A DGLAP based second order *x* evolution equation of quarks and gluon distribution at small *x* and their solutions

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The coupled DGLAP equations at small *x* is Taylor approximated upto the second order O(x^2) and are solved analytically. Assuming a plausible relation between quark and gluon distribution we demonstrate that the two different sets of quark and gluon distributions are possible from the two coupled equations not reported earlier. Using the proper matching condition, we then obtain the range of (x, Q^2) where they will be identical.

1. Introduction

This paper reports the solution of Taylor approximated $O(x^2)$ DGLAP [1, 2, 3, 4] equations at small x using the Method of separation of Variables [5]. We demonstrate that such DGLAP equations are of elliptic nature under plausible assumptions relating quark and gluon distributions and the coupled version of them have got two alternate solutions for quark and gluon distributions a feature not noticed earlier[6]. It gives a very restrictive range of x and Q^2 for this validity, which we study numerically as well. Taylor approximated upto $O(x^2)$ form of the coupled DGLAP equations [7] are:

$$t\frac{\partial}{\partial t}q(x,t) = \frac{\alpha_s(t)}{2\pi}[J_1(x) + J_4(x)C(t)]q(x,t) + \frac{\alpha_s(t)}{2\pi}x[J_2(x) + J_5(x)C(t)]\frac{\partial}{\partial x}q(x,t) + \frac{\alpha_s(t)}{2\pi}x^2[J_3(x) + J_6(x)C(t)]\frac{\partial^2}{\partial x^2}q(x,t), \quad (1)$$

$$t\frac{\partial}{\partial t}(C(t)q(x,t)) = \frac{\alpha_s(t)}{2\pi}[J_7(x) + J_{10}(x)C(t)]q(x,t) + \frac{\alpha_s(t)}{2\pi}x[J_8(x) + J_{11}(x)C(t)]\frac{\partial}{\partial x}q(x,t) + \frac{\alpha_s(t)}{2\pi}x^2[J_9(x) + J_{12}(x)C(t)]\frac{\partial^2}{\partial x^2}q(x,t). \quad (2)$$

Where J_i's (i = 1, 2, ...12) are explicit calculable functions of x. We assumed the relation between quark and gluon distribution [8, 9] given by,

$$xg(x,t) = C(t)x\sum_i \left\{q_i(x,t) + \overline{q_i}(x,t)\right\}. \quad (3)$$

Neglecting the flavor and antiquark degrees of freedom equation (3) is can be simplified to

$$xg(x,t) = C(t)xq(x,t), \quad (4)$$

where $C(t)$ is a 't' dependent function.

1.1 Solution $q^I(x,t)$ from quark distribution equation

In standard QCD quark and gluon distributions, are in general not factorizable in x and t. To fecilitate our solution analytically, we however use the method of separation of variables[5]. We assume the solution of equation (1) to be $q^I(x,t) = X(x)T(t)$. Substituting this in equation(1) we get the solution for $T(t)$ as:

$$T(t) = \exp[-k^2\log t + D], \quad (5)$$

where $-k^2$ is separation constant and D is an integration constant. To obtain the solution for $X(x)$ we define,

$$A(x,t) = \frac{\alpha_s(t)}{2\pi} x^2 [J_3(x) + J_6(x)C(t)],$$
$$B(x,t) = \frac{\alpha_s(t)}{2\pi} x [J_2(x) + J_5(x)C(t)],$$
$$C(x,t) = \frac{\alpha_s(t)}{2\pi} [J_1(x) + J_4(x)C(t)] + k^2. \tag{6}$$

Assuming them to be nearly x and t independent such that, $A(x,t) \simeq A,\ B(x,t) \simeq B,\ C(x,t) \simeq C$. One gets the solution for three separate cases depending on the sign of the discriminant $(B^2 - 4AC)$. To that end we also need to define: $m_1 = \frac{-B+\sqrt{B^2-4.A.C}}{2.A},\ \ m_2 = \frac{-B-\sqrt{B^2-4.A.C}}{2.A}$.

CASE I: $B^2 - 4AC > 0$ (Hyperbolic equation): If m_1 and m_2 are real and unequal i.e $m_1 \neq m_2$, then the solution $q^I(x,t)$ for quark distribution of equation (1) is

$$q^I(x,t) = \exp[-k^2 \log t + D].\,[c_1 e^{m_1.x} + c_2 e^{m_2.x}]\,, \tag{7}$$

where c_1 and c_2 are arbitrary constants.

CASE II: $B^2 - 4AC = 0$ (Parabolic equation): Here m_1 and m_2 are equal ($\sim m$), then the quark distribution solution $q^I(x,t)$ for equation (1) is

$$q^I(x,t) = \exp[-k^2 \log t + D].\,[c_1 e^{m.x} + c_2 x e^{m.x}]\,. \tag{8}$$

CASE III: $B^2 - 4AC < 0$ (Elliptic equation): If $m_1 = \alpha + i\beta$, and $m_2 = \alpha - i\beta$ are complex numbers, where α and β are respectively $\frac{-B}{2A}$ and $\frac{\sqrt{4AC-B^2}}{2A}$, then the solution $q^I(x,t)$ of equation (1) would be

$$q^I(x,t) = \exp[-k^2 \log t + D].\,[e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)]\,. \tag{9}$$

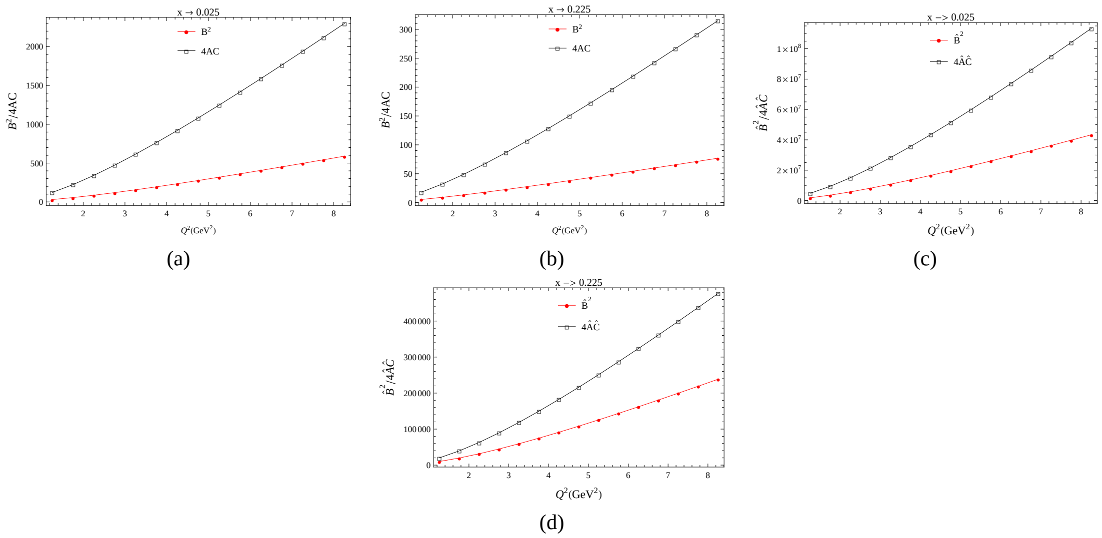

Figure 1: Plots (a), (b) are for $B^2/4AC$ versus Q^2 and (c), (d) are for $\hat{B}^2/4\hat{A}\hat{C}$ versus Q^2 at different fixed x.

1.2 Solution $q^{II}(x,t)$ from gluon distribution equation

For equation (2) we write the solution as $q^{II}(x,t)) = \hat{X}(x)\hat{T}(t)$. The analysis yields again three plausible solutions as follows:

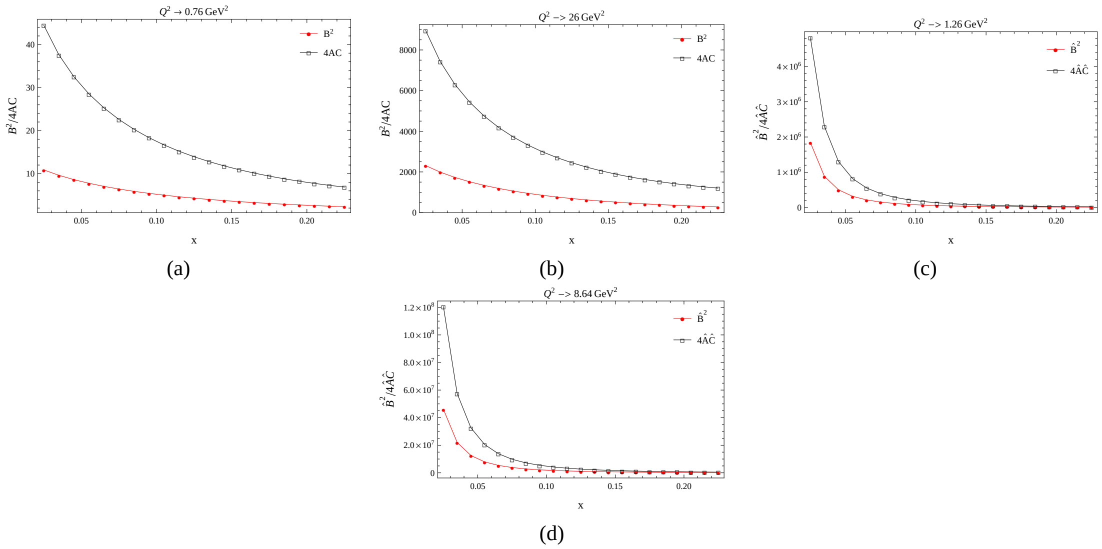

Figure 2: Plots (a), (b) are for $B^2/4AC$ versus x and (c), (d) are for $\hat{B}^2/4\hat{A}\hat{C}$ versus x at different fixed Q^2.

CASE I: $\hat{B}^2 - 4\hat{A}\hat{C} > 0$ (Hyperbolic equation):

$$q^{II}(x,t) = \exp\left[\int \left(\frac{-\hat{k}^2}{tC(t)} - \frac{1}{C(t)}\frac{\partial C(t)}{\partial t}\right) dt + F\right] . \left[\hat{c}_1 e^{\frac{-\hat{B}+\sqrt{\hat{B}^2-4\hat{A}\hat{C}}}{2\hat{A}}.x} + \hat{c}_2 e^{\frac{-\hat{B}-\sqrt{\hat{B}^2-4\hat{A}\hat{C}}}{2\hat{A}}.x}\right], \tag{10}$$

where, $\hat{A}, \hat{B}, \hat{C}$ are the corresponding parameters similar to equation (6). $-\hat{k}^2$ is separation constant and $\hat{c}_1$, $\hat{c}_2$ are arbitrary constants.

CASE II: $\hat{B}^2 - 4\hat{A}\hat{C} = 0$ (Parabolic equation):

$$q^{II}(x,t) = \exp\left[\int \left(\frac{-\hat{k}^2}{tC(t)} - \frac{1}{C(t)}\frac{\partial C(t)}{\partial t}\right) dt + F\right] . [\hat{c}_1 e^{m.x} + \hat{c}_2 x e^{m.x}] . \tag{11}$$

CASE III: $\hat{B}^2 - 4\hat{A}\hat{C} < 0$ (Elliptic equation):

$$q^{II}(x,t) = \exp\left[\int \left(\frac{-\hat{k}^2}{tC(t)} - \frac{1}{C(t)}\frac{\partial C(t)}{\partial t}\right) dt + F\right] . \left[e^{\hat{\alpha}x}(\hat{c}_1 \cos\hat{\beta}x + \hat{c}_2 \sin\hat{\beta}x)\right] . \tag{12}$$

From equations (7), (8), (9) and equations (10), (11), (12) we establish that $q^I(x,t) \neq q^{II}(x,t)$. For completeness we also record the solution of DGLAP equations taking only O(x) terms from equation (1) and (2). Here too we obtain two non unique solutions $q^I(x,t)$ and $q^{II}(x,t)$ using the same method as follows:

$$q^I(x,t) = \exp[-k^2 \log t + D] . \exp[\frac{-C}{B}x + c_1], \tag{13}$$

$$q^{II}(x,t) = \exp\left[\int \left(\frac{-\hat{k}^2}{tC(t)} - \frac{1}{C(t)}\frac{\partial C(t)}{\partial t}\right) dt + F\right] . \exp[\frac{-\hat{C}}{\hat{B}}x + \hat{c}_1], \tag{14}$$

where $B, C, \hat{B}, \hat{C}$ are the counterterms as in $O(x^2)$ and c_1, $\hat{c}_1$ are constants of integration.

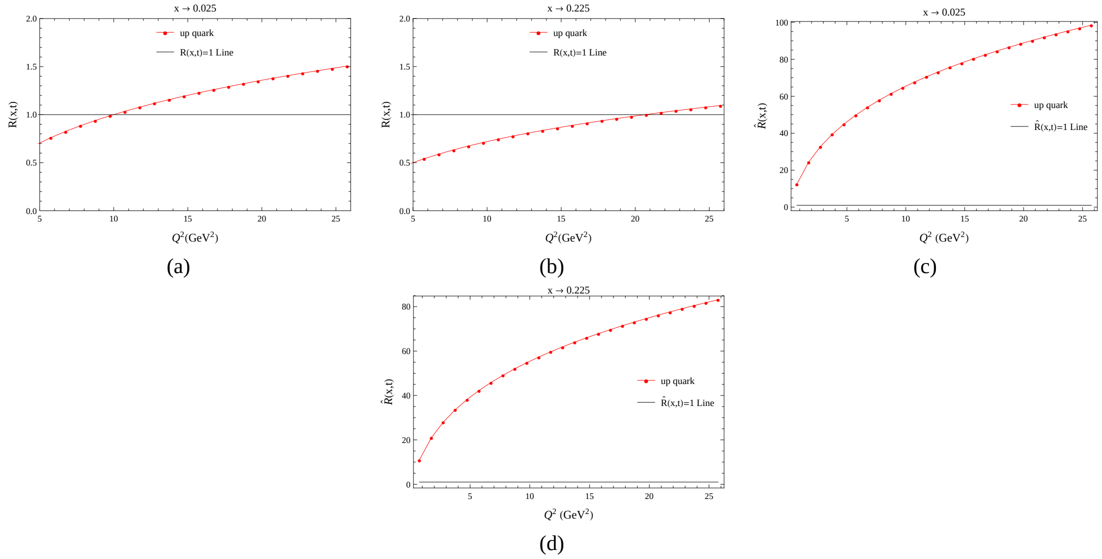

Figure 3: Plots (a), (b) are for $R(x,t)$ versus Q^2 and (c), (d) are for $\hat{R}(x,t)$ versus Q^2 at certain fixed x.

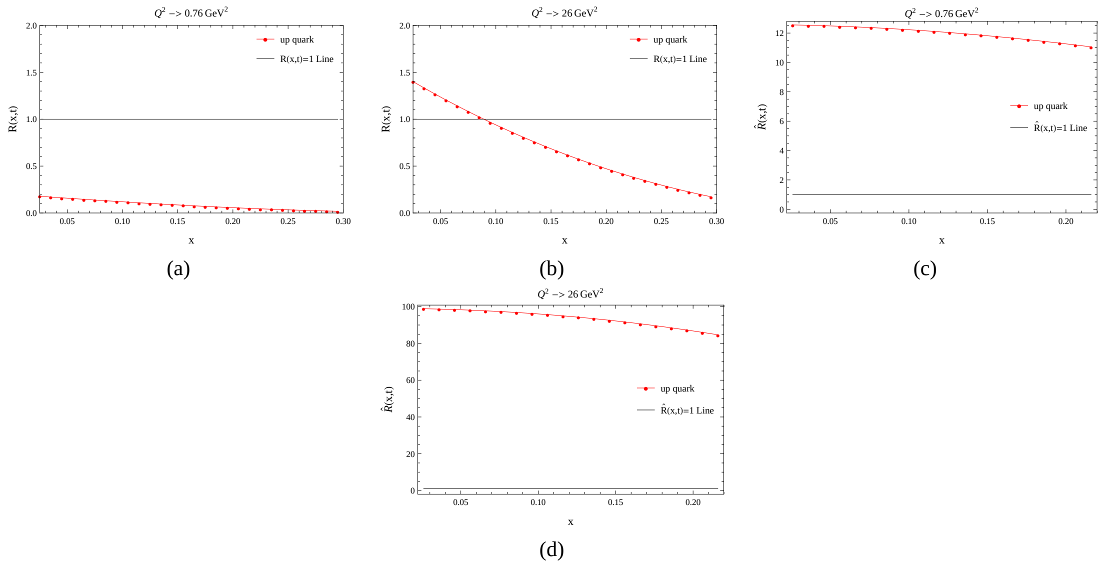

Figure 4: Plots (a), (b) are for $R(x,t)$ versus x and (c), (d) are for $\hat{R}(x,t)$ versus x at certain fixed Q^2.

2. Results

2.1 Nature of the equation

To ascertain the nature of the above equations, we numerically determine $(B^2 - 4AC)$ and $(\hat{B}^2 - 4\hat{A}\hat{C})$ by taking $\frac{\alpha_s(t)}{2\pi} = \frac{6}{(33-2N_f)}\frac{1}{t}$, $N_f = 1$, $C(t) = \log(\frac{Q^2}{\Lambda^2})^\sigma, \sigma = 2.5$ [10], $\Lambda = 220$ MeV [11] and approximating $k^2 = \hat{k}^2 = 0$. Then we see graphically in which class they belong. Fig. 1and Fig. 2 show $B^2 < 4AC$ and $\hat{B}^2 < 4\hat{A}\hat{C}$ indicating elliptic nature. Fig. 1 shows that such nature is always true for any Q^2, while Fig. 2 indicates that for large x, it might have a tendency to transform into a parabolic nature.For graphical representation of $q^I(x,t) \neq q^{II}(x,t)$ we approximate $D = 0, c_1 + c_2 = U$ (say) for equation (9) and $F = 0, \hat{c_1} + \hat{c_2} = \hat{U}$ (say) for equation (12). U and $\hat{U}$ are found to be 7.9×10 and 4597 respectively, taking the input of MSTW2008 LO [12] for up quarks with the value of $Q^2 = 2$ GeV^2 and $x = 0.005$ and $q(x,t) = 78.44$ [12]. We obtain the ratio $R(x,t) = \frac{q^I(x,t)}{q^{II}(x,t)}$ numerically. Correspondingly we obtain the counterterm $\hat{R}(x,t) = \frac{q^I(x,t)}{q^{II}(x,t)}$ numerically as well, taking similar approximations as in O(x^2) for $O(x)$ and assuming $c_1 = 0$ and $\hat{c}_1 = 0$. We show it graphically $R(x,t)$ and $\hat{R}(x,t)$ as in Fig. 3 and Fig. 4 for a certain range of (x, Q^2) i.e, $0.025 \leq x \leq 0.225$ and $0.76 \leq Q^2 \leq 26$ GeV^2. From both the figures we observe that for various x and Q^2, the two quark distributions are identical only at certain x and Q^2. However it is not obvious that one will obtain such intersecting point for any value of x and Q^2. Fig. 3 and Fig. 4 illustrate that there is no allowed range of x and Q^2 that will coincide.

3. Conclusion

We highlight the Elliptic nature of leading order $O(x^2)$ DGLAP equations at small *x* neglecting the flavor and antiquark degrees of freedom.The incorporation of flavor as well as antiquark degrees of freedom and testing with data are currently in progress.

Acknowledgment

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