_{A_\omega} m_\omega + \frac{1}{2}[p_\omega, p_\omega] + \frac{1}{2}[q_\omega, q_\omega] \rangle + \langle D(A_\omega), m_\omega \rangle + \frac{1}{2}\langle D(p_\omega), p_\omega \rangle + \frac{1}{2}\langle D(q_\omega), q_\omega \rangle \end{aligned} \tag{F.15}$$

and substituting $\star m_\omega$ in the first term with (4.105) the terms can be rearranged into

$$\begin{aligned} S_\omega = \int_\Sigma & \frac{1}{2}\langle m_\omega, \mathrm{d}\tilde{\Lambda}_\mathfrak{m} - D_{\tilde{\Lambda}_\mathfrak{m}}(A_\omega) \rangle + \langle \tilde{\Lambda}_\mathfrak{h}, F_{A_\omega} \rangle + \\ & + \frac{1}{2}\langle p_\omega, \nabla_{A_\omega} \tilde{\Lambda}_q + \{ \tfrac{1}{2} q_\omega - D_{\tilde{\Lambda}_\mathfrak{m}}(p_\omega) - [\tilde{\Lambda}_\mathfrak{p}, m_\omega] - [\tilde{\Lambda}_\mathfrak{h}, q_\omega] + \nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{q} \} \rangle + \\ & + \frac{1}{2}\langle q_\omega, \nabla_{A_\omega} \tilde{\Lambda}_p + \{ -\tfrac{1}{2} p_\omega - D_{\tilde{\Lambda}_\mathfrak{m}}(q_\omega) - [\tilde{\Lambda}_\mathfrak{q}, m_\omega] - [\tilde{\Lambda}_\mathfrak{h}, p_\omega] + \nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{p} \} \rangle . \end{aligned} \tag{F.16}$$

Now the curly brackets are precisely the equations of motion (4.106) in the subspaces $\mathfrak{p}$ and $\mathfrak{q}$, hence vanish identically and one can proceed by substituting the solution (4.107) for p_ω and q_ω

$$\begin{aligned} S_\omega = \int_\Sigma & \frac{1}{2}\langle m_\omega, \mathrm{d}\tilde{\Lambda}_\mathfrak{m} - D_{\tilde{\Lambda}_\mathfrak{m}}(A_\omega) \rangle + \langle \tilde{\Lambda}_\mathfrak{h}, F_{A_\omega} \rangle + \\ & + \frac{1}{2}\langle \nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{p}, R_{21}(\nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{p}) + R_{22}(\nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{q}) \rangle + \frac{1}{2}\langle \nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{q}, R_{11}(\nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{p}) + R_{12}(\nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{q}) \rangle + \\ & + \frac{1}{2}\langle (R_{11} \circ \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}} + R_{12} \circ \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}) m_\omega, \nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{q} \rangle + \frac{1}{2}\langle (R_{21} \circ \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{q}} + R_{22} \circ \mathrm{ad}_{\tilde{\Lambda}_\mathfrak{p}}) m_\omega, \nabla_{A_\omega} \tilde{\Lambda}_\mathfrak{p} \rangle . \end{aligned} \tag{F.17}$$

At this point one needs to recombine the last line with the first term on the first line. This can be achieved by noting that $\forall X, Y \in \mathfrak{g}$ the entries R_{ij} of the matrix (4.108) satisfy the following relations

$$\langle R_{11}(X), Y\rangle = -\langle X, R_{22}(Y)\rangle \qquad \langle R_{12}(X), Y\rangle = -\langle X, R_{12}(Y)\rangle \qquad \langle R_{21}(X), Y\rangle = -\langle X, R_{21}(Y)\rangle \tag{F.18}$$

which allow to bring the operators acting on m_ω to the other side of the inner product. Combining the two terms one then recovers the definition of T given in (4.110)

$$\begin{aligned} S_\omega = \int_\Sigma & \frac{1}{2}\langle m_\omega, T\rangle + \langle \tilde{\Lambda}_{\mathfrak{h}}, F_{A_\omega}\rangle + \\ & + \frac{1}{2}\langle \nabla_{A_\omega}\tilde{\Lambda}_{\mathfrak{p}}, R_{21}(\nabla_{A_\omega}\tilde{\Lambda}_{\mathfrak{p}}) + R_{22}(\nabla_{A_\omega}\tilde{\Lambda}_{\mathfrak{q}})\rangle + \frac{1}{2}\langle \nabla_{A_\omega}\tilde{\Lambda}_{\mathfrak{q}}, R_{11}(\nabla_{A_\omega}\tilde{\Lambda}_{\mathfrak{p}}) + R_{12}(\nabla_{A_\omega}\tilde{\Lambda}_{\mathfrak{q}})\rangle\,. \end{aligned} \tag{F.19}$$

Finally, exploiting the expression (4.111) one can immediately rearrange the first term as $\langle m_\omega, T\rangle = 2\langle T, \frac{1}{1-S}P_+T\rangle$, thus recovering the hybrid action (4.112). We finally notice that the above relations for the components R_{ij} of (4.108) are easily derived after using the definition (4.109) of $c_\pm$ to find that $\langle c_\pm(X), Y\rangle = \langle X, c_\mp(Y)\rangle$. For example, this implies that

$$\langle c_+ \circ D_{\tilde{\Lambda}_{\mathfrak{m}}} \circ c_- \circ D_{\tilde{\Lambda}_{\mathfrak{m}}}(X), Y\rangle = \langle X, D_{\tilde{\Lambda}_{\mathfrak{m}}} \circ c_+ \circ D_{\tilde{\Lambda}_{\mathfrak{m}}} \circ c_-(Y)\rangle \qquad \forall X, Y \in \mathfrak{g} \tag{F.20}$$

and in turn allows to find

$$\begin{aligned} \langle R_{11}(X), Y\rangle &= -\langle \sum_{k=0}^{\infty}(-1)^k (c_+ \circ D_{\tilde{\Lambda}_{\mathfrak{m}}} \circ c_- \circ D_{\tilde{\Lambda}_{\mathfrak{m}}})^k \circ c_+(X), Y\rangle = \\ &= -\langle X, c_- \circ \sum_{k=0}^{\infty}(-1)^k (D_{\tilde{\Lambda}_{\mathfrak{m}}} \circ c_+ \circ D_{\tilde{\Lambda}_{\mathfrak{m}}} \circ c_-)^k (Y)\rangle = \\ &= -\langle X, \sum_{k=0}^{\infty}(-1)^k (c_- \circ D_{\tilde{\Lambda}_{\mathfrak{m}}} \circ c_+ \circ D_{\tilde{\Lambda}_{\mathfrak{m}}})^k \circ c_-(Y)\rangle = -\langle X, R_{22}(Y)\rangle\,. \end{aligned} \tag{F.21}$$

The remaining relations are obtained using analogous reasonings after noting that

$$R_{12} = R_{11} \circ D_{\tilde{\Lambda}_{\mathfrak{m}}} \circ c_- \qquad\qquad R_{21} = -R_{22} \circ D_{\tilde{\Lambda}_{\mathfrak{m}}} \circ c_+\,. \tag{F.22}$$

Equations of motion. To vary the action (4.112) with respect to the multipliers and A_ω it is important to notice that for a generic variation the first term in the action gives

$$\delta\left(\langle T, \frac{1}{1-S}P_+T\rangle\right) = -\langle \delta T, \tilde{m}\rangle - \langle \delta S \circ \frac{1}{1-S}T, \frac{1}{1+S}P_-T\rangle\,, \tag{F.23}$$

where $\tilde{m} := m_\omega$ as in (4.111) and we used the relation

$$\delta\left(\frac{1}{1\pm S}\right) = \mp\frac{1}{1\pm S}\circ\delta S\circ\frac{1}{1\pm S}\ , \tag{F.24}$$

and exploited (F.18) to find

$$\langle X, \frac{1}{1\pm S}Y\rangle = \langle\frac{1}{1\mp S}X, Y\rangle \qquad \forall X, Y\in\mathfrak{g}\ . \tag{F.25}$$

The only non-trivial terms to vary are the entries R_{ij} of (4.108). These only depend on $\tilde{\Lambda}_\mathfrak{h}, \tilde{\Lambda}_\mathfrak{m}$ and have variations

$$\begin{aligned}
\delta_{\tilde{\Lambda}_\mathfrak{m}} R_{11} &= R_{12}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{11} + R_{11}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{21}\\
\delta_{\tilde{\Lambda}_\mathfrak{m}} R_{22} &= R_{21}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{22} + R_{22}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{12}\\
\delta_{\tilde{\Lambda}_\mathfrak{m}} R_{12} &= R_{12}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{12} + R_{11}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{22}\\
\delta_{\tilde{\Lambda}_\mathfrak{m}} R_{21} &= R_{21}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{21} + R_{22}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{11}
\end{aligned} \tag{F.26}$$

$$\begin{aligned}
\delta_{\tilde{\Lambda}_\mathfrak{h}} R_{11} &= R_{11}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{h}}\circ R_{11} + R_{12}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{h}}\circ R_{21}\\
\delta_{\tilde{\Lambda}_\mathfrak{h}} R_{22} &= R_{22}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{h}}\circ R_{22} + R_{21}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{h}}\circ R_{12}\\
\delta_{\tilde{\Lambda}_\mathfrak{h}} R_{12} &= R_{11}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{h}}\circ R_{12} + R_{12}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{h}}\circ R_{22}\\
\delta_{\tilde{\Lambda}_\mathfrak{h}} R_{21} &= R_{22}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{h}}\circ R_{21} + R_{21}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{h}}\circ R_{11}\ .
\end{aligned} \tag{F.27}$$

In deriving the latter one uses that $c_\pm$ only depends on $\tilde{\Lambda}_\mathfrak{h}$ and from (4.20) has variations $\delta_{\tilde{\Lambda}_\mathfrak{h}} c_\pm = \mp c_\pm\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{h}}\circ c_\pm$. For example, also using $\delta_{\tilde{\Lambda}_\mathfrak{m}} D_{\tilde{\Lambda}_\mathfrak{m}} = \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}$ and exploiting again (4.20), one has

$$\begin{aligned}
\delta_{\tilde{\Lambda}_\mathfrak{m}} R_{11} = &\Bigg(\frac{1}{1+c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}}\circ c_+\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ\frac{1}{1+c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}} +\\
&+\frac{1}{1+c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}}\circ c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ\frac{1}{1+c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}}\Bigg)\circ c_+ =\\
= &-R_{11}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ\frac{1}{1+c_+\circ D_{\tilde{\Lambda}_\mathfrak{m}}\circ c_-\circ D_{\tilde{\Lambda}_\mathfrak{m}}}\circ c_+ + R_{12}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{11} =\\
= &\ R_{11}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{21} + R_{12}\circ \mathrm{ad}_{\delta\tilde{\Lambda}_\mathfrak{m}}\circ R_{11}\ ,
\end{aligned} \tag{F.28}$$

where in the last step we used a rearrangement analogous to that in the last step of (F.21).

Similarly, varying with respect to $\tilde\Lambda_{\mathfrak h}$ and using $\delta_{\tilde\Lambda_{\mathfrak h}} c_\pm = \mp c_\pm \circ \mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}} \circ c_\pm$ one finds

$$
\begin{aligned}
\delta_{\tilde\Lambda_{\mathfrak h}} R_{11} = &\left(\frac{1}{1+c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}}\circ \delta_{\tilde\Lambda_{\mathfrak h}} c_+ \circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}\circ \frac{1}{1+c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}}+\right.\\
&\left.+\frac{1}{1+c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}}\circ c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ \delta_{\tilde\Lambda_{\mathfrak h}} c_-\circ D_{\tilde\Lambda_{\mathfrak m}}\circ\frac{1}{1+c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}}\right)\circ c_+ +\\
&+\frac{1}{1+c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}}\circ\delta_{\tilde\Lambda_{\mathfrak h}} c_+ =\\
&= R_{11}\circ \mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}\circ c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}\circ\frac{1}{1+c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}}\circ c_+ +\\
&- R_{12}\circ \mathrm{ad}_{\tilde\Lambda_{\mathfrak h}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}\circ\frac{1}{1+c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}}\circ c_+ - R_{11}\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}\circ c_+ =\\
&= R_{11}\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}\circ\left(-1+\circ c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}\circ\frac{1}{1+c_+\circ D_{\tilde\Lambda_{\mathfrak m}}\circ c_-\circ D_{\tilde\Lambda_{\mathfrak m}}}\right)\circ c_+ +\\
&+ R_{12}\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}\circ R_{21} =\\
&= R_{11}\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}\circ R_{11} + R_{12}\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}\circ R_{21}\ .
\end{aligned}
\tag{F.29}
$$

In the last but one step we used a rearrangement analogous to that in the last step of (F.21). The variations of the hybrid action can then be quickly computed by exploiting the above findings and making use of the notation (4.118). This way one has

$$
\begin{aligned}
&\begin{cases}
\delta_{\tilde\Lambda_{\mathfrak h}} S = \mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}} + \mathcal{O}_2^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}\circ\mathcal{O}_1 + \mathcal{O}_1^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}\circ\mathcal{O}_2\\
\delta_{\tilde\Lambda_{\mathfrak h}} T = \mathcal{O}_1^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}(R_{21}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak p} + R_{22}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak q}) + \mathcal{O}_2^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak h}}(R_{11}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak p} + R_{12}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak q})
\end{cases}\\
&\begin{cases}
\delta_{\tilde\Lambda_{\mathfrak p}} S = \mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak p}}\circ\mathcal{O}_1 + \mathcal{O}_1^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak p}}\\
\delta_{\tilde\Lambda_{\mathfrak p}} T = \mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak p}}(R_{11}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak p} + R_{12}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak q}) + \mathcal{O}_2^\dagger\nabla_{A_\omega}\delta\tilde\Lambda_{\mathfrak p}
\end{cases}\\
&\begin{cases}
\delta_{\tilde\Lambda_{\mathfrak m}} S = \mathcal{O}_1^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak m}}\circ\mathcal{O}_1 + \mathcal{O}_2^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak m}}\circ\mathcal{O}_2\\
\delta_{\tilde\Lambda_{\mathfrak m}} T = \nabla_{A_\omega}\delta\tilde\Lambda_{\mathfrak m} + \mathcal{O}_1^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak m}}(R_{11}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak p} + R_{12}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak q}) + \mathcal{O}_2^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak m}}(R_{21}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak p} + R_{22}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak q})
\end{cases}\\
&\begin{cases}
\delta_{\tilde\Lambda_{\mathfrak q}} S = \mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak q}}\circ\mathcal{O}_2 + \mathcal{O}_2^\dagger\circ\mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak q}}\\
\delta_{\tilde\Lambda_{\mathfrak q}} T = \mathrm{ad}_{\delta\tilde\Lambda_{\mathfrak q}}(R_{21}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak p} + R_{22}\nabla_{A_\omega}\tilde\Lambda_{\mathfrak q}) + \mathcal{O}_1^\dagger\nabla_{A_\omega}\delta\tilde\Lambda_{\mathfrak q}
\end{cases}\\
&\begin{cases}
\delta_{A_\omega} S = 0\\
\delta_{A_\omega} T = -(D_{\tilde\Lambda_{\mathfrak m}} + \mathcal{O}_2^\dagger\circ\mathrm{ad}_{\tilde\Lambda_{\mathfrak p}} + \mathcal{O}_1^\dagger\circ\mathrm{ad}_{\tilde\Lambda_{\mathfrak q}})\delta A_\omega\ .
\end{cases}
\end{aligned}
\tag{F.30}
$$

Upon using the latter it is sufficient to recognise that, for $i = 1, 2$, one has

$$\frac{1}{2}[\tilde{m}, \tilde{m}] = -\left[\frac{1}{1-S}T, \frac{1}{1+S}P_-T\right]$$
$$[\mathcal{O}_1(\tilde{m}), \mathcal{O}_2(\tilde{m})] = -\left[\mathcal{O}_1 \circ \frac{1}{1+S}T, \mathcal{O}_2 \circ \frac{1}{1-S}P_+T\right] - \left[\mathcal{O}_1 \circ \frac{1}{1-S}T, \mathcal{O}_2 \circ \frac{1}{1+S}P_-T\right] \tag{F.31}$$
$$[\tilde{m}, \mathcal{O}_i(\tilde{m})] = -\left[\frac{1}{1+S}T, \mathcal{O}_i \circ \frac{1}{1-S}P_+T\right] - \left[\frac{1}{1-S}T, \mathcal{O}_i \circ \frac{1}{1+S}P_-T\right]$$
$$\frac{1}{2}[\mathcal{O}_i(\tilde{m}), \mathcal{O}_i(\tilde{m})] = -\left[\mathcal{O}_i \circ \frac{1}{1-S}T, \mathcal{O}_i \circ \frac{1}{1+S}P_-T\right]$$

and use the definitions of $\tilde{p}, \tilde{q}$ from (4.107) to recover the equations of motion (4.113). It is then straightforward to check that the equations (4.115) hold true upon substituting the equations (4.106), satisfied by $\tilde{p}, \tilde{m}, \tilde{q}$, and exploiting the T-dual equations of motion (4.113) together with Jacobi identities.

F.3. Examples

$\mathbf{S}^3 \simeq \mathrm{SO}(4)/\mathrm{SO}(3)$. For symmetric spaces, upon dualising the whole isometry group and setting $D = 0$, the operators W and Z from (4.122) reduce to

$$W := \mathrm{ad}_{\Lambda_\mathfrak{h}} + \mathrm{ad}_{\Lambda_\mathfrak{m}} \circ \sum_{k=0}^{\infty} \mathrm{ad}^{2k+1}_{\Lambda_\mathfrak{h}} \circ \mathrm{ad}_{\Lambda_\mathfrak{m}} \qquad Z := \mathrm{ad}_{\Lambda_\mathfrak{m}} \circ \sum_{k=0}^{\infty} \mathrm{ad}^{2k}_{\Lambda_\mathfrak{h}} \circ \mathrm{ad}_{\Lambda_\mathfrak{m}} \ . \tag{F.32}$$

Hence, given the commutators

$$[H_i, H_j] = \frac{i}{2}\varepsilon_{ij}{}^k H_k \qquad [M_i, H_j] = \frac{i}{2}\varepsilon_{ij}{}^k M_k \qquad [M_i, M_j] = \frac{i}{2}\varepsilon_{ij}{}^k H_k \ , \tag{F.33}$$

and the expansions $A_\omega = A^i_\omega H_i$ and $\Lambda = y^i H_i + x^i M_i$, it is simple to compute

$$\mathrm{ad}_{\Lambda_\mathfrak{h}}(A_\omega) = \frac{i}{2} y^i A^j_\omega \varepsilon_{ij}{}^k H_k \qquad \mathrm{ad}_{\Lambda_\mathfrak{m}}(A_\omega) = \frac{i}{2} x^i A^j_\omega \varepsilon_{ij}{}^k M_k \ . \tag{F.34}$$

After a few more commutators, using $\varepsilon_{ijp}\varepsilon_{kl}{}^p = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$, one finds the following pattern

$$\mathrm{ad}^{2k}_{\Lambda_\mathfrak{h}} \circ \mathrm{ad}_{\Lambda_\mathfrak{m}}(A_\omega) = -\frac{i}{8}\left(\frac{y^2}{4}\right)^{k-1} \left[(y \cdot x)A^i_\omega - (y \cdot A_\omega)x^i\right] y^j \varepsilon_{ij}{}^l M_l \qquad \text{for} \qquad k \geqslant 1$$
$$\tag{F.35}$$
$$\mathrm{ad}^{2k+1}_{\Lambda_\mathfrak{h}} \circ \mathrm{ad}_{\Lambda_\mathfrak{m}}(A_\omega) = +\frac{1}{4}\left(\frac{y^2}{4}\right)^{k} \left[(y \cdot x)A^l_\omega - (y \cdot A_\omega)x^l\right] M_l \qquad \text{for} \qquad k \geqslant 0$$

so that using $\sum_{k=0}^{\infty}(y^2/4)^k = \frac{4}{4-y^2}$ one can easily re-sum both as

$$\begin{aligned}\sum_{k=0}^{\infty}\mathrm{ad}^{2k}_{\Lambda_{\mathfrak{h}}}\circ\mathrm{ad}_{\Lambda_{\mathfrak{m}}}(A_\omega) &= \mathrm{ad}_{\Lambda_{\mathfrak{m}}}(A_\omega) + \sum_{k=1}^{\infty}\mathrm{ad}^{2k}_{\Lambda_{\mathfrak{h}}}\circ\mathrm{ad}_{\Lambda_{\mathfrak{m}}}(A_\omega) = \\ &= \frac{i}{2}x^i A^j_\omega \varepsilon_{ij}{}^k M_k - \frac{i}{2(4-y^2)}\big[(y\cdot x)A^i_\omega - (y\cdot A_\omega)x^i\big]y^j\varepsilon_{ij}{}^k M_k = \\ &= \frac{i}{2(4-y^2)}\big[(y\cdot A_\omega)x^i y^j - (y\cdot x)A^i_\omega y^j - (4-y^2)A^i_\omega x^j\big]\varepsilon_{ij}{}^k M_k \\ \sum_{k=0}^{\infty}\mathrm{ad}^{2k+1}_{\Lambda_{\mathfrak{h}}}\circ\mathrm{ad}_{\Lambda_{\mathfrak{m}}}(A_\omega) &= \frac{1}{4-y^2}\big[(y\cdot x)A^k - (y\cdot A)x^k\big]M_k\ .\end{aligned}\tag{F.36}$$

Acting once more with $\mathrm{ad}_{\Lambda_{\mathfrak{m}}}$ and rearranging, the expressions (4.133) are obtained. One can then take the sum and difference of $W_k{}^l$ and $Z_k{}^l$ and construct an ansatz for the their inverses

$$\begin{aligned}[(W\pm Z)^{-1}]_k{}^l = a_1^\pm\delta_k{}^l + x_k(a_2^\pm x^l + a_3^\pm y^l) + y_k(a_4^\pm x^l + a_5^\pm y^l) + \varepsilon_{ak}{}^l(a_6^\pm x^a + a_7^\pm y^a) + \\ + x^a y^b\varepsilon_{ab}{}^l(a_8^\pm x^k + a_9^\pm y^k) + x^a y^b\varepsilon_{abk}(a_{10}^\pm x^l + a_{11}^\pm y^l) + a_{12}^\pm x^a y^b\varepsilon_{abk}x^c y^d\varepsilon_{cd}{}^l\ .\end{aligned}\tag{F.37}$$

This leads to a set of conditions for the coefficients of the ansatz which can be solved in both cases and, defining $z := x\cdot y$ for shortness, take the explicit form

$$\begin{aligned}
a_1^+ &= -\frac{4(y^2-4)[z^2-x^2(y^2-4)]}{(x^4-4y^2)(y^2-4)^2-2z^2[x^2(y^2-2)-4(y^2-4)]+z^4} = -a_1^- \\
a_2^+ &= \frac{x^6(y^2-4)^2-2z^2[2y^2(y^2-4)-4x^2(y^2-4)+x^4(y^2-2)]+x^2z^4}{(x^2y^2-z^2)[z^4-2z^2\{x^2(y^2-2)-4(y^2-4)\}+(x^4-4y^2)(y^2-4)^2]} = -a_2^- \\
a_3^+ &= \frac{z^5-2z^3[8-2y^2+x^2(y^2-2)]+z(4x^2+x^4-4y^2)(y^2-4)^2}{(x^2y^2-z^2)[z^4-2z^2\{x^2(y^2-2)-4(y^2-4)\}+(x^4-4y^2)(y^2-4)^2]} = -a_3^- \\
a_5^+ &= -\frac{(16+x^4-4y^2)(y^3-4y)^2-2z^2(y^2-4)[16+y^2(x^2-4)]+z^4(y^2-4)}{(x^2y^2-z^2)[z^4-2z^2\{x^2(y^2-2)-4(y^2-4)\}+(x^4-4y^2)(y^2-4)^2]} = -a_5^- \\
a_6^+ &= \frac{8iz(y^2-4)}{(x^4-4y^2)(y^2-4)^2-2z^2[x^2(y^2-2)-4(y^2-4)]+z^4} = a_6^- \\
a_7^+ &= -\frac{8i(y^2-4)^2}{(x^4-4y^2)(y^2-4)^2-2z^2[x^2(y^2-2)-4(y^2-4)]+z^4} = a_7^- \\
a_8^+ &= -\frac{2i\{x^4(y^2-4)^2-2z^2[8-2y^2+x^2(y^2-2)]+z^4\}}{(x^2y^2-z^2)[z^4-2z^2\{x^2(y^2-2)-4(y^2-4)\}+(x^4-4y^2)(y^2-4)^2]} = a_8^- \\
a_9^+ &= \frac{8iz(y^2-4)(y^2-x^2-4)}{(x^2y^2-z^2)[z^4-2z^2\{x^2(y^2-2)-4(y^2-4)\}+(x^4-4y^2)(y^2-4)^2]} = a_9^- \\
a_{12}^+ &= \frac{4(y^2-4)[z^2-x^2(y^2-4)]}{(x^2y^2-z^2)[z^4-2z^2\{x^2(y^2-2)-4(y^2-4)\}+(x^4-4y^2)(y^2-4)^2]} = -a_{12}^-\ ,
\end{aligned}\tag{F.38}$$

with $a_4^+ = a_3^+ = -a_4^- \quad a_{10}^+ = -a_8^+ = a_{10}^- \quad a_{11}^+ = -a_9^+ = a_{11}^-$.

OSp(1|2)/SO(1,1). For this semi-symmetric coset, upon dualising the whole group of isometry and setting $D=0$, the operators W and Z from (4.122) reduce to

$$W := \mathrm{ad}_{\Lambda_\mathfrak{h}} + N + (ad_{\Lambda_\mathfrak{m}} - M^\dagger)\circ \sum_{k=0}^{\infty} S^{2k+1}\circ(\mathrm{ad}_{\Lambda_\mathfrak{m}} + M) \qquad Z := (\mathrm{ad}_{\Lambda_\mathfrak{m}} - M^\dagger)\circ \sum_{k=0}^{\infty} S^{2k}\circ(ad_{\Lambda_\mathfrak{m}} + M)\ .$$

Recall then the expansion of the gauge field and multipliers

$$A_\omega = AL_{+-} \qquad \Lambda = \Lambda_\mathfrak{h} + \Lambda_\mathfrak{p} + \Lambda_\mathfrak{m} + \Lambda_\mathfrak{q} = yL_{+-} + \theta^+ Q_+ + x^{++}L_{++} + x^{--}L_{--} + \theta^- Q_-\ , \tag{F.39}$$

as well as the $\mathfrak{osp}(1|2)$ algebra in lightcone notation

$$\begin{aligned} &\{Q_\pm, Q_\pm\} = L_{\pm\pm} && \{Q_+, Q_-\} = L_{+-} && [L_{\pm\pm}, Q_\mp] = \mp i Q_\pm \\ &[L_{+-}, L_{\pm\pm}] = \pm i L_{\pm\pm} && [L_{++}, L_{--}] = -2iL_{+-} && [L_{+-}, Q_\pm] = \pm \tfrac{i}{2} Q_\pm\ . \end{aligned} \tag{F.40}$$

We start computing the action of R_{ij} defined in (4.108) on the fermionic subspaces. These act as

$$R_{11} : \mathfrak{p} \to \mathfrak{p} \qquad R_{12} : \mathfrak{q} \to \mathfrak{p} \qquad R_{21} : \mathfrak{p} \to \mathfrak{q} \qquad R_{22} : \mathfrak{q} \to \mathfrak{q} \tag{F.41}$$

and are nothing but combined actions of $\mathrm{ad}_{\Lambda_\mathfrak{m}}$ and $c_\pm$, defined in (4.109). We can thus exploit the above algebra to notice that

$$\mathrm{ad}_{\Lambda_\mathfrak{h}}(Q_\pm) = \pm\tfrac{i}{2} y(Q_\pm) \qquad \mathrm{ad}_{\Lambda_\mathfrak{m}}(Q_\pm) = \pm i x^{\mp\mp}(Q_\mp)\ , \tag{F.42}$$

and use these to extract the eigenvalues of $c_\pm$ as

$$c_+(Q_\pm) = 2\sum_{k=0}^{\infty}(-2)^k \mathrm{ad}^k_{\Lambda_\mathfrak{h}}(Q_\pm) = 2\sum_{k=0}^{\infty}(-2)^k(\pm\tfrac{i}{2}y)^k(Q_\pm) = \frac{2}{1\pm iy}(Q_\pm) =: a_\pm(Q_\pm) \tag{F.43}$$

and similarly for $c_-(Q_\pm) = a_\mp(Q_\pm)$. We can now proceed with R_{ij}. For example, noting that $-c_+\circ \mathrm{ad}_{\Lambda_\mathfrak{m}} \circ c_- \circ \mathrm{ad}_{\Lambda_\mathfrak{m}}(Q_+) = -x^{++}x^{--}a_+^2(Q_+)$, one can easily extract

$$R_{11}(Q_+) = -\sum_{k=0}^{\infty}(-c_+\circ \mathrm{ad}_{\Lambda_\mathfrak{m}}\circ c_-\circ \mathrm{ad}_{\Lambda_\mathfrak{m}})^k \circ c_+(Q_+) = -a_+\sum_{k=0}^{\infty}(-x^{++}x^{--}a_+^2)^k(Q_+) =: r_{11}(Q_+)$$

and after repeating the same for the other operators we find

$$R_{11}(Q_+) = r_{11}(Q_+) \quad R_{12}(Q_-) = r_{12}(Q_+) \quad R_{21}(Q_+) = r_{21}(Q_-) \quad R_{22}(Q_-) = r_{22}(Q_-) \tag{F.44}$$

where we defined

$$r_{11} := \frac{-a_+}{1 + x^{++}x^{--}a_+^2} = -r_{22} \qquad r_{12} := \frac{ix^{++}a_+^2}{1 + x^{++}x^{--}a_+^2} \qquad r_{21} := \frac{-ix^{--}a_+^2}{1 + x^{++}x^{--}a_+^2} \ . \tag{F.45}$$

We can now proceed in computing the action of the various operators involved in W and Z on the gauge field $A_\omega := AL_{+-}$. First of all we notice that

$$\mathrm{ad}_{\Lambda_\mathfrak{h}}(A_\omega) = 0 \qquad \mathrm{ad}_{\Lambda_\mathfrak{m}}(A_\omega) = -iA(x^{++}L_{++} - x^{--}L_{--}) \ , \tag{F.46}$$

where the first relation follows from the fact that $\mathfrak{h}$ is Abelian. Then, to compute $N(A_\omega)$ and $M(A_\omega)$ we exploit the relations

$$\mathrm{ad}_{\Lambda_\mathfrak{p}}(A_\omega) = -\tfrac{i}{2}A\theta^+Q_+ \qquad \mathrm{ad}_{\Lambda_\mathfrak{q}}(A_\omega) = \tfrac{i}{2}A\theta^-Q_- \tag{F.47}$$

and nilpotency of the multipliers $(\theta^+)^2 = 0 = (\theta^-)^2$, which implies the vanishing of any term containing more than once the operators $\mathrm{ad}_{\Lambda_\mathfrak{p}}$ or $\mathrm{ad}_{\Lambda_\mathfrak{q}}$. This leads to

$$\begin{aligned} N(A_\omega) &= (\mathrm{ad}_{\Lambda_\mathfrak{q}} \circ \mathcal{O}_3 + \mathrm{ad}_{\Lambda_\mathfrak{p}} \circ \mathcal{O}_4)(A_\omega) = \\ &= \big\{\mathrm{ad}_{\Lambda_\mathfrak{q}} \circ (R_{12} \circ \mathrm{ad}_{\Lambda_\mathfrak{q}} + R_{11} \circ \mathrm{ad}_{\Lambda_\mathfrak{p}}) + \mathrm{ad}_{\Lambda_\mathfrak{p}} \circ (R_{22} \circ \mathrm{ad}_{\Lambda_\mathfrak{q}} + R_{21} \circ \mathrm{ad}_{\Lambda_\mathfrak{p}})\big\}(A_\omega) = \\ &= [\Lambda_\mathfrak{q}, -\tfrac{i}{2}r_{11}A\theta^+Q_+] + [\Lambda_\mathfrak{p}, \tfrac{i}{2}r_{22}A\theta^-Q_-] = \\ &= -\tfrac{i}{2}r_{11}A\theta^2L_{+-} - \tfrac{i}{2}r_{22}A\theta^2L_{+-} = 0 \ , \end{aligned} \tag{F.48}$$

where in the last line we defined $\theta^2 := \theta^+\theta^-$ and used $r_{22} = -r_{11}$ to cancel the two terms. Proceeding similarly for M we then find

$$M(A_\omega) = -\tfrac{i}{2}r_{12}A\theta^2L_{++} - \tfrac{i}{2}r_{21}A\theta^2L_{--} \tag{F.49}$$

and can thus start collecting the following terms

$$\begin{aligned} &(\mathrm{ad}_{\Lambda_\mathfrak{h}} + N)(A_\omega) = 0 \qquad (\mathrm{ad}_{\Lambda_\mathfrak{m}} + M)(A_\omega) := \lambda^{++}L_{++} + \lambda_{--}L_{--} =: \lambda \in \mathfrak{m} \\ &\text{with} \qquad \lambda^{++} := -ix^{++}\big[1 + \tfrac{r_{12}}{2x^{++}}\theta^2\big]A \qquad \lambda^{--} := ix^{--}\big[1 - \tfrac{r_{21}}{2x^{--}}\theta^2\big]A \ . \end{aligned} \tag{F.50}$$

To compute W, Z we need now to repeatedly act with $S := \mathrm{ad}_{\Lambda_\mathfrak{h}} + L$ defined in (4.123) on $\lambda \in \mathfrak{m}$

$$\begin{aligned} \mathrm{ad}_{\Lambda_\mathfrak{h}}(\lambda) &= iy\lambda^{++}L_{++} - iy\lambda^{--}L_{--} \\ L(\lambda) &= [\Lambda_\mathfrak{p}, ir_{11}\lambda^{++}\theta^2Q_+] + [\Lambda_\mathfrak{q}, -ir_{22}\lambda^{--}\theta^+Q_+] = -ir_{11}\lambda^{++}\theta^2L_{++} - ir_{22}\lambda^{--}\theta^2L_{--} \ , \end{aligned} \tag{F.51}$$

so that we find

$$S(\lambda) = \lambda^{++}S^{++}L_{++} + \lambda^{--}S^{--}L_{--} \qquad \text{with} \qquad \begin{cases} S^{++} := iy\big[1 - \frac{r_{11}}{y}\theta^2\big] \\ S^{--} := -iy\big[1 + \frac{r_{22}}{y}\theta^2\big] \end{cases} . \tag{F.52}$$

Recalling that $r_{22} = -r_{11}$ we see that $S^{--} = -S^{++}$ and we can thus easily compute

$$\begin{gathered} \sum_{k=0}^{\infty} S^{2k}(\lambda) = \lambda^{++}\sum_{k=0}^{\infty}(S^{++})^{2k}L_{++} + \lambda^{--}\sum_{k=0}^{\infty}(S^{--})^{2k}L_{--} = s_e(\lambda^{++}L_{++} + \lambda^{--}L_{--}) =: \lambda_e \in \mathfrak{m} \\ \sum_{k=0}^{\infty} S^{2k+1}(\lambda) = \lambda^{++}\sum_{k=0}^{\infty}(S^{++})^{2k+1}L_{++} + \lambda^{--}\sum_{k=0}^{\infty}(S^{--})^{2k+1}L_{--} = s_o(\lambda^{++}L_{++} - \lambda^{--}L_{--}) =: \lambda_o \in \mathfrak{m} \end{gathered} \tag{F.53}$$

where we defined

$$s_e := \sum_{k=0}^{\infty}(S^{++})^{2k} \qquad\qquad s_o := \sum_{k=0}^{\infty}(S^{++})^{2k+1} . \tag{F.54}$$

We shall later compute the latter sums, but for the moment we can directly proceed in computing the final ingredient, namely the action of $(\mathrm{ad}_{\Lambda_\mathfrak{m}} - M^\dagger)\lambda_x$ with $x = \{e, o\}$. We easily find

$$\begin{aligned} \mathrm{ad}_{\Lambda_\mathfrak{m}}(\lambda_x) &= 2i(x^{--}\lambda_x^{++} - x^{++}\lambda_x^{--})L_{+-} \\ M^\dagger(\lambda_x) &= (ir_{12}\lambda_x^{--} + ir_{21}\lambda_x^{++})\theta^2 L_{+-} , \end{aligned} \tag{F.55}$$

so that we can combine the two terms into

$$(\mathrm{ad}_{\Lambda_\mathfrak{m}} - M^\dagger)(\lambda_x) = 2i\big(x^{--}\big[1 - \tfrac{r_{21}}{2x^{--}}\theta^2\big]\lambda_x^{++} - x^{++}\big[1 + \tfrac{r_{12}}{2x^{++}}\theta^2\big]\lambda_x^{--}\big)L_{+-} \tag{F.56}$$

and rewrite the operators W, Z as

$$W(A_\omega) = (\mathrm{ad}_{\Lambda_\mathfrak{m}} - M^\dagger)(\lambda_o) \qquad\qquad Z(A_\omega) = (\mathrm{ad}_{\Lambda_\mathfrak{m}} - M^\dagger)(\lambda_e) . \tag{F.57}$$

Recalling the definitions (F.53) of λ_x and (F.50) of $\lambda^{\pm\pm}$ we then easily find

$$\begin{aligned} W(A_\omega) &= 2i\big(x^{--}\big[1 - \tfrac{r_{21}}{2x^{--}}\theta^2\big]\lambda_o^{++} - x^{++}\big[1 + \tfrac{r_{12}}{2x^{++}}\theta^2\big]\lambda_o^{--}\big)L_{+-} = \\ &= 2is_o\big(x^{--}\big[1 - \tfrac{r_{21}}{2x^{--}}\theta^2\big]\lambda^{++} - x^{++}\big[1 + \tfrac{r_{12}}{2x^{++}}\theta^2\big]\lambda^{--}\big)L_{+-} = \\ &= 2s_o x^{++}x^{--}\big(\big[1 - \tfrac{r_{21}}{2x^{--}}\theta^2\big]\big[1 + \tfrac{r_{12}}{2x^{++}}\theta^2\big] - \big[1 + \tfrac{r_{12}}{2x^{++}}\theta^2\big]\big[1 - \tfrac{r_{21}}{2x^{--}}\theta^2\big]\big)AL_{+-} = 0 \end{aligned} \tag{F.58}$$

and

$$
\begin{aligned}
Z(A_\omega) &= 2i\big(x^{--}\big[1-\tfrac{r_{21}}{2x^{--}}\theta^2\big]\lambda_e^{++} - x^{++}\big[1+\tfrac{r_{12}}{2x^{++}}\theta^2\big]\lambda_e^{--}\big)L_{+-} = \\
&= 2is_e\big(x^{--}\big[1-\tfrac{r_{21}}{2x^{--}}\theta^2\big]\lambda^{++} - x^{++}\big[1+\tfrac{r_{12}}{2x^{++}}\theta^2\big]\lambda^{--}\big)L_{+-} = \\
&= 2s_e x^{++}x^{--}\big(\big[1-\tfrac{r_{21}}{2x^{--}}\theta^2\big]\big[1+\tfrac{r_{12}}{2x^{++}}\theta^2\big] + \big[1+\tfrac{r_{12}}{2x^{++}}\theta^2\big]\big[1-\tfrac{r_{21}}{2x^{--}}\theta^2\big]\big)AL_{+-} = \\
&= 4x^{++}x^{--}s_e\big[1+\tfrac{r_{12}}{x^{++}}\theta^2\big]AL_{+-}\ .
\end{aligned}
\tag{F.59}
$$

Where in the last line of the computation for Z we exploited (F.45) to notice that $\frac{r_{12}}{x^{++}} = -\frac{r_{21}}{x^{--}}$ and combine the θ^2 prefactors. All we are now left with is the computation of $s_e := \sum_{k=0}^{\infty}(S^{++})^{2k}$. Recalling that $S^{++} := iy\big[1+\frac{r_{22}}{y}\theta^2\big]$ one can easily find

$$
(S^{++})^k = (iy)^k + k(iy)^{k-1} i r_{22}\theta^2\ , \tag{F.60}
$$

so that upon defining $z := iy$ it is immediate to obtain

$$
s_e := \sum_{k=0}^{\infty}\big(z^{2k} + 2k(z)^{2k-1} i r_{22}\theta^2\big) = \big(1 + i r_{22}\theta^2\partial_z\big)\sum_{k=0}^{\infty}(z^2)^k = \frac{1}{1+y}\big[1-\tfrac{2yr_{22}}{1+y^2}\theta^2\big]\ . \tag{F.61}
$$

Substituting the latter into $Z(A_\omega)$ and recalling the definitions (F.45) one finally recovers (4.138).

References

[1] D. Bielli, S. Penati, D. Sorokin, and M. Wolf, *Super Non-Abelian T-Duality*, Nucl. Phys. B **983** (2022) 115904 [2112.12168 [hep-th]].

[2] D. Bielli, *On Super Non-Abelian T-Duality of Symmetric and Semi-Symmetric Coset Sigma Models*, 2209.07336 [hep-th].

[3] K. Kikkawa and M. Yamasaki, *Casimir Effects In Superstring Theories*, Phys. Lett. B **149** (1984) 357.

[4] N. Sakai and I. Senda, *Vacuum Energies Of String Compactified On Torus*, Prog. Theor. Phys. **75** (1986) 692.

[5] B. Sathiapalan, *Duality In Statistical Mechanics And String Theory*, Phys. Rev. Lett. **58** (1987) 1597.

[6] T. H. Buscher, *A Symmetry Of The String Background Field Equations*, Phys. Lett. B **194** (1987) 59.

[7] T. H. Buscher, *Path Integral Derivation Of Quantum Duality In Nonlinear Sigma Models*, Phys. Lett. B **201** (1988) 466.

[8] M. Rocek and E. P. Verlinde, *Duality, Quotients, And Currents*, Nucl. Phys. B **373** (1992) 630 [hep-th/9110053].

[9] A. Giveon and M. Rocek, *Generalized Duality In Curved String Backgrounds*, Nucl. Phys. B **380** (1992) 128 [hep-th/9112070].

[10] X. C. de la Ossa and F. Quevedo, *Duality Symmetries From NonAbelian Isometries In String Theory*, Nucl. Phys. B **403** (1993) 377 [hep-th/9210021].

[11] C. R. Nappi, *Some Properties Of An Analog Of The Nonlinear σ Model*, Phys. Rev. D **21** (1980) 418.

[12] B. E. Fridling and A. Jevicki, *Dual Representations And Ultraviolet Divergences In Nonlinear σ Models*, Phys. Lett. B **134** (1984) 70.

[13] E. S. Fradkin and A. A. Tseytlin, *Quantum Equivalence Of Dual Field Theories*, Annals Phys. **162** (1985) 31.

[14] A. Giveon and M. Rocek, *On NonAbelian Duality*, Nucl. Phys. B **421** (1994) 173 [hep-th/9308154].

[15] E. Alvarez, L. Alvarez-Gaume, J. L. F. Barbon, and Y. Lozano, *Some Global Aspects Of Duality In String Theory*, Nucl. Phys. B **415** (1994) 71 [hep-th/9309039].

[16] K. Sfetsos, *Gauged WZW Models And NonAbelian Duality*, Phys. Rev. D **50** (1994) 2784 [hep-th/9402031].

[17] E. Alvarez, L. Alvarez-Gaume, and Y. Lozano, *On NonAbelian Duality*, Nucl. Phys. B **424** (1994) 155 [hep-th/9403155].

[18] A. Giveon, M. Porrati, and E. Rabinovici, *Target Space Duality In String Theory*, Phys. Rept. **244** (1994) 77 [hep-th/9401139].

[19] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer, and G. Veneziano, *Remarks On NonAbelian Duality*, Nucl. Phys. B **435** (1995) 147 [hep-th/9409011].

[20] E. Alvarez, L. Alvarez-Gaume, and Y. Lozano, *A Canonical Approach To Duality Transformations*, Phys. Lett. B **336** (1994) 183 [hep-th/9406206].

[21] T. Curtright and C. K. Zachos, *Currents, Charges, And Canonical Structure Of Pseudodual Chiral Models*, Phys. Rev. D **49** (1994) 5408 [hep-th/9401006].

[22] Y. Lozano, *NonAbelian Duality And Canonical Transformations*, Phys. Lett. B **355** (1995) 165 [hep-th/9503045].

[23] C. Klimcik and P. Severa, *Dual NonAbelian Duality And The Drinfeld Double*, Phys. Lett. B **351** (1995) 455 [hep-th/9502122].

[24] C. Klimcik, *Poisson-Lie T-Duality*, Nucl. Phys. B Proc. Suppl. **46** (1996) 116 [hep-th/9509095].

[25] S. Demulder, F. Hassler, and D. C. Thompson, *An Invitation To Poisson-Lie T-Duality in Double Field Theory And Its Applications*, PoS C **ORFU2018** (2019) 113 [1904.09992 [hep-th]].

[26] D. C. Thompson, *An Introduction To Generalised Dualities And Their Applications To Holography And Integrability*, PoS C **ORFU2018** (2019) 099 [1904.11561 [hep-th]].

[27] E. Bergshoeff, C. M. Hull, and T. Ortin, *Duality In The Type II Superstring Effective Action*, Nucl. Phys. B **451** (1995) 547 [hep-th/9504081].

[28] S. F. Hassan, *T-Duality, Space-Time Spinors And RR Fields In Curved Backgrounds*, Nucl. Phys. B **568** (2000) 145 [hep-th/9907152].

[29] M. Cvetic, H. Lu, C. N. Pope, and K. S. Stelle, *T-Duality In The Green-Schwarz Formalism, And The Massless / Massive IIA Duality Map*, Nucl. Phys. B **573** (2000) 149 [hep-th/9907202].

[30] B. Kulik and R. Roiban, *T-Duality Of The Green-Schwarz Superstring*, JHEP **09** (2002) 007 [hep-th/0012010].

[31] R. Benichou, G. Policastro, and J. Troost, *T-Duality In Ramond-Ramond Backgrounds*, Phys. Lett. B **661** (2008) 192 [0801.1785 [hep-th]].

[32] N. Berkovits and J. Maldacena, *Fermionic T-Duality, Dual Superconformal Symmetry, And The Amplitude/Wilson Loop Connection*, JHEP **09** (2008) 062 [0807.3196 [hep-th]].

[33] L. F. Alday and R. Roiban, *Scattering Amplitudes, Wilson Loops And The String/Gauge Theory Correspondence*, Phys. Rept. **468** (2008) 153 [0807.1889 [hep-th]].

[34] N. Beisert, R. Ricci, A. A. Tseytlin, and M. Wolf, *Dual Superconformal Symmetry from $AdS_5 \times S^5$ Superstring Integrability*, Phys. Rev. D **78** (2008) 126004 [0807.3228 [hep-th]].

[35] R. Ricci, A. A. Tseytlin, and M. Wolf, *On T-Duality And Integrability For Strings On AdS Backgrounds*, JHEP **12** (2007) 082 [0711.0707 [hep-th]].

[36] E. O Colgain, *Fermionic T-duality: A Snapshot Review*, Int. J. Mod. Phys. A **27** (2012) 1230032 [1210.5588 [hep-th]].

[37] I. Adam, A. Dekel, and Y. Oz, *On Integrable Backgrounds Self-Dual Under Fermionic T-Duality*, JHEP **04** (2009) 120 [0902.3805 [hep-th]].

[38] C.-g. Hao, B. Chen, and X.-c. Song, *On Fermionic T-duality Of Sigma Models On AdS Backgrounds*, JHEP **12** (2009) 051 [0909.5485 [hep-th]].

[39] A. Dekel and Y. Oz, *Self-Duality Of Green-Schwarz Sigma-Models*, JHEP **03** (2011) 117 [1101.0400 [hep-th]].

[40] M. C. Abbott, J. Murugan, S. Penati, A. Pittelli, D. Sorokin, P. Sundin, J. Tarrant, M. Wolf, and L. Wulff, *T-Duality Of Green-Schwarz Superstrings On $AdS_d \times S^d \times M^{10-2d}$*, JHEP **12** (2015) 104 [1509.07678 [hep-th]].

[41] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, C. Ratti, and A. Santambrogio, *From Correlators To Wilson Loops In Chern-Simons Matter Theories*, JHEP **06** (2011) 118 [1103.3675 [hep-th]].

[42] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, and A. Santambrogio, *Scattering Amplitudes/Wilson Loop Duality In ABJM Theory*, JHEP **01** (2012) 056 [1107.3139 [hep-th]].

[43] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, and A. Santambrogio, *Scattering In ABJ Theories*, JHEP **12** (2011) 073 [1110.0738 [hep-th]].

[44] B. Basso and A. V. Belitsky, *ABJM Flux-Tube And Scattering Amplitudes*, JHEP **09** (2019) 116 [1811.09839 [hep-th]].

[45] G. Arutyunov and S. Frolov, *Superstrings On $AdS_4 \times \mathbb{C}P^3$ As A Coset Sigma-Model*, JHEP **09** (2008) 129 [0806.4940 [hep-th]].

[46] B. Stefanski, jr, *Green-Schwarz Action For Type IIA Strings On $AdS_4 \times \mathbb{C}P^3$*, Nucl. Phys. B **808** (2009) 80 [0806.4948 [hep-th]].

[47] J. Gomis, D. Sorokin, and L. Wulff, *The Complete $AdS_4 \times \mathbb{C}P^3$ Superspace For The Type IIA Superstring And D-Branes*, JHEP **03** (2009) 015 [0811.1566 [hep-th]].

[48] D. Sorokin and L. Wulff, *Evidence For The Classical Integrability Of The Complete $AdS_4 \times \mathbb{C}P^3$ Superstring*, JHEP **11** (2010) 143 [1009.3498 [hep-th]].

[49] J. A. Minahan and K. Zarembo, *The Bethe Ansatz For Superconformal Chern-Simons*, JHEP **09** (2008) 040 [0806.3951 [hep-th]].

[50] D. Bak and S.-J. Rey, *Integrable Spin Chain In Superconformal Chern-Simons Theory*, JHEP **10** (2008) 053 [0807.2063 [hep-th]].

[51] P. A. Grassi, D. Sorokin, and L. Wulff, *Simplifying Superstring And D-Brane Actions In $AdS_4 \times \mathbb{C}P^3$ Superbackground*, JHEP **08** (2009) 060 [0903.5407 [hep-th]].

[52] I. Adam, A. Dekel, and Y. Oz, *On The Fermionic T-Duality Of The $AdS_4 \times \mathbb{C}P^3$ Sigma-Model*, JHEP **10** (2010) 110 [1008.0649 [hep-th]].

[53] I. Bakhmatov, *On $AdS_4 \times \mathbb{C}P^3$ T-Duality*, Nucl. Phys. B **847** (2011) 38 [1011.0985 [hep-th]].

[54] E. O. Colgáin and A. Pittelli, *A Requiem for $AdS_4 \times \mathbb{C}P^3$ Fermionic Self-T-Duality*, Phys. Rev. D **94** (2016) 106006 [1609.03254 [hep-th]].

[55] A. Eghbali and A. Rezaei-Aghdam, *Poisson-Lie T-dual Sigma Models On Supermanifolds*, JHEP **09** (2009) 094 [0901.1592 [hep-th]].

[56] A. Eghbali and A. Rezaei-Aghdam, *String Cosmology From Poisson-Lie T-Dual Sigma Models On Supermanifolds*, JHEP **01** (2012) 151 [1107.2041 [hep-th]].

[57] A. Eghbali and A. Rezaei-Aghdam, *Super Poisson-Lie Symmetry Of The $GL(1|1)$ WZNW Model And Worldsheet Boundary Conditions*, Nucl. Phys. B **866** (2013) 26 [1207.2304 [hep-th]].

[58] A. Eghbali and A. Rezaei-Aghdam, *WZW Models As Mutual Super Poisson-Lie T-dual Sigma Models*, JHEP **07** (2013) 134 [1303.4069 [hep-th]].

[59] A. Eghbali, *Solution Of The Equations Of Motion For A Super Non-Abelian Sigma Model In Curved Background By The Super Poisson-Lie T-Duality*, JHEP **02** (2015) 025 [1409.3933 [hep-th]].

[60] P. A. Grassi and A. Mezzalira, *Aspects Of Quantum Fermionic T-Duality*, JHEP **05** (2011) 019 [1101.5969 [hep-th]].

[61] K. Sfetsos and D. C. Thompson, *On Non-Abelian T-Dual Geometries With Ramond Fluxes*, Nucl. Phys. B **846** (2011) 21 [1012.1320 [hep-th]].

[62] Y. Lozano, E. O Colgain, K. Sfetsos, and D. C. Thompson, *Non-Abelian T-Duality, Ramond Fields And Coset Geometries*, JHEP **06** (2011) 106 [1104.5196 [hep-th]].

[63] Y. Lozano and C. Núñez, *Field Theory Aspects Of Non-Abelian T-duality And $\mathcal{N} = 2$ Linear Quivers*, JHEP **05** (2016) 107 [1603.04440 [hep-th]].

[64] Y. Lozano, N. T. Macpherson, J. Montero, and C. Nunez, *Three-Dimensional $\mathcal{N}=4$ Linear Quivers And Non-Abelian T-Duals*, JHEP **11** (2016) 133 [1609.09061 [hep-th]].

[65] Y. Lozano, C. Núñez, and S. Zacarias, *BMN Vacua, Superstars And Non-Abelian T-Duality*, JHEP **09** (2017) 008 [1703.00417 [hep-th]].

[66] G. Itsios, Y. Lozano, J. Montero, and C. Núñez, *The AdS_5 Non-Abelian T-Dual Of Klebanov-Witten As A $\mathcal{N}=1$ Linear Quiver From M5-Branes*, JHEP **09** (2017) 038 [1705.09661 [hep-th]].

[67] Y. Lozano, N. T. Macpherson, C. Núñez, and A. Ramirez, *AdS_3 Solutions In Massive IIA, Defect CFTs And T-Duality*, JHEP **12** (2019) 013 [1909.11669 [hep-th]].

[68] R. R. Metsaev and A. A. Tseytlin, *Type IIB Superstring Action In $AdS_5 \times S^5$ Background*, Nucl. Phys. B **533** (1998) 109 [hep-th/9805028].

[69] I. Bena, J. Polchinski, and R. Roiban, *Hidden Symmetries Of The $AdS_5 \times S^5$ Superstring*, Phys. Rev. D **69** (2004) 046002 [hep-th/0305116].

[70] F. Delduc, M. Magro, and B. Vicedo, *An Integrable Deformation Of The $AdS_5 \times S^5$ Superstring Action*, Phys. Rev. Lett. **112** (2014) 051601 [1309.5850 [hep-th]].

[71] T. J. Hollowood, J. L. Miramontes, and D. M. Schmidtt, *An Integrable Deformation Of The $AdS_5 \times S^5$ Superstring*, J. Phys. A **47** (2014) 495402 [1409.1538 [hep-th]].

[72] T. J. Hollowood, J. L. Miramontes, and D. M. Schmidtt, *Integrable Deformations Of Strings On Symmetric Spaces*, JHEP **11** (2014) 009 [1407.2840 [hep-th]].

[73] C. Klimcik, *Yang-Baxter Sigma Models And dS/AdS T-Duality*, JHEP **12** (2002) 051 [hep-th/0210095].

[74] K. Sfetsos, *Integrable Interpolations: From Exact CFTs To Non-Abelian T-Duals*, Nucl. Phys. B **880** (2014) 225 [1312.4560 [hep-th]].

[75] C. Klimcik, *η And λ Deformations As $\mathcal{E}$-Models*, Nucl. Phys. B **900** (2015) 259 [1508.05832 [hep-th]].

[76] G. Arutyunov, R. Borsato, and S. Frolov, *Puzzles Of η-Deformed $AdS_5 \times S^5$*, JHEP **12** (2015) 049 [1507.04239 [hep-th]].

[77] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban, and A. A. Tseytlin, *Scale Invariance Of The η-Deformed $AdS_5 \times S^5$ Superstring, T-Duality And Modified Type II Equations*, Nucl. Phys. B **903** (2016) 262 [1511.05795 [hep-th]].

[78] L. Wulff and A. A. Tseytlin, *Kappa-Symmetry Of Superstring Sigma Model And Generalized 10d Supergravity Equations*, JHEP **06** (2016) 174 [1605.04884 [hep-th]].

[79] R. Borsato and L. Wulff, *Target Space Supergeometry Of η And λ-Deformed Strings*, JHEP **10** (2016) 045 [1608.03570 [hep-th]].

[80] D. Osten and S. J. van Tongeren, *Abelian Yang–Baxter Deformations And TsT Transformations*, Nucl. Phys. B **915** (2017) 184 [1608.08504 [hep-th]].

[81] B. Hoare and A. A. Tseytlin, *Homogeneous Yang-Baxter Deformations As Non-Abelian Duals Of The AdS_5 σ-Model*, J. Phys. A **49** (2016) 494001 [1609.02550 [hep-th]].

[82] R. Borsato and L. Wulff, *Integrable Deformations Of T-Dual σ Models*, Phys. Rev. Lett. **117** (2016) 251602 [1609.09834 [hep-th]].

[83] R. Borsato and L. Wulff, *On Non-Abelian T-Duality And Deformations Of Supercoset String Sigma-Models*, JHEP **10** (2017) 024 [1706.10169 [hep-th]].

[84] R. Borsato and L. Wulff, *Non-Abelian T-Duality And Yang-Baxter Deformations Of Green-Schwarz Strings*, JHEP **08** (2018) 027 [1806.04083 [hep-th]].

[85] S. J. van Tongeren, *Unimodular Jordanian Deformations Of Integrable Superstrings*, SciPost Phys. **7** (2019) 011 [1904.08892 [hep-th]].

[86] R. Borsato, A. A. Tseytlin, and L. Wulff, *Supergravity Background Of λ-Deformed Model For $AdS_2 \times S^2$ Supercoset*, Nucl. Phys. B **905** (2016) 264 [1601.08192 [hep-th]].

[87] L. Astrakhantsev, I. Bakhmatov, and E. T. Musaev, *Non-Abelian Fermionic T-Duality In Supergravity*, JHEP **09** (2021) 135 [2101.08206 [hep-th]].

[88] L. Astrakhantsev, I. Bakhmatov, and E. T. Musaev, *Fermionic T-Duality Of Double Field Theory*, Phys. Rev. D **107** (2023) 066028 [2212.09312 [hep-th]].

[89] A. Ramirez, *AdS_2 Geometries And Non-Abelian T-Duality In Non-Compact Spaces*, JHEP **10** (2021) 020 [2106.09735 [hep-th]].

[90] M. Bugden. *A Tour Of T-Duality: Geometric And Topological Aspects Of T-Dualities*. PhD thesis, Australian Natl. U., Canberra 2018.

[91] E. Plauschinn, *T-Duality Revisited*, JHEP **01** (2014) 131 [1310.4194 [hep-th]].

[92] A. Torrielli, *Yangians, S-matrices And AdS/CFT*, J. Phys. A **44** (2011) 263001 [1104.2474 [hep-th]].

[93] L. Frappat, P. Sorba, and A. Sciarrino, *Dictionary On Lie Superalgebras*, hep-th/9607161.

[94] S. M. Kuzenko, U. Lindstrom, and G. Tartaglino-Mazzucchelli, *Off-Shell Supergravity-Matter Couplings In Three Dimensions*, JHEP **03** (2011) 120 [1101.4013 [hep-th]].

[95] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, *Three-Dimensional N=2 (AdS) Supergravity And Associated Supercurrents*, JHEP **12** (2011) 052 [1109.0496 [hep-th]].

[96] S. M. Kuzenko, U. Lindstrom, and G. Tartaglino-Mazzucchelli, *Three-Dimensional (p,q) AdS Superspaces And Matter Couplings*, JHEP **08** (2012) 024 [1205.4622 [hep-th]].

[97] S. M. Kuzenko, U. Lindstrom, M. Rocek, I. Sachs, and G. Tartaglino-Mazzucchelli, *Three-Dimensional $\mathcal{N} = 2$ Supergravity Theories: From Superspace To Components*, Phys. Rev. D **89** (2014) 085028 [1312.4267 [hep-th]].

[98] J. Wess and B. Zumino, *Superspace Formulation Of Supergravity*, Phys. Lett. B **66** (1977) 361.

[99] R. Grimm, J. Wess, and B. Zumino, *Consistency Checks On The Superspace Formulation Of Supergravity*, Phys. Lett. B **73** (1978) 415.

[100] R. Grimm, J. Wess, and B. Zumino, *A Complete Solution Of The Bianchi Identities In Superspace*, Nucl. Phys. B **152** (1979) 255.

[101] N. Dragon, *Torsion And Curvature In Extended Supergravity*, Z. Phys. C **2** (1979) 29.

[102] E. Bergshoeff, E. Sezgin, and P. K. Townsend, *Superstring Actions In D=3, 4, 6, 10 Curved Superspace*, Phys. Lett. B **169** (1986) 191.

[103] I. Adam, A. Dekel, L. Mazzucato, and Y. Oz, *Integrability Of Type II Superstrings On Ramond-Ramond Backgrounds In Various Dimensions*, JHEP **06** (2007) 085 [hep-th/0702083].

[104] A. Torrielli, *Lectures On Classical Integrability*, J. Phys. A **49** (2016) 323001 [1606.02946 [hep-th]].

[105] P. Dorey, G. Korchemsky, N. Nekrasov, V. Schomerus, D. Serban, and L. Cugliandolo, *Integrability: From Statistical Systems To Gauge Theory*, Oxford University Press, 2019.

[106] O. Babelon, D. Bernard, and M. Talon, *Introduction To Classical Integrable Systems*, Cambridge University Press, 2003.

[107] S. Driezen, *Modave Lectures on Classical Integrability in 2d Field Theories*, PoS M **odave2021** (2022) 002 [2112.14628 [hep-th]].

[108] M. Schottenloher, *A Mathematical Introduction To Conformal Field Theory*, Springer, 2008.

[109] G. G. A. Bäuerle and E. A. de Kerf, *Lie Algebras. Pt. 1: Finite And Infinite Dimensional Lie Algebras And Applications In Physics*, North-Holland, 1990.

[110] E. A. de Kerf, G. G. A. Bäuerle, and A. P. E. ten Kroode, *Lie Algebras. Pt. 2: Finite And Infinite Dimensional Lie Algebras And Applications In Physics*, North-Holland, 1997.

[111] M. Scheunert and R.-b. Zhang, *Cohomology Of Lie Superalgebras And Of Their Generalizations*, J. Math. Phys. **39** (1998) 5024 [q-alg/9701037].

[112] R. Catenacci, C. A. Cremonini, P. A. Grassi, and S. Noja, *Cohomology Of Lie Superalgebras: Forms, Integral Forms And Coset Superspaces*, 2012.05246 [hep-th].

[113] C. A. Cremonini and P. A. Grassi, *Cohomology Of Lie Superalgebras: Forms, Pseudoforms, And Integral Forms*, 2106.11786 [hep-th].

[114] R. L. Arnowitt, P. Nath, and B. Zumino, *Superfield Densities And Action Principle in Curved Superspace*, Phys. Lett. B **56** (1975) 81.

[115] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, *Progress Toward A Theory Of Supergravity*, Phys. Rev. D **13** (1976) 3214.

[116] S. Deser and B. Zumino, *Consistent Supergravity*, Phys. Lett. B **62** (1976) 335.

[117] P. Nath and R. L. Arnowitt, *Supergravity And Gauge Supersymmetry*, Phys. Lett. B **65** (1976) 73.

[118] V. V. Serganova, *Classification Of Real Simple Lie Superalgebras And Symmetric Superspaces*, Funct. Anal. Appl. **17** (1983) 200.

[119] K. Zarembo, *Strings On Semisymmetric Superspaces*, JHEP **05** (2010) 002 [1003.0465 [hep-th]].

[120] C. A. S. Young, *Non-Local Charges, Z(m) Gradings And Coset Space Actions*, Phys. Lett. B **632** (2006) 559 [hep-th/0503008].

[121] G. Arutyunov and S. Frolov, *Foundations Of The $AdS_5 \times S^5$ Superstring. Part I*, J. Phys. A **42** (2009) 254003 [0901.4937 [hep-th]].

[122] I. N. McArthur, *Kappa Symmetry Of Green-Schwarz Actions In Coset Superspaces*, Nucl. Phys. B **573** (2000) 811 [hep-th/9908045].

[123] H. L. Verlinde, *Superstrings On AdS_2 And Superconformal Matrix Quantum Mechanics*, hep-th/0403024.

[124] I. Bars and K. Sfetsos, *Global Analysis Of New Gravitational Singularities In String And Particle Theories*, Phys. Rev. D **46** (1992) 4495 [hep-th/9205037].

[125] I. Bars and K. Sfetsos, *Conformally Exact Metric And Dilaton In String Theory On Curved Space-Time*, Phys. Rev. D **46** (1992) 4510 [hep-th/9206006].

[126] C. M. Hull and B. J. Spence, *The Gauged Nonlinear σ Model With Wess-Zumino Term*, Phys. Lett. B **232** (1989) 204.

[127] A. Fontanella and L. Romano, *Lie Algebra Expansion and Integrability in Superstring Sigma-Models*, JHEP **07** (2020) 083 [2005.01736 [hep-th]].

[128] A. Fontanella and J. M. Nieto García, *Extending the nonrelativistic string AdS coset*, Phys. Rev. D **106** (2022) L121901 [2208.02295 [hep-th]].

[129] S. M. Kuzenko and I. L. Buchbinder, *Ideas And Methods Of Supersymmetry And Supergravity Or A Walk Through Superspace*, Taylor & Francis, 1998.

[130] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, hep-th/0108200.

[131] J. Wess and J. Bagger, *Supersymmetry And Supergravity*, Princeton University Press, 1992.

[132] P. S. Howe, *Supergravity In Superspace*, Nucl. Phys. B **199** (1982) 309.

[133] V. G. Kac, *Lie Superalgebras*, Adv. Math. **26** (1977) 8.

[134] E. I. Buchbinder, J. Hutomo, S. M. Kuzenko, and G. Tartaglino-Mazzucchelli, *Two-Form Supergravity, Superstring Coupling And Goldstino Superfields In Three Dimensions*, Phys. Rev. D **96** (2017) 126015 [1710.00554 [hep-th]].

[135] R. Kallosh, J. Rahmfeld, and A. Rajaraman, *Near Horizon Superspace*, JHEP **09** (1998) 002 [hep-th/9805217].

[136] P. Claus and R. Kallosh, *Superisometries Of The AdS$\times$S Superspace*, JHEP **03** (1999) 014 [hep-th/9812087].