A spherically symmetric stiff fluid spacetime in light of cosmic structure formation

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We will report here a critical inspection of the Penrose conjecture according to which the gravitational entropy should be quantified via the Weyl curvature, with the Clifton-Ellis-Tavakol entropy being one specific realization of this proposal. In fact, we will show that in some exact inhomogeneous and anisotropic cosmological models which arise as exact solutions in general relativity with either closed and open topologies, the Clifton-Ellis-Tavakol gravitational entropy is increasing in time despite the decrease of the magnitude of the Weyl curvature: this is possible thanks to the growth of the spatial shearing effects. The matter content driving the dynamics of this class of models comes in the form of a stiffened fluid which can be relevant in the early universe. We choose the values of the free parameters entering the metric tensor consistently with the holographic principle and the second law of thermodynamics. Our study can be of interest in light of the modeling of the formation of some primordial structures, like the Large Quasar Groups, as suggested by the growth of gravitational entropy, and whose existence cannot be accounted for by standard perturbation methods over a homogeneous background.

Keywords: Exact solution; stiff matter; gravitational entropy.

1. Introduction

Our Universe contains a plethora of different astrophysical structures with different sizes such as galaxies, clusters of galaxies, filaments, voids,... The formation process for cosmic structures of sizes smaller than 150 Mpc, which is the length scale above which the universe is considered to be homogeneous,[1] can be described with perturbation schemes applied to a homogeneous background as the Friedman-Lemaître-Robertson-Walker one.[2,3] However, there is observational evidence of the existence of astrophysical structures like the one known as the "axis of evil"[4] which challenge the Copernican principle because they come with an alignment of matter along a preferred direction and their sizes are larger than 150 Mpc. Other examples are the Large Quasar Groups with sizes in the range of 70-350 Mpc.[5–8] Accounting for their existence by means of perturbation methods is problematic because the cosmic material would not cluster quickly enough for forming a bound system of this size: this is one among few other open questions affecting the standard model of cosmology.[12] Inhomogeneous cosmology may offer an alternative route for developing theoretical frameworks which can describe the evolution of matter perturbations undergoing a local collapse which can lead to the formation of an astrophysical structure. For example, in a dust Szekeres spacetime the density contrast grows eight times quicker than in the linear perturbation regime making the process of formation of astrophysical structures more efficient.[13] There is also observational evidence of exotic astrophysical structures with non-standard size which should have formed in the early universe, as a supermassive black hole at redshift $z \sim 10$,[9,10] and the Huge Quasar Group at redshift $z \sim 1.27$,[11] whose existence cannot be explained by invoking effects played by an inhomogeneous spacetime supported by dust (which can be interpreted as pressureless dark matter). Therefore, the issue becomes to provide an appropriate theoretical framework in which spatial shear effects and tidal effects can trigger a local collapse in an overall expanding background supported by a matter content consistent with the early stages of evolution of the universe.

In the primordial universe shearing effects may play an important role, as compared to the one of the matter content, even though they are almost negligible at the present day.[14] Moreover, it has been argued that in early cosmic epochs the matter content may come in the form of a stiff fluid[a],[15–19] which is the effective hydrodynamical realization of a massless scalar field according to the canonical formalism.[20,21] More in detail, in[18,19] it has been argued that the early universe should be filled with cold baryons which fulfill the equation of state $p = \rho$, and that if cosmic matter is described as a relativistic self-gravitating Bose-Einstein condensate, then the cosmic evolution would experience a stiff-matter dominated era. Indeed the development of specific algorithms for integrating the Einstein field equations when the matter content is a stiff fluid has received attention in the literature;[22–28] then a number of exact and analytical solutions expressible in terms of elementary functions has been found obeying to a variety of symmetry groups,[29–32] just to mention a few examples.

One specific solution to the Einstein equations that we have adopted in our own research,[33] and which was found and investigated from the mathematical perspective by a number of different authors,[34–38] reads as (see also page 261 in[39]):

$$ds^2 = -\left(\frac{r}{2}\right)^2 dt^2 + \frac{dr^2}{\epsilon + Cr^2} + r^2\left[\frac{\epsilon}{2} + h(t)\right](d\theta^2 + \sin^2\theta d\phi^2), \qquad (1)$$

where the "generalized scale factor" can either be

$$h(t) = A\sin(t) + B\cos(t) \qquad \text{if} \qquad \epsilon = -1\,, \qquad (2)$$

$$h(t) = -\left(\frac{t}{2}\right)^2 + 2At + B \qquad \text{if} \qquad \epsilon = 0\,, \qquad (3)$$

$$h(t) = Ae^t + Be^{-t} \qquad \text{if} \qquad \epsilon = 1\,, \qquad (4)$$

according to the topology of the spacetime. We can note that the model (1) depends on three free parameters A, B, and C, and that only its angular part, and not the

[a] A stiff fluid is a perfect fluid, e.g. it is fully described by its adiabatic pressure p and energy density ρ, whose equation of state in natural units is $p = \rho$, and with matter-energy tensor $T^{\mu}{}_{\nu} = \mathrm{diag}[-\rho, p, p, p]$.

radial one, is expanding. The spacetime is supported by a fluid obeying to a stiffened equation of state

$$p = \rho + \frac{3C}{4\pi}\,. \tag{5}$$

which reduces to the case of a stiff fluid, i.e. of a massless scalar field, in the limit of $C \rightarrow 0$. Thus, this latter parameter, which does not affect directly the time evolution of the generalized scale factor $h(t)$ as we can appreciate from (2)-(3)-(4), would instead make the cosmic fluid pressure to be non-zero but equal to a constant in an empty space for which $\rho = 0$; more in general th equation of state (5) can be written in the form $p = \omega(\rho)\rho$ in which the density-dependent equation of state parameter can be interpreted as a chameleon field.[40, 41] The Hubble function, invariant shear, cosmic energy density and Weyl curvature of the spacetime (1) are given by

$$H = \frac{4\dot{h}(t)}{3(\epsilon + 2h(t))r}\,, \tag{6}$$

$$\sigma^2 = \frac{4\dot{h}(t)^2}{3(\epsilon + 2h(t))^2 r^2} = \frac{3H^2}{4}\,, \tag{7}$$

$$\rho = -\frac{c^2(R + 18C)}{16\pi G}\,, \qquad R = -2\frac{\epsilon + \mathcal{R} + 6Cr^2(\epsilon + 2h(t))^2}{r^2(\epsilon + 2h(t))^2}\,, \tag{8}$$

$$\Psi_2 = -\frac{\mathcal{R} + \epsilon}{3r^2(2h(t) + \epsilon)^2}\,, \tag{9}$$

where an overdot denotes a derivative with respect to the cosmic time. In the computation of the latter quantity we have applied the Newman-Penrose formalism,[42] and the following notation has been introduced:

$$\begin{aligned} \mathcal{R} &= 4(A^2 + B^2) \qquad \text{for} \qquad \epsilon = -1\,, \\ \mathcal{R} &= 4(4A^2 + B) \qquad \text{for} \qquad \epsilon = 0\,, \\ \mathcal{R} &= -16AB \qquad \text{for} \qquad \epsilon = 1\,. \end{aligned} \tag{10}$$

Since we have a nonzero Weyl scalar Ψ_2, some tidal forces are present in the cosmological model (1),[43] and they can trigger a local gravitational collapse whose evolution then experiences the non-standard spatial shearing effects described by σ^2, potentially taming the previously mentioned problems of accounting for the existence of some astrophysical structures observed in the early universe. For exploring this topic, in our research[33] we have computed the gravitational entropy for the cosmology (1) by following the Clifton-Ellis-Tavakol proposal,[44] in which we have implemented some numerical values for the free parameters A, B, and C which are consistent with the thermodynamical requirements formulated through the cosmological holographic principle and the second law of thermodynamics. We have obtained that the gravitational entropy can be increasing even in the time intervals in which the strength of the Weyl curvature is decreasing, and therefore we have claimed that *Thermodynamics of shearing massless scalar field spacetimes* *is inconsistent with the Weyl curvature hypothesis* according to which the conformal curvature can be adopted as a measure for the gravitational entropy.[45] Our result is due to the effects played by the spatial shear which accounts for the spatial anisotropies.

This *MG16 conference proceeding* is organized as follows: in Sect. 2 we impose the holographic principle and the second law of thermodynamics to the spacetime (1) showing how they can set some constraints on the behavior of various relevant physical quantities. In Sect. 3 we introduce the Clifton-Ellis-Tavakol entropy, discuss its importance in the modeling of the formation of astrophysical structures, and report on our specific results for the spacetime under investigation. In Sect. 4 we formulate the two messages that according to us the reader should take at home regardless the technicalities involved in this project. Finally in Sect. 5 we put our research into the larger perspective of the studies of gravitational entropy in inhomogeneous cosmology.

2. Imposing the thermodynamical requirements

According to the cosmological holographic principle, as formulated by Bousso, the matter entropy S_m inside a region bounded by a "horizon" should be smaller than the area A_H of the horizon itself, or more precisely $S_m \leq A_H/4$.[46] We can see that the celebrated Bekenstein-Hawking entropy for static black holes[47,48] constitutes the limiting case of this more general principle. For a dynamical spacetime we consider appropriate to work with the dynamical apparent horizon as the boundary of the region we are interested in; by introducing the areal radius

$$\tilde{r} = r\sqrt{\frac{\epsilon}{2} + h(t)}\,, \tag{11}$$

the location $\tilde{r}_H$ of the dynamical apparent horizon is such that[49]

$$||\nabla\tilde{r}||^2_{\tilde{r}=\tilde{r}_H} = 0\,. \tag{12}$$

In a time-evolving configuration, the dynamical apparent horizon is not the unique choice for the boundary of the region to which the cosmological holographic principle can be imposed; however in the homogeneous and isotropic Friedman universe assuming the space to be filled with some form of dark energy the first and second law of thermodynamics have been shown to hold for the dynamical apparent horizon, but not if one works with the cosmic horizon,[50] providing a motivation for choosing our route.[51,52]

Since the fluid energy density should be non-negative all along the time evolution of the system, we found that C should be non-positive; under this restriction, equation (12) admits a solution only for the topology $\epsilon = 1$. Setting the parameters A and B to be positive guarantees both a well-defined cosmic energy density and the existence of a solution for the equation determining the location of the dynamical apparent horizon. Taking into account that $A_H = 4\pi\tilde{r}_H^2$ and that $S_m = \alpha\tilde{r}_H^3$, with

α an overall proportionality constant, it can be seen after some computations that the holographic principle is further constraining the value of C, i.e. of the deviations of the equation of state of the cosmic matter from a stiff fluid. The cosmological consequences of these restrictions on the values of the arbitrary model parameters are:

- If we compute the deceleration parameter as in,[53] it would be negative;
- Since $\epsilon + 2h(t) \neq 0$, there would be no initial singularity in this cosmology because there would not exist a time t_B at which the energy density (8) would diverge.

On the other hand, imposing the second law of thermodynamics to the matter content would require $\dot{S}_m > 0$, which can be translated into the equivalent condition $\frac{d\tilde{r}_H}{dt} > 0$. After some algebraic manipulations, we could recast this condition as $Ae^t - Be^{-t} > 0$; thus, the cosmological consequences within the model (1) of imposing the second law are:

- The lower limit on the size of the universe $h(t) > 2Be^{-t}$ is established;
- The lower limit on the age of universe $t > \frac{1}{2} \ln \frac{B}{A}$ is established;
- The Weyl curvature Ψ_2 is a time-decreasing quantity.

3. Computing the gravitational entropy

"Entropy" can be naively regarded as a measure of the disorder within a system (as related to the existence of an arrow of time), or as a quantification of the number of different microscopic realizations compatible with the same macroscopic system (statistical entropy). Therefore, in cosmology, as astrophysical structures form in one specific spatial regions rather than in others with the whole universe remaining homogeneous on large enough length scales, the gravitational entropy should increase. Since when an observer looks at the universe from its location it may or may not detect an astrophysical structure, the growth of gravitational entropy should be accompanied by an increase of the spatial anisotropy. One specific way of computing the density s_{grav} of gravitational entropy in Petrov type D spacetime, as for example (1), is due to Clifton-Ellis-Tavakol[44]:

$$T_{\text{grav}} \dot{s}_{\text{grav}} = -dV \sigma_{\mu\nu} \left(\pi^{\mu\nu}_{\text{grav}} + \frac{(\rho + p)}{3\rho_{\text{grav}}} E^{\mu\nu} \right) . \tag{13}$$

The ingredients of this formula are the following:

- The temperature of the free gravitational field defined as

$$T_{\text{grav}} = \frac{|u_{a;b} l^a n^b|}{\pi} = \frac{r}{8\pi\sqrt{Cr^2 + \epsilon}} , \tag{14}$$

  where u^a is the matter-comoving observer four-velocity, and l^a and n^a are part of the null coframe containing the dt and dr contributions.

- The volume element

$$dV = \frac{r^2 \sin\theta(\epsilon + 2h(t))}{2\sqrt{Cr^2 + \epsilon}}\, dr d\theta d\phi\,. \tag{15}$$

- The shear tensor $\sigma_{\mu\nu}$ which describes the rate of distortion of a given region with fixed volume during the cosmic evolution.
- The gravitational energy defined in terms of the Weyl curvature as:

$$\rho_{\text{grav}} := 16\pi|\Psi_2|\,. \tag{16}$$

- The gravitational anisotropic pressure also defined in terms of the Weyl curvature as:

$$\pi^{\text{grav}}_{\mu\nu} := \frac{|\Psi_2|}{16\pi}(-x_a x_b + y_a y_b + z_a z_b + u^a u^b)\,, \tag{17}$$

  in which x_a, y_a, z_a constitute the spacelike vectors in the orthonormal base for the metric (1).
- The gravito-electric tensor $E_{\mu\nu}$ which accounts for tidal effects, which are indeed necessary for having a gravitational collapse.

We have considered the approach (13) to be a sound proposal for computing the gravitational entropy because, as pointed out in,[44] it delivers a non-negative entropy, it is consistent with the Hawking-Bekenstein entropy of black holes, it provides a vanishing entropy in and only in conformally flat spacetimes (as for example in the homogeneous and isotropic Friedman-Lemaître-Robertson-Walker universe in which astrophysical structures cannot form due to the lack of any matter density contrast), and it is related to the growth of anisotropies in the spacetime (as they indeed increase during the structure formation phase).

Thus, it does not come as a surprise that the conjecture of a connection between gravitational entropy and Weyl curvature has been regarded as a useful tool for the description of the formation of astrophysical structures in various inhomogeneous models. For example, in[54] a relationship between the growth of gravitational entropy and the amplitude of initial fluctuations of spatial curvature at the last scattering time has been found, while the possible saturation of its value at a certain time has been investigated in[55] for the Lemaître-Tolman-Bondi universe, in[56] more in general for the class of silent universes for which the gravitomagnetic part of the Weyl curvature vanishes, and in[57] for a perturbed Friedman spacetime. Furthermore, a gravitational entropy quantified via the Weyl curvature is consistent with the generalized second law in the void Lemaître-Tolman-Bondi universe,[58] and the specific Clifton-Ellis-Tavakol entropy has been shown to be a time increasing function in a number of cosmological models.[59]

Our explicit computations for the Clifton-Ellis-Tavakol gravitational entropy delivered the result

$$T_{\text{grav}}\dot{s}_{\text{grav}} = dV\frac{64\pi\dot{h}(t)(1-16AB)}{3(2h(t)+\epsilon)^3 r^3}\,. \tag{18}$$

Here we have restricted ourselves to the spacetime (4) because, as mentioned in Sect. 2, it is the only case within (1) for which a thermodynamical investigation based on the cosmological holographic principle was possible. Then, implementing the constraints on the free parameters obtained from such analysis and those arising from the second law of thermodynamics, we have found that a challenge to the Weyl curvature conjecture proposed by Penrose[45] is posed. In fact, the gravitational entropy is increasing even when the gravito-electric curvature as measured by Ψ_2 in (9) is not, and thus the latter cannot be considered too naively as a measure of the former.

4. Discussion

Quantifying the effects that the interactions between small-scale inhomogeneities (e.g. galaxies, galaxy clusters, voids, filaments, etc...) have on the large-scale evolution of the Universe is not a trivial task, and the debate in the literature about their importance is still open.[61] There exists in fact a range of completely different claims from that they have no effects at all to the one that they can bring an energy budget which can be responsible for the observed accelerated expansion of the universe without the need of a cosmological constant; see[62] for a review of different approaches and their predictions. Once endowed with a cosmological model formulated in terms of a metric tensor solution of the field equations of the underlying gravitational theory, it is necessary to investigate its physical applicability. Some widely applied cosmological tests which can assess the suitability of a mathematical solution of the Einstein equations as a cosmological model rely on supernovae data,[63–65] cosmic microwave background data,[66,67] or baryon acoustic oscillations data,[68,69] and clearly require data-analysis skills. For example, the inhomogeneous dust Lemaître-Tolman-Bondi model has been shown to be in tension with the kinematic Sunyaev-Zel'dovich effect[70] although it is not with observations related to the luminosity distance of supernovae, without the need of dark energy but assuming that the observer is located inside a giant void;[71] more in general the development of "model independent" tests for inhomogeneous cosmology constitutes a line of current research.[72–74] However, in Sect. 2 we have explained how "theoretical" requirements can also be exploited for constraining the values of the free model parameters and for extracting cosmological information about the model under investigation. We had previously applied the same way of thinking to the Stephani cosmological model by estimating the strength of spatial inhomogeneities, the matter abundance and the size of the universe by imposing the cosmological holographic principle and the second law of thermodynamics[75] (however this latter spacetime is conformally flat and therefore it does not offer any viable possibility for an investigation of the gravitational entropy). Thus, the first take at home message from our research is:

- By working with solely pen and paper, we can constrain a given cosmological model known in terms of its metric tensor by imposing the thermodynamical requirements which follow from the cosmological holographic principle and the second law of thermodynamics.

In Sect. 3 we have mentioned that the Clifton-Ellis-Tavakol gravitational entropy is increasing in time for the spacetime (1) although the Weyl curvature is not thanks to the presence of a term given by shear tensor which is inserted by hands in its formulation. The fact that spatial shearing effects are developing in the same time interval in which the gravitational entropy is increasing is a fundamental consistency check for the mathematical formula adopted for the latter. In fact, both these quantities are expected to increase during the structure formation phase because the universe is becoming anisotropic and inhomogeneous, with the former being measured by the shear tensor while the latter can be regarded as an increase of the level of "disorder" of the system which should be quantified via an appropriate entropy quantity. Thus, our second take at home message, whose consequences will be further explored in the next Sect. 5 is:

- Implementing the Weyl curvature conjecture according to which the Weyl curvature should serve as a measure for the gravitational entropy is not as simple as it may look like at a first sight and other ingredients may be necessary in the construction of a mathematical formula for the gravitational entropy.

5. Outlook

The search for an appropriate inhomogeneous cosmological model which can provide a viable framework for addressing the issue of the formation of astrophysical structures with sizes larger than 150 Mpc (and which is also in agreement with other datasets), and the formulation of an appropriate measure for the "gravitational entropy" in the spirit of the Weyl curvature conjecture are two related problems. Therefore, we consider appropriate to investigate various notions of gravitational entropy as the following:

- $$S = \frac{C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}{R_{\mu\nu}R^{\mu\nu}} \tag{19}$$

  where $C_{\mu\nu\rho\sigma}$ and $R_{\mu\nu}$ are respectively the Weyl and Ricci tensor, and its spatially integrated version

  $$\tilde{S} = \int dV S\,. \tag{20}$$

  This formula has arisen in the context of cosmologies exhibiting an anisotropic singularity.[76] However this proposal cannot be applied to vacuum spacetimes because of the vanishing of the Ricci tensor.

- Eq. (19) but multiplied by the square root of the determinant of the spatial metric h as suggested in[77]:

$$S = \sqrt{h}\frac{C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}{R_{\mu\nu}R^{\mu\nu}}\,, \tag{21}$$

  which would be something in between the two proposals of the previous step.
- Applying a spatially averaging scheme and writing the entropy as[78,79]

$$S = V_{\mathcal{D}}\left\langle \rho \ln \frac{\rho}{\langle\rho\rangle_{\mathcal{D}}}\right\rangle_{\mathcal{D}}\,, \tag{22}$$

  where angular brackets denote an average over the spatial domain of interest $\mathcal{D}$.
- The Clifton-Ellis-Tavakol entropy that we have already talked about in Sect. 3.

These inequivalent ways of computing the gravitational entropy should be applied to the general line element describing a spherically symmetric dynamical spacetime (see page 251 in[39])

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + Y^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2) \tag{23}$$

in a set of future projects. Intuitively, we expect that demanding the simultaneous growth of shear, Weyl curvature and gravitational entropy would provide some information on the increasing/decreasing and concavity properties of the functions $\nu(t,r)$, $\lambda(t,r)$ and $Y(t,r)$, which then will be implemented into the Einstein equations for understanding which types of matter content permit to achieve this goal.

All the previously mentioned proposals for computing the gravitational entropy are directly sensitive to the matter content of the spacetime. In fact in some of them the energy density (Hosoya-Buchert-Morita) and possibly also the pressure (Clifton-Ellis-Tavakol) appear explicitly; in other cases as in (19)-(20)-(21) they are inserted via the Ricci tensor, which is connected to the matter content of the universe as we can see from the Einstein equations written in the form

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\,. \tag{24}$$

A more conservative proposal which has been formulated recently in the context of black hole physics argues that it is possible to write the density s of gravitational entropy just in terms of the frame component of the Weyl tensor W and of its Newman-Penrose derivative DW as[80]

$$s = \left|\frac{DW}{W}\right|\,. \tag{25}$$

The applicability of such formalism in inhomogeneous cosmology and in particular the investigation of its relationship to the spatial anisotropies quantified by the shear tensor would constitute a subject of future studies.

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References

1. V. Springel, et al., Simulating the joint evolution of quasars, galaxies and their large-scale distribution, *Nature* **435**, 629 (2005).
2. E. Lifshitz, Republication of: On the gravitational stability of the expanding universe, *J. Phys. (USSR)* **10** (1946) 116; *Gen. Rel. Grav.* **49**, 18 (2017).
3. P. J. E. Peebles, and J. T. Yu, Primeval adiabatic perturbation in an expanding universe, *Astrophys. J.* **162**, 815 (1970).
4. K. Land, and J. Magueijo, Examination of evidence for a preferred axis in the cosmic radiation anisotropy, *Phys. Rev. Lett.* **95**, 071301 (2005).
5. M. Scrimgeour, et al., The WiggleZ Dark Energy Survey: the transition to large-scale cosmic homogeneity, *Mon. Not. Roy. Astron. Soc.* **425**, 116 (2012).
6. J. K. Yadav, J. S. Bagla, and N. Khandai, Fractal dimension as a measure of the scale of homogeneity, *Mon. Not. Roy. Astron. Soc.* **405**, 2009 (2010).
7. F. Sylos Labini, Super-homogeneity and inhomogeneities in the large scale matter distribution, *PoS Cosmology* **002**, 2009 (2009).
8. F. Sylos Labini, N. L. Vasilyev, Y. V. Baryshev, and M. Lopez-Corredoira, Absence of anti-correlations and of baryon acoustic oscillations in the galaxy correlation function from the Sloan Digital Sky Survey DR7, *Astron. Astrophys.* **505**, 981 (2009).
9. D. J. Mortlock, et al., A luminous quasar at a redshift of $z = 7.085$, *Nature* **474**, 616 (2011).
10. Y. Matsuoka et al., Subaru high-z exploration of lowluminosity quasars (SHELLQs). I. Discovery of 15 quasars and bright galaxies at $5.7 < z < 6.9$, *Astrophys. J.* **828**, 26 (2016).
11. R. G. Clowes, et al., A structure in the early universe at $z \sim 1.3$ that exceeds the homogeneity scale of the R-W concordance cosmology, *Mon. Not. Roy. Astron. Soc.* **429**, 2910 (2013).
12. A. D. Dolgov, Beasts in Lambda-CDM Zoo, *Physics of Atomic Nuclei* **80**, 987 (2017).
13. K. Bolejko, Structure formation in the quasispherical Szekeres model, *Phys. Rev. D* **73**, 123508 (2006).
14. A. Coley, and W. C. Lim, Spikes and matter inhomogeneities in massless scalar field models, *Class. Quantum Grav.* **33**, 015009 (2016).
15. E. M. Lifshitz, and I. M. Khalatnikov, Investigations in relativistic cosmology, *Adv. Phys.* **12**, 185 (1963).
16. V. A. Belinski, I. M. Khalatnikov, and E. M. Lifshitz, Oscillatory approach to the singular point in relativistic cosmology, *Adv. Phys.* **19**, 525 (1970).
17. V. A. Belinski, I. M. Khalatnikov, and E. M. Lifshitz, A general solution of the Einstein equations with a time singularity, *Adv. Phys.* **31**, 639 (1982).
18. J. B. Zel'dovic, A hypothesis, unifying the structure and the entropy of the Universe, *Mont. Not. Roy. Astron. Soc.* **160**, 1 (1972).
19. P. -H. Chavanis, Cosmology with a stiff matter era, *Phys. Rev. D* **92**, 103004 (2015).
20. D. H. Lyth, and A. Riotto, Particle physics models of inflation and the cosmological density perturbation, *Phys. Rep.* **314**, 1 (1999).
21. C. Armendariz-Picon, T. Damour, and V. F. Mukhanov, k-Inflation, *Phys. Lett. B* **458**, 209 (1999).
22. R. Geroch, A method for generating solutions of Einstein's equations, *J. Math. Phys.* **12**, 918 (1971).

23. R. Geroch, A method for generating new solutions of Einstein's equation. II., *J. Math. Phys.* **13**, 394 (1972).
24. D. Garfinkle, E. N. Glass, and J. P. Krisch, Solution generating with perfect fluids, *Gen. Rel. Grav.* **29**, 467 (1997).
25. H. Stephani, Symmetries of Einstein's field equations with a perfect fluid source as examples of Lie-Bäcklund symmetries, *J. Math. Phys.* **29**, 1650 (1988).
26. R. Hirota, Direct method of finding exact solutions of nonlinear evolution equations, in "Bäcklund transformations, the inverse scattering method, solitons, and their applications", *Lect. Not. Math. R.M. Miura eds.* **515** (1976).
27. V. A. Belinski, and V. E. Zakharov, Steady-state gravitational solitons with axial symmetry, *Sov. Phys. JETP* **50**, 1 (1979); *Zh. Eksp. Teor. Fiz.* **77**, 3 (1979).
28. J. Wainwright, W. C. W. Ince, and B. J. Marshman, Spatially homogeneous and inhomogeneous cosmologies with equation of state $p = \mu$, *Gen. Rel. Grav.* **10**, 259 (1979).
29. D. Gregoris, W. C. Lim, and A. Coley, Stiff fluid spike solutions from Bianchi type V seed solutions, *Class. Quantum Grav.* **34**, 235013 (2017).
30. M. Marklund, and M. Bradley, Invariant construction of solutions to Einstein's field equations - LRS perfect fluids II, *Class. Quantum Grav.* **16**, 1577 (1999).
31. A. Coley, D. Gregoris, and W. C. Lim, On the first G_1 stiff fluid spike solution in general relativity, *Class. Quantum Grav.* **33**, 215010 (2016).
32. D. Gregoris, On the connection between cosmological parameters and peculiar motion in a G_2 massless scalar field spacetime, *Int. J. Mod. Phys. D* 2150099 (2021).
33. D. Gregoris, Y. C. Ong, and B. Wang, Thermodynamics of shearing massless scalar field spacetimes is inconsistent with the Weyl curvature hypothesis, *Phys. Rev. D* **102** 023539 (2020).
34. C. Leibovitz, Time-dependent solutions of Einstein's equations, *Phys. Rev. D* **4**, 2949 (1971).
35. K. Lake, Remark concerning spherically symmetric nonstatic solutions to the Einstein equations in the comoving frame, *Gen. Rel. Grav.* **15**, 357 (1983).
36. N. van den Bergh and P. Wils, Exact solutions for nonstatic perfect fluid spheres with shear and an equation of state, *Gen. Rel. Grav.* **17**, 223 (1985).
37. B. Collins and J. M. Lang, A class of self-similar perfect fluid spacetimes, and a generalisation, *Class. Quantum Grav.* **4**, 61 (1987).
38. S. D. Maharaj, R. Maartens, and M. Maharaj, A note on a class of spherically symmetric solutions, *Nuovo Cimento B* **108**, 1 (1993).
39. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations* (Cambridge University Press, Cambridge, England, 2002).
40. J. Khoury, and A. Weltman, Chameleon fields: Awaiting surprises for tests of gravity in space, *Phys. Rev. Lett.* **93**, 171104 (2004).
41. J. Khoury, and A. Weltman, Chameleon cosmology, *Phys. Rev. D* **69**, 044026 (2004).
42. E. Newman, and R. Penrose, An approach to gravitational radiation by a method of spin coefficients, *J. Math. Phys* **3**, 566 (1962).
43. J. C. Baez, and E. F. Bunn, The meaning of Einstein's equation, *Amer. Jour. Phys.* **73**, 644 (2005).
44. T. Clifton, G. F. R. Ellis, and R. Tavakol, A gravitational entropy proposal, *Class. Quantum Grav.* **30**, 125009 (2013).
45. R. Penrose, Singularities and time-asymmetry, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
46. R. Bousso, The holographic principle, *Rev. Mod. Phys.* **74**, 825 (2002).

47. S. W. Hawking, Particle creation by black holes, *Comm. Math. Phys.* **43**, 199 (1975).
48. J. D. Bekenstein, Black holes and entropy, *Phys. Rev. D* **7**, 2333, (1973).
49. A. Ashtekar, and B. Krishnan, Isolated and dynamical horizons and their applications, *Living Rev. Relativity* **7**, 10 (2004).
50. B. Wang, Y. Gong, and E. Abdalla, Thermodynamics of an accelerated expanding universe, *Phys. Rev. D* **74**, 083520 (2006).
51. V. Faraoni, Cosmological apparent and trapping horizons, *Phys. Rev. D* **84**, 024003 (2011).
52. D. Bak, and S.-J. Rey, Cosmic holography, *Class. Quantum Grav.* **17**, L83 (2000).
53. G. F. R. Ellis, and H. van Elst, Cosmological models, *Cargése lectures 1998*, [arXiv:gr-qc/9812046].
54. R. A. Sussman, and J. Larena, Gravitational entropy of local cosmic voids, *Class. Quantum Grav.* **32**, 165012 (2015).
55. R. A. Sussman, and J. Larena, Gravitational entropies in LTB dust models, *Class. Quantum Grav.* **31**, 075021 (2014).
56. K. Bolejko, Gravitational entropy and the cosmological no-hair conjecture, *Phys. Rev. D* **97**, 083515 (2018).
57. G. Marozzi, J.-P. Uzan, O. Umeh, and C. Clarkson, Cosmological evolution of the gravitational entropy of the large-scale structure, *Gen. Rel. Grav.* **47**, 114 (2015).
58. P. Mishra, and T. P. Singh, Thermodynamics and Lemaitre-Tolman-Bondi void models, *Phys. Rev. D* **89**, 123007 (2014).
59. S. Chakraborty, S. Guha, and R. Goswami, An investigation on gravitational entropy of cosmological models, *Int. J. Mod. Phys. D* **30**, 2150051 (2021).
60. T. Buchert, et al., Is there proof that backreaction of inhomogeneities is irrelevant in cosmology?, *Class. Quantum Grav.* **32**, 215021 (2015).
61. T. Buchert, et al., Is there proof that backreaction of inhomogeneities is irrelevant in cosmology?, *Class. Quantum Grav.* **32**, 215021 (2015).
62. C. Clarkson, G. Ellis, J. Larena, and O. Umeh, Does the growth of structure affect our dynamical models of the universe? The averaging, backreaction, and fitting problems in cosmology, *Rep. Prog. Phys.* **74**, 112901 (2011).
63. A. G. Riess, et al., Observational evidence from supernovae for an accelerating universe and a cosmological constant, *Astron. J.* **116**, 1009 (1998).
64. S. Perlmutter, et al. (The Supernova Cosmology Project), Measurements of Omega and Lambda from 42 high-redshift supernovae, *Astrophys. J.* **517**, 565 (1999).
65. D. M. Scolnic, The complete light-curve sample of spectroscopically confirmed type Ia supernovae from Pan-STARRS1 and cosmological constraints from the combined pantheon sample, *Astrophys. J.* **859**, 101 (2018).
66. E. Komatsu, et al., Seven-year Wilkinson microwave anisotropy probe (WMAP) observations: cosmological interpretation, *Astrophys. J. Suppl.* **192**, 18 (2011).
67. R. Keisler, et al., A measurement of the damping tail of the cosmic microwave background power spectrum with the South Pole Telescope, *Astrophys. J.* **743**, 28 (2011).
68. D. J. Eisenstein, et al., Detection of the baryon acoustic peak in the large-scale correlation function of SDSS luminous red galaxies, *Astrophys. J.* **633**, 560 (2005).
69. S. Cole, et al. (The 2dFGRS Team), The 2dF Galaxy Redshift Survey: Power-spectrum analysis of the final dataset and cosmological implications, *Mon. Not. R. Astron. Soc.* **362**, 505 (2005).
70. P. Bull, T. Clifton, and P. G. Ferreira, The kSZ effect as a test of general radial inhomogeneity in LTB cosmology, *Phys. Rev. D* **85**, 024002 (2012).
71. T. Clifton, P. G. Ferreira, and K. Land, Living in a void: Testing the Copernican principle with distant supernovae, *Phys. Rev. Lett.* **101**, 131302 (2008).

72. L. H. Dam, A. Heinesen, and D. L. Wiltshire, Apparent cosmic acceleration from type Ia supernovae, *Mon. Not. R. Astron. Soc.* **472**, 835 (2017).
73. A. Heinesen, C. Blake, Y.-Z. Li, and D. L. Wiltshire, Baryon acoustic oscillation methods for generic curvature: Application to the SDSS-III baryon oscillation spectroscopic survey, *J. Cosmol. Astropart. Phys.* **03**, 003 (2019).
74. J. Trost Nielsen, A. Guffanti, and S. Sarkar, Marginal evidence for cosmic acceleration from Type Ia supernovae, *Sci. Rep.* **6**, 35596 (2016).
75. D. Gregoris, Y. C. Ong, B. Wang, Holographic principle and the second law in Stephani cosmology revisited, *Eur. Phys. J. Plus* **135**, 246 (2020).
76. J. Wainwright, and P. J. Anderson, Isotropic singularities and isotropization in a class of Bianchi Type-VI$_h$ cosmologies, *Gen. Rel. Grav.* **16**, 609 (1984).
77. Ø. Grøn, and S. Hervik, Gravitational entropy and quantum cosmology, *Class. Quantum Grav.* **18**, 601 (2001).
78. A. Hosoya, T. Buchert, M. Morita, Information entropy in cosmology, *Phys. Rev. Lett.* **92**, 141302 (2004).
79. R. Sussman, Weighed scalar averaging in LTB dust models: Part I. Statistical fluctuations and gravitational entropy, *Class. Quantum Grav.* **30**, 065015 (2013).
80. D. Gregoris, and Y. C. Ong, Understanding Gravitational Entropy of Black Holes: A New Proposal via Curvature Invariants, arXiv:gr-qc/3945089.