Aspects of Trace Anomaly in Perturbation Theory and Beyond

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Abstract

In this thesis we study the connection between conformal symmetry breaking and the the renormalization group. In the first chapter we review the main properties of conformal field theories (CFTs), Wilsonian RG and describe how renormalization induces a flow between different CFTs. The prominent role is given to the trace of energy-momentum tensor (TEMT) as a measure for conformal symmetry violation. Scaling properties of supersymmetric gauge theories are also reviewed . In the second chapter the quantum action principle is introduced as a scheme for renormalizing composite operators. The framework is then applied to derive conditions for UV finiteness of two-point correlators of composite operators with special emphasis on TEMT. We then proceed to discuss the application of the Feynman-Hellmann theorem to evaluate gluon condensates. In the third chapter the basic elements the Trace anomaly on curved space are examined. The finiteness results from Chapter 2 are given physical meaning in relation with the RG flow of the geometrical quantity $\tilde{d}$ (coefficient of $\Box R$ in the anomaly). The last chapter is dedicated to the a-theorem. First we apply some of the results derived in Chapter 3 to extend the known perturbative calculation for the flow of the central charge β_a for gauge theories with Banks-Zaks fixed point. In the last part we review the main ideas of the recent proof of the a-theorem by Komargodski and Schwimmer and apply their formalism to re-derive the known non-perturbative formula for $\Delta\beta_a$ of SUSY conformal window theories.

Lay Summary

Our description of the world around us depends on the scale at which we observe it. Looking at the table we don't see the atoms that it is made of. As we zoom in we start seeing these atoms and their interactions. Particle physicists zoom even further inside the nucleus to reveal its constituents - quarks. In theoretical physics the process of zooming in (changing the typical distance scale) can be described mathematically using the so called renormalization. Theories which are invariant under renormalization have a special kind of symmetry called conformal. In this thesis we have investigated the connection between breaking of this conformal symmetry and renormalization which is quantified in terms of the conformal or trace anomaly.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Parts of this work have appeared in [1–4] .

(*Vladimír Procházka, September 2016*)

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List of Acronyms and Short-cuts

AF	Asymptotically Free
AS	Asymptotically Safe
CFT	Conformal Field Theory
DR	Dimensional Regularization
EMT	Energy-Momentum Tensor
EOM	Equation of Motion
IRFP	Infrared Fixed Point
LL	Leading Logarithm
LO	Leading Order
NLO	Next to Leading Order
OPE	Operator-Product Expansion
PT	Perturbation Theory
QAP	Quantum Action Principle
QCD	Quantum Chromodynamics
QFT	Quantum Field Theory
RG	Renormalization Group
RGE	Renormalization Group Equation
SUSY	Supersymmetry
SYM	Super Yang-Mills
TEMT	Trace of Energy-Momentum Tensor
UVFP	Ultraviolet Fixed Point
VEV	Vacuum Expectation Value
YM	Yang-Mills

Chapter 1

Introduction

The renormalization group remains to be one of the most useful and universal theoretical frameworks in the modern physics. The connection between renormalization and conformal symmetry has been known since the advent of RG in 1970s [5, 6] and continues to be an active research field. On the one hand the high precision calculations of beta functions have allowed for the theoretical discoveries of weakly coupled conformal fixed points in both IR [7] and more recently also in the UV [8]. On the other hand the lattice QCD shed some light on the possibility of strongly coupled conformal gauge theories [9, 10].
Despite this progress there is still a lot of work to be done. For example it is not clear under what conditions (number of flavours etc.) can a strongly coupled gauge theory become conformal in IR (boundary of conformal window) or whether the standard model has a conformal UV completion. Resolving these questions could have profound phenomenological consequences similar to the impact of the discovery of asymptotic freedom on high-energy QCD.
Since the world we live in is not conformal (at the energies accessible to our accelerators), it is also important to understand how the conformal symmetry becomes broken and what are the consequences of its violation. The breaking of conformal symmetry is very efficiently quantified through the trace anomaly. It is the aim of this thesis to investigate the trace anomaly using a range of modern tools and techniques.

The thesis is structured as follows. In the rest of the present chapter we will introduce the basic concepts related to conformal symmetry and RG flows. In the second chapter the quantum action principle will be presented and we will demonstrate its application to define renormalized operators. We then continue the Chapter 2 by examining UV finiteness of 2-point correlators of composite operator and the chapter finishes with the discussion of gluon condensates. Chapter 3 deals with the trace anomaly on curved spaces and its relation to the fourth moment. Finally, in Chapter 4 we review the current status of the a-theorem and present our calculations of Δa for weakly coupled gauge theories and $\mathcal{N}=1$ SUSY in the conformal window.

1.1 Conformal Field Theories

1.1.1 Basics

One possible way of extending the Poincaré group is to consider the conformal symmetry. The conformal group consists of coordinate transformations that preserve the surface of an infinitesimal lightcone:

$$ds^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu = 0\ . \tag{1.1}$$

It clearly includes the Poincaré group as its subgroup and also additional transformations that rescale the distances $ds^2 \to \Omega^2 ds^2$. These are dilatations and special conformal transformations. The infinitesimal version of these read

$$\delta x^\mu = \lambda x^\mu \qquad \text{Dilatations} \tag{1.2}$$

$$\delta x^\mu = (x^2 f^\mu - 2x^\mu x^\nu f_\nu) \qquad \text{Special conformal transformations}\ , \tag{1.3}$$

which correspond to the following set of generators

$$D = i x^\mu \partial_\mu \qquad \text{Dilatation} \tag{1.4}$$

$$K_\mu = i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu) \qquad \text{Special conformal transformation}\ . \tag{1.5}$$

Together with generators of Poincaré group ($M_{\mu\nu}, P_\mu = i\partial_\mu$) they form the conformal Lie algebra isomorphic to that of $SO(4,2)$. The additional commutation relations are

$$\begin{aligned}
[D, K_\mu] &= iK_\mu \\
[D, P_\mu] &= -iP_\mu \\
[K_\mu, K_\nu] &= 0 \\
[P_\mu, K_\mu] &= 2i\eta_{\mu\nu}D - 2iM_{\mu\nu} \\
[K_\mu, M_{\rho\sigma}] &= i(\eta_{\mu\rho}K_\sigma - \eta_{\mu\sigma}K_\rho) \ .
\end{aligned} \tag{1.6}$$

Representations of this algebra are formed using radial quantization wherein the dilation operator D plays the role of the Hamiltonian with eigenvalues being the scaling dimensions and eigenvectors the corresponding operators

$$[D, O] = i\Delta O \ . \tag{1.7}$$

Operators then map to states through acting on the vacuum (state-operator correspondence). The lowest weight state/operator is called primary with momentum and special conformal generators acting as raising and lowering operators respectively by increasing/decreasing the scaling dimensions.
Unitarity imposes strict bounds on possible scaling dimensions of spin s states in CFT [11]. For example it can be shown that for spins $0, \frac{1}{2}, 1$ in $d = 4$ the following inequalities hold

$$\Delta_0 \geq 1 \tag{1.8}$$

$$\Delta_{\frac{1}{2}} \geq \frac{3}{2} \tag{1.9}$$

$$\Delta_1 \geq 3 \ . \tag{1.10}$$

1.1.2 The Energy Momentum Tensor

For scale invariant field theories (SFTs) which are invariant under dilations the following conserved Noether's current called *dilatation current* exists [12] :

$$j^\mu_D = x^\nu T^\mu_\nu + V^\mu \ , \tag{1.11}$$

where $T_{\mu\nu}$ is the conserved ($\partial^\mu T_{\mu\nu} = 0$) energy-momentum tensor and V^μ is the Virial current. The scale invariance implies :

$$T^\mu{}_\mu = -\partial_\mu V^\mu \ . \tag{1.12}$$

For full conformal invariance we also need the current corresponding special conformal transformations to be conserved:

$$K^{\mu\nu} = (x^2\eta^{\nu\alpha} - 2x^\nu x^\alpha)T^\mu_\alpha - 2x^\nu V^\mu + 2L^{\mu\nu} \ , \tag{1.13}$$

where the conservation of the above current ($\partial_\mu K^{\mu\nu} = 0$) together with (1.12) is satisfied if [13]

$$T^\mu_{\ \mu} = -\partial_\mu\partial_\nu L^{\mu\nu} \ . \tag{1.14}$$

for some operator $L_{\mu\nu}$.[1] In practice $\partial_\mu\partial_\nu L^{\mu\nu}$ can be cancelled by local term called *an improvement term* [2] to satisfy

$$\boxed{T^\mu_\mu = 0} \tag{1.15}$$

for conformal invariance. Throughout this paper we will use the terms *scale* and *conformal invariance* interchangeably. Whether the scale invariance is enough to guarantee conformal invariance in $d = 4$ is still an open question subject to ongoing research [14].

As an example we can consider free massless fermions with the following energy-momentum tensor

$$T_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi \ . \tag{1.16}$$

The trace T^μ_μ vanishes by virtue of fermion equation of motion $\gamma^\mu\partial_\mu\psi = 0$ so the theory is conformal.

The energy momentum of free massless scalar reads:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 \ . \tag{1.17}$$

Tracing this equation yields

$$T^\mu_\mu = -\partial^\mu\phi\partial_\mu\phi = -\frac{1}{2}\Box\phi^2 + \phi\Box\phi \ . \tag{1.18}$$

Note that the second term $\phi\Box\phi$ vanishes by e.o.m and we end up with

$$T^\mu_\mu = -\frac{1}{2}\Box\phi^2 \ , \tag{1.19}$$

[1] To prove (1.14) one also needs to assume symmetry of $T_{\mu\nu}$ which follows from the Lorentz invariance.

[2] We will describe how this is done later on in Section 3.1.1

which is of the form (1.14) with $L_{\mu\nu} = \frac{1}{2}\phi^2\eta_{\mu\nu}$ and therefore the theory is conformal.

The conformal symmetry can be broken *classically* for example by adding masses to particles. For example for fermions and scalars this introduces explicit symmetry breaking terms $T^\mu_\mu = m\bar{\psi}\psi$ and $T^\mu_\mu = m^2\phi^2$ respectively. In this thesis we will be mostly interested in *quantum* breaking of conformal symmetry induced by scale dependence of couplings. The next section is devoted to explaining basics of this process.

1.2 The Renormalization Group

1.2.1 The traditional approach

Correlation functions in QFT involve UV divergences. For a theory to be well defined these need to be removed by addition of local counterterms to the Lagrangian. These counterterms will correspond to new interactions or modify the existing ones. It is important to note that divergences in QFT are local (polynomials in external momenta) otherwise we would not be able to introduce corresponding local counterterm in the Lagrangian.

The systematic process of adding such counterterms to the Lagrangian to make the correlators finite is called *renormalization*. The procedure will necessarily introduce a scale dependence (running) of the couplings.

To see this let us consider an example of ϕ^4 with

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \ , \tag{1.20}$$

where m and λ are *finite* parameters of the theory. As it stands this theory contains some UV divergent diagrams. For example to deal with divergences in the 4-point function of ϕ one introduces the following counterterm

$$\mathcal{L} \to \mathcal{L} + \frac{1}{4!}\lambda_{ct}\phi^4 \ , \tag{1.21}$$

where λ_{ct} is designed to absorb the UV divergences of the 4-point function

$$\Gamma(p_1, ..., p_4, \lambda) + i\lambda_{ct} = \Gamma(p_1, ..., p_4, \lambda)_{ren} \equiv [\text{finite}] \ , \tag{1.22}$$

where $\Gamma(p_1, ..., p_4, \lambda) = i\lambda + \lambda^2 \times$ (divergent) contains UV divergent Feynman diagrams (in particular the "fish" diagram at order λ^2). [3] This means that λ_{ct} will be a function of coupling λ and an arbitrary mass scale that was introduced during the regularization. For example this could be the UV cutoff or μ^ϵ in dimensional regularization. To lowest order in perturbation theory we can write

$$\lambda_{ct} = c_1 \lambda^2 + O(\lambda^3) \; . \tag{1.23}$$

In more standard notation one writes

$$\lambda_{ct} = \lambda(Z_4 - 1) \; , \tag{1.24}$$

where

$$Z_4 = 1 + c_1 \lambda + O(\lambda^2) \; . \tag{1.25}$$

Similarly by requiring a finite two-point function introduces mass and kinetic counterterms

$$m^2(Z_m - 1)\phi^2, \quad (Z_2 - 1)(\partial\phi)^2 \; . \tag{1.26}$$

It is seen that the original Lagrangian now becomes equivalent

$$\mathcal{L}_0 = \frac{1}{2} Z_2 (\partial\phi)^2 + \frac{1}{2} Z_m m^2 \phi^2 + \frac{1}{4!} Z_4 \lambda \phi^4 \; . \tag{1.27}$$

To make things bit cleaner one defines a *bare field* with canonically normalised kinetic terms

$$\phi_0 = \sqrt{Z_2}\phi \; . \tag{1.28}$$

The *bare mass* and *bare coupling* are then introduced

$$m_0^2 = Z_2^{-1} Z_m m^2 \tag{1.29}$$

$$\lambda_0 = Z_2^{-2} Z_4 \lambda \tag{1.30}$$

so that the bare Lagrangian retains the original form

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2} m_0^2 \phi_0^2 + \frac{1}{4!} \lambda_0 \phi_0^4 \tag{1.31}$$

This *bare* Lagrangian now depends on UV divergent parameters as opposed to finite *renormalized* parameters (ϕ, m, λ). The Lagrangian itself is not an observable so this is not a problem as long as it yields finite physical quantities.

[3] At this stage the splitting in (1.22) is schematic. At higher orders λ_{ct} feeds back into $\Gamma(p_1, ..., p_4, \lambda + \lambda_{ct})$ and an iterative renormalization procedure has to be set up.

The bare Lagrangian cannot depend on the arbitrary scale μ. This means that we need to choose the μ dependence of renormalized quantities so that the bare ones are μ independent. By differentiting bare parameters with respect to μ we obtain *renormalization group equations* (RGEs):

$$\frac{d}{d\ln\mu}\lambda_0 = 0 \qquad \Longrightarrow \quad \frac{d}{d\ln\mu}\lambda = \beta_\lambda \ , \tag{1.32}$$

where

$$\beta_\lambda = \lambda \frac{d}{d\ln\mu}\ln(Z_2^2 Z_4^{-1}) \tag{1.33}$$

is called the *beta function.* Similarly one can introduce the *field anomalous dimension*

$$\gamma_\phi \equiv -\frac{1}{2}\frac{d}{d\ln\mu}\ln Z_2 \tag{1.34}$$

and *mass anomalous dimension*

$$\gamma_m \equiv -\frac{d\ln m}{d\ln\mu} = \frac{1}{2}\frac{d}{d\ln\mu}\ln(Z_2^{-1}Z_m) \ . \tag{1.35}$$

Since any correlator can be calculated in terms of (ϕ, m, λ) we see that $(\gamma_\phi, \gamma_m, \beta_\lambda)$ encode the scaling properties of the entire theory. This is the idea behind *Callan-Symanzik* equations which assert that bare correlators (correlators formed from bare fields) are independent of μ.

$$\frac{d}{d\ln\mu}\langle\phi_0(p_1)...\phi_0(p_n)\rangle = 0 \ . \tag{1.36}$$

Which then implies the following RGE by virtue of equations (1.33),(1.34) and (1.35)

$$\left(\frac{\partial}{\partial\ln\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} - \gamma_m\frac{\partial}{\partial\ln m} - n\gamma_\phi\right)\langle\phi(p_1)...\phi(p_n)\rangle = 0 \ . \tag{1.37}$$

This equation can be used to study behaviour of correlators under dilations (1.4) in momentum space [15]. A simple dimensional analysis shows that under rescaling $p_i \to e^{-t}p_i$

$$\left(\frac{\partial}{\partial\ln\mu} + \frac{\partial}{\partial\ln m} + \frac{\partial}{\partial t} + nd_\phi\right)\langle\phi(e^{-t}p_1)...\phi(e^{-t}p_n)\rangle = 0 \ . \tag{1.38}$$

Plugging this back to (1.37) we get

$$\left(\frac{\partial}{\partial t} + \beta_\lambda\frac{\partial}{\partial\lambda} - (1+\gamma_m)\frac{\partial}{\partial\ln m} - n(\gamma_\phi + d_\phi)\right)\langle\phi(e^{-t}p_1)...\phi(e^{-t}p_n)\rangle = 0 \ . \tag{1.39}$$

This equation is solved along characteristics defined by $\frac{d\lambda}{dt} = \beta_\lambda$ and $\frac{d\ln m}{dt} = -(1+\gamma_m)$

$$\langle\phi(e^{-t}p_1)...\phi(e^{-t}p_n)\rangle|_t = \langle\phi(p_1)...\phi(p_n)\rangle|_{t=0}\exp\left[nd_\phi t + n\int_0^t \gamma_\phi(\lambda(t'))dt'\right], \tag{1.40}$$

where the '$|_t$' subscript indicates that the solution is a function of the coupling $\lambda(t)$ and mass $m(t)$ evolving along the characteristics starting at $\lambda(t=0) \equiv \lambda(\mu)$, $m(t=0) \equiv m(\mu)$. Upon taking $p_i \to p_i e^t$ in the above solution one finds that

$$\langle\phi(e^{t}p_1)...\phi(e^{t}p_n)\rangle|_{t=0} = \langle\phi(p_1)...\phi(p_n)\rangle|_{t}\exp\left[-\left(nd_\phi t + n\int_0^t \gamma_\phi(\lambda(t'))dt'\right)\right]. \tag{1.41}$$

By studying (1.41) in the large t limit, some information about the UV behaviour of correlators can be revealed, which is something that will be exploited later on in this thesis.

Before we proceed we need to discuss the issue of *scheme dependence*. One may always choose a different $\lambda_{ct} \to \lambda'_{ct}$ in (1.22) to define a different (finite) renormalized correlator. The new counterterm λ'_{ct} will differ from the old one by a finite (possibly $\mu-$dependent) constant. The choice of this finite constant defines the *renormalization scheme*. The choice of scheme is entirely arbitrary as long as the physical observables and bare quantities remain unchanged. The renormalized Green's functions themselves are not observables, only the resulting cross-sections, masses of particles etc. are truly scheme independent quantities. Typically one picks a scheme that suits purposes of given calculation. For example the *on-shell scheme* is defined so that the renormalized mass m is equal to the physical pole mass of the full propagator $\Delta(-k^2 = m^2)|_{on-shell} \equiv 0$. Perhaps the most popular is the *minimal subtraction scheme* (MS) where the counterterm is chosen to include divergences only.

It is worth mentioning that beta functions and anomalous dimensions do depend on scheme[4] since they are entirely determined by the choice of counterterms. This does not necessarily mean that they are devoid of physical content as we will see in the next section.

[4]It can be shown that the LO coefficient γ_0 of γ_m and first two coefficients β_0, β_1 of the beta function are scheme-independent

1.2.2 An intuitive approach- Wilsonian RG

Wilsonian RG represents an alternative point of view on the issues discussed above where the scaling of fields and couplings arises naturally from coarse graining of the underlining UV physics. In this formalism there always exists a cutoff Λ which is much larger than masses or momenta of IR degrees of freedom and therefore effectively separates the potentially unknown UV physics from observable IR. This idea has lead to very successful program of *effective field theories* where one can study low-energy regime of field theories without the knowledge of corresponding UV Lagrangian.

To introduce some basic ideas of Wilsonian RG let us consider again the example of scalar field theory where momentum modes with $p^2 \geq \Lambda^2$ have been integrated out

$$\phi_\Lambda(x) = \int_{|k|\leq\Lambda} \frac{d^4k}{(2\pi)^4} e^{ikx}\phi(k) \ . \tag{1.42}$$

The couplings and renormalization factors of this theory will depend on Λ with

$$\mathcal{L}_\Lambda(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2(\Lambda)\phi^2 + \frac{1}{4!}\lambda(\Lambda)\phi^4 \ . \tag{1.43}$$

This Lagrangian describes UV physics at the scale Λ. To get an effective Lagrangian at some lower scale Λ' the modes with energies $\Lambda \geq |k| \geq \Lambda'$ need to be integrated out using the Euclidean path integral. One defines the effective Lagrangian $\mathcal{L}_{\Lambda'}$ through

$$e^{-\int \mathcal{L}_{\Lambda'} d^4x} = \int \mathcal{D}\phi_{\Lambda\geq|k|\geq\Lambda'} e^{-\int \mathcal{L}_\Lambda d^4x} \ . \tag{1.44}$$

After performing this path integral one obtains

$$\mathcal{L}_{\Lambda'} = \frac{1}{2}Z(\Lambda,\Lambda')(\partial\phi)^2 + \frac{1}{2}m^2(\Lambda')\phi_\Lambda^2 + \frac{1}{4!}\lambda(\Lambda')\phi^4 + \sum_{n>2} c_n(\Lambda')\phi^{2n} \ , \tag{1.45}$$

The new couplings c_n correspond to operators of higher dimension allowed by the global symmetry $\phi \to -\phi$ and will appear suppressed by inverse powers of Λ'. For low energy processes with the typical energy $E \ll \Lambda'$ these operators will contribute factors $\frac{E}{\Lambda'}$. In principle one should also include higher derivative corrections such as $\phi\Box^2\phi$ but these will be again suppressed by inverse powers of Λ'.

In practice the path integral (1.44) is computed using the background field method. In the background field formalism one splits the field $\phi_\Lambda = \phi_{\Lambda'} + \delta\phi$, where $\phi_{\Lambda'}$ is treated as background field and $\delta\phi \equiv \phi_{\Lambda \geq |k| \geq \Lambda'}$ is path integrated over. To lowest order this can be done explicitly (see A.1) giving results:

$$\begin{aligned} Z(\Lambda, \Lambda') &= O(\lambda(\Lambda)^2) \\ m^2(\Lambda') &= m^2(\Lambda) + \frac{1}{16\pi^2}\lambda(\Lambda)(\Lambda^2 - \Lambda'^2) + \frac{1}{16\pi^2}\lambda(\Lambda)m^2 \ln(\frac{\Lambda}{\Lambda'}) + O(\lambda(\Lambda)^3) \\ \lambda(\Lambda') &= \lambda(\Lambda) - \frac{3}{16\pi^2}\lambda(\Lambda)^2 \ln(\frac{\Lambda}{\Lambda'}) + O(\lambda(\Lambda)^2) \,. \end{aligned} \tag{1.46}$$

Next step is to rescale the field $\phi \to Z(\Lambda, \Lambda')^{-\frac{1}{2}}\phi$ and define the renormalized parameters

$$\lambda(\Lambda') \to \quad Z^{-2}(\Lambda, \Lambda')\lambda(\Lambda') \tag{1.47}$$

$$m^2(\Lambda') \to \quad Z^{-1}(\Lambda, \Lambda')m^2(\Lambda') \tag{1.48}$$

to bring (1.45) to the same form as the original Lagrangian (1.43) so that the entire procedure can be repeated by integrating out modes between $\Lambda' \geq |k| \geq \Lambda''$ etc. [5]

In the last step the RG equations are obtained by taking Λ' infinitesimally close to Λ and studying the change in effective Lagrangian. This is done by introducing a *blocking parameter* α so that $\Lambda' = e^{-\alpha}\Lambda$ and then differentiating $\frac{d}{d\alpha}\mathcal{L}_{\Lambda'}|_{\alpha=0}$. It is then used to define for example the beta function through

$$\beta_\lambda = \Lambda\frac{d}{d\Lambda}\lambda(\Lambda) = -\frac{d}{d\ln\alpha}\lambda(\Lambda')|_{\alpha=0} \,. \tag{1.49}$$

To LO (1.46) may be used to give $\beta_\lambda = \frac{3}{16\pi^2}\lambda^2(\Lambda) + O(\lambda^3)$. Note that in (1.49) it was implicitly assumed that $\frac{d}{d\ln\alpha}\lambda(\Lambda) = 0$. This is reminiscent of (1.32) and indeed, in Wilsonian context one can think of bare couplings as some initial UV values that evolve into IR under RG flow. A natural question to ask is whether the far UV and deep IR limits exist. We will now turn to this question following closely the reasoning presented in [16].

The main question is whether the cutoff Λ can be safely taken to infinity to define a *continuum limit*. To study this one may wish to invert (1.49):

$$\exp\left(\int_{\lambda(\mu)}^{\lambda(\Lambda)} \frac{d\lambda}{\beta_\lambda}\right) = \frac{\Lambda}{\mu} \,, \tag{1.50}$$

[5] Note that the mass also requires additive renormalization to tame the quadratic divergence in (1.46), which is related to the *naturalness problem*.

where μ represents some IR cutoff scale (for example it could be the mass of a particle). First, we may analyse whether the UV limit $\Lambda \to \infty$ exists. Clearly this can happen only if the integral

$$\int_{\lambda(\mu)}^{\lambda(\infty)} \frac{d\lambda}{\beta_\lambda} \tag{1.51}$$

diverges. This in turn implies that β_λ has zero at $\lambda(\infty)$. If this happens we say that theory has an *ultraviolet fixed point* and becomes scale invariant for high energies. The UV scale invariance of such theory follows from observation that in the far UV any finite mass can be neglected since $m \ll \Lambda$ so the only source of scale breaking is the running coupling λ. In general we recognize two types of scale invariant UV behaviour.

- **Asymptotic Freedom**
  The theory becomes free in the UV. This property is desirable since it allows for reliable perturbative calculations for high-energy processes. A most famous example of such theory is QCD [17, 18] which consists of free quarks in UV. Assuming that $\lambda > 0$ this means that coupling will gradually decrease towards zero as we increase Λ and the LO beta function takes the following form close to UV fixed point:

$$\beta_\lambda = -b_0 \lambda^{r+1} + O(\lambda^{r+2}) \,, \tag{1.52}$$

  with $r, b_0 \geq 0$. The solution to this equation reads:

$$\lambda(\Lambda) = \lambda(\mu)(1 + r b_0 \lambda(\mu)^r t)^{-\frac{1}{r}} \,, \tag{1.53}$$

  where the RG time $t \equiv \ln \Lambda/\mu$ was used. This solution decays for large t so the use of perturbation theory in this regime is justified.

- **Asymptotic Safety**
  The theory is scale invariant and interacting in the UV. Such fixed point was first found by Wilson [19] using ϵ expansion about 4D. More recently nontrivial UV fixed points were discovered in gauge-Yukawa theories [8]. If the beta function is positive λ will increase until it reaches the critical value λ^* when the beta function vanishes. Close to this point

$$\beta_\lambda = -|a|(\lambda - \lambda^*) \,, \tag{1.54}$$

for some constant $|a|$. The solution reads

$$(\lambda(\Lambda) - \lambda^*) = (\lambda(\mu) - \lambda^*) e^{-|a|t} \,. \tag{1.55}$$

Hence the coupling approaches its asymptotic value λ^* exponentially fast.

In both of the above cases the theory becomes conformal, which means that all the scaling dimensions become physical observables (c.f (1.7)). Thus for the example at hand (1.34) would need to approach a constant γ^* for large μ.
The remaining possibility is that β_λ does not vanish in the UV and the coupling continues to grow until it blows up at a scale Λ_{LP} called *Landau pole* which can be determined from (1.50)

$$\Lambda_{LP} = \mu \exp \left(\int_{\lambda(\mu)}^{\infty} \frac{d\lambda}{\beta_\lambda} \right) \,. \tag{1.56}$$

Existence of Landau poles does not imply that the theory is wrong. It simply means that theory should be understood as some effective low energy description which breaks down for energies comparable with Λ_{LP} where more input is needed (new states, interactions etc.). An example of theory with a Landau pole is QED, however here the small value of fine structure constant forces $\Lambda_{LP} \gg M_{Planck}$ so the high energy QED computations are justified for all practical purposes.
It now remains to discuss the deep IR behaviour when all the masses are set to zero so that theory doesn't have any explicit IR cutoff. The question now is what happens when $\mu \to 0$ in (1.50). As before the l.h.s needs to diverge which happens when there is an *infrared fixed point* where $\beta_\lambda(\lambda(0)) = 0$. Just like in the case of UV fixed point it can be Gaussian (non-interacting) and the previous discussion applies with $b_0 \to -b_0$ in (1.52) so that $\lambda(\mu) \to 0$ for $\mu \to 0$. The above example of ϕ^4 in 4 dimensions falls into this category. Also, for example QED has this property - interaction between electrons is screened at large distances. The examples of interacting IR fixed points will be studied in more detail later on in the Section 1.3.4.
In case the theory doesn't possess an IR fixed point, the limit $\mu \to 0$ can't be taken, the coupling $\lambda \to \infty$ at some finite scale μ_c which can be obtained from (1.50)

$$\mu_c = \mu \exp \left(\int_{\lambda(\mu)}^{\infty} \frac{d\lambda}{\beta_\lambda} \right) \,. \tag{1.57}$$

Usually this scale signals confinement phase transition where the strong dynamics takes over and physics needs to be described in terms of corresponding bound states (e.g. in QCD these would be pions, glueballs etc.).

1.2.3 Trace Anomaly and Composite Operator Renormalization

In this section we will present yet another conceptional view on RG flows. Let us summarize what we have learnt so far. We saw that renormalization generates scale dependence of the couplings. For theories with continuum limit this flow begins at a UV fixed point with $\lambda = \lambda^*$ where the theory is conformal. The position of this point can be inferred from zeros of beta functions.
Let us now revert this reasoning so that instead of deducing the UV fixed point from RG flow we will generate a flow by deforming a CFT in the UV. This is done for example by adding a primary operator O to the CFT action [6]

$$S = S_{CFT} + \lambda \int d^4x O(x) \quad . \tag{1.58}$$

In a generic case, this won't be a CFT any more and the coupling λ will acquire dependence on RG scale. As we saw in Section 1.1 the operator O has a definite scaling dimensions $\Delta_O = d_O + \gamma^*_O$ where d_O denotes its naive *engineering dimension* and γ^*_O is its anomalous dimension. In the generic case this anomalous dimension won't be just the sum of anomalous dimensions of its components (e.g $\phi(x)$ in $O(x) = \phi^4(x)$) as additional renormalization is needed to deal with UV divergences when all the components are defined at the same point.
Based on dimensional analysis one can see that close to the UV conformal theory lowering the cutoff $\Lambda \to \Lambda'$ the deformation will behave as $\lambda \sim \left(\frac{\Lambda'}{\Lambda}\right)^{(\Delta_O - 4)}$. We then distinguish between three cases. If $\Delta_O > 4$, the deformation will vanish as the cutoff is lowered and thus it won't affect the IR so the corresponding operator is called *irrelevant*. If $\Delta_O < 4$ the operator is called *relevant* and the deformation will grow in the IR to drive the flow away from the original CFT. An example of such flow can be seen in deforming massless Yang-Mills theory by a quark mass $m\bar{q}q$ which will flow between a theory with N_f quarks to the one with $N_f - 1$ quarks as we change m from 0 to ∞. The deformation with $\Delta_O = 4$ is called *marginal* and it runs logarithmically so that no further information can be inferred without the knowledge of its beta function. Usually such theories possess an IR fixed point where the corresponding beta function vanishes. Only

[6] Conformal symmetry can also be broken spontaneously via vev of some operator but here we don't consider that case.

the latter two cases lead to an interesting IR behaviour since theories that differ by an irrelevant operator are in the same *universality class* and are expected to flow to the same IR CFT. In fact in the remainder of this thesis we will be mostly interested in the case of marginal deformations.

As we already mentioned above, the running coupling $\lambda(\mu)$ breaks the conformal symmetry. This breaking of conformal symmetry condition (1.15) is quantified by the *trace anomaly*:

$$T^\mu_\mu = \beta_\lambda [O] + \dots \quad , \tag{1.59}$$

where the dots represent equations of motion and possibly other operators that vanish when applied to physical states so these will be neglected in the present thesis. A rather peculiar example is massless QCD [20] whose coupling appears in the combination $\frac{1}{4}\frac{1}{g^2}(G^a_{\mu\nu})^2$ so that the deformation corresponds to $\lambda = \frac{1}{g^2}$ with

$$T^\mu_\mu = -\frac{1}{2}\frac{\beta}{g^2}[(G^a_{\mu\nu})^2] + \dots \ , \tag{1.60}$$

where $\beta = \frac{d \ln g}{d \ln \mu}$ is the logarithmic QCD beta function and the notation $[\dots]$ in (1.59) and (1.60) indicates that the operator $[O]$ is renormalized so that physical operator T^μ_μ remains finite. The renormalization of a *composite operator* is quite difficult task in general. One needs to ensure that single insertions of the renormalized operator into correlation functions remain finite and this can induce mixing with operators [21].

Assuming that O does not mix with other operators its renormalization is reflected through the appearance of the anomalous dimension γ_O defined through

$$-\frac{d}{d \ln \mu}[O] = \gamma_O [O] \quad . \tag{1.61}$$

In the case at hand one might deduce the renormalization of $[O]$ directly from (1.59) and the physicality of T^μ_μ which implies the following important condition:

$$\boxed{\frac{d}{d \ln \mu} T^\mu_\mu = 0} \quad . \tag{1.62}$$

Together with (1.59) this implies that:

$$\gamma_O = \frac{d \ln \beta_\lambda}{d \ln \mu} \quad . \tag{1.63}$$

The above can be straight-forwardly generalized to the case of multiple marginal couplings $\{g^A\}$ with the following trace anomaly

$$\Theta \equiv T^\mu_\mu = \beta^A [O_A] \quad , \tag{1.64}$$

where $\beta^A = \frac{d}{d\ln\mu} g^A$. Due to mixing of operators under renormalization, the anomalous dimension becomes a matrix γ^B_A. More explicitly

$$-\frac{d}{d\ln\mu}[O_A] = \gamma^B_A [O_B] \quad . \tag{1.65}$$

We can use (1.62) again to derive a condition

$$\dot\beta^A = \beta^B \gamma^A_B \, . \tag{1.66}$$

For the cases where the mixing matrix can be diagonalized $\gamma^B_A \to \gamma_{O_A}$, the equation (1.66) simplifies drastically and the relation (1.63) can be generalized

$$\gamma_{O_A} = \frac{d\ln\beta^A}{d\ln\mu} \quad . \tag{1.67}$$

For non-diagonal case (1.66) can be solved by writing $\dot\beta^A = \beta^B \frac{\partial\beta^A}{\partial g^B}$ to give

$$\gamma^B_A = \frac{\partial\beta^A}{\partial g^B} \quad , \tag{1.68}$$

The above analysis is only valid for exactly marginal operators, which is the main focus of this thesis. Deforming the UV CFT by relevant operator amounts to explicit breaking of conformal symmetry by the dimension of the coupling (see the discussion at the end of Section 1.1.2). Thus for CFT deformed by relevant coupling λ to operator O with the UV scaling dimension Δ_{UV} we get:

$$T^\mu_\mu = (4 - \Delta_{UV} + \gamma_\lambda)\lambda O \, , \tag{1.69}$$

where $\gamma_\lambda = -\frac{d\ln\lambda}{d\ln\mu}$. A well known example is obtained by deforming massless QCD (1.60) by quark mass operator $\bar q q$ which leads to

$$T^\mu_\mu = -\frac{1}{2}\frac{\beta}{g^2}[(G^a_{\mu\nu})^2] + (1+\gamma_m) m [\bar q q] \, . \tag{1.70}$$

1.3 Renormalization in SUSY

Supersymmetric gauge theory is constructed by writing down a general gauge invariant action that is simultaneously invariant under the action of supersymmetry generators $Q, \bar{Q}$. This can be done in an elegant manner by using superfields (see Appendix B.2). Projecting to $\theta\theta$ component of a chiral superfield the SUSY invariance is achieved since it gets annihilated by the SUSY raising operator Q and becomes a total space-time derivative under the action of $\bar{Q}$. [7] Similarly it can be seen that the highest component ($\theta^2\bar{\theta}^2$) of any superfield is mapped to a total derivative under the action of $Q, \bar{Q}$. Using the property of Grassmann variables that the integral $d\theta^2$ picks up the θ^2 term (and similarly for $d\bar{\theta}^2$ and $\bar{\theta}^2$) one can now write a SUSY, gauge-invariant Lagrangian

$$\mathcal{L} = \frac{1}{4g^2}\int d\theta^2 TrW^2 + (h.c) + \frac{1}{8}\int d\theta^2 d\bar{\theta}^2 \mathbf{\Phi}^\dagger e^{-V}\mathbf{\Phi} + \int d\theta^2 \mathcal{W}(\mathbf{\Phi}) + (h.c) \quad , \tag{1.71}$$

where the superfields $(\mathbf{\Phi}, \mathbf{\Phi}^\dagger, V)$ together with their gauge transformation properties have been defined in the Appendix B.2. The holomorphic function $\mathcal{W}$ is called the *superpotential.* In addition to the SUSY and gauge invariance one also requires invariance under the global R-symmetry[8] that rotates θs and fields simultaneously:

$$R\mathbf{\Phi}(\theta, x) = e^{2ir\alpha}\mathbf{\Phi}(e^{-i\alpha}\theta, x) \quad . \tag{1.72}$$

The R-charge r has to be determined in such a way that leaves (1.71) invariant. For example, with $\mathcal{W} = m\mathbf{\Phi}^2$ we need $r = \frac{1}{2}$ (N.B. the measure also transforms $d\theta^2 \to e^{-2i\alpha}d\theta^2$). This symmetry (together with other $U(1)$ symmetries) becomes an important tool to constrain the form of the low energy effective actions.

Now we proceed to discuss the perturbative renormalization properties of (1.71). We will follow the reasoning of [22]. The non-renormalization can be also understood diagramatically from supergraphs [23].

The usual statement of non-renormalization theorem is that the superpotential $\mathcal{W}$ ($\theta\theta$ components of superfields are called F-terms) doesn't renormalize in the perturbation theory. This can be readily understood by looking at the action (1.71). Notice that the interaction vertices carry an integral over all superspace

[7] this follows from $\bar{Q} = \bar{D} - 2i\theta\partial$

[8] In SUSY there exists a combination of U(1) charges that does not commute with the SUSY charges. If we go to higher SUSY theories the R-symmety also rotates the SUSY charges (say in $\mathcal{N} = 2$ we have $SU(2) \times U(1)$ R-symmetry).

$\int d\theta^2 d\bar{\theta}^2$. Hence also the gauge vertices (V^3, V^4) can be written in this way after expanding the W^2 in terms of V using (B.24) and using that $\bar{D}^2$ behaves as $d\bar{\theta}^2$ in combination with the $d\theta^2$ measure. But since the superpotential is of the form $d^2\theta\mathcal{W}$ and the superfield propagators only contain superspace delta functions [23] which can't remove the extra $d\bar{\theta}^2$ we conclude that the vertices can't contribute to the renormalization of F-terms. This can be directly applied to the renormalization of the mass

$$Z_m m \mathbf{\Phi}^2 \ , \tag{1.73}$$

where one concludes that $Z_m = 1$. Note that by (1.29), this implies a non-trivial relation

$$\gamma = -\frac{1}{2}\gamma_m \tag{1.74}$$

between the field and mass anomalous dimension in SUSY theories.

The non-renormalization theorem does not apply to the matter kinetic term with factor of $d\theta^2 d\bar{\theta}^2$ (such terms are called D-terms), which do indeed receive perturbative corrections contributing to the anomalous dimension γ. Similarly, as we have seen the W^2 can actually be written as a D-term and thus can have a running coefficient in front of it in perturbation theory which generates the beta function of the theory as will be seen shortly.

1.3.1 The Konishi Anomaly

We start by reviewing some basic knowledge about chiral anomalies in non-supersymmetric YM theories. Any theory containing a chiral fermion has anomalous axial current [24, 25] with

$$\partial_\mu j^\mu_A = \frac{T_R}{16\pi^2}\mathrm{tr}(G_{\mu\nu}\tilde{G}^{\mu\nu}) \quad , \tag{1.75}$$

where $\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$ is the Hodge dual of $G^{\mu\nu}$ and T_R is the Dynkin index of the representation of R of the chiral fermion. The chiral anomaly coefficient in (1.75) can be calculated from the massless fermion triangle with j_A insertion at one of its tips.

Alternatively, this can be seen as a quantum anomaly of the Jacobian under a chiral rotation [26]

$$\psi \to e^{i\gamma_5\alpha}\psi$$

$$\bar{\psi} \to \bar{\psi} e^{i\gamma_5 \alpha} \tag{1.76}$$

which does not leave the path-integral measure invariant

$$D(\psi)D(\bar{\psi}) \to D(\psi)D(\bar{\psi}) \exp[\mathrm{tr} \ln J + \mathrm{tr} \ln \bar{J}] \ . \tag{1.77}$$

Here $J, \bar{J}$ correspond Jacobians under rescaling (1.76). The equation (1.76) can be understood as a shift in the Lagrangian:

$$\mathcal{L} \to \mathcal{L} + \alpha \frac{T_R}{16\pi^2} Tr(G\tilde{G}) \ . \tag{1.78}$$

To preserve the invariance of the path integral one needs to to include a new interaction $\mathrm{tr}(G\tilde{G})$ by means of the following term in the action:

$$-\frac{\theta_{YM}}{32\pi^2} Tr(G\tilde{G}) \ . \tag{1.79}$$

The Yang-Mills angle θ_{YM} transforms under the chiral rotations ($\psi \to e^{i\alpha}\psi$)

$$\theta_{YM} \to \theta_{YM} - 2\alpha T_R \ . \tag{1.80}$$

In non-trivial topological backgrounds the anomaly (1.75) can be related to the fermion zero modes through the Aytiah-Singer index theorem [27]:

$$n_\psi - n_{\bar{\psi}} = \frac{1}{32\pi^2} \int d^4x Tr(G\tilde{G}) \ . \tag{1.81}$$

Where n_ψ and $n_{\bar{\psi}}$ count the fermion zero modes and antifermion zero modes respectively.
The integer in (1.81) is a topological invariant called index. In particular if the R.H.S of (1.81) is nonzero it means that $G_{\mu\nu}$ doesn't vanish at the boundary of space-time which is true for example for an instanton configuration. For instanton configurations, which are self-dual ($\tilde{G} = \pm G$) we have:

$$e^{-S_{inst}} = e^{\frac{-1}{4g^2}\int G^2} = e^{-n\frac{8\pi^2}{g^2}} \ . \tag{1.82}$$

The equation (1.82) is then understood as $n-$instanton configuration amplitude and clearly, it becomes more significant as the coupling grows larger so it is a non-perturbative effect. It has been shown by Adler and Bardeen [28] that the chiral anomaly (1.78) does not receive any higher loop corrections in perturbation theory. In addition the topological character of the anomaly prevents it from any

non-perturbative corrections.
All of the above extends to SUSY quite naturally. Here one considers the *Konishi anomaly* [29] under a superfield rescaling

$$\begin{aligned} \mathbf{\Phi} \to & \quad e^{\alpha}\mathbf{\Phi} \\ \mathbf{\Phi}^\dagger \to & \quad e^{\alpha^*}\mathbf{\Phi}^\dagger \ . \end{aligned} \tag{1.83}$$

As before, under such rescaling the path-integral measure $\mathcal{D}(\mathbf{\Phi})\mathcal{D}(\mathbf{\Phi}^\dagger)$ pick up a contribution from anomalous Jacobian under (1.83) which amounts to a shift in the Lagrangian

$$\mathcal{L}_{SYM} \to \mathcal{L}_{SYM} + \frac{\alpha T_R}{16\pi^2}\int d\theta^2 W^2 + (h.c) \ . \tag{1.84}$$

Note that one can also recast the Konishi anomaly into an operator equation using the resulting anomalous Ward identity [29]:

$$\frac{1}{8}\bar{D}^2\mathbf{\Phi}^\dagger e^{-V}\mathbf{\Phi} = \frac{1}{16\pi^2}T_R W^2 + \mathbf{\Phi}\frac{\delta\mathcal{W}}{\delta\mathbf{\Phi}} \tag{1.85}$$

which is valid as an operator insertion up to contact terms.
Few remarks about this SUSY generalization of (1.78) are in order. First, notice that taking imaginary $\alpha \to i\alpha$ amounts to chiral transformation. Using (B.25) we see that (1.86) reduces to (1.78) (the extra factor of 1/2 comes from the fact that superfield $\mathbf{\Phi}$ contains a Weyl fermion so we need two superfields to form a Dirac fermion). Next observation is that taking a real α amounts to rescaling of matter fields by real parameter, which is closely related to conformal transformation of fields. By using (1.83) and (B.25) again with α real we get

$$\mathcal{L}_{SYM} \to \mathcal{L}_{SYM} - \frac{\alpha T_R}{16\pi^2}G^2 \ , \tag{1.86}$$

where the fermionic part was omitted. This is indeed reminiscent of QCD trace anomaly. The remarkable property is exactness of Konishi rescaling anomaly for general α. For real α a similar rescaling anomaly can be defined in the absence of SUSY [30], where it receives further loop corrections. SUSY allows one to extend the exactness of chiral anomaly to a general field rescaling which is a very powerful tool as will be seen later on in this thesis. A more thorough investigation of the Konishi anomaly has been carried out in [31] for certain classes of gauge groups such as the adjoint representation of $SU(N)$ and Sp(N) groups using the Wess-Zumino consistency conditions.

Exploiting same ideas as above SUSY allows for very natural incorporation of θ_{YM} by complexifying the inverse gauge coupling $\frac{1}{g^2}$. Looking at (B.25) it is readily seen that the topological term from (1.79) can be written as $Im(d\theta^2 W^2)$. Thus complexifying $\frac{1}{g^2}$ in (1.71) we can write Lagrangian with θ_{YM} more compactly as

$$\mathcal{L}_{gauge} = \frac{1}{16\pi i}\int d\theta^2 \tau W^2 + (h.c) = \frac{1}{8\pi} Im(\int d\theta^2 \tau W^2) \ . \tag{1.87}$$

where the *holomorphic coupling*

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta_{YM}}{2\pi} \tag{1.88}$$

was introduced. This parametrization will be important when deriving the standard nonrenormalization theorem for coupling which is reviewed in the next section.

1.3.2 SUSY Beta Functions

The discussion of this section is based on ideas in [32]. Before we proceed to discuss SUSY beta functions, there is one further important concept that we have to introduce - the holomorphy [33, 34] . The main idea is that one can treat the couplings as background fields (expectation values of some very heavy fields). Then the couplings λ_i that appear holomorphically [9] in the UV action (couplings of the F-terms) also have to appear holomorphically in the low energy effective theory. In combination with the R-symmetry this puts severe constraint on the form of low energy Lagrangians and it can be used to determine the form of the SUSY beta function.

At the one loop level the Wilsonian evolution of $\frac{1}{g^2}$ from Λ to a lower scale Λ' is given by

$$\frac{1}{g^2(\Lambda')} = \frac{1}{g^2(\Lambda)} - \frac{b_0}{8\pi^2}\ln\frac{\Lambda}{\Lambda'} \ , \tag{1.89}$$

where $b_0 = 3T_G - \sum T_{R_i}$ is calculable by adding up one loop contributions from all the superfield components. Going to the complexified coupling (1.88) we note that the RG evolution of τ is holomorphic. Also, since rotating θ_{YM} by 2π leaves the physics invariant we require:

$$\tau \to \tau + 1 \tag{1.90}$$

[9] As in the theory of complex of functions, the holomorphy requires explicit dependence on λ_i only, whereas the anti-holomorphic function depends only on $\lambda_i^\dagger$

to be a symmetry of the theory. The Wilsonian beta function of τ is then defined through (1.49)

$$\frac{d}{d\Lambda}\tau = f(\tau) \ , \tag{1.91}$$

where the function $f(\tau)$ needs to be periodic in τ. This allows one to expand f in Fourier series

$$f(\tau) = \sum_{n\geq 0} a_n e^{i2\pi n\tau} \ . \tag{1.92}$$

This sum is to taken only over $n \geq 0$ since $Re(2\pi i\tau) = -\frac{8\pi^2}{g^2(\Lambda)}$ so that $f(\tau(\Lambda)) \to \frac{ib_0}{2\pi}$ as $\Lambda \to \infty$ which is expected from (1.89) (in another words we expect the beta function to be consistent in the weak coupling limit). Note that $a_0 = \frac{ib_0}{2\pi}$ is the all-loop perturbative running of $\frac{1}{g^2}$ since $n > 0$ terms are proportional to $e^{-\frac{1}{g^2}}$ which cannot arise in perturbation theory (c.f instantons in (1.82)). Hence as far as perturbative corrections go, the evolution (1.89) seems to be exact. Just as for Konishi anomaly the one loop exactness arises because we can treat $\frac{1}{g^2}$ and θ_{YM} as parts of the same coupling in SUSY. However this reasoning is not quite complete.

Since neither the holomorphy, nor the nonrenormalization don't disallow the matter kinetic term to run, the matter fields do renormalize (at all loops) and this has to be taken into account in Wilsonian approach. After integrating out the physics between Λ and Λ' as in (1.45) the effective IR Lagrangian will take the form

$$\mathcal{L}_{\Lambda'} = \frac{1}{4g^2(\Lambda')}\int d\theta^2 TrW^2 + (h.c) + \frac{1}{8}Z(\Lambda',\Lambda)\int d\theta^2 d\bar{\theta}^2 \mathbf{\Phi}^\dagger e^{-V}\mathbf{\Phi} + \int d\theta^2 \mathcal{W}(\mathbf{\Phi}) + (h.c) \quad , \tag{1.93}$$

where $\frac{1}{g^2(\Lambda')}$ is given by (1.89). In Section 1.2.2 we saw that in a Wilsonian scheme one works with canonically normalized fields and the rescaling

$$\mathbf{\Phi} \to \frac{1}{\sqrt{Z}}\mathbf{\Phi} \tag{1.94}$$

needs to be performed before calculating the beta function. Such rescaling comes at a prize of Konishi anomaly (1.86) which clearly contributes to the running of $\frac{1}{g^2}$ in (1.93). In fact by using (1.86) with $\alpha = -\frac{1}{2}\ln Z$ one can easily see that the rescaling (1.94) leads to a new, shifted effective coupling

$$\frac{1}{g^2(\Lambda')} = \frac{1}{g^2(\Lambda)} - \frac{b_0}{8\pi^2}\ln\frac{\Lambda}{\Lambda'} - \frac{T_R}{8\pi^2}\ln Z(\Lambda',\Lambda) \ . \tag{1.95}$$

By making infinitesimal variation of the cutoff and using $\beta_g = -g^3 \beta_{\frac{1}{g^2}}$ one obtains the NSVZ beta function

$$\beta_{NSVZ} = -\frac{g^3}{16\pi^2}(b_0 + \gamma T_R) \ , \tag{1.96}$$

where $\gamma = \frac{d}{d \ln \Lambda} \ln Z$. Usually, the NSVZ beta function comes with the denominator $\frac{1}{1-\frac{T_G g^2}{8\pi^2}}$ which arises because of the choice to work in a particular normalization of the gauge field with $\frac{1}{g^2}$ in front of W^2. To get the action to the canonical form one needs to take $V \to gV$, which again leads to an extra contribution to $\frac{1}{g^2}$ through the Konishi anomaly. In this thesis a non-canonical normalization is assumed so that (1.96) remains the correct beta function of the theory.

1.3.3 The Anomaly Multiplet

The main idea is that in SUSY, the anomaly currents form a supermultiplet [35], $\mathcal{J}_{\alpha\dot{\alpha}}$ whose lowest component is a current j_{R_μ} [10] , the lowest fermionic components are the the conserved SUSY currents $s^\alpha_\mu, \bar{s}^{\dot{\alpha}}_\mu$ and the second bosonic component involves the energy momentum tensor $T_{\mu\nu}$. Omitting the auxiliary fields [11]:

$$\mathcal{J}_\mu = j_{R_\mu} + \theta_\alpha(s^\alpha_\mu - \frac{1}{3}\sigma_\mu \bar{s}) + \bar{\theta}^{\dot{\alpha}}_\mu \bar{s}_{\dot{\alpha}} + \theta\sigma^\nu\bar{\theta}(2T_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}T^\rho_\rho + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\rho j^\sigma_R) + ... \tag{1.97}$$

where the Pauli mapping $\mathcal{J}_\mu = \bar{\sigma}^{\alpha\dot{\alpha}}_\mu \mathcal{J}_{\alpha\dot{\alpha}}$ to map from between $SL(2,\mathbb{C})$ and Lorentz indices. One also needs to impose the following condition

$$\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha X \ , \tag{1.98}$$

where X is a chiral superfield $\bar{D}_{\dot{\alpha}} X \equiv 0$. Equations (1.97) and (1.98) determine the form of X. In particular by using that $X|_{\theta^2} = \frac{1}{4}D^2 X|_{\theta=\bar{\theta}=0}$ we get

$$X|_{\theta^2} = \frac{1}{4}D^\alpha \bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}}|_{\theta=\bar{\theta}=0} = \frac{2}{3}T^\mu_\mu + i\partial \cdot j_R \ . \tag{1.99}$$

Thus we see that the supersymmetry forces the trace anomaly to be on a par with chiral anomaly. This is something that has been already seen in Section 1.3.1 in the context of Konishi anomaly.

[10]The notation might be slightly confusing here, since for non-conformal theory the current j_R does not need to correspond to an R-symmetry in general.

[11]These are necessary to keep the number of fermionic and bosonic d.o.f. equal

If either $X = 0$ or $X = \bar{D}^2\bar{Y}$ for some anti-chiral superfield Y, the theory is *superconformal*. By examining (1.99) we see that $X = 0$ corresponds to

$$T^\mu_\mu = 0 \; ; \quad \partial \cdot j_R = 0 \; ; \tag{1.100}$$

which is consistent with the conformal symmetry condition (1.15) as expected. [12]

Thus we see that in a superconformal there exists a conserved current j_R which will correspond to a well defined R-charge of the theory. The fact that this R-current and the energy-momentum tensor (and therefore the dilatation current (1.11)) are related by supersymmetry means that for each operator there will be a relation between its scaling dimension Δ (dilatation charge) and its R-charge R. Using the properties of superconformal algebra an explicit relation can be derived [36],[37]

$$\Delta = \frac{3}{2}R \; ; \tag{1.102}$$

the relative numerical factor $\frac{3}{2}$ nicely corresponds to the one in (1.99) as one would intuitively expect. The equation (1.102) represents a very powerful constraint on the field theory, in that it equates the internal abelian symmetry with spacetime symmetry. Since the R-charge is simply additive quantity just like an electric charge, in practice (1.102) implies that the scaling dimension of composite operators (c.f Section 1.2.3)is equal to the sum of dimensions of its constituents. From (1.7) the superconformal symmetry implies the following relation

$$[D, O_1O_2] = i(\Delta_1 + \Delta_2)O_1O_2 \; , \tag{1.103}$$

where Δ_1 and Δ_2 are the scaling dimensions of O_1 and O_2 respectively.
It is not the purpose of this thesis to review all the beautiful consequences of superconformal symmetry, however the relation (1.102) will be essential for the discussion in the next section.
At the end of this section we to turn to a relevant example of SUSY gauge theories. For SUSY Yang-Mills theories the form of X is well known [38]

$$X = \frac{2}{3}[3\mathcal{W} - \sum_i \mathbf{\Phi}_i \frac{\delta\mathcal{W}}{\delta\mathbf{\Phi_i}} + \frac{b_0}{16\pi^2}\mathrm{tr}W^2 + \frac{1}{8}\sum_i \gamma_i \bar{D}^2\mathbf{\Phi}_i^\dagger e^{-V}\mathbf{\Phi}_i] \; , \tag{1.104}$$

[12] The latter case $X = \bar{D}^2\bar{Y}$ corresponds to

$$T^\mu_\mu = 4\Box\bar{y} \; ; \tag{1.101}$$

where $\bar{y}$ is the scalar component of $\bar{Y}$. This can be derived via useful identity $D^2\bar{D}^2\bar{Y} = 16\Box\bar{Y}$ true for any antichiral superfield Y and it is indeed consistent with (1.14).

Here the first two terms represent the explicit scale violation by the superpotential the second term is the one-loop trace anomaly contribution with b_0 defined as in the previous section and the third term represents the contribution from anomalous dimensions of the matter fields. The following exercise can give some insight into the NSVZ beta function. If we take zero superpotential $\mathcal{W} = 0$ and apply the Konishi anomaly (1.85) the (1.104) becomes:

$$X = \frac{2}{3}\frac{1}{16\pi^2}\mathrm{tr}W^2(b_0 + \sum_i T_{R_i}\gamma_i) \ . \tag{1.105}$$

By projecting on the θ^2 component using (B.25) one obtains

$$T^\mu_\mu = \frac{1}{2}\frac{1}{16\pi^2}(b_0 + \sum_i T_{R_i}\gamma_i)G^2 \ . \tag{1.106}$$

This is perfectly consistent with the gauge theory trace anomaly formula (1.60) if

$$\beta = \frac{d\ln g}{d\ln\mu} = -\frac{g^2}{16\pi^2}(b_0 + \sum_i T_{R_i}\gamma_i) \ , \tag{1.107}$$

which is in accordance with (1.96).

1.3.4 The Conformal Window

Let us now consider $SU(N_c)$ gauge theory with $SU(N_f) \times SU(N_f)$ flavour symmetry. The matter sector contains chiral superfields $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ transforming in fundamental and anti-fundamental representation representations respectively. This theory has $T_R = 1/2$, $T_G = N_c$ and the NSVZ beta function reads:

$$\beta = -\frac{g^3}{16\pi^2}[(3N_c - N_f) + N_f\gamma] \ . \tag{1.108}$$

For the asymptotically free case with $N_f < 3N_c$ the above beta function has an IR fixed point with:

$$\gamma^* = 1 - \frac{3N_c}{N_f} \quad . \tag{1.109}$$

The above condition determines the anomalous dimension of composite operators made out of chiral superfields $\mathbf{\Phi}, \tilde{\mathbf{\Phi}}$. This follows from the discussion in the previous section namely (1.103). For example the dimension $\Delta_{\mathbf{\Phi}\tilde{\mathbf{\Phi}}}$ of the composite s-meson $\mathbf{\Phi}\tilde{\mathbf{\Phi}}$ is given by

$$\Delta_{\mathbf{\Phi}\tilde{\mathbf{\Phi}}} = 2 + \gamma^* = 3 - \frac{3N_c}{N_f} \ . \tag{1.110}$$

Here it was used that the scaling dimension of superfields $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ is $1 + \frac{\gamma^*}{2}$. Note that the scaling dimension of a superfield $\mathbf{\Phi} = \phi + ...$ is equal to the scaling dimension of its lowest component (s-quark) ϕ. Therefore the dimension of composite s-quark $\phi\tilde{\phi}$ is equal to $\Delta_{\mathbf{\Phi}\tilde{\mathbf{\Phi}}}$. In Section 1.1 we saw that unitarity imposes constraints on dimensions of fields in CFT. In particular for scalar operators the relation (1.8) holds which in turn imposes condition

$$2 \geq \frac{3N_c}{N_f} \ . \tag{1.111}$$

Together with asymptotic freedom bounds $N_f \leq 3N_c$ the bound (1.111) defines the so called *conformal window*

$$\frac{3}{2}N_c < N_f < 3N_c \ , \tag{1.112}$$

which comprises the SUSY gauge theories with IR fixed points.

Chapter 2

Quantum Action Principle, Contact Terms and Condensates

This chapter is devoted to study of matrix elements of composite operators and their products. In the previous chapter we have already touched upon the topic of renormalization of composite operators in Section 1.2.3 when considering the important physical example of trace anomaly. In fact, composite operators are often related to physical observables (e.g. currents or the energy-momentum tensor in QCD) so it is important to be able to define their matrix elements systematically. In essence, the present chapter introduces two important theoretical tools to achieve this goal - the *quantum action principle* (QAP) and the *Feynman-Hellmann theorem*. Both of these are based on the idea of differentiation wrt finite external parameter and thus can be seen as complementary. A particular emphasis will be given to explicit examples to illustrate usefulness and beauty of these ideas in practical computations.

This chapter is divided into three parts. In the first section we introduce the concept of *Quantum Action Principle* (QAP) in the presence of local couplings and use it to define renormalized two-point functions of composite operators. This part does not contain any new results *per se*, but it aims at providing an explicit pedagogical application of the local coupling formalism in QFT. As a warm up exercise, the formalism will be first applied to free fields and then we will move on to more physically relevant examples of scalar and pseudoscalar glueball operators in QCD. The discussion presented in Section 2.1 is based on notes written together with my supervisor Dr. Roman Zwicky.

The second part of this chapter contains the bulk of new results contained in

[3] that have to do with finiteness of two-point functions of gauge invariant composite operators with particular emphasis on correlators of trace of the energy-momentum tensor (TEMT). The main tool here are RGEs corresponding to contact terms and the QAP is utilized directly to show the finiteness of TEMT correlators in the presence of condensates.
Finally in the last section the Feynman-Hellmann theorem is used to define gluon condensates in various theories. The problem is approached from Hamiltonian point of view rather than conventional Lagrangian formalism. Again, the explicit examples of relevant computations are presented to clarify the formalism. This section is based on a published paper [39].

2.1 OPE, contact terms and QAP

Contact terms appear in practical computations involving the operator product expansion (OPE) [40–43] in configuration space whenever the two coincidence limit is considered where they correspond to delta functions $\delta(x-y)$ and derivatives thereof. They do play an important role as they are associated with anomalies (see [44–46] for recent progress), which will be discussed in more details in the next chapter. For now let's just mention the example from the previous chapter - the chiral (triangle) anomaly which can be understood as a contact term in the 3-point function of currents. On the other hand contact terms are not important when studying the (massive) spectrum of 2-point functions (e.g. QCD sum rules [47] or lattice QCD [48]) since they bear no relation to the spectrum. At the technical level one may view contact terms as the counterterms of the short distance singularities $1/(x-y)^n$ of the OPE. In euclidean space, assumed throughout, two operators at small distance allow for an OPE [40] which is schematically given by

$$[\mathcal{O}_A(x)][O_B(0)] \overset{x\to 0}{\simeq} \sum_I \hat{c}_{AB}^C(x)[\mathcal{O}_C(0)] + \mathcal{O}(x^2) , \tag{2.1}$$

where as before the [..]-brackets denote the renormalization of the operators or normal products in the language of the original papers [41–43]. In these works the OPE was shown to hold in perturbation theory and has also enjoyed phenomenological success in the non-perturbative regime where it is expected to hold [49].
For a theory close to a trivial fixed point, such as QCD in the ultraviolet (UV),

it makes sense to parametrize

$$\hat{c}_{AB}^{C}(x) = (x^2)^{(d_C - d_A - d_B)/2} \hat{\mathbb{C}}_{AB}^{C}(x) \ . \tag{2.2}$$

Above d_C is the engineering dimension [1] of O_C and $\mathbb{C}_{AB}^{C}(x)$ are the dimensionless Wilson coefficients subject to logarithmic corrections in perturbation theory. The OPE (2.1) is well-defined for $x \neq 0$ but exhibits local divergences in the limit $x \to 0$ requiring additional renormalization.[2] This becomes necessary for example when the 2-point function is analysed in momentum space which amounts to integrating over d^4x. Since the divergences are *local* the prescription to renormalise $O_A O_B$ is the same as for any other composite operator [41, 42]. Namely, add all operators of equal or lower dimension with the same quantum numbers to the bare Lagrangian and allow them to appear only once in the perturbative expansion. These operators are precisely the operators O_A appearing on the right-hand side in (2.1).
We will see that this renormalization procedure becomes automatic when the local quantum action principle (QAP) is used to generate n-point functions since finite answers are guaranteed by construction (differentiating finite quantities with respect to finite parameters). The cancellation of divergences becomes explicit, in the sense of distributions, upon Fourier transformation into momentum space. We now proceed to review the consequences of QAP applied to the 2-point functions followed by examples of free scalar, fermion and a QCD-like gauge theories.

2.1.1 Quantum Action Principle with local couplings

Schwinger's QAP [50] ensures that functional differentiation of the renormalised (finite) generating functional and parameter is finite. This has been exploited to define a scheme for renormalised composite operators [51–54] with particular important applications in the context of the trace anomaly in QED [55], QCD [20, 56] and curved space [57, 58]. The idea to work with *local* couplings emerged from at least two avenues. First in the mathematical renormalization program of Epstein-Glaser [59–61] where fields are interpreted as operator-valued distributions in coordinate space. Second through quantum field theory in curved space time from the synthesis of functional differentiation wrt the metric

[1] For a non-trivial UV fixed point one needs to replace $d_C \to \Delta_C^{UV}$ where Δ_C^{UV} is the scaling dimension of O_C at UVFP.

[2] The OPE (2.1) itself might be viewed as a (point-splitting) regularisation of the operator $O_A O_B$.

and Callan-Symanzik equations [62–65]. The local renormalization group (RG) served as the basis of a thorough perturbative investigation of the a-theorem [66] and continues to evolve into new aspects [67, 68]. Crucially, finiteness of the QAP is extended to local couplings provided all terms in the coupling including derivatives allowed by symmetry and dimension are added to the bare action [63]. Differentiating the generating functional wrt two local renormalised couplings is finite and also generates the 2-point function as well as a local *contact* term. Hence the contact term is to cancel the local divergences of the 2-point functions discussed in the context of the OPE.

This work relies on the renormalised QAP which states that functional differentiation wrt to a parameter and renormalization as well as path integration commute. This has been shown to hold in $d = 4 - 2\epsilon$ regularisation (DR) in the MS-scheme [52]. Let the bare action be defined by $S^{\rm tot} = S_O + S^{\rm ct}$

$$
\begin{aligned}
S_O &= \int d^d x g_0^A(x) O_A(x) \;, \\
S^{\rm ct} &= \int d^d x \mu^{d-4} \frac{1}{2} L^{\mathbb{1}}_{AB} \left(\Box g^A(x) \Box g^B(x) \right) + O((\partial^n g^A)^3) \;. \qquad (2.3)
\end{aligned}
$$

where the couplings g_0^A are dimensionless (this assumption can be lifted) and summation over repeated indices, such as A, is implied unless otherwise stated hereafter. This is similar to the procedure in Section 1.2.3. The vacuum graph counterterm action $S^{\rm ct}$ contains up to $(\partial g^A)^4$-terms which are omitted since only 2-point functions are studied in this part of the thesis. In a curved background one needs to add terms like Ricci scalar squared and mixed terms of the metric and couplings [66]. The QAP defines the VEV ($\langle ... \rangle = \langle 0 | ... | 0 \rangle$) of a local renormalised composite operator

$$
\langle [O_A(y)] \rangle = (-\delta_{A(y)}) \ln \mathcal{Z} = [\text{finite}] \;, \qquad (2.4)
$$

via a single variational derivative[3]

$$
\delta_{A(x)} \equiv \frac{\delta}{\delta g^A(x)} \;, \quad \partial_A \equiv \frac{\partial}{\partial g^A} \;. \qquad (2.5)
$$

[3] The partial derivatives are distinct from spatial partial derivatives in that indices run over roman letters. Moreover, in the examples where the couplings and operators are not as simply related as in (2.3) a modified variational derivative $\bar{\delta}_{A(x)}$ will be defined such that $[O_A(x)] = \bar{\delta}_{A(x)} S$ holds and Eq. (2.7) below will be adapted to $\delta \to \bar{\delta}$ enforcing a covariant like notation.

The couplings $g^A(x)$ and the operators $O^A(x)$ are conjugate to each other and take on roles reminiscent from statistical physics such as an external magnetic field and the magnetisation in the Ising model. In fact the QAP can be applied to any parameter of the theory, not just couplings (see footnote 3). Double variational differentiation, with c standing for the connected component, leads to

$$\begin{aligned}\hat{\Gamma}^{(ren)}_{AB}(x-y) &= (-\delta_{B(y)})(-\delta_{A(x)})\ln\mathcal{Z} = (-\delta_{B(y)})\langle[O_A(x)]\rangle \\ &= \langle[O_A(x)][O_B(y)]\rangle_c - \langle\big(\delta_{B(y)}[O_A(x)]\big)\rangle = [\text{finite}]\ , \end{aligned} \quad (2.6)$$

a sum of expressions which is finite (hence the superscript *ren* stands for renormalized). We will study the specifics of this renormalization in Sec. 2.2. The derivative of the operator is the contact term which can be written in terms of delta functions and derivatives thereof (to be discussed in the next section). More concretely, following some of the notation in [66],

$$\langle\big(\delta_{B(y)}[O_A(x)]\big)\rangle\big|_{\mathcal{J}_0} = K^C_{AB}\langle[O_C]\rangle\delta(x-y) + \mu^{d-4}L^{\mathbb{1}}_{AB}\langle\mathbb{1}\rangle\Box^2\delta(x-y)\ , \quad (2.7)$$

with $\Box = \partial_\mu\partial^\mu$, $\delta(x) \equiv \delta^{(d)}(x)$ and $\big|_{\mathcal{J}_0}$ indicating that couplings and sources assume constant and zero values respectively. An explicit formula for K^C_{AB} in terms of renormalization constants is given further below and $L^{\mathbb{1}}_{AB}$ does correspond to the term in the vacuum graph counterterm action S^{ct} (2.3). Eq. (2.7) is correct for a theory where all couplings are dimensionless. The UV-singularities of the 2-point function, in (2.6), can be analysed in terms of the OPE (2.1) which for an asymptotically free theory read (cf. (2.1) and (2.2))

$$\langle[O_A(x)][O_B(0)]\rangle_c = \frac{\hat{\mathbb{C}}^C_{AB}(x)}{x^d}\langle[O_C]\rangle + \frac{\hat{\mathbb{C}}^{\mathbb{1}}_{AB}(x)}{x^{2d}}\langle\mathbb{1}\rangle + \dots\ , \quad x \neq 0\ , \quad (2.8)$$

with additional contact terms at $x \to 0$. I.e. terms proportional to the $\delta(x)$ and derivatives thereof with coefficient function which can be either finite or infinite. The $\mathbb{1}$ ($\langle\mathbb{1}\rangle = 1$) denotes the unit operator which corresponds to the perturbative contribution. Upon Fourier transformation, at large p^2 the following OPE-like expression can be defined

$$\Gamma_{AB}(p^2) = \int d^dx e^{ip\cdot x}\langle[O_A(x)][O_B(0)]\rangle_c = \mathbb{C}^{\mathbb{1}}_{AB}(p)p^d\langle\mathbb{1}\rangle + \mathbb{C}^C_{AB}(p)\langle[O_C]\rangle \quad (2.9)$$

where the unity term and the $\langle[O_C]\rangle$-condensate terms now include UV divergences from integrating over $x = 0$. By power counting higher condensate terms are free from divergences in perturbation theory. From the viewpoint of the QAP

(2.6)[4] the divergences that arise upon computation are cancelled by

$$\begin{aligned}\mathbb{C}^{\mathbb{1}}_{AB} - L^{\mathbb{1}}_{AB} &= [\text{finite}]\ , \\ \mathbb{C}^{C}_{AB} - K^{C}_{AB} &= [\text{finite}]\ ,\end{aligned} \tag{2.10}$$

the $\Box^2\delta(x)$ and $\delta(x)$ contact terms K^C_{AB}, $L^{\mathbb{1}}_{AB}$ in Eq. (2.6). In the remaining part of this work these cancellations will be shown to work in explicit examples. It is self-understood that these principles work for higher point functions as well albeit in more complicated ways.

Formulae for Composite Operator renormalization

The aim of this section is to clarify the meaning of the renormalised operators appearing in (2.6) as well as the functional derivative thereof. For a bare action as in (2.3) the relation between bare and renormalised couplings is parameterised

$$g_0^A(x) = g_0(\{g^B(x)\}, \{\partial^n g^B(x)\}) \quad , \tag{2.11}$$

indicating a possible derivative dependence with n being a positive integer.[5] Writing (2.4) in more detail

$$\begin{aligned}\langle [O_A(y)]\rangle &= (-\delta_{A(y)}) \ln \mathcal{Z} = \langle \delta_{A(y)} S\rangle \\ &= \left\langle \delta_{A(y)} \int d^dx (g_0^B(x) O_B(x) + \frac{1}{2} L^{\mathbb{1}}_{BC}(\Box g^B(x) \Box g^C(x)))\right\rangle\end{aligned} \tag{2.12}$$

and the ansatz

$$\langle [O_A(x)]\rangle = \mathbb{Z}_A{}^B \langle O_B(x)\rangle + \mathbb{Z}_A{}^{\mathbb{1}}(x)\langle O_{\mathbb{1}}\rangle\ , \quad O_{\mathbb{1}} = \mathbb{1}\ , \tag{2.13}$$

and one obtains

$$\begin{aligned}\mathbb{Z}_A{}^B &= \langle \delta_{A(x)} S_O\rangle\big|_{O_B(x)} = \partial_A(g_0^B)\ , \\ \mathbb{Z}_A{}^{\mathbb{1}}(x) &= \langle \delta_{A(x)} S^{\text{ct}}\rangle\big|_{O_{\mathbb{1}}} = L^{\mathbb{1}}_{AB} \Box^2 g^B(x)\end{aligned} \tag{2.14}$$

where the bars stand for projection on the corresponding operator. Note that $\mathbb{Z}_A{}^B$ in the constant coupling limit corresponds to RG mixing matrix of O_A. The

[4] In a practical computation, using perturbation theory to perform the OPE, one needs to check that the remaining divergences are local e.g. [69–71].

[5] Given the scaling dimension of O_A one expects n to be bounded from above by power-counting.

formula for $\mathbb{Z}_A{}^B$ has been presented in [54, 72] for example, although the mixing with identity $\mathbb{Z}_A{}^{\mathbb{1}}(x)$ was not included in these works. Eqn. (2.14) then allows for explicit formulae of the contact terms in (2.7) by differentiating (2.13) wrt local coupling

$$\begin{aligned} K^C_{AB} &= \delta_{A(x)}[O_B(y)]\big|_{[O_C]\delta(x-y)} = \left(\partial_A \partial_B\, g_0^D\right)(\mathbb{Z}^{-1})_D{}^C \\ &= (\partial_A \mathbb{Z}_B{}^D)(\mathbb{Z}^{-1})_D{}^C = (\partial_B \mathbb{Z}_A{}^D)(\mathbb{Z}^{-1})_D{}^C \,, \\ L^{\mathbb{1}}_{AB} &= \delta_{A(x)}[O_A(y)]\big|_{\mu^{d-4}\mathbb{1}\Box^2\delta(x-y)} = \delta_{A(x)}\mathbb{Z}_B{}^{\mathbb{1}}(y) O_{\mathbb{1}}\big|_{\mu^{d-4}\mathbb{1}\Box^2\delta(x-y)} \,. \end{aligned} \tag{2.15}$$

2.1.2 Free scalar example

In the following two subsections we will illustrate some of the above ideas on the simplest possible models - free massive scalar and fermion. Even in such models divergences arise (for example by closing propagators) and we will demonstrate how QAP can be used to define finite insertions of the mass operator in both cases.
Consider a free field theory of a single real scalar field of mass dimension $(d-2)/2$ described by the following partition function[6]

$$\mathcal{Z}_m = \int D\phi e^{-\int d^dx \left(\frac{1}{2}(\partial_\mu\phi(x))^2 + \frac{1}{2}m^2(x)\phi^2(x) + \frac{1}{d(d-2)}\Lambda^{\mathbb{1}}_{mm} m^d(x)\right)} \,, \tag{2.16}$$

where $m^2(x)$ is regarded as a source term and $\Lambda^{\mathbb{1}}_{mm}$ contains a counterterm and a vacuum energy contribution. The vacuum energy counterterm $\Lambda^{\mathbb{1}}_{mm}$ is determined by requirement of the finiteness of the functional derivative

$$m^2\langle[\phi^2]\rangle = (-\bar{\delta}_{m(x)})\big|_{\mathcal{J}_0} \ln \mathcal{Z}_m = m^2\langle\phi^2\rangle + \frac{\Lambda^{\mathbb{1}}_{mm}}{(d-2)} m^d = [\text{finite}] \,. \tag{2.17}$$

The symbol $\bar{\delta}_{m(x)}$ denotes

$$\bar{\delta}_{m(x)} \equiv 2\frac{\delta}{\delta \ln m^2(x)} \,, \tag{2.18}$$

the conjugate derivative to $O_m \equiv m^2\phi^2$ in the sense of Eqs. (2.4,2.5).

[6] Above $m_0 = m$ was silently assumed since the theory is free. Moreover, a term of the form $\delta\mathcal{L}(x) = k(\partial m)^2(x)$ ought to be added as a counterterm, resulting in a $m^2\Box\delta(x-y)$-term in (2.7), if an OPE with quadratically divergent terms was considered, e.g. $\phi^4(x)\phi^2(y)$-term. Since only a $\phi^2(x)\phi^2(y)$-term is examined, which is logarithmically divergent, there is no need to discuss the above mentioned counterterm.

Evaluating the condensate appearing in Eq. (2.17) in $d = 4 - 2\epsilon$

$$\langle \phi^2 \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} = \frac{m^{d-2}}{16\pi^2} \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \ , \tag{2.19}$$

fixes the divergent part of $\Lambda^{\mathbb{1}}_{mm}$

$$\Lambda^{\mathbb{1}}_{mm} = -\frac{2}{16\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \ . \tag{2.20}$$

Choosing $\Lambda^{\mathbb{1}}_{mm} = -2/(16\pi^2\epsilon)$ correspond to the MS-prescription.

$\phi^2(x)\phi^2(y)$-OPE and Contact Terms from Quantum Action Principle

In the following the OPE of the two dimension two operators

$$\langle [\phi^2(x)][\phi^2(0)] \rangle_c = \frac{\hat{\mathbb{C}}^{\mathbb{1}}_{mm}}{x^{2(d-2)}} \langle \mathbb{1} \rangle + \frac{\hat{\mathbb{C}}^{m}_{mm}}{x^{(d-2)}} \langle [\phi^2] \rangle + \ldots \ , \quad x \neq 0 \ , \tag{2.21}$$

is considered and in the case at hand $\langle [\phi^2(x)][\phi^2(0)] \rangle_c = \langle \phi^2(x)\phi^2(0) \rangle_c$ holds since the ϕ^2 only mixes with the identity. The x^{2-d}-term is not divergent upon integration and we shall therefore not discuss it any further but focus on the unit operator term which diverges logarithmically.

Applying the idea of section 2.1.1, in particular Eq. (2.6), leads to

$$\begin{aligned} \Gamma_{mm}(x-y) &\equiv (-\bar{\delta}_{m(y)})\big|_{\mathcal{J}_0}(-\bar{\delta}_{m(x)}) \ln \mathcal{Z}_m = (-\bar{\delta}_{m(y)})\big|_{\mathcal{J}_0} m^2 \langle [\phi^2(x)] \rangle \\ &= m^4 \langle [\phi^2(x)][\phi^2(y)] \rangle_c - \delta(x-y) 2 (m^2 \langle [\phi^2] \rangle + m^d \Lambda^{\mathbb{1}}_{mm}) \\ &= 2m^4 \langle \phi(x)\phi(y) \rangle^2 + \delta(x-y) \frac{m^d}{8\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) = [\text{finite}] \ , \end{aligned} \tag{2.22}$$

finite expression. It is noted that formally $K^m_{mm} = 2$ (2.7) is finite which is consistent with the previously mentioned x^{2-d}-term being finite. The euclidean propagator is given by

$$\langle \phi(x)\phi(0) \rangle = \frac{1}{4\pi^2}\frac{1}{x^{d-2}} \left(1 + \mathcal{O}(\epsilon, \ln(mx)mx)\right) \ , \quad x \equiv \sqrt{x^2} \ . \tag{2.23}$$

Finiteness in Eq. (2.22) means that the Fourier transform exists in $d = 4 - 2\epsilon$ and the limit $\epsilon \to 0$ can be taken smoothly. Somewhat symbolically

$$\frac{1}{x^4}\Big|_{\text{reg}} = \frac{1}{x^4} + \frac{\pi^2}{\epsilon}\delta(x) \ , \tag{2.24}$$

where the reg stands for regularised. This can be established explicitly by using the d-dimensional Fourier transformation formula. It is seen that the Fourier transform of the singular part of the propagators

$$\int d^d x e^{iq\cdot(x-y)} \frac{1}{8\pi^4} \frac{1}{(x-y)^d} = -\frac{1}{8\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0 \ln q^2) \ , \tag{2.25}$$

cancels the contact term pole (2.22) exactly. This establishes that $S^{(2)}(x-y)$ is finite in the limit $x \to y$ in the sense of distributions. Note that potential infrared (IR) divergences in the limit $q \to 0$ are regularised by the mass m in the complete theory. This establishes $\mathbb{C}^{\mathbb{1}}_{mm} - \Lambda^{\mathbb{1}}_{mm} = [\text{finite}]$ as a special case of (2.10) with $\mathbb{C}^{m}_{mm}$ & K^{m}_{mm} being separately finite.

2.1.3 Free fermion example

This section parallels the discussion of the free scalar field in Section 2.1.2 and will be kept rather short. Consider a free field theory of a single Dirac fermion described by a generating functional

$$\mathcal{Z}_{m_f} = \int D\Psi D\bar{\Psi} e^{-\int d^d x (\bar{\Psi}(x)(\not{\partial} - m_f(x))\Psi(x) + \frac{1}{4} L^{\mathbb{1}}_{mm} m_f(x) \Box m_f(x) + \frac{1}{d(d-1)} \Lambda^{\mathbb{1}}_{m_f m_f} m_f^d(x))} \ , \tag{2.26}$$

with local mass parameter $m_f(x)$. Notice that in this case a counterterm $m_f \Box m_f$ needs to be included to cancel the quadratic divergence in the two point function. [7]

The vacuum graph counterterm $\Lambda^{\mathbb{1}}_{m_f m_f}$ is determined by requiring

$$m_f \langle [\bar{\Psi}\Psi] \rangle = (\bar{\delta}_{m_f(x)}) \ln Z_{m_f}|_{m_f = const.} = m_f \langle \bar{\Psi}\Psi \rangle - \Lambda^{\mathbb{1}}_{m_f m_f} \frac{1}{(d-1)} m_f^d = [\text{finite}] \ , \tag{2.27}$$

the single variation to be finite and the variational derivative is defined by

$$\bar{\delta}_{m_f(x)} \equiv \frac{\delta}{\delta \ln m_f(x)} \ . \tag{2.28}$$

The value $\Lambda^{\mathbb{1}}_{m_f m_f}|_{\text{MS}} = -\frac{3}{4\pi^2}\frac{1}{\epsilon}$ follows from (2.26) and

$$m_f \langle \bar{\Psi}\Psi \rangle = m_f \int \frac{d^d k}{(2\pi)^d} \frac{\text{tr}[i\not{k} - m_f]}{k^2 + m_f^2} = -\frac{m_f^4}{4\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \ . \tag{2.29}$$

[7] Of course, if we were to consider higher point functions we would need to include $O(\Box, m_f^3)$ counterterms.

To determine $L^{\mathbb{1}}_{mm}$ we need to apply two scale derivatives on the functional (2.26) and set the mass to constant

$$\begin{aligned}
\Gamma_{m_f m_f}(x-y)|_{m_f=const.} &= (-\bar{\delta}_{m_f(y)})\big|_{\mathcal{J}_0}(-\bar{\delta}_{m_f(x)})\ln \mathcal{Z}_{m_f} \\
&= m_f^2\langle[\bar{\Psi}\Psi(x)][\bar{\Psi}\Psi(y)]\rangle_c - L^{\mathbb{1}}_{mm} m_f^2 \Box\delta(x-y) \\
&\quad +\delta(x-y)\left(m_f\langle[\bar{\Psi}\Psi]\rangle - \frac{d}{(d-1)} m_f^d \Lambda^{\mathbb{1}}_{m_f m_f}\right) \\
&= -m_f^2\langle\Psi(x)\bar{\Psi}(y)\rangle\langle\Psi(y)\bar{\Psi}(x)\rangle - L^{\mathbb{1}}_{mm} m_f^2\Box\delta(x-y) \\
&\quad +\delta(x-y) m_f^4 \frac{3}{4\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \\
&= [\text{finite}]\ , \qquad (2.30)
\end{aligned}$$

leads to a finite expression if we choose $L^{\mathbb{1}}_{mm} = \frac{2}{16\pi^2}\frac{1}{\epsilon}$ and it is noted that formally $K^{m_f}_{m_f m_f} = 1$ (2.7). The finiteness of the expression above is assessed through the Fourier transform

$$\begin{aligned}
\int d^d x e^{iq\cdot(x-y)}\Gamma_{m_f m_f}(x-y) = &-m_f^2 \int \frac{d^d k}{(2\pi)^d}\frac{\text{tr}[(i\not{k}+m_f)(i(\not{k}+\not{q})+m_f)]}{(k^2+m_f^2)((k+q)^2+m_f^2)} \\
&-\frac{2m_f^2 q^2}{16\pi^2}\frac{1}{\epsilon} + \frac{3m_f^4}{4\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \\
=&12 m_f^2\langle\phi^2\rangle - \frac{3m_f^d}{4\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \overset{(2.19)}{=} \mathcal{O}(\epsilon^0)\ , \qquad (2.31)
\end{aligned}$$

where $\langle\phi^2\rangle$ is a shorthand for the bubble integral (2.19) with mass $m = m_f$, whose divergent part does indeed render (2.31) finite as expected from the QAP.
It is interesting to see that both $L^{\mathbb{1}}_{mm}$ and $\Lambda^{\mathbb{1}}_{m_f m_f}$ can be obtained directly via background field calculation. Indeed, the path-integral (2.26) can be performed

$$\ln\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{-\int d^d x(\bar{\Psi}(x)(\not{\partial}-m_f(x))\Psi(x))} = \frac{1}{2}\text{tr}\ln\Delta_m^{(\frac{1}{2})}\ , \qquad (2.32)$$

where $\Delta_m^{(\frac{1}{2})} = -(\not{\partial}-m_f(x))(\not{\partial}+m_f(x))$. The above determinant can be evaluated by heat kernel methods (see Appendix for details of such calculations)

$$\frac{1}{2}\text{tr}\ln\Delta_m^{(\frac{1}{2})} = \frac{1}{2}\frac{1}{16\pi^2}\frac{1}{\epsilon} m_f \Box m_f - \frac{1}{16\pi^2} m_f^4 \frac{1}{\epsilon} + [\text{finite}]\ . \qquad (2.33)$$

The counterterms in (2.26) need to be chosen to cancel these divergences to yield finite effective action which implies $L^{\mathbb{1}}_{mm} = \frac{2}{16\pi^2}\frac{1}{\epsilon}$ and $\Lambda^{\mathbb{1}}_{m_f m_f} = \frac{3}{4\pi^2}\frac{1}{\epsilon}$ in accordance with the results obtained by direct calculations (2.29), (2.31). Note that for a fermion with $SU(N_c)$ (global) symmetry the trace (2.33) picks up a factor of N_c.

2.1.4 $G^2(x)G^2(y)$-gauge theory correlator - CP-even sector

CP-even sector with local gauge coupling

Consider a gauge theory of the QCD type with N_f massless fermions and gauge bosons in the fundamental and adjoint representations of some semi-simple compact gauge group respectively. The theory is described by the generating functional (or partition function)

$$\mathcal{Z}_g = \int DADqD\bar{q}\, e^{-S_{\text{even}}} \ , \quad S_{\text{even}} = S_g + S_f + S_g^{\text{ct}} \ , \tag{2.34}$$

where the quark fermion term, $S_f = \int d^4x \bar{q}(\not{D})q$, with zero mass will play no further role since it is proportional to EOM (this will be explained in more detail in Section 3.1.1). The gauge kinetic part is given by[8]

$$S_g \;=\; \frac{1}{4}\int d^dx O_g \ , \quad O_g \equiv \frac{1}{g_0(x)^2} G^2(x) \ , \tag{2.35}$$

where bare and renormalised (local) coupling g_0 and g are related by (2.11)

$$g_0(x) = Z_g g(x) \mu^{(4-d)/2} \ , \tag{2.36}$$

and $G^2 = G^a_{\mu\nu} G_a^{\mu\nu}$, $G_{\mu\nu} = i[D_\mu, D_\nu]$, covariant derivative $D_\mu = \partial_\mu - iA^a_\mu T^a$ and T^a a Lie algebra valued matrix in the fundamental representation of the gauge group. The counterterm action parameterises all dimension four terms as function of the local coupling

$$S_g^{\text{ct}} = \int d^dx \mu^{d-4} (\frac{1}{8} L^{\mathbb{1}}_{gg} (\Box \ln g)^2 + \mathcal{O}(\ln^3 g)) \ , \tag{2.37}$$

up to total derivative terms. Below the determination of $L^{\mathbb{1}}_{gg}$ and K^g_{gg} is discussed

- The divergent part of $L^{\mathbb{1}}_{gg}$ is determined by requiring a finite generating functional $W(g)$, in particular for the coefficient of the $\Box \ln g^2$-term. The generating functional $W(g)$ is evaluated using the background field method

[8] Composite operators of gauge theories are classified into three types [73]: Class I [IIa] gauge invariant (GI) operators non-vanishing [vanishing] by virtue of the equation of motion (EOM) and non-GI operators of class IIb (gauge fixing). Since class I does not get admixtures of class II operators in the MS-scheme [54] and class II operators do not contribute to 2-point function, unique focus on class I operators is justified for this work. Class I consists of O_g and $m_q \bar{q} q$ with the latter vanishing for massless quarks as assumed here.

for the local coupling $g(x)$. At leading order this is equivalent to using the heat kernel method cf. Appendix A.2.1 figure A.1

$$W(g) = \int d^d x \mu^{d-4} \left[\left(\frac{1}{2\epsilon} \frac{n_g}{16\pi^2} \right) - \frac{1}{8} L^{\mathbb{1}}_{gg} \right] (\Box \ln g)^2 + \mathcal{O}(\partial^n \ln^3 g, \epsilon^0) \;, \tag{2.38}$$

where $n_g|_{SU(N_c)} = N_c^2 - 1$ is the dimension of adjoint representation. Finiteness of the generating functional requires $L^{\mathbb{1}}_{gg} = \frac{1}{2\epsilon}\frac{8n_g}{16\pi^2}$ in the MS-scheme at leading order (LO). The result for $L^{\mathbb{1}}_{gg}$ is identical to the one of Jack and Osborn [66] [Eq.5.5] (taking into account that these authors use the $d = 4 - \epsilon$ convention). For later comparison the next-LO (NLO) result from [66] [Eq.5.8] is also quoted

$$L^{\mathbb{1}}_{gg}|_{\mathrm{MS}} = \frac{8n_g}{16\pi^2} \left(\frac{1}{2\epsilon} + a_s \left(\frac{1}{4\epsilon} \left(17N_c - \frac{10}{3} N_F \right) - \frac{\beta_0}{2\epsilon^2} \right) \right) + \mathcal{O}({a_s}^4) \;, \tag{2.39}$$

where $a_s \equiv g^2/(4\pi)^2$ and β_0 is the first coefficient of the beta function given explicitly in Appendix C.1.

- The quantity K^g_{gg} follows from the renormalization factor ${\mathbb{Z}_g}^g$ (2.15) where $[O_g] = {\mathbb{Z}_g}^g O_g + \ldots$ denotes the renormalised composite operator in the absence of local couplings and the dots stand for class II operators which can be discarded for the purpose of this work (cf.footnote 8). To discuss ${\mathbb{Z}_g}^g$ it is advantageous to introduce renormalised composite operator following (2.12)

$$\langle [O_g(x)] \rangle \;\; = \;\; (-\bar{\delta}_{g(x)}) \ln \mathcal{Z}_g \;, \tag{2.40}$$

with conjugate functional derivative

$$\bar{\delta}_{g(x)} \equiv 4 \frac{\delta}{\delta \ln 1/g^2(x)} = -2 \frac{\delta}{\delta \ln g(x)} \;. \tag{2.41}$$

The renormalization factor ${\mathbb{Z}_g}^g$ is deduced from Eq. (2.13)

$${\mathbb{Z}_g}^g = \frac{\partial}{\partial \ln 1/g^2} \ln \left(\frac{1}{g^2 Z_g^2} \right) = (1 + \frac{\partial \ln Z_g}{\partial \ln g}) \;, \tag{2.42}$$

with the difference that the variation of the logarithm is taken since the term $(g^2 Z_g^2)^{-1}$ is part of the definition of the operator (2.35). Using the beta functions β and $\hat{\beta}$ (cf. Appendix C.1 for definitions) ${\mathbb{Z}_g}^g$ can be transformed into its most commonly used form

$$\mathbb{Z}_g{}^g = \frac{(d-4)}{2\hat{\beta}} = \left(1 + \frac{2\beta}{(d-4)}\right)^{-1} . \tag{2.43}$$

The expression above agrees with the original result in [20]. Finally, the quantity K^g_{gg} follows from $\mathbb{Z}_g{}^g$ and (2.15)

$$\begin{aligned} K^g_{gg} &= \left(\frac{\mathbb{Z}_g{}^g}{g^2 Z_g^2}\right)^{-1} (\bar{\partial}_g \frac{\mathbb{Z}_g{}^g}{g^2 Z_g^2}) \\ &= -2\left((1+\frac{\partial \ln Z_g}{\partial \ln g})^{-1} \frac{\partial^2 \ln Z_g}{\partial \ln g^2} - 2(1+\frac{\partial \ln Z_g}{\partial \ln g})\right) \\ &= \frac{2\partial_{\ln g}\hat{\beta}}{\hat{\beta}} + \frac{2(d-4)}{\hat{\beta}} \end{aligned} \tag{2.44}$$

with the amendment $\mathbb{Z}_g{}^g \to \mathbb{Z}_g{}^g/(g^2 Z_g^2)$ for the same reason as mentioned bellow Eq. (2.42). The latter quantity is expanded in a_s to extract the perturbative $\frac{1}{\epsilon}$ poles [9]

$$\begin{aligned} K^g_{gg} &= 4\left({a_s}^2\left[\frac{\beta_1}{\epsilon}\right] + {a_s}^3\left[-\frac{\beta_0\beta_1}{\epsilon^2} + \frac{2\beta_2}{\epsilon}\right]\right) \\ &\quad + 4{a_s}^4\left[\frac{\beta_0^2\beta_1}{\epsilon^3} - \frac{\beta_1(\beta_1+2\beta_0)}{\epsilon^2} + \frac{3\beta_3}{\epsilon}\right] \\ &\quad + \mathcal{O}({a_s}^5, \epsilon^0) \end{aligned} \tag{2.45}$$

The anomalous dimension of the d-dimensional operator O_g is deduced from (2.43)

$$\hat{\gamma}_g{}^g = -\frac{d \ln \mathbb{Z}_g{}^g}{d \ln \mu} = -\frac{\partial \ln \mathbb{Z}_g{}^g}{\partial \ln g}\frac{d \ln g}{d \ln \mu} = \partial_{\ln g}\hat{\beta} , \tag{2.46}$$

and is indeed the correct anomalous dimension known in the literature. The definition in (2.46) is equivalent to $\hat{\gamma}_g{}^g = -d\ln[O_g]/d\ln\mu$ since the bare coupling and the bare operator are independent of the renormalization scale.

$G^2(x)G^2(y)$-**OPE and Contact Terms from QAP**

Here we will verify by comparison with known results from the literature that the counterterms we found by applying local coupling QAP, indeed correspond

[9] The results for K^g_{gg} in (2.44) and $\frac{1}{4}Z^L_{11}/Z_{11}$ in [74] [Eq.4.7] are identical up to finite terms.

to divergences of the $G^2(x)G^2(y)$ OPE. The coordinate space OPE of the gauge kinetic term reads

$$\langle [O_g(x)][O_g(0)]\rangle_c = \frac{\hat{\mathbb{C}}^{\mathbb{1}}_{gg}(x)}{x^{2d}}\langle \mathbb{1}\rangle + \frac{\hat{\mathbb{C}}^{g}_{gg}(x)}{x^{d}}\langle [O_g]\rangle + \dots\,, \quad x \neq 0\,, \tag{2.47}$$

where the dots stand for higher order condensate terms. The divergent parts, of the corresponding momentum space OPE (2.9) ($d = 4 - 2\epsilon$-convention), are extracted from [71]

$$\mathbb{C}^{\mathbb{1}}_{gg}\big|_{\rm div} \;=\; 4\,\frac{n_g}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{a_s}{\epsilon}\left((\frac{17}{2}N_c - \frac{5}{3}N_F) - \frac{\beta_0}{\epsilon}\right)\right) + \mathcal{O}({a_s}^2, \epsilon^0)\,, \tag{2.48}$$

and [69, 70]

$$\mathbb{C}^{g}_{gg}\big|_{\rm div} \;=\; 4\left({a_s}^2\left[\frac{\beta_1}{\epsilon}\right] + {a_s}^3\left[-\frac{\beta_0\beta_1}{\epsilon^2} + \frac{2\beta_2}{\epsilon}\right]\right) + \mathcal{O}({a_s}^4, \epsilon^0)\,. \tag{2.49}$$

From (2.10) it follows that $\mathbb{C}^{\mathbb{1}}_{gg} - L^{\mathbb{1}}_{gg}$ and $\mathbb{C}^{g}_{gg} - K^{g}_{gg}$ are finite. This is easily verified from the explicit expression of $\mathbb{C}^{\mathbb{1}}_{gg}$ (2.48), $L^{\mathbb{1}}_{gg}$ (2.39) and $\mathbb{C}^{g}_{gg}$ (2.49), K^{g}_{gg} (2.44) respectively.

2.1.5 $G\tilde{G}(x)G\tilde{G}(y)$-gauge theory correlator - CP-odd sector

CP-odd sector with local $\theta(x)$-angle

The aim of this section is to investigate the same matters as in the previous section for the CP-odd sector of the gauge theory and to demonstrate how the renormalization of the the topological density works. The latter mixes with the derivative of the axial singlet current (e.g. [75]) [10] under renormalization. The two operators discussed in this section are

$$O_\theta \equiv i\mu^{d-4}G\tilde{G} \equiv i\mu^{d-4}\epsilon^{\alpha\beta\gamma\delta}G_{\alpha\beta}G_{\gamma\delta}\,, \quad O_{\partial J_5} \equiv -\partial_\mu J_5^\mu\,, \tag{2.50}$$

where $J_5^\mu \equiv \bar{q}\gamma^\mu\gamma_5 q$ and it is noted that the minus sign in the definition of $O_{\partial J_5}$ is the result of two factors of i. In this section we show how the corresponding counterterms follow from the background field computations when using a local

[10] The axial singlet current also renormalises ${\mathbb{Z}_{\partial J_5}}^{\partial J_5} \neq 1$ (e.g. [75]). This would be easy to integrate into the formalism of this work but is beyond the specific scope of this section and therefore omitted.

theta angle. To study CP-odd correlators only, it is enough to localise theta angle $\theta \to \theta(x)$ since CP-odd and CP-even sectors don't mix under renormalization.

In the CP-odd sector the generating functional is augmented

$$\mathcal{Z}_{g,g_{\partial J_5},\theta} = \int DADqD\bar{q}\, e^{-(S_{\rm even}+S_{\rm odd})} \;, \qquad S_{\rm odd} \equiv S_\theta + S^{\rm ct}_\theta + S_{g_{\partial J_5}} \;, \tag{2.51}$$

by the CP-odd terms[11 12]

$$\begin{aligned} S_\theta = & \int d^dx\; (\theta(x)\, O_\theta(x)) \;, \quad S_{g_{\partial J_5}} = \int d^dx\, g^0_{\partial J_5}\, (\theta(x)\, O_{\partial J_5}) \;, \\ S^{\rm ct}_\theta = & \int d^dx\, \mu^{d-4}(\frac{1}{2} L^{\mathbb{1}}_{\theta\theta} (\Box\theta)^2 + \mathcal{O}(\theta^3) + \text{total derivatives}) \;, \end{aligned} \tag{2.52}$$

where both $g^0_{\partial J_5}, L^{\mathbb{1}}_{\theta\theta}$ parametrise divergent counterterms and are functions of the gauge coupling. Renormalized single insertion is then given in terms of formula (2.13)

$$\langle [O_\theta] \rangle = (-\delta_{\theta(x)})\big|_{\mathcal{J}_0} \ln \mathcal{Z}_{g,g_{\partial J_5},\theta} = \langle O_\theta \rangle + \mathbb{Z}_\theta^{\;\partial J_5} \langle O_{\partial J_5} \rangle \;, \quad \mathbb{Z}_\theta^{\;\partial J_5} = g^0_{\partial J_5} \;. \tag{2.53}$$

with the conjugate derivative as in (2.5). Note the presence of purely background coupling term

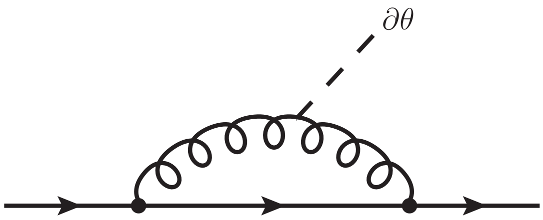
$$S_{g_{\partial J_5}} = \int d^dx\, g^0_{\partial J_5}\, (\theta(x)\, O_{\partial J_5}) = -\int d^dx\, g^0_{\partial J_5}\, (\theta(x)\, \partial\cdot J_5) = \int d^dx\, g^0_{\partial J_5} (\partial_\mu \theta)\, J_5^\mu \tag{2.54}$$

is necessary to cancel a divergent contribution to fermion self-energy proportional to $\partial\!\!\!/\theta\gamma_5$. Other potential counterterms involving higher derivatives of θ are excluded on the grounds of power-counting (there is no dimension 2 operator in this theory). Bellow we show that indeed a background field contribution (2.54) appears already at one loop level and outline its calculation. Afterwards the determination of $L^{\mathbb{1}}_{\theta\theta}$ and $K^g_{\theta\theta}$ will be discussed.

- Writing $O_\theta = \partial \cdot K_{\rm CS}$ with $K^\mu_{\rm CS}$ (with $K^\mu_{\rm CS} = 4i\epsilon^{\mu\alpha\beta\gamma} A^a_\alpha \partial_\beta A^a_\gamma + \mathcal{O}(A^3)$ being the Chern-Simons current) the topological term with local $\theta(x)$ assumes the

[11]Liberty is taken in choosing the normalisation of the S_θ action. The parameter $\theta_{\rm QCD}$ with $\theta = -1/(64\pi^2)\theta_{\rm QCD}$ is the proper angular variable with periodicity of 2π. The normalisation is of no particular concern to us and chosen on grounds of convenience. The factor of i in front of $G\tilde{G}$ is consistent Minkowski space continuation and assures that the real part of θ is an angular variable.

[12]The θ-parameter does not renormalise in QCD-like theories in perturbation theory (cf. however, footnote 13) and this is why no distinction between bare and renormalised parameter is made. The equation below serves as the definition of $g^0_{\partial J_5}$.



$$- \quad g^0_{\partial J_5} \not{\partial}\theta\gamma_5 \quad = [\text{finite}]$$

Figure 2.1 *Diagram corresponding to the one loop counterterm computation in Eq. (2.56) using the background field method. The modification of the gluon propagator is discussed in Appendix A.2.2.*

form

$$S_\theta = \int d^d x \theta(x) O_\theta = -\int d^d x (\partial_\mu \theta(x))\, K^\mu_{\text{CS}} \,. \tag{2.55}$$

This introduces a correction to gluon propagator (A.18). To simplify the computation of $g^0_{\partial J_5}$ we can choose $\partial\theta = \mathit{const.}$, which allows us to use standard Feynman diagram techniques. In particular the LO part of $g^0_{\partial J_5}$ can be obtained from the divergent contribution of the diagram on Fig. 2.1 to the fermion self-energy

$$\begin{aligned}
\Sigma^{(\theta)} &= +8C_F g^2 \epsilon^{\alpha\gamma\omega\delta} \gamma_\alpha \gamma_\beta \gamma_\gamma \partial_\omega \theta \int \frac{d^d k}{(2\pi)^d} \frac{k_\beta k_\delta}{(k^2+P^2)^3} + \mathcal{O}(\epsilon^0) \\
&= +12 C_F g^2 \gamma_\omega \gamma_5 \partial_\omega \theta \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+P^2)^2} + \mathcal{O}(\epsilon^0) \\
&= \frac{1}{\epsilon} 12 C_F a_s \not{\partial}\theta \gamma_5 + \mathcal{O}(\epsilon^0) \,,
\end{aligned} \tag{2.56}$$

where P is proportional to external momentum and a combination of Feynman parameters irrelevant for the determination of $\frac{1}{\epsilon}$ pole at this order. Thus by using (2.53) we find that

$$\mathbb{Z}_\theta^{\partial J_5\,\text{MS}} = g^0_{\partial J_5}|_{\text{divergent}} = \frac{1}{\epsilon} 12 C_F a_s + \mathcal{O}(a_s{}^2) \,, \tag{2.57}$$

where a MS-scheme was chosen. This result is manifestly gauge invariant to this order due to the presence of $\epsilon^{\alpha\gamma\omega\delta}$ in (2.56). The value in (2.57) is consistent with the result in the literature [75] ($\mathbb{Z}_\theta^{\partial J_5} = Z_{GJ}$ in [75]) which was obtained through direct renormalization of the single insertion of topological density operator. In principle the method of this section can be extended to higher orders in perturbation theory, however a special care has to be taken when defining γ_5 in dimensional regularization beyond leading order. Above it was implicitly used that in QCD the topological term does not renormalise in perturbation theory i.e. $\mathbb{Z}_\theta{}^\theta = 1$.

- The $L^{\mathbb{1}}_{\theta\theta}$ counterterm in (2.52) is determined, as previously, by requiring the VEV to be finite for local couplings. At LO the generating functional $W(\theta)$ is evaluated, using the heat kernel cf. Appendix A.2.2 figure A.1, and yields

$$W(\theta) = \int d^d x \mu^{d-4} \left[\left(\frac{-1}{2\epsilon} \frac{16 n_g}{16\pi^2} \right) - \frac{1}{2} L^{\mathbb{1}}_{\theta\theta} \right] (\Box\theta)^2 + \mathcal{O}(a_s, \partial^n \theta^3, \epsilon^0) \ . \quad (2.58)$$

Finiteness of the generating functional requires

$$L^{\mathbb{1}}_{\theta\theta}|_{\rm MS} = \frac{-2}{2\epsilon} \frac{16 n_g}{16\pi^2} + \mathcal{O}(a_s) \ . \quad (2.59)$$

Note that the $\mathcal{O}(\theta^2)$ expression for $W(\theta)$ given in the Appendix A.2.2 agrees with results in the literature (c.f. the $\mathcal{O}(\theta^2)$ part in Eqs. (23),(24) of [76]).

- The quantity $K^g_{\theta\theta}$ is formally given by (2.15) $K^g_{\theta\theta} = (\partial_\theta \mathbb{Z}_\theta{}^I)(\mathbb{Z}^{-1})_I{}^g$. However, in perturbation theory O_g does not mix into $O_\theta = \partial \cdot K$ since the latter is effectively a dimension three operator from the viewpoint of power counting. This means that O_g cannot appear as a divergent operator on the right-hand side in (2.7) and $\mathbb{Z}_\theta{}^\theta|_{\rm MS} = 1$. Since $\mathbb{Z}_\theta{}^\theta = 1$ in perturbation theory it follows that $K^g_{\theta\theta}|_{\rm MS} = 0$.[13]

$G\tilde{G}(x)G\tilde{G}(y)$-OPE and Contact Terms from QAP

Again we would like to compare our local coupling calculation with known OPE results. Consider the OPE of the topological density

$$\langle [O_\theta(x)][O_\theta(0)] \rangle_c \ = \ \frac{\hat{\mathbb{C}}^{\mathbb{1}}_{\theta\theta}(x)}{x^{2d}} \langle \mathbb{1} \rangle + \frac{\hat{\mathbb{C}}^{g}_{\theta\theta}(x)}{x^{d}} \langle [O_g] \rangle + \dots \ , \quad x \neq 0 \ , \quad (2.60)$$

and the divergent part of the momentum space OPE can be extracted from[71][14]

$$\mathbb{C}^{\mathbb{1}}_{\theta\theta}\big|_{\rm div} \ = \ -16 \frac{n_g}{(4\pi)^2} \left(\frac{1}{\epsilon} + \frac{a_s}{\epsilon} \left((\frac{25}{2} N_c - \frac{5}{3} N_F) - \frac{\beta_0}{\epsilon} \right) \right) + \mathcal{O}(a_s{}^2, \epsilon^0) \quad (2.61)$$

[13] These arguments are not necessarily expected to hold non-perturbatively. For example in $\mathcal{N} = 2$ supersymmetric gauge theories [77] instanton contributions lead to β-functions of θ and g depending on both parameters (with the θ variable remaining 2π-periodic). This implies mixing between O_g and O_θ which affects the renormalization.

[14] We have checked the LO result. The NLO result in [71] ought to be correct despite not considering the mixing of the $\partial \cdot J_5$ with O_θ under renormalization since $\langle \partial \cdot J_5(x) O_\theta(y) \rangle$ and $\langle \partial \cdot J_5(x) \partial \cdot J_5(y) \rangle$ themselves vanish at LO and therefore do not affect the NLO result itself. At NNLO one has to take the mixing affects the result and has to be taken into account cf.[78].

By (2.10) it is to be concluded that $\mathbb{C}^{\mathbb{1}}_{\theta\theta} - L^{\mathbb{1}}_{\theta\theta}$ is finite, which is easily verified from the explicit expressions in (2.59) and (2.61) at LO. The finiteness of $K^g_{\theta\theta}$ and therefore $\mathbb{C}^g_{\theta\theta}$ warrants some discussion since the Fourier transform of the x^{-d}-term by itself is divergent $\int d^dx e^{ip\cdot x} x^{-d} \sim 1/\epsilon$. The resolution is that $\hat{\mathbb{C}}^g_{\theta\theta} \to 0$ and that it is a contact term mentioned after (2.8) that are responsible for the contribution. In this case the contact term is finite which we explicitly demonstrate in Appendix D.1.

2.1.6 Summary and Discussion

In this section we illustrated the appearance of contact terms in practical computation of the OPE with use of QAP. This is closely related to the use of the QAP in defining a renormalization-scheme for composite operators (e.g. [20, 51–58, 72]). Examples discussed include a free scalar field theory as well as the $\langle G^2(x)G^2(y)\rangle$ and $\langle G\tilde{G}(x)G\tilde{G}(y)\rangle$ gauge theory correlation functions (in Section 2.1.2 and 2.1.4 respectively). The counterterms were determined by localising the couplings and imposing finiteness on the generating functional. This automatically renders all (multiple) variational derivatives finite with the contact terms in the OPEs being cancelled by local counterterms depending on local couplings. The principle was hereby explicitly verified for the double variations in free field theories and interacting gauge theories by comparing with results from the literature. The new results include an explicit expression of the contact term in (2.44) in terms of YM β-function. The same result was simultaneously obtained in [74], although observations of the finiteness (see Sec. 2.2.6 of this thesis) were not made in this work. Explicit calculations using of the local coupling $\theta(x)$ to renormalize $G\tilde{G}(x)$ and its products that we presented in Section 2.1.5 provides and extra consistency check to the formalism of [66, 79].

2.2 Contact terms of two point functions

We start this discussion by reviewing briefly some of the ideas presented in Section 2.1 and introducing some new notation. Consider a two-point function[15]

$$\Gamma_{AB}(p^2) = \int d^4x e^{ip\cdot x} \langle [O_A(x)][O_B(0)] \rangle_c = \mathbb{C}^{\mathbb{1}}_{AB}(p^2) p^4 \tag{2.62}$$

in a 4-dimensional euclidean space where c stands for the connected component, $\langle ... \rangle$ for the vacuum expectation value (VEV), $[O_{A,B}]$ are renormalised (composite) operators of dimension four and $\mathbb{C}^{\mathbb{1}}_{AB}$ therefore a dimensionless function. We saw in 2.1.1 that the latter might be thought of as the Wilson coefficient of the identity operator. The coefficient $\mathbb{C}^{\mathbb{1}}_{AB}(p^2)$ is potentially logarithmically divergent by power counting. In coordinate space this divergence results from $x \to 0$ divergences which can be removed by local counterterms within the standard renormalization program. The (UV-finite) renormalized correlation function $\Gamma^{\mathcal{R}}_{AB}$ is obtained from the bare one Γ_{AB} by splitting the bare Wilson coefficient $\mathbb{C}^{\mathbb{1}}_{AB}(p^2)$ into renormalised $\mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}(p^2)$ and a counterterm $L^{\mathbb{1},\mathcal{R}}_{AB}$ part

$$\mathbb{C}^{\mathbb{1}}_{AB}(p^2) = \mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}(p^2) + L^{\mathbb{1},\mathcal{R}}_{AB} \ , \tag{2.63}$$

where it follows from the local character of UV divergences that $L^{\mathbb{1},\mathcal{R}}_{AB}$ doesn't depend on momentum. The splitting (2.63) defines a subtraction scheme, which we denoted by superscript $\mathcal{R}$. In the coordinate space this translates into

$$\hat{\Gamma}^{\mathcal{R}}_{AB}(x^2) = \underbrace{\langle [O_A(x)][O_B(0)] \rangle_c}_{\equiv \hat{\Gamma}_{AB}(x^2)} - L^{\mathbb{1},\mathcal{R}}_{AB} \Box^2 \delta(x) \ , \tag{2.64}$$

which is the familiar form (2.6). The QAP analysis from previous section then allows for interpretation of the scheme $\mathcal{R}$ in (2.64) as a choice of a local coupling counterterm $L^{\mathbb{1},\mathcal{R}}_{AB} \Box g^A(x) \Box g^B(x)$ in (2.3). The correlators $\hat{\Gamma}^{\mathcal{R}}_{AB}$ and $\hat{\Gamma}_{AB}$ clearly agree at nonzero separation ($x \neq 0$) so the 'bare' function $\mathbb{C}^{\mathbb{1}}_{AB}(p^2)$ in (2.63) is nothing but the Fourier transform of the correlator at a nonzero separation and as such it contains UV divergences in general. In Section 2.1.4 and Section 2.1.5 we saw that in perturbation theory these divergences appear order by order as poles in ϵ. It is a purpose of this section to elucidate under what conditions these poles resum and $L^{\mathbb{1},\mathcal{R}}_{AB}$ (and therefore $\mathbb{C}^{\mathbb{1}}_{AB}(p^2)$) becomes finite.

[15] An extension of the discussion of condensate corrections is deferred to Section 2.2.6 within the OPE of QCD-like theories.

2.2.1 Dimensional Regularisation with one Coupling

At first we restrict ourselves to one coupling $a_s = a_s(\mu)$ whose scale dependence will frequently be suppressed throughout. In the MS-scheme with DR ($d = 4-2\epsilon$) the counterterm [16]

$$L^{\mathbb{1},\mathrm{MS}}_{QQ}(\mu) \equiv \sum_{n\geq 1} \frac{Q^{\mathbb{1}}_n(a_s)}{\epsilon^n} \,, \qquad Q^{\mathbb{1}}_1(a_s) = Q^{\mathbb{1}}_{1,0} + Q^{\mathbb{1}}_{1,1} a_s + \mathcal{O}(a_s^2) \tag{2.65}$$

is a Laurent series in ϵ with dimensionless residues which are functions of the running coupling only. Other subtraction schemes are defined by including an arbitrary finite remainder function $f_{\mathcal{R}}(a_s)$ so that $L^{\mathbb{1},\mathcal{R}}_{QQ} = L^{\mathbb{1},\mathrm{MS}}_{QQ} + f_{\mathcal{R}}(a_s)$. We start by deriving a RGE for $L^{\mathbb{1},\mathcal{R}}_{QQ}$ in a generic scheme, particularizing to MS later on. The starting point is the renormalised correlation function

$$\mathbb{C}^{\mathbb{1}}_{QQ}(p^2)p^{-2\epsilon} = \left(\mathbb{C}^{\mathbb{1},\mathcal{R}}_{QQ}(p^2) + L^{\mathbb{1},\mathcal{R}}_{QQ}\right)\mu^{-2\epsilon} \,, \tag{2.66}$$

which follows from adapting (2.62) and (2.63) to DR. Suppose that $[O_Q]$ can be made RG-invariant by multiplying by a function $\kappa_Q(a_s)$, i.e.

$$\frac{d}{d\ln\mu}\kappa_Q[O_Q(x)] = 0 \,, \tag{2.67}$$

which will be the case for the upcoming examples. It is then convenient to define the operator

$$\mathcal{D}_Q = \frac{1}{\kappa_Q^2}\frac{d}{d\ln\mu}\kappa_Q^2 = 2\gamma_Q + \frac{d}{d\ln\mu} \,, \tag{2.68}$$

with

$$\gamma_Q = \frac{d}{d\ln\mu}\ln\kappa_Q = 2\hat{\beta}\partial_{\ln a_s}\ln\kappa_Q \,, \tag{2.69}$$

where $\hat{\beta} = \frac{d\ln g_s}{d\ln\mu} = -\epsilon + \beta$ is the d-dimensional logarithmic β-function. In the last equality in (2.69) mass-independence of the MS-scheme was used. Applying $\mathcal{D}_Q$ to (2.66) and using the scale independence of the bare Wilson coefficient, $\mathcal{D}_Q\mathbb{C}^{\mathbb{1}}_{QQ}(p^2) = 0$, leads to the following RGE

$$\mu^{2\epsilon}(2\gamma_Q + \frac{d}{d\ln\mu})\mu^{-2\epsilon}L^{\mathbb{1},\mathcal{R}}_{QQ} = -2\chi^{\mathcal{R}}_{QQ} \,, \tag{2.70}$$

[16] Extension to the non-diagonal case will be given at the end of the section, several couplings will be discussed in Section 2.2.5 and we comment on scheme (in)dependence further on.

where $2\chi^{\mathcal{R}}_{QQ} \equiv \mathcal{D}_Q \mathbb{C}^{\mathbb{1},\mathcal{R}}_{QQ}$ is a finite function of a_s. Noting that $L^{\mathbb{1},\mathcal{R}}_{QQ}$ depends on μ only through the running coupling the equation above can be written as

$$\left((\epsilon - \gamma_Q) - \hat{\beta}\partial_{\ln a_s}\right) L^{\mathbb{1},\mathcal{R}}_{QQ} = \chi^{\mathcal{R}}_{QQ} \ . \tag{2.71}$$

In particular in the MS-scheme (2.65) one gets

$$\chi^{\text{MS}}_{QQ} = \partial_{a_s}(a_s Q^{\mathbb{1}}_1) \ . \tag{2.72}$$

From now on we will mostly focus on MS$-$ scheme (we will discuss the effects of scheme-change in Sec. 2.2.5). The ordinary differential equation (ODE) (2.71) is solved by

$$L^{\mathbb{1},\text{MS}}_{QQ} = -\int_{a_s^{\text{UV}}}^{a_s} \frac{\chi^{\text{MS}}_{QQ}(u)}{\hat{\beta}(u)} I_{\text{UV}}(a_s,u) I_{\gamma_Q}(a_s,u) \frac{du}{u} \ , \tag{2.73}$$

which shows the MS-property that all higher pole residues of $L^{\mathbb{1},\text{MS}}_{QQ}$ follow from the first one (encoded in χ^{MS}_{QQ}). Above $a_s^{\text{UV}} \equiv a_s(\infty)$ is the coupling at the UV fixed point and $I_{\text{UV}} I_{\gamma_Q}$ is the integrating factor

$$I_{\text{UV}}(a_s,u) = \left(\frac{\mu(a_s)}{\mu(u)}\right)^{2\epsilon} \ , \quad I_{\gamma_Q}(a_s,u) = \exp\left(-\int_u^{a_s} \frac{\gamma_Q(v)}{\hat{\beta}(v)} \frac{dv}{v}\right) \ . \tag{2.74}$$

It is the function I_{γ_Q} which decides on whether or not the integral diverges for $u \to a_s^{\text{UV}}$ and I_{UV} serves as a potential UV-regulator. A more refined analysis requires to distinguish whether the UVFP is of the asymptotically free (AF) $a_s^{\text{UV}} = 0$ or asymptotically safe (AS) type $a_s^{\text{UV}} \neq 0$.

Asymptotically free theory

In order to analyse the integrals in (2.74) it is convenient to perform the change of variable from u (recall that u corresponds to a_s) to the RG-time $t \equiv \ln \mu'/\mu$

$$\frac{d \ln u}{dt} = 2\hat{\beta}(u) \ , \tag{2.75}$$

with μ playing the role of reference scale and μ' being integrated over. In the new variable $I_{\text{UV}} \to e^{-2\epsilon t}$ and $L^{\mathbb{1},\text{MS}}_{QQ}$ become

$$L^{\mathbb{1},\text{MS}}_{QQ} = 2\int_0^\infty \chi^{\text{MS}}_{QQ}(t) I_{\gamma_Q}(t) e^{-2\epsilon t} dt \ , \quad I_{\gamma_Q}(t) = e^{2\int_0^t \gamma_Q(t')dt'} \ . \tag{2.76}$$

In the asymptotic regime a leading log (LL) analysis is sufficient. Assuming $\hat{\beta}(u) = -\epsilon + \beta = -\epsilon - \beta_0 u + \mathcal{O}(u^2)$ the LL relation (1.53) is given by [17]

$$u(t) = \frac{\epsilon a_s e^{-2\epsilon t}}{\epsilon + \beta_0 a_s(1 - e^{-2\epsilon t})} = \frac{a_s}{1 + 2\beta_0 \alpha_s t} + \mathcal{O}(\epsilon) \; , \tag{2.77}$$

with the initial value $u(0) = a_s$ and UV-value $u(\infty) = 0$. The anomalous dimension is parameterised by $\gamma_Q = a_s \gamma_{Q,0} + \mathcal{O}(a_s{}^2)$ implying the asymptotic behaviour $I_{\gamma_Q}(t) \sim t^\eta$ with $\eta = \gamma_{Q,0}/\beta_0$. Assuming a perturbative $\chi_{QQ}^{\rm MS} \sim t^{-n}$ for $t \to \infty$ with $n \geq 0$ ($n = 0$, i.e. $\chi_{QQ}^{\rm MS} = \mathcal{O}(a_s{}^0)$, being the nominal case) the condition for UV finiteness is

$$1 + \frac{\gamma_{Q,0}}{\beta_0} \leq n \quad \Leftrightarrow \quad \bar{L}_{QQ}^{\mathbb{1},\rm MS} \stackrel{\epsilon/(\beta_0 a_s) \to 0}{\longrightarrow} \bar{L}_{QQ}^{\mathbb{1},\rm MS} = [\text{finite}] \; . \tag{2.78}$$

It is instructive to consider the formula for $n = 0$

$$\begin{aligned} L_{QQ}^{\mathbb{1},\rm MS} &\simeq 2Q_{1,0}^{\mathbb{1}} \int_0^\infty \frac{e^{-2\epsilon t} \epsilon^{-\frac{\gamma_{Q,0}}{\beta_0}} dt}{(\epsilon + \beta_0 a_s(1 - e^{-2\epsilon t}))^{\frac{\gamma_{Q,0}}{\beta_0}}} = Q_{1,0}^{\mathbb{1}} \frac{(1 + \frac{a_s \beta_0}{\epsilon})^{1+\frac{\gamma_{Q,0}}{\beta_0}} - 1}{a_s(\beta_0 + \gamma_{Q,0})} \\ &= Q_{1,0}^{\mathbb{1}} \left(\frac{1}{\epsilon} + \frac{\gamma_{Q,0} a_s}{2\epsilon^2} + \frac{(-\beta_0 \gamma_{Q,0} + (\gamma_{Q,0})^2) a_s{}^2}{6\epsilon^3} \right) + \mathcal{O}(a_s{}^3) \; , \end{aligned} \tag{2.79}$$

which leads to divergent terms when expanded in a_s. Provided (2.78) is met for $n = 0$ the $\epsilon \to 0$ limit is finite and gives

$$\bar{L}_{QQ}^{\mathbb{1},\rm MS} = -\frac{Q_{1,0}^{\mathbb{1}}}{a_s(\beta_0 + \gamma_{Q,0})} \; . \tag{2.80}$$

Two important remarks are in order. First when (2.79) is expanded in powers of a_s then $1/\epsilon$-poles appear irrespective of whether condition (2.78) is met. This is an example where perturbation theory gives the wrong indication. In other words the a_s- and ϵ-expansion do *not* commute. The $\epsilon \to 0$ followed by $a_s \to 0$ limit does not exist for the O_Q-correlation function. In cases where the correlation function is related to a physical observable, such as the TEMT correlation function, there are a_s-dependent prefactors which assure a smooth limit (or uniformly converging $\epsilon \to 0$ limit).

[17] For $\hat{\beta}(u) = -\epsilon + \beta = -\epsilon - \beta_0 u^r + \mathcal{O}(u)$ this leads to $u(t) = (a_s e^{-2\epsilon t} \epsilon^{1/r})/(\epsilon + \beta_0 a_s{}^r(1 - e^{-2r\epsilon t}))^{1/r} \to a_s(1 + 2r\beta_0 a_s{}^r t)^{-1/r}$ for $\epsilon \to 0$ provided $r > 0$. This does not change conclusions in this paragraph.

Asymptotically safe theory

The non-trivial fixed point is characterised by generally non-vanishing anomalous dimensions $\gamma_Q = \gamma_Q^* + (a_s - a_s^{\rm UV})\gamma_{Q,0} + \ldots$. The integrating factor assumes the form $I_{\gamma_Q}(t) \sim e^{2\gamma_Q^* t}$. The exponential behaviour dominates over the polynomial behaviour of $\chi_{QQ}^{\rm MS}$. Hence the sign of γ_Q^* decides on the convergence

$$\gamma_Q^* < 0 \quad \Rightarrow \quad L_{QQ}^{\mathbb{1},{\rm MS}} \stackrel{\epsilon/a_s \to 0}{\longrightarrow} [\text{finite}] \,. \tag{2.81}$$

If $\gamma_Q^* > 0$ $L_{QQ}^{\mathbb{1},{\rm MS}}$ diverges and if $\gamma_Q^* = 0$ then the analysis of AF in the previous section applies.

2.2.2 Some further remarks

In summary the presence or absence of UV divergences depends on the anomalous dimension γ_Q and the leading power behaviour of the χ_{QQ}. A more detailed comparison is instructive. In the AF-case (2.78) the condition depends on both quantities mentioned above whereas in the AS-case (2.81) it only depends on the anomalous dimension at the FP. The polynomial behaviour of χ_{QQ} is overruled by the exponential behaviour of the anomalous dimension. This is reminiscent of marginal flows requiring specific analysis in order to determine whether or not they are relevant or exactly marginal, whereas relevant and irrelevant flows are settled from the start. The behaviour of the AS-case is similar to the case of a scale or conformaly invariant field theory. The 2-point function of operators, of scaling dimension $\Delta_O = d_O + \gamma_O$, is given by $\langle O(x)O(0)\rangle \sim (x^2)^{-\Delta_O}$. In our case $d_O = 4$ and the Fourier transform of the p^4-structure is convergent provided $\gamma_O < 0$ in accordance with the criteria for an AS theory (2.81).

A priori the divergent structure of 2-point function of dimension four operators in momentum space reads ($d = 4$)

$$\Gamma_{AB} \sim a\,\Lambda_{\rm UV}^4 + b\,p^2\Lambda_{\rm UV}^2 + c\,p^4 \ln\Lambda_{\rm UV} + [\text{finite}] \,, \tag{2.82}$$

for a cut-off regularisation. Above a, b, c are dimensionless functions of $\Lambda_{\rm UV}/\mu_0$ where μ_0 is some reference scale. In this section it was shown under what conditions $c_{\rm DR}(\Lambda_{\rm UV}/\mu_0)\ln\Lambda_{\rm UV} = [\text{finite}]$ holds for $\Lambda_{\rm UV} \to \infty$ in DR (symbolically $\ln\Lambda_{\rm UV} \leftrightarrow 1/\epsilon$). Since DR is defined only in perturbation theory one might question as to whether the result holds outside this framework. An argument in favour is that perturbation theory is trustworthy in the UV and that the LL approximation should therefore be sufficient. One assumption though is that the UV divergences can be captured as a Laurent expansion in powers of $1/\epsilon$. Whether or not this is valid outside perturbation theory is unknown since DR is only defined perturbatively. It is well-known that DR is blind to power divergences since no explicit scale is introduced into the integral regularisation other than the pre-factor $\mu^{-2\epsilon}$. Hence $a_{\rm DR} = b_{\rm DR} = 0$ is built into DR rather than being a result.[18]

2.2.3 QCD-like gauge theory as an example

We consider a QCD-like gauge theory, i.e. N_f massless fermions in a fundamental representation coupled to gluons in the adjoint representation (cf. Section 2.1.4). In particular this means that $\beta \neq 0$ at least at some length scales. We start this section by discussing the finiteness of the $\langle G^2 G^2 \rangle$- and the closely related $\langle \Theta\Theta \rangle$-correlator is established[19] followed by a discussion of the physical consequences: unsubtracted dispersion relation and scale independence. In Section 2.2.6 the discussion is extended to the QCD OPE with condensates.

Correlation functions of the field strength tensor

We consider the two-point function of the field strength correlation function, with

$$[O_g] = [\frac{1}{g_0^2} G^2] \, , \tag{2.83}$$

where the above operator is the field strength tensor squared defined in Section 2.1.4. From (2.43),(2.67) it follows that $\kappa_g = \hat{\beta}$ (cf. Appendix C.1 for the QCD β-function conventions) and therefore we have that $\gamma_g = 2\partial_{\ln a_s} \hat{\beta} = \gamma_{g,0} a_s + \mathcal{O}({a_s}^2)$. This leads to a simple form of the integrating factor (2.74)

$$I_{\gamma_g}(a_s, u) = \left(\frac{\hat{\beta}(u)}{\hat{\beta}(a_s)} \right)^2 \, . \tag{2.84}$$

[18] Let us mention, in passing, that it has been argued by Bardeen [80] that cut-off regularisations are not a natural choice for renormalizable theories. For example when a theory exhibits a global chiral symmetry one would preferably use a chirally invariant regularisation as otherwise the Ward Identities need to be fixed by adding local counterterms.

[19] The $\bar{q}q$-correlator, as an example which is generally not finite, is discussed in Section 2.2.4.

The corresponding Laurent series (2.73) takes on the form

$$L_{gg}^{\mathbb{1},\mathrm{MS}}(\epsilon) = -\frac{1}{\hat{\beta}^2(a_s)} \int_0^{a_s} \chi_{gg}^{\mathrm{MS}}(u)\hat{\beta}(u) \left(\frac{\mu(a_s)}{\mu(u)}\right)^{2\epsilon} \frac{du}{u} \ . \tag{2.85}$$

For this theory the relation (2.72) gives

$$\chi_{gg}^{\mathrm{MS}}(a_s) = \frac{\partial}{\partial a_s}(a_s g_1^{\mathbb{1}}) = \frac{n_g}{4\pi^2} + \mathcal{O}(a_s) \sim \mathcal{O}(a_s{}^0) \ , \tag{2.86}$$

where we used the LO expression for the first pole $g_1^{\mathbb{1}} = g_{1,0}^{\mathbb{1}} + \mathcal{O}(a_s)$ that was calculated in (2.39) (see Appendix G.1 for higher order contributions to $g_1^{\mathbb{1}}$) . From this and $\gamma_{g,0} = -2\beta_0$ we therefore conclude that the inequality (2.78) is satisfied with $1 - 2 < 0$. This means that

$$L_{gg}^{\mathbb{1},\mathrm{MS}}(\epsilon) \stackrel{\epsilon/\beta \to 0}{\longrightarrow} \bar{L}_{gg}^{\mathbb{1},\mathrm{MS}} = [\text{finite}] \ . \tag{2.87}$$

It is instructive to consider this constant at LL explicitly

$$L_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}} = \frac{g_{1,0}^{\mathbb{1}}}{\epsilon + \beta_0 a_s} \stackrel{\epsilon/(\beta_0 a_s) \to 0}{\longrightarrow} \bar{L}_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}} \equiv \frac{g_{1,0}^{\mathbb{1}}}{\beta_0 a_s} = [\text{finite}] \ , \tag{2.88}$$

as it becomes apparent that the correlation function is not finite for $a_s \to 0$ either. One should keep in mind that the field strength correlation function is not a physical quantity unlike the closely related correlation function of the TEMT to which we turn to now. Before doing so let us emphasise that in perturbation theory divergent contact terms appear and by expanding (2.85) we reproduce the divergent terms in [69, 71] at NLO and NNLO respectively. To obtain agreement it is important to expand the term to the power 2ϵ in the integrand.[20]

Correlation functions of the trace of the energy momentum tensor

Using (1.60) TEMT takes the following form in d dimensions

$$T^{\mu}{}_{\mu} = \Theta + \Theta_{\mathrm{eom}} + \Theta_{\mathrm{gf}} \ , \tag{2.89}$$

[20] One uses that $(\frac{\mu(a_s)}{\mu(u)})^{2\epsilon} = \exp(\epsilon \int_u^{a_s} \frac{du'}{(u'\hat{\beta}(u'))}) = \frac{u}{a_s(\mu)} + \mathcal{O}(\frac{1}{\epsilon})$.

where for the YM action with (2.83) we get

$$\Theta = -\frac{\hat{\beta}}{2}[O_g] \tag{2.90}$$

and $\Theta_{\rm eom}, \Theta_{\rm gf}$ correspond to EOM and gauge fixing part of the trace of the energy momentum tensor. The $\Theta_{\rm gf}$-part does not contribute to physical observables, $\Theta_{\rm eom}$ contributes to the $(p^2)^0$ structure so we can therefore concentrate on Θ. [21] Adapting the notation $[O_T] = \Theta$ [22] in analogy with (2.83), (2.62) together with (2.90) amounts to

$$\mathbb{C}^{\mathbb{1}}_{TT}(p^2) = \frac{\hat{\beta}^2}{4}\mathbb{C}^{\mathbb{1}}_{gg}(p^2) \ , \tag{2.91}$$

which in turn implies relation between the corresponding pole series

$$L^{\mathbb{1},\rm MS}_{TT} = \frac{\hat{\beta}^2}{4}L^{\mathbb{1},\rm MS}_{gg} + [\text{finite}] \ . \tag{2.92}$$

The quantity $L^{\mathbb{1},\rm MS}_{TT}$ is then obtained from (2.85) by multiplying by the pre-factor, partial integration and subtracting the finite constant in (2.92) (see the footnote 20)

$$L^{\mathbb{1},\rm MS}_{TT}(\epsilon) = \frac{1}{4}\int_0^{a_s} \partial_u\left(\frac{\beta}{u}\right) u \left(\left(\frac{\mu(a_s)}{\mu(u)}\right)^{2\epsilon} - \frac{u}{a_s}\right) g^{\mathbb{1}}_1(u) du \ . \tag{2.93}$$

The limiting expression $\bar{L}^{\mathbb{1},\rm MS}_{TT}$ is manifestly finite and well-behaved in the limit $\beta \to 0$ and $a_s \to 0$. For instance, the LL-expression is given by $\bar{L}^{\mathbb{1},\rm MS}_{TT}|_{\rm LL} = g^{\mathbb{1}}_{1,0}/4\beta_0 a_s$. It is to be concluded that both $\mathbb{C}^{\mathbb{1}}_{gg}(p^2)$ and $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)$ are finite.

There are two immediate consequences of the finiteness of $L^{\mathbb{1},\rm MS}_{gg}$ and $L^{\mathbb{1},\rm MS}_{TT}$ that we would like to discuss in the following subsections.

- The bare Wilson coefficients $\mathbb{C}^{\mathbb{1}}_{gg}(p^2)$ and $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)$ are finite. This means that both $\mathbb{C}^{\mathbb{1}}_{gg}(p^2)$ and $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)$ satisfy an unsubtracted dispersion relation since the dispersion integral has to converge as otherwise it would result in ϵ-divergent terms. This is illustrated at LL in Section 2.2.3.
- The crucial difference is that finiteness of $\mathbb{C}^{\mathbb{1}}_{gg}(p^2)$ only seems to be true if the theory isn't conformal with $\beta \neq 0$ whereas $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)$ is well defined also for $\beta = 0$. This suggests that the latter is the fundamental, physical

[21] Where the fermions are massive we can use (1.70) so that $\Theta \to \Theta + N_f m_f(1+\gamma_m)\bar{q}q$.

[22] This notation is consistent since as we saw in Section 1.2.3 the trace anomaly doesn't renormalize $[\Theta] = \Theta$.

quantity. It might not be so surprising because of the vanishing of Θ, which tends to smoothen the short-distance behaviour of its correlators.

Explicit convergent dispersion representation for leading log

The starting point is the LL expression (2.88). The associated logarithms can be obtained from $L_{gg}^{\mathbb{1},\mathrm{MS}}$ by replacing $\epsilon^{-n} \leftrightarrow -\ln^n(1/\mu^2)$ (which is derived in Appendix E.1 from the bare correlation function) and by dimensional analysis this implies $\epsilon^{-n} \leftrightarrow -\ln^n(-p^2/\mu^2)$. Considering the LL expression one gets

$$\begin{aligned}\mathbb{C}_{gg}^{\mathbb{1}}(p^2)\big|_{\mathrm{LL}} &= \mathbb{C}_{gg}^{\mathbb{1},\mathrm{MS}}(p^2)|_{\mathrm{LL}} + \left(L_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}}\right)_{\epsilon\to 0} \\ &= \left(L_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}}\right)_{\epsilon^{-n}\to -\ln^n\left(-\frac{p^2}{\mu^2}\right)} + \left(L_{gg}^{\mathbb{1},\mathrm{MS}}|_{\mathrm{LL}}\right)_{\epsilon\to 0} \\ &= -\frac{g_{1,0}^{\mathbb{1}}\ln(-p^2/\mu^2)}{1+a_s\beta_0\ln(-p^2/\mu^2)} + \frac{g_{1,0}^{\mathbb{1}}}{a_s\beta_0} = \frac{g_{1,0}^{\mathbb{1}}}{a_s\beta_0}x(p^2)\,, \end{aligned} \tag{2.94}$$

with

$$x(p^2) = \frac{1}{1+a_s\beta_0\ln(-p^2/\mu^2)}\,, \tag{2.95}$$

Since $x(p^2)$ is finite for $p^2 \to 0$, it obeys an unsubtracted dispersion relation of the form

$$x(p^2) = \frac{1}{2\pi i}\int_\Gamma \frac{x(s)}{s-p^2}\,, \tag{2.96}$$

where Γ is such that no singularities are crossed. The singularities of $x(p^2)$ are a branch cut $p^2 > 0$ and a pole in the euclidean domain at $p^2 = p_0^2 \equiv -\mu^2 e^{-\frac{1}{b_0 a}}$. The latter is an IR effect having to do with breakdown of LL approximation at lower p where subleading effects need to be considered, therefore not relevant for the UV finiteness discussion. It is convenient to split the dispersion representation into the pole part

$$x(p^2) = \frac{-1}{1-p^2/p_0^2} + \hat{x}(p^2)\,, \tag{2.97}$$

and the integration over the cut

$$\hat{x}(p^2) = \frac{1}{\pi}\int_0^\infty ds\frac{\mathrm{Im}[x(s)]}{s-p^2-i0} = \int_0^\infty \frac{ds}{s-p^2-i0}\frac{1}{(1+a\beta_0\ln(s/\mu^2))^2+(a\beta_0\pi)^2}\,. \tag{2.98}$$

Above it was used that $x(s) \to 0$ for $s \to \infty$ as otherwise the arc at infinity would contribute to the dispersion integral. This is the formal solution and it is easily seen that for finite p^2 the integrand behaves $\int_0^\infty ds/(s\ln(s/\mu^2)^2) < 0$ which is finite. The integral (2.98) is explicitly evaluated in the Appendix E.2 to reproduce the expression in (2.97). The dispersion relation for the TEMT-part simply $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)\big|_{\rm LL} = \beta_0^2/4{a_s}^2\mathbb{C}^{\mathbb{1}}_{gg}(p^2)\big|_{\rm LL}$.

The quantity $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)$ is scale independent and physical

In a theory with one scale a quantity with one momentum is given by $\varphi(p^2/\mu^2, a_s(\mu/\mu_0))$ where μ_0 is a reference scale, e.g. $\Lambda_{\rm QCD}$, which we suppress further below. In the case where φ is a physical quantity and therefore independent of the renormalization scale the functional dependence simplifies to

$$\frac{d}{d\ln\mu}\varphi(p^2/\mu^2, a_s(\mu^2)) = 0 \quad \Leftrightarrow \quad \varphi = \tilde{\varphi}(a_s(p^2)) \ . \tag{2.99}$$

Furthermore we expect $\tilde{\varphi}(a_s) \propto a_s$ from vanishing of TEMT in far UV. We will demonstrate that this is the case for $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)$ at LL starting with (2.97) one gets

$$\mathbb{C}^{\mathbb{1}}_{TT}(p^2)|_{\rm LL} = \frac{g^{\mathbb{1}}_{1,0}}{4}\frac{a_s(\mu^2)\beta_0}{1 + a_s(\mu^2)\beta_0\ln(-p^2/\mu^2)} = \frac{g^{\mathbb{1}}_{1,0}}{4}\beta_0 a_s(p^2) + \mathcal{O}(\beta_1) \ , \tag{2.100}$$

indeed a function which depends on $a(p^2)$ only and vanishes fast enough for large p^2. Note if we had subtracted $\bar{L}^{\mathbb{1}}_{gg}$ from $\mathbb{C}^{\mathbb{1}}_{gg}(p^2)$ then we would have obtained $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)|_{\rm LL} = g^{\mathbb{1}}_{1,0}\beta_0(a_s(p^2) - a_s(\mu^2))$ which is not physical in the sense that there's μ-dependence.
Hence the following dispersion representation holds

$$\mathbb{C}^{\mathbb{1}}_{TT}(p^2) = \frac{1}{\pi}\int_0^\infty ds \frac{{\rm Im}[\mathbb{C}^{\mathbb{1}}_{TT}(s)]}{s - p^2 - i0} \ . \tag{2.101}$$

which is distinct from the LL expression in that there are no spurious singularities on the negative axis. Note that in principle one may still add an arbitrary, but $a_s(\mu)$-independent, constant on the RHS by changing the theory by a local term in the UV. This constant should not impact on any physical prediction cf. Section 3.2.

2.2.4 $\bar{q}q$-gauge theory correlation function

Finally we consider the example of bifermion scalar operator

$$[O_m] = [\bar{q}q] \ , \quad \kappa_m = m \quad , \tag{2.102}$$

for which $m[\bar{q}q] = m_0\bar{q}q$ is an RG-invariant. The parameter m does not enter the dynamics and is regarded as a source term only. The relevant input to (2.71) or the criteria (2.78) is given by $\gamma_{m,0}$, $\chi^{\rm MS}_{mm}$ and β_0. The leading order of the mass anomalous dimension is given by

$$\gamma_m = \gamma_{m,0} a_s + \mathcal{O}({a_s}^2) \;, \quad \gamma_{m,0} = -6C_F \tag{2.103}$$

where $C_F = (N_c^2-1)/(2N_c)$, $\chi^{\rm MS}_{mm} = \partial_{a_s}(a_s m_1)$, $m_1(a_s) = m_{1,0} + \mathcal{O}(a_s)$, $m_{1,0} = \frac{N_c N_f}{8\pi^2}$ (cf. $L^{\mathbb{1}}_{mm}$ in Section 2.1.3) and β_0 is given in Appendix C.1. This means that in the formula (2.78) we take $n = 0$

$$L^{\mathbb{1},\rm MS}_{mm} = [\text{finite}] \quad \Leftrightarrow \quad -\frac{\gamma_{m,0}}{\beta_0}\Big|_{SU(N_c)} = \frac{3(N_c^2-1)/(4N_c)}{11/3N_c - 2/3N_f} > 1 \;. \tag{2.104}$$

This criteria is satisfied for $N_f > (9+2N_c^2)/(2N_c)$ which for $N_c = 3$ implies a value for $N_f > 4.5$.

The leading order expression for the pole function is given by

$$L^{\mathbb{1},\rm MS}_{mm}|_{\rm LL} = m_{1,0}\frac{(1+\frac{a_s\beta_0}{\epsilon})^{1+\frac{\gamma_{m,0}}{\beta_0}} - 1}{a_s(\beta_0+\gamma_{m,0})} \overset{(2.104)}{\longrightarrow} \bar{L}^{\mathbb{1},\rm MS}_{mm}|_{\rm LL} = -\frac{m_{1,0}}{a_s(\gamma_{m,0}+\beta_0)} \;. \tag{2.105}$$

From this expression it is seen that an $\epsilon \to 0$ limit exists for $1 + \gamma_{m,0}/\beta_0 < 0$ only as indicated on the right. For QCD with three massless flavours $N_f = 3$ and $N_c = 3$ the expression is divergent. Expanding in a_s one obtains

$$L^{\mathbb{1}}_{mm}|_{\rm LL} = m_{1,0}\left(\frac{1}{\epsilon} - \frac{\gamma_{m,0}a_s}{2\epsilon^2} + \frac{(\beta_0\gamma_{m,0} + (\gamma_{m,0})^2){a_s}^2}{6\epsilon^3}\right) + \mathcal{O}({a_s}^3) \;, \tag{2.106}$$

from where the leading poles in [81, 82] are recovered.
For the sake illustration let us quote the LL result, obtained by replacing $\frac{1}{\epsilon^n} \to -\ln^n\left(\frac{-p^2}{\mu^2}\right)$,

$$\begin{aligned}\Gamma^{\rm MS}_{mm}|_{LL}(p^2) &= \int d^4x e^{ip\cdot x}\langle 0|[\bar{q}q(x)][\bar{q}q(0)]|0\rangle_{\rm LL} \\ &= -p^2 m_{1,0}\frac{\left(1 - a_s\beta_0\ln\left(\frac{\mu^2}{-p^2}\right)\right)^{1+\frac{\gamma_{m,0}}{\beta_0}} - 1}{a_s(\beta_0+\gamma_{m,0})} + \ldots\end{aligned} \tag{2.107}$$

where the dots stand for condensate contributions. Expanding in $a_s\ln\left(\frac{\mu^2}{-p^2}\right)$ the $\mathcal{O}({a_s}^3)$-LL expression matches the result in [81].

Following Section 2.2.3 we explicitly demonstrate at LL that the bare correlator, multiplied by $\kappa_m^2 = m^2(\mu)$, is μ-independent in the following sense

$$
\begin{aligned}
m^2(\mu)\Gamma_{mm}(p^2,\mu) &= \mu_0^4 f(a_s(\mu^2/\mu_0^2), m/\mu_0, p^2/\mu_0^2) \\
&= p^2 m^2(p^2) F(a_s(p^2/\mu_0^2)) \ , \qquad (2.108)
\end{aligned}
$$

and μ_0 being an arbitrary reference scale. First we note that the renormalised correlator

$$
m^2(\mu)\Gamma_{mm}^{\rm MS}|_{\rm LL}(p^2) = p^2 \left(-m_{1,0}\frac{m^2(p^2)}{a_s(p^2)(\beta_0+\gamma_{m,0})} + m_{1,0}\frac{m^2(\mu)}{a_s(\mu)(\beta_0+\gamma_{m,0})} \right) \ , \qquad (2.109)
$$

splits into a μ-independent non-local and a μ-dependent local term. If we now restrict to the convergent case satisfying (2.104), then the second term is equal to (2.105) and in the $\epsilon \to 0$ limit

$$
\begin{aligned}
m^2(\mu)\Gamma_{mm}(p^2) &= m^2(\mu)\Gamma_{mm}^{\rm MS}(p^2) + \bar{L}_{mm}^{\mathbb{1},\rm MS} \\
&\overset{\rm LL}{=} -p^2 m_{1,0}\frac{m^2(p^2)}{a_s(p^2)(\beta_0+\gamma_{m,0})} \ , \qquad (2.110)
\end{aligned}
$$

which satisfies (2.108) in analogy with (2.100).

2.2.5 Multiple couplings and finiteness of TEMT correlators

In this subsection we proceed to show finiteness of $\langle\Theta\Theta\rangle$-correlator for a general field theory with UV fixed point. We start by considering an RG flow generated by deforming the UV theory by some marginal operators $\{O_A\}$ with corresponding couplings $\{g^A\}$. This will induce a trace anomaly[23]

$$
\Theta = \hat{\beta}^A [O_A] \ , \qquad (2.111)
$$

[23] Three possible structures are neglected. EOM and gauge-fixing terms can be omitted for the same reasons as before. It is assumed that no virial currents $\Theta = \partial \cdot V + \dots$ are present implicit in the assumption that the UVFP is conformal (no non-trivial unitary scale but not conformally invariant theories are known to date). Terms of the form $\Theta = -\Box\phi^2 + \dots$ originating from non-conformally coupled scalars can be improved (cf. Section 1.1.2 for the definition of improvement term). An exception is the chirally broken phase but since the term is relevant in the IR and not the UV we do not need to consider it for the purposes of this section.

where beta functions for couplings g^A of the theory read

$$\hat{\beta}^A = \frac{d}{d\ln\mu} g^A = \beta^A - \epsilon\xi^A g^A \ , \tag{2.112}$$

with ξ^A being the explicit (evanescent) mass dimension of g^A in $4-2\epsilon$ dimensions . The the theory has an UV fixed point so that $\beta^A \to 0$ for $\mu \to \infty$. The generalisation of (2.66) to the non-diagonal case is straightforward and given by

$$\mathbb{C}^{\mathbb{1}}_{AB}(p^2) p^{-2\epsilon} = \left(\mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}(p^2) + L^{\mathbb{1},\mathcal{R}}_{AB}\right)\mu^{-2\epsilon} \ . \tag{2.113}$$

The multiple coupling generalisation of (2.71) reads

$$(\mathcal{L}_\beta - 2\epsilon) L^{\mathbb{1},\mathcal{R}}_{AB} = -2\chi^{\mathcal{R}}_{AB} \ , \tag{2.114}$$

where $\chi^{\mathcal{R}}_{AB}$ is a finite function of couplings of the theory and $\mathcal{L}_\beta$ denotes the Lie derivative on a 2 tensor in coupling space

$$\mathcal{L}_\beta L^{\mathbb{1},\mathcal{R}}_{AB} = \partial_A \hat{\beta}^C L^{\mathbb{1},\mathcal{R}}_{CB} + \partial_B \hat{\beta}^C L^{\mathbb{1},\mathcal{R}}_{AC} + \hat{\beta}^C \partial_C L^{\mathbb{1},\mathcal{R}}_{AB} \ , \tag{2.115}$$

(∂_A defined in (2.5)). The generalization of (2.71) is seen through the anomalous dimension formula

$$\begin{aligned} \hat{\gamma}_A{}^B &= \partial_A \hat{\beta}^B \\ &= \partial_A \beta^B - \delta_A{}^B \xi^A \epsilon = \gamma_A{}^B - \delta_A{}^B \xi^A \epsilon \ , \end{aligned} \tag{2.116}$$

which follows from $\frac{d}{d\ln\mu}\langle\Theta\rangle = 0$ in flat space. The above equation is the analogue of $\gamma_g = \hat{\gamma}_g = 2\partial_{\ln a_s}\hat{\beta}$ stated below (2.83). The reason for $\gamma_g = \hat{\gamma}_g$ is that we used the logarithmic β-function for QCD-like theories for which the $\mathcal{O}(\epsilon)$-term is coupling independent. The quantity χ^{MS}_{AB} generalising (2.72) is then given by

$$\chi^{\text{MS}}_{AB} = \left(1 + \frac{1}{2}(\xi^A + \xi^B) + \frac{1}{2}\xi^Q g^Q \partial_Q\right) r^{\mathbb{1}(1)}_{AB} \ , \quad L^{\mathbb{1},\text{MS}}_{AB} = \sum_{n\geq 1} \frac{r^{\mathbb{1}(n)}_{AB}}{\epsilon^n} \ , \tag{2.117}$$

The RGE (2.114) can be solved by the method of characteristics in terms of the anomalous dimension matrices $\hat{\gamma}_A{}^B$

$$L^{\mathbb{1},\mathcal{R}}_{AB}(\mu) = 2\int_0^\infty I_A{}^C(t)\chi^{\mathcal{R}}_{CD}(t) I_B{}^D(t) e^{-2\epsilon t} dt \ , \tag{2.118}$$

where the t-dependence of $\chi^{\mathcal{R}}_{CD}$ inside the above integral comes in through the running of couplings defined at the scale μe^t and

$$I_A{}^B(t) = (\exp\left(\int_0^t \hat{\gamma}(t')dt'\right))_A{}^B \ . \tag{2.119}$$

It can be shown that[24]

$$\hat{\beta}^A(\mu) I(t)_A{}^B = \hat{\beta}^B(t) \ . \tag{2.120}$$

As previously $\mathbb{C}^{\mathbb{1}}_{TT}(p^2) = \hat{\beta}^A\hat{\beta}^B\mathbb{C}^{\mathbb{1}}_{AB}(p^2)$ and the generalisation of (2.92) reads

$$L^{\mathbb{1},\mathcal{R}}_{TT}(\mu) = \hat{\beta}^A\hat{\beta}^B L^{\mathbb{1},\mathcal{R}}_{AB}(\mu) + [\text{finite}] = 2\int_0^\infty \hat{\beta}^A(t)\hat{\beta}^B(t)\chi^{\mathcal{R}}_{AB}(t)e^{-2\epsilon t}dt + [\text{finite}] \ . \tag{2.121}$$

We will now argue that the $\epsilon \to 0$ limit of the above expression can be safely taken. Assuming $\chi_{AB} = \mathcal{O}(t^{-n_{AB}})$ with $n_{AB} \geq 0$ the integrand of (2.121) is controlled by the β-functions for large t which tend to 0 by the UVFP-assumption.

The criteria for finiteness of $L^{\mathbb{1},\mathcal{R}}_{TT}$ are easily generalised. For the cases of AF (1.53) and AS (1.55), close to UVFP we have $\beta^Q_{\text{AF}} = -\beta^Q_0/(4\pi)^2(g^Q)^{1+r_Q} + \ldots$ and $r_Q > 0$ and $\beta^Q_{\text{AS}} = |a_Q|(g^{Q^{\text{UV}}} - g^Q) + \ldots$. Expressed in the RG time variable t (assuming large t) this reads

$$\beta^Q_{\text{AF}} \sim \frac{1}{t^{(1+\frac{1}{r_Q})}} \ , \quad \hat{\beta}^Q_{\text{AS}} \sim e^{-|a_Q|t} \ . \tag{2.122}$$

This means that the terms in the integrand in (2.121) vanish at least as $t^{-2-|\alpha|}$ or are exponentially suppressed which guarantees convergence of the t-integral. Hence the $\epsilon \to 0$ limit can be taken safely and $L^{\mathbb{1},\mathcal{R}}_{TT}$ is finite which is the aimed result. Moreover the limit is uniform defining a smooth function of running couplings.

Is $\mathbb{C}^{\mathbb{1}}_{TT}$ renormalized? Just as we argued in Section 2.2.3 for one coupling, finiteness and μ-independence implies that $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)$ is physical and only depends on p^2 through running of couplings (assuming the theory in question is renormalizable).

[24] This follows by writing $\hat{\beta}^A(\mu)I(t)_A{}^B = f^B(t)$ which satisfies the differential equation $\partial_t f^B = f^C\hat{\gamma}_C{}^B(t)$ with initial condition $f^B(t=0) = \hat{\beta}^B(\mu)$. It is easy to show using (2.116) that $f^B(t) = \hat{\beta}^B(t)$ is the unique solution to to the initial value problem.

It is still possible to define a splitting

$$\mathbb{C}^{\mathbb{1}}_{TT}(p^2) = \hat{\beta}^A\hat{\beta}^B\mathbb{C}^{\mathbb{1}}_{AB}(p^2,\mu) \overset{\epsilon\to 0}{=} \mathbb{C}^{\mathbb{1},\mathcal{R}}_{TT}(p^2,\mu) + \beta^A\beta^B L^{\mathbb{1},\mathcal{R}}_{AB} \ , \tag{2.123}$$

where we defined $\mathbb{C}^{\mathbb{1},\mathcal{R}}_{TT} = \beta^A\beta^B\mathbb{C}^{\mathbb{1},\mathcal{R}}_{AB}$ and $\epsilon \to 0$ limit is assumed in the last equality. Using the μ-independence of $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)$ and (2.114) one can verify that

$$\frac{d}{d\ln\mu}\mathbb{C}^{\mathbb{1},\mathcal{R}}_{TT}(p^2,\mu) = -\frac{d}{d\ln\mu}\beta^A\beta^B L^{\mathbb{1},\mathcal{R}}_{AB} = 2\beta^A\beta^B\chi^{\mathcal{R}}_{AB}(\mu) \tag{2.124}$$

The above equation is consistent with

$$\beta^A\beta^B L^{\mathbb{1},\mathcal{R}}_{AB} = 2\int_{\mu}^{\infty}\beta^A(\mu')\beta^B(\mu')\chi^{\mathcal{R}}_{AB}(\mu')\frac{d\mu'}{\mu'} \ , \tag{2.125}$$

which is obtained by taking $\epsilon \to 0$ limit of the integral in (2.121) and changing the integration variable via $\frac{\mu'}{\mu} = e^t$.

Scheme dependence Finally we would like to discuss the scheme dependence of $L^{\mathbb{1},\mathcal{R}}_{AB}$, $\chi^{\mathcal{R}}_{AB}$. First, let us consider a change in subtraction scheme (2.63), obtained by adding an arbitrary finite term to the counterterm part

$$L^{\mathbb{1},\mathcal{R}'}_{AB} = L^{\mathbb{1},\mathcal{R}}_{AB} + \omega_{AB} \ , \tag{2.126}$$

where ω_{AB} is an arbitrary finite function of the couplings. From (2.114) we find the transformation law of $\chi^{\mathcal{R}}_{AB}$ under (2.126)

$$\chi^{\mathcal{R}'}_{AB} = \chi^{\mathcal{R}}_{AB} - \frac{1}{2}\mathcal{L}_\beta\omega_{AB} \ . \tag{2.127}$$

Using the definition (2.115) of the Lie derivative $\mathcal{L}_\beta$ together with the above equation it is easily verified that

$$\hat{\beta}^A\hat{\beta}^B\chi^{\mathcal{R}'}_{AB} = \hat{\beta}^A\hat{\beta}^B\chi^{\mathcal{R}}_{AB} - \frac{1}{2}\beta^C\partial_C(\hat{\beta}^A\hat{\beta}^B\omega_{AB}) \ . \tag{2.128}$$

Second, we might consider a scheme change for the renormalized couplings

$$g^A \to g'^A \ , \tag{2.129}$$

which is analogous to a coordinate transformation in General Relativity. Using (2.6) it is easy to see that $L^{\mathbb{1},\mathcal{R}}_{AB}$ transforms as a tensor under the 'coordinate' change (2.129)

$$(L^{\mathbb{1},\mathcal{R}}_{AB})' = (\partial'_A g^C)(\partial'_B g^D) L^{\mathbb{1},\mathcal{R}}_{CD} \ . \qquad (2.130)$$

After some algebra it can be shown that just as in Riemannian geometry the Lie derivative (2.115) commutes with coordinate transformations (2.129) and

$$(\chi^{\mathcal{R}}_{AB})' = (\partial'_A g^C)(\partial'_B g^D)\chi^{\mathcal{R}}_{CD} \qquad (2.131)$$

transforms covariantly.

2.2.6 OPE-extension with condensates

So far we have treated the correlation function (2.62) within the framework of perturbation theory in the sense that the vacuum condensates were neglected. To include non-perturbative contributions to OPE we need to extend (2.62) to include the condensates. A detailed discussion of OPE with condensates was given in Section 2.1.1 with explicit results for QCD in the Section 2.1.4. In this section, some of these results will be restated with particular emphasis on finiteness of contact terms.
We start by writing (2.9) for our QCD example

$$\Gamma_{gg}(p^2) = \mathbb{C}^{\mathbb{1}}_{gg}(p^2)p^4\langle\mathbb{1}\rangle + \mathbb{C}^{g}_{gg}(p^2)\langle[O_g]\rangle \ . \qquad (2.132)$$

It is our aim to investigate whether or not the Wilson coefficient $\mathbb{C}^{g}_{gg}(p^2)$, in analogy to $\mathbb{C}^{\mathbb{1}}_{gg}(p^2)$ is finite or not. In Section 2.1.4 we found that contact divergence of $\mathbb{C}^{g}_{gg}(p^2)$ is given by (2.44)

$$K^g_{gg} = \frac{2\partial_{\ln g}\hat{\beta}}{\hat{\beta}} + \frac{2(d-4)}{\hat{\beta}} \overset{\epsilon/\beta\to 0}{\to} [\text{finite}] \ . \qquad (2.133)$$

Thus we see again, that $\frac{1}{\epsilon}$ poles resum to yield a finite expression assuming $\beta \neq 0$. From (2.139) it then follows that $\mathbb{C}^{g}_{gg}(p^2)$ is finite in the limit $\epsilon \to 0$ but divergent in each order in perturbation theory just as $\mathbb{C}^{\mathbb{1}}_{gg}$. It is again instructive to write down the LL expression

$$\mathbb{C}^{g}_{gg}(p^2)|_{\mathrm{LL}} = \frac{-1}{1 + a_s\beta_0 \ln(-p^2/\mu^2)} \ , \qquad (2.134)$$

which has a (UV)-convergent dispersion integral.
We now proceed to discuss correlator of EMTs

$$\Gamma_{TT}(p^2) = \mathbb{C}^{\mathbb{1}}_{TT}(p^2)p^4\langle \mathbb{1} \rangle + \mathbb{C}^{T}_{TT}(p^2)\langle \Theta \rangle \;, \tag{2.135}$$

where $\mathbb{C}^{\mathbb{1}}_{TT}$ is given by (2.91) and

$$\mathbb{C}^{T}_{TT}(p^2) = \frac{\hat{\beta}}{2}\mathbb{C}^{g}_{gg}(p^2) \;. \tag{2.136}$$

It follows that $\mathbb{C}^{T}_{TT}$ is RG invariant. The contact divergences of $\mathbb{C}^{T}_{TT}$ are given by $K^{T}_{TT}(p^2) = \frac{\hat{\beta}}{2}K^{g}_{gg} = (d-4) + \partial_{\ln g}\hat{\beta}$ which is manifestly finite at each order in perturbation theory as well when every quantity, e.g. beta-function, is treated consistently in d-dimensions. Hence one can write down convergent dispersion relations for both $\mathbb{C}^{g}_{gg}(p^2)$ and $\mathbb{C}^{T}_{TT}(p^2)$ as done in Section 2.2.3. The LL expression

$$\mathbb{C}^{T}_{TT}(p^2)|_{\mathrm{LL}} = \frac{1}{2}\frac{\beta_0 a_s}{1 + a_s\beta_0 \ln(-p^2/\mu^2)} = \frac{1}{2}\beta_0 a_s(p^2) \;, \tag{2.137}$$

which is RG invariant as expected.
We can generalise this argument by using (2.1) and (1.64)

$$\Gamma_{TT}(p^2) = \mathbb{C}^{\mathbb{1}}_{TT}(p^2)p^4\langle \mathbb{1} \rangle + \sum_{C}\mathbb{C}^{C}_{TT}(p^2)\langle [O_C] \rangle \;, \tag{2.138}$$

where $\mathbb{C}^{C}_{TT}(p^2) = \beta^A\beta^B\mathbb{C}^{C}_{AB}(p^2)$. In the language of Section 2.1.1 each of the operators O_A corresponds to a deformation by a local coupling g^A and above we have assumed that all these deformations are marginal operators. In Section 2.1.1 it was argued that divergences of $\mathbb{C}^{C}_{AB}(p^2)$ are cancelled by local counterterm K^{C}_{AB} so that

$$\mathbb{C}^{C}_{AB}(p^2) - K^{C}_{AB} = [\text{finite}] \;, \tag{2.139}$$

where K^{C}_{AB} was given in (2.15) in terms of RG mixing matrix $\mathbb{Z}_A{}^I$. Using (1.65) we get

$$\beta^A K^{C}_{AB} = -\gamma^{C}_{B} = [\text{finite}] \;. \tag{2.140}$$

Therefore we also have

$$\beta^A\beta^B K^{C}_{AB} = [\text{finite}] \;, \tag{2.141}$$

which guarantees the finiteness of $\mathbb{C}^{C}_{TT}$.
With the finiteness of the condensate contribution to EMT OPE we have concluded this section. So far we have only discussed behaviour of the Wilson coefficients. To completely determine the short distance behaviour of two point

function, the knowledge of condensates is needed. Section 2.3 is devoted to this topic.

2.2.7 Summary and Discussion

In the second part of this chapter we re-examined the contact terms as solutions to their respective RGEs. Bulk of this discussion (in particular the formulas (2.73),(2.93),(2.118), (2.121)) and the finiteness analysis represent new results that appeared in [3]. In the literature it usually assumed (see for example [57], [69]) that the correlator $\langle\Theta\Theta\rangle$ has short-distance divergences that appear in the small coupling expansion. We have shown how these apparent divergences arise by expanding the finite expression (2.93) in a_s. Physical consequences of these results will be considered in the next chapter of this thesis.
The RGE solutions we provided were given in terms of closed form integrals with $\frac{1}{\epsilon}$ poles resummed. A particular care was given to their scheme-dependence. Asymptotic analysis of these integrals allowed us to draw conclusions about UV-convergence of OPE beyond the conventional perturbative investigations. We derived a condition on anomalous dimensions of operators that guarantees well defined $\epsilon \to 0$ limit of contact terms. This was then related to the convergence of corresponding dispersion relations. The physically most interesting case was the two point function of trace of the energy-momentum tensor (TEMT). Our momentum space analysis of this correlator has shown that large logarithms resum into RG evolution of the coupling constant leaving an expression satisfying unsubtracted dispersion relation. The reason for this is that the short-distance behaviour of TEMT correlators is influenced by the presence of UVFP where the vanishing of TEMT counteracts the contact divergences. The UV finiteness was shown to hold even in the multiple coupling case and with non-perturbative condensate contribution included. As a possible extension of this work one could consider redoing the same analysis with different regularization from DR, which could be helpful to understand the issue of potential power divergences.

2.3 Gluon Condensates in Hamiltonian Formalism

In this section we will delve further into the idea that differentiating finite/-physical quantity wrt to renormalized coupling produces an insertion of the renormalized operator. Only this time the couplings will be kept constant and emphasis will be on matrix elements of physical states rather than correlators. The main tool here is the *Feynman-Hellmann theorem* [83], which was originally derived in quantum mechanics, but it applies straightforwardly to quantum field theory in the case where the relevant part of the Hamiltonian is known. One such example is the fermion mass term of a gauge theory $\mathcal{H}_m = m\bar{q}q$ e.g. [84]. The Hamiltonian formalism of gauge theories is not straightforward because of the elimination of two degrees of freedom from the vector potential one of which is associated with the gauge freedom.

In [85] a Feynman Hellman relation for the gauge coupling constant was obtained by combining the trace anomaly, renormalization group equation (RGE) and the Feynman Hellmann theorem for the fermion mass. The relations read [85]:

$$g\frac{\partial}{\partial g}E_\varphi^2 = -\frac{1}{2}\langle\varphi|\frac{1}{g^2}[G^2]|\varphi\rangle_c \,, \tag{2.142}$$

$$g\frac{\partial}{\partial g}\Lambda_{\rm GT} = -\frac{1}{2}\langle\frac{1}{g^2}[G^2]\rangle_0 \tag{2.143}$$

where $G^2 = G_{\mu\nu}G^{\mu\nu}$ is the field strength tensor squared, the subscript c stands for the connected part, φ denotes a physical state (normalisation to be specified below) and $\langle X\rangle_0 \equiv \langle 0|X|0\rangle$ corresponds to the vacuum expectation value throughout. The scheme dependence of the matrix elements on the right hand side is determined by the scheme dependence of the couplings on the left hand side. The partial derivatives are understood in the sense of the RGE. That is to say implicit dependencies of other parameters on the coupling are not considered by definition. In Eq. (2.142) the momentum is taken to be independent of M_φ as in [85].[25] Relation (2.142) is valid for the following normalisation of states,

$$\langle\varphi(E',\vec{p'})|\varphi(E,\vec{p})\rangle = 2E_\varphi(2\pi)^{D-1}\delta^{(D-1)}(\vec{p}-\vec{p'}) \,, \tag{2.144}$$

where D stands for the space-time dimension. The cosmological constant $\Lambda_{\rm GT}$ contribution in (2.143) was defined as $\langle T^\mu{}_\mu\rangle_0 = D\Lambda_{\rm GT}$. The goal of this section is to derive these relations, after all, using a Hamiltonian formalism. The key observation is that by a canonical transformation (rescalings in the gauge coupling constant), one can obtain a suitable form of the Hamiltonian.

The following presentation is organised as follows. In Section 2.3.1 we pursue

[25] The latter is of significance (Section 2.3.3) for the derivation of the trace anomaly matrix element from an RGE for the Energy.

the derivation of relations (2.142,2.143) within the Hamiltonian formalism. In Sections 2.3.2, 2.3.2 and 2.3.2 we illustrate the formula within the Schwinger Model and the $\mathcal{N} = 2$ super Yang Mills theory (Seiberg-Witten theory). Relevant comments on the transformation of the measure under the canonical transformation can be found in Appendix D.2.

2.3.1 (Re)derivation in the Hamiltonian formalism

The suitable canonical transformation of the Hamiltonian

In the Hamiltonian formalism of a (non-abelian) gauge theory $\vec{\pi} = \vec{E}$ and $\vec{A}$ are the independent canonically conjugate variables. (e.g. [86]).[26] The Hamiltonian reads,

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_g + \mathcal{H}_C + \mathcal{H}_\mathcal{G} \;, \\ \mathcal{H}_g &= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) - \overline{q}(i\vec{\gamma}\cdot\vec{D} - m)q \;, \end{aligned} \tag{2.145}$$

where $\vec{D} = \vec{\partial} + ig\vec{A}$ is the gauge covariant derivative and q stands for fermions (quarks) in some representation of the gauge group. The magnetic field is defined as $2B_k = \epsilon_{kij}G_{ij} = \epsilon_{kij}(\partial_i A_j - \partial_j A_i + ig[A_i, A_j])$. The term $\mathcal{H}_\mathcal{G} = A_0^a \mathcal{G}^a$ with $\mathcal{G}^a = ((\vec{D}\cdot\vec{E})^a + \bar{q}t^a\gamma_0 q)$ corresponds to Gauss' law (i.e. one of Maxwell's equations). The expression $\mathcal{H}_C$ is associated with primary and secondary constraints (resulting in gauge transformation). Both $\mathcal{H}_\mathcal{G}$ and $\mathcal{H}_C$ vanish on matrix elements of physical states and shall therefore be omitted hereafter.

Our strategy is to make the dependence on the coupling g as simple as possible through the canonical transformation,

$$\begin{aligned} \vec{A} &\to \frac{1}{g}\vec{A} \\ \vec{E} &\to g\vec{E} \;. \end{aligned} \tag{2.146}$$

This leads to a Hamiltonian of the form,

$$\mathcal{H}_g = \frac{1}{2}(g^2\vec{E}^2 + \frac{1}{g^2}\vec{B}^2) - \overline{q}(i\vec{\gamma}\cdot\vec{D} + m)q \;, \tag{2.147}$$

[26] The variable A_0 is degraded to be a Lagrangian multiplier imposing Gauss' law in (cf.. $\mathcal{H}_\mathcal{G}$ below) and $\pi_0 = 0$ is at the heart of all the difficulties with the Hamiltonian formalism of gauge theories (parameterised by $\mathcal{H}_C$ below).

where, crucially, the only g-dependence is in front of the electric and magnetic field terms. It is important to note that the transformation in Eq. (2.146) leaves the measure of the path integral $\mathcal{D}\vec{E}\mathcal{D}\vec{A}$ invariant. First the transformation (2.146) does not affect the equal time canonical commutation relation, $[A^k(x_0,\vec{x}), E_l(x_0,\vec{y})] = i\delta^k_l \delta^{(D-1)}(\vec{x}-\vec{y})$; the (simple) Jacobian is therefore trivial. Second the measure is not affected by a rescaling anomaly of the type [87] since the two transformations in (2.146) exactly cancel each other (as outlined in Appendix D.2).

Gluon condensates from Hamiltonian

The Feynman-Hellmann theorem [83] in quantum mechanics (here $\langle\varphi|\varphi\rangle = 1$) states that

$$\frac{\partial}{\partial\lambda} E_\varphi(\lambda) = \langle\varphi|\frac{\partial}{\partial\lambda} H(\lambda)|\varphi\rangle \;, \tag{2.148}$$

where λ is a parameter. It is crucial that $|\varphi\rangle$ is an eigenstate of the Hamiltonian H. The rest follows from the normalisation being independent on the parameter λ. The adaption to quantum field theory solely involves the incorporation of the specific normalisation convention (2.144). The right hand side of (2.148), in our case, is obtained by differentiating (2.147)

$$g\frac{\partial}{\partial g}\mathcal{H}_g = g^2 \vec{E}^2 - \frac{1}{g^2}\vec{B}^2 = -\frac{1}{2}\frac{1}{g^2} G_{\mu\nu}G^{\mu\nu} \;. \tag{2.149}$$

This form is very close to Eqs. (2.142,2.143). In particular a Lorentz invariant result has emerged from the non-covariant Hamilton formalism as is usually the case. Note, the Hamiltonian is a physical quantity and is therefore not renormalized. Below we shall write the Hamiltonian in terms of renormalized quantities which is natural since the physical quantities are matrix elements thereof. Identifying $\langle\mathcal{H}\rangle_0 = \Lambda_{\rm GT}$ one gets (2.143) from (2.149):

$$g\frac{\partial}{\partial g}\Lambda_{\rm GT} = \langle g\frac{\partial}{\partial g}\mathcal{H}\rangle_0 + \Lambda_{\rm GT}\, g\underbrace{\frac{\partial}{\partial g}\langle 0|0\rangle}_{=0} \stackrel{(2.149)}{=} -\frac{1}{2}\langle\frac{1}{g^2}[G^2]\rangle_0 \;. \tag{2.150}$$

For the derivation of (2.142) the factor E_φ in the normalisation (2.144) complicates the algebra and we shall use $\sqrt{2E_\varphi}|\tilde{\varphi}\rangle = |\varphi\rangle$ below restoring the factor in the end.

$$g\frac{\partial}{\partial g}E_\varphi = g\frac{\partial}{\partial g}\langle\tilde{\varphi}|\mathcal{H}|\tilde{\varphi}\rangle_c = \langle\tilde{\varphi}|g\frac{\partial}{\partial g}\mathcal{H}|\tilde{\varphi}\rangle_c + \frac{E_\varphi}{V}\, g\underbrace{\frac{\partial}{\partial g}\langle\tilde{\varphi}|\tilde{\varphi}\rangle_c}_{=0} = \langle\tilde{\varphi}|\frac{1}{g^2}[G]^2|\tilde{\varphi}\rangle_c$$

where V is the volume. Above we have identified $(2\pi)^{D-1}\delta^{(D-1)}(\vec{p}-\vec{p'}) = \int d^{D-1}x$ (in the sense of distributions) since the Hamiltonian is given by $H = \int d^{D-1}x\mathcal{H}$. Restoring the normalisation (2.144) we get an expression,

$$2E_\varphi g\frac{\partial}{\partial g}E_\varphi = \langle\varphi|\frac{1}{g^2}[G^2]|\varphi\rangle_c\ , \tag{2.151}$$

which is equivalent to (2.142). We have therefore rederived Eqs. (2.142,2.143) in a Hamiltonian framework which was the main goal of our work. We proceed to illustrate the formula in three models where exact results are known.

2.3.2 Examples

The relation (2.142) was used [88] to derive the scaling corrections to the hadron masses in two alternative ways. It therefore constitutes one independent check. Below we provide three further examples.

Photon mass in the Schwinger Model

Two dimensional quantum electrodynamics, known as the Schwinger model [89, 90] (for a review cf.. [91]), has served as a test ground for many formal approaches and lattice simulations. A curious feature of the Schwinger model is that the photon acquires a mass through the chiral anomaly as the η' in quantum chromodynamics. This is sometimes referred to as a dynamical Higgs mechanism. The photon mass is:

$$M_\gamma^2 = \frac{e^2}{\pi}\ . \tag{2.152}$$

The relation (2.142) adapted to the Schwinger model, for a massive photon state at rest, reads:

$$e\frac{\partial}{\partial e}M_\gamma^2 = -\frac{1}{2}\langle\gamma|G^2|\gamma\rangle_c\ . \tag{2.153}$$

Above $G^2 = G_{\mu\nu}G^{\mu\nu}$ is the electromagnetic field strength tensor squared and e is the charge of mass dimension one. The latter does not receive any renormalization (vanishing beta function).

In order to obtain (2.152) from (2.153) we have to evaluate the matrix element $\langle\gamma|G^2|\gamma\rangle_c$ for which we resort to the operator solution of the Schwinger model [92] (e.g. chapter 10 [91]). The field strength tensor is given by

$$G_{\mu\nu} = \frac{\sqrt{\pi}}{e}\epsilon_{\mu\nu}\Box\Sigma \ , \tag{2.154}$$

where $\Box = \partial_\mu\partial^\mu$ is the Laplacian and Σ is a canonically normalised free field of mass e^2/π. Choosing the connected part automatically fixes the scheme of the matrix element, which incidentally corresponds to normal ordering as used in ordinary perturbation theory: $\langle G^2\rangle_0 = 0$. This is not surprising since there is no scheme ambiguity on the left hand side as the coupling does not run. Through an explicit computation in terms of creation and annihilation operators one gets,

$$\langle\gamma|G^2|\gamma\rangle_c = \frac{\pi^2}{e^2}\epsilon_{\mu\nu}\epsilon^{\mu\nu}2(-M_\gamma^2)^2 = -4\frac{e^2}{\pi} \ , \tag{2.155}$$

where the factor of 2 is of combinatorial nature and we have replaced $\Box \to -q^2 = -M_\gamma^2$. Inserting (2.155) into (2.153) we get:

$$e\frac{\partial}{\partial e}M_\gamma^2 = 2\frac{e^2}{\pi} \quad \Rightarrow \quad M_\gamma^2 = \frac{e^2}{\pi} + C \ , \tag{2.156}$$

where C is a constant. From the limit $e \to 0$, where we expect $M_\gamma \to 0$, we infer $C = 0$ and therefore (2.156) corresponds to the exact result (2.152) known in the literature. In essence we have shown that (2.154) and (2.153) implies the Photon mass (2.152).

As an additional, but not necessary test, we can verify whether (2.153) is compatible with an RGE. The trace of the energy momentum tensor in massless QED, in terms of bare quantities, reads $T^\mu{}_\mu = -(D-4)\mathcal{L} + \text{cf.}$, where EOM stands for terms which vanish by the equation of motions. The latter are not of interest for us as we shall evaluate the trace on physical states. Using $D = 2$ we get

$$\langle\gamma|T^\mu{}_\mu|\gamma\rangle_c = -\frac{1}{2}\langle\gamma|G^2|\gamma\rangle_c \ , \tag{2.157}$$

and since $2M_\gamma^2 = \langle\gamma|T^\mu{}_\mu|\gamma\rangle_c$ it can be combined with (2.153) into

$$(e\frac{\partial}{\partial e} - 2)M_\gamma^2 = 0 \quad \Rightarrow \quad M_\gamma^2 = C'\, e^2 \tag{2.158}$$

where C' is a constant ($C' = 1/\pi$ according to (2.152)) and the equation on the right hand side corresponds to an RGE. In fact the latter is equivalent to an equation based on dimensional analysis on grounds of the fact that there are no running quantities in the Schwinger model.

Vacuum energy in massive mutliflavour Schwinger model

The Schwinger Model with N_f massive fermions has aspects which are known exactly (cf.. [93] and references therein). The model has got a global $SU_L(N_f) \times SU_R(N_f)$ flavour symmetry which is explicitly broken down to $SU_V(N_F)$ by the fermion mass term. The spectrum consists of one massive boson (the massive photon of the proceeding section) and $N_f^2 - 1$ quasi Goldstone boson, similar to the η' and the octet π, K, η in quantum chromodynamics. The situation is though distinct in that the quark condensate does not form in the massless case and the quasi Goldsone bosons show scaling behaviour of a critical theory. The vacuum energy is proportional to the mass gap squared (for $m \ll e$ cf.. [93] and references therein):

$$\Lambda_{\rm GT} \propto M_{\rm gap}^2 \propto m^{\eta_m} e^{\eta_e} \; , \quad \eta_m = \frac{2N_f}{N_f+1} \; , \;\; \eta_e = \frac{2}{N_f+1} \; . \tag{2.159}$$

From the trace anomaly equation one gets:

$$2\Lambda_{\rm GT} = -\frac{1}{2}\langle G^2\rangle_0 + N_f m \langle \bar{q}q\rangle_0 \; . \tag{2.160}$$

The analogous equation for four dimension is given in [85]. The adaption of the G^2-term to two dimensions has been discussed in the previous section and the anomalous dimension of the mass is zero. Using (2.143) and $N_f m_f \langle \bar{q}q\rangle_0 = m\frac{\partial}{\partial m}\Lambda_{\rm GT}$ one gets

$$2\Lambda_{\rm GT} = e\frac{\partial}{\partial e}\Lambda_{\rm GT} + m\frac{\partial}{\partial m}\Lambda_{\rm GT} = \underbrace{(\eta_e + \eta_m)}_{=2}\Lambda_{\rm GT} \; , \tag{2.161}$$

a consistent result. Summarising we obtain $\langle G^2\rangle_0 = -2\eta_e\Lambda_{\rm GT}$ and $N_f m\langle \bar{q}q\rangle_0 = \eta_m\Lambda_{\rm GT}$. Again (2.161) reveals itself directly equivalent to an RGE for $\Lambda_{\rm GT} = \Lambda_{\rm GT}(m,e)$

$$\left(e\frac{\partial}{\partial e} + m\frac{\partial}{\partial m} - \Delta_{\Lambda_{\rm GT}} \right) \Lambda_{\rm GT}(m,e) = 0 \; . \tag{2.162}$$

Above $\Delta_{\Lambda_{\rm GT}} = 2$ is the scaling dimension of the $\Lambda_{\rm GT}$ which is free from anomalous scaling as it is an observable. As (2.158) Eq. (2.162) is merely an equation that follows from dimensional analysis since all the scale breaking is explicit and not anomalous.

Magnetic monopole in Seiberg-Witten theory

The $\mathcal{N} = 2$ pure super Yang-Mills theory (with gauge group $SU(2)$), known as Seiberg-Witten theory [94], has features which are known exactly. In particular it is known that BPS states obey [94],

$$M_{(n_e,n_m)} = 2|Z|^2 \quad \text{with} \quad Z = n_e a + n_m a_D \ , \tag{2.163}$$

where n_e and n_m count the units of electric and magnetic charges. Exact solutions for a and a_D along with the effective coupling constant $\tau(a)$ constitute part of the work of Seiberg and Witten [94]. First we are going to derive Eq. (2.149) for the BPS sector. In the magnetic BPS sector the relevant part of the Hamiltonian reads [94]

$$\mathcal{H}_{\text{BPS}} = \frac{1}{g^2}\vec{D}\phi\cdot\vec{D}\phi + \frac{1}{2}\frac{1}{g^2}\vec{B}^2 \ , \tag{2.164}$$

where we shall comment on the (non-)significance of the extra $1/g^2$-factor in front of the scalar kinetic term shortly below. Note, Maxwell's equations imply $\vec{E} = 0$ for static solution with $\vec{B} \neq 0$ (magnetic monopole). The fermionic terms are absent by construction of what is known as a BPS state in supersymmetry. Using the BPS equation,

$$\vec{D}\phi|\text{BPS}\rangle = \frac{1}{\sqrt{2}}\vec{B}|\text{BPS}\rangle \ , \tag{2.165}$$

the total Hamiltonian becomes,

$$\mathcal{H}_{\text{BPS}} = \frac{1}{g^2}\vec{B}^2 \ , \tag{2.166}$$

and the $\mathcal{N} = 2$ supersymmetry, which is responsible for the $1/g^2$-factor in front of the kinetic term in (2.164), effectively introduces a factor of 2 in the relation (2.142). This can be seen explicitly by differentiating, with respect to the coupling constant (2.149),

$$g\frac{\partial}{\partial g}\mathcal{H}_{\text{BPS}} = -2\frac{1}{g^2}\vec{B}^2 \overset{\vec{E}=0}{=} -\frac{1}{g^2}G^2 \tag{2.167}$$

and comparing with Eq. (2.167). In summary we have shown that in Seiberg-Witten theory (2.142) holds on the BPS subspace. Conversely assuming that the formula (2.142) is true we know that (2.165) has to hold for $\mathcal{H}_{\mathrm{BPS}}$ in (2.164).

Unlike in the Schwinger model we cannot compute the matrix elements in (2.167) on the BPS states directly. We may turn things around and use the formula to express the matrix elements for the magnetic monopole in terms of a_D which is known explicitly in terms of the coupling constant. Formula (2.142) adapted for $\mathcal{N} = 2$ supersymmetry (with factor of two difference as explained above) reads:

$$\langle (0, n_m)| \frac{1}{g^2} G^2 |(0, n_m)\rangle_c = -g \frac{\partial M^2_{(0,n_m)}}{\partial g} \, , \tag{2.168}$$

In order to evaluate the right hand side we use $M^2_{(0,n_m)} = 2n_m^2 |a_D|^2$ (2.163) and $g\frac{\partial}{\partial g} = -\frac{1}{2}\omega \frac{\partial}{\partial \omega}$ where $\omega \equiv \frac{1}{g^2}$,[27]

$$\begin{aligned} \frac{1}{n_m^2} \frac{\partial M^2_{(0,n_m)}}{\partial \omega} &= 2[a_D^* \frac{\partial a_D}{\partial \omega} + a_D \frac{\partial a_D^*}{\partial \omega}] = 8\pi i [a_D^* \frac{\partial a_D}{\partial \tau} - a_D \frac{\partial a_D^*}{\partial \tau^*}] \\ &= -16\pi \, \mathrm{Im}[a_D^* \frac{\partial a_D}{\partial \tau}] \, . \end{aligned} \tag{2.169}$$

This leads to

$$\langle (0, n_m)| \frac{1}{g^2} G^2 |(0, n_m)\rangle_c = 8\pi \frac{n_m^2}{g^2} \, \mathrm{Im}[a_D^* \frac{\partial a_D}{\partial \tau}] \, . \tag{2.170}$$

The function a_D is known [94]

$$a_D(\tau) = \frac{\sqrt{2}\Lambda}{\pi} \int_1^{v(\tau)} \frac{dx \sqrt{x - v(\tau)}}{\sqrt{x^2 - 1}} \, , \quad v(\tau) = -1 + \frac{2}{\lambda(\tau)} \, , \tag{2.171}$$

with $v = u/\Lambda^2$ where $u = \langle \phi^2 \rangle_0$ is a modulus and Λ is a dynamical scale and constitute important parameters of the theory. The function $\lambda(\tau)$ is given in [95]. We have checked numerically that the condensate is zero for $g_D \propto 1/g \to 0$ and increases monotonically as a function of g_D. The coupling g_D corresponds to the magnetic coupling and is dual to the electric coupling g. Loosely speaking the magnetic monopole condensate is governed by the magnetic coupling g_D.

[27] In doing so use the fact that a_D is a holomorphic function of holomorphicity in $\tau = 4\pi i/g^2 + \frac{1}{2\pi}\theta$.

2.3.3 Trace anomaly and the Hamiltonian

In this section we show how the matrix element of the QCD trace anomaly (1.70) follows from an RGE of the Hamiltonian matrix elements. We consider

$$h(g,m,\mu,p) = \langle H(p)|\mathcal{H}|H(p)\rangle \stackrel{(2.144)}{=} 2(E_\varphi(p))^2 \ , \qquad (2.172)$$

where $p = |\vec{p}|$ denotes the spatial angular momentum which is considered to be an external parameter. By the latter we mean that it is in particular *independent* on M_φ in accordance with the remark below Eq. (2.142). This type of matrix element satisfies an RGE of the form (e.g. [15])

$$(\beta g\frac{\partial}{\partial g} - m(1+\gamma)\frac{\partial}{\partial m} + \Delta_h - p\frac{\partial}{\partial p})h(g,m,\mu,p) = 0 \ , \qquad (2.173)$$

where $\Delta_h = 2$ is the scaling dimension of (2.172) which corresponds to the engineering dimension since E_φ is a physical observable. Using the fact that the p-dependence is known exactly, $h = 2E_\varphi^2 = 2(M_\varphi^2 + \vec{p}^2)$, one can rewrite (4.26) as

$$(\beta g\frac{\partial}{\partial g} - m(1+\gamma)\frac{\partial}{\partial m} + \Delta^{\rm eff}_{E^2})E_\varphi^2 = 0 \ , \quad \Delta^{\rm eff}_{E^2} \equiv 2\frac{M_\varphi^2}{E_\varphi^2} \ . \qquad (2.174)$$

The two derivatives in (2.174) can be substituted by the relation (2.142) and $m\frac{\partial}{\partial m}E_\varphi^2 = m\langle\bar{q}q\rangle_\varphi$. One obtains,

$$2M_\varphi^2 = -\frac{\beta}{2}\langle\frac{1}{g^2}[G^2]\rangle_\varphi + (1+\gamma)m\langle[\bar{q}q]\rangle_\varphi \ , \qquad (2.175)$$

which corresponds to the well-known matrix element of the trace anomaly (1.70) between a physical state (e.g. [85]).

Note that this section corresponds to the, almost, backwards derivation of [85] where the Feynman-Hellmann relation (2.142) is derived from the trace anomaly. Furthermore it is also closely related to heuristic derivation of the trace anomaly using $T^\alpha_{\ \alpha} \propto \frac{d}{d\mu}\mathcal{L}(\mu)$. The main reason for presenting the derivation is to clarify how matters work out for states with non-zero spatial momenta (i.e. $M_\varphi^2 \neq E_\varphi^2$). The latter necessitate an RGE where the external momenta are taken into account.

2.3.4 Summary and Discussion

In the final section of this chapter we have derived the relations in Eqs. (2.142,2.143), previously obtained in [85] through the trace anomaly, the Feynman-Hellmann theorem and an RGE, in a Hamiltonian formulation of gauge theories. This derivation is new and it extends the original proof [85] in that it can be directly applied to gauge theories with multiple couplings.

The derivation presented here contains two ingredients. First, we start by eliminating the terms which vanish as matrix elements from the Hamiltonian. In this way we bypass the notoriously difficult problem of gauge fixing. The second step is a canonical transformation which arranges the Hamiltonian in such a way that only the $\vec{E}^2$ and $\vec{B}^2$-terms depend on the gauge coupling. The derivative with respect to the gauge coupling then gives rise to the explicitly Lorentz invariant result. A subtle point, which we have verified in Appendix D.2, is that the canonical transformation is free from rescaling anomalies of the Konishi type. One possible advantage of the Hamiltonian derivation is that it makes it clear that the relations holds for gauge theories with more than one gauge coupling. Furthermore we have tested the relation within the Schwinger Model and the $\mathcal{N} = 2$ super Yang Mills theory (Seiberg-Witten theory). An interesting extension of this work would be further independent verification of relations (2.142) within the AdS/CFT framework or lattice simulations. Another possible venue could be application to cosmological models, where gluon condensate may parametrise QCD contribution to cosmological constant through (2.142).

Chapter 3

Curved Space and Moments

Large part of the preceding chapter was devoted to correlators of composite operators. In particular we studied the correlators of the energy-momentum trace (EMT) operator. In this chapter we will give those correlators more geometrical meaning when considering a QFT in curved space. We will start this chapter discussing how by coupling the theory to a curved background the metric can serve as a source for the energy-momentum tensor. This does not come without the price since introducing the curvature introduces and extra scale which means that even if the theory is CFT, there will still be a c-number violation of the conformal symmetry through curved space trace anomaly [96]

$$T^{\rho}{}_{\rho} = -\beta_a E_4 - \beta_c W^2 + 4\tilde{d}\Box H \ , \quad H \equiv \frac{1}{(d-1)} R \ , \tag{3.1}$$

where the constants $\beta_a, \beta_c, \tilde{d}$ in front of the above geometrical terms (to be defined later in this chapter) have physical meaning and are called *central charges*. In this chapter (and the next one) we will study how some of these central charges change along the RG flow.

Correlators of the EMT are then renormalized by adding suitable gravitational counterterms to the action [97]. We saw how this works in Section 2.1 for local couplings. This time the metric will take a role of local coupling. The resulting counterterms can then be seen as a source of the trace anomaly (3.1) through their non-invariance under Weyl transformations. In the first part of this chapter we will review process focusing on the R^2 counterterm.

Finally the connection with the results from Section 2.2 will be made. The finiteness of EMT correlators will gain concrete physical meaning in association with the flow of central charge $\tilde{d}$. The result in Section 3.1.3 and the analysis presented in Section 3.2 are part of [4].

3.1 Trace anomaly in curved space

This section is mostly meant as a review of the work of Freeman [58] and Hathrell [57]. In particular we will concentrate the connection between gravitational counterterms and the trace anomaly in curved space. It will be very useful introducing the basic technology and terminology that is needed when treating gauge theories coupled to curved metric.

3.1.1 EMT and the Weyl transformations

We have already studied the energy-momentum tensor in Section 1.1.2 where it was associated with the Noether's current of coordinate transformations. The alternative definition follows when we couple the theory to a metric $g_{\mu\nu}$ and consider the variation of the action

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}(x)} \ , \tag{3.2}$$

where $g \equiv \det g_{\mu\nu}$. This suffices to define insertions of $T_{\mu\nu}$ into Green's functions

$$\langle T_{\mu\nu}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}(x)} \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle \ , \tag{3.3}$$

which holds up to potential contact terms that may arise if operators $\mathcal{O}_i$ depend on the metric.
The TEMT can then be written

$$T^{\mu}{}_{\mu}(x) = \frac{2}{\sqrt{g}} g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}(x)} \ . \tag{3.4}$$

This equivalent to varying the action w.r.t.. to an infinitesimal transformation of the metric

$$g_{\mu\nu} \to e^{-2\alpha} g_{\mu\nu} \ . \tag{3.5}$$

This transformation is called *Weyl rescaling.* We saw in Section 1.1.2 that EMT can be understood as a response to a dilation. Indeed, the Weyl rescaling can be viewed as a *local* rescaling of distances with

$$\sqrt{g}T^{\mu}{}_{\mu}(x) = \frac{\delta S}{\delta\alpha(x)} \ . \tag{3.6}$$

In fact the connection of Weyl rescaling with dilatations and special conformal transformations can be made more precise. In the language of Riemannian geometry the conformal transformations (1.2) are diffeomorphisms that rescale the metric

$$\begin{aligned} \delta x_\mu &= \xi_\mu \\ \delta g_{\mu\nu} &= -(\frac{1}{2}\nabla^\rho \xi_\rho) g_{\mu\nu} \ . \end{aligned} \tag{3.7}$$

Where ξ^μ are the Killing fields corresponding the Lie algebra of the conformal group SO(4,2). From (1.2) we have

$$\begin{array}{lll} \xi^\mu = & \lambda x^\mu & \text{Dilatations} \\ \xi^\mu = & (x^2 f^\mu - 2x^\mu x^\nu f_\nu) & \text{Special conformal transformations} \end{array} \tag{3.8}$$

To relate the above with infinitesimal Weyl rescaling

$$\delta g_{\mu\nu} = -2\alpha(x) g_{\mu\nu} \ ,$$

one makes the following association

$$\alpha(x) = \frac{1}{4}\nabla^\mu \xi_\mu \ . \tag{3.9}$$

Substituting (3.8) in (3.9) we get the respective Weyl parameters

$$\begin{array}{lll} \alpha(x) = & \lambda & \text{Dilatations} \\ \alpha(x) = & -2(x \cdot f) & \text{Special conformal transformations} \end{array} \tag{3.10}$$

Next we are going to consider few examples of how (3.6) can be used to find the EMT.

Free scalar field

A general action for free, massless scalar field coupled to curved space reads

$$S^{(0)} = \frac{1}{2}\int d^dx\sqrt{g}\phi[-\Box + \zeta R]\phi \ , \tag{3.11}$$

where $\Box \equiv \nabla^2$ and R is the Ricci scalar. A Weyl variation (3.6) of this action this action reads

$$\frac{\delta S^{(0)}}{\delta\alpha(x)} = \frac{1}{2}(2-d)\sqrt{g}\phi[-\Box + \zeta R]\phi + \frac{1}{2}\sqrt{g}\left[-\frac{(d-2)}{2} + 2(d-1)\zeta\right]\Box\phi^2 \ , \tag{3.12}$$

where we used the identities from Appendix F.1. The first term in the above equation is proportional to the equation of motion so it can be ignored in the final expression for EMT

$$T^{\mu}{}_{\mu} = \frac{1}{2}\left[-\frac{(d-2)}{2} + 2(d-1)\zeta\right]\Box\phi^2 \ , \tag{3.13}$$

where we have taken the flat limit $g_{\mu\nu} \to \delta_{\mu\nu}$. Note that for $d = 4$, $\zeta = 0$ the above expression reduces to the canonical trace (1.19). We can now see how the improvement procedure from Section 1.1.2 works. By choosing $\zeta = \frac{(d-2)}{4(d-1)}$ the trace (3.13) can be made to vanish. In fact this procedure can be defined for a generic theory with

$$T^{\mu}{}_{\mu} = \Box\mathcal{O} \ , \tag{3.14}$$

with $\mathcal{O}$ being a dimension 2 operator. This precisely the form required for conformal invariance(1.14). By including a counterterm $-\frac{1}{2(d-1)}R\mathcal{O}$ to the Lagrangian, the total Weyl variation will vanish leaving $T^{\mu}{}_{\mu} = 0$.

Fermion field

The action of free, massless fermion coupled to curved space reads

$$S^{(\frac{1}{2})} = \int d^dx\sqrt{g}\bar{\psi}\not{D}\psi \ , \tag{3.15}$$

where $\not{D}$ is the curved space Dirac operator defined using spin connection ω. The Weyl variation of this action can again be evaluated using formulas in the

Appendix F.1

$$T^{\mu}{}_{\mu} = \frac{\delta S^{(\frac{1}{2})}}{\delta\alpha(x)} = \frac{(d-1)}{2}\sqrt{g}\bar{\psi}\overleftrightarrow{\not{D}}\psi \ , \tag{3.16}$$

where $\overleftrightarrow{\not{D}} = \overrightarrow{\not{D}} - \overleftarrow{\not{D}}$. The above operator (3.16) is proportional to the fermion equation of motion. Note that this conclusion does not change when we include gauge interaction $\bar{\psi}\not{A}\psi$ - one simply replaces the Dirac operators in (3.16) with gauge covariant derivatives. Thus the contribution of fermions to gauge theory EMT is

$$\Theta_{\psi} = \frac{(d-1)}{2}(\psi\frac{\delta}{\delta\psi} + \bar{\psi}\frac{\delta}{\delta\bar{\psi}})S \ . \tag{3.17}$$

The above form implies that Θ_{ψ} does not contribute to correlators of the full gauge theory EMT (2.89). To show this one starts from the observation that using functional integration by parts it can be shown [21]

$$\langle\Theta_{\psi}(x_1)...\Theta_{\psi}(x_N)\rangle \propto \sum_i \delta^d(x_1 - x_i)\langle\Theta_{\psi}(x_2)...\Theta_{\psi}(x_N)\rangle \ . \tag{3.18}$$

One can now continue this procedure until left with single insertion of Θ_{ψ} which gives 0. The same arguments hold if we include arbitrary number of insertions of $\Theta \propto [G^2]$ in (3.18) since $\frac{\delta}{\delta\psi}G^2 = 0$.

Gauge theory in dimensional regularization

The curved space action for gauge field can be written as

$$S = \frac{1}{4}\int d^dx\sqrt{g}\frac{1}{g_0^2}G^2 \ , \tag{3.19}$$

where $G^2 = G^a_{\mu\nu}G^{\mu\nu}_a$ with rank 2 antisymmetric field strength tensor $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}A^b_\mu A^c_\nu$. Rather remarkably the connection has cancelled in the expression of curved space field strength, which leaves it independent of the metric. The Weyl variation then gives

$$T^{\mu}{}_{\mu} = (4-d)\frac{1}{4}\sqrt{g}\frac{1}{g_0^2}G^2 \ . \tag{3.20}$$

Taking the flat space limit and using the notation of Section 2.1.4 we get

$$T^{\mu}{}_{\mu} = (4-d)\frac{1}{4}O_g \ . \tag{3.21}$$

In Section 2.1.4 it was shown that up to physically irrelevant operators $O_g = \mathbb{Z}_g{}^{g-1}[O_g]$ where $\mathbb{Z}_g{}^g$ was given in (2.43). Plugging this back to (3.22) we get

$$T^\mu{}_\mu = \Theta = -\frac{1}{2}\hat{\beta}[O_g] \; , \tag{3.22}$$

which is perfectly consistent with (2.90).
Note that we have chosen to ignore the gauge fixing terms

$$\sqrt{g}\frac{1}{\alpha_0}(\nabla_\mu A^{\mu a})^2 + \sqrt{g}\bar{c}\nabla^\mu(D_\mu c) \; . \tag{3.23}$$

As already discussed in Section 2.2.3 these do contribute to the total $T^\mu{}_\mu$. This contribution obtained by the Weyl variation of the above term involves the ghost EOM and a BRS variation, hence it has to vanish when inserted into physical matrix elements [20, 56].
Up until now we have only been discussing operator contributions to the trace anomaly ignoring the purely background terms allowed by symmetries (e.g R^2 etc.). In the next section we will address these contributions.

3.1.2 Gravitational counterterms

We now proceed consider correlators of (3.6). As was demonstrated in Section 2.1, in traditional perturbative approach products of operators require additional renormalization that involves adding countererms to cancels the contact term divergences. In this section we will follow the same recipe for products of EMT. The QAP (cf. Section 2.1.1) is applicable if the metric is treated as local source. The main point of this section is to motivate that on general grounds one expects the trace anomaly in curved space to take the form

$$T^\mu{}_\mu(x) = \Theta_{\text{dyn}} + \Theta_{\text{grav}} \; , \tag{3.24}$$

where $\Theta_{\text{dyn}} = \hat{\beta}^A[\mathcal{O}_A]$ represents the *dynamical* breaking of conformal symmetry discussed in Section 1.2.3 and

$$\Theta_{\text{grav}} = -\beta_a E_4 - \beta_b H^2 - \beta_c W^2 + 4\tilde{d}H \; , \quad H \equiv \frac{1}{(d-1)}R \; , \tag{3.25}$$

is the pure gravitational contribution. Our aim now is to elucidate how the central charges $\beta_a, \beta_b, \beta_c, \tilde{d}$ arise.
Working in dimensional regularization with $d = 4 - 2\epsilon$ we add all the possible

metric dependent,local counterterms allowed by diffeomorphism symmetry. These will be parametrised by the following Lagrangian

$$\mathcal{L}_{\text{grav}} = a_0 E_4 + b_0 H^2 + c_0 W^2 \ , \tag{3.26}$$

where $H = \frac{1}{(d-1)} R$,

$$E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 \tag{3.27}$$

and

$$W^2 = R^2_{\mu\nu\rho\sigma} - \frac{4}{(d-2)} R^2_{\mu\nu} + \frac{2}{(d-1)(d-2)} R^2 \tag{3.28}$$

is the square of $d-$dimensional Weyl tensor. For massless theory, these are the only possible local terms consistent with diffeomorphism invariance.[1]
Just as we did in Chapter 2 we split the bare coefficients a_0, b_0, c_0 into a finite part and a pole series $L_x = \sum_{n>0} \frac{x_n}{\epsilon^n}$ (implicitly assuming a MS-scheme). Thus

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \mu^{(d-4)} \begin{pmatrix} a + L_a \\ b + L_a \\ c + L_c \end{pmatrix} . \tag{3.29}$$

The respective beta functions are defined

$$\begin{pmatrix} \hat{\beta}_a \\ \hat{\beta}_b \\ \hat{\beta}_c \end{pmatrix} = \frac{d}{d \ln \mu} \begin{pmatrix} a \\ b \\ c \end{pmatrix} . \tag{3.30}$$

Using that $\frac{d}{d\ln\mu} a_0 = ((d-4) + \hat{\beta}^A \partial_A + \hat{\beta}_a \frac{\partial}{\partial a}) a_0 = 0$ etc. and (3.29) we can derive the following relation

$$\begin{aligned} \begin{pmatrix} \hat{\beta}_a \\ \hat{\beta}_b \\ \hat{\beta}_c \end{pmatrix} &= -(d-4) \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} \beta_a \\ \beta_b \\ \beta_c \end{pmatrix} \quad \text{with} \\ \begin{pmatrix} \beta_a \\ \beta_b \\ \beta_c \end{pmatrix} &= (2\epsilon - \hat{\beta}^A \partial_A) \begin{pmatrix} L_a \\ L_b \\ L_c \end{pmatrix} . \end{aligned} \tag{3.31}$$

Thus we see that scale dependence of a, b, c is determined from their respective pole functions. In the following section we will expose the intimate relationship

[1]In general one should also include the Einstein-Hilbert term $\Lambda^2 H$ and the cosmological constant Λ^4, however as there are no power divergences in dimensional regularization we can set these to 0

between these pole functions and the dynamics of the theory. Before we proceed we would briefly like to turn to the issue of scheme dependence of $\beta_{a,b,c}$. So far the MS-scheme was implicitly assumed for $L_{a,b,c}$. Let us choose a different subtraction scheme

$$L'_{a,b,c} = L_{a,b,c} + f_{a,b,c} \ , \tag{3.32}$$

with $f_{a,b,c}$ being some finite functions of the couplings. Since the bare constants (3.29) should remain μ-independent for any scheme we take $(a', b', c') = (a, b, c) - f_{a,b,c}$ (up to possibly μ-independent constants) so that the relations (3.31) are preserved with corresponding beta functions transforming as

$$\beta'_{a,b,c} = \beta_{a,b,c} - \beta^A \partial_A f_{a,b,c} \ . \tag{3.33}$$

Determination of $L_{a,b,c}$

The counterterterms $L_{a,b,c}$ are determined via the QAP as in Section 2.1.1. We start again from the generating functional

$$\mathcal{Z} = \int \mathcal{D}(\phi_i) e^{-S(\phi_i, g_{\mu\nu})} \ , \tag{3.34}$$

for the action

$$S = S_{\text{dyn}}(\phi_i, g_{\mu\nu}) - S_{\text{grav}}(g_{\mu\nu}) \tag{3.35}$$

with counterterms (3.26) included $S_{\text{grav}} = \int d^d x \sqrt{g} \mathcal{L}_{\text{grav}}$. The QAP is applied via differentiation of $\ln \mathcal{Z}$ w.r.t. α and setting the background to be flat at the end. In particular taking two derivatives we obtain

$$\langle \frac{\delta S}{\delta\alpha(0)} \frac{\delta S}{\delta\alpha(x)} \rangle - \langle \frac{\delta^2 S}{\delta\alpha(0)\delta\alpha(x)} |_{\text{flat}} \rangle = [\text{finite}] \ , \tag{3.36}$$

where the $\langle\rangle$ are taken in the flat space. From (3.6) and (3.24) we obtain

$$\langle \Theta_{\text{dyn}}(0) \Theta_{\text{dyn}}(x) \rangle + \langle \frac{\delta^2 S_{\text{grav}}}{\delta\alpha(0)\delta\alpha(x)} |_{\text{flat}} \rangle = [\text{finite}] \ . \tag{3.37}$$

In the above equation we set the flat space condensates to 0 so that for example the contact term $\langle \frac{\delta^2 S_{\text{dyn}}}{\delta\alpha(0)\delta\alpha(x)} \rangle = 0$.[2] To evaluate the variations of S_{grav} we can use

[2] The condensate contact terms for EMT have been discussed in the previous chapter (see 2.2.6)

the following trick. Setting the metric to be conformally flat

$$g_{\mu\nu} = e^{-2s(x)}\delta_{\mu\nu} \tag{3.38}$$

means that doing the variations w.r.t. α become simply functional derivatives w.r.t. s. Expressions for individual geometric terms in this space are give in the Appendix F.2. This approach has a downside that we will not be able to evaluate the c_0 contribution this way as the the invariant W^2 vanishes on conformally flat spaces with (3.38).

Hence to find the second term in (3.37) we need to evaluate S_{grav} on the background and expand to $O(s^2)$. This can be readily done using the expressions for E_4 and R from Appendix F.2 and we see that if we discard total derivatives the only $O(s^2)$ comes from expanding $\sqrt{g}H^2$ giving [3]

$$S_{\text{grav}} = \int d^dx e^{(4-d)s} 4b_0(\Box s)^2 + O(s^3)\ , \tag{3.39}$$

Taking the two functional derivatives is now straight-forward exercise and (3.37) can be readily finished

$$\langle T^\mu{}_\mu(x) T^\nu{}_\nu(0)\rangle = \langle \Theta(x)\Theta(0)\rangle + 8b_0\Box^2\delta^{(d)}(x) = [\text{finite}]\ , \tag{3.40}$$

where we suppressed the 'dyn' subscript defining $\Theta_{\text{dyn}} \equiv \Theta$ to make connection with previous chapters of this thesis (cf. (1.64)). Above we have *defined* the renormalized correlator

$$\langle T^\mu{}_\mu(x) T^\nu{}_\nu(0)\rangle \equiv \frac{\delta^2}{\delta s(x)\delta s(0)} \ln \mathcal{Z}|_{s=0} \tag{3.41}$$

through functional derivatives of the path integral (3.34) (the higher point functions $\langle T^\mu{}_\mu T^\nu{}_\nu T^\rho{}_\rho\rangle$ etc. are defined analogically). From (3.40) we see that the correlators $\langle T^\mu{}_\mu T^\nu{}_\nu\rangle$ and $\langle\Theta\Theta\rangle$ agree at $x \neq 0$, with the only difference being the purely gravitational contribution from $b_0 H^2$ supported at $x = 0$. This also means that the correlator $\langle T^\mu{}_\mu T^\nu{}_\nu\rangle$ is not unique since under the addition of a local term $\delta\mathcal{L}_{\text{grav}} = \omega_0 H^2$ it acquires an extra contribution $8\omega_0\Box^2\delta(x)$. On contrary, the 'bare' correlator $\langle\Theta\Theta\rangle$ remains unchanged under such a shift.

[3] The advantage of using $H = \frac{1}{(d-1)}R$ becomes clear when expanded in terms of s where the extra d-dependent prefactor drops out.

We can now take the Fourier transform of the above equation

$$\Gamma_{ss} \equiv \int d^d x e^{ipx} \langle T^{\mu}{}_{\mu}(x) T^{\nu}{}_{\nu}(0) \rangle = \Gamma_{TT}(p^2) + 8 b_0 p^4 = [\text{finite}] \tag{3.42}$$

written in the familiar notation of Section 2.2 (we take again $[\mathcal{O}_T] = \Theta$).[4] By comparing the respective pole functions in (3.42) (see (2.66) and (3.29)) we can obtain the non-trivial, known [58], relation

$$L_b = -\frac{1}{8} L_{TT}^{\mathbb{1},\mathrm{MS}} , \tag{3.43}$$

which quantifies the advertised relationship between gravitational counterterms and the dynamics which studied in the previous chapter. A similar relationship can be found between the three-point function and L_a and we will return to it in the next chapter. At last we only state the L_c relation given in [58] for completeness. It can be obtained from $\frac{\delta}{\delta g_{\mu\nu}(0)} \frac{\delta}{\delta g_{\rho\sigma}(x)} \ln \mathcal{Z}$ for which one needs non-conformally flat metric to get

$$\int d^d x e^{ipx} \langle \Theta_{\mu\nu}(0) \Theta^{\mu\nu}(x) \rangle + 4(d-3)(d+1) c_0 p^4 + \frac{8}{(d-1)} b_0 p^4 = [\text{finite}] , \tag{3.44}$$

where $\Theta_{\mu\nu}$ is the dynamical energy-momentum tensor. A similar equation involving a_0 can be derived by considering the three point function of TEMT.

Application to gauge theory

The discussion so far has been completely general, but from now on we will specialize to the Yang-Mills case with one marginal coupling g (although most of the arguments can be readily generalized to multiple couplings). Starting again from the action[5] (3.19), we include the purely gravitational part

$$S = \frac{1}{4} \int d^d x \sqrt{g} \frac{1}{g_0^2} G^2 + \int d^4 x \sqrt{g} \mathcal{L}_{\text{grav}} . \tag{3.45}$$

The TEMT is again obtained from (3.6) using the identities given in the Appendix F.1

$$T^{\mu}{}_{\mu} = (4-d) \frac{1}{4} O_g - (4-d)(a_0 E_4 + b_0 H^2 + c_0 W^2) - 4 b_0 \Box H . \tag{3.46}$$

[4] The choice of subscript s in (3.42) is justified through derivatives w.r.t. s.

[5] We neglect the fermionic and gauge fixing parts for the reasons discussed around (3.18) and (3.19) respectively.

This is not yet the desired form (3.24) of TEMT. To complete the exercise one needs to factor in the mixing of O_g with the background terms $E_4, H^2, W^2, \Box H$ under renormalization. To determine the relevant mixing coefficients the QAP of previous section will be useful (the equation (2.14) in Section 2.1.1 can be directly applied). The contribution of E_4, H^2, W^2 can be found by differentiating the $\mathcal{L}_{grav}$ w.r.t. coupling $2\ln g$ (cf. (2.41)). Thus we get [57]

$$[O_g] = \frac{2(d-4)}{\hat{\beta}} O_g + \mathbb{Z}_g{}^a E_4 + \mathbb{Z}_g{}^b H^2 + \mathbb{Z}_g{}^c W^2 + \mu^{(d-4)} \frac{8}{\hat{\beta}} (\sigma + L_\sigma) \Box H \ , \quad (3.47)$$

where $\hat{\beta}$ is defined by using the conventions of Appendix C.1 and

$$\mathbb{Z}_g{}^{a,b,c} = -\mu^{(d-4)} \frac{2(d-4)}{\hat{\beta}} \left(L_{a,b,c} + \frac{\beta_{a,b,c}}{(d-4)} \right) . \quad (3.48)$$

These equations were derived using (3.31) and $\frac{\partial}{\partial \ln g} L_{a,b,c} = \frac{1}{\hat{\beta}} \frac{d}{d\ln\mu} L_{a,b,c}$. In the last section the we described a method for finding $L_{a,b,c}$ in terms of 2,3-point functions of energy-momentum tensor. Notice that the last term $\sigma + L_\sigma$ [6] can't be obtained by this method as there is no counterterm proportional to $\Box H$ in the Lagrangian. Instead, we will apply the QAP directly to $\langle [O_g] \rangle$ using

$$\frac{\delta}{\delta\alpha(x)} \langle [O_g] \rangle \big|_{\text{flat}} = \langle -\Theta(x) [O_g](0) \rangle + \langle \frac{\delta}{\delta\alpha(x)} [O_g] \rangle \big|_{\text{flat}} = [\text{finite}] \ , \quad (3.49)$$

where Θ is the dynamical trace anomaly (3.22). The variation $\frac{\delta}{\delta\alpha(x)}[O_g]$ can be worked by applying identities from Appendix F.1 directly to (3.47). Neglecting the flat space condensate contribution one obtains

$$-\int d^d x e^{ipx} \langle \Theta(x) [O_g(0)] \rangle + \mu^{(d-4)} \frac{16}{\hat{\beta}} (\sigma + L_\sigma) p^4 = [\text{finite}] \ . \quad (3.50)$$

Multiplying this equation through $\frac{\hat{\beta}}{2}$ and comparing with (3.42) we obtained a consistency relation

$$L_b = L_\sigma \quad (3.51)$$

The consistency condition for σ can be found by applying the finite operator $\frac{1}{\hat{\beta}} \frac{d}{d\ln\mu} \hat{\beta}$ to (3.50). Invoking the RG invariance of $\langle \Theta\Theta \rangle$ then yields

$$\frac{1}{\hat{\beta}} \frac{d}{d\ln\mu} \mu^{(d-4)} (\sigma + L_\sigma) = [\text{finite}] \ . \quad (3.52)$$

[6] The coefficient $\frac{1}{\hat{\beta}}$ in front of $\sigma + L_\sigma$ is chosen out of convenience anticipating that O_g appears in the combination $\hat{\beta}[O_g]$ in TEMT

Recalling the definitions of $\beta_{a,b,c}$ in (3.31) together with (3.51) implies that $\frac{d}{d\ln\mu}\mu^{(d-4)}L_\sigma = -\beta_b$. Since σ is finite function of the coupling we have $\frac{d}{d\ln\mu}\sigma = \hat{\beta}\frac{\partial}{\partial\ln g}\sigma$ the condition (3.52) becomes equivalent to

$$\frac{1}{\left(1-\frac{\beta}{\epsilon}\right)}\left(-2\sigma - \frac{\beta_b}{\epsilon}\right) = [\text{finite}] \ . \tag{3.53}$$

We can expand this Laurent series and demanding means the poles need to vanish. For example vanishing of the first pole finally implies

$$\sigma = -\frac{1}{2}\frac{\beta_b}{\beta} \ . \tag{3.54}$$

This implies that all the information about the gravitational part of the trace anomaly (3.46) is stored in the three functions $L_{a,b,c}$. This relation will also be important later on when discussing the fourth moment.

Finally we can use (3.47) to substitute for the first term in (3.46). Together with the consistency condition (3.51) this yields

$$T^\mu{}_\mu = -\frac{1}{2}\hat{\beta}[O_g] - \hat{\beta}_a E_d - \hat{\beta}_b H^2 - \hat{\beta}_c W^2 - 4(b-\sigma)\Box H \ , \tag{3.55}$$

where the beta functions $\hat{\beta}_{a,b,c}$ have been defined in (3.31). It is readily observed that (3.55) has the desired form (3.24).

This concludes our rather lengthy review of the trace anomaly in curved space. Throughout this section we did not make any assumptions about the finiteness of TEMT correlators or resummation of the poles. The main purpose was to introduce the counterterms a_0,b_0 and their associated beta functions. We are now ready to apply the results of Section 2.2 to the quantities we have just introduced. Our discussion in this chapter will revolve around the H^2 term and the consequences of finiteness of L_b. The detailed analysis of β_a term will be given in Chapter 4. Before we move on let us comment on the apparent tension between the existence of anomalies and the finiteness of EMT two-point functions proven in Section 2.2.

3.1.3 Finiteness and the R^2 anomaly

As was demonstrated above, the anomalies can be associated with UV-divergences and so one might wonder whether this means that the corresponding R^2-anomaly in (3.25) is absent. In a generic example this will not be the case since it is the

$\ln\mu$-terms which signal the presence of the anomaly which are of course present despite the ϵ-poles resumming to a finite expression. We will show however, that it is always possible to choose a subtraction scheme where the anomaly vanishes. Let us first look at the QCD example again. We can obtain the expression for β_b by substituting the explicit form of $L^{\mathbb{1}}_{TT}$ from (2.93) (see the Appendix G.1 for the definition of $g^{\mathbb{1}}_1$) into (3.43)

$$\beta_b = -\mu^{2\epsilon}\frac{d}{d\ln\mu}(\mu^{-2\epsilon}L_b) = \frac{1}{16}\frac{\beta(a_s)}{a_s}\int_0^{a_s}\partial_u\left(\frac{\beta(u)}{u}\right)u^2 g^{\mathbb{1}}_1(u)du\ , \tag{3.56}$$

which is clearly non-zero. [7] In fact we see that (3.56) vanishes only at fixed point with $\beta = 0$ which is consistent with the observation that R^2 vanishes in CFT [96, 98].

Note that (3.56) can be written as $\beta_b = \beta\frac{\partial}{\partial\ln g}B$, where

$$B = \frac{1}{32}\int_0^{a_s}\partial_u\left(\frac{\beta}{u}\right)u\left(1-\frac{u}{a_s}\right)g^{\mathbb{1}}_1(u)du\ . \tag{3.57}$$

This is not a coincidence since by inspecting (2.93) we find that $B = \frac{1}{8}L^{\mathbb{1},\mathrm{MS}}_{TT}(\epsilon = 0) = -L_b(\epsilon = 0)$, which is well defined as was shown in the Section 2.2.3. By choosing $f_b = B$ in (3.33) we then conclude that there exists a scheme where $\beta'_b = 0$ along the flow. Using the finiteness of $L^{\mathbb{1},\mathrm{MS}}_{TT}$ that we proved in Section 2.2.5 we can generalize this argument to multiple couplings. From the relation (3.31)

$$\beta_b = \lim_{\epsilon\to 0}(2\epsilon - \hat{\beta}^A\partial_A)L_b(\epsilon) = -\beta^A\partial_A L_b(\epsilon = 0) \tag{3.58}$$

where we used the finiteness of L_b that follows from the finiteness of $L^{\mathbb{1},\mathrm{MS}}_{TT}$ and the relation (3.43).[8] Thus we can always find a scheme with $\beta'_b = 0$ by choosing $f_b = -L_b(\epsilon = 0)$.

This result is non-trivial especially in theories with multiple couplings, where it is not a priory clear that β_b can be written as a scale derivative of some finite function. Here the finiteness of $\langle\Theta\Theta\rangle$ was crucial in defining such a function (3.58). To the author's knowledge this has not been demonstrated in the literature. Another side result is the explicit expression (3.56) and its application to extend the QCD R^2-anomaly to $\mathcal{O}(a_s^5)$ by using the recent update of the $\langle G^2G^2\rangle$ in Appendix G.1.

[7] From (3.56) one infers that $\beta_b = \mathcal{O}({a_s}^3)$ since $g^{\mathbb{1}}_1 = \mathcal{O}({a_s}^0)$ and that the R^2-anomaly-term is absent for theories with $\beta = -\beta_0 a_s$ which is the case for pure $\mathcal{N} = 1$ supersymmetric QCD. Both facts are consistent with the explicit computations in the literature.

[8] We also assume that $\lim_{\epsilon\to 0} L_b$ is uniform and smooth (i.e with well-defined derivatives) so that the $\epsilon \to 0$ limit and the derivatives in (3.58) commute.

3.2 Flow of $\Box R$ anomaly from $\langle\Theta(x)\Theta(0)\rangle$

The correlation function of the TEMT[9]

$$\Gamma_{TT}(p^2) = \int d^4x e^{ip\cdot x}\langle\Theta(x)\Theta(0)\rangle_c = \mathbb{C}^{\mathbb{1}}_{TT}(p^2)p^4 \tag{3.59}$$

is an object that contains a wealth of information on the RG-flow in the form of moments

$$\langle x^n\rangle \equiv \int d^4x\,(x^2)^{\frac{n}{2}}\langle\Theta(x)\Theta(0)\rangle_c \tag{3.60}$$

In particular in two dimensions the c-theorem [99] can be written as the second moment of energy momentum correlation function [100], using the normalisation $\langle T^\rho{}_\rho\rangle_{\rm CFT} = -(\beta_c/(24\pi)R$, where $\beta_c = 1$ for a free scalar field,

$$\Delta\beta_c = \beta_c^{\rm UV} - \beta_c^{\rm IR} = 3\pi\int d^2x\,x^2\langle\Theta(x)\Theta(0)\rangle_c \geq 0\ . \tag{3.61}$$

Positivity then follows from reflection positivity of $\langle\Theta(x)\Theta(0)\rangle$ for $x \neq 0$ as well as the finiteness when integrating over d^2x. This is equivalent to having a convergent dispersion relation as discussed in previous sections. This type of argument is sketched in the original paper discussing spectral representations and c-theorems [101]. We recall from Section 3.1.2 that in four dimensions the VEV of the TEMT in curved space reads

$$\langle T^\rho{}_\rho\rangle = -\frac{\beta}{2}\langle[G^2]\rangle - \beta_a E_4 - \beta_c W^2 - \beta_b H^2 + 4\tilde{d}\Box H\ , \quad H \equiv \frac{1}{(d-1)}R\ , \tag{3.62}$$

with E_4, W^2 and R being the Euler, the Weyl squared and the Ricci scalar and the cosmological constant is set to zero. The analogue of the c-theorem in four dimensions is known as the a-theorem $\Delta\beta_a \geq 0$ [102] and it will be the main subject of Chapter 4. The $\tilde{d}$-term has not received much attention since it can be shifted by a local counterterm in the action $\mathcal{L} = \delta b\cdot H^2$ results in $\tilde{d} \to \tilde{d} - \delta b$.

[9] The restrictive structure of (3.59) follows from the flat-space translational Ward Identity $\int d^4x e^{ip\cdot x}\langle\Theta_{\alpha\beta}(x)\Theta_{\gamma\delta}(0)\rangle_c = P^{(0)}_{\alpha\beta\gamma\delta}\Gamma^{(0)} + P^{(2)}_{\alpha\beta\gamma\delta}\Gamma^{(2)} + P^{(CT)}_{\alpha\beta\gamma\delta}\langle\Theta\rangle$. From the traces of the spin 0 and 2 structures, $P^{(0)}_{\alpha\gamma\alpha\gamma} \sim p^4$ and $P^{(2)}_{\alpha\gamma\alpha\gamma} = 0$ (note $P^{(CT)}_{\alpha\gamma\alpha\gamma} = c_T$), one infers that $\Gamma^{(0)}(p) \sim \mathbb{C}^{\mathbb{1}}_{TT}(p^2)$.

In what follows we aim to show that

$$\Delta\tilde{d} = \tilde{d}^{\rm UV} - \tilde{d}^{\rm IR} = \frac{1}{2^9\,3}\langle x^4\rangle \geq 0 \,, \tag{3.63}$$

within the framework used in this thesis. Furthermore we will show that the quantities in (3.63) can be unambiguously defined and provide explicit scheme-independent formulas for calculating them. Positivity will follow from the reflection positivity of (3.60) and the previously discussed finiteness of the ΘΘ-correlator in momentum space. In Section 3.2.1 we will present a derivation of (3.63) and related it to the contact term formulas from Section 2.2. An explicit MS computation of $\Delta\tilde{d}$ valid for weakly coupled gauge theories in Section 3.2.2 serves as a check using the results of Section 2.2.3. The delicate issue is the convergence of $\langle x^4\rangle$, which is necessary for $\Delta\tilde{d}$ to be physical, is discussed in Section 3.2.3. These two sections are of help to discuss the status of $\Delta\tilde{d}$ as a physical observable in Section 3.2.4 supplemented with comments on the earlier literature.

3.2.1 $\Delta\tilde{d}$ as the fourth moment of $\langle\Theta(x)\Theta(0)\rangle$

The link between the two quantities in Eq. (3.63) is provided by the QAP (differentiation wrt. a Weyl parameter (3.38)). This will used in Section 3.2.1 by extracting the anomaly from the IR effective action. In Section 3.2.1 the formalism of Section 2.2 will be used to provide formulas useful for explicit evaluation of $\Delta\tilde{d}$.

Derivation using the IR effective action

The IR effective action takes the following form

$$\ln\mathcal{Z} = -\tilde{d}^{\rm IR}\int d^4x\sqrt{g}H^2 + \dots \,, \tag{3.64}$$

where the dots stand for non-local and Weyl-invariant contributions. The local part of (3.64) is dictated by the IR trace anomaly. To eliminate the Weyl-invariant (e.g W^2) terms in (3.64) we choose a conformally flat background

$$g_{\mu\nu} = e^{-2s(x)}\delta_{\mu\nu} \,, \tag{3.65}$$

in which the effective action takes the form

$$\ln \mathcal{Z} = -4\tilde{d}^{\mathrm{IR}} \int d^4x (\Box s)^2 + \mathcal{O}(s^3) \; . \tag{3.66}$$

On the other hand $\ln \mathcal{Z}$ is the Euclidean path integral of

$$\mathcal{Z} = \left(\int \mathcal{D}\phi_i e^{-S_{dyn}(\phi_i, g_{\mu\nu}) + b_0 \int d^4x \sqrt{g} H^2} \right) \; , \tag{3.67}$$

depends on dynamical fields ϕ_i and b_0 is the gravitational counterterm cf. (3.26). In the conformally flat background (3.65)

$$\mathcal{Z} = \left(\int \mathcal{D}\phi_i e^{-S_{dyn}(\phi_i, s) + 4b_0 \int d^4x (\Box s)^2 + \mathcal{O}(s^3)} \right) \; . \tag{3.68}$$

To find $\hat{b}^{\mathrm{IR}}$ we need to perform a derivative expansion of the above expression to extract the $(\Box s)^2$ and compare it with (3.66).

$$S_{dyn}(\phi_i, s) = S_{dyn}(\phi_i) + \int d^4x s(x) \Theta(x) + \dots \; , \tag{3.69}$$

where $S_{dyn}(\phi_i)$ is the flat-space action and $\dots$ stand for terms which do not contribute to $(\Box s)^2$ in the derivative (and s) expansion. The dynamical contribution is extracted by first expanding the exponential about $s = 0$

$$\begin{aligned} \int \mathcal{D}(\phi_i) e^{-S_{dyn}(\phi_i, s)} &= 1 - \int d^4x s(x) \langle \Theta(x) \rangle + \frac{1}{2} \int \int d^4x d^4y s(x) s(y) \langle \Theta(x) \Theta(y) \rangle \\ &\quad + \mathcal{O}(s^3) \; , \end{aligned} \tag{3.70}$$

where $\langle \quad \rangle$ stand for flat-space VEVs. The four derivative term (3.66) is matched by Taylor expanding the double integral term in (3.70) by

$$s(y) = s(x) + \dots + \frac{1}{4!} (x-y)^\mu (x-y)^\nu (x-y)^\rho (x-y)^\sigma \partial_\mu \partial_\nu \partial_\rho \partial_\sigma s(x) + \mathcal{O}(\partial^5) \; . \tag{3.71}$$

Using the Euclidean rotational symmetry the following replacement

$$(x-y)^\mu (x-y)^\nu (x-y)^\rho (x-y)^\sigma \to \frac{1}{24} (x-y)^4 (\delta^{\mu\nu} \delta^{\rho\sigma} + \delta^{\mu\rho} \delta^{\nu\sigma} + \delta^{\mu\sigma} \delta^{\rho\nu}) \; , \tag{3.72}$$

is valid under the integral. Changing the integration variable to $y = z + x$ one gets

$$\frac{1}{2} \int \int d^4x d^4y s(x) s(y) \langle \Theta(x) \Theta(y) \rangle = \frac{1}{2} \int d^4x s(x)^2 \int d^4z \langle \Theta(z) \Theta(0) \rangle + \dots$$

$$+\frac{1}{3.2^7}\int d^4xs(x)\Box s(x)\int d^4zz^4\langle\Theta(z)\Theta(0)\rangle\ . \tag{3.73}$$

Substituting (3.70) in (3.68) and using the derivative expansion (3.73) leads to

$$-\tilde{d}^{\mathrm{IR}}=\ln\mathcal{Z}\big|_{4\int(\Box s)^2}=b_0+\frac{1}{3.2^9}\int d^4zz^4\langle\Theta(z)\Theta(0)\rangle\ . \tag{3.74}$$

Next we use the result of Section (2.2) that the bare coupling $b_0=$ [finite], so we can match it with the UV anomaly. Indeed, had we not deformed the UV CFT with $\Theta=0$ we would have no flow with $\tilde{d}^{\mathrm{IR}}=\tilde{d}^{\mathrm{UV}}$ so that (3.74) implies

$$b_0=-\tilde{d}^{\mathrm{UV}}\ . \tag{3.75}$$

and the desired result

$$\Delta\tilde{d}=\frac{1}{3.2^9}\int d^4zz^4\langle\Theta(z)\Theta(0)\rangle \tag{3.76}$$

follows by subtracting b_0 from both sides of (3.74). The integral in (3.76) is UV finite and it involves the bare correlator so it can be understood as integral over all space with an infinitesimal sphere removed from the origin, which guarantees its scheme-independence.

Flow of $\tilde{d}$ from the RG analysis

From (3.59) it follows that the fourth moment is related to the IR limit of $\mathbb{C}^{\mathbb{1}}_{TT}$ by [10]

$$\mathbb{C}^{\mathbb{1}}_{TT}(0)=\Box_p^2\Gamma_{TT}(p^2)|_{p^2=0}=\frac{1}{2^6\,3}\int d^4x\,x^4\langle\Theta(x)\Theta(0)\rangle_c\ , \tag{3.77}$$

with $\Box_p^2=\partial_{p_\alpha}\partial_{p^\alpha}$. Combining this with (3.76) we get

$$\Delta\tilde{d}=\frac{1}{8}\mathbb{C}^{\mathbb{1}}_{TT}(0)\ . \tag{3.78}$$

[10] It is assumed that $\mathbb{C}^{\mathbb{1}}_{TT}$ is regular at small momentum with $\mathbb{C}^{\mathbb{1}}_{TT}(p^2)\overset{p^2\to 0}{=}\mathbb{C}^{\mathbb{1}}_{TT}(0)+\mathcal{O}(\beta(p^2))$. This is justified by the observation made in Section 2.2.3, that $\mathbb{C}^{\mathbb{1}}_{TT}$ depends on momentum only through running couplings at p^2.

The relation allows us to use the methods of Section 2.2 to find way to calculate $\Delta\tilde{d}$. First, we recall that

$$\mathbb{C}^{\mathbb{1}}_{TT}(p^2) = \beta^A\beta^B\mathbb{C}^{\mathbb{1}}_{AB}(p^2) \ , \tag{3.79}$$

where $\mathbb{C}^{\mathbb{1}}_{AB}$ is defined as in (2.63). Next, we define a MOM-type scheme [103] defined implicitly by the condition

$$\mathbb{C}^{\mathbb{1},\mathrm{MOM}}_{AB}(p^2 = \mu^2) \equiv 0 \ . \tag{3.80}$$

From (2.124) we have

$$\frac{d}{d\ln\mu}\mathbb{C}^{\mathbb{1},\mathrm{MOM}}_{TT}(p^2,\mu) \equiv 2\chi^{\mathrm{MOM}}_{TT}(\mu) = 2\beta^A\beta^B\chi^{\mathrm{MOM}}_{AB}(\mu) \ , \tag{3.81}$$

with $\mathbb{C}^{\mathbb{1},\mathrm{MOM}}_{TT}$ defined as bellow (2.123). This equation can be formally solved subject to the boundary condition (3.80). Together with the solution (2.125) for $\beta^A\beta^B L^{\mathbb{1},\mathrm{MOM}}_{AB}$ we get

$$\begin{aligned}
\mathbb{C}^{\mathbb{1}}_{TT}(p^2) &= \underbrace{2\int_p^\mu \chi^{\mathrm{MOM}}_{TT}(\mu')\frac{d\mu'}{\mu'}}_{\mathbb{C}^{\mathbb{1},\mathrm{MOM}}_{TT}(p^2,\mu)} + \underbrace{2\int_\mu^\infty \chi^{\mathrm{MOM}}_{TT}(\mu')\frac{d\mu'}{\mu'}}_{\beta^A\beta^B L^{\mathbb{1},\mathrm{MOM}}_{AB}} \\
&= 2\int_p^\infty \chi^{\mathrm{MOM}}_{TT}(\mu')\frac{d\mu'}{\mu'} \\
&= 2\int_p^\infty \beta^A\beta^B\chi^{\mathrm{MOM}}_{AB}(\mu')\frac{d\mu'}{\mu'} \ .
\end{aligned} \tag{3.82}$$

Above p denotes the positive square root of p^2. Finally, we take the $p \to 0$ limit of (3.82) and use (3.78) to find that

$$\Delta\tilde{d} = \frac{1}{8}\mathbb{C}^{\mathbb{1}}_{TT}(0) = \frac{1}{4}\int_0^\infty \beta^A\beta^B\chi^{\mathrm{MOM}}_{AB}(\mu')\frac{d\mu'}{\mu'} \ . \tag{3.83}$$

At first glance, (3.83) appears scheme-dependent. Choosing a different scheme MOM $\to \mathcal{R}$, the integrand of (3.83) transforms following (2.128). However it is easy to see that the difference is proportional to a total scale derivative $\mu'\frac{d}{d\mu'}\beta^A\beta^B\omega_{AB}$, which doesn't contribute at both limits of the integral where the beta functions vanish (assuming the existence of UV and IR fixed points). We can therefore finally write a scheme-independent expression

$$\Delta\tilde{d} = \frac{1}{8}\mathbb{C}^{\mathbb{1}}_{TT}(0) = \frac{1}{4}\int_0^\infty \beta^A\beta^B\chi^{\mathcal{R}}_{AB}(\mu')\frac{d\mu'}{\mu'} \ , \tag{3.84}$$

which links the flow of $\tilde{d}$ to calculable quantities like $\chi^{\mathcal{R}}_{AB}$ and beta functions.

3.2.2 $\Delta\tilde{d}$ in QCD-like Theories

In this section the formula (3.84) will be verified for QCD-like theories by direct use of the MS-scheme expressions for β_b & L_b from Section 3.1.2. The relation between b and $\tilde{d}$ was established in (3.55)

$$\tilde{d}(\mu) = \sigma(\mu) - b(\mu) \ , \quad \sigma^{\mathrm{UV}} = 0 \tag{3.85}$$

where $\sigma(\mu) = \sigma(a_s(\mu))$ is a quantity related to the renormalisation of G^2 in a curved background (see (3.54) and the related discussion). In some more detail the bare b_0 in the Lagrangian (3.26) (with $\epsilon \to 0$ allowed by finiteness of L_b (3.43)) is

$$b_0 \equiv b^{\mathrm{UV}} = -\tilde{d}^{\mathrm{UV}} = b(\mu) + L_b(\mu) \ , \tag{3.86}$$

where we remind the reader that the μ-dependence arises from $a_s(\mu)$. From the explicit expression of $L^{\mathbb{1},\mathrm{MS}}_{TT}$ given in (2.92) and (3.43), it is observed that ($\epsilon \to 0$ implied)

$$\begin{aligned} L_b(\mu) &= -\frac{1}{32}\int_0^{a_s} \partial_u\left(\frac{\beta}{u}\right)u\left(1-\frac{u}{a_s}\right)g^{\mathbb{1}}_1(u)du \\ &= \frac{\beta_b}{2\beta} - \frac{1}{32}\int_0^{a_s} \partial_u\left(\frac{\beta}{u}\right)ug^{\mathbb{1}}_1(u)du \\ &= -\sigma^{\mathrm{MS}} - \frac{1}{32}\int_0^{a_s} \partial_u\left(\frac{\beta}{u}\right)ug^{\mathbb{1}}_1(u)du \ , \end{aligned} \tag{3.87}$$

where in the last line the formula $\sigma = -\beta_b/(2\beta)$ (3.54) was used along with the formula for β_b (3.56). Taking the IR fixed point limit ($a_s \to {a_s}^{\mathrm{IR}}$) in (3.87) we get

$$L_b(a_s^{\mathrm{IR}}) = -\sigma^{\mathrm{IR,MS}} - \frac{1}{32}\int_0^{{a_s}^{\mathrm{IR}}} \partial_u\left(\frac{\beta}{u}\right)ug^{\mathbb{1}}_1(u)du \ . \tag{3.88}$$

Further using $L_b(a_s^{\rm IR}) = b^{\rm UV} - b^{\rm IR}$ (3.86) and taking into account $\sigma^{\rm UV} = 0$ one arrives at

$$\Delta\tilde{d} = \frac{1}{32}\int_0^{a_s{}^{\rm IR}} \partial_u \left(\frac{\beta}{u}\right) u g_1^{\mathbb{1}} du = -\frac{1}{32}\int_0^{a_s{}^{\rm IR}} \beta\partial_u(u g_1^{\mathbb{1}})\frac{du}{u} = \frac{1}{16}\int_0^{\infty} \beta^2 \chi_{gg}^{\rm MS}\frac{d\mu'}{\mu'} \ , \tag{3.89}$$

where we used a change of variables $\frac{du}{u} = 2\frac{d\mu'}{\mu'}\beta$ and the definition (2.86) $\chi_{gg}^{\rm MS}(u) = \partial_u(u g_1^{\mathbb{1}})$ in the last equality. The equation (3.89) is in agreement with (3.84) ($\beta^A \to \frac{\beta}{2}$), which constitutes a check of the formalism.

We finish this section by a concrete example of QCD-like theory with a weakly-coupled Banks-Zaks fixed point.

$\Delta\tilde{d}$ to $\mathcal{O}(a_s{}^4)$ at the Banks-Zaks fixed point Finally, let us compute $\Delta\tilde{d}$ for explicit example of the Banks-Zaks FP using the formula (3.89). We assume the theory is in the conformal window (asymptotically free with IR FP). To justify perturbative treatment one needs to have evidence that there is a weakly-coupled FP and the result of $g_1^{\mathbb{1}}$ to a given order in PT. The Banks-Zaks FP appears for theories with the combination of N_c and N_f such that $-\beta_1/\beta_0$ is very small. It turns out that in this regime $a_s{}^{\rm IR}$ is small (see Appendix C.2 for detailed analysis) and that there is a weakly-coupled FP which can be determined from $\beta(a_s{}^{\rm IR}) = 0$. The first pole residue $g_1^{\mathbb{1}}$ is known at NNLO [69] and given in the Appendix G.1. Using (3.89) with expressions for beta function and $g_1^{\mathbb{1}}$ from Appendix C.1 and Appendix G.1 respectively we get

$$\begin{aligned}\Delta\tilde{d} &= \frac{-\beta_1 g_{1,0}^{\mathbb{1}}}{64}(a_s{}^{\rm IR})^2 \\ &- \frac{1}{96}(2\beta_2 g_{1,0}^{\mathbb{1}} + \beta_1 g_{1,0}^{\mathbb{1}})(a_s{}^{\rm IR})^3 \\ &- \frac{1}{64}\left(\frac{3}{2}\beta_3 g_{1,0}^{\mathbb{1}} + \beta_2 g_{1,1}^{\mathbb{1}} + \frac{1}{2}\beta_1 g_{1,2}^{\mathbb{1}}\right)(a_s{}^{\rm IR})^4 + \mathcal{O}(a_s{}^5) \ . \end{aligned} \tag{3.90}$$

It is observed that the first term is positive since $\beta_1 < 0$ and $g_{1,0}^{\mathbb{1}} > 0$ in accordance with $\Delta\tilde{d} \geq 0$. Positivity of $\Delta\tilde{d}$ is a consistency check for the existence of weakly coupled FP. In Section 4.2 we will return to this computation and give an explicit manifestly positive expression in terms of the parameter $\kappa = -\beta_1/\beta_0$.

3.2.3 UV and IR convergence of the fourth moment $\langle x^4 \rangle$

For (3.63) being a useful way to compute $\Delta\tilde{d}$ the moment integral needs to be finite. The latter can either diverge in the IR (for $x \to \infty$) or in the UV ($x \to 0$). The discussion of the UV-convergence parallels the one in Section 2.2. It is convenient to analyse this question using a Källén-Lehmann spectral representation

$$\langle x^4 \rangle = \int_0^\infty ds \frac{\rho(s)}{s^{n-1}} \, . \tag{3.91}$$

The integration variable s is of mass dimension two and the spectral function ρ,

$$\rho(p^2)\theta(p_0) = (2\pi)^3 \sum_n \delta(p_n - p) |\langle n(p_n)|\Theta|0\rangle|^2 \, , \tag{3.92}$$

is defined as a formal sum over the complete set of physical states. For clarity we would like to emphasise that $\theta(p_0)$ on the LHS is the step-function.
The spectral function behaves like

$$\rho(s) \sim s^{\Delta-2} \, , \tag{3.93}$$

close to the FPs where Δ is the scaling dimension of the most relevant operator.[11]
As in section 2.2 it is useful to distinguish the cases of a non-trivial (AS) and trivial (AF) FP. The case where there is spontaneous breaking of chiral symmetry is subtle and needs a special attention.
For non-trivial FPs $\Delta_{\rm UV} > 4$ and $\Delta_{\rm IR} < 4$ (a similar analysis has been done in [104]) . Then the fourth moment $\langle x^4 \rangle$ converges both in the UV and in the IR. For the trivial FP $\Delta = 0$ which is potentially both divergent in the IR and UV and requires a refined discussion taking into account the logarithmic behaviour. For the trivial UV FP the criteria in section (2.2) were sufficient to show finiteness for QCD-like theory and the same conclusion holds for multiple couplings. An asymptotically free theory in the IR behaves in the same way with $s \to s^{-1}$ which leads to the same integral as in the UV. Finally we remark that possible power divergences of the type $c_1\Lambda^4 + c_2 p^2\Lambda^2 \in \Gamma_{TT} = \int e^{ipx}\langle\Theta\Theta\rangle$ vanish when plugged into (3.77).

The case of a spontaneously broken symmetry, such as chiral symmetry in QCD, is more cumbersome [104, 105] since the trace anomaly contains a term $\Theta = \Box\pi^2 + ..$

[11] The identity operator (cosmological constant) which is an IR effect has to be cancelled by a UV-counterterm as otherwise $\rho(s) \sim s^{-2}$. Strictly speaking such a term must be added to (2.89).

at the classical level (e.g. [105]) with π being the pion field. This EMT cannot undergo the improvement proposed in [106] which removes the term above, since the improvement term is incompatible with chiral symmetry [105].[12] Hence one has to take this term into consideration. In the limit of free pions the critical term comes from a bubble graph $\Gamma_{TT}(p^2) \sim p^4 \ln(4m_\pi^2 - p^2) + \ldots$ which leads to $\langle x^4 \rangle \sim \ln(4m_\pi^2) + \ldots$. The latter diverges in the limit $m_\pi \to 0$. This observation seems to invalidate the (3.63) as a basis for a flow variable in the case of a chirally broken IR phase. Unlike in the UV-case it does not seem possible that this behaviour is improved by resumming interactions since corrections necessarily come with additional powers of p^2/f_π^2 where f_π is the pion decay constant. A series of the form $\ln(4m_\pi^2 - p^2) \sum_{n \geq 0} x_n (p^2/f_\pi^2 \ln(4m_\pi^2 - p^2))^n$ does not resum to anything which is finite in the limit $p^2, m_\pi^2 \to 0$ since each coefficient $n \geq 1$ of the series vanishes in this limit and the $x_0 \neq 0$ term therefore leads to a divergence.

Comments on lower moments

Other moments are also of interest. The zeroth moment $\langle x^0 \rangle$ is related to the cosmological constant and has been put forward in connection with a proof of the a-theorem using a spherical background [107]. The second moment $\langle x^2 \rangle$ is related to the induced Ricci scalar in the action [108] as well as the entanglement entropy [109]. The moments $\langle x^{0,2} \rangle$ are better behaved in the IR so we do not need to discuss them any further. In the UV $\langle x^{0,2} \rangle$ are worse behaved. Generally $\langle x^n \rangle \sim (\Lambda_{\mathrm{UV}}^2)^{\Delta_{\mathrm{UV}} - n}$. Hence $\langle x^0 \rangle$ and $\langle x^2 \rangle$ are therefore quartically and quadratically dependent on the UV-cut off for a trivial FP. In our analysis using dimensional regularisation (DR) and the $\overline{\mathrm{MS}}$-scheme no such divergences have shown up. This is of no surprise since it is well understood that dimensional regularisation is blind to power divergences (c.f. the discussion around (2.82) in Section 2.2). Let us mention that it has been argued by Bardeen [80] that in theories cut-off regularisation are inappropriate for theories with classical conformal symmetry.

[12] Another way to see this is to note that the the TEMT in QCD (formulated in terms of quarks and gluons) is free from scalar ambiguities and therefore one would expect the same to be true for the TEMT in chiral perturbation theory since the two are equivalent at low energy [105].

3.2.4 $\Delta\tilde{d}$ as a physical observable and comments on the literature

A necessary condition for $\Delta\tilde{d}$ to be physical is the convergence of the moment representation. Should the $\Delta\tilde{d}$ diverge in the UV then this would necessitate a subtraction constant for the spectral representation

$$\Delta\tilde{d} = \frac{1}{2^9\,3}\int_0^\infty ds\frac{\rho(s)}{s^3} \tag{3.94}$$

and since $\Delta\tilde{d}$ is a constant only it would then be void of any useful information. Whereas it was possible to show UV convergence, the IR convergence in the case of a spontaneously broken continuous global symmetry remains unclear. Assuming convergence it is then clear from the spectral representation that $\Delta\tilde{d}$ is scheme independent. Furthermore in Section 3.2.1 we showed directly how (3.84) is scheme-invariant.

This deserves some further discussion since it is well known that $\tilde{d}$ itself is scheme dependent since a Weyl variation of R^2 induces $\Box R$ so that $\tilde{d}$ receives contribution from a local R^2 in the action [110]. Such a modification of the path integral needs to be independent of any scale dependent coupling and is therefore a true constant. We refer the reader to the concluding section in Hathrell's paper [57] for a nice discussion from the viewpoint of an RG equation analysis. This suggests that it is the free field theory (the UV fixed point) values which are ambiguous. Indeed for free field theory the ζ- and dimensional-regularisation yield $d = -18$ and $d = 12$ (in units of $(6840\pi^2)^{-1}$) respectively [110]. From the viewpoint of the flow this undetermined constant is an initial conditions which cancels in $\Delta\tilde{d}$ (3.94) which is as stated before a scheme independent quantity provided we can assure convergence. In order for $\Delta\tilde{d}$ to be a useful observable one would need to able to compute $\tilde{d}$, in a given scheme, at each FP separately. It would seem that if the particle content of the UV and IR theory remain the same as is the case in the conformal window with massless degrees of freedom, the ambiguity should drop out in a concrete computation. The discussion of the Banks-Zaks FP at the end of Section 3.2.2 is an encouraging example in this direction. In the case where the UV and IR particle content is different we do not have a way to compute $\Delta\tilde{d}$ through $\tilde{d}^{\rm UV}$ minus $\tilde{d}^{\rm IR}$ since the ambiguity seems to depend on the field content as the free field theory result of the ζ- dimensional-regularisation suggests.

At last we take the chance to comment on the literature. The connection of $\langle x^4 \rangle$ with induced R^2-gravity was made in [111] without linking it an interpolation of central charges of CFTs. It was Anselmi who puts forward the fourth moment $\langle x^4 \rangle$ as a candidate for measuring the irreversibility of the flow [112] who also offers a derivation. He initially proposes that $\Delta\beta_a$ and $\Delta\tilde{d}$ are proportional to each other but later states that this is only possible for marginal deformations [113]. In section 4.2 we will return to this conjecture and asses its validity.

3.2.5 Summary and Discussion

The central part of this section was a proof of the equation (3.63). This equation is a sum rule relating the flow of the coefficient $\tilde{d}$ in (3.25) to the fourth moment (3.60). In Section 3.2.1 the relation (3.63) was proved using the IR effective action for background (conformal) gravity. Although the result was already known in the literature [111, 112], our independent analysis elucidated some of the questions regarding scheme-dependence and finiteness of the fourth moment. In doing this the language and results of Section 2.2 were instrumental. The methods we used in the Section 3.2.1 (for example the use of MOM-scheme in (3.82)) were novel in this context and helped to establish the main result (3.84). The equation (3.84) relates $\Delta\tilde{d}$ to calculable quantities such as beta functions and χ^{MS}_{AB} (cf. (2.117)). Explicit dependence of (3.84) on known quantities was not only useful in calculating $\Delta\tilde{d}$ in theories with weakly coupled Banks-Zaks fixed point, but also to demonstrate directly its scheme independence.

The finiteness of $\langle\Theta\Theta\rangle$ correlators was again crucial. First, it allowed us to take the $\epsilon \to 0$ limit of L_b and therefore interpret the bare coefficient b_0 as an arbitrary, finite initial condition for the flow of $\tilde{d}$. The infamous scheme dependence of $\tilde{d}$ was then translated into the choice of the constant b_0, which has no influence on the physics (it corresponds to a finite pure background counterterm).

Secondly, the UV finiteness allows one to deduce the positivity of $\Delta\tilde{d}$. This can be seen as follows. We can define a quantity

$$I(\delta) = \frac{1}{2^9\,3}\int_{|x|>\delta} d^4x x^4 \langle\Theta(x)\Theta(0)\rangle \geq 0 \tag{3.95}$$

which is manifestly positive-definite by reflection-positivity. Since the $\delta \to 0$ limit of (3.95) is well defined, it follows by basic real analysis that

$$\Delta\tilde{d} = \lim_{\delta\to 0} I(\delta) \geq 0 \tag{3.96}$$

which proves the asserted positivity.

Finally, let us comment on the IR convergence of the fourth moment. We saw that for theories with IRFP the IR finiteness was guaranteed by vanishing of TEMT in this regime. For chirally broken theories this was not the case and we were not able find a resummation argument similar to the one in Section 2.2. We plan to study this issue further in the future.

Chapter 4

The a-theorem

In this chapter we will discuss the so called a-theorem (although we use the term β_a-theorem which is more consistent with the notation of the present thesis adopted from [65]). The main idea is very intuitive. In Section 1.2.2 we discussed how in the Wilsonian picture renormalization amounts to integrating out high-energy modes. This implies that the RG flow is *irreversible* in the sense that the massive modes are 'dissipated' along the flow. A rigorous argument would require finding an effective measure for degrees of freedom and proving that such function is strictly *decreasing* along the flow. As the RG flows stops at a conformally invariant fixed point, we would expect this function to be stationary there and reduce to some calculable quantity. Finding such a function in four dimensions is equivalent to proving the a-theorem. It could provide constraints on IR degrees of freedom in ways similar to the t'Hooft anomaly matching [114].

In the language of the present thesis the a-theorem amounts to finding a function $a(\{g^A(\mu)\})$ of running couplings $\{g^A(\mu)\}$ that is decreasing under the RG flow and reduces to the central charge β_a (cf. (3.1)) at a fixed point. In relation to the monotonicity properties of RG flows we recognize three 'strengths' of the a-theorem ordered from the bottom (lower one implies the upper)

$$\begin{aligned} a^{\mathrm{UV}} - a^{\mathrm{IR}} \geq &\quad 0 && \text{Weak } a\text{-theorem} \\ \dot{a} \leq &\quad 0 && \text{Strong } a\text{-theorem} \\ \dot{a} = &\quad -\beta^B\beta^B\chi^g_{AB} && \text{Gradient flow of } a \end{aligned} \tag{4.1}$$

where $\dot{a} = -\beta^A \frac{\partial}{\partial g^A} a$ and χ^g_{AB} is positive definite (and scheme-dependent in general). We will start the chapter by reviewing the progress up to date concentrating on $d = 2$ and $d = 4$ with particular focus on the recent proof of 4D a-theorem by Komargodski and Schwimmer (K-S) [115]. In Section 4.2, we will provide a formula analogical to (3.84) for $\Delta\beta_a$. Using the results from Section 3.2.2 and this formula, we will extend the known results for $\Delta\beta_a$ of gauge theories with Banks-Zaks fixed point. The last section is based on a published paper [2], where the non-perturbative expression for $\Delta\beta_a$ of $\mathcal{N} = 1$ SUSY in the conformal window was derived by calculating the dilaton effective action.

4.1 Historical development of the a-theorem

A function with correct monotonicity properties was found for the first time in two dimensions in famous Zamolodchikov's c-theorem papers [99, 116]. The result defined a function of couplings called c, which was proven monotonically decreasing along the RG flow and becomes equal to the central charge c[1] at fixed point. The proof involves correlators of EMT and relies on unitarity/reflection positivity. The physical interpretation of the result as a spectral measure of the two point function of EMT came later through the work of [101, 117].

We already discussed in Section 3.2 that another viable candidate for the c-function is the coefficient β_c of R in the $d = 2$ trace anomaly, which satisfies the weak c-theorem as a consequence of positivity of the second moment (3.61).

The success and countless verifications of the c-theorem motivated people to look for an analogical quantity in higher dimensions. An obvious candidate would be one of the coefficients in (3.1). The coefficient $\tilde{d}$ was initially discarded due to its scheme dependence, although some works, including this thesis provided an evidence for $\tilde{d}$-theorem (see Section 3.2). The coefficient β_c was found to be non-decreasing in some cases [101]. One then remains with β_a coefficient. In [102] Cardy proposed a candidate for the a-function in $d = 4$

$$a \propto \int_{S^4} d^4x \sqrt{g} \langle T^{\mu}{}_{\mu} \rangle \tag{4.2}$$

which uses the property that the other two terms (W^2 and $\Box R$) in (3.1) vanish on spherical backgrounds and therefore the function reduces to β_a at fixed points. Here the connection with d.o.f. comes by writing association of the free-field

[1]For a correlator of 2 energy-momentum tensors in $d = 2$ CFT the central charge is defined through $\langle T(z)T(0)\rangle = \frac{c/2}{(z^4)} + \dots$

trace anomaly with the number of zero modes of the Laplacian [118][2]. Despite the amount of activity in the field during 1990s, a proof of Cardy's conjecture appeared only 10 years later [107]. This proof was later shown to be incomplete [119] due to a possible contact term ambiguity. To the best of our knowledge this issue still remains unresolved.

In the meantime, the major breakthrough came through the work of Jack and Osborn [66, 120] who calculated directly the flow of β_a in perturbation theory. Using local coupling techniques (cf. Section 2.1.1) on curved background they managed to prove that β_a is decreasing in perturbation theory. The main tool that allowed the proof were the consistency relations between coefficients of various local coupling/gravitational counterterms in the effective action. More precisely they defined a function $\beta_{\tilde{a}}$ which reduces to β_a at fixed point and satisfies

$$\dot{\beta}_{\tilde{a}} = -\beta^B \beta^B \chi^g_{AB} \,, \tag{4.3}$$

where χ^g_{AB}[3] is positive-definite to LO in perturbation theory, which proves the strongest version (4.1) of a-theorem in perturbation theory (in vicinity of a Gaussian fixed point). By utilizing Local Callan-Symanzik equations the authors of [67] recently extended this result to include nearly marginal perturbations about non-trivial fixed points.

The advent of AdS/CFT lead to non-perturbative evaluation of the trace anomaly in theories which admitted gravity dual [121]. This eventually paved the way for holographic c-theorem [122], where the relation between the central charge β_a and the entanglement entropy of the dual was established. Other non-perturbative computations of the flow of β_a came from SUSY and we will review these in Section 4.3.

The full proof of the weak a-theorem (the first line of (4.1)) appeared few years ago in the work of Komargodski and Schwimmer [115] and we will summarize the main aspects of this proof in the following section.

Overview of K-S proof

In this section we will review main points the recent proof of the a-theorem of [123] with particular emphasis on the techniques and ideas. One starts by imposing

[2]Recall from appendix A.3 that the free field trace anomaly corresponds the Heat Kernel expansion $\lim_{t\to\infty} \mathrm{tr}\exp(-t\Delta) = \mathrm{tr}\mathbb{1}$

[3]More concretely χ^g_{AB} is the MS-scheme coefficient of $G_{\mu\nu}\partial_\mu g^A \partial_\nu g^B$ in the trace anomaly with local couplings.

invariance under Weyl transformations via the background compensator field (dilaton) τ which transforms as $\tau(x) \to \tau(x) + \alpha(x)$ under Weyl transformations so that the combination

$$\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu} \tag{4.4}$$

remains Weyl invariant[4]. In addition all the mass scales are promoted to fields $M \to Me^{-\tau}$ to compensate for explicit violations of scale symmetry. The theory coupled to dilaton is therefore scale/Weyl-invariant and anomaly matching techniques may be applied to the conformal anomaly. The theory coupled to (background) dilaton is formally scale invariant and therefore the total anomalies in UV and IR have to match[5]. It can be then shown [115] that the effective action in IR takes the form

$$\begin{aligned} S^{\mathrm{IR}}[g_{\mu\nu}, \tau] = CFT^{\mathrm{IR}}[g_{\mu\nu}] + f^2 \int d^4x \sqrt{-\hat{g}} \hat{H} + \kappa \int d^4x \hat{H}^2 + \kappa' \int d^4x \sqrt{-\hat{g}} \hat{W}^2 \\ - (\beta_a^{\mathrm{UV}} - \beta_a^{\mathrm{IR}}) \int d^4x \sqrt{-g} \left(\tau E_4 + 4G^{\mu\nu} \partial_\mu \tau \partial_\nu \tau - 4(\partial\tau)^2 \Box\tau + 2(\partial\tau)^4 \right) \\ - (\beta_c^{\mathrm{UV}} - \beta_c^{\mathrm{IR}}) \int d^4x \sqrt{-g} \tau W^2 + (\bar{b}^{\mathrm{UV}} - \bar{b}^{\mathrm{IR}}) \int d^4x \sqrt{-g} H^2 + \dots , \end{aligned} \tag{4.5}$$

where the hatted quantities correspond to geometric terms formed from the metric (4.4) and thus are manifestly Weyl-invariant (ergo they don't contribute to the anomaly). The three terms in last two lines of (4.5) are defined so that their Weyl variation balances the Weyl variation of CFT^{IR}[6]. This effective action is directly analogous to the Wess-Zumino term [124] of pions in connection with the the axial anomaly (here gravity plays the role of background gauge field).
In order to simplify the analysis it is useful to choose a specific background. In this case one picks the flat background $g_{\mu\nu} = \eta_{\mu\nu}$, which conveniently eliminates the last line of (4.5) and all the quantities depending on derivatives of the unhatted metric. Thus (4.5) reduces to

$$\begin{aligned} S^{\mathrm{IR}}[g_{\mu\nu}, \tau] = CFT^{\mathrm{IR}}[\eta_{\mu\nu}] + 3f^2 \int d^4x e^{-2\tau} \left(\Box\tau - (\partial\tau)^2 \right) + 6\kappa \int d^4x \left(\Box\tau - (\partial\tau)^2 \right)^2 \\ (\beta_a^{\mathrm{UV}} - \beta_a^{\mathrm{IR}}) \int d^4x \left(4(\partial\tau)^2 \Box\tau - 2(\partial\tau)^4 \right) \quad . \end{aligned} \tag{4.6}$$

[4] Note that K-S use $g_{\mu\nu} \to e^{2\alpha} g_{\mu\nu}$ convention for Weyl transformations. Also, since the proof uses unitarity, the Minkowski space conventions will be used in this subsection.

[5] This is in direct analogy with the t'Hooft anomaly matching argument where the dilaton has taken the role of spectator fermions which cancel the anomaly.

[6] This can be seen by using $-\frac{\delta}{\delta\alpha} CFT^{IR} = \beta_a^{\mathrm{IR}} E_4 + \dots$.

The term proportional to $\Delta\beta_a$ can be isolated using the constraint

$$\Box\tau = (\partial\tau)^2 \ , \tag{4.7}$$

which is equivalent to the EOM $\Box\chi = 0$ for the scalar field $\chi = 1 - e^{-\tau}$, which simplifies the dilaton-dependent part of (4.6) to

$$-2(\beta_a^{\mathrm{UV}} - \beta_a^{\mathrm{IR}}) \int d^4x (\partial\chi)^4 + O(\chi^5) \ . \tag{4.8}$$

From this K-S conclude that $\Delta\beta_a$ can be extracted from the four point, on-shell ($p_i^2 = 0$), forward ($t = 0$ and $s = -u$) amplitude $\mathcal{A}^{\chi\chi\to\chi\chi}(s)$

$$\mathcal{A}^{\chi\chi\to\chi\chi}(s) = 2\Delta\beta_a s^2 + \dots \quad , \tag{4.9}$$

where dots stand for the sub-leading contributions proportional to beta functions etc. Finally, the leading s^2 term is disentangled from the rest of the amplitude through the dispersion relation [7]

$$\Delta\beta_a = \frac{1}{\pi}\int_{s>0} ds \frac{\mathrm{Im}\mathcal{A}^{\chi\chi\to\chi\chi}(s)}{s^3} \geq 0 \ , \tag{4.10}$$

where the positivity of above integral follows from the applying the optical theorem to $\mathrm{Im}\mathcal{A}^{\chi\chi\to\chi\chi}(s)$ [123]. This proves the weak a-theorem conjecture in $d = 4$.

4.2 Formula for the flow of β_a

In this very short section we will apply the formula

$$\Delta\beta_a = \frac{1}{2}\int_0^\infty \beta^A\beta^B \left(\chi^{\mathcal{R}}_{AB} - \beta^C \chi^{\mathcal{R}}_{ABC}\right) \frac{d\mu'}{\mu'} \ . \tag{4.11}$$

to calculate $\Delta\beta_a$ for gauge theories with Banks-Zaks fixed point. The function $\chi^{\mathcal{R}}_{AB}$ was defined in (2.114) and $\chi^{\mathcal{R}}_{ABC}$ represents its three-point counterpart. A derivation of this relation along the lines of our discussion in Section 3.2.1 is presented in Appendix G.2. It can be shown (cf. the paragraph bellow (G.19) in Appendix G.2) that (4.11) is scheme independent, finite and hence well-defined

[7]To derive this relation one actually needs to assume analyticity of the amplitude ($\mathcal{A}(s)^* = \mathcal{A}(s^*)$), which might be problematic in the absence of a mass gap [65].

relation. This formula is new, although as we show at the end of Appendix G.2.2, the MS-scheme version of it can be obtained from the formulas of [66]. Where this work extends the known literature is the use of generic scheme $\mathcal{R}$ and the related discussion of the scheme-(in)dependence.

By comparison with (3.84) we see that the first term in (4.11) is equal to $2\Delta\tilde{d}$. The three-point contribution in (4.11) is subleading in the (conformal) perturbation theory, so close to the fixed point (at the lowest order in PT) $\Delta\beta_a \approx 2\Delta\tilde{d}$. For the relation $\Delta\beta_a = 2\Delta\tilde{d}$ to be exact the second term has to vanish which would put some non-trivial constraints on the 3-point functions of the theory. It would be tempting to assume such a relation as it would give an alternative Euclidean proof of the a-theorem similar to the $d = 2$ argument (3.61). For example, in curved space $\Delta\beta_a$ can be linked to a 2-point function evaluated on the space of constant curvature [107]. In the paper [112] it was proposed that for classically conformal theories $2\Delta\tilde{d} = \Delta\beta_a$ which was verified for a QCD-like theory up to order $\mathcal{O}(a_s^3)$ (or $\mathcal{O}(\kappa^3)$ as explained bellow). Here we use (4.11) to extend this QCD result to few more orders.

We will consider again the case of weakly-coupled QCD-like gauge theories (see Section 3.2.2) with $[O_g] \propto [G^2]$ (see (2.83) and (2.86)) for which the difference $\Delta\beta_a - 2\Delta\tilde{d}$ is proportional to

$$\int_0^\infty \beta^3 \chi^{\mathrm{MS}}_{ggg} \frac{d\mu'}{\mu'} = -\frac{1}{2}\int_0^{a_s^{\mathrm{IR}}} \beta^2 \chi^{\mathrm{MS}}_{ggg} \frac{da_s}{a_s} \; . \tag{4.12}$$

To see at which order this correction contributes one needs to count powers of the Banks-Zaks parameter $\kappa = -\frac{3}{2}\frac{\beta_0}{N_c} \ll 1$ rather than a_s. Hence $a_s \sim \mathcal{O}(\kappa)$ and $\beta \sim \mathcal{O}(\kappa^2)$. From (4.11) it can be seen that $\mathcal{O}(\kappa^n)$ corresponds to n loop beta function and $(n-1)$ loop χ^{MS}_{gg}. This means that if $\chi^{\mathrm{MS}}_{ggg} \sim \mathcal{O}(\kappa^r)$ the correction (4.12) contributes at $O(\kappa^{r+4})$. This implies that the correction (4.12) can potentially appear at $\mathcal{O}(\kappa^4)$ which corresponds to 4 loop beta function, 3 loop χ^{MS}_{gg} and 1 loop χ^{MS}_{ggg} calculation. We checked [8]

$$\chi^{\mathrm{MS}}_{ggg} = \mathcal{O}(\kappa^2) \; , \tag{4.13}$$

which seems to support Anselmi's conjecture to $\mathcal{O}(\kappa^5)$. As a side result we get $O(\kappa^4)$ expression for $\Delta\beta_a$ directly from (3.90). To this end we will need the zero

[8] We checked the LO by direct calculation of the G^2 three-point function in momentum space and projecting on the Källén function. The NLO in κ can be deduced from the background field calculation result of Jack and Osborn (cf. eq [5.8] in [66]).

of four-loop beta function (cf. Appendix C.2)

$$a_s^{\mathrm{IR}} = -\frac{\beta_0}{\beta_1}\left(1 + \frac{\beta_0\beta^2}{\beta_1^2} + \beta_0^2\frac{(2\beta_2^2 - \beta_1\beta_3)}{\beta_1^4}\right) + O(\beta_0^4)\ . \tag{4.14}$$

Plugging this expression to (3.90) and using $\Delta\beta_a = 2\Delta\tilde{d} + O(\kappa^6)$ we get

$$\Delta\beta_a = \frac{1}{3600\pi^2}N_c^2\kappa^2\left(1 + 2\left(\frac{7}{25}\right)^2\kappa + \frac{53\cdot 4231}{3^3\cdot 25^4}\kappa^2\right) + O(\kappa^5)\ . \tag{4.15}$$

Note that LO and NLO agrees with the expression in [66]. Note that (4.15) also extends these results by one more order in κ. The $O(\kappa^4)$ term is new and rather reassuringly the factor of ζ_3 has dropped from the final expression. With the knowledge of the four loop expression $g^{\mathbb{1}}_{1,3}$ one could easily extend this expression to $O(\kappa^5)$.
We therefore validated the $2\Delta\tilde{d} = \Delta\beta_a$ conjecture of [112] to another two orders in κ. From our discussion it follows that an all-order proof would follow from vanishing of χ^{MS}_{ggg} at $O(\kappa^3)$ and beyond, which we could not verify. In any case, a satisfactory non-perturbative argument for this is not known to us.

4.3 $\mathcal{N} = 1$ anomaly the from the dilaton effective action

Let us start by reviewing the progress on a-theorem in SUSY. An exact expression for the difference of the ultraviolet (UV) and infrared (IR) Euler anomaly $\Delta a \equiv a_{\mathrm{UV}} - a_{\mathrm{IR}}$ was derived for $\mathcal{N} = 1$ supersymmetric gauge theories by Anselmi, Freedman, Grisaru, and Johansen (AFGJ) [125]. Thereafter it has served as a fruitful laboratory for testing different techniques by rederiving the result. Examples include verification up to fourth loop order [126], the use of the local renormalization group (RG) [67] and employing superspace techniques assuming a gradient flow equation [68]. In the latter case an expression valid outside the fixed point has been obtained [68] of a form conjectured earlier by a perturbative approach [127].

In this section $\Delta\beta_a|_{\mathcal{N}=1}$ is derived by using the techniques of conformal anomaly matching and dilaton effective action. The latter were used by Komargodski and

Schwimmer (KS) [115, 123] to derive the a-theorem $\Delta a \geq 0$ as conjectured in 1988 by Cardy [102]. A crucial ingredient is the introduction of an *external* field called the dilaton by coupling it to the renormalization scale $\mu \to \mu e^{\tau(x)}$, thereby introducing a local scale interpretation analogous to the the local RG pioneered by Shore [128, 129]. The locality of the approach is crucial and served Jack and Osborn to derive a proof the a-theorem at weak coupling (i.e. perturbation theory) by using it as a source term in a field theory in a generic curved background. KS and later Komargodski [123] focused on the four point dilaton function and were able to prove the a-theorem based on analyticity assumptions. Details of the proof were outlined in Section 4.1. In the next few paragraphs we present a literature review of some of the things that we already mentioned in this thesis.
In essence the dilaton serves as a compensator field to the Weyl-rescaling

$$g_{\mu\nu} \to e^{-2\alpha(x)} g_{\mu\nu} \ . \tag{4.16}$$

The transformation (4.16) corresponds to changing distances locally and implies that coordinate and momenta invariants change as $x^2 \to e^{-2\alpha(x)}x^2$ and $p^2 \to e^{2\alpha(x)}p^2$. Variation of the logarithm of the partion function with respect to the Weyl-parameter results in the vacuum expectation value (VEV) of the trace of the energy momentum tensor (TEMT). For a theory on a curved space, with no explicit scale symmetry breaking, the TEMT is parametrised by [130, 131]

$$\Theta_{grav} = -\beta_a \, E_4 - \beta_b \, W^2 - \beta_c \, H^2 + 4\tilde{d}\Box H \ , \tag{4.17}$$

where as before the abbreviations

$$\Theta \equiv T^{\rho}{}_{\rho} \ , \quad H \equiv \frac{R}{d-1} \ , \tag{4.18}$$

are used throughout. The quantities $E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$, $W^2 = R^2_{\mu\nu\alpha\beta} - \frac{4}{(d-2)}R^2_{\mu\nu} + \frac{2}{(d-1)(d-2)}R^2$ and R are the Euler density, the Weyl tensor squared and the Ricci-scalar; and $R_{\mu\nu\alpha\beta}$ and $R_{\mu\nu}$ denote the Riemann and Ricci tensors. The Euler density E_4 is a topological quantity and the Weyl tensor squared W^2 vanishes on a conformally flat space. The absence of β_c, and therefore the H^2-term, in a 4D conformal field theory (CFT) was shown in [98]. For the detailed discussion of $\Box R$ we refer the reader to Section 3.2. The constants $\beta_a, \beta_b, \beta_c$ and $\tilde{d}$ depend on the dynamics of the theory. Their free field values for various spins were computed in [130]. Note, the non-vanishing of β_a and β_b therefore establish the conformal or Weyl anomaly in 4D [130, 131]. This discussion is structured as follows. In Section 4.3.1 the general framework is outlined by restating some of the results of [115] in a language appropriate for this work. The specific construction is presented and illustrated in sections 4.3.2 and 4.3.3 respectively. In Section 4.3.4, in particular 4.3.5, the AFGJ Euler anomaly result is rederived within our framework using the Konishi anomaly. This result has been published in [2].

4.3.1 General framework

Consider a massless theory with fields ϕ and a coupling g. The path integral is given by

$$e^{W(g(\mu),\mu)} = \int [\mathcal{D}\phi]_\mu \, e^{-S_{\rm W}(g(\mu),\mu,\phi)} \;, \tag{4.19}$$

where the action $S_{\rm W}$ is to be understood in a Wilsonian sense and W is proportional to the negative free energy. For the purposes of this work $S_{\rm W}$ is interpreted to be on a renormalization trajectory from the UV to an IR fixed point. In massless theories correlation functions depend on ratios of q^2/μ^2 where q denotes an external momentum. Hence the renormalization scale transforms as $\mu \to e^{-\alpha}\mu$ under the Weyl-rescaling. An *external* field, known as the dilaton τ, is introduced in the action

$$S_{\rm W}(g(\mu),\mu,\phi) \to S_\tau \equiv S_{\rm W}(g(\mu e^\tau),\mu e^\tau,\phi) \;. \tag{4.20}$$

transforming under Weyl-rescaling (4.16) as

$$\tau \to \tau + \alpha \;, \tag{4.21}$$

such that the product μe^τ is Weyl-invariant. The dilaton therefore serves as a spurion (or compensator) formally restoring scale invariance. In this work no dynamic nature is attributed to the dilaton field which is in line with [123] but not the first paper [115] on the a-theorem in 2011. The dilaton serves as a source term for the TEMT and when made a local field the (yet to-be-defined) Wess-Zumino term carries the information on the Euler anomaly. Promoting the dilaton to a local field $\tau \to \tau(x)$ requires *local* Weyl invariance and demands changes similar to passing from global to local gauge invariance. The specific implementation will be discussed in the the explicit examples. The space-dependence of τ augments the couplings to local objects $g(\mu) \to g(\mu e^{\tau(x)})$. Note that the functional form μe^τ renders local Weyl-rescaling equivalent to a local RG transformation. The path integral becomes τ-dependent,

$$e^{W_\tau} = \int [\mathcal{D}\phi]_\mu e^{-S_\tau} = \int [\mathcal{D}\phi]_\mu e^{-S_{\rm W}(g(\mu e^\tau),\mu e^\tau,\phi)} \ . \tag{4.22}$$

The quantity W_τ corresponds to the generating functional of the correlation function (connected component) of the traces of the EMT. The Wess-Zumino action can be read off from (4.5) and (4.6)

$$S_{\rm WZ} = \int d^4x \, 2\left(2\Box\tau(\partial\tau)^2 - (\partial\tau)^4\right) + \mathcal{O}(R) \ , \tag{4.23}$$

where we saw it to be the source term of the Euler anomaly β_a. Above R stands for the non-dilaton curvature background. More precisely, using arguments of conformal anomaly matching it was shown that the difference of the UV and IR dilaton effective action, with with $g^*_{\rm (UV,IR)} \equiv g(\infty,0)$,

$$\Delta W_\tau \equiv \int_{g^*_{\rm IR}}^{g^*_{\rm UV}} dg\, \partial_g W_\tau = -\int_{-\infty}^{\infty} d\ln\mu\, \partial_{\ln\mu} W_\tau = W_\tau(g^*_{\rm UV}) - W_\tau(g^*_{\rm IR}) = \Delta\beta_a \, S_{\rm WZ} + \dots \ , \tag{4.24}$$

contains a term proportional to $S_{\rm WZ}$ times the sought after quantity $\Delta\beta_a \equiv \beta_a(\mu_{\rm UV}) - \beta_a(\mu_{\rm IR})$ [115]. Hence determining $\Delta\beta_a$ reduces to finding $\partial_{\ln\mu} W_\tau$. Note that the second equality in (4.24) follows from (F.19) in the limit $m \to 0$ and using $dg/\beta = d\ln\mu$.

4.3.2 Dilaton dependent conformal factor

In this work a theory is considered which can be reinterpreted as a free field theory in a conformally flat background

$$\tilde{\rm g}_{\rho\lambda} = e^{-2s(\tau)}\delta_{\rho\lambda} \ , \tag{4.25}$$

which carries the information on the RG flow parameters. Above $\delta_{\rho\lambda}$ denotes the flat Euclidean metric with positive signature. The adaption to Minkowski space is straightforward as it results in the appearance of various factors of i only. By using (F.23) from the appendix[9] [10]

$$\partial_{\ln\mu} W_\tau = \int d^4x \sqrt{\tilde{g}}\ \langle\Theta\rangle_\tau \ , \tag{4.26}$$

and (4.24) we may write a more explicit formula for the difference of the Euler anomaly Δa

$$\Delta\beta_a = -\int_{-\infty}^{\infty} d\ln\mu \ \int d^4x \sqrt{\tilde{g}} \langle\Theta\rangle_\tau|_{S_{\rm WZ}} \ , \tag{4.27}$$

where $|_{S_{\rm WZ}}$ denotes the following projection

$$\langle\Theta\rangle_\tau = -\tilde{\beta}_a \tilde{E}_4 - \tilde{\beta}_b \tilde{H}^2 = \langle\Theta\rangle_\tau|_{S_{\rm WZ}} \, S_{\rm WZ} + \ldots \ . \tag{4.28}$$

The coefficients $\tilde{\beta}_a$ and $\tilde{\beta}_b$ depend on the dynamics of the theory. We have omitted the $\tilde{\Box}\tilde{H}$ which can be removed by suitable counterterm and therefore won't be relevant to the discussion. The Wess-Zumino action $S_{\rm WZ}$ is defined in (4.23) and we follow the rule that the tilde denotes geometric quantities, e.g. $\tilde{E}_4$ and $\tilde{H}^2$, evaluated in the background metric $\tilde{g}_{\mu\nu}$[11]. The quantity Δa is determined once $\tilde{\beta}_a$ and $\tilde{\beta}_b$ are known. In the next section we will discuss a very simple toy model that illustrates these ideas and will serve as a stepping stone for the $\mathcal{N} = 1$-computation.

4.3.3 Weyl anomaly of free scalar in conformally flat space

We consider a scalar field theory on a flat space, focusing solely on the kinetic term

$$S_{\rm W}(\mu) = \int d^4x Z(\mu) \delta^{\rho\lambda} \partial_\rho\phi \partial_\lambda\phi \ , \tag{4.29}$$

thereby ignoring other contributions. The usefulness of this construction will, hopefully, become clear in the following sections. Taking $Z(\mu) \to Z(\mu e^{\tau(x)})$ amounts to passing to S_τ (4.20). The factor $Z(\mu e^{\tau(x)})$ can be absorbed into the metric by a local Weyl-rescaling by choosing $\alpha = s$ in (4.16) with

$$s(\mu e^\tau) = -\frac{1}{2} \ln Z(\mu e^{\tau(x)}) \quad \Rightarrow \quad \tilde{g}_{\rho\lambda} = Z \delta_{\rho\lambda} \ . \tag{4.30}$$

[9]Note that metric (4.25) is not a physical or geometric metric as it does not transform like (4.16) under Weyl-rescaling $s(\tau) \to s(\tau + \alpha)$ unless $s(x) = x$. The latter is the case in [115], $\hat{g}_{\rho\lambda} = e^{-2\tau(x)}\delta_{\rho\lambda}$, and constitutes one of the differences with respect to our approach.

[10]The subscript τ refers to the VEV of the trace of the EMT with respect to the partition function (4.22).

[11]Conformal flatness of $\tilde{g}$ implies that $\tilde{W}^2 = 0$.

The theory then becomes a field theory on a conformally flat space with metric $\tilde{g}_{\mu\nu}$ (4.30)

$$S_\tau(\mu) = \int d^4x \sqrt{\tilde{g}} \tilde{g}^{\rho\lambda} D_\rho^{(s)} \phi D_\lambda^{(s)} \phi \, . \tag{4.31}$$

Above $D_\rho^{(s)} = \partial_\rho - (\partial_\rho s)$ denotes the Weyl-covariant derivative[12] analogous to the covariant derivative in gauge theories. The coefficients of the trace anomaly (4.17) for a theory with metric $\tilde{g}_{\rho\lambda}$ and one free scalar field are given by (cf. [110, 130] or the explicit computation in appendix A.3)

$$\tilde{\beta}_a = a_{(0)}^{\text{free}} \, , \quad \tilde{\beta}_b = 0 \, , \quad \tilde{c}' = -2a_{(0)}^{\text{free}} \, , \qquad a_{(0)}^{\text{free}} = \frac{1}{360}\frac{1}{16\pi^2} = \frac{1}{5760\pi^2} \, , \tag{4.32}$$

or equivalently $\langle \Theta \rangle_\tau = -a_{(0)}^{\text{free}}(\tilde{E}_4 - 2\tilde{\Box}\tilde{R})$. As stated earlier, the coefficient c' is of no importance for this work and is therefore discarded[13]. The Euler density in terms of s is given by

$$\sqrt{\tilde{g}}\tilde{E}_4 = -8(\frac{1}{2}\Box(\partial s)^2 - \partial \cdot (\partial s(\Box s - (\partial s)^2))) \, . \tag{4.33}$$

Using the explicit form $s = -\frac{1}{2}\ln Z(\mu e^\tau)$ the Euler term becomes

$$\sqrt{\tilde{g}}\tilde{E}_4 = -[\gamma^2\Box(\partial\tau)^2 + (2\gamma\dot{\gamma} - 2\gamma^2)\partial^\lambda(\partial_\lambda\tau\Box\tau) - \gamma^3\partial^\lambda(\partial_\lambda\tau(\partial\tau)^2) - \\ 6\gamma\dot{\gamma}(\partial\tau)^2\Box\tau - 3\gamma^2\dot{\gamma}(\partial\tau)^4] \, , \tag{4.34}$$

where here and below we use the abbreviation $\dot{\gamma} \equiv \frac{d}{d\log\mu}\gamma$ and the following expressions

$$\partial_\rho\gamma = \dot{\gamma}\,\partial_\rho\tau \, , \quad \partial_\rho s = -\frac{1}{2}\frac{\partial \ln Z(\mu e^\tau)}{\partial(\mu e^\tau)}\partial_\rho(\mu e^\tau) = -\frac{1}{2}\gamma\,\partial_\rho\tau \, , \quad \gamma = \frac{\partial \ln Z(\mu)}{\partial \ln \mu} \, , \tag{4.35}$$

have been used. The quantity $\Delta\beta_a$ is obtained by integrating over $d\ln\mu$ and projecting on S_{WZ}. In doing so γ and $\dot{\gamma}$ can be treated as being space-independent, since expanding $\gamma(\mu e^\tau) = \gamma(\mu) + O(\tau(x))$ leads to terms which are not contained in S_{WZ}. Furthermore it is then clear that the first line in (4.34) can be discarded since it is a total derivative and therefore inequivalent to the S_{WZ} (4.23) bulk-term. In order to project the second line of (4.34) on S_{WZ} (4.23) it is convenient

[12] Adding the Weyl-covariant derivatives is equivalent to the replacement $\Box \to \Box - \frac{1}{6}\tilde{R}$ which is the usual conformally coupled scalar in a curved space of metric $\tilde{g}_{\rho\lambda}$.

[13] Reader recall that in Section 3.2 we concluded that the UV (bare) value of $\Box R$ is ambiguous and therefore devoid of physical content.

(following [115],[123])[14]

$$\Box\tau = (\partial\tau)^2 \ , \tag{4.36}$$

under which all four-derivative invariants vanish, except for

$$S_{\rm WZ}|_{(4.36)} = \int d^4x\, 2(\Box\tau)^2 \, . \tag{4.37}$$

Using (4.27) and performing the integral over $d\ln\mu$ we get

$$\Delta a = \frac{1}{2} a^{\rm free}_{(0)} \left[3A_1 + A_2\right] \ , \tag{4.38}$$

where

$$\begin{aligned} A_1 &= \int_{-\infty}^{\infty} d\ln\mu \; 2\gamma\dot{\gamma} \ = \int_{\gamma_{\rm IR}}^{\gamma_{\rm UV}} d\gamma \; 2\gamma \ = (\gamma_{\rm UV}^2 - \gamma_{\rm IR}^2) \ , \\ A_2 &= \int_{-\infty}^{\infty} d\ln\mu \; 3\gamma^2\dot{\gamma} = \int_{\gamma_{\rm IR}}^{\gamma_{\rm UV}} d\gamma \; 3\gamma^2 = (\gamma_{\rm UV}^3 - \gamma_{\rm IR}^3) \ , \end{aligned} \tag{4.39}$$

and $\gamma_{\rm IR,UV} \equiv \gamma(g^*_{\rm IR,UV})$ are the values of the anomalous dimensions at the respective fixed points. For further reference the final result (4.38) is stated with explicit coefficients A_1 and A_2

$$\Delta\beta_a \ = \ \frac{1}{2}\left((\gamma_{\rm UV}^3 - \gamma_{\rm IR}^3) + 3(\gamma_{\rm UV}^2 - \gamma_{\rm IR}^2)\right) a^{\rm free}_{(0)} \ . \tag{4.40}$$

This result constitutes an important intermediate result for the derivation of $\Delta a|_{\mathcal{N}=1}$.

4.3.4 $\mathcal{N} = 1$ supersymmetric gauge theory

The theory considered in this section is a $\mathcal{N} = 1$ supersymmetric gauge theory with flavour symmetry $SU(N_f) \times SU(N_f)$ and gauge group $SU(N_c)$. The action can be written in terms of the usual vector superfield V and matter superfields $(\Phi_f, \tilde{\Phi}_f)$ as, e.g. [132],

$$S_{\rm W}(\mu) = \int d^6z \frac{1}{g^2(\mu)} {\rm tr} W^2 + {\rm h.c.} + \frac{1}{8} Z(\mu,\mu') \sum_f \Big[\int d^8z \Phi_f^\dagger e^{-2V} \Phi_f + \int d^8z \tilde{\Phi}_f^\dagger e^{-2V} \tilde{\Phi}_f \Big] \ , \tag{4.41}$$

[14] We note in passing that Eq. (4.36) is the lowest order equation of motion [115] when a dynamic nature is attributed the dilaton.

where W^2 is the supersymmetric gauge field kinetic term, g is referred to as the holomorphic coupling constant parametrisation and d^6z and d^8z include integration over the fermionic superspace variables.

The main tool in deriving $\Delta\beta_a|_{\mathcal{N}=1}$ is the use of the Konishi anomaly [29, 133, 134]. The latter is illustrated in appendix D.2 as a method to derive the NSVZ beta function. In Section 4.3.5 the Konishi anomaly is used to write the Wilsonian action such that the RG flow can be absorbed into the metric. This procedure makes it amenable to the free field theory computation in the dilaton background discussed in Section 4.3.3.

4.3.5 $\Delta\beta_a|_{\mathcal{N}=1}$ from Dilaton effective Action

We consider the $\mathcal{N} = 1$ supersymmetric gauge theory with Wilsonian effective action given in (B.26). Choosing a rescaling factor, with $\gamma_* = -b_0/N_f$ (B.31),

$$(\Phi_f, \tilde{\Phi}_f) \to \left(\frac{\mu'}{\mu}\right)^{\gamma_*/2} (\Phi_f, \tilde{\Phi}_f) \tag{4.42}$$

on the matter fields the Konishi turns the action into the following form

$$\begin{aligned} S_{\mathrm{W}}(\mu) \quad &= \quad \int d^6z \frac{1}{g(\mu')^2} \mathrm{tr} W^2 + \text{h.c.} + \\ & \frac{1}{8}\sum_f \Big[\int d^8z \hat{Z}(\mu) \Phi_f^\dagger e^{-2V} \Phi_f + \int d^8z \hat{Z}(\mu) \tilde{\Phi}_f^\dagger e^{-2V} \tilde{\Phi}_f \Big] \,, \end{aligned} \tag{4.43}$$

with

$$\hat{Z}(\mu) \equiv Z(\mu, \mu') \left(\frac{\mu'}{\mu}\right)^{\gamma_*} \,, \tag{4.44}$$

where $\mu' > \mu$ is an arbitrary scale which can be thought off as a UV cut-off Λ_{UV}. Crucially, the RG flow[15] is absorbed into the precoefficient $\hat{Z}(\mu)$ in front of the matter term. Eq. (4.43) is the analogue of the action (4.29) for the scalar field to the degree that the running of the theory is parametrised by a coefficient in front of the matter kinetic term. As previously mentioned (4.42) inverts, to some degree, the rescaling trick to derive the NSVZ beta function in Appendix D.2.

[15] by that we mean that all μ-dependence in the action (4.43) is now in $\hat{Z}(\mu)$ leaving the coefficient of $\mathrm{tr}W^2$ depending only on the UV scale μ'.

Again following the procedure in (4.20) a dilaton is introduced through

$$\hat{Z}(\mu) \to \hat{Z}_\tau(\mu) = \hat{Z}(\mu e^{\tau(x)}) = Z(\mu e^{\tau(x)}, \mu') \left(\frac{\mu'}{\mu e^{\tau(x)}} \right)^{\gamma_*} . \tag{4.45}$$

To keep the procedure manifestly supersymmetric, following [135], the dilaton is promoted to a (chiral) superfield T such that

$$T| = \tau + i\omega \;, \quad \hat{Z}(\mu e^T)| = \hat{Z}(\mu e^\tau) \;. \tag{4.46}$$

Above ω is the axion and the bar stands for projection on to the lowest component of the multiplet. It will be seen that $\hat{Z}$ in (4.43) can be absorbed into the background geometry by a local Weyl-rescaling. To preserve local SUSY invariance the Weyl transformations are promoted to super-Weyl transformations. Under the latter, the $\mathrm{tr}W^2$-term is invariant whereas the matter term transforms as follows (cf. [136])

$$\int d^8 z \Phi^\dagger e^{-2V} \Phi \to \int d^8 z e^{-A} e^{-A^\dagger} \Phi^\dagger e^{-2V} \Phi \;, \tag{4.47}$$

with the superfield $A = \alpha + i\beta + \ldots$ being the super-Weyl parameter corresponding to α in (4.16). Note that such a formalism is automatically local Weyl-invariant and that there is no need to introduce the Weyl-covariant derivatives as in (4.30). Furthermore, the transformation (4.47) amounts to a Weyl-rescaling of the vielbein[16]

$$e^a_\rho \to e^{-\frac{A}{2}} e^{-\frac{A^\dagger}{2}} | \; e^a_\rho = e^{-\alpha} e^a_\rho \;. \tag{4.48}$$

Upon identifying $\alpha = s$ in Eq. (4.30)

$$e^a_\rho \to \tilde{e}^a_\rho = \sqrt{\hat{Z}_\tau(\mu)} e^a_\rho \;. \tag{4.49}$$

The action $S_W(\mu e^T)$ (4.43) can then be written in a manifestly locally supersymmetric form; cf. Section 6.3 in [137]. Eq. (4.49) results in

$$\tilde{g}_{\rho\lambda} = \tilde{e}^a_\rho \tilde{e}^a_\lambda = e^{-2s(\mu e^\tau)} \delta_{\rho\lambda} \;, \quad s(\mu e^\tau) = -\frac{1}{2} \ln \hat{Z}(\mu e^{\tau(x)}) \;. \tag{4.50}$$

Notice that the UV scale μ' is arbitrary and that therefore a physical quantity

[16]Under a super-Weyl transformation, the supersymmetric generalization of vielbein transforms as $E^a_\mu \to e^{-\frac{A}{2}} e^{-\frac{A^\dagger}{2}} E^a_\mu$, which corresponds to the standard Weyl transformation $e^a_\mu \to e^{-\alpha} e^a_\mu$ after projecting on the lowest component of E^a_μ. In the interest of clarity we would like to add that $e^a_\mu = \delta^a_\mu$ on flat space.

like $\langle\Theta\rangle_\tau$ should not depend on it. Since the geometric terms $\tilde{E}_4$ and $\tilde{H}^2$ are independent of μ',[17] the form of (4.28) implies that $\tilde{a}$ and $\tilde{c}$ are μ'-independent and therefore constants. This means that $\tilde{a}$ and $\tilde{c}$ assumed the values at the (free) UV fixed point and the geometric quantities are to be evaluated in the background metric $\tilde{g}_{\rho\lambda}$ carrying the dynamic information. This allows to recycle, in large parts, the computation in Section 4.3.3 as outlined below.

A free theory in a curved background is in particular conformal and therefore free of the $\tilde{R}^2$-term (i.e. $\tilde{c} = 0$). Since the dilaton couples to the matter part only, the trace anomaly is exhausted by the free field theory computation of the matter-fields in the curved background with metric (4.50). Equivalence to the example in the previous section is achieved through the formal replacement $Z \to \hat{Z}$ (following from (4.50)) which implies $\gamma \to \delta\gamma \equiv \gamma - \gamma_*$ *and* the change in the number of degrees of freedom ν. More precisely the matter superfield consists of a complex scalar and a Weyl fermion which contribute [130]

$$\nu \equiv 2\Big|_{\mathbb{C}\text{-scalar}} + \frac{11}{2}\Big|_{\text{Weyl-fermion}} = \frac{15}{2} \tag{4.51}$$

in units of a real scalar field. This number has to be multiplied by the number of colours $2N_f$ (two matter-field per flavour) and N_c (the $SU(N_c)$ Casimir of the adjoint representation). Hence Δa is given by $2N_f N_c \nu \Delta a|^{(4.40)}_{\gamma_{\text{UV,IR}} \to \delta\gamma_{\text{UV,IR}}}$. Now, $(\gamma_{\text{UV}}, \gamma_{\text{IR}}) = (0, \gamma_*)$ implies $(\delta\gamma_{\text{UV}}, \delta\gamma_{\text{IR}}) = (-\gamma_*, 0)$ and therefore

$$\Delta\beta_a|_{\mathcal{N}=1} = \frac{15}{2} N_c N_f (-\gamma_*^3 + 3\gamma_*^2) a^{\text{free}}_{(0)} \,. \tag{4.52}$$

We note that (4.52) is indeed the same as the non-perturbative result quoted in (Eq.4.18) in [125] when taking into account the explicit form of γ_* (B.31). The formula above is valid in the conformal window $3/2N_c < N_f < 3N_c$, where the UV theory is asymptotically free and the IR theory acquires a non-trivial fixed point (see Section 1.3.4 for more details). Within these boundaries the anomalous dimension γ_* takes on the values -1 to 0 and the quantity Δa is therefore manifestly positive in accordance with the a-theorem. The latter has been proven for $\mathcal{N} = 1$ supersymmetric theories by using R-symmetries and is known as a-maximization [138]. The adaptation to gauge groups other than $SU(N_c)$, provided they are asymptotically free, amounts to replacing the $SU(N_c)$-Casimir N_c by the corresponding Casimir of the group.

[17] To see this notice that these terms depend on derivatives of s only (cf. (4.33)). The latter are related to the anomalous dimension $\gamma(\mu e^\tau)$ through the relation (4.35) which is independent of μ'.

4.3.6 Summary and Discussion

The main result of this section was derivation the difference of the Euler term $\Delta\beta_a$ in $\mathcal{N} = 1$ supersymmetric gauge theories (4.52) in the conformal window. We obtained an explicit expression (4.52) as a function of the anomalous dimension at the fixed point and the free-field Euler anomaly for matter sector. This result was known [125], but our derivation is novel in the use of dilaton effective action, which can serve as a direct non-perturbative check of the formalism of[115].
By an appropriate rescaling of the matter superfield *and* choosing the Weyl-parameter α (4.16) to equal the logarithm of the matter prefactor (4.50), the computation was shown to be equivalent to one of the (free) UV theory in a curved background carrying the information on the flow. This allowed for $\Delta\beta_a|_{\mathcal{N}=1}$ to be computed from the free field theory example, in Section 4.3.3, with a simple formal replacement for γ_{IR} and γ_{UV}. It is noted that the structure of $\Delta\beta_a|_{\mathcal{N}=1}$ is completely given by the Wess-Zumino term of the dilaton effective action. The aspect of matching the computation with a free theory bears some resemblance with the original AFGJ-derivation [125] in that independence on an RG-scale is exploited in evaluating certain quantities in the UV where they correspond to free field theory computations. An extension to the non-supersymmetric case is not straightforward because it relies on the one-loop exactness of the rescaling anomaly in supersymmetric gauge theories. From sections D.2 and 4.3.5 it is seen that an exact expression of Δa in non-supersymmetric theories is related to finding an exact beta function. The Konishi anomaly is a rescaling anomaly which in $\mathcal{N} = 1$ supersymmetric theories is, by holomorphicity, bound to the axial anomaly. The latter is generally one-loop exact by topological protection of the axial charge. In non-supersymmetric theories there is no holomorphicity and the Konishi anomaly is an unknown function which could be determined order by order in perturbation theory. We end the section with remarks of the speculative and qualitative kind. Reformulating a gauge theory as a free theory in a curved background is reminiscent of the anti-de Sitter space/conformal field theory duality which has given rise to a lot of work and inspiration over the past two decades. The extension to theories with more than one relevant coupling is not immediate. One might wonder whether bi-gravity, whose renormalization group flow has been studied in [139], might be a possible avenue for a theory with two relevant couplings. A practical requirement is that the UV theory is asymptotically free in order to retain computability.

Conclusions

In this thesis we explored the properties of the trace anomaly in various contexts. The topics we have covered range from the renormalization of composite operators and their matrix elements to QFT on curved space. Whereas the latter two might seem as totally unrelated areas, we have tried to paint a coherent picture where each chapter was built on top of the previous ones using the same tools. Among the most prominent of the tools permeating this thesis were the quantum action principle (QAP) and renormalization group equations (RGEs). The QAP served us as a guideline to relate the information on correlators of composite operators to the coefficients of certain background field counterterms in the action. RGEs were then used to study scaling properties of these coefficients and their contribution to the trace anomaly.

Our initial pedagogical discussion of QAP in Chapter 2 involved localizing the couplings in space. We demonstrated by independent background field calculations how the resulting effective action for local couplings reproduces the divergences of two-point functions found by direct OPE evaluation. The entire procedure was mimicked later on in Chapter 3 to recover the divergences in the gravitational effective action by treating the metric as a local coupling associated with the trace of energy-momentum tensor (TEMT). In section 4.3 we explored the analogy between local couplings and metric even further by absorbing the renormalization of the gauge coupling inside the metric and therefore treating both as formally equal. Given the holographical nature of this construction (short-distance physics projected on long distances via geometry) it would be interesting to understand if it is in way connected to AdS/CFT.

The Feynman-Hellmann theorem discussed in Section 2.3 and QAP might be seen as related concepts in that both are prescriptions to obtain renormalized quantities by differentiation of finite/physical quantity with respect to a finite parameter.

In Section 2.2 RGEs were used to show that the correlators of TEMT are UV finite when the divergences/logarithms are resumed. The argument can be in principle extended to higher point correlators of TEMT (cf. Appendix G.2 for the three-point function discussion). In particular the covariant language and scheme transformations (2.128) introduced in Section 2.2.5 equipped us with a universal way to keep track of scheme-dependent quantities throughout this thesis. In turn this allowed us to conclude the scheme-independence of formulas (3.84), (4.11).
The UV finiteness of TEMT correlators was crucial to ensure a well-defined, positive expression for $\Delta\tilde{d}$ in terms of the fourth moment. As an extra corollary of finiteness we have found a general way to define a scheme where the R^2 anomaly vanishes. The question of IR finiteness of the fourth moment and therefore the validity of relation (3.63) for the theories with phase transitions (e.g. chirally broken QCD) remains open for future research. Perhaps clarifying the IR convergence would also bring some new insights about anomalies.
Analysis of RGE for the generating functional in Section 4.3 lead to a key equation (4.26) which relates the trace anomaly on a special curved background (4.25) to the flow of the dilaton effective action. This allowed us to calculate $\Delta\beta_a$ as a coefficient of the dilaton Wess-Zumino term in the effective action and thus demonstrate the validity of the abstract a-theorem proof [115] on an explicit non-perturbative example of supersymmetric conformal window. More examples of such kind would be desirable.

Appendix A

Background Field Calculations

A.1 The background field calculation of effective Lagrangian

We would like to perform a path integral over (1.43). First, let us drop Λ indices for clarity and make substitution $\phi = \phi_{\Lambda'} + \delta\phi$ in the Lagrangian (1.43). Expanding to quadratic order in $\delta\phi$ we get:

$$\mathcal{L}(\phi) = \mathcal{L}(\phi_{\Lambda'}) + \frac{1}{2}(\partial\delta\phi)^2 + \frac{1}{2}(\delta\phi)^2[m^2 + \frac{1}{2}\lambda\phi_{\Lambda'}^2] + O((\delta\phi)^3) \;, \tag{A.1}$$

used that the background field $\phi_{\Lambda'}$ is assumed to satisfy the classical equation of motion $\frac{\delta}{\delta\phi}\mathcal{L}|_{\phi=\phi_{\Lambda'}} = 0$. Neglecting the self-interaction of $\delta\phi$ the path integral of $\mathcal{L}$ becomes Gaussian and

$$\mathcal{L}_{\Lambda'} = -\ln\int \mathcal{D}\delta\phi e^{-\mathcal{L}(\phi_{\Lambda'}+\delta\phi)} \approx \mathcal{L}(\phi_{\Lambda'}) - \ln\int \mathcal{D}\delta\phi e^{-\delta\phi\Delta\delta\phi} \;, \tag{A.2}$$

where the kinetic operator of (A.1) reads

$$\Delta = \frac{1}{2}(-\Box + m^2 + \frac{1}{2}\lambda\phi_{\Lambda'}^2) \;. \tag{A.3}$$

The Gaussian path integral can be done

$$\int \mathcal{D}\delta\phi e^{-\delta\phi\Delta\delta\phi} = (\det\Delta)^{-\frac{1}{2}} = \exp\left(-\frac{1}{2}\mathrm{Tr}\log\Delta\right) \;. \tag{A.4}$$

This trace can be easily computed in momentum space if we take $\phi_{\Lambda'} = const.$ for simplicity

$$\mathrm{Tr}\log\Delta = \int_{\Lambda'}^{\Lambda} \frac{d^4k}{(2\pi)^4} \ln\left(k^2 + m^2 + \frac{1}{2}\lambda\phi_{\Lambda'}^2\right) . \tag{A.5}$$

To obtain dependence on $\phi_{\Lambda'}$ we expand the logarithm

$$\ln\left(k^2 + m^2 + \frac{1}{2}\lambda\phi_{\Lambda'}^2\right) = \ln\left(k^2 + m^2\right) + \frac{1}{2}\frac{\lambda\phi_{\Lambda'}^2}{(k^2+m^2)} - \frac{1}{8}\frac{\lambda^2\phi_{\Lambda'}^4}{(k^2+m^2)^2} + \ldots . \tag{A.6}$$

Substituting this expansion into the expression for $\mathcal{L}_{\Lambda'} = \mathcal{L}(\phi_{\Lambda'}) + \frac{1}{2}\mathrm{Tr}\log\Delta$, so the LO contributions to $\phi_{\Lambda'}^2$ and $\phi_{\Lambda'}^4$ in (1.45) will read

$$m^2(\Lambda') = m^2(\Lambda) + \frac{1}{2}\lambda(\Lambda)\int_{\Lambda'}^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m(\Lambda)^2)} \tag{A.7}$$

$$\lambda(\Lambda') = \lambda(\Lambda) - \frac{3}{2}\lambda(\Lambda)^2\int_{\Lambda'}^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m(\Lambda)^2)^2} . \tag{A.8}$$

These integrals are doable giving [1]

$$m^2(\Lambda') = m^2(\Lambda) + \frac{1}{16\pi^2}\lambda(\Lambda)(\Lambda^2 - \Lambda'^2) + \frac{1}{16\pi^2}\lambda(\Lambda)m^2\ln\left(\frac{\Lambda}{\Lambda'}\right) \tag{A.9}$$

$$\lambda(\Lambda') = \lambda(\Lambda) - \frac{3}{16\pi^2}\lambda(\Lambda)^2\ln\left(\frac{\Lambda}{\Lambda'}\right) . \tag{A.10}$$

The LO contribution to the kinetic term $(\partial\phi_{\Lambda'})^2$ can be found by using derivative expansion of (A.4), however it turns out this contribution is proportional to a total derivative $\Box\phi_{\Lambda'}^2$ and therefore vanishes.

A.2 Heat Kernel evaluation of the generating functional W

A.2.1 CP-even sector $W(g)$

In this appendix we outline the computation of the LO generating functional $W(g) = \ln\mathcal{Z}_g$ (2.34). There are a number of simplifications at LO. First in the chosen normalisation $\frac{1}{g^2}F^2$ the fermion term is independent of g and we can be

[1] We choose to neglect inverse powers of Λ, Λ' in the following equation.

discarded. Only quadratic terms need to be retained

$$S_g \to \int d^d x A_a^\mu \tilde{\Delta}^{(1)ab}_{\mu\nu} A_b^\nu \ , \quad S_{gf} \to \int d^d x A_a^\mu \bar{\Delta}^{(1)ab}_{\mu\nu} A_b^\nu \ , \tag{A.11}$$

where at LO Z_g can be neglected. The inverse spin 1 propagators read

$$\begin{aligned} \tilde{\Delta}^{(1)ab}_{\mu\nu} &= \frac{1}{2}\frac{1}{g(x)^2}[-\Box\delta_{\mu\nu} + \partial_\nu\partial_\mu + 2(\partial \ln g \cdot \partial) - 2\partial_\nu \ln g \partial_\mu]\delta^{ab} \ , \\ \bar{\Delta}^{(1)ab}_{\mu\nu} &= \frac{1}{2}\frac{1}{g(x)^2}[-\partial_\mu\partial_\nu + 2\partial_\mu \ln g \partial_\nu]\delta^{ab} \ , \end{aligned} \tag{A.12}$$

with the latter being the localised version of the Feynman gauge. Finally the generating functional at LO is given by ($\Delta^{(1)}_{\mu\nu} \equiv \tilde{\Delta}^{(1)}_{\mu\nu} + \bar{\Delta}^{(1)}_{\mu\nu}$)

$$e^{W(g)} = \int DADqD\bar{q} e^{-(S_g+S_f+S_{gf})}\Big|_{\text{LO}} = (\det \Delta^{(1)}_{\mu\nu})^{-1/2} = \exp -\frac{1}{2}\text{Tr}\ln \Delta^{(1)}_{\mu\nu} \ , \tag{A.13}$$

which is amenable to a heat kernel expansion by using Schwinger's formula

$$W = -\frac{1}{2}\text{Tr}\ln \Delta^{(1)}_{\mu\nu} = \frac{1}{2}\int_0^\infty \frac{dt}{t}\text{Tr}(e^{-t\Delta^{(1)}_{\mu\nu}}) \ . \tag{A.14}$$

The heat kernel can be evaluated in DR using the plane-wave basis expansion [140] $\text{Tr}(e^{-t\Delta^{(1)}_{\mu\nu}}) = \sum_{n\geq 0} b_n t^{\frac{n-d}{2}}$. Its divergent part is given by

$$W_{\text{div}}(g) = \frac{1}{2\epsilon} b_4 \ , \tag{A.15}$$

where

$$b_4 = \frac{n_g}{16\pi^2}\int d^d x \left[(\Box \ln g)^2 + \mathcal{O}(\ln^3 g) + \partial b_4 \right] \ . \tag{A.16}$$

with total derivative term

$$\partial b_4 = \frac{8}{15}\Box^2 \ln g + \frac{22}{15}\partial(\partial \ln g \, \Box \ln g) - \frac{4}{15}\Box(\partial \ln g)^2 \ . \tag{A.17}$$

The latter is irrelevant for this thesis since it vanishes for a functional derivative w.r.t. coupling but is given for the sake of completeness only.

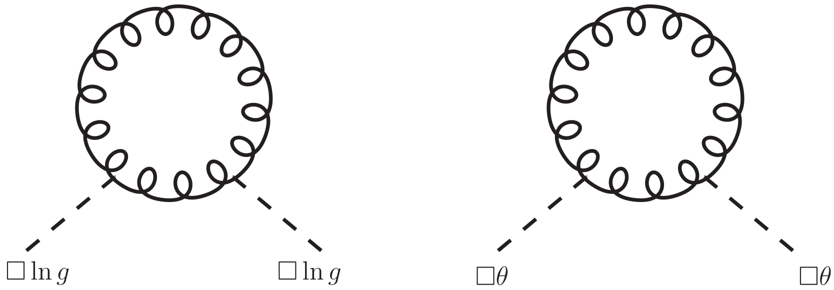


Figure A.1 *Evaluation of the generating functional W at LO, by the use of the heat kernel method, CP-even (left) and CP-odd (right) respectively. The relevant divergent parts are given below Eqs.*(A.15) *and* (A.19).

A.2.2 CP-odd sector $W(\theta)$

To calculate the contribution from O_θ at LO it is convenient to perform a partial integration on $\theta(x)O_\theta$ for which the quadratic kernel is given by

$$(\Delta^\theta)^{ab}_{\mu\nu} = -4i\epsilon^{\mu\nu\alpha\beta}(\partial_\alpha\theta)\partial_\beta\delta^{ab} \ . \tag{A.18}$$

The θ-dependent contribution to the effective action is obtained by replacing $\Delta^{(1)}_{\mu\nu} \to \Delta^{(1)}_{\mu\nu} + \Delta^\theta_{\mu\nu}$ with regard to the previous section. To simplify matter $g = 1$ is assumed since the purely θ-dependent terms are sufficient. The heat kernel expansion yields

$$W_{\text{div}}(\theta) = \frac{b_4^\theta}{2\epsilon} \ , \tag{A.19}$$

with

$$b_4^\theta = -\frac{16 n_g}{16\pi^2}\int d^dx \left[(\Box\theta)^2 + \mathcal{O}(\theta^3) + \partial b_4^\theta\right] \ , \tag{A.20}$$

and as previously an irrelevant total derivative term

$$\partial b_4^\theta = -\frac{2}{3}\left[\partial(\partial\theta\,\Box\theta) + \Box(\partial\theta)^2\right] \ . \tag{A.21}$$

A.3 Free theory trace anomaly in conformally flat background

In this appendix the trace anomaly of a free scalar field theory (4.31) is evaluated on a conformally flat background $\tilde{g}_{\rho\lambda} = e^{-2s(x)}\delta_{\rho\lambda}$. The path integral is Gaussian and evaluates to

$$e^W = \int \mathcal{D}\phi e^{-S^{(0)}} = \sqrt{\det \Delta^{(0)}} = \exp -\frac{1}{2}\mathrm{Tr}\ln\Delta^{(0)} \ , \tag{A.22}$$

where $\Delta^{(0)} = [-\tilde{\Box} + \frac{1}{6}\tilde{R}^2]$ is the conformal Laplacian obtained from (4.31) by integration by parts. The contribution can be evaluated using Schwinger's formula

$$W = -\frac{1}{2}\mathrm{Tr}\ln\Delta^{(0)} = \frac{1}{2}\int_0^\infty \frac{dt}{t}\mathrm{Tr}(e^{-t\Delta^{(0)}}) \ . \tag{A.23}$$

The trace anomaly is obtained from W by using (F.22) and Weyl covariance of $\Delta^{(0)}$ [2]

$$\begin{aligned}\int d^4x\sqrt{\tilde{g}}\langle\Theta\rangle &= \int_0^\infty dt\ \mathrm{Tr}(\Delta^{(0)}e^{-t\Delta^{(0)}}) = -\int_0^\infty dt\frac{d}{dt}\mathrm{Tr}(e^{-t\Delta^{(0)}}) \\ &= \lim_{t\to 0}\mathrm{Tr}(e^{-t\Delta^{(0)}}) = b_4 \ ,\end{aligned} \tag{A.24}$$

where b_4 is a coefficient of the asymptotic Heat Kernel expansion [3]

$$\mathrm{Tr}(e^{-t\Delta^{(0)}}) = \sum_{n\geq 0} b_n t^{\frac{n-d}{2}} \ . \tag{A.25}$$

Using the plane-wave basis (see A.3.1 for details of this calculation) to evaluate the trace we obtain

$$\int d^4x\sqrt{\tilde{g}}\langle\Theta\rangle = \frac{1}{16\pi^2}\frac{1}{90}\int d^4x\big[3\Box^2 s - 2\Box(\partial s)^2 - 4\partial\cdot(\partial s((\partial s)^2 - \Box s))\big] \ , \tag{A.26}$$

which decomposes into the the following invariants

$$\int d^4x\sqrt{\tilde{g}}\langle\Theta\rangle = -\frac{1}{16\pi^2}\frac{1}{90}\int d^4x\sqrt{\tilde{g}}(-\frac{1}{2}\tilde{\Box}\tilde{R} + \frac{1}{4}\tilde{E}_4) \ , \tag{A.27}$$

where the geometric quantities $\tilde{R}$ and $\tilde{E}_4$ are defined with respect to the metric $\tilde{g}_{\rho\lambda}(s)$ given above. The result quoted in (4.32) follows by comparing the equation above to (4.17).

[2]By covariance we mean the following transformation property $2\int g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\Delta^{(0)} = 2\Delta^{(0)}$.

[3]Here we have ignored the quadratic and quartic divergences which need to be subtracted by suitable counterterms. In a supersymmetric theory those divergences cancel to zero.

A.3.1 Asymptotic expansions of the heat kernel

We wish to perform the expansion (A.25) explicitly by using the Fujikawa's method [140]. To expand the kernel in flat space basis we first need to transform the variables $\phi \to g^{-1/4}\phi$ so that functional metric $\langle\phi|\phi\rangle = \int d^4x\sqrt{g}\phi(x)\phi(x)$ becomes the flat-space one. In this basis the kinetic operator becomes $\bar{\Delta}^{(0)} = g^{-1/4}\sqrt{g}\Delta^{(0)}g^{-1/4}$. Hence for the conformally flat metric $\tilde{g}_{\rho\lambda} = e^{-2s(x)}\delta_{\rho\lambda}$ one needs to use the following operator

$$\bar{\Delta}^{(0)} = -e^{2s}[\Box + 2\partial s \cdot \partial + (\Box s + (\partial s)^2)] \ , \tag{A.28}$$

where $\Box$ is the flat-space Laplacian. The asymptotic behaviour of the trace (A.25) now amounts to large $M = t^{-1/2}$ expansion of

$$\int d^4x \langle x| \exp(-\frac{\bar{\Delta}^{(0)}}{M^2})|x\rangle \ . \tag{A.29}$$

This trace can now be computed in the plane-wave basis with $\langle k|x\rangle = e^{ikx}$ and use the fact that for any regular function f one has $f(\partial)e^{ikx} = e^{ikx}f(\partial + ik)$

$$\int d^4x \langle x| \exp(-\frac{\bar{\Delta}^{(0)}}{M^2})|x\rangle = \int d^4x \int d^4k \exp(-\frac{\bar{\Delta}^{(0)}}{M^2})\big|_{\partial\to\partial+ik} \ . \tag{A.30}$$

From (A.28) we have

$$\bar{\Delta}^{(0)}\big|_{\partial\to\partial+ik} = -e^{2s}[-k^2 + 2ik(\partial + \partial s) + \Box + 2\partial s \cdot \partial + (\Box s + (\partial s)^2)] \ . \tag{A.31}$$

After the rescaling $k \to Mk$ the trace (A.29) becomes

$$M^4 \int d^4x \int d^4k e^{4\tau} \ \exp(-e^{-2\tau}k^2 + \frac{1}{M}\hat{O}) \ , \tag{A.32}$$

where

$$\hat{O} = 2ike^{2s}(\partial + \partial s) + \frac{e^{2s}}{M}[\Box + 2\partial s \cdot \partial + (\Box s + (\partial s)^2)] \ . \tag{A.33}$$

We see that the large M expansion amounts to expanding the exponential in (A.32) about $\exp(-e^{-2s}k^2)$ up to fourth order in $\frac{1}{M}$. We will focus on the fourth-order which corresponds to b_4 in (A.25). Since the operators in the exponent are not commuting we need to use Dyson series-type expansion

$$\exp(-e^{2s}k^2 + \frac{1}{M}\hat{O}) = \exp(-e^{2s}k^2)(1 + \int_0^1 dt \ e^{-\mathrm{ad}_{H_0 t}}\frac{1}{M}\hat{O}$$

$$+\int_0^1 dt\ e^{-\mathrm{ad}_{H_0 t}} \frac{1}{M}\hat{O}\ \int_0^t dt'\ e^{-\mathrm{ad}_{H_0 t'}} \frac{1}{M}\hat{O} + ... \tag{A.34}$$

where

$$e^{-\mathrm{ad}_X} Y = e^{-X} Y e^X \tag{A.35}$$

In this expansion we will neglect all the terms having derivatives at the end since those correspond to total derivatives.
We perform the momentum integrals first. After taking into account the rotational symmetry of the integrals (e.g $\int d^4k\ k^\mu k^\nu \propto \delta^{\mu\nu} \int d^4k\ k^2$) we end up with expressions of the form:

$$\int d^4k k^{2\alpha} \exp(-e^{2s}k^2) = e^{-(4+2\alpha)s} \int d^4k\ k^{2\alpha} \exp(-k^2) = \pi^2 \Gamma(\alpha+2) e^{-(4+2\alpha)s} \tag{A.36}$$

Due to rather large number of integrals, the actual calculation was automatized using Mathematica with package `xAct` to obtain the final result (A.26).

Appendix B

Superfield Methods

B.1 Useful SUSY identities

$$v^\alpha = \epsilon^{\alpha\beta} v_\beta \tag{B.1}$$

$$uv = u_\alpha v^\alpha \tag{B.2}$$

$$v_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} v_\mu \tag{B.3}$$

Grassmannian variables $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$:

$$\theta^2 = \theta_\alpha \theta^\alpha \tag{B.4}$$

$$d\theta^2 = d\theta_\alpha d\theta^\alpha \tag{B.5}$$

$$\int d\theta^2 = \int d\bar{\theta}^2 = 0 \tag{B.6}$$

$$\int d\theta^2 \theta^2 = \int d\theta^2 \bar{\theta}^2 \tag{B.7}$$

$$\theta^2 = \delta^2(\theta) \tag{B.8}$$

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\bar{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \tag{B.9}$$

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha \partial_{\alpha\dot{\alpha}} \tag{B.10}$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \tag{B.11}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\partial_{\alpha\dot{\alpha}} \quad \text{(B.12)}$$

With the following anticommutation relations:

$$\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\} = i\partial_{\alpha\dot{\alpha}} \quad \text{(B.13)}$$

$$\{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} = -i\partial_{\alpha\dot{\alpha}} \quad \text{(B.14)}$$

B.2 Superfields

We will try to stick to Wess and Bagger [136] conventions (see Appendix A). A most concise description can be found in [141]. Superfields can be thought of as representations of the supermultiplets in the superspace $(x_{\mu}, \theta_{\alpha}, \bar{\theta}_{\dot{\alpha}})$. Generally we have:

$$G(x, \theta.\bar{\theta}) = g(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + ... \quad \text{(B.15)}$$

This works because the θ_{α} coordinates themselves carry spin $\frac{1}{2}$ index, thus provide a natural representation of SUSY so that g has spin 0, ψ has spin $\frac{1}{2}$, v spin 1 etc. The SUSY transformations $Q, \bar{Q}$ that can be expressed as differential operators in the superspace (Appendix A) acting on (B.15) reshuffle the components of G (e.g $Q_{\alpha}g(x) = i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}g$). In the $\mathcal{N} = 1$ language one encounters two types of multiplets. We have the chiral supermultiplet $(\phi, \psi_{\alpha}, \mathcal{F})$ (complex scalar ϕ, Weyl fermion ψ and another complex scalar $\mathcal{F}$ [1]) represented by chiral superfield $\mathbf{\Phi}$ with:

$$\bar{D}_{\dot{\alpha}}\mathbf{\Phi} = 0 \quad \text{(B.16)}$$

We can always choose a coordinate system $y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta}$, where $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}$ so that (B.16) simply tells us that $\mathbf{\Phi}$ does not depend on the $\bar{\theta}$:

$$\mathbf{\Phi} = \phi(y) + \theta\psi(y) + \theta^2\mathcal{F}(y) \quad \text{(B.17)}$$

Clearly, the product and the sum of two chiral superfields is again a chiral superfield (this follows from the Leibnitz rule and linearity of D).

To construct a gauge theory one needs to define the *vector supermultiplet* (A_{μ}, λ, D) with the gauge field A_{μ},gaugino λ^a and a real scalar D^a that we

[1] The scalar field $\mathcal{F}$ really ensures that we have the same number of bosonic and fermionic degrees of freedom off-shell. For the on-shell states we integrate $\mathcal{F}$ using equations of motion

integrate out using equations of motion. It is represented by a real superfield:

$$V = V^\dagger = V^a T^a \quad , \tag{B.18}$$

where T^a is the generator of gauge group. In the Wess-Zumino gauge ($V^3 = 0$) one has:

$$V = -\theta\sigma^\mu\bar{\theta}A^\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D \quad , \tag{B.19}$$

Matter superfield $\mathbf{\Phi}$ transforms under a fundamental representation of the gauge group and SUSY invariance is achieved by promoting the gauge parameter $\Lambda = \Lambda^a T^a$ to a chiral superfield

$$\bar{D}_{\dot{\alpha}}\Lambda = 0 \tag{B.20}$$

so that the transformation

$$\mathbf{\Phi} \rightarrow e^{i\Lambda}\mathbf{\Phi} \tag{B.21}$$

preserves the condition (B.16). Under the same transformation the gauge superfield transforms as

$$e^V \rightarrow e^{i\Lambda^\dagger} e^V e^{-i\Lambda} \quad . \tag{B.22}$$

Thus a gauge invariant interaction with a gauge coupling g is achieved via

$$\mathbf{\Phi}^\dagger e^{-gV}\mathbf{\Phi} \quad . \tag{B.23}$$

Another gauge invariant term is formed from the gauge strength superfield W_α (precise expression in the Appendix A)

$$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V \quad . \tag{B.24}$$

Which itself is a chiral superfield that transforms under the adjoint representation of the gauge group and thus we can form a gauge invariant term of the form $TrW_\alpha W^\alpha$. Note that the $\theta\theta$ component of this combination contains the usual gauge kinetic terms:

$$\mathrm{tr}W_\alpha W^\alpha\big|_{\theta\theta} = -\frac{1}{2}\mathrm{tr}\ G_{\mu\nu}^2 + \frac{i}{4}\mathrm{tr}\ G_{\mu\nu}\tilde{G}^{\mu\nu} - 2i\lambda\sigma^\mu\mathcal{D}_\mu\bar{\lambda} + (D^a)^2 \quad , \tag{B.25}$$

where the first two terms involving the field strength tensor G and its dual $\tilde{G}$ represent the usual gauge field part of QCD and the latter two are the superpartners.

B.3 $\mathcal{N}=1$ effective action and the Konishi anomaly

In this appendix we will repeat some of the arguments from section 1.3.2 with the notation more suitable for section 4.3. Using arguments of holomorphicity it can be argued that the running of the coupling g, of the Wilsonian effective action of the supersymmetric gauge theory (4.41), is one-loop exact [32, 132] and reads

$$\begin{aligned} S_{\rm W}(\mu) &= \left(\frac{1}{g^2(\mu')} - \frac{b_0}{8\pi^2}\ln\frac{\mu'}{\mu}\right)\int d^6z {\rm tr}W^2 + {\rm h.c.} \\ &+ \frac{1}{8}Z(\mu)\sum_f \left[\int d^8z \Phi_f^\dagger e^{-2V}\Phi_f + \int d^8z \tilde{\Phi}_f^\dagger e^{-2V}\tilde{\Phi}_f\right] , \end{aligned} \tag{B.26}$$

where $b_0 \equiv 3N_c - N_f$ and $\mu' > \mu$ is an arbitrary scale which can be identified with the UV cut-off $\Lambda_{\rm UV}$. Rescaling the matter fields by

$$(\Phi_f, \tilde{\Phi}_f) \to Z^{-1/2}(\Phi_f, \tilde{\Phi}_f) , \tag{B.27}$$

is accompanied by the Konishi anomaly [29, 133, 134], and leads to the effective action [32]

$$S_{\rm W}(\mu) = \int d^6z \frac{1}{g^2(\mu)}{\rm tr}W^2 + {\rm h.c.} + \sum_f \left[\int d^8z \Phi_f^\dagger e^{-2V}\Phi_f + \int d^8z \tilde{\Phi}_f^\dagger e^{-2V}\tilde{\Phi}_f\right] , \tag{B.28}$$

where the running has been removed from the matter term and all the running is absorbed in front of the gauge field term which in this case defines the running gauge coupling to be

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu')} - \frac{1}{8\pi^2}\left(b_0 \ln\frac{\mu'}{\mu} - N_f \ln Z\right) . \tag{B.29}$$

The μ'-independence of $g(\mu)$ implies an RGE which solves to the holomorphic NSVZ beta function [142–144]

$$\beta \equiv \frac{d}{d\ln\mu}g = -g^3\frac{N_f}{16\pi^2}(\gamma - \gamma_*) . \tag{B.30}$$

Above we have used the following notation

$$\gamma \equiv \frac{\partial \ln Z(\mu)}{\partial \ln \mu} , \quad \gamma_* \equiv -b_0/N_f = 1 - 3N_c/N_f . \tag{B.31}$$

In the range $3/2N_c < N_f < 3N_c$ the theory is in the so-called conformal window (see section1.3.4 for more details).

Appendix C

Beta Functions

C.1 Conventions for QCD beta function

In this work the bare beta function $\hat{\beta}$ of DR is defined as

$$\hat{\beta} = \frac{d \ln g}{d \ln \mu} = \frac{(d-4)}{2} + \beta \; . \qquad \text{(C.1)}$$

We draw the reader's attention to the fact that the logarithmic β-function (C.1) is used throughout in order to keep the formulae more compact. Explicitly

$$\beta = -\beta_0 a_s - \beta_1 {a_s}^2 - \beta_2 {a_s}^3 - \beta_3 {a_s}^4 \dots \; , \quad a_s = \frac{\alpha_s}{4\pi} = \frac{g^2}{(4\pi)^2} \qquad \text{(C.2)}$$

where β_{0-3} in $\overline{\text{MS}}$-scheme can be found in [145]. The first two coefficients read

$$\beta_0 = (\frac{11}{3} C_A - \frac{4}{3} N_F T_F) \; , \quad \beta_1 = (\frac{34}{3} C_A^2 - \frac{20}{3} N_c N_F T_F - 4 C_F T_F N_F) \; ,$$

where C_F, C_A are quadratic Casimir operators of the fundamental (quark) and adjoint (gluons) representations, N_F the number of quarks and $\text{tr}[T^a T^b] = T_F \delta^{ab}$ is a Lie algebra normalisation factor of the fundamental representation. For $SU(N_c)$ these factors are given by

$$C_A = N_c \; , \quad C_F = \frac{N_c^2 - 1}{2 N_c} \; , \quad T_F = \frac{1}{2} \; . \qquad \text{(C.3)}$$

C.2 Banks-Zaks type Fixed Point at four loops

In this appendix we summarise the BZ fixed point. We consider N_f quarks in the fundamental representation of $SU(N_c)$ coupled to gluons in the adjoint representations. The four loop beta function is parametrised by

$$\beta \equiv \frac{d\ln g}{d\ln\mu} = -(\beta_0 a_s + \beta_1 {a_s}^2 + \beta_2 {a_s}^3 + \beta_3 {a_s}^4) \;, \quad a_s \equiv \frac{g^2}{(4\pi)^2} \;. \tag{C.4}$$

The $O(\kappa^4)$ calculation in section 3.2.4 corresponds to in the $\overline{\text{MS}}$-scheme [145]

$$\begin{aligned} \beta_0 &= -\frac{2}{3}\kappa N_c \;; \quad \beta_1 = -(\frac{25}{2} - \frac{13}{3}\kappa)N_c^2 \;; \quad \beta_2 = -(\frac{701}{12} - \frac{53}{6}\kappa)N_c^3 \;; \\ \beta_3 &= (\frac{14731}{144} + 275\zeta_3)N_c^4 \;; \end{aligned} \tag{C.5}$$

with

$$\kappa \equiv \frac{\frac{11}{2}N_c - N_f}{N_c} \;. \tag{C.6}$$

The parameterisation of the coupling in terms of a_s is convenient as it avoids additional factors of π. In the limit $\kappa \ll 1$, which is achieved by taking N_c to infinity with $N_f = (11/2 - \kappa)N_c$, the beta function develops a perturbative IR fixed point at

$$ {a_s}^{\text{IR}} = -\frac{\beta_0}{\beta_1}\left(1 + \frac{\beta_0\beta^2}{\beta_1^2} + \beta_0^2\frac{(2\beta_2^2 - \beta_1\beta_3)}{\beta_1^4}\right) + O(\beta_0^4) \;, \tag{C.7}$$

where we expand in $\beta_0 \propto \kappa$. Plugging this expression back to (C.4) yields $\beta({a_s}^{\text{IR}}) = O(\kappa^5)$.

Appendix D

Miscellaneous Calculations

D.1 OPE of topological density - the $\hat{\mathbb{C}}^g_{\theta\theta}$-term

In this appendix we sketch the computation of the $\hat{\mathbb{C}}^g_{\theta\theta}$-term of the OPE (2.60). In performing the OPE at the tree level, the following propagator (Feynman gauge assumed) is needed

$$\langle \partial_\alpha A_\beta(x) \partial_\gamma A_\delta(0) \rangle = -\delta_{\beta\delta} \partial_\alpha \partial_\gamma \langle \phi(x)\phi(0) \rangle \; , \tag{D.1}$$

where $\langle \phi(x)\phi(0) \rangle$ is the free field propagator as given in (2.23). The contribution to the coordinate and momentum space Wilson coefficient reads

$$\hat{\mathbb{C}}^g_{\theta\theta} = -2^4 \Box \langle \phi(x)\phi(0) \rangle = 2^4 \delta(x) \quad \Rightarrow \mathbb{C}^g_{\theta\theta} = 2^4 \; , \tag{D.2}$$

where the Green's function property $\Box\langle \phi(x)\phi(0) \rangle = -\delta(x)$ was used. This is an example where contact terms appear in the x-space OPE as discussed below (2.8). The result (D.2) is consistent with [146] when taking into account the continuation to Minkowski space.

D.2 Rescaling Anomaly in Hamiltonian language

In section 2.3.1 we have used a particular canonical transformation (2.146) and one might wonder whether the measure is anomalous under this transformation.

Generally any rescaling of a field which is gauged, produces anomalous term proportional to the kinetic term of the corresponding gauge field [87]. We shall see that for the transformation (2.146) the effect cancels. Let us write (2.146) for a generic transformation

$$\begin{aligned} \vec{A} &\to \frac{1}{f(g)}\vec{A} \, , \\ \vec{\Pi} &\to f(g)\vec{E} \, . \end{aligned} \tag{A.3}$$

The anomalous Jacobian of the $\mathcal{D}\vec{E}\mathcal{D}\vec{A}$ measure is

$$\begin{aligned} &\ln\det\frac{\delta Q'(x)}{\delta Q(y)} = \ln\det\begin{pmatrix} f(g)^{-1}\delta(x-y) & 0 \\ 0 & f(g)\delta(x-y) \end{pmatrix} = \\ &\ln\det\begin{pmatrix} f(g)^{-1} & 0 \\ 0 & f(g) \end{pmatrix}\delta(x-y) = \ln\det\delta(x-y) \, . \end{aligned} \tag{A.4}$$

It is proportional to an expression independent of $f(g)$ and therefore justifies our manipulations in section 2.3.1. The second equality sign is the crucial step where we use the fact that the $\vec{A}$ and $\vec{E}$ can be expanded in the same set of eigenfunctions. For the chiral anomaly this is not the case since left and right handed fermions have different eigenfunction, or more precisely a different number of zero modes. For an arbitrary rescaling the two dimensional matrix on the second line has not got unit determinant and will therefore depend on the transformation [87].

Appendix E

LL QCD Results

Note: Contents of this appendix appear in [3].

E.1 Form of leading logs of $\mathbb{C}^{\mathbb{1}}_{gg}$

The leading terms in the bare correlation function take the form

$$\int d^d x e^{ip\cdot x} \langle 0|G^2(x)G^2(0)|0\rangle_{\text{LL}} = k \sum_{n\geq 0} \frac{(\beta_0 a_{s0})^{n-1}}{\epsilon^n} \left(\frac{\mu^2}{-p^2}\right)^{n\epsilon} \tag{E.1}$$

with k being a constant which is immaterial for the argument. Upon renormalising the operator $[G^2] = Z_{G^2} G^2$ and the coupling $a_{s0} = a_s Z_{a_s}$ with $Z_{G^2} = Z_{a_s}$ in the LL approximation one finds

$$\int d^d x e^{ip\cdot x} \langle 0|[G^2(x)][G^2(0)]|0\rangle_{\text{LL}} = k \sum_{n\geq 0} \frac{f_n (\beta_0 a_s)^{n-1}}{\epsilon^n} \tag{E.2}$$

where

$$f_n = \sum_{j=0}^{n-1} (-1)^j \left(\frac{\mu^2}{-p^2}\right)^{(n-j)\epsilon} \binom{n}{n-j} \tag{E.3}$$

This sum evaluates to

$$f_n = \sum_{j=0}^{n-1} (-1)^j \binom{n}{n-j} + \frac{\epsilon^n}{n!} \ln^n \left(\frac{\mu^2}{-p^2}\right) \sum_{j=0}^{n} (-1)^j (n-j)^n \binom{n}{n-j}$$

$$= (-1)^{n+1} + \epsilon^n \ln^n\left(\frac{\mu^2}{-p^2}\right) , \tag{E.4}$$

confirming the rule $\epsilon^{-n} \leftrightarrow -\ln^n(-p^2/\mu^2)$ used in section 2.2.3.

Note that non-local divergent terms in (E.4) are avoided since the sum, somewhat magically,

$$\sum_{j=0}^{n-1}(-1)^j(n-j)^l\binom{n}{n-j} = 0 , \quad 0 < l < n . \tag{E.5}$$

only contributes for $l = 0$ and $l = n$. We note that such non-local terms could not be eliminated by local counterterm in perturbation theory.

E.2 Explicit evaluation of dispersion integral

As a check the integral (2.98) is integrated explicitly. This is best done by changing variables to $s = \mu^2 e^y$ which results in an integral

$$\hat{x}(p^2) = \int_{-\infty}^{\infty} dy \frac{dye^y}{e^y - p^2/\mu^2}\frac{1}{((1+a_s\beta_0 y)^2 + (a\beta_0\pi)^2} , \tag{E.6}$$

with a poles at $y_\pm = -1/(a_s\beta_0) \pm i\pi$ and a series of poles $y_{n_\pm} = \ln(-p^2/\mu^2) \pm i\pi(2n+1)$ for $n \geq 0$. The integration contour can, for example, be closed in the upper half plane. The y_+ pole result in the pole term in (2.97) and the series of poles $y_{n_+} = \ln(-p^2/\mu^2) + i\pi(2n+1)$ for $n \geq 0$ leads to a series

$$\begin{aligned}\hat{x}(p^2) &= \frac{1}{1-p^2/p_0^2} - (2\pi i)\sum_{n\geq 0}\frac{1}{(1+a_s\beta_0(\ln(-p^2/\mu^2)+i\pi(2n+1))^2 + (a\beta_0\pi)^2} \\ &= \frac{1}{1-p^2/p_0^2} + x(p^2) \end{aligned} \tag{E.7}$$

which can be resumed into an analytic form. The final result is consistent with Eq.(2.97) which was the aim of this appendix.

Appendix F

Curved Space Identities

F.1 Weyl rescaling identities

Weyl rescaling is defined

$$g_{\mu\nu} \to e^{-2\alpha} g_{\mu\nu} \ . \tag{F.1}$$

Volume rescaling

$$\sqrt{g} \to e^{-d\alpha} \sqrt{g} \tag{F.2}$$

The variation identity

$$\frac{\delta}{\delta\alpha} = 2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \ . \tag{F.3}$$

Trace of energy-momentum tensor with action S_E reads

$$\sqrt{g} T^{\mu}{}_{\mu} = 2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} S_E = \frac{\delta}{\delta\alpha} S_E \ . \tag{F.4}$$

Weyl variation of Ricci tensor and Ricci scalar

$$\begin{aligned} \frac{\delta}{\delta\alpha(y)} R_{\mu\nu}(x) &= (d-2)\nabla_\mu \partial_\nu \delta^{(d)}(x-y) + g_{\mu\nu} \Box \delta^{(d)}(x-y) \\ \frac{\delta}{\delta\alpha(y)} R(x) &= 2R(x)\delta^{(d)}(x-y) + 2(d-1)\Box\delta^{(d)}(x-y) \ . \end{aligned} \tag{F.5}$$

Variation of Weyl tensor squared

$$\frac{\delta}{\delta\alpha(y)} W^2(x) = 4W^2(x)\delta^{(d)}(x-y) \ . \tag{F.6}$$

Euler density:

$$\frac{\delta}{\delta\alpha(y)}E_d(x) = 4E_d(x)\delta^{(d)}(x-y) - 8(d-3)G^{\mu\nu}\nabla_\mu\partial_\nu\delta^{(d)}(x-y) \ , \tag{F.7}$$

where we used the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.
The variation of $\Box$ and Dirac operator read

$$\begin{aligned} \frac{\delta}{\delta\alpha(y)}\Box &= 2\delta^{(d)}(x-y)\Box - (d-2)\partial^\mu\delta^{(d)}(x-y)\partial_\mu \\ \frac{\delta}{\delta\alpha(y)}\not{D} &= \frac{1}{2}(d+1)\delta^{(d)}(x-y)\not{D} - \frac{1}{2}(d-1)\not{D}\delta^{(d)}(x-y) \ . \end{aligned} \tag{F.8}$$

Weyl variation of conformally coupled scalar

$$\frac{\delta}{\delta\alpha(x)}\int d^dx\sqrt{g}\phi(\Box - \frac{1}{(d-1)}R)\phi = -(d-2)\sqrt{g}\phi(x)(\Box - \frac{1}{(d-1)}R)\phi(x) \ . \tag{F.9}$$

Weyl variation of fermion

$$\frac{\delta}{\delta\alpha(x)}\int d^dx\sqrt{g}\bar{\psi}\not{D}\psi = -\frac{1}{2}(d-1\sqrt{g})\bar{\psi}(x)\overleftrightarrow{\not{D}}\psi(x) \ , \tag{F.10}$$

where $\overleftrightarrow{\not{D}} = \overrightarrow{\not{D}} - \overleftarrow{\not{D}}$. And finally the gauge field action variation

$$\frac{\delta}{\delta\alpha(x)}\int d^dx\sqrt{g}G^2_{\mu\nu} = -(d-4)\sqrt{g}G^2_{\mu\nu}(x) \ . \tag{F.11}$$

F.2 Conformally flat identities

Use a conformally flat metric

$$g_{\mu\nu} = e^{-2\alpha}\delta_{\mu\nu} \ . \tag{F.12}$$

The Ricci tensor reads

$$R_{\mu\nu} = (d-2)(\partial_\mu\partial_\nu s + \partial_\mu s\partial_\nu s) + \delta_{\mu\nu}(\Box s - (d-2)(\partial s)^2) \ , \tag{F.13}$$

yielding the Ricci scalar

$$R = 2(d-1)e^{-2s}\left(\Box s - \frac{1}{2}(d-2)(\partial s)^2\right) \ . \tag{F.14}$$

On conformally flat manifold $W^2 = 0$ and the Euler density close to $d = 4 - 2\epsilon$ reads

$$\begin{aligned}\sqrt{\tilde{g}}\tilde{E}_d &= -4k_d \left[\frac{1}{2}\Box(e^{2\epsilon s}(\partial s)^2) + \partial(e^{2\epsilon s}\partial s((1-\epsilon)(\partial s)^2 - \Box s))\right] \\ &\quad +2\epsilon k_d e^{2\epsilon s}(-2\Box s(\partial s)^2 + (\partial s)^4 - 2\epsilon(\partial s)^4) \;. \end{aligned} \tag{F.15}$$

where $k_d = (d-3)(d-2)$. Note that continuing beyond $d = 4$, (G.10) receives an evanescent contribution which is not a total derivative.
The scalar Laplacian takes the form

$$\nabla^2 = \Box - (d-2)\partial s \cdot \partial \;. \tag{F.16}$$

Spin connection becomes

$$\omega_\mu^{ij} = \delta_\mu^j \partial^i s - \delta_\mu^i \partial^j s \tag{F.17}$$

yielding a very simple form of the Dirac operator

$$\not{D} = e^{\frac{5}{2}s} \not{\partial} e^{-\frac{3}{2}s} \tag{F.18}$$

F.3 Renormalisation group equations for W

In this appendix we summarise the RG equation obeyed by W and how they relate to the trace anomaly. For future reference and completeness an explicit scale symmetry breaking term in form of a matter mass term is added. The quantum vacuum transition amplitude W obeys an RG equation

$$\left(\frac{\partial}{\partial \ln \mu} + \beta \frac{\partial}{\partial g} - \gamma_m \frac{\partial}{\partial \ln m}\right) W = 0 \;, \quad \gamma_m \equiv -\frac{\partial \ln m}{\partial \ln \mu} \;, \tag{F.19}$$

which follows from $\frac{dW}{d\mu} = 0$.[1] Assuming a space-dependent metric, dimensional analysis gives an equation of the form

$$\left(\frac{\partial}{\partial \ln \mu} + \frac{\partial}{\partial \ln m} + 2\int d^4x \; g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)}\right) W = 0 \;. \tag{F.20}$$

[1]Throughout this paper $\gamma = -\gamma_m$ is the anomalous dimension of the squark composite operator whereas other authors [68, 127] use the anomalous dimension of the superfield Φ as $\gamma = \gamma_\Phi$. The relation between the two is $-2\gamma_\Phi = \gamma_m$; i.e. $\gamma|_{\text{this work}} = 2\gamma|_{[68,\ 127]}$.

Equations (F.19,F.20) can be combined into an RG equation with no explicit μ-derivative

$$\left(\beta\frac{\partial}{\partial g} - (1+\gamma_m)\frac{\partial}{\partial \ln m} - 2\int d^4x \; \mathrm{g}^{\mu\nu}(x)\frac{\delta}{\delta \mathrm{g}^{\mu\nu}(x)}\right) W = 0 \; . \tag{F.21}$$

The adaption of these equation to W_τ involves replacing $\mu \to \mu e^\tau$ everywhere. Note, if τ is made space-dependent then $g(\mu e^\tau)$ and $m(\mu e^\tau)$ and the partial derivatives are to be replaced by functional derivatives $\frac{\partial}{\partial g} \to \int d^4x \frac{\delta}{\delta g(x)}$ and $\frac{\partial}{\partial m} \to \int d^4x \frac{\delta}{\delta m(x)}$ respectively.

A definition of the trace of the EMT is given by

$$\langle \Theta \rangle = -2 \; \frac{\mathrm{g}^{\mu\nu}(x)}{\sqrt{\mathrm{g}(x)}}\frac{\delta}{\delta \mathrm{g}^{\mu\nu}(x)} W \tag{F.22}$$

where $\mathrm{g}(x)$ denotes the determinant of the metric. Combining (F.22) with (F.20) the following equations are obtained

$$\int d^4x \sqrt{\mathrm{g}} \; \langle \Theta \rangle_{\mathrm{anom}} = \frac{\partial}{\partial \ln \mu} W \; , \quad \int d^4x \sqrt{\mathrm{g}} \; \langle \Theta \rangle_{\mathrm{expl}} = \frac{\partial}{\partial \ln m} W \; , \tag{F.23}$$

where the subscripts "anom" and "expl" refer to anomalous and explicit scale breaking respectively.

Appendix G

Central Charge Calculations

G.1 Extension of R^2-anomaly term to $\mathcal{O}({a_s}^5)$

Note: Contents of this appendix appear in [3].

The $\mathcal{O}({a_s}^5)$-term is new due to the effort of [74] in computing $g^{\mathbb{1}}_{1,2}$. Recall that $g^{\mathbb{1}}_1$ is a coefficient of the first pole in the Laurent expansion

$$L^{\mathbb{1},\mathrm{MS}}_{gg} = g^{\mathbb{1}}_1(a_s)\frac{1}{\epsilon} + g^{\mathbb{1}}_2(a_s)\frac{1}{\epsilon^2} + \ldots \tag{G.1}$$

of the $\langle G^2 G^2 \rangle$ contact term .
The relevant expression of $g^{\mathbb{1}}_1(a_s) = g^{\mathbb{1}}_{1,0} + g^{\mathbb{1}}_{1,1} a_s + g^{\mathbb{1}}_{1,2} {a_s}^2 + \mathcal{O}({a_s}^3)$ is taken from [74]

$$\begin{aligned}
g^{\mathbb{1}}_{1,0} &= \frac{n_g}{4\pi^2} \ , \\
g^{\mathbb{1}}_{1,1} &= g^{\mathbb{1}}_{1,0}\left(\frac{17}{2}C_A - \frac{10}{3}N_F T_F\right) \ , \\
g^{\mathbb{1}}_{1,2} &= 4 g^{\mathbb{1}}_{1,0}\big(C_A^2(\frac{11}{6}\zeta_3 + \frac{22351}{1296}) - C_A N_F T_F(\frac{14}{3}\zeta_3 + \frac{799}{81}) \\
&\quad + n_g N_F T_F(2\zeta_3 - \frac{107}{36}) + \frac{49}{81}T_F^2 N_F^2\big) \ ,
\end{aligned} \tag{G.2}$$

with ζ_3 being the Riemann zeta function at the value 3 and n_g is the number of gluons or the dimension of the adjoint representation $n_g = C_A C_F / T_F|_{SU(N_c)} = N_c^2 - 1$. The $\mathcal{O}({a_s}^3)$ contribution agrees with [58] [Eq.7.7] at the level of β_0 and β_1 which straightforwardly extends to QCD-like theories.

From (3.56) one then obtains β_b up to $\mathcal{O}({a_s}^5)$. We give the result in terms of the first pole in L_b by the MS-type relation[1]

$$\beta_b = 2\partial_{a_s}(a_s b_1) = 8B_{1,3}{a_s}^3 + 10B_{1,4}{a_s}^4 + 12B_{1,5}{a_s}^5 + \mathcal{O}({a_s}^6) \;, \tag{G.3}$$

where

$$\begin{aligned} B_{1,3} &= \frac{1}{24\cdot 16}\beta_0\beta_1 g^{\mathbb{1}}_{1,0} \;, \quad B_{1,4} = \frac{1}{120\cdot 16}(4\beta_1^2 g^{\mathbb{1}}_{1,0} + 6\beta_0\beta_2 g^{\mathbb{1}}_{1,0} + 3\beta_0\beta_1 g^{\mathbb{1}}_{1,1}) \;, \\ B_{1,5} &= \frac{1}{720\cdot 16}(12\beta_0\beta_1 g^{\mathbb{1}}_{1,2} + 50\beta_0\beta_2 g^{\mathbb{1}}_{1,0} + (24\beta_0\beta_2 + 15\beta_1^2) g^{\mathbb{1}}_{1,1}) \;. \end{aligned} \tag{G.4}$$

Comparing with [57, 58, 66] we find agreement with [57, 58] to the computed order of $\mathcal{O}({a_s}^3)$ and with [66] to order $\mathcal{O}({a_s}^4)$. The fact that β_b is proportional to the beta function is consistent with β_b being zero in conformal field theories [96, 98].

G.2 Extension to three-point functions

In this appendix we discuss the extension of finiteness discussion to 3-point functions and the consequences for β_a anomaly. The main goal is the derivation of (G.20). Contents of this appendix are based on author's unpublished work.

G.2.1 Finiteness of ϵL_a and the Relation $\epsilon L_a^{\mathrm{UV}} = \beta_a^{\mathrm{UV}}/2$

As shown in [3], the contact terms of TEMT 3-point function

$$M^{(\lambda_3)}_{TTT}(p_x, p_y) = \hat{P}_{\lambda_3} \int d^4x d^4y e^{i(p_x\cdot x + p_y\cdot y)} \langle \Theta(x)\Theta(y)\Theta(0)\rangle_c \;, \tag{G.5}$$

decompose into two structures

$$\begin{aligned} \lambda_3 &= p_x^4 + p_y^4 + p_z^4 - 2(p_x^2 p_y^2 + p_x^2 p_z^2 + p_y^2 p_z^2) \;, \\ P_3 &= p_x^4 + p_y^4 + p_z^4 \;, \end{aligned} \tag{G.6}$$

with λ_3 being the true 3-point structure and P_3 being related to the 2-point function by a variational derivative. Both structures were shown to be finite.

[1]The extra factor of 2 in the first equality w.r.t. [57, 58] originates from the $d = 4 - 2\epsilon$ versus $d = 4 - \epsilon$ convention.

Below we exploit the finiteness of the λ_3-structure to conclude finiteness of ϵL_a, similar to the steps leading to finiteness of L_b in Chapter 3. Assuming momentum conservation $p_z = -(p_x + p_y)$, $\lambda_3 = 4\left[(p_x \cdot p_y)^2 - p_x^2 p_y^2\right]$ and the the associated projector $\hat{P}_{\lambda_3}\lambda_3 = 1$ is

$$\hat{P}_{\lambda_3} = \frac{1}{96}\left[(\partial_{p_x} \cdot \partial_{p_y})^2 - \partial^2_{p_x}\partial^2_{p_y}\right] , \tag{G.7}$$

for which the P_3-structure automatically vanishes ($\hat{P}_{\lambda_3} P_3 = 0$).

Taking three Weyl variations and Fourier transforming one obtains

$$\begin{aligned} M^{(\lambda_3)}_{sss}(p_x, p_y) &\equiv \hat{P}_{\lambda_3} \int d^4x d^4y e^{i(p_x \cdot x + p_y \cdot y)} \frac{\delta^3}{\delta s(x)\delta s(y)\delta s(0)} \ln \mathcal{Z} \\ &= (2k_d a_0 - 8b_0 - M^{(\lambda_3)}_{TTT}(p_x, p_y)) = [\text{finite}] , \end{aligned} \tag{G.8}$$

where $M^{(\lambda_3)}_{TTT}$ is defined in (G.5) and the abbreviation $k_d \equiv (d-4)(d-3)(d-2)$ is introduced.[2] The finiteness of $M^{(\lambda_3)}_{TTT}$ ensures finiteness of $(2k_d a_0 - 8b_0)$. Since b_0 has been shown to be finite [3] it is to be concluded that the quantity $k_d a_0$ is finite. Like for b_0 in (3.86) this means that the $\epsilon \to 0$ limit $k_d a_0$ is meaningful

$$\lim_{\epsilon \to 0} k_d a_0 \equiv \lim_{\epsilon \to 0} k_d (L_a^{\mathrm{UV}} + a^{\mathrm{UV}}) = -2\beta_a^{\mathrm{UV}} . \tag{G.9}$$

In the last step we used that a^{UV} is finite and that $L_a^{\mathrm{UV}} = \frac{\beta_a^{\mathrm{UV}}}{2\epsilon}$. The latter follows from $\beta_a = -(\frac{d}{d\ln\mu} - 2\epsilon)L_a$ and the stationarity property $\frac{d}{d\ln\mu} L_a^{\mathrm{UV}} = 0$ at FPs (which can be seen by writing $L_a \sim x_1 + x_2(g^Q - g^{Q,\mathrm{UV}})$ with $x_{1,2}$ constants and using $\beta^{Q,\mathrm{UV}} = 0$). Eq. (G.9) is a relevant observation as this implies finiteness of the corresponding term in the dilaton effective action.

G.2.2 Flow of $\Delta\beta_a$ the IR effective action

Let us start by writing the d-dimensional Euler term as a sum of a four dimensional and an evanescent term

$$\sqrt{\mathrm{g}} E_4 = \partial O - k_d e^{2\epsilon s}(-2\Box s(\partial s)^2 + (\partial s)^4 - 2\epsilon(\partial s)^4) , \tag{G.10}$$

[2]The specific combination $2k_d a_0 - 8b_0$ comes from differentiating (3.26) three times w.r.t. s in d-dimensions.

where we have assumed the conformally flat metric $\mathrm{g}_{\alpha\beta} = e^{-2s(x)}\delta_{\alpha\beta}$ and $k_d \sim \epsilon$ is defined below (G.8). The total derivative term

$$\partial O = -4(d-3)(d-2)\left[\frac{1}{2}\Box(e^{2\epsilon s}(\partial s)^2) + \partial(e^{2\epsilon s}\partial s((1-\epsilon)(\partial s)^2 - \Box s))\right] \ , \quad \text{(G.11)}$$

corresponds to a topological term that survives the $d \to 4$ limit. The evanescent part of the gravitational counterterms (3.26) becomes the Wess-Zumino term of the dilaton effective action in [115]

$$\begin{aligned} \mathcal{L}_{\text{gravity}} \quad &\supset \quad a_0 \int d^dx\sqrt{g}(E_4 - \partial O) = -k_d a_0 \int d^dx(-2\Box s(\partial s)^2 + (\partial s)^4 - 2\epsilon(\partial s)^4) \\ &\stackrel{\epsilon\to 0}{\to} \quad 2\beta_a^{\text{UV}} \int d^4x(-2\Box s(\partial s)^2 + (\partial s)^4) = 2\beta_a^{\text{UV}} S_{WZ} \ , \end{aligned} \quad \text{(G.12)}$$

where we have used (G.9). In the preceding argument the finiteness of $k_d a_0$ (and b_0) was essential to ensure UV finiteness of the dilaton effective action and match the the bare coefficient of Wess-Zumino term to the Euler anomaly β_a^{UV}.

Similarly, the IR effective action contains the term $2\beta_a^{\text{IR}} S_{WZ}$ which contributes to (3.66) at $\mathcal{O}(s^3)$

$$\ln \mathcal{Z} = -4\tilde{d}^{\text{IR}} \int d^4x(\Box s)^2 - (4\beta_a^{\text{IR}} - 8\tilde{d}^{\text{IR}}) \int d^4x(\partial s)^2\Box s \ + \dots \quad . \quad \text{(G.13)}$$

We are now ready to put all the pieces together. By Fourier transforming the third functional derivative with respect to s of (G.13) we see that at low momenta the LHS of (G.8) behaves as

$$-(4\beta_a^{\text{IR}} - 8\tilde{d}^{\text{IR}})\lambda_3 + \dots \ , \quad \text{(G.14)}$$

where the dots stand for nonlocal contributions subleading in the momentum expansion. Applying $\hat{P}_{\lambda_3}$ to the RHS of (G.8) one gets

$$-M_{TTT}^{(\lambda_3)}(p_x, p_y)|_{p_x=p_x=0} - (4\beta_a^{\text{UV}} - 8\tilde{d}^{\text{UV}}) = -(4\beta_a^{\text{IR}} - 8\tilde{d}^{\text{IR}}) \ , \quad \text{(G.15)}$$

where we used that $(2k_d a_0 - 8b_0) \to -(4\beta_a^{\text{UV}} - 8\tilde{d}^{\text{UV}})$ for $\epsilon \to 0$. The 3-point sum rule in momentum space follows

$$\Delta\beta_a = 2\Delta\tilde{d} - \frac{1}{4}M_{TTT}^{(\lambda_3)}(p_x, p_y)|_{p_x=p_y=0} \ . \quad \text{(G.16)}$$

Assuming regularity of $M_{TTT}^{(\lambda_3)}$ the $p_x, p_y \to 0$ limit can be taken from any direction.

In particular if we choose $p_x = -p_y = p \to 0$ we can define a one-variable function

$$\mathbb{C}^{\mathbb{1}}_{TTT}(p^2) \equiv M^{(\lambda_3)}_{TTT}(p_x, -p_x) \tag{G.17}$$

satisfying

$$\Delta\beta_a = 2\Delta\tilde{d} - \frac{1}{4}\mathbb{C}^{\mathbb{1}}_{TTT}(0) \ . \tag{G.18}$$

It turns that all the properties and proofs related to $\mathbb{C}^{\mathbb{1}}_{TT}$ that we discussed in Section 3.2.1 can be easily extended to $\mathbb{C}^{\mathbb{1}}_{TTT}$. In particular repeating the steps leading to (3.82),(3.83) we get

$$\mathbb{C}^{\mathbb{1}}_{TTT}(0) = 2\int_0^\infty \beta^A\beta^B\beta^C\chi^{\text{MOM}}_{ABC}(\mu')\frac{d\mu'}{\mu'} \ , \tag{G.19}$$

where χ^{MOM}_{ABC} is obtained from the contact term corresponding to λ_3 structure in (G.6) using a covariant extension of (2.115) to 3-points. Scheme dependence follows again from the property that the integrand transforms as a total scale derivative $\beta^D\partial_D(\beta^A\beta^B\beta^C\omega_{ABC})$, for some finite function ω_{ABC} parametrizing the 3-point scheme change. Clearly, this doesn't contribute at the UV/IR limits of the above integral.
Substituting this back into (G.18) we get the final scheme-independent result

$$\Delta\beta_a = \frac{1}{2}\int_0^\infty \beta^A\beta^B\left(\chi^{\mathcal{R}}_{AB} - \beta^C\chi^{\mathcal{R}}_{ABC}\right)\frac{d\mu'}{\mu'} \ . \tag{G.20}$$

Note that the MS-scheme version of this equation can be obtained from the result (3.17b) of [66] for $\Delta\beta_a$ which reads

$$\Delta\beta_a = \frac{1}{8}\int_0^\infty \beta^A(\mu')\beta^B(\mu')\chi^{g,\text{MS}}_{AB}(\mu')\frac{d\mu'}{\mu'}, \tag{G.21}$$

where $\chi^{g,\text{MS}}_{AB}$ is corresponds to a specific counterterm composed of local couplings and the Einstein tensor. From (3.23) of [66] one gets[3]

$$(\chi^{g,\text{MS}}_{AB} - 4\chi^{\text{MS}}_{AB} + 4\chi^{\text{MS}}_{ABC}\beta^C)\beta^A\beta^B = \frac{d}{d\ln\mu}(-V_{AB}\beta^A\beta^B) \ , \tag{G.22}$$

where V_{AB} is some scheme-dependent function. The term on the RHS vanishes

[3]Note that χ^a_{ij} and and $\bar{\chi}_{ijk}$ of [66] corresponds to χ^{MS}_{AB} and χ^{MS}_{ABC} respectively in the notation of this thesis.

when inserted into (G.21) and one gets

$$\Delta\beta_a = \frac{1}{2}\int_0^\infty \left(\chi^{\rm MS}_{AB}\beta^A\beta^B - \chi^{\rm MS}_{ABC}\beta^A\beta^B\beta^C\right)\frac{d\mu'}{\mu'}\ , \tag{G.23}$$

which is equivalent to (G.20) evaluated in the MS-scheme.

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