Published for SISSA by Springer

Received: November 6, 2024
Accepted: November 28, 2024
Published: December 27, 2024

Non-holomorphic modular A_5 symmetry for lepton masses and mixing

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Abstract: We perform a comprehensive bottom-up study of all the simplest lepton models based on non-holomorphic A_5 modular flavor symmetry, in which neutrinos are assumed to be Majorana particles and their masses are generated by the Weinberg operator or the type I seesaw mechanism. In the case that the generalized CP (gCP) symmetry is not considered, we find that 21 Weinberg operator models and 174 seesaw models can accommodate the experimental data in lepton sector, and all of them depend on six dimensionless free parameters and two overall scales. If gCP symmetry compatible with A_5 modular symmetry is imposed, one more free parameter would be reduced. Then only 4 of the 21 Weinberg operator models and 100 of the 174 seesaw models agree with the experimental data on lepton masses and mixing parameters. Furthermore, we perform a detailed numerical analysis for two example models for illustration.

Keywords: CP Violation, Discrete Symmetries, Flavour Symmetries, Neutrino Mixing

ArXiv ePrint: 2410.24103

https://doi.org/10.1007/JHEP12(2024)189

Contents

1 Introduction

The vast majority of free parameters in Standard Model (SM) arise from the flavor sector, including the masses, mixing angles and CP violation phases of quarks and leptons [1]. There are huge hierarchies among the quark masses, for example, the top quark is about five orders of magnitude heavier than the up quark. The quark flavor mixing described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix also exhibits a hierarchical pattern, and all the three quark mixing angles are small. The mass hierarchies in the lepton sector is more dramatic. The neutrinos have a tiny mass and its absolute mass scale is of the order of eV which is approximately nine orders of magnitude smaller than the tau lepton mass. The neutrino oscillation experiments show that the lepton mixing differs significantly from quark mixing. The solar mixing angle θ_{12} and the atmospheric mixing angle θ_{23} are large, while the reactor angle θ_{13} is much smaller and similar in magnitude to the Cabibbo angle which is the largest quark mixing angle.

The above peculiar flavor structure calls for a dynamical explanation, and lots of effort has been devoted to search a fundamental principle governing the flavor sector of SM. It was found that the large lepton mixing angles can be understood by extending the SM with a non-abelian discrete flavor symmetry [2–6]. In the past years, the flavor symmetry based on the modular group opened up a new promising approach to address the flavor structure of SM [7]. In this paradigm, the three generations of quarks and leptons transform non-trivially under the modular symmetry, and Yukawa couplings are assumed to be modular forms of level N which are holomorphic functions of the complex modulus τ and non-trivially transform under the action of the modular group. The modular flavor symmetry is formulated in the framework of supersymmetry to ensure the holomorphicity of the Yukawa couplings as functions of τ [7–10]. The modular symmetry provides a novel origin of the discrete flavor symmetry, and the inhomogeneous (homogeneous) finite modular group Γ_N (Γ'_N) plays the role of flavor symmetry and it is isomorphic to the (double covering of) permutation group for $N \leq 5$, e.g., $\Gamma_2 \cong S_3$, $\Gamma_3 \cong A_4$, $\Gamma_4 \cong S_4$, $\Gamma_5 \cong A_5$ [11]. The modular flavor symmetry has been intensively studied to explain the quark and lepton flavor structure in the literature, please see the recent reviews of the modular flavor symmetry [12, 13] and references therein. In the minimal modular invariant model to date, the experimental data of all lepton masses and mixing angles can be described in terms of only six free real parameters including $\Re(\tau)$ and $\Im(\tau)$ [14, 15], and fourteen parameters are invoked to accommodate the masses and mixing parameters of both quarks and leptons [15], in which the complex modulus τ is a portal so that the flavor observables in quark and lepton sectors are strongly correlated [16]. It is notable that modular symmetry models exhibit a universal behavior in the vicinity of fixed points, independently from details of the models [17, 18]. The universality also holds true in multiple moduli models and the same universal scaling behavior as the single moduli case is found [19]. The small departure of the modulus from the fixed points can produce the quark and lepton mass hierarchies [20–29]. Moreover, it was shown that the invariance under the modular group offers an axion-less solution to the strong CP problem [30–33], and the modulus τ can play the role of inflaton in the early Universe [34, 35].

The modular symmetry is ubiquitous in higher dimensional theories such as superstring theory. For instance, the complex modulus τ parameterizes the shape of the torus in torus or its orbifold compactification, and the modular symmetry is the geometrical symmetry of the compact space. The top-down approach generally gives rise to both traditional flavor symmetry and modular symmetry. This leads to the idea of eclectic flavor group which is the nontrivial product of the traditional flavor symmetry and modular symmetry [36–43]. The traditional flavor symmetry severely restricts the Kähler potential such that plenty of terms compatible with modular symmetry [44] are suppressed and the flavor universal Kähler potential can be produced at leading order. However, one has to reintroduce the notorious flavon fields besides the modulus to break the traditional flavor symmetry. The eclectic flavor group combined the advantages of modular symmetries and traditional flavor symmetries, and some predictive models with eclectic flavor symmetry have been proposed [45–49].

Recently the non-holomorphic formulation of the modular flavor symmetry has been developed in ref. [50], and lepton flavor models based on the finite modular group $\Gamma_3 \cong A_4$ have been built. The modular invariance is inherited while the assumption of holomorphicity is superseded by the harmonic condition. As a consequence, the Yukawa couplings are polyharmonic Maaß forms of level N, and supersymmetry is not a requisite anymore. Although the modular weights of non-vanishing modular forms are non-negative,[1] the weights of polyharmonic Maaß forms can be negative integers. Moreover, there exist non-holomorphic polyharmonic Maaß forms apart from the known holomorphic modular forms at weight one and weight two. This formalism can be extended to consistently include generalized CP (gCP) symmetry which constrains the phases of the coupling constants, thus the predictive power of the modular invariant models are improved further, as in the supersymmetric modular flavor symmetry [51].

The non-holomorphic modular invariant approach to the flavor problem has been explored at levels $N = 3$ [50, 52] and $N = 4$ [53], and some viable lepton flavor models were constructed.

[1]The modular form of weight zero is a constant which is τ-independent.

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In the present work, we shall consider the level $N = 5$ finite modular group $\Gamma_5 \cong A_5$ in the framework of non-holomorphic modular flavor symmetry. The three generations of the left-handed (LH) leptons are assigned to an irreducible triplet of A_5, and right-handed (RH) leptons are invariant or transform as an irreducible triplet. The non-vanishing polyharmonic Maaß forms transforming under Γ_N require even integer weights. Utilizing the level 5 polyharmonic Maaß forms with weights $k = \pm 4, \pm 2, 0$, we construct all possible lepton models invariant under the A_5 modular symmetry. We consider the most economical modular invariant lepton models, no flavon other than τ is introduced and the light neutrino masses are generated through the Weinberg operator or the type-I seesaw mechanism. Furthermore, the corresponding predictions for lepton mixing parameters and neutrino masses are discussed.

The remaining parts of this paper are organized as follows. We review the concept of modular group and the framework of non-holomorphic modular flavor symmetry in section 2. We construct the non-holomorphic lepton flavor models based on the finite modular group $\Gamma_5 \cong A_5$ in a systematical way, and the phenomenological predictions for the lepton mixing angles, CP violation phases and neutrino masses are studied in section 3. Two example models are presented and a thorough numerical analysis is performed to discuss their phenomenology in section 4. Finally we summarize the results and draw the conclusion in section 5. The irreducible representations and the related Clebsch-Gordan (CG) coefficients are given in appendix A, they are indispensable elements when constructing A_5 flavor symmetry models.

2 Non-holomorphic modular flavor symmetry

In this section, we shall give a brief review of the non-holomorphic modular flavor symmetry. The homogeneous modular group $\Gamma \equiv \mathrm{SL}(2, \mathbb{Z})$ is the group of two-by-two matrices with integer entries and unit determinant,

$$\mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}. \tag{2.1}$$

The modular group can be generated by two matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

One can check immediately that S and T satisfy the following relations

$$S^2 = -\mathbb{1}, \quad S^4 = (ST)^3 = \mathbb{1}, \quad S^2 T = T S^2, \tag{2.3}$$

which implies $(TS)^3 = \mathbb{1}$, where $\mathbb{1}$ refers to the two-dimensional unit matrix. The modular group acts on the complex upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} | \Im(\tau) > 0\}$ by linear fractional transformation

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{2.4}$$

The actions of the generators S and T on the complex modulus τ read as

$$S\tau = -\frac{1}{\tau}, \qquad T\tau = \tau + 1. \tag{2.5}$$

By applying the modular T transformation in succession, one can map any value of modulus τ in the upper half plane into the region $-1/2 \leq \Re(\tau) \leq 1/2$, and it can be further mapped into the fundamental domain,

$$\mathcal{D} = \left\{ \tau \in \mathcal{H} \middle| -\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2},\ |\tau| \geq 1 \right\} . \tag{2.6}$$

Each point in the upper half plane can be mapped into the fundamental domain $\mathcal{D}$ by some modular transformation, nevertheless no two points in the interior of $\mathcal{D}$ are related under modular group. Let N be a positive integer, the principal congruence subgroup of level N is the subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ N) \right\} . \tag{2.7}$$

Obviously we see $\Gamma(1) = \Gamma$. The quotients $\Gamma_N \equiv \Gamma / \pm \Gamma(N)$ isomorphic to the permutation groups S_3, A_4, S_4 and A_5 for $N = 2$, 3, 4 and 5 respectively [11]. Moreover, the quotients $\Gamma'_N \equiv \Gamma/\Gamma(N)$ are double covering groups of Γ_N [54].

The modular flavor symmetry is originally formulated in the framework of supersymmetry, in which modular symmetry and supersymmetry constrain the Yukawa couplings to be holomorphic modular forms [7]. Recently the formalism of the non-holomorphic modular flavor symmetry was proposed and supersymmetry is unnecessary in principle [50]. The modular invariant Lagrangian is built from matter fields and polyharmonic Maaß forms of level N. The polyharmonic Maaß forms $Y(\tau)$ of weight k at level N are functions of the complex modulus τ, and their transformation under the modular symmetry is

$$Y(\tau) \mapsto Y(\gamma\tau) = (c\tau + d)^k Y(\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) . \tag{2.8}$$

Moreover, they have to satisfy the following Laplacian condition and proper growth condition [50],

$$\begin{aligned} &\left(-4y^2 \frac{\partial}{\partial\tau} \frac{\partial}{\partial\bar{\tau}} + 2iky \frac{\partial}{\partial\bar{\tau}} \right) Y(\tau) = 0 , \\ &Y(\tau) = \mathcal{O}(y^\alpha) \quad \text{as } y \to +\infty, \text{ uniformly in } x , \end{aligned} \tag{2.9}$$

where α is some real parameter and $\tau \equiv x + iy$, and the Laplacian condition can be equivalently expressed as $\partial_\tau \left[y^k \partial_{\bar{\tau}} Y(\tau) \right] = 0$. The polyharmonic Maaß form $Y(\tau)$ can contain both holomorphic part and non-holomorphic part, and its Fourier expansion is of the following form

$$Y(\tau) = \sum_{n=0}^{+\infty} c_n^+ q_N^n + c_0^- y^{1-k} + \sum_{n=1}^{+\infty} c_n^- \,\Gamma(1-k, 4\pi n y/N)\, q_N^{-n} , \quad q_N = e^{2\pi i\tau/N} , \tag{2.10}$$

where $\Gamma(s,z)$ is the incomplete gamma function:

$$\Gamma(s,z) = \int_z^{+\infty} e^{-t} t^{s-1}\, dt . \tag{2.11}$$

It is straightforward to determine $\Gamma(1,z) = e^{-z}$. For other integer values of s, one can obtain the analytical expression of the incomplete gamma function by using the following recursion relation,

$$\Gamma(s+1,z) = s\Gamma(s,z) + z^s\, e^{-z}\,. \tag{2.12}$$

It is known that the modular weights of modular forms are non-negative integers, while the modular weights of polyharmonic Maaß forms can be generic integers which can be positive, negative or zero. Note that the product of two polyharmonic Maaß forms of weights k and k' generally is not a weight $k+k'$ polyharmonic Maaß form, because the Laplacian condition could be spoiled for $k, k' < 0$. The weight k polyharmonic Maaß forms of level N can be lifted from the weight $2-k$ modular forms of level N [50]. The non-holomorphic part is vanishing at weights $k \geq 3$ while there exist non-holomorphic polyharmonic Maaß forms at weights $k \leq 2$. The weights k polyharmonic Maaß forms of level N are invariant up to the automorphic factor $(c\tau+d)^k$ under $\Gamma(N)$, nevertheless they transform under the finite modular group Γ'_N for generic integer k and Γ_N for even k. One can choose a basis in which this transformation is described by irreducible representations of Γ'_N (or Γ_N for even k), so that the polyharmonic Maaß forms of weight k and level N can be arranged into irreducible multiplets of Γ'_N (or Γ_N) [50].

The modular symmetry can strongly constrain the Yukawa couplings and the fermion mass structure. The transformation property of the matter field is fully specified by its modular weight and its transformation under the finite modular group Γ_N or Γ'_N. To be more specific, adopting the two-component spinor notation, the Weyl spinors ψ, ψ^c and the Higgs field H transform under the modular group as

$$\begin{aligned}
\psi(x) &\mapsto (c\tau+d)^{-k_\psi}\rho_\psi(\gamma)\psi(x)\,,\\
\psi^c(x) &\mapsto (c\tau+d)^{-k_{\psi^c}}\rho_{\psi^c}(\gamma)\psi^c(x)\,,\\
H(x) &\mapsto (c\tau+d)^{-k_H}\rho_H(\gamma)H(x)\,,
\end{aligned} \tag{2.13}$$

where the modular weights k_ψ, k_{ψ^c}, k_H are integers, $\rho_\psi(\gamma)$, $\rho_{\psi^c}(\gamma)$ and $\rho_H(\gamma)$ are unitary representations of Γ_N (or Γ'_N). Then the Yukawa interaction can be written as

$$\mathcal{L}_Y = -Y^{(k_Y)}(\tau)\psi^c\psi H + \text{h.c.}\,, \tag{2.14}$$

where the Higgs field H should be its complex conjugate H^* for the down-type quarks and charged leptons, and we drop the gauge indices. Requiring modular invariance of the above Lagrangian, the field coupling $Y^{(k_Y)}(\tau)$ should be a multiplet of weight k_Y and level N polyharmonic Maaß forms, and its modular transformation is given by

$$Y^{(k_Y)}(\tau) \mapsto Y^{(k_Y)}(\gamma\tau) = (c\tau+d)^{k_Y}\rho_Y(\gamma)Y^{(k_Y)}(\tau)\,, \tag{2.15}$$

where ρ_Y is a representation of Γ'_N (or Γ_N for even k_Y), and the modular weight k_Y and the representation ρ_Y should obey the following conditions

$$k_Y = k_{\psi^c} + k_\psi + k_H\,, \qquad \rho_Y \otimes \rho_{\psi^c} \otimes \rho_\psi \otimes \rho_H \ni \mathbf{1}\,. \tag{2.16}$$

Weight	Polyharmonic Maaß forms $Y_{\boldsymbol{r}}^{(k)}$
$k=-4$	$Y_{\mathbf{1}}^{(-4)}\,,\,Y_{\mathbf{3}}^{(-4)}\,,\,Y_{\mathbf{3'}}^{(-4)}\,,\,Y_{\mathbf{5}}^{(-4)}$
$k=-2$	$Y_{\mathbf{1}}^{(-2)}\,,\,Y_{\mathbf{3}}^{(-2)}\,,\,Y_{\mathbf{3'}}^{(-2)}\,,\,Y_{\mathbf{5}}^{(-2)}$
$k=0$	$Y_{\mathbf{1}}^{(0)}\,,\,Y_{\mathbf{3}}^{(0)}\,,\,Y_{\mathbf{3'}}^{(0)}\,,\,Y_{\mathbf{5}}^{(0)}$
$k=2$	$Y_{\mathbf{1}}^{(2)}\,,\,Y_{\mathbf{3}}^{(2)}\,,\,Y_{\mathbf{3'}}^{(2)}\,,\,Y_{\mathbf{5}}^{(2)}$
$k=4$	$Y_{\mathbf{1}}^{(4)}\,,\,Y_{\mathbf{3}}^{(4)}\,,\,Y_{\mathbf{3'}}^{(4)}\,,\,Y_{\mathbf{4}}^{(4)}\,,\,Y_{\mathbf{5}I}^{(4)},\,Y_{\mathbf{5}II}^{(4)}$

Table 1. Polyharmonic Maaß form multiplets of level 5 and weights $k=\pm4,\pm2,\,0$, where the subscript $\boldsymbol{r}$ denotes the transformation property under finite group A_5. Here $Y_{\mathbf{5}I}^{(k)}$ and $Y_{\mathbf{5}II}^{(k)}$ stand for two linear independent quintuplets of Maaß forms at weight 4. The explicit forms of these polyharmonic Maaß form multiplets can be found in ref. [50].

The gCP symmetry can be consistently included in the above framework of non-holomorphic modular symmetry [50]. It acts on a field multiplet $\varphi(x)$ as $\varphi(x)\mapsto X_{\boldsymbol{r}}\varphi^*(x_{\mathcal{P}})$ with $x_{\mathcal{P}}=(t,-\mathbf{x})$. It turns out that the gCP transformation has the canonical form $X_{\boldsymbol{r}}=\mathbb{1}_{\boldsymbol{r}}$ in the basis where both modular generators S and T are represented by unitary and symmetric matrices [51]. As a result, the gCP invariance enforces the coupling constants accompanying each invariant singlet in the Lagrangian to be real. In the present work, we consider the finite modular group $\Gamma_5\cong A_5$ of level $N=5$. We shall employ the representation basis for the A_5 generators S and T shown in appendix A. One see that T is represented by diagonal matrices in different A_5 irreducible representations, and the representation matrices of S are real and symmetric. Hence the gCP symmetry amounts to requiring real couplings in our case. In the framework of non-holomorphic modular flavor symmetry, modular invariance constrains the Yukawa couplings to be polyharmonic Maaß forms of level 5. The explicit forms of their Fourier expansion have been presented in ref. [50]. Here we do not show these lengthy expressions and only summarize the polyharmonic Maaß form multiplets of level $N=5$ and weights $k=\pm4,\pm2,\,0$ in table 1.

3 General model building

In this section, we shall perform a systematic classification of all minimal lepton mass models based on the finite modular symmetry $\Gamma_5\cong A_5$. It is realized from non-supersymmetric case with the polyharmonic Maaß forms of level $N=5$ which can be decomposed into the irreducible multiplets of the finite group A_5 [50], as is shown in table 1. In the present work, neutrinos are assumed to be Majorana particles and their masses are considered to arise from the Weinberg operator or the type I seesaw mechanism with three RH neutrinos. No flavon other than complex modulus τ will be introduced, and the modular symmetry is broken when τ obtains a vacuum expectation value. The Higgs doublet field H is assumed to be A_5 singlet with vanishing modular weight. We assume that the three generations of LH lepton doublets $L=(L_1,L_2,L_3)^T$ with weight k_L and the three RH neutrinos $N^c=(N_1^c,N_2^c,N_3^c)^T$

Cases	$C_1\,(C'_1)$	$C_2\,(C'_2)$	$C_3\,(C'_3)$	$C_4\,(C'_4)$	$C_5\,(C'_5)$
$(k_1+k_L, k_2+k_L, k_3+k_L)$	$(-4,-2,0)$	$(-4,-2,2)$	$(-4,-2,4)$	$(-4,0,2)$	$(-4,0,4)$
Cases	$C_6\,(C'_6)$	$C_7\,(C'_7)$	$C_8\,(C'_8)$	$C_9\,(C'_9)$	$C_{10}\,(C'_{10})$
$(k_1+k_L, k_2+k_L, k_3+k_L)$	$(-4,2,4)$	$(-2,0,2)$	$(-2,0,4)$	$(-2,2,4)$	$(0,2,4)$

Table 2. The 20 different possible cases for the assignments that the LH lepton fields L transform as a triplet under A_5 and the RH charged leptons are A_5 singlets. The ten cases C_i (C'_i) are for assignment $L \sim \mathbf{3}$ ($L \sim \mathbf{3'}$) and $E^c_{1,2,3} \sim \mathbf{1}$ with different modular weights, and the corresponding charged lepton mass matrices take the form in eq. (3.2) (eq. (3.4)).

with weight k_N in the type I seesaw models transform as triplet $\mathbf{3}$ or $\mathbf{3'}$. The three RH charged leptons $E^c_{1,2,3}$ are assigned to singlet $\mathbf{1}$ or a triplet $\mathbf{3}$ or $\mathbf{3'}$ under A_5 modular group. For the former assignment, they are distinguished by their modular weights $k_{1,2,3}$. For the latter assignment, the modular weight of $E^c = (E^c_1, E^c_2, E^c_3)^T$ is denoted as k_E. For each representation assignment, there are in principle infinitely possible weight assignments for lepton fields, and the number of independent couplings of the charged lepton mass terms and neutrino mass terms generally increases with the weight of the involved modular forms. We shall consider the polyharmonic Maaß forms of even weight $k = -4$ to $k = 4$, and employ potentially the lowest weight modular forms as much as possible in order to reduce free parameters.

3.1 Charged lepton sector

Firstly, let us consider the assignments that the three RH charged leptons are all singlet $\mathbf{1}$ of A_5. Then if the LH lepton fields L are embedded into the triplet $\mathbf{3}$ (or $\mathbf{3'}$), modular form multiplet in the representation $\mathbf{3}$ (or $\mathbf{3'}$) should be invoked in the charged lepton mass terms. As a consequence, there are only two different assignments for lepton fields and we will list the Lagrangian and charged lepton mass matrices for the two cases in the following.

(i) $L \sim \mathbf{3}$ and $E^c_{1,2,3} \sim \mathbf{1}$

The Lagrangian for the charged lepton Yukawa coupling reads as

$$-\mathcal{L}_e = \alpha \left(E^c_1 L Y^{(k_1+k_L)}_{\mathbf{3}} H^*\right)_{\mathbf{1}} + \beta \left(E^c_2 L Y^{(k_2+k_L)}_{\mathbf{3}} H^*\right)_{\mathbf{1}} + \gamma \left(E^c_3 L Y^{(k_3+k_L)}_{\mathbf{3}} H^*\right)_{\mathbf{1}} + \text{h.c.}, \tag{3.1}$$

where $Y^{(k_i+k_L)}_{\mathbf{3}}$ ($i = 1, 2, 3$) stand for level 5 polyharmonic Maaß forms transforming as $\mathbf{3}$ under A_5 up to automorphy factor. Notice that if there are several linearly independent polyharmonic Maaß form multiplets in the same representation $\mathbf{3}$ at a given weight $k_i + k_L$, their contributions to the Lagrangian are of similar form which can be straightforwardly read out from the general results in the following. Guided by the principle of minimality and simplicity, the polyharmonic Maaß forms of those weights which only involve one independent triplet polyharmonic Maaß form in the representation $\mathbf{3}$ are imposed in the charged lepton sector. From table 1, we find that the possible values of the modular weights $k_i + k_L$ are $\pm 4, \pm 2$ and 0. Notice that any two rows of a charged lepton mass matrix should not be proportional to each other, otherwise at least one charged lepton would be massless. Furthermore, the permutations of any two rows of the charged lepton mass matrix amounts a redefinition of the RH charged lepton fields, and thus the predictions for charged lepton masses and lepton mixing matrix are unchanged. Without loss of generality, we take $k_1+k_L<k_2+k_L<k_3+k_L$. In the end, we find there are totally 10 possible charged lepton mass matrices from the 10 possible independent assignments of the values of k_i+k_L. The 10 cases are labelled as C_j $(j=1,2,\cdots,10)$ and summarized in table 2. The three terms in eq. (3.1) have similar form, and they contribute to three rows of the charged lepton mass matrix, respectively. We can read out the charged lepton mass matrices for the 10 cases as follows

$$M_e(k_1+k_L,k_2+k_L,k_3+k_L)=\begin{pmatrix}\alpha Y^{(k_1+k_L)}_{\mathbf{3},1} & \alpha Y^{(k_1+k_L)}_{\mathbf{3},3} & \alpha Y^{(k_1+k_L)}_{\mathbf{3},2}\\ \beta Y^{(k_2+k_L)}_{\mathbf{3},1} & \beta Y^{(k_2+k_L)}_{\mathbf{3},3} & \beta Y^{(k_2+k_L)}_{\mathbf{3},2}\\ \gamma Y^{(k_3+k_L)}_{\mathbf{3},1} & \gamma Y^{(k_3+k_L)}_{\mathbf{3},3} & \gamma Y^{(k_3+k_L)}_{\mathbf{3},2}\end{pmatrix}v\,, \tag{3.2}$$

where the charged lepton mass matrix M_e is given in the right-left basis $E^c\,M_e\,L$ with $v=\langle H\rangle$, and we denote $Y^{(k_i+k_L)}_{\mathbf{3}}\equiv(Y^{(k_i+k_L)}_{\mathbf{3},1},Y^{(k_i+k_L)}_{\mathbf{3},2},Y^{(k_i+k_L)}_{\mathbf{3},3})^T$. For all the 10 cases, the phases of all the three couplings α, β and γ can be absorbed into the RH charged leptons E^c_1, E^c_2 and E^c_3 respectively, and they can be taken to be real.

(ii) $L\sim\mathbf{3'}$ and $E^c_{1,2,3}\sim\mathbf{1}$

Similarly, the charged lepton mass terms of the Lagrangian are

$$-\mathcal{L}_e=\alpha\left(E^c_1LY^{(k_1+k_L)}_{\mathbf{3'}}H^*\right)_{\mathbf{1}}+\beta\left(E^c_2LY^{(k_2+k_L)}_{\mathbf{3'}}H^*\right)_{\mathbf{1}}+\gamma\left(E^c_3LY^{(k_3+k_L)}_{\mathbf{3'}}H^*\right)_{\mathbf{1}}+\text{h.c.}\,, \tag{3.3}$$

and the corresponding lepton mass matrix takes the following form

$$M_e(k_1+k_L,k_2+k_L,k_3+k_L)=\begin{pmatrix}\alpha Y^{(k_1+k_L)}_{\mathbf{3'},1} & \alpha Y^{(k_1+k_L)}_{\mathbf{3'},3} & \alpha Y^{(k_1+k_L)}_{\mathbf{3'},2}\\ \beta Y^{(k_2+k_L)}_{\mathbf{3'},1} & \beta Y^{(k_2+k_L)}_{\mathbf{3'},3} & \beta Y^{(k_2+k_L)}_{\mathbf{3'},2}\\ \gamma Y^{(k_3+k_L)}_{\mathbf{3'},1} & \gamma Y^{(k_3+k_L)}_{\mathbf{3'},3} & \gamma Y^{(k_3+k_L)}_{\mathbf{3'},2}\end{pmatrix}v\,, \tag{3.4}$$

where $k_i+k_L=\pm4,\pm2,0$ with $k_1+k_L<k_2+k_L<k_3+k_L$, and parameters α, β and γ can be taken to be real. There are also 10 independent weight assignments which are labelled as C'_j $(j=1,2,\cdots,10)$ and are given in table 2.

In the following, we proceed to consider the assignments that the three RH charged leptons E^c also transform as a triplet $\mathbf{3}$ or $\mathbf{3'}$ under A_5. Note that LE^c decomposes as $\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{3}_{\boldsymbol{A}}\oplus\mathbf{5}_{\boldsymbol{S}}$, $\mathbf{3'}\otimes\mathbf{3'}=\mathbf{1}\oplus\mathbf{3'}_{\boldsymbol{A}}\oplus\mathbf{5}_{\boldsymbol{S}}$ or $\mathbf{3}\otimes\mathbf{3'}=\mathbf{4}\oplus\mathbf{5}$, using the CG coefficients shown in table 11, we can write out the modular invariant charged lepton mass terms and extract the mass matrix for each assignment as following.

(iii) $L\sim\mathbf{3}$ and $E^c\sim\mathbf{3}$

In this case, the most general Lagrangian $\mathcal{L}_e$ for the charged lepton masses is given by:

$$-\mathcal{L}_e=\alpha\left(E^cLY^{(k_E+k_L)}_{\mathbf{1}}H^*\right)_{\mathbf{1}}+\beta\left(E^cLY^{(k_E+k_L)}_{\mathbf{3}}H^*\right)_{\mathbf{1}}+\gamma\left(E^cLY^{(k_E+k_L)}_{\mathbf{5}}H^*\right)_{\mathbf{1}}+\text{h.c.}\,, \tag{3.5}$$

where the coupling α is real, and the couplings β and γ are generally complex in the case that gCP symmetry isn't considered. If the modular group A_5 is extended to combine with the gCP symmetry, all the couplings are further constrained to be real in our working basis. For the simplest weight assignment $w = k_E + k_L = -4, -2, 0, 2$, the charged lepton mass matrix reads as,

$$M_e(w) = \begin{pmatrix} \alpha Y_{\mathbf{1}}^{(w)} + 2\gamma Y_{\mathbf{5},1}^{(w)} & -\beta Y_{\mathbf{3},3}^{(w)} - \sqrt{3}\gamma Y_{\mathbf{5},5}^{(w)} & \beta Y_{\mathbf{3},2}^{(w)} - \sqrt{3}\gamma Y_{\mathbf{5},2}^{(w)} \\ \beta Y_{\mathbf{3},3}^{(w)} - \sqrt{3}\gamma Y_{\mathbf{5},5}^{(w)} & \sqrt{6}\gamma Y_{\mathbf{5},4}^{(w)} & \alpha Y_{\mathbf{1}}^{(w)} - \beta Y_{\mathbf{3},1}^{(w)} - \gamma Y_{\mathbf{5},1}^{(w)} \\ -\beta Y_{\mathbf{3},2}^{(w)} - \sqrt{3}\gamma Y_{\mathbf{5},2}^{(w)} & \alpha Y_{\mathbf{1}}^{(w)} + \beta Y_{\mathbf{3},1}^{(w)} - \gamma Y_{\mathbf{5},1}^{(w)} & \sqrt{6}\gamma Y_{\mathbf{5},3}^{(w)} \end{pmatrix} v \,. \tag{3.6}$$

Note that the above mass matrix involves two more parameters than $C_j(C'_j)$ in table 2 when gCP symmetry is not imposed. Generally one needs to tune the values of couplings α, β, γ to reproduce the charged lepton mass hierarchy for this type of assignment,

(iv) $L \sim \mathbf{3'}$ and $E^c \sim \mathbf{3'}$

Similar to previous case, the charged lepton Lagrangian invariant under the A_5 modular symmetry reads as

$$-\mathcal{L}_e = \alpha \left(E^c L Y_{\mathbf{1}}^{(k_E+k_L)} H^*\right)_{\mathbf{1}} + \beta \left(E^c L Y_{\mathbf{3'}}^{(k_E+k_L)} H^*\right)_{\mathbf{1}} + \gamma \left(E^c L Y_{\mathbf{5}}^{(k_E+k_L)} H^*\right)_{\mathbf{1}} + \text{h.c.}\,, \tag{3.7}$$

which leads to the following charged lepton mass matrix

$$M_e(w) = \begin{pmatrix} \alpha Y_{\mathbf{1}}^{(w)} + 2\gamma Y_{\mathbf{5},1}^{(w)} & -\beta Y_{\mathbf{3'},3}^{(w)} - \sqrt{3}\gamma Y_{\mathbf{5},4}^{(w)} & \beta Y_{\mathbf{3'},2}^{(w)} - \sqrt{3}\gamma Y_{\mathbf{5},3}^{(w)} \\ \beta Y_{\mathbf{3'},3}^{(w)} - \sqrt{3}\gamma Y_{\mathbf{5},4}^{(w)} & \sqrt{6}\gamma Y_{\mathbf{5},2}^{(w)} & \alpha Y_{\mathbf{1}}^{(w)} - \beta Y_{\mathbf{3'},1}^{(w)} - \gamma Y_{\mathbf{5},1}^{(w)} \\ -\beta Y_{\mathbf{3'},2}^{(w)} - \sqrt{3}\gamma Y_{\mathbf{5},3}^{(w)} & \alpha Y_{\mathbf{1}}^{(w)} + \beta Y_{\mathbf{3'},1}^{(w)} - \gamma Y_{\mathbf{5},1}^{(w)} & \sqrt{6}\gamma Y_{\mathbf{5},5}^{(w)} \end{pmatrix} v \,, \tag{3.8}$$

with the modular weight $w = k_E + k_L$.

(v) $L \sim \mathbf{3}$ and $E^c \sim \mathbf{3'}$

From the multiplication rule $\mathbf{3} \otimes \mathbf{3'} = \mathbf{4} \oplus \mathbf{5}$, we find that the modular invariant Lagrangian $\mathcal{L}_e$ for the charged lepton masses can be written as:

$$-\mathcal{L}_e = \alpha \left(E^c L Y_{\mathbf{5}}^{(k_E+k_L)} H^*\right)_{\mathbf{1}} + \beta \left(E^c L Y_{\mathbf{4}}^{(k_E+k_L)} H^*\right)_{\mathbf{1}} + \text{h.c.}\,. \tag{3.9}$$

When the modular weight $w = k_E + k_L$ is taken to be -4, -2, 0 and 2, the charged lepton mass matrix is given by

$$M_e(w) = \alpha \begin{pmatrix} \sqrt{3} Y_{\mathbf{5},1}^{(w)} & Y_{\mathbf{5},5}^{(w)} & Y_{\mathbf{5},2}^{(w)} \\ Y_{\mathbf{5},4}^{(w)} & -\sqrt{2} Y_{\mathbf{5},3}^{(w)} & -\sqrt{2} Y_{\mathbf{5},5}^{(w)} \\ Y_{\mathbf{5},3}^{(w)} & -\sqrt{2} Y_{\mathbf{5},2}^{(w)} & -\sqrt{2} Y_{\mathbf{5},4}^{(w)} \end{pmatrix} v \,, \tag{3.10}$$

which only involves one real input parameter α. The reason is that there is no quartet modular form at these modular weights. It turns out to be very difficult to accommodate the three charged lepton mass with only one coupling α. For $k_E + k_L = 4$, the charged lepton mass matrix is given by

$$M_e = \left[\begin{pmatrix} \sqrt{3}\alpha_1 Y^{(4)}_{\mathbf{5}I,1} + \sqrt{3}\alpha_2 Y^{(4)}_{\mathbf{5}II,1} & \alpha_1 Y^{(4)}_{\mathbf{5}I,5} + \alpha_2 Y^{(4)}_{\mathbf{5}II,5} & \alpha_1 Y^{(4)}_{\mathbf{5}I,2} + \alpha_2 Y^{(4)}_{\mathbf{5}II,2} \\ \alpha_1 Y^{(4)}_{\mathbf{5}I,4} + \alpha_2 Y^{(4)}_{\mathbf{5}II,4} & -\sqrt{2}\alpha_1 Y^{(4)}_{\mathbf{5}I,3} - \sqrt{2}\alpha_2 Y^{(4)}_{\mathbf{5}II,3} & -\sqrt{2}\alpha_1 Y^{(4)}_{\mathbf{5}I,5} - \sqrt{2}\alpha_2 Y^{(4)}_{\mathbf{5}II,5} \\ \alpha_1 Y^{(4)}_{\mathbf{5}I,3} + \alpha_2 Y^{(4)}_{\mathbf{5}II,3} & -\sqrt{2}\alpha_1 Y^{(4)}_{\mathbf{5}I,2} - \sqrt{2}\alpha_2 Y^{(4)}_{\mathbf{5}II,2} & -\sqrt{2}\alpha_1 Y^{(4)}_{\mathbf{5}I,4} - \sqrt{2}\alpha_2 Y^{(4)}_{\mathbf{5}II,4} \end{pmatrix}\right.$$
$$\left. +\beta \begin{pmatrix} 0 & \sqrt{2} Y^{(4)}_{\mathbf{4},4} & \sqrt{2} Y^{(4)}_{\mathbf{4},1} \\ -\sqrt{2} Y^{(4)}_{\mathbf{4},3} & -Y^{(4)}_{\mathbf{4},2} & Y^{(4)}_{\mathbf{4},4} \\ -\sqrt{2} Y^{(4)}_{\mathbf{4},2} & Y^{(4)}_{\mathbf{4},1} & -Y^{(4)}_{\mathbf{4},3} \end{pmatrix}\right] v\,, \tag{3.11}$$

where parameter α_1 can taken to be real, while parameters α_2 and β are generally complex. This mass matrix also involves more input parameters than the mass matrices of $C_i(C'_i)$ in table 2.

(vi) $L \sim \mathbf{3'}$ and $E^c \sim \mathbf{3}$

For this assignment, the charged lepton mass matrices are the transpose of those in case (v).

In general, the viable charged lepton mass matrices for triplet RH charged lepton assignment involve more free parameters than those for three singlet RH charged leptons when gCP symmetry is not imposed. Hence we shall analyze the scenario that the three RH charged leptons are A_5 singlets in the following.

3.2 Neutrino sector

In the present work, the neutrinos are assumed to be Majorana particles. The light neutrino masses are described by the effective Weinberg operator or arise from the type I seesaw mechanism. Let us first consider that the general form of the modular invariant Majorana neutrino mass terms are described by Weinberg operator. Under the assumption that the LH lepton doublets L transform as a triplet $\mathbf{3}$ or $\mathbf{3'}$ under A_5, the LL decomposes as $\mathbf{3}\otimes\mathbf{3} = \mathbf{1}\oplus\mathbf{3_A}\oplus\mathbf{5_S}$ or $\mathbf{3'}\otimes\mathbf{3'} = \mathbf{1}\oplus\mathbf{3'_A}\oplus\mathbf{5_S}$, and the anti-symmetric contractions $(LL)_{\mathbf{3_A}}$ for $L\sim\mathbf{3}$ and $(LL)_{\mathbf{3'_A}}$ for $L\sim\mathbf{3'}$ are vanishing. Thus the singlet polyharmonic Maaß form $Y^{(2k_L)}_{\mathbf{1}}$ or the quintuplet Maaß form $Y^{(2k_L)}_{\mathbf{5}}$ is necessary to obtain nonzero neutrino masses. Then the most general modular invariant Lagrangian for neutrino masses can be written as

$$-\mathcal{L}_\nu = \frac{g_1}{2\Lambda}\left(LLHHY^{(2k_L)}_{\mathbf{1}}\right)_{\mathbf{1}} + \frac{g_2}{2\Lambda}\left(LLHHY^{(2k_L)}_{\mathbf{5}}\right)_{\mathbf{1}} + \text{h.c.}\,. \tag{3.12}$$

For the sake of simplicity, we are concerned with the modular weight $k_L = -2, -1, 0, 1$. Then there are 8 independent cases differing in the representation assignment and weight of L, and they are labelled as W_i and W'_i $(i = 1, 2, 3, 4)$ for $L\sim\mathbf{3}$ and $L\sim\mathbf{3'}$ respectively, as summarized in table 3. Applying the decomposition rules of the finite modular group A_5 given

Weinberg operator				
Cases	$W_1(W_1')$	$W_2(W_2')$	$W_3(W_3')$	$W_4(W_4')$
k_L	-2	-1	0	1
Seesaw				
Cases	$D_1\,(D_1')$	$D_2\,(D_2')$	$D_3\,(D_3')$	$D_4\,(D_4')$
k_L+k_N	-4	-2	0	2
Cases	$N_1(N_1')$	$N_2(N_2')$	$N_3(N_3')$	$N_4(N_4')$
k_N	-2	-1	0	1

Table 3. The values of the modular weights k_L and k_N of the LH lepton L and RH neutrinos N^c. The cases $W_i(W_i')$ are for the assignment $L\sim\mathbf{3}$ ($\mathbf{3'}$) with different value of k_L, and the light neutrino masses are described by the Weinberg operator. The Majorana mass terms of $N_i(N_i')$ are for the assignment $N^c\sim\mathbf{3}$ ($\mathbf{3'}$) with different weight k_N, and the Dirac mass terms of D_i (D_i') correspond to the assignments $(L,N^c)\sim(\mathbf{3},\mathbf{3'})$ ($(L,N^c)\sim(\mathbf{3'},\mathbf{3})$) with different weight k_L+k_N. The light neutrino mass matrices of $W_i(W_i')$ are given in eq. (3.13). The Majorana mass matrices of cases $N_i(N_i')$ can be obtained from M_N in eqs. (3.15), (3.16). The Dirac mass matrices for cases D_i (D_i') are shown in eq. (3.17).

in table 11, we find that $\mathcal{L}_\nu$ in eq. (3.12) gives rise to the following light neutrino mass matrices

$$M_\nu(k_L)=\frac{v^2}{\Lambda}\begin{pmatrix} g_1Y_{\mathbf{1}}^{(2k_L)}+2g_2Y_{\mathbf{5},1}^{(2k_L)} & -\sqrt{3}g_2Y_{\mathbf{5},5}^{(2k_L)} & -\sqrt{3}g_2Y_{\mathbf{5},2}^{(2k_L)} \\ -\sqrt{3}g_2Y_{\mathbf{5},5}^{(2k_L)} & \sqrt{6}g_2Y_{\mathbf{5},4}^{(2k_L)} & g_1Y_{\mathbf{1}}^{(2k_L)}-g_2Y_{\mathbf{5},1}^{(2k_L)} \\ -\sqrt{3}g_2Y_{\mathbf{5},2}^{(2k_L)} & g_1Y_{\mathbf{1}}^{(2k_L)}-g_2Y_{\mathbf{5},1}^{(2k_L)} & \sqrt{6}g_2Y_{\mathbf{5},3}^{(2k_L)} \end{pmatrix},$$

$$M_\nu(k_L)=\frac{v^2}{\Lambda}\begin{pmatrix} g_1Y_{\mathbf{1}}^{(2k_L)}+2g_2Y_{\mathbf{5},1}^{(2k_L)} & -\sqrt{3}g_2Y_{\mathbf{5},4}^{(2k_L)} & -\sqrt{3}g_2Y_{\mathbf{5},3}^{(2k_L)} \\ -\sqrt{3}g_2Y_{\mathbf{5},4}^{(2k_L)} & \sqrt{6}g_2Y_{\mathbf{5},2}^{(2k_L)} & g_1Y_{\mathbf{1}}^{(2k_L)}-g_2Y_{\mathbf{5},1}^{(2k_L)} \\ -\sqrt{3}g_2Y_{\mathbf{5},3}^{(2k_L)} & g_1Y_{\mathbf{1}}^{(2k_L)}-g_2Y_{\mathbf{5},1}^{(2k_L)} & \sqrt{6}g_2Y_{\mathbf{5},5}^{(2k_L)} \end{pmatrix}, \qquad (3.13)$$

for $L\sim\mathbf{3}$ and $L\sim\mathbf{3'}$ respectively, where the phase of parameter g_1 can be set to be zero, while the phase of g_2 can not be removed by a field redefinition. Both g_1 and g_2 would be real if gCP symmetry is included.

When the neutrino masses are generated by the type I seesaw mechanism, the three RH neutrinos N^c are assigned to triplet $\mathbf{3}$ or $\mathbf{3'}$ of A_5. Then the Lagrangian for the neutrino masses can be generally written as

$$-\mathcal{L}_\nu=\sum_{\boldsymbol{r_1}}g\left(N^cLY_{\boldsymbol{r_1}}^{(k_L+k_N)}H\right)_{\mathbf{1}}+\sum_{\boldsymbol{r_2}}\frac{h}{2}\Lambda\left(N^cN^cY_{\boldsymbol{r_2}}^{(2k_N)}\right)_{\mathbf{1}}+\text{h.c.}\,, \qquad (3.14)$$

where the polyharmonic Maaß form multiplets $Y_{\boldsymbol{r_1}}^{(k_L+k_N)}$ and $Y_{\boldsymbol{r_2}}^{(2k_N)}$ are required to ensure modular invariance, and the explicit forms of them are determined by the weights and representation assignments of L and N^c. Similar to the Weinberg operator in eq. (3.12), the representation $\boldsymbol{r_2}$ can be $\mathbf{1}$ and $\mathbf{5}$ of A_5. We are concerned with the models with minimal number of parameters, consequently we focus on the low weights $k_N=-2,-1,0,1$. Then the 8 possible assignments of N^c and its weight are labelled as N_i and N_i' $(i = 1, 2, 3, 4)$ which are summarized in table 3, and the corresponding mass matrices for the Majorana neutrinos N^c are given by

$$M_N(k_N) = \Lambda \begin{pmatrix} h_1 Y_{\mathbf{1}}^{(2k_N)} + 2h_2 Y_{\mathbf{5},1}^{(2k_N)} & -\sqrt{3}h_2 Y_{\mathbf{5},5}^{(2k_N)} & -\sqrt{3}h_2 Y_{\mathbf{5},2}^{(2k_N)} \\ -\sqrt{3}h_2 Y_{\mathbf{5},5}^{(2k_N)} & \sqrt{6}h_2 Y_{\mathbf{5},4}^{(2k_N)} & h_1 Y_{\mathbf{1}}^{(2k_N)} - h_2 Y_{\mathbf{5},1}^{(2k_N)} \\ -\sqrt{3}h_2 Y_{\mathbf{5},2}^{(2k_N)} & h_1 Y_{\mathbf{1}}^{(2k_N)} - h_2 Y_{\mathbf{5},1}^{(2k_N)} & \sqrt{6}h_2 Y_{\mathbf{5},3}^{(2k_N)} \end{pmatrix}, \tag{3.15}$$

for $N^c \sim \mathbf{3}$ and

$$M_N(k_N) = \Lambda \begin{pmatrix} h_1 Y_{\mathbf{1}}^{(2k_N)} + 2h_2 Y_{\mathbf{5},1}^{(2k_N)} & -\sqrt{3}h_2 Y_{\mathbf{5},4}^{(2k_N)} & -\sqrt{3}h_2 Y_{\mathbf{5},3}^{(2k_N)} \\ -\sqrt{3}h_2 Y_{\mathbf{5},4}^{(2k_N)} & \sqrt{6}h_2 Y_{\mathbf{5},2}^{(2k_N)} & h_1 Y_{\mathbf{1}}^{(2k_N)} - h_2 Y_{\mathbf{5},1}^{(2k_N)} \\ -\sqrt{3}h_2 Y_{\mathbf{5},3}^{(2k_N)} & h_1 Y_{\mathbf{1}}^{(2k_N)} - h_2 Y_{\mathbf{5},1}^{(2k_N)} & \sqrt{6}h_2 Y_{\mathbf{5},5}^{(2k_N)} \end{pmatrix}, \tag{3.16}$$

for $N^c \sim \mathbf{3'}$.

The Dirac neutrino mass terms are similar to the charged lepton mass terms discussed in cases (iii) to (vi) of section 3.1. Firstly, let us consider the assignments that L and N^c transform as different triplets of A_5, then $Y_{\boldsymbol{r_1}}^{(k_L+k_N)}$ must be quartet $\mathbf{4}$ or quintuplet $\mathbf{5}$. When the modular weight $w = k_L + k_N$ takes the values of -4, -2, 0 and 2, only the quintuplet Maaß form $Y_{\mathbf{5}}^{(k_L+k_N)}$ contributes to the Dirac neutrino mass matrix reading as

$$\begin{aligned} M_D(w) &= g \begin{pmatrix} \sqrt{3}Y_{\mathbf{5},1}^{(w)} & Y_{\mathbf{5},5}^{(w)} & Y_{\mathbf{5},2}^{(w)} \\ Y_{\mathbf{5},4}^{(w)} & -\sqrt{2}Y_{\mathbf{5},3}^{(w)} & -\sqrt{2}Y_{\mathbf{5},5}^{(w)} \\ Y_{\mathbf{5},3}^{(w)} & -\sqrt{2}Y_{\mathbf{5},2}^{(w)} & -\sqrt{2}Y_{\mathbf{5},4}^{(w)} \end{pmatrix} v\,, \quad \text{for} \quad (L, N^c) \sim (\mathbf{3}, \mathbf{3'})\,, \\ M_D(w) &= g \begin{pmatrix} \sqrt{3}Y_{\mathbf{5},1}^{(w)} & Y_{\mathbf{5},4}^{(w)} & Y_{\mathbf{5},3}^{(w)} \\ Y_{\mathbf{5},5}^{(w)} & -\sqrt{2}Y_{\mathbf{5},3}^{(w)} & -\sqrt{2}Y_{\mathbf{5},2}^{(w)} \\ Y_{\mathbf{5},2}^{(w)} & -\sqrt{2}Y_{\mathbf{5},5}^{(w)} & -\sqrt{2}Y_{\mathbf{5},4}^{(w)} \end{pmatrix} v\,, \quad \text{for} \quad (L, N^c) \sim (\mathbf{3'}, \mathbf{3})\,, \end{aligned} \tag{3.17}$$

which depend on a unique overall coupling g. The 8 possible Dirac neutrino mass matrices in eq. (3.17) correspond to the 8 possible independent cases D_i and D_i' listed in table 3.

If both L and N^c are assigned to a same triplet $\mathbf{3}$ (or $\mathbf{3'}$) of A_5, the polyharmonic Maaß form $Y_{\boldsymbol{r_1}}^{(k_L+k_N)}$ can be A_5 singlet $\mathbf{1}$, triplet $\mathbf{3}$ (or $\mathbf{3'}$) and quintuplet $\mathbf{5}$. Since all these three modular multiplets are present at every modular weight, the Dirac neutrino mass matrix involves at least three couplings and the corresponding Dirac neutrino mass matrix is given by

$$M_D(w) = \begin{pmatrix} g_1 Y_{\mathbf{1}}^{(w)} + 2g_3 Y_{\mathbf{5},1}^{(w)} & -g_2 Y_{\mathbf{3},3}^{(w)} - \sqrt{3}g_3 Y_{\mathbf{5},5}^{(w)} & g_2 Y_{\mathbf{3},2}^{(w)} - \sqrt{3}g_3 Y_{\mathbf{5},2}^{(w)} \\ g_2 Y_{\mathbf{3},3}^{(w)} - \sqrt{3}g_3 Y_{\mathbf{5},5}^{(w)} & \sqrt{6}g_3 Y_{\mathbf{5},4}^{(w)} & g_1 Y_{\mathbf{1}}^{(w)} - g_2 Y_{\mathbf{3},1}^{(w)} - g_3 Y_{\mathbf{5},1}^{(w)} \\ -g_2 Y_{\mathbf{3},2}^{(w)} - \sqrt{3}g_3 Y_{\mathbf{5},2}^{(w)} & g_1 Y_{\mathbf{1}}^{(w)} + g_2 Y_{\mathbf{3},1}^{(w)} - g_3 Y_{\mathbf{5},1}^{(w)} & \sqrt{6}g_3 Y_{\mathbf{5},3}^{(w)} \end{pmatrix} v\,, \tag{3.18}$$

for $L, N^c \sim \mathbf{3}$ and

$$M_D(w) = \begin{pmatrix} g_1 Y_{\mathbf{1}}^{(w)} + 2g_3 Y_{\mathbf{5},1}^{(w)} & -g_2 Y_{\mathbf{3'},3}^{(w)} - \sqrt{3}g_3 Y_{\mathbf{5},4}^{(w)} & g_2 Y_{\mathbf{3'},2}^{(w)} - \sqrt{3}g_3 Y_{\mathbf{5},3}^{(w)} \\ g_2 Y_{\mathbf{3'},3}^{(w)} - \sqrt{3}g_3 Y_{\mathbf{5},4}^{(w)} & \sqrt{6}g_3 Y_{\mathbf{5},2}^{(w)} & g_1 Y_{\mathbf{1}}^{(w)} - g_2 Y_{\mathbf{3'},1}^{(w)} - g_3 Y_{\mathbf{5},1}^{(w)} \\ -g_2 Y_{\mathbf{3'},2}^{(w)} - \sqrt{3}g_3 Y_{\mathbf{5},3}^{(w)} & g_1 Y_{\mathbf{1}}^{(w)} + g_2 Y_{\mathbf{3'},1}^{(w)} - g_3 Y_{\mathbf{5},1}^{(w)} & \sqrt{6}g_3 Y_{\mathbf{5},5}^{(w)} \end{pmatrix} v\,. \tag{3.19}$$

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Observable	bf $\pm 1\sigma$	3σ region	Observable	bf $\pm 1\sigma$	3σ region
$\sin^2\theta_{13}$	$0.02215^{+0.00056}_{-0.00058}$	$[0.02030, 0.02388]$	$\frac{\Delta m^2_{31}}{10^{-3}\mathrm{eV}^2}$	$2.513^{+0.021}_{-0.019}$	$[2.451, 2.578]$
$\sin^2\theta_{12}$	$0.308^{+0.012}_{-0.011}$	$[0.275, 0.345]$	$\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$	$0.0298^{+0.00079}_{-0.00079}$	$[0.0268, 0.0328]$
$\sin^2\theta_{23}$	$0.470^{+0.017}_{-0.013}$	$[0.435, 0.585]$	m_e/m_μ	0.004737	—
δ_{CP}/π	$1.178^{+0.144}_{-0.228}$	$[0.689, 2.022]$	m_μ/m_τ	0.05882	—
$\frac{\Delta m^2_{21}}{10^{-5}\mathrm{eV}^2}$	$7.49^{+0.19}_{-0.19}$	$[6.92, 8.05]$	m_e/MeV	0.469652	—

Table 4. The global best fit values, 1σ ranges and 3σ ranges for mixing parameters and lepton mass ratios, where the experimental data and errors of the lepton mixing parameters and neutrino masses for NO neutrino mass spectrum are obtained from NuFIT 6.0 with Super-Kamiokande atmospheric data [55]. The 1σ error of the charged lepton mass ratios are taken to be 0.1% of their central values in χ^2 function.

for $L, N^c \sim \mathbf{3'}$ with $w = k_L + k_N = -4, -2, 0, 2$, respectively. We shall not consider these cases in the following, since we are only interested in the models with less free parameters.

In short, we have obtained 8 minimal cases labelled as W_i and W'_i when neutrino masses are generated by the Weinberg operator, and 32 minimal combinations $D_i - N'_j$ and $D'_i - N_j$ with $i, j = 1, 2, 3, 4$ are obtained if neutrino masses are generated by seesaw mechanism. The values of the modular weights k_L and k_N for each case are listed in table 3. All these resulting light neutrino mass matrices contain 3 real input parameters besides the complex modulus τ in the case of without gCP symmetry. One more real parameter would be reduced if gCP symmetry is imposed.

3.3 Numerical results

In short, taking into account the possible structures of the models with the charged lepton mass matrices summarized in table 2 and neutrino mass matrices summarized in table 3, we find that there are a total of 400 minimal non-supersymmetric lepton models based on the finite modular symmetry A_5: 80 different models $C_i - W_j$ and $C'_i - W'_j$ in which the neutrino masses are described by the effective Weinberg operator, and 320 different models $C_i - D_j - N'_k$ and $C'_i - D'_j - N_k$ in which neutrinos gain masses via type I seesaw mechanism, where $i = 1, 2, \cdots, 10$ and $j, k = 1, 2, 3, 4$. In each model, the modular weight assignments of the matter fields can be fixed uniquely. Here we do not show them. It is notable that coupling constants α, β and γ in the charged lepton mass matrices can be taken to be positive. If the neutrino masses are generated by the effective Weinberg operator, the light neutrino mass matrices have two independent real parameters $|g_2/g_1|$, $\arg(g_2/g_1)$ besides the overall factor $g_1 v^2/\Lambda$ and the modulus τ. If the light neutrino masses originate from the type I seesaw mechanism, the light neutrino mass matrices for all models are determined by $|h_2/h_1|$ and $\arg(h_2/h_1)$ up to an overall factor $g^2 v^2/(h_1 \Lambda)$ and the modulus τ. Hence the number of independent real free parameters of all the 400 models is eight in the case without gCP symmetry. If gCP symmetry compatible with A_5 is imposed, all couplings are further constrained to be real in our working basis and the number of free parameters reduces to seven.

In the following, we shall perform a comprehensive numerical analysis for all the 400 models both with and without gCP symmetry. In order to quantitatively assess how well a model can describe the current experiment data, we define a χ^2 function to estimate the goodness-of-fit of a set of chosen values of the input parameters,

$$\chi^2 = \sum_{i=1}^{7} \left(\frac{P_i - O_i}{\sigma_i} \right)^2 , \tag{3.20}$$

where O_i and σ_i represent the global best values and 1σ deviations respectively of the following seven dimensionless observable quantities for normal ordering (NO) neutrino masses:

$$m_e/m_\mu, \quad m_\mu/m_\tau, \quad \sin^2\theta_{12}, \quad \sin^2\theta_{13}, \quad \sin^2\theta_{23}, \quad \delta_{\rm CP}, \quad \Delta m^2_{21}/\Delta m^2_{31} , \tag{3.21}$$

where the contribution of the Dirac phase $\delta_{\rm CP}$ is also included in the χ^2 function. The global best fit values and 1σ uncertainties are summarized in table 4. P_i in eq. (3.20) are the theoretical predictions for the above seven physical observable quantities, and they depend on the values of the following six dimensionless input parameters

$$\Re\tau, \quad \Im\tau, \quad \beta/\alpha, \quad \gamma/\alpha, \quad |g_2/g_1| \ (\text{or } |h_2/h_1|) , \quad \arg(g_2/g_1) \ (\text{or } \arg(h_2/h_1)) , \tag{3.22}$$

for Weinberg operator (seesaw) models. When gCP symmetry is included, the phase parameters $\arg(g_2/g_1)$ and $\arg(h_2/h_1)$ are equal to 0 or π. Note that the overall parameters αv of charged lepton mass matrices and $g_1 v^2/\Lambda$ (or $g^2 v^2/(h_1\Lambda)$) of light neutrino mass matrices are determined by the measured electron mass and the solar neutrino mass square difference Δm^2_{21}, respectively.

For each set values of the input parameters, we can obtain the predicted values of lepton masses, mixing parameters and the corresponding χ^2. Then we can find out the lowest χ^2 by extensively scanning the parameter space. Here it is sufficient to limit the modulus τ in the fundamental domain $\mathcal{D} = \{\tau \in \mathcal{H} | |\Re(\tau)| \leq 1/2, |\tau| \geq 1\}$. When we scan over the input parameter space of the 80 Weinberg operator models and the 320 seesaw models without gCP symmetry, we find that 21 Weinberg operator models and 174 seesaw models which contain 76 models $C_i - D_j - N'_k$ and 98 models $C'_i - D'_j - N_k$ can accommodate the experimental data, and all the fitting results for input parameters and predictions for mixing angles, CP violation phases, neutrino masses, the effective mass $m_{\beta\beta}$ in neutrinoless double beta decay ($0\nu\beta\beta$-decay) and the kinematical mass m_β in beta decay of these models are summarized in tables 5, 7 and 9, respectively. When the finite modular group A_5 is extended to combine with the gCP symmetry, only 4 of the 21 viable Weinberg operator models, 48 of 76 seesaw models $C_i - D_j - N'_k$ and 52 of 98 seesaw models $C'_i - D'_j - N_k$ can give results in agreement with the experimental data on mixing parameters and lepton masses. The best fit values of the input parameters and the observable quantities of them are listed in table 6, table 8 and table 10, respectively. Henceforth we focus on the predictions of models that give the three mixing angles θ_{13}, θ_{12}, θ_{23}, the Dirac CP phase $\delta_{\rm CP}$ and $\Delta m^2_{21}/m^2_{31}$ in their 3σ ranges of global data analysis [55], the mass ratios m_e/m_μ, m_μ/m_τ in the experimentally favored intervals shown in table 4, and the three neutrino mass sum below the current most stringent limit $\sum m_\nu < 120\,\text{meV}$ from Planck + lensing + BAO [58].

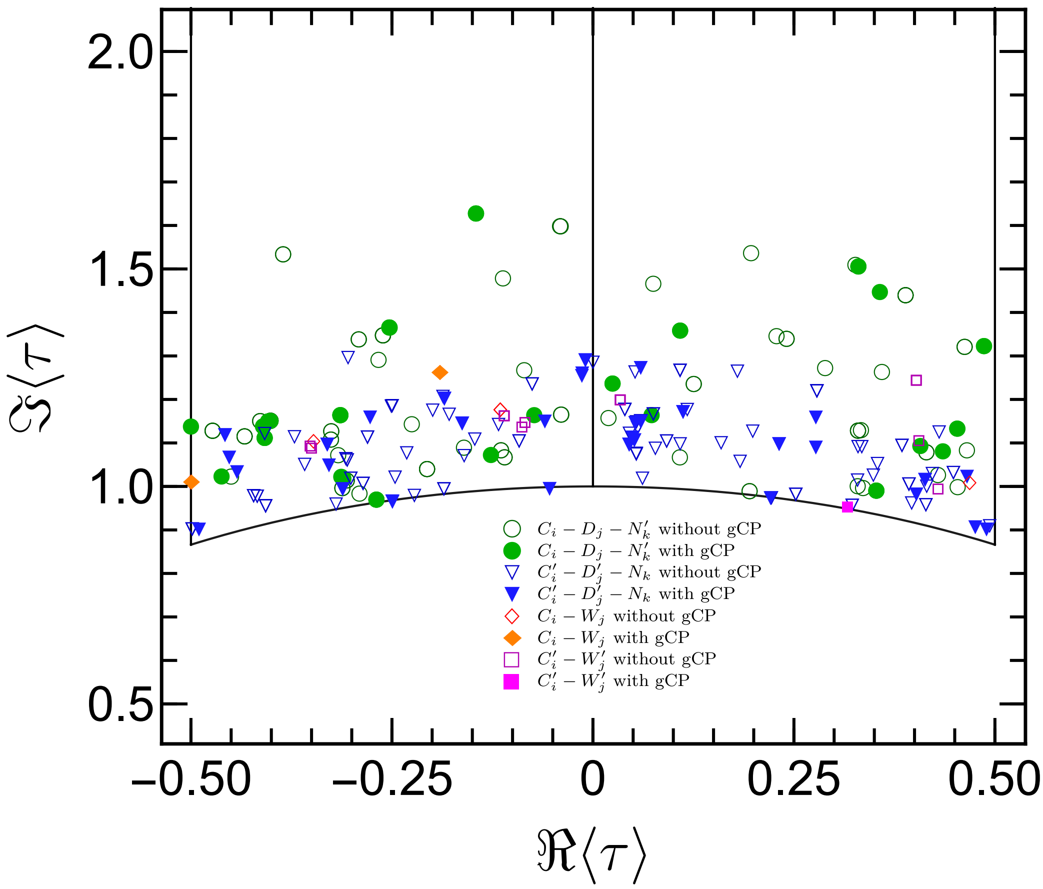

Figure 1. The best fit values of τ for the 21 (4) viable Weinberg operator models $C_i - W_j$ and $C'_i - W'_j$, and the 174 (100) viable seesaw models $C_i - D_j - N'_k$ and $C'_i - D'_j - N_k$ in the case of without (with) gCP symmetry.

In all these models, the modulus τ is treated as an extra free parameter, varied to maximize the agreement between data and theoretical predictions. The best fit values of the complex τ in the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ for all the 195 (104) viable models without (with) gCP are displayed in figure 1. It is obvious that the VEV of τ is not preferred to lie in the vicinity of modular fixed points. Furthermore, we can pin down the minimal value of χ^2 for each model relevant to three lepton mixing angles, three CP violation phases, the three neutrino mass sum $\sum m_\nu$, the effective mass in $0\nu\beta\beta$-decay $m_{\beta\beta}$ and the kinematical mass in beta decay m_β. The results are shown in figure 2 and figure 3. From figure 2, we find that the best fit values of the three mixing angles and Dirac CP phase of the most viable models are predicted to lie in their experimentally allowed 1σ regions [55]. These models can agree with the experimental data very well. In principle, if a very precise measurement of the lepton mixing parameters and neutrino masses is derived from the forthcoming neutrino oscillation experiments and cosmic surveys, these data combined with a very accurate determination of $m_{\beta\beta}$ might provide a good opportunity for probing different modular models. The next generation medium baseline reactor neutrino experiment JUNO [56] can make very precise measurement of the solar angle θ_{12}. The prospective 3σ range of $\sin^2\theta_{12}$ after 6 years of

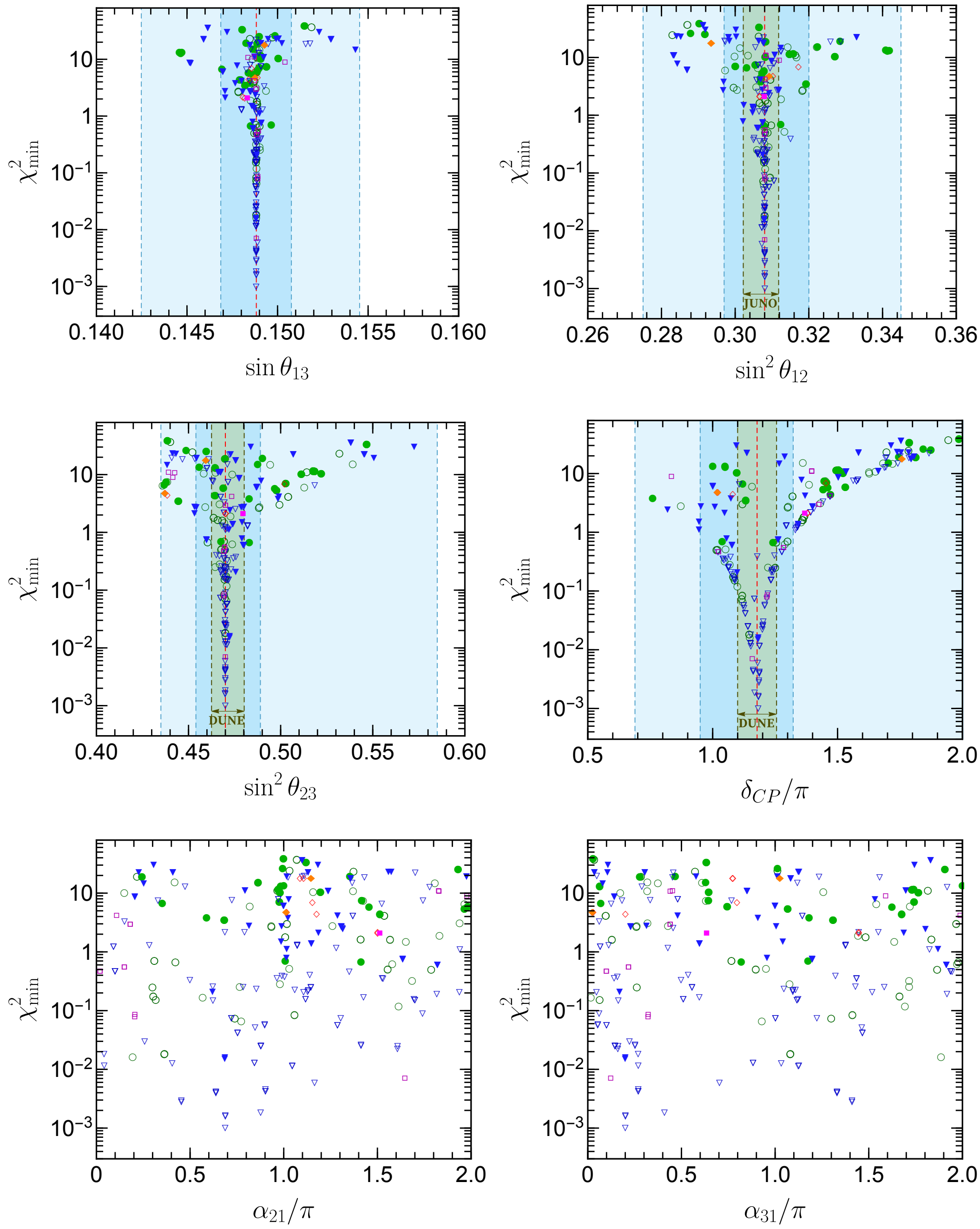

Figure 2. The results of the best fit values of the minimum value of χ^2, the three lepton mixing angles and three CP violation phases for all 195 (104) viable models without (with) gCP. The red dashed lines in the first four panels represent the best fit values, and the light blue bounds represent the 1σ and 3σ ranges from NuFIT 6.0 with Super-Kamiokande atmospheric data [55]. The lighter green band in the panel of $\sin^2\theta_{12}$ is the prospective 3σ range after 6 years of JUNO running [56]. The lighter green regions in the panels of $\sin^2\theta_{23}$ and $\delta_{\rm CP}$ are the resolution in degrees after 15 years of DUNE running [57] for true values of them corresponding to their best fit values given by NuFIT 6.0.

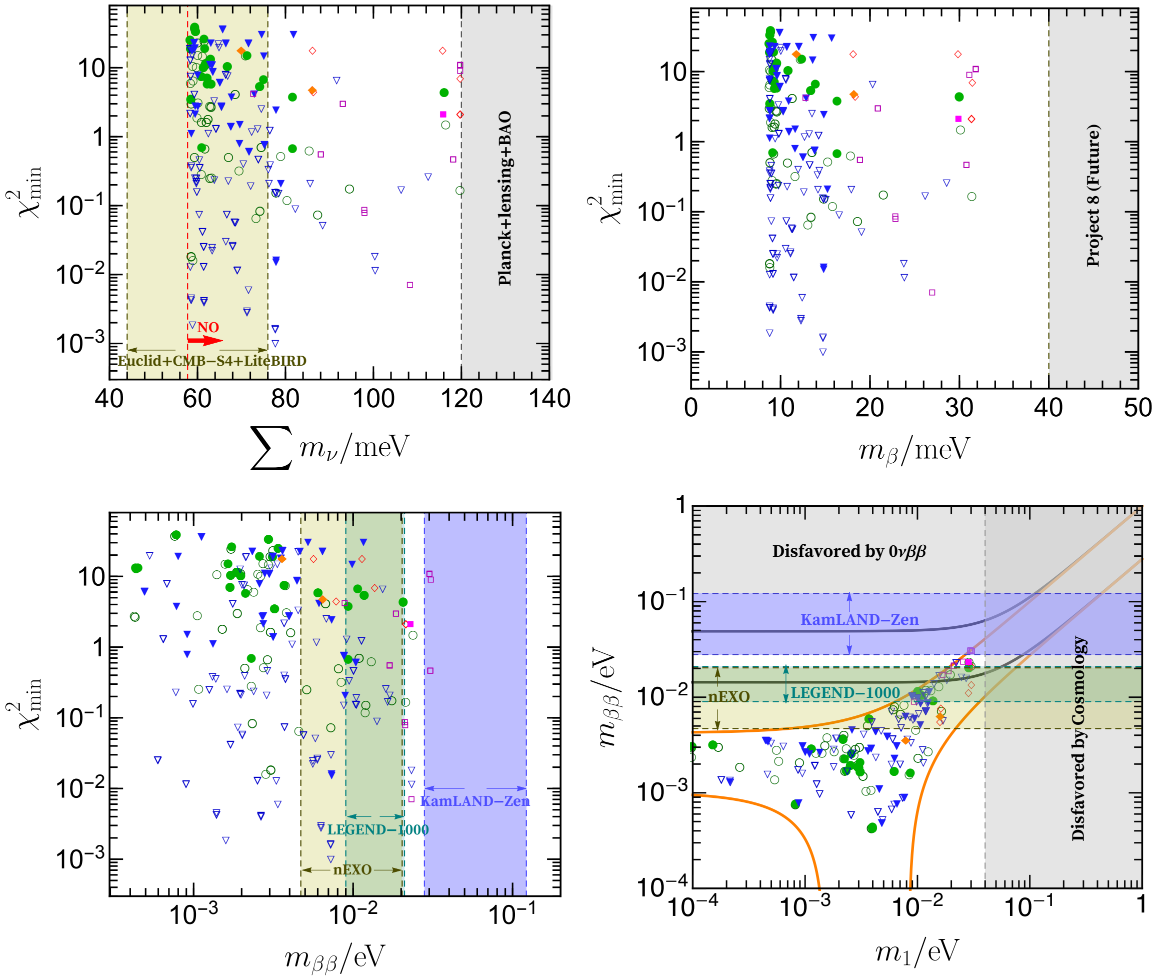

Figure 3. The best fit values of the minimum value of χ^2, the effective mass in $0\nu\beta\beta$-decay $m_{\beta\beta}$, the kinematical mass in beta decay m_β and the three neutrino mass sum $\sum m_\nu$. In the panel of the neutrino mass sum $\sum m_\nu$, the vertical bands indicate the current most stringent limit $\sum m_\nu < 120\,\mathrm{meV}$ from the Planck + lensing + BAO [58], the next generation experiments sensitivity ranges $\sum m_\nu < (44-76)\,\mathrm{meV}$ of Euclid+CMB-S4+LiteBIRD [59], and the dashed line represents the limitation of the normal mass ordering ($\sum m_\nu \geq 57.75\,\mathrm{meV}$). In the panel of the kinematical mass in beta decay m_β, the gray region represents Project 8 future bound ($m_\beta < 0.04\,\mathrm{meV}$) [60]. In the panel of the effective Majorana mass $m_{\beta\beta}$, the vertical bands indicate the latest result $m_{\beta\beta} < (28-122)\,\mathrm{meV}$ of KamLAND-Zen [61], and the next generation experiments sensitivity ranges $m_{\beta\beta} < (9-21)\,\mathrm{meV}$ from LEGEND-1000 [62] and $m_{\beta\beta} < (4.7-20.3)\,\mathrm{meV}$ from nEXO [63].

JUNO running are displayed in figure 2. Critical tests of the viability of these models will be provided by the planned high-precision measurements of θ_{23} and $\delta_{\rm CP}$ at the future long baseline experiments DUNE [57] and T2HK [64]. After 15 years of DUNE running, the resolution in degrees of them are plotted in figure 2. We see that the synergy between JUNO and long baseline neutrino experiments DUNE and T2HK will be extremely powerful for testing a large number of these modular models.

From figure 3, we find that the predictions for the neutrino mass sum $\sum m_\nu$ in most of the viable models can be tested by the next generation experiments sensitivity ranges $\sum m_\nu < (44 - 76)\,\text{meV}$ of Euclid+CMB-S4+LiteBIRD [59]. The best fit values of m_β in all viable models are below the Project 8 future bound 0.04eV [60]. The predictions for $m_{\beta\beta}$ in almost all viable models are compatible with the latest result $m_{\beta\beta} < (28 - 122)\,\text{meV}$ of KamLAND-Zen [61], and most of the viable models can be tested by the next generation experiments such as LEGEND-1000 [62] which aims to improve the sensitivity to $m_{\beta\beta} < (9 \sim 21)\,\text{meV}$ and nEXO [63] which expects to achieve $m_{\beta\beta} < (4.7 \sim 20.3)\,\text{meV}$. The relationship between the best fit values of the lightest neutrino mass m_1 and $m_{\beta\beta}$ in each viable model are also displayed in figure 3.

4 Typical models

From the discussion above, we see that the non-holomorphic A_5 modular models are quite predictive, particularly models with gCP symmetry. The reason is that the number of input parameters is less than the number of observable quantities. Hence the free parameters, mixing parameters and neutrino masses are generally correlated with each other. In order to show the predictions of non-holomorphic A_5 modular invariant models, we shall give detailed numerical results of Weinberg operator model $C_2' - W_4'$ and seesaw model $C_6' - D_1' - N_1$ as examples for illustration. The assignments of the two models are

$$L \sim \mathbf{3'}, \qquad E^c_{1,2,3} \sim \mathbf{1}, \qquad k_L = 1, \qquad k_1 = -5, \qquad k_2 = -3, \qquad k_3 = 1\,, \tag{4.1}$$

and

$$N^c \sim \mathbf{3}, \qquad L \sim \mathbf{3'}, \qquad E^c_{1,2,3} \sim \mathbf{1}, \qquad k_N = k_L = k_1 = -2, \qquad k_2 = 4, \qquad k_3 = 6\,. \tag{4.2}$$

For the two example models without gCP and with gCP, the best fit values of the free parameters, the neutrino masses and mixing parameters are summarized in tables 5, 6, 9 and 10, respectively. After we comprehensively scan the parameter space of each model, and require all the observables lie in their experimentally preferred 3σ regions, some interesting correlations among the input parameters and observables for the two example models are obtained. For model $C_2' - W_4'$, there are six independent and disconnected regions compatible with experiment data in the parameter space, and the corresponding correlations among input parameters and observables are plotted in figure 4, where the results without gCP and with gCP are displayed in green and red, respectively. When gCP invariance is required in the model, all couplings are real and $\Re\tau$ is the unique source of CP violation. As a consequence, the three CP violation phases and the allowed regions of the input parameters are constrained to lie in narrow ranges in figure 4. We find that the predicted ranges of the

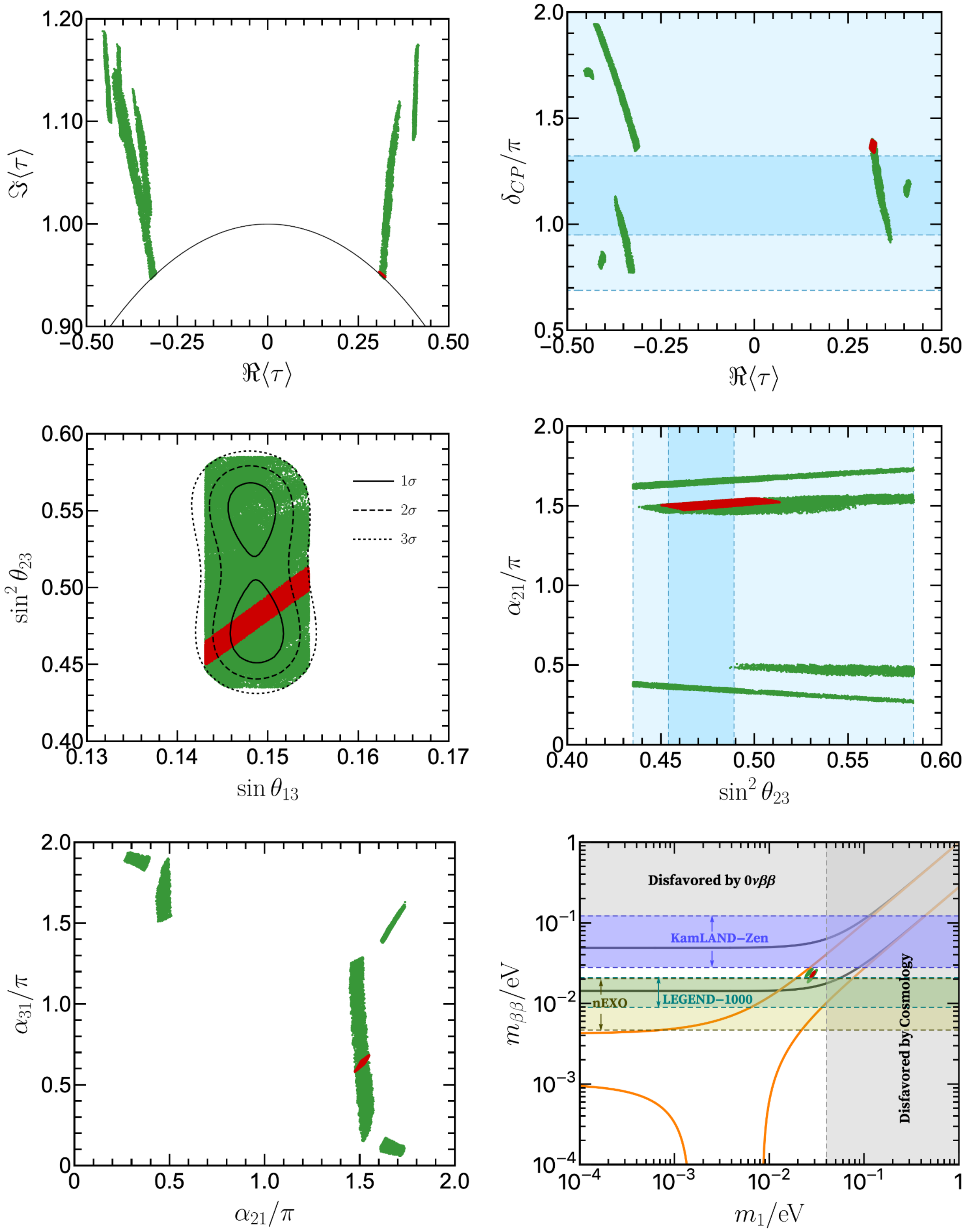

Figure 4. The predicted correlations among the input free parameters, neutrino mixing angles and CP violation phases in the model $C_2' - W_4'$ with (red) and without (green) gCP symmetry.

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atmospheric mixing angle $\sin^2\theta_{23}$ and the Dirac CP violation phase $\delta_{\rm CP}$ are $[0.4502, 0.5134]$ and $[1.3367\pi, 1.399\pi]$, respectively. These predictions may be tested at future long-baseline experiments DUNE [57] and T2HK [64], and at the discussed ESSνSB experiment [65]. The possible values of the sum of neutrino masses $\sum m_\nu$ lie in the interval $[112\,\text{meV}, 120\,\text{meV}]$ which is below the current most stringent limit $\sum m_\nu < 120\,\text{meV}$ from the Planck + lensing + BAO collaboration [58], and the allowed range of the effective Majorana neutrino mass $m_{\beta\beta}$ is $[21.39\,\text{meV}, 24.79\,\text{meV}]$ which may be tested by the KamLAND-Zen experiment [61], and future large-scale $0\nu\beta\beta$-decay experiments, such as LEGEND-1000 [62] and nEXO [63]. Furthermore, the two Majorana CP violation phases are predicted to be in narrow regions $\alpha_{21} \in [1.473\pi, 1.549\pi]$ and $\alpha_{31} \in [0.5821\pi, 0.6832\pi]$.

After we perform an extensive numerical scan over the parameter space of the seesaw model $C_6' - D_1' - N_1$, the numerical results are obtained. We find that the allowed regions of the ratio β/α are $[1.189, 2.899]$, $[17.32, 49.30]$, $[0.00245, 0.00314]$ and $[0.0255, 0.0386]$, and the corresponding viable regions of the ratio γ/α are $[12.68, 32.36]$, $[0.636, 1.938]$, $[1.43, 1.78]$ and $[0.000349, 0.000476]$, respectively. Note that hierarchical values of the parameters α, β and γ are unnecessary to obtained the charged lepton mass hierarchies. This appears to be a lucky coincidence in this model. We display allowed region of the modulus τ and the correlations among different observables in figure 5. From figure 5, we find that the atmospheric angle $\sin^2\theta_{23}$ are strongly correlated with the solar angle $\sin^2\theta_{12}$ when gCP symmetry consistent with non-holomorphic modular symmetry is imposed. It can be tested by JUNO experiment [56] in combination with DUNE [57] or T2HK [64]. Moreover, the three CP violation phases are all restricted to the narrow intervals $0.9339\pi \leq \delta_{\rm CP} \leq 1.066\pi$, $0.9749\pi \leq \alpha_{21} \leq 1.026\pi$ and $0.9114\pi \leq \alpha_{31} \leq 1.087\pi$. Hence model can be tested at future long baseline neutrino oscillation experiments DUNE [57] or T2HK [64]. The neutrino mass sum $\sum m_\nu$ and $m_{\beta\beta}$ are found to lie in narrow intervals $[66.87\,\text{meV}, 75.21\,\text{meV}]$ and $[0.8438\,\text{meV}, 0.9706\,\text{meV}]$, respectively. The prediction for the former is compatible with the upper bound on neutrino mass sum from Planck + lensing + BAO [58] and may be tested by Euclid+CMB-S4+LiteBIRD [59], and the prediction for latter is much below the current most stringent limit from the KamLAND-Zen [61], and next generation $0\nu\beta\beta$-decay experiments LEGEND-1000 [62] and nEXO [63].

5 Conclusion

Modular symmetry is a promising approach for predicting both the hierarchical masses and flavor mixing parameters of fermions [7]. In the present work, we have performed a comprehensive analysis of lepton models based on $\Gamma_5 \cong A_5$ modular flavor symmetry in the framework of non-supersymmetric [50], and all the simplest lepton models without any other flavon except τ have been constructed. In these models, the neutrinos are assumed to be Majorana particles. They are considered gaining masses via either the Weinberg operator or the type I seesaw mechanism. For all these models, the LH lepton doublets L and the three RH charged leptons $E^c_{1,2,3}$ are assigned to a triplet and singlet $\mathbf{1}$ of A_5 respectively, and the RH neutrinos N^c transform as triplet $\mathbf{3}$ or $\mathbf{3'}$ for seesaw models. The Yukawa couplings come from the polyharmonic Maaß forms of weights $k = \pm4, \pm2, 0$ and level 5. Then 80 independent minimal Weinberg operator models and 320 independent minimal seesaw models

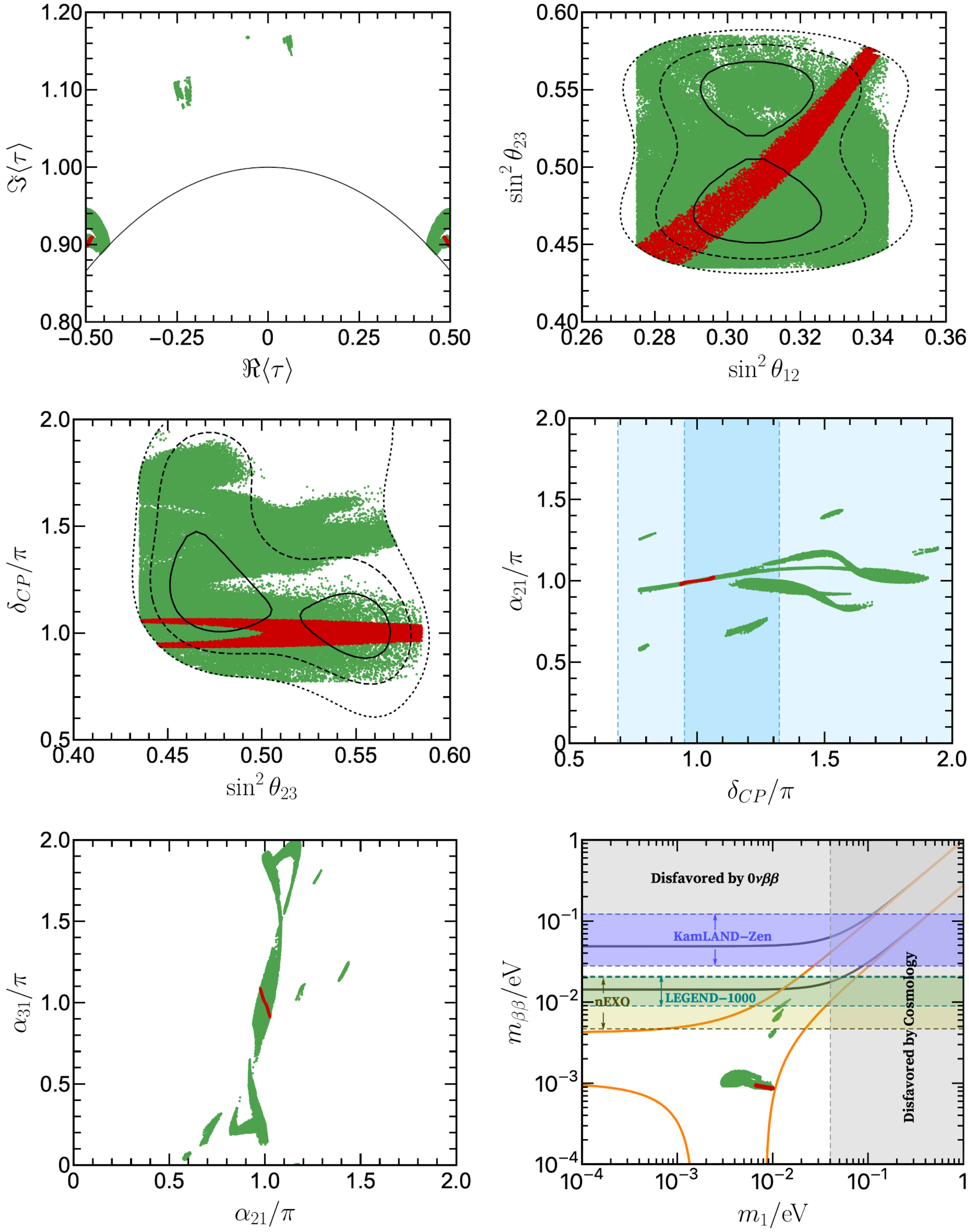

Figure 5. The experimentally allowed values of the complex modulus τ and the correlations between lepton mixing parameters in the model $C_6' - D_1' - N_1$ with (red) and without (green) gCP symmetry.

are obtained, and the corresponding charged lepton and neutrino masses matrices are shown in table 2 and table 3, respectively. All the 400 models depend on six real dimensionless free parameters (including the real and imaginary parts of τ) in eq. (3.22) and two overall parameters in the case of without gCP symmetry. It is known that the gCP symmetry can be consistently combined with the non-holomorphic modular flavor symmetry [50]. When the non-holomorphic modular flavor symmetry is extended to combine with the gCP symmetry, all coupling constants are enforced to be real in our working basis.

After performing an exhaustive numerical analysis for all the 400 models, we find out 21 simplest phenomenologically viable Weinberg operator models and 174 simplest phenomenologically viable seesaw models in the case of without gCP symmetry. The best fit values of input parameters, lepton mixing angles, CP violation phases, neutrino masses, the $0\nu\beta\beta$-decay effective Majorana mass and the kinematical mass in beta decay are listed in tables 5, 7 and 9. We find that the most viable models agree with the experimental data very well, in which all mixing parameters are predicted to lie in their experimentally allowed 1σ regions. When gCP symmetry is imposed, then one more free parameter would be reduced for all the 195 viable models. As a consequence, only 4 of the 21 viable Weinberg operator models and 100 of the 174 viable seesaw models can accommodate the experimental data in lepton sector, as shown in tables 6, 8 and 10. The future medium baseline reactor neutrino oscillation experiments should measure the solar mixing angle $\sin^2\theta_{12}$ with very good precision. A high precision determination of the atmospheric mixing $\sin^2\theta_{23}$ and the Dirac CP phase $\delta_{\rm CP}$ can be performed by the next generation long baseline neutrino experiments DUNE and T2HK. We find that the synergy between the forthcoming neutrino experiments JUNO and DUNE (or T2HK) will be extremely powerful for testing a large number of these viable modular models, particularly viable models with gCP symmetry. Additional important tests of the models will be provided by precision measurements of the neutrino mass sum $\sum m_\nu$ from the next generation experiments Euclid+CMB-S4+LiteBIRD and the effective mass $m_{\beta\beta}$ from the future large-scale $0\nu\beta\beta$-decay experiments LEGEND-1000 and nEXO. Then the number of viable models will further be reduced.

Guided by the analysis of viable models, we present detailed numerical results of Weinberg operator model $C_2' - W_4'$ and seesaw model $C_6' - D_1' - N_1$ as examples for illustration. The result predictions for lepton mixing parameters, neutrino masses and the $0\nu\beta\beta$-decay effective mass of the two models are obtained. When gCP symmetry is imposed, the allowed regions of the input parameters and mixing parameters are constrained to lie in narrow ranges, and some interesting correlations among the input parameters and observables for the two example models are obtained. These correlations could be tested by the next generation neutrino experiments. The Weinberg operator model $C_2' - W_4'$ can be tested by the future large-scale $0\nu\beta\beta$-decay experiments LEGEND-1000 and nEXO.

Acknowledgments

CCL is supported by Natural Science Basic Research Program of Shaanxi (Program No. 2024JC-YBQN-0004), and the National Natural Science Foundation of China under Grant No. 12247103. JNL is supported by the Grants No. NSFC-12147110 and the China Post-doctoral Science Foundation under Grant No. 2021M70. GJD is supported by the National Natural Science Foundation of China under Grant Nos. 12375104.

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Models	Best fit results for 8 viable models $C_i - W_j$ without gCP							
$\{i,j\}$	$\Re\langle\tau\rangle$	$\Im\langle\tau\rangle$	β/α	γ/α	$\lvert g_2/g_1\rvert$	$\arg(g_2/g_1)/\pi$	$\alpha v/\mathrm{MeV}$	$(g_1 v^2/\Lambda)/\mathrm{meV}$	$\chi^2_{\min}$
{1,1}	−0.115	1.18	489.1	8.29	0.441	0.296	6.74	67.4	2.20
{1,3}	−0.190	1.27	11.4	0.00448	0.725	0.170	242.8	27.1	18.3
{2,1}	−0.348	1.11	10.3	0.000907	0.246	0.151	362.9	116.8	7.11
{2,3}	−0.190	1.27	11.4	0.00149	0.545	0.276	242.8	36.0	18.3
{3,1}	0.469	1.01	0.00517	3.26	0.464	1.95	581.0	139.0	4.55
{5,1}	−0.500	1.01	0.000625	3.28	0.468	2.00	580.0	138.2	4.89
{6,1}	−0.500	1.01	0.000801	3.28	0.468	2.00	580.0	137.9	4.91
{8,1}	−0.115	1.18	0.0169	0.00177	0.441	0.296	3295.	67.4	2.15

$\{i,j\}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	α_{21}/π	α_{31}/π	m_1/meV	m_2/meV	m_3/meV	$m_{\beta\beta}/\mathrm{meV}$	m_β/meV
{1,1}	0.02193	0.307	0.470	1.38	1.50	1.45	30.1	31.3	58.4	21.4	31.4
{1,3}	0.02228	0.293	0.459	1.76	1.09	0.775	15.9	18.0	52.3	5.65	18.1
{2,1}	0.02207	0.317	0.502	1.45	1.15	0.797	30.2	31.4	58.2	13.7	31.4
{2,3}	0.02228	0.293	0.459	1.76	1.11	0.773	28.5	29.8	57.4	11.4	29.9
{3,1}	0.02203	0.309	0.438	1.08	1.18	0.201	16.1	18.3	51.9	7.84	18.3
{5,1}	0.02213	0.309	0.437	1.02	1.02	0.0289	16.0	18.1	52.0	6.47	18.2
{6,1}	0.02217	0.310	0.437	1.02	1.01	0.0236	16.0	18.1	52.0	6.42	18.2
{8,1}	0.02196	0.307	0.470	1.38	1.50	1.45	30.1	31.3	58.3	21.4	31.4

Models	Best fit results for 13 viable models $C'_i - W'_j$ without gCP							
$\{i,j\}$	$\Re\langle\tau\rangle$	$\Im\langle\tau\rangle$	β/α	γ/α	$\lvert g_2/g_1\rvert$	$\arg(g_2/g_1)/\pi$	$\alpha v/\mathrm{MeV}$	$(g_1 v^2/\Lambda)/\mathrm{meV}$	$\chi^2_{\min}$
{1,1}	0.429	0.992	104.8	4007.	0.548	1.86	2.55	124.2	4.14
{2,4}	0.405	1.10	8.53	0.00875	0.182	1.23	361.8	107.5	0.00695
{3,1}	−0.0883	1.13	0.000770	0.0528	0.499	0.312	4688.	75.6	8.87
{4,1}	−0.110	1.16	0.00650	0.0328	0.964	1.74	4271.	42.4	0.545
{5,1}	0.0338	1.20	37.7	49.3	0.670	1.72	31.9	47.3	0.0850
{5,4}	0.402	1.24	127.9	325.2	0.252	1.25	4.76	86.4	0.461
{6,1}	−0.110	1.16	0.0328	0.00108	0.964	1.74	4271.	42.3	0.542
{6,2}	−0.350	1.09	161.7	8.39	0.288	1.28	10.2	150.1	10.9
{7,1}	−0.0846	1.15	1741.	39.6	0.899	1.72	3.28	47.8	2.93
{8,1}	0.0341	1.20	397.1	518.3	0.670	1.72	3.03	47.3	0.0778
{8,4}	0.402	1.24	183.0	465.1	0.252	1.25	3.33	86.4	0.461
{10,1}	−0.0845	1.15	0.0227	0.000796	0.899	1.72	5707.	47.8	2.93
{10,2}	−0.352	1.09	304.1	15.9	0.299	1.28	5.43	144.0	10.7

$\{i,j\}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	α_{21}/π	α_{31}/π	m_1/meV	m_2/meV	m_3/meV	$m_{\beta\beta}/\mathrm{meV}$	m_β/meV
{1,1}	0.02212	0.308	0.473	1.47	0.108	1.99	9.32	12.7	50.7	8.87	12.8
{2,4}	0.02215	0.308	0.470	1.16	1.65	0.122	25.5	26.9	55.9	23.3	27.0
{3,1}	0.02262	0.312	0.442	0.835	1.98	1.59	29.8	31.0	59.0	30.8	31.1
{4,1}	0.02218	0.308	0.470	1.28	0.149	0.218	16.7	18.8	52.5	17.0	18.9
{5,1}	0.02218	0.308	0.469	1.22	0.206	0.323	21.1	22.8	54.1	21.2	22.8
{5,4}	0.02215	0.308	0.469	1.02	0.0196	0.0998	29.5	30.8	57.9	30.5	30.8

Table 5. The best fit values of the input parameters, three mixing angles θ_{12}, θ_{13}, θ_{23}, Dirac CP violation phase δ_{CP}, Majorana CP violation phases α_{21}, α_{31}, three light neutrino masses $m_{1,2,3}$, the effective mass $m_{\beta\beta}$ in $0\nu\beta\beta$-decay and the kinematical mass m_β in beta decay at the minimum values of χ^2 for viable models $C_i - W_j$ and $C'_i - W'_j$ which are labelled as $\{i, j\}$ in the case of without gCP symmetry. The best fit values of the charged lepton mass ratios m_e/m_μ and m_μ/m_τ are both the global best fit values of them, i.e. $m_e/m_\mu = 0.004737$ and $m_\mu/m_\tau = 0.05882$. The best fit values of the mass sum $\sum m_\nu$ and $\Delta m^2_{21}/\Delta m^2_{31}$ can be easy obtained from the three neutrino masses. In the following tables, we would not show them as the same reason (*continues . . . .*).

{6,1}	0.02218	0.308	0.470	1.28	0.149	0.217	16.7	18.8	52.5	17.0	18.9
{6,2}	0.02212	0.308	0.439	1.40	1.83	0.453	30.6	31.8	57.3	30.2	31.8
{7,1}	0.02215	0.308	0.470	1.43	0.180	0.440	18.9	20.8	53.3	18.6	20.9
{8,1}	0.02218	0.308	0.469	1.22	0.205	0.322	21.1	22.8	54.1	21.2	22.9
{8,4}	0.02215	0.308	0.469	1.02	0.0196	0.0998	29.5	30.8	57.9	30.5	30.8
{10,1}	0.02215	0.308	0.470	1.43	0.180	0.440	19.0	20.8	53.3	18.6	20.9
{10,2}	0.02201	0.308	0.443	1.40	1.83	0.440	30.7	31.9	57.2	30.2	31.8

Table 5. The best fit values of the input parameters, three mixing angles θ_{12}, θ_{13}, θ_{23}, Dirac CP violation phase $\delta_{\rm CP}$, Majorana CP violation phases α_{21}, α_{31}, three light neutrino masses $m_{1,2,3}$, the effective mass $m_{\beta\beta}$ in $0\nu\beta\beta$-decay and the kinematical mass m_β in beta decay at the minimum values of χ^2 for viable models $C_i - W_j$ and $C'_i - W'_j$ which are labelled as $\{i, j\}$ in the case of without gCP symmetry. The best fit values of the charged lepton mass ratios m_e/m_μ and m_μ/m_τ are both the global best fit values of them, i.e. $m_e/m_\mu = 0.004737$ and $m_\mu/m_\tau = 0.05882$. The best fit values of the mass sum $\sum m_\nu$ and $\Delta m^2_{21}/\Delta m^2_{31}$ can be easy obtained from the three neutrino masses. In the following tables, we would not show them as the same reason.

`Models`	Best fit results for 3 viable models $C_i - W_j$ with gCP						
$\{i,j\}$	$\Re\langle\tau\rangle$	$\Im\langle\tau\rangle$	β/α	γ/α	g_2/g_1	αv/MeV	$(g_1 v^2/\Lambda)$/meV	$\chi^2_{\rm min}$
{1,3}	−0.190	1.27	11.4	0.00448	0.842	242.8	23.3	18.3
{2,3}	−0.190	1.27	11.4	0.00149	0.842	242.8	23.3	18.3
{3,1}	−0.499	1.01	0.00511	3.28	0.468	579.9	138.1	4.90

$\{i,j\}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\delta_{\rm CP}/\pi$	α_{21}/π	α_{31}/π	m_1/meV	m_2/meV	m_3/meV	$m_{\beta\beta}$/meV	m_β/meV
{1,3}	0.02228	0.293	0.459	1.76	1.15	1.03	7.87	11.7	50.4	3.58	11.8
{2,3}	0.02228	0.293	0.459	1.76	1.15	1.03	7.87	11.7	50.4	3.58	11.8
{3,1}	0.02213	0.309	0.437	1.02	1.01	0.0243	16.0	18.1	52.0	6.46	18.2

`Model`	Best fit values for the viable model $C'_2 - W'_4$ with gCP							
$\{i,j\}$	$\Re\langle\tau\rangle$	$\Im\langle\tau\rangle$	β/α	γ/α	g_2/g_1	αv/MeV	$(g_1 v^2/\Lambda)$/meV	$\chi^2_{\rm min}$
{2,4}	0.317	0.951	368.3	10.8	−0.0356	10.4	436.7	2.07

$\{i,j\}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\delta_{\rm CP}/\pi$	α_{21}/π	α_{31}/π	m_1/meV	m_2/meV	m_3/meV	$m_{\beta\beta}$/meV	m_β/meV
{2,4}	0.02200	0.308	0.480	1.37	1.51	0.635	28.6	29.9	57.4	22.9	29.9

Table 6. The best fit values of the input parameters, lepton masses and mixing parameters at the minimum values of χ^2 for viable models $C_i - W_j$ and $C'_i - W'_j$ with gCP symmetry.

`Models`	Best fit results for 76 viable models $C_i - D_j - N'_k$ without gCP							
$\{i,j,k\}$	$\Re\langle\tau\rangle$	$\Im\langle\tau\rangle$	β/α	γ/α	$\lvert h_2/h_1\rvert$	$\arg(h_2/h_1)/\pi$	αv/MeV	$(g_1^2 v^2/(h_1\Lambda))$/meV	$\chi^2_{\rm min}$
{1,1,1}	0.289	1.28	829.6	19.7	1.01	1.97	3.27	47.0	10.3
{1,1,2}	0.465	1.09	1237.	17.2	2.54	0.640	3.15	428.0	0.288
{1,1,3}	−0.0397	1.17	323.0	5.23	0.949	0.0132	10.5	159.6	0.508
{1,1,4}	−0.326	1.11	10.6	0.00156	0.897	0.308	352.7	331.1	0.680
{1,3,1}	−0.267	1.29	781.0	19.7	0.928	0.0158	3.34	69.1	0.170
{1,3,2}	0.415	1.08	9.44	0.00101	2.58	1.06	415.2	163.0	0.155
{1,3,4}	0.330	1.13	68.1	237.8	0.586	0.414	3.50	70.0	0.716
{2,1,1}	−0.225	1.15	11.7	0.00107	0.397	1.78	298.4	63.5	10.7

Table 7. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C_i - D_j - N'_k$ which are labelled as $\{i, j, k\}$. In this table, the gCP symmetry compatible with A_5 is not imposed (*continues...*).

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{2,1,4}	−0.326	1.13	0.000269	0.0142	0.304	1.81	4719.	71.7	1.64
{2,2,2}	−0.450	1.03	0.00175	1.11	1.45	0.0100	749.2	234.2	15.3
{2,3,1}	0.389	1.44	143.1	1177.	1.57	1.80	1.02	125.2	1.66
{2,3,2}	0.0751	1.47	0.0129	12.2	0.881	0.154	100.6	63.6	0.258
{2,3,3}	0.329	1.00	93.9	279.8	1.31	0.929	2.91	21.6	6.64
{3,1,1}	0.125	1.24	708.3	33.8	0.934	1.99	4.14	58.9	0.253
{3,1,2}	−0.160	1.09	0.0506	0.000533	0.837	0.0656	4875.	154.9	0.0665
{3,1,3}	0.0193	1.16	448.2	14.1	0.924	0.0147	7.65	170.0	0.328
{3,1,4}	−0.326	1.11	10.6	0.00448	0.897	0.308	352.7	331.0	0.681
{3,3,1}	−0.0855	1.27	710.5	38.0	0.874	0.0266	3.87	69.9	1.50
{3,3,2}	0.415	1.08	9.44	0.00329	2.57	1.06	415.3	162.9	0.155
{3,3,4}	−0.317	1.07	79.8	404.5	0.457	1.55	4.23	63.5	0.121
{4,1,1}	−0.385	1.54	14.1	340.2	0.976	1.99	3.76	27.2	0.513
{4,1,2}	−0.0401	1.60	16.9	417.2	0.857	1.99	3.19	13.4	0.0185
{4,1,4}	−0.326	1.13	0.000630	0.0142	0.304	1.81	4719.	71.7	1.64
{4,2,1}	−0.110	1.07	0.0596	0.000279	0.866	0.000609	4778.	44.5	24.8
{4,2,2}	−0.473	1.13	2.53	46.5	1.16	1.96	19.8	130.3	2.80
{4,2,3}	−0.114	1.09	24.7	1.56	0.916	2.00	33.8	210.8	0.0859
{4,3,2}	−0.112	1.48	0.0235	12.6	0.836	1.84	97.7	61.1	2.82
{5,1,1}	−0.290	0.987	21.7	365.7	0.655	0.0478	4.22	69.0	11.0
{5,1,3}	−0.306	1.02	23.3	416.8	3.35	0.990	3.90	161.3	19.3
{5,1,4}	0.454	1.00	18.8	556.9	0.123	1.48	3.37	71.2	14.9
{5,2,1}	−0.110	1.07	0.0596	0.000478	0.866	0.000611	4779.	44.5	24.9
{5,3,1}	−0.312	0.999	212.9	41.7	0.948	1.89	3.96	30.3	0.178
{5,3,2}	0.429	1.03	0.000743	3.35	1.37	1.19	555.0	89.4	0.637
{5,3,4}	0.359	1.27	57.8	734.3	0.280	0.176	2.41	9.00	15.5
{6,1,1}	0.197	1.54	273.3	26.7	0.955	0.00878	4.69	34.6	1.92
{6,1,2}	0.336	0.999	0.0000483	0.0155	0.0514	1.25	7990.	15.1	0.0164
{6,1,3}	0.228	1.35	248.7	25.0	3.43	1.69	4.50	129.8	6.07
{6,1,4}	−0.206	1.04	159.3	16.6	0.458	1.69	5.35	256.1	5.33×10^{-6}
{6,2,2}	0.462	1.32	330.5	34.8	1.03	0.0106	3.31	67.4	37.7
{6,2,3}	0.108	1.07	0.0105	0.000487	0.919	0.000383	5072.	245.8	0.524
{6,3,1}	−0.414	1.15	250.0	29.4	0.0706	0.629	3.77	21.1	11.2
{6,3,2}	−0.433	1.12	230.0	28.0	0.114	1.36	3.96	11.7	1.82
{6,3,3}	−0.261	1.35	271.0	27.4	1.02	0.0987	4.13	77.3	4.21
{6,3,4}	−0.401	1.15	249.9	29.2	0.00272	1.06	3.77	10.1	11.7
{7,1,1}	−0.385	1.54	48.6	1171.	0.976	1.99	1.09	27.2	0.512
{7,1,2}	−0.0409	1.60	49.1	1211.	0.857	1.99	1.10	13.4	0.0187
{7,1,3}	−0.0394	1.17	0.0162	0.0000941	0.949	0.0132	3377.	159.6	0.509
{7,2,2}	−0.473	1.13	41.8	767.1	1.16	1.96	1.20	130.1	2.71
{7,2,3}	−0.114	1.09	643.9	40.8	0.916	2.00	1.29	210.7	0.0846
{7,3,1}	0.389	1.44	0.00268	8.22	1.57	1.80	145.8	125.2	1.66
{7,3,2}	0.326	1.51	0.00326	9.45	0.405	1.99	133.1	39.7	7.60
{7,3,4}	0.330	1.13	3.49	0.00119	0.586	0.414	238.1	70.0	0.716
{8,1,1}	0.125	1.24	0.000808	0.0477	0.935	1.99	2934.	58.9	0.254
{8,1,3}	−0.306	1.02	18.0	322.9	3.35	0.990	5.03	161.3	19.3
{8,1,4}	0.333	1.13	0.0154	0.00101	0.403	1.61	3586.	116.7	3.15
{8,3,1}	−0.312	0.999	157.1	30.8	0.948	1.89	5.37	30.3	0.178
{8,3,4}	0.330	1.13	3.49	0.0152	0.586	0.414	238.1	70.0	0.716
{9,1,1}	0.241	1.34	731.9	73.8	1.03	1.78	1.52	66.6	3.05
{9,1,3}	0.195	0.992	30.4	881.9	0.935	0.00581	1.57	350.8	0.0744
{9,1,4}	−0.206	1.04	561.9	58.5	0.458	1.69	1.52	256.1	8.05×10^{-6}
{9,2,2}	0.462	1.32	590.0	62.2	1.03	0.0105	1.85	67.3	37.8

Table 7. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C_i - D_j - N'_k$ which are labelled as $\{i, j, k\}$. In this table, the gCP symmetry compatible with A_5 is not imposed (*continues...*).

{9,2,3}	−0.291	1.34	695.7	70.9	0.852	2.00	1.60	51.1	23.5
{9,3,1}	0.389	1.44	8.22	0.00982	1.57	1.80	145.8	125.2	1.65
{9,3,2}	−0.433	1.12	424.7	51.8	0.114	1.36	2.14	11.7	1.82
{9,3,3}	−0.261	1.35	721.5	72.9	1.02	0.0990	1.55	77.3	4.21
{9,3,4}	−0.401	1.15	482.4	56.3	0.00272	1.06	1.95	10.1	11.6
{10,1,1}	0.241	1.34	1174.	118.4	1.03	1.78	0.948	66.6	3.06
{10,1,2}	−0.0404	1.60	24.7	0.0717	0.857	1.99	54.0	13.4	0.0186
{10,1,3}	0.195	0.992	71.1	2063.	0.935	0.00581	0.671	350.8	0.0744
{10,1,4}	−0.206	1.04	1263.	131.6	0.458	1.69	0.675	256.1	6.40×10^{-6}
{10,2,2}	−0.473	1.13	18.4	0.0187	1.16	1.96	50.2	130.2	2.72
{10,2,3}	−0.291	1.34	1302.	132.7	0.852	2.00	0.854	51.1	23.6
{10,3,1}	−0.414	1.15	2016.	236.8	0.0705	0.629	0.467	21.1	11.2
{10,3,2}	−0.433	1.12	2181.	265.9	0.114	1.36	0.418	11.7	1.82
{10,3,3}	−0.261	1.35	1218.	123.1	1.02	0.0989	0.919	77.3	4.21
{10,3,4}	−0.401	1.15	1983.	231.5	0.00271	1.05	0.476	10.1	11.7

$\{i,j,k\}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\delta_{\rm CP}/\pi$	α_{21}/π	α_{31}/π	m_1/meV	m_2/meV	m_3/meV	$m_{\beta\beta}$/meV	m_β/meV
{1,1,1}	0.02209	0.317	0.474	1.63	0.149	0.173	1.25	8.69	49.8	2.64	8.90
{1,1,2}	0.02221	0.309	0.467	1.07	0.980	1.08	2.21	8.88	49.8	2.30	9.07
{1,1,3}	0.02215	0.308	0.470	1.02	1.79	1.87	6.25	10.6	50.2	8.11	10.8
{1,1,4}	0.02221	0.310	0.460	1.25	0.422	1.98	2.81	9.04	49.8	2.91	9.23
{1,3,1}	0.02215	0.308	0.470	1.08	0.567	0.0167	30.1	31.3	58.2	21.4	31.4
{1,3,2}	0.02215	0.308	0.470	1.09	0.317	0.0638	11.9	14.7	51.2	12.0	14.8
{1,3,4}	0.02214	0.308	0.469	1.30	0.310	0.767	10.3	13.4	50.8	10.9	13.5
{2,1,1}	0.02216	0.300	0.458	1.63	1.16	1.26	0.537	8.61	49.8	1.40	8.77
{2,1,4}	0.02216	0.309	0.468	1.36	1.57	0.129	2.22	8.88	49.8	4.07	9.06
{2,2,2}	0.02216	0.322	0.494	1.68	0.863	0.610	8.45	12.1	50.4	1.58	12.2
{2,3,1}	0.02217	0.307	0.466	1.36	1.41	1.83	3.04	9.12	49.9	3.04	9.29
{2,3,2}	0.02221	0.306	0.464	1.22	0.765	1.72	3.86	9.42	50.0	1.08	9.60
{2,3,3}	0.02217	0.306	0.436	1.39	1.96	1.54	0.583	8.62	49.8	2.18	8.80
{3,1,1}	0.02215	0.308	0.470	1.25	0.299	0.262	3.60	9.32	49.9	5.13	9.49
{3,1,2}	0.02215	0.308	0.471	1.12	0.773	0.930	9.66	12.9	50.7	4.57	13.1
{3,1,3}	0.02215	0.308	0.471	1.05	1.69	1.72	7.61	11.5	50.4	8.43	11.6
{3,1,4}	0.02221	0.310	0.460	1.25	0.422	1.98	2.81	9.04	49.8	2.90	9.23
{3,3,1}	0.02215	0.307	0.469	1.35	1.65	1.64	28.8	30.1	57.5	23.7	30.1
{3,3,2}	0.02215	0.308	0.470	1.09	0.316	0.0638	11.9	14.7	51.2	12.0	14.8
{3,3,4}	0.02215	0.308	0.471	1.10	1.61	1.69	13.1	15.7	51.5	11.9	15.8
{4,1,1}	0.02212	0.309	0.468	1.02	1.56	0.897	0.894	8.64	49.8	2.10	8.83
{4,1,2}	0.02215	0.308	0.470	1.15	0.362	1.08	0.169	8.60	49.8	3.05	8.79
{4,1,4}	0.02216	0.309	0.468	1.36	1.57	0.129	2.22	8.88	49.8	4.07	9.06
{4,2,1}	0.02242	0.283	0.452	1.78	1.03	1.01	2.26	8.89	50.4	1.72	9.07
{4,2,2}	0.02185	0.318	0.489	1.30	0.934	0.261	3.90	9.44	49.7	0.425	9.59
{4,2,3}	0.02215	0.310	0.469	1.12	1.06	1.41	10.1	13.3	50.8	2.34	13.4
{4,3,2}	0.02231	0.300	0.459	0.872	1.26	0.428	3.49	9.28	50.0	1.06	9.46
{5,1,1}	0.02216	0.303	0.465	1.65	1.37	0.317	2.72	9.02	49.9	2.53	9.19
{5,1,3}	0.02217	0.308	0.470	1.81	0.217	0.284	0.0978	8.60	49.7	3.02	8.78
{5,1,4}	0.02220	0.301	0.459	1.72	1.81	0.375	2.33	8.91	49.9	2.95	9.07
{5,2,1}	0.02242	0.283	0.452	1.78	1.03	1.01	2.26	8.89	50.4	1.72	9.07
{5,3,1}	0.02215	0.308	0.471	1.08	0.303	1.66	19.6	21.4	53.5	17.5	21.5
{5,3,2}	0.02216	0.308	0.474	1.29	1.58	1.97	15.5	17.7	52.2	13.0	17.8
{5,3,4}	0.02213	0.314	0.539	1.36	0.417	0.318	1.82	8.79	49.4	2.73	8.95
{6,1,1}	0.02206	0.312	0.473	1.37	1.83	1.44	1.42	8.71	49.8	2.63	8.90
{6,1,2}	0.02215	0.308	0.470	1.15	0.193	1.89	0.557	8.61	49.8	2.84	8.80
{6,1,3}	0.02213	0.297	0.513	1.16	0.818	1.72	0.0976	8.60	49.7	2.38	8.73

Table 7. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C_i - D_j - N'_k$ which are labelled as $\{i,j,k\}$. In this table, the gCP symmetry compatible with A_5 is not imposed (*continues...*).

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{6,1,4}	0.02215	0.308	0.470	1.18	0.278	0.701	6.09	10.5	50.2	7.50	10.7
{6,2,2}	0.02306	0.287	0.440	1.94	1.07	0.0349	0.821	8.64	49.9	0.763	8.88
{6,2,3}	0.02216	0.313	0.468	1.05	1.03	0.951	12.5	15.1	51.3	2.75	15.3
{6,3,1}	0.02223	0.316	0.514	1.51	0.966	1.82	2.93	9.08	49.8	1.97	9.30
{6,3,2}	0.02216	0.308	0.464	1.36	1.01	1.48	3.28	9.20	49.9	1.52	9.38
{6,3,3}	0.02212	0.310	0.504	1.32	1.54	0.0968	7.10	11.2	50.3	6.73	11.3
{6,3,4}	0.02222	0.315	0.518	1.50	0.973	1.73	3.00	9.10	49.8	1.86	9.31
{7,1,1}	0.02212	0.309	0.468	1.02	1.55	0.897	0.893	8.64	49.8	2.10	8.83
{7,1,2}	0.02215	0.308	0.470	1.15	0.364	1.08	0.169	8.60	49.8	3.05	8.79
{7,1,3}	0.02215	0.308	0.470	1.02	1.79	1.86	6.25	10.6	50.2	8.11	10.8
{7,2,2}	0.02186	0.318	0.488	1.30	0.935	0.259	3.90	9.44	49.7	0.430	9.59
{7,2,3}	0.02215	0.309	0.469	1.12	1.06	1.41	10.1	13.3	50.8	2.33	13.4
{7,3,1}	0.02217	0.307	0.466	1.36	1.41	1.83	3.04	9.12	49.9	3.03	9.29
{7,3,2}	0.02232	0.305	0.439	1.45	1.42	0.631	2.17	8.87	49.8	3.76	9.06
{7,3,4}	0.02214	0.308	0.469	1.30	0.310	0.767	10.3	13.4	50.8	10.9	13.5
{8,1,1}	0.02215	0.308	0.470	1.25	0.299	0.262	3.60	9.32	49.9	5.12	9.49
{8,1,3}	0.02217	0.308	0.470	1.81	0.216	0.284	0.0978	8.60	49.7	3.02	8.78
{8,1,4}	0.02215	0.306	0.467	1.43	1.94	0.319	1.57	8.74	49.8	3.83	8.91
{8,3,1}	0.02215	0.308	0.471	1.08	0.303	1.66	19.6	21.4	53.5	17.5	21.5
{8,3,4}	0.02214	0.308	0.469	1.30	0.310	0.767	10.3	13.4	50.8	10.9	13.5
{9,1,1}	0.02212	0.300	0.499	1.12	1.02	1.97	0.262	8.60	49.7	1.88	8.74
{9,1,3}	0.02215	0.308	0.470	1.12	0.739	1.30	16.4	18.5	52.4	7.26	18.6
{9,1,4}	0.02215	0.308	0.470	1.18	0.278	0.701	6.09	10.5	50.2	7.50	10.7
{9,2,2}	0.02307	0.287	0.440	1.94	1.07	0.0347	0.820	8.64	49.9	0.762	8.88
{9,2,3}	0.02205	0.306	0.532	1.70	1.37	0.572	1.50	8.73	49.8	3.14	8.89
{9,3,1}	0.02217	0.307	0.466	1.36	1.41	1.83	3.04	9.12	49.9	3.03	9.29
{9,3,2}	0.02216	0.308	0.464	1.36	1.01	1.48	3.28	9.20	49.9	1.52	9.38
{9,3,3}	0.02212	0.310	0.504	1.32	1.54	0.0952	7.09	11.1	50.3	6.71	11.3
{9,3,4}	0.02222	0.315	0.518	1.50	0.972	1.73	3.00	9.10	49.8	1.86	9.31
{10,1,1}	0.02212	0.300	0.499	1.12	1.02	1.97	0.263	8.60	49.7	1.88	8.74
{10,1,2}	0.02215	0.308	0.470	1.15	0.363	1.08	0.169	8.60	49.8	3.05	8.79
{10,1,3}	0.02215	0.308	0.470	1.12	0.739	1.30	16.4	18.5	52.4	7.26	18.6
{10,1,4}	0.02215	0.308	0.470	1.18	0.278	0.701	6.09	10.5	50.2	7.50	10.7
{10,2,2}	0.02186	0.318	0.488	1.30	0.934	0.259	3.90	9.44	49.7	0.430	9.59
{10,2,3}	0.02205	0.306	0.532	1.70	1.37	0.573	1.50	8.73	49.8	3.14	8.89
{10,3,1}	0.02223	0.316	0.514	1.51	0.966	1.82	2.93	9.08	49.8	1.97	9.30
{10,3,2}	0.02216	0.308	0.464	1.36	1.01	1.48	3.28	9.20	49.9	1.52	9.38
{10,3,3}	0.02212	0.310	0.504	1.32	1.54	0.0964	7.10	11.1	50.3	6.72	11.3
{10,3,4}	0.02222	0.315	0.518	1.50	0.972	1.73	3.00	9.10	49.8	1.86	9.31

Table 7. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C_i - D_j - N'_k$ which are labelled as $\{i,j,k\}$. In this table, the gCP symmetry compatible with A_5 is not imposed.

`Models`	Best fit results for 48 viable models $C_i - D_j - N'_k$ with gCP						
$\{i,j,k\}$	$\Re\langle\tau\rangle$	$\Im\langle\tau\rangle$	β/α	γ/α	h_2/h_1	αv/MeV	$(g_1^2 v^2/(h_1\Lambda))$/meV	$\chi^2_{\rm min}$
{1,1,2}	0.435	1.08	1219.	16.8	−5.18	3.20	877.7	0.714
{1,1,3}	0.108	1.36	537.5	16.9	2.83	4.23	49.3	25.8
{1,3,1}	0.0244	1.24	390.3	7.96	0.808	7.51	67.2	4.49
{1,3,2}	−0.0730	1.17	0.00123	0.586	1.77	1315.	111.5	3.88
{2,2,2}	−0.462	1.03	0.00173	1.09	1.45	758.8	235.5	15.6

Table 8. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C_i - D_j - N'_k$ with gCP symmetry (*continues...*).

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{2,3,1}	0.357	1.45	144.5	1206.	2.19	1.00	167.6	5.91
{2,3,2}	0.330	1.51	144.5	1358.	0.400	0.924	39.4	7.64
{3,1,1}	0.353	0.994	84.5	343.8	1.20	4.93	117.8	6.00
{3,3,2}	0.407	1.10	9.67	0.00350	−3.12	394.8	193.6	6.87
{4,1,2}	−0.145	1.63	18.2	452.4	0.868	3.00	11.8	3.59
{4,2,2}	0.453	1.14	2.63	48.5	1.11	19.1	124.0	13.6
{5,1,1}	−0.313	1.03	23.8	434.7	0.639	3.80	59.7	19.6
{5,1,3}	−0.310	1.02	23.6	427.4	−3.39	3.84	161.0	19.4
{5,2,1}	−0.127	1.08	0.0515	0.000423	0.864	4812.	41.9	27.0
{5,3,2}	0.0730	1.17	0.586	0.00260	1.77	1315.	111.5	0.689
{6,1,3}	−0.314	1.17	242.3	26.9	0.901	3.95	118.9	10.6
{6,1,4}	−0.269	0.973	154.8	17.0	0.165	5.11	163.3	5.50
{6,2,2}	0.486	1.33	332.7	35.0	1.03	3.29	67.0	39.6
{6,2,3}	−0.253	1.37	270.5	27.2	0.842	4.20	46.5	34.3
{6,3,1}	−0.404	1.15	249.5	29.2	−0.0191	3.78	21.2	11.8
{6,3,2}	−0.408	1.12	228.9	27.7	−0.0297	3.97	10.3	7.22
{6,3,3}	−0.410	1.14	243.6	28.8	−0.133	3.82	50.3	10.4
{6,3,4}	−0.401	1.15	249.9	29.2	−0.00273	3.77	10.1	11.7
{7,1,2}	−0.146	1.63	49.7	1234.	0.868	1.10	11.8	3.54
{7,1,3}	0.108	1.36	0.0315	0.000176	2.83	2274.	49.2	25.8
{7,2,2}	0.453	1.14	41.9	772.6	1.11	1.20	124.0	13.2
{7,3,1}	0.0244	1.24	0.0204	0.000118	0.807	2931.	67.2	4.45
{7,3,2}	0.330	1.51	0.00323	9.40	0.400	133.6	39.4	7.64
{8,1,1}	0.353	0.994	0.00272	4.07	1.20	416.4	117.8	6.01
{8,1,3}	−0.310	1.02	18.0	326.8	−3.39	5.02	161.0	19.4
{9,1,1}	0.353	0.994	0.000588	4.07	1.20	416.5	117.8	6.00
{9,1,3}	−0.314	1.17	570.6	63.3	0.901	1.68	118.9	10.6
{9,1,4}	−0.269	0.973	440.2	48.3	0.165	1.80	163.3	5.50
{9,2,2}	0.486	1.33	588.1	61.9	1.03	1.86	66.9	39.7
{9,2,3}	−0.253	1.37	735.6	73.9	0.843	1.54	46.6	34.2
{9,3,1}	−0.404	1.15	479.2	56.1	−0.0190	1.97	21.2	11.8
{9,3,2}	−0.408	1.12	440.6	53.3	−0.0296	2.06	10.3	7.22
{9,3,3}	−0.410	1.14	464.0	54.9	−0.133	2.01	50.3	10.4
{9,3,4}	−0.401	1.15	482.2	56.3	−0.00272	1.96	10.1	11.6
{10,1,2}	−0.145	1.63	24.8	0.0687	0.868	54.6	11.8	3.57
{10,1,3}	−0.314	1.17	1659.	183.9	0.901	0.577	118.9	10.6
{10,1,4}	−0.269	0.973	1709.	187.3	0.165	0.463	163.3	5.50
{10,2,2}	−0.500	1.14	18.5	0.0192	1.11	50.2	123.9	13.7
{10,2,3}	−0.253	1.37	1181.	118.6	0.842	0.962	46.5	34.3
{10,3,1}	−0.404	1.15	1993.	233.3	−0.0190	0.473	21.2	11.8
{10,3,2}	−0.408	1.12	2130.	257.6	−0.0297	0.427	10.3	7.22
{10,3,3}	−0.410	1.14	2046.	242.2	−0.133	0.455	50.3	10.4
{10,3,4}	−0.401	1.15	1983.	231.6	−0.00272	0.475	10.1	11.7

$\{i,j,k\}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\delta_{\rm CP}/\pi$	α_{21}/π	α_{31}/π	m_1/meV	m_2/meV	m_3/meV	$m_{\beta\beta}$/meV	m_β/meV
{1,1,2}	0.02240	0.312	0.468	1.04	1.01	1.18	2.24	8.88	49.8	2.30	9.12
{1,1,3}	0.02220	0.292	0.460	1.87	1.93	1.91	0.0495	8.60	49.7	3.40	8.71
{1,3,1}	0.02188	0.307	0.464	1.47	1.51	1.68	28.7	29.9	57.5	20.6	30.0
{1,3,2}	0.02205	0.307	0.483	0.759	0.588	1.18	13.8	16.2	51.6	9.31	16.3

Table 8. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C_i - D_j - N'_k$ with gCP symmetry (*continues...*).

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{2,2,2}	0.02210	0.322	0.488	1.70	0.864	0.629	8.66	12.2	50.4	1.64	12.4
{2,3,1}	0.02202	0.308	0.497	1.46	1.46	1.62	3.09	9.13	50.0	2.12	9.30
{2,3,2}	0.02232	0.305	0.438	1.45	1.42	0.644	2.18	8.87	49.8	3.70	9.06
{3,1,1}	0.02216	0.308	0.469	1.53	1.98	0.744	3.72	9.37	50.0	6.03	9.55
{3,3,2}	0.02159	0.303	0.437	1.12	0.351	0.0694	10.9	13.9	50.2	10.7	13.9
{4,1,2}	0.02196	0.319	0.444	1.13	0.683	1.31	0.151	8.60	49.7	3.22	8.80
{4,2,2}	0.02091	0.342	0.464	1.05	0.984	0.0634	3.99	9.48	49.5	0.432	9.56
{5,1,1}	0.02198	0.328	0.490	1.75	1.35	0.467	2.59	8.98	50.1	2.59	9.24
{5,1,3}	0.02217	0.308	0.470	1.81	0.243	0.278	0.0986	8.60	49.7	3.12	8.78
{5,2,1}	0.02246	0.288	0.449	1.84	0.994	1.01	2.33	8.91	50.3	1.73	9.10
{5,3,2}	0.02206	0.308	0.483	1.24	1.41	0.819	13.8	16.2	51.6	9.32	16.3
{6,1,3}	0.02251	0.327	0.522	1.10	1.20	0.645	6.21	10.6	50.0	1.71	10.8
{6,1,4}	0.02210	0.307	0.498	1.44	1.96	1.07	10.1	13.3	50.8	11.7	13.4
{6,2,2}	0.02295	0.290	0.439	1.99	0.999	0.0280	0.824	8.64	49.9	0.780	8.88
{6,2,3}	0.02192	0.306	0.547	1.79	1.12	0.633	1.14	8.67	49.8	2.95	8.81
{6,3,1}	0.02222	0.316	0.518	1.50	0.971	1.74	2.94	9.09	49.8	1.90	9.30
{6,3,2}	0.02220	0.300	0.503	1.46	0.977	1.74	3.09	9.14	50.0	1.69	9.30
{6,3,3}	0.02222	0.308	0.511	1.52	0.980	1.76	2.54	8.96	49.9	1.97	9.15
{6,3,4}	0.02222	0.315	0.518	1.49	0.971	1.73	3.00	9.10	49.8	1.87	9.31
{7,1,2}	0.02197	0.319	0.445	1.13	0.682	1.31	0.151	8.60	49.7	3.22	8.80
{7,1,3}	0.02220	0.292	0.460	1.87	1.93	1.91	0.0495	8.60	49.7	3.40	8.71
{7,2,2}	0.02092	0.341	0.465	1.05	0.984	0.0639	3.98	9.47	49.5	0.436	9.56
{7,3,1}	0.02189	0.307	0.465	1.47	1.51	1.68	28.7	30.0	57.5	20.6	30.0
{7,3,2}	0.02232	0.305	0.438	1.45	1.42	0.644	2.18	8.87	49.8	3.70	9.06
{8,1,1}	0.02216	0.308	0.469	1.53	1.98	0.744	3.72	9.37	50.0	6.03	9.55
{8,1,3}	0.02217	0.308	0.470	1.81	0.243	0.278	0.0986	8.60	49.7	3.12	8.78
{9,1,1}	0.02216	0.308	0.469	1.53	1.98	0.744	3.72	9.37	50.0	6.03	9.55
{9,1,3}	0.02251	0.327	0.522	1.10	1.20	0.645	6.21	10.6	50.0	1.71	10.8
{9,1,4}	0.02210	0.307	0.498	1.44	1.96	1.07	10.1	13.3	50.8	11.7	13.4
{9,2,2}	0.02295	0.290	0.439	1.99	0.999	0.0277	0.823	8.64	49.9	0.778	8.88
{9,2,3}	0.02192	0.306	0.547	1.79	1.12	0.633	1.14	8.67	49.8	2.95	8.82
{9,3,1}	0.02222	0.316	0.518	1.50	0.971	1.74	2.94	9.09	49.8	1.90	9.30
{9,3,2}	0.02220	0.300	0.503	1.46	0.976	1.74	3.09	9.14	50.0	1.69	9.30
{9,3,3}	0.02222	0.308	0.511	1.52	0.980	1.76	2.54	8.96	49.9	1.97	9.15
{9,3,4}	0.02222	0.315	0.518	1.49	0.971	1.73	3.00	9.10	49.8	1.87	9.31
{10,1,2}	0.02197	0.319	0.444	1.13	0.683	1.31	0.151	8.60	49.7	3.22	8.80
{10,1,3}	0.02251	0.327	0.522	1.10	1.20	0.645	6.21	10.6	50.0	1.71	10.8
{10,1,4}	0.02210	0.307	0.498	1.44	1.96	1.07	10.1	13.3	50.8	11.7	13.4
{10,2,2}	0.02094	0.341	0.456	1.0	1.00	2.00	3.99	9.48	49.5	0.448	9.57
{10,2,3}	0.02192	0.306	0.547	1.79	1.12	0.633	1.14	8.67	49.8	2.95	8.81
{10,3,1}	0.02222	0.316	0.518	1.50	0.971	1.74	2.94	9.09	49.8	1.90	9.30
{10,3,2}	0.02220	0.300	0.503	1.46	0.977	1.74	3.09	9.14	50.0	1.69	9.30
{10,3,3}	0.02222	0.308	0.511	1.52	0.980	1.76	2.54	8.96	49.9	1.97	9.15
{10,3,4}	0.02222	0.315	0.518	1.49	0.971	1.73	3.00	9.10	49.8	1.87	9.31

Table 8. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C_i - D_j - N'_k$ with gCP symmetry.

`Models`	Best fit results for 98 viable models $C'_i - D'_j - N_k$ without gCP							
$\{i,j,k\}$	$\Re\langle\tau\rangle$	$\Im\langle\tau\rangle$	β/α	γ/α	$\lvert h_2/h_1\rvert$	$\arg(h_2/h_1)/\pi$	αv/MeV	$(g_1^2v^2/(h_1\Lambda))$/meV	$\chi^2_{\rm min}$
{1,1,1}	−0.232	1.08	40.3	1215.	0.763	2.00	5.67	146.7	0.0920
{1,1,2}	0.422	1.03	8.96	0.0122	0.317	1.97	389.5	98.4	2.60
{1,1,3}	0.354	1.06	79.7	2381.	1.62	1.98	3.30	161.8	7.76
{1,1,4}	0.183	1.06	28.1	930.9	0.220	0.310	7.54	55.5	0.470
{1,2,1}	0.393	1.01	91.6	3288.	0.537	1.19	2.85	18.8	0.00437
{1,2,4}	0.415	1.00	100.4	3650.	0.0460	1.29	2.65	12.2	0.00189
{1,3,1}	−0.187	1.21	39.2	702.6	1.03	2.00	7.02	41.4	0.196
{1,3,2}	−0.407	0.959	93.7	4088.	0.120	0.865	2.71	3.38	0.00419
{1,3,3}	−0.251	1.19	0.0649	0.00231	1.53	0.0600	3698.	26.0	0.363
{1,4,1}	−0.301	1.02	56.6	2013.	3.28	1.34	4.11	17.6	2.23
{2,1,1}	0.493	0.913	4.47	26.1	0.180	0.0900	56.1	39.2	0.234
{2,1,2}	0.349	1.03	388.0	13.9	0.360	1.90	8.92	136.9	0.0528
{2,1,3}	−0.286	1.01	42.8	218.4	1.57	1.99	7.10	208.6	0.00286
{2,2,4}	0	1.29	0.0110	5.75	0.442	0.00895	302.4	36.2	15.3
{2,3,1}	−0.359	1.05	394.8	15.4	1.32	1.81	8.45	49.1	0.400
{2,3,2}	0.0454	1.13	376.2	882.1	1.16	1.95	1.81	48.6	0.213
{2,3,4}	0.0616	1.02	1421.	139.8	0.174	1.80	2.26	11.1	20.1
{3,1,1}	0.0595	1.15	0.00286	1.61	0.762	0.0000972	947.1	102.3	0.00168
{3,1,2}	0.0917	1.11	0.00537	1.84	0.967	1.95	795.2	162.6	0.0584
{3,1,3}	−0.179	1.17	26.5	179.6	1.32	1.99	8.55	97.6	22.8
{3,1,4}	−0.185	0.998	12.0	56.8	0.113	1.86	21.2	40.0	0.0133
{3,2,4}	0.180	1.27	0.00926	4.31	0.409	0.184	381.2	44.0	19.1
{3,3,1}	−0.319	0.963	321.0	11.0	1.23	0.128	11.7	31.8	1.20
{3,3,3}	−0.250	1.19	0.0652	0.000759	1.53	0.0609	3686.	26.3	0.362
{3,3,4}	0.0556	1.14	0.00267	1.44	0.316	0.484	1040.	29.8	0.0272
{4,1,1}	−0.422	0.982	635.0	6.42	0.110	1.46	16.5	43.9	0.162
{4,1,2}	0.396	0.966	636.0	5.90	0.0358	1.09	16.8	16.4	0.259
{4,1,3}	−0.117	1.15	262.7	6.07	1.36	0.00988	21.8	104.1	0.0308
{4,1,4}	−0.418	0.981	637.0	6.40	0.00892	1.34	16.5	23.2	0.256
{4,2,4}	0.0396	1.18	15.9	59.4	0.408	0.0794	28.3	68.1	4.20
{4,3,1}	0.0751	1.17	213.1	5.59	0.712	0.0795	25.3	39.6	1.26
{4,3,3}	0.117	1.18	266.6	7.56	0.977	0.0474	19.8	44.1	3.40
{4,3,4}	0.414	0.961	620.5	5.56	0.0502	0.948	17.9	4.02	0.0426
{4,4,2}	0.384	1.10	609.5	13.1	0.229	1.47	11.4	4.37	7.66
{5,1,1}	0.0595	1.15	0.110	1.61	0.762	0.0000902	944.0	102.3	0.00102
{5,1,2}	0.0917	1.11	0.0409	1.84	0.967	1.95	794.9	162.6	0.0584
{5,1,3}	−0.0755	1.24	24.5	38.8	1.26	1.99	41.6	62.2	0.0228
{5,1,4}	−0.160	1.07	149.3	193.4	0.197	1.84	7.08	36.2	0.0755
{5,2,4}	−0.222	0.983	1769.	28.2	0.0821	1.97	4.52	5.73	0.387
{5,3,1}	−0.305	1.30	0.0679	6.04	1.14	1.90	271.5	75.4	0.342
{5,3,2}	−0.147	1.11	113.6	156.4	0.446	0.300	9.22	41.4	0.172
{5,3,3}	0.448	1.04	269.3	450.3	1.04	1.08	2.64	62.0	0.0756
{5,3,4}	−0.280	1.12	166.4	318.1	0.0981	1.75	4.40	24.4	0.0186
{5,4,1}	0.431	1.13	1915.	41.3	1.32	0.0757	3.32	16.5	13.0
{5,4,2}	0.329	1.02	15.4	0.00105	0.128	0.786	558.8	1.97	0.0131
{5,4,3}	0.330	1.10	1909.	37.8	1.52	0.0231	3.61	6.24	19.1
{6,1,1}	0.0594	1.15	0.00283	1.61	0.762	0.0000943	947.3	102.3	0.00162
{6,1,2}	0.108	1.10	0.0221	0.000854	0.876	1.93	5320.	160.6	0.00610
{6,1,3}	−0.306	1.06	143.6	7.23	1.43	0.00324	11.3	217.7	0.156
{6,1,4}	−0.246	1.02	0.00687	2.86	0.199	1.79	431.6	93.9	0.241
{6,2,3}	0.252	0.985	6.16	77.5	1.18	0.0150	14.7	190.3	1.35

Table 9. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C'_i - D'_j - N_k$ without gCP symmetry (*continues...*).

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{6,2,4}	0.0539	1.08	46.1	2.53	0.309	0.196	34.6	80.7	0.227
{6,3,1}	0.108	1.27	12.6	145.4	0.762	0.0789	11.5	49.6	0.476
{6,3,3}	0.279	1.22	18.1	210.0	1.50	1.90	7.47	37.6	0.0256
{6,3,4}	0.0557	1.14	0.00254	1.44	0.316	0.484	1038.	29.8	0.0261
{7,1,1}	−0.422	0.982	7323.	74.1	0.110	1.46	1.43	43.9	0.161
{7,1,2}	0.0777	1.09	1911.	32.2	0.903	1.95	3.35	153.3	0.412
{7,1,3}	−0.286	1.01	0.0127	5.11	1.57	1.99	303.6	208.6	0.00305
{7,1,4}	−0.418	0.981	7251.	72.9	0.00892	1.34	1.45	23.2	0.255
{7,2,1}	0.393	1.01	35.9	0.00261	0.537	1.19	260.7	18.8	0.00436
{7,2,4}	0.0397	1.18	134.6	499.6	0.408	0.0798	3.37	68.3	4.18
{7,3,1}	0.0752	1.17	1634.	42.9	0.712	0.0796	3.30	39.6	1.26
{7,3,2}	−0.407	0.959	43.6	0.00223	0.120	0.865	253.6	3.38	0.00403
{7,3,3}	−0.199	1.18	20.4	0.00645	1.25	0.133	261.9	45.7	1.13
{7,3,4}	0.414	0.961	7750.	69.4	0.0502	0.948	1.43	4.02	0.0430
{7,4,1}	−0.301	1.02	35.6	0.00402	3.28	1.34	232.9	17.6	2.23
{7,4,2}	0.384	1.10	3919.	84.3	0.229	1.47	1.77	4.37	7.65
{8,1,1}	0.0542	1.15	446.4	498.2	0.761	0.000119	3.03	103.4	0.153
{8,1,2}	0.0494	1.12	536.5	474.3	1.08	0	3.05	159.3	2.78
{8,1,3}	−0.0757	1.24	335.1	530.4	1.26	1.99	3.04	62.2	0.0253
{8,1,4}	−0.185	0.998	0.0511	4.72	0.113	1.86	254.2	39.9	0.0132
{8,2,1}	0.393	1.01	35.9	0.00587	0.537	1.19	260.7	18.8	0.00465
{8,2,4}	0.222	0.977	2283.	35.8	0.0816	0.0221	3.51	5.75	0.366
{8,3,1}	0.0524	1.27	331.4	534.6	0.687	1.98	3.06	47.1	1.32
{8,3,2}	−0.407	0.959	43.6	0.00628	0.120	0.865	253.5	3.38	0.00430
{8,3,3}	0.448	1.04	163.0	272.7	1.04	1.08	4.36	62.0	0.0758
{8,3,4}	−0.280	1.12	223.3	426.4	0.0979	1.75	3.28	24.4	0.0118
{8,4,1}	−0.408	1.13	1802.	38.6	2.13	0.158	3.56	24.3	13.3
{8,4,3}	0.334	1.10	1982.	38.8	1.52	0.0231	3.48	6.24	19.4
{9,1,1}	−0.499	0.906	32.0	1.17	0.156	0.0804	45.5	39.2	0.0117
{9,1,2}	−0.0918	1.11	28.6	410.1	0.979	0.0521	3.59	163.9	0.0579
{9,1,3}	−0.307	1.07	285.8	14.5	1.43	0.00296	5.67	214.2	0.159
{9,1,4}	0.322	0.960	203.6	8.63	0.165	0.195	7.36	144.8	8.12
{9,2,3}	0.252	0.986	16.9	212.4	1.18	0.0149	5.38	190.2	1.33
{9,2,4}	0.0539	1.08	447.5	24.6	0.306	0.199	3.56	80.4	0.215
{9,3,1}	0.108	1.27	40.3	463.5	0.763	0.0789	3.61	49.6	0.476
{9,3,3}	0.279	1.22	28.8	334.5	1.50	1.90	4.69	37.6	0.0260
{9,3,4}	−0.371	1.12	0.00397	3.13	0.0919	1.88	434.4	20.0	6.72
{10,1,1}	−0.498	0.906	354.3	13.2	0.156	0.0813	4.11	39.3	0.0119
{10,1,2}	−0.0918	1.11	5.93	84.9	0.979	0.0522	17.4	163.9	0.0596
{10,1,3}	−0.306	1.06	283.2	14.3	1.43	0.00322	5.71	217.4	0.156
{10,1,4}	0.323	0.960	301.7	12.8	0.165	0.195	4.97	144.9	8.10
{10,2,3}	0.252	0.986	15.1	189.7	1.18	0.0149	6.03	190.2	1.31
{10,2,4}	0.0538	1.08	56.4	3.10	0.309	0.196	28.2	80.7	0.225
{10,3,1}	0.108	1.27	7.54	86.8	0.763	0.0789	19.3	49.6	0.476
{10,3,2}	0.159	1.10	0.00105	1.41	0.392	1.70	1012.	39.4	0.265
{10,3,3}	0.279	1.22	17.4	202.0	1.50	1.90	7.76	37.6	0.0256
{10,3,4}	0.199	1.13	0.00132	1.69	0.137	0.338	859.1	23.8	0.213

$\{i,j,k\}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	δ_{CP}/π	α_{21}/π	α_{31}/π	m_1/meV	m_2/meV	m_3/meV	$m_{\beta\beta}$/meV	m_β/meV
{1,1,1}	0.02215	0.308	0.470	1.22	1.74	0.0609	14.0	16.5	51.7	14.1	16.6
{1,1,2}	0.02213	0.306	0.475	1.41	1.31	0.456	10.3	13.4	50.9	6.81	13.5
{1,1,3}	0.02223	0.307	0.463	1.56	0.723	0.333	5.88	10.4	50.6	2.01	10.6
{1,1,4}	0.02215	0.308	0.470	1.02	1.35	1.95	1.80	8.78	49.8	2.90	8.97
{1,2,1}	0.02215	0.308	0.470	1.16	0.903	0.270	0.180	8.60	49.8	1.39	8.79

Table 9. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C'_i - D'_j - N_k$ without gCP symmetry (*continues...*).

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{1,2,4}	0.02215	0.308	0.470	1.17	0.877	0.410	0.464	8.61	49.8	1.59	8.80
{1,3,1}	0.02215	0.308	0.470	1.08	1.82	1.92	9.04	12.5	50.6	10.5	12.6
{1,3,2}	0.02215	0.308	0.470	1.19	0.638	1.33	2.65	9.00	49.9	2.65	9.18
{1,3,3}	0.02214	0.308	0.470	1.04	1.52	1.43	1.88	8.80	49.8	3.76	8.98
{1,4,1}	0.02216	0.309	0.469	1.39	0.791	1.55	0	8.60	49.8	3.70	8.79
{2,1,1}	0.02211	0.306	0.473	1.24	1.06	1.11	6.21	10.6	50.2	0.774	10.7
{2,1,2}	0.02215	0.308	0.470	1.13	0.847	1.46	16.9	19.0	52.6	5.80	19.1
{2,1,3}	0.02215	0.308	0.470	1.19	0.453	1.41	8.59	12.2	50.5	6.29	12.3
{2,2,4}	0.02206	0.305	0.471	1.74	1.37	1.12	0.500	8.61	49.7	3.36	8.77
{2,3,1}	0.02222	0.315	0.467	1.18	0.990	1.32	11.3	14.2	51.0	2.09	14.3
{2,3,2}	0.02214	0.307	0.470	1.24	1.11	1.52	11.2	14.1	51.0	2.87	14.2
{2,3,4}	0.02218	0.297	0.441	1.75	1.11	0.0478	4.91	9.90	50.1	0.530	10.0
{3,1,1}	0.02215	0.308	0.470	1.18	0.688	0.201	11.9	14.6	51.2	7.25	14.8
{3,1,2}	0.02215	0.308	0.471	1.12	1.29	0.101	7.16	11.2	50.3	4.89	11.3
{3,1,3}	0.02215	0.308	0.472	1.87	1.70	0.457	4.76	9.83	50.1	4.53	10.0
{3,1,4}	0.02215	0.308	0.470	1.15	0.844	0.882	2.58	8.97	49.9	2.52	9.15
{3,2,4}	0.02235	0.297	0.461	1.79	1.02	0.452	0.642	8.62	49.9	3.10	8.80
{3,3,1}	0.02216	0.307	0.471	1.33	1.28	0.428	9.14	12.5	50.6	5.91	12.7
{3,3,3}	0.02213	0.308	0.469	1.04	1.53	1.44	1.91	8.81	49.8	3.78	8.99
{3,3,4}	0.02215	0.308	0.470	1.20	1.41	0.252	6.80	11.0	50.3	5.97	11.1
{4,1,1}	0.02215	0.309	0.470	1.09	1.12	0.510	1.35	8.70	49.8	0.993	8.89
{4,1,2}	0.02215	0.308	0.470	1.06	0.784	1.58	0.947	8.65	49.8	1.62	8.84
{4,1,3}	0.02215	0.308	0.470	1.20	0.258	0.222	5.88	10.4	50.1	7.27	10.6
{4,1,4}	0.02215	0.309	0.470	1.06	1.15	0.557	1.27	8.69	49.8	1.26	8.88
{4,2,4}	0.02205	0.306	0.468	1.47	1.37	1.15	0.866	8.64	49.8	1.76	8.80
{4,3,1}	0.02218	0.308	0.470	1.34	0.0918	0.0663	11.2	14.2	51.1	11.3	14.3
{4,3,3}	0.02228	0.308	0.470	1.44	0.150	0.0615	4.21	9.57	50.0	4.53	9.76
{4,3,4}	0.02215	0.308	0.470	1.13	0.754	1.47	2.38	8.92	49.9	1.70	9.10
{4,4,2}	0.02223	0.306	0.471	1.58	1.74	0.183	0	8.60	49.9	2.08	8.80
{5,1,1}	0.02215	0.308	0.470	1.18	0.688	0.201	11.9	14.7	51.2	7.26	14.8
{5,1,2}	0.02215	0.308	0.471	1.12	1.29	0.100	7.16	11.2	50.3	4.89	11.3
{5,1,3}	0.02214	0.308	0.470	1.20	1.61	0.161	3.93	9.45	50.0	5.59	9.62
{5,1,4}	0.02216	0.308	0.470	1.22	1.30	1.22	1.80	8.78	49.8	1.89	8.97
{5,2,4}	0.02218	0.305	0.476	1.07	1.00	0.158	0.217	8.60	49.8	1.31	8.78
{5,3,1}	0.02216	0.308	0.471	1.05	0.499	0.498	11.1	14.0	51.0	9.47	14.1
{5,3,2}	0.02217	0.308	0.469	1.09	1.43	0.595	24.7	26.1	55.6	16.5	26.2
{5,3,3}	0.02216	0.311	0.467	1.17	0.721	1.00	6.02	10.5	50.2	3.71	10.7
{5,3,4}	0.02215	0.308	0.469	1.15	0.0414	0.269	22.1	23.7	54.5	23.3	23.8
{5,4,1}	0.02219	0.312	0.460	1.69	1.78	1.92	0	8.60	49.8	1.96	8.81
{5,4,2}	0.02215	0.308	0.470	1.15	0.405	1.03	0	8.60	49.8	3.30	8.78
{5,4,3}	0.02297	0.326	0.447	1.69	1.43	1.56	0	8.60	49.7	2.12	8.96
{6,1,1}	0.02215	0.308	0.470	1.18	0.688	0.201	11.9	14.6	51.2	7.25	14.8
{6,1,2}	0.02217	0.307	0.470	1.18	1.16	0.703	8.77	12.3	50.6	3.38	12.4
{6,1,3}	0.02214	0.308	0.472	1.23	1.70	1.12	11.9	14.7	51.2	10.5	14.8
{6,1,4}	0.02216	0.308	0.470	1.25	0.854	1.05	4.27	9.60	50.0	2.38	9.77
{6,2,3}	0.02191	0.305	0.482	1.30	1.07	1.38	4.45	9.68	50.1	0.645	9.82
{6,2,4}	0.02212	0.306	0.468	1.24	1.13	1.10	1.22	8.68	49.8	1.96	8.86
{6,3,1}	0.02216	0.308	0.470	1.28	0.0999	1.94	9.93	13.1	50.8	10.1	13.3
{6,3,3}	0.02215	0.308	0.470	1.14	0.878	0.145	2.57	8.97	49.9	0.598	9.15
{6,3,4}	0.02215	0.308	0.470	1.20	1.41	0.252	6.80	11.0	50.3	5.98	11.1
{7,1,1}	0.02215	0.309	0.470	1.09	1.12	0.510	1.35	8.70	49.8	0.991	8.89
{7,1,2}	0.02223	0.305	0.465	1.25	1.14	0.536	8.14	11.8	50.5	3.70	12.0
{7,1,3}	0.02215	0.308	0.470	1.19	0.453	1.41	8.59	12.2	50.5	6.29	12.3

Table 9. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C'_i - D'_j - N_k$ without gCP symmetry (*continues. . .*).

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{7,1,4}	0.02215	0.309	0.470	1.06	1.15	0.557	1.27	8.69	49.8	1.25	8.88
{7,2,1}	0.02215	0.308	0.470	1.16	0.903	0.270	0.180	8.60	49.8	1.39	8.79
{7,2,4}	0.02205	0.306	0.468	1.47	1.37	1.15	0.868	8.64	49.8	1.76	8.80
{7,3,1}	0.02218	0.308	0.470	1.34	0.0916	0.0661	11.2	14.2	51.1	11.3	14.3
{7,3,2}	0.02215	0.308	0.470	1.19	0.638	1.33	2.65	9.00	49.9	2.65	9.18
{7,3,3}	0.02216	0.308	0.470	1.33	1.94	1.82	3.33	9.22	49.9	4.15	9.40
{7,3,4}	0.02215	0.308	0.470	1.13	0.754	1.47	2.38	8.92	49.9	1.70	9.10
{7,4,1}	0.02216	0.309	0.469	1.39	0.686	1.45	0	8.60	49.8	3.70	8.79
{7,4,2}	0.02223	0.306	0.472	1.58	0.329	0.768	0	8.60	49.9	2.08	8.80
{8,1,1}	0.02214	0.308	0.471	1.09	0.622	0.163	12.2	15.0	51.3	8.76	15.1
{8,1,2}	0.02206	0.311	0.480	1.40	1.31	0.229	5.39	10.1	49.9	3.97	10.3
{8,1,3}	0.02214	0.308	0.470	1.20	1.61	0.162	3.93	9.45	50.0	5.60	9.62
{8,1,4}	0.02215	0.308	0.470	1.15	0.844	0.883	2.57	8.97	49.9	2.51	9.15
{8,2,1}	0.02215	0.308	0.470	1.16	0.903	0.270	0.180	8.60	49.8	1.39	8.79
{8,2,4}	0.02217	0.307	0.474	1.05	0.963	0.141	0.215	8.60	49.8	1.38	8.78
{8,3,1}	0.02225	0.308	0.468	1.34	0.282	0.112	9.85	13.1	50.8	9.12	13.2
{8,3,2}	0.02215	0.308	0.470	1.19	0.638	1.33	2.65	9.00	49.9	2.65	9.18
{8,3,3}	0.02216	0.311	0.467	1.17	0.720	1.00	6.02	10.5	50.2	3.72	10.7
{8,3,4}	0.02215	0.308	0.470	1.15	0.0420	0.270	22.2	23.8	54.5	23.3	23.8
{8,4,1}	0.02219	0.312	0.460	1.69	0.473	0.289	0	8.60	49.8	3.07	8.81
{8,4,3}	0.02304	0.329	0.453	1.70	1.10	1.23	0	8.60	49.7	2.14	8.98
{9,1,1}	0.02214	0.307	0.471	1.19	1.05	1.12	7.38	11.3	50.3	0.897	11.5
{9,1,2}	0.02214	0.309	0.471	1.21	0.898	0.0534	7.03	11.1	50.3	2.02	11.3
{9,1,3}	0.02215	0.308	0.471	1.24	1.69	1.12	11.8	14.6	51.2	10.4	14.7
{9,1,4}	0.02205	0.311	0.463	1.58	0.972	0.480	5.91	10.4	50.1	0.680	10.6
{9,2,3}	0.02191	0.305	0.482	1.29	1.07	1.38	4.45	9.68	50.1	0.646	9.82
{9,2,4}	0.02211	0.306	0.468	1.24	1.13	1.09	1.23	8.68	49.8	1.98	8.86
{9,3,1}	0.02216	0.308	0.470	1.28	0.0996	1.94	9.93	13.1	50.8	10.1	13.3
{9,3,3}	0.02215	0.308	0.470	1.14	0.878	0.145	2.57	8.97	49.9	0.596	9.15
{9,3,4}	0.02199	0.309	0.519	1.11	1.52	0.125	18.3	20.2	53.1	15.3	20.3
{10,1,1}	0.02214	0.307	0.471	1.19	1.05	1.13	7.41	11.3	50.3	0.899	11.5
{10,1,2}	0.02214	0.309	0.471	1.21	0.898	0.0539	7.03	11.1	50.3	2.02	11.3
{10,1,3}	0.02214	0.308	0.472	1.23	1.70	1.12	11.9	14.7	51.2	10.5	14.8
{10,1,4}	0.02205	0.311	0.463	1.58	0.972	0.480	5.91	10.4	50.1	0.682	10.6
{10,2,3}	0.02191	0.305	0.482	1.29	1.07	1.38	4.46	9.68	50.1	0.646	9.82
{10,2,4}	0.02211	0.306	0.468	1.24	1.13	1.10	1.22	8.68	49.8	1.95	8.86
{10,3,1}	0.02216	0.308	0.470	1.28	0.0998	1.94	9.93	13.1	50.8	10.1	13.3
{10,3,2}	0.02217	0.308	0.468	1.06	0.621	1.43	27.2	28.5	56.7	16.1	28.6
{10,3,3}	0.02215	0.308	0.470	1.14	0.878	0.145	2.57	8.97	49.9	0.597	9.15
{10,3,4}	0.02215	0.308	0.470	1.07	1.94	0.0463	15.6	17.8	52.2	17.0	17.9

Table 9. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C'_i - D'_j - N_k$ without gCP symmetry.

`Models`	Best fit results for 52 models $C'_i - D'_j - N_k$ with gCP							
$\{i,j,k\}$	$\Re\langle\tau\rangle$	$\Im\langle\tau\rangle$	β/α	γ/α	h_2/h_1	$\alpha v/\mathrm{MeV}$	$(g_1^2 v^2/(h_1\Lambda))/\mathrm{meV}$	$\chi^2_{\min}$
{1,1,2}	0.465	1.03	9.04	0.0117	0.289	392.0	90.6	4.30
{1,1,4}	0.402	0.986	0.00273	20.0	−0.00824	509.2	22.4	8.04
{1,3,1}	−0.185	1.21	38.3	703.0	1.02	7.10	40.7	0.620
{1,3,3}	−0.277	1.16	0.0585	0.00187	1.63	4024.	15.6	2.20
{2,1,1}	0.476	0.910	4.88	28.2	0.194	51.7	42.7	3.89

Table 10. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C'_i - D'_j - N_k$ with gCP symmetry (*continues. . .*)

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{2,2,4}	−0.00936	1.29	0.0110	5.77	0.444	301.7	34.0	18.4
{3,1,1}	−0.0598	1.15	0.00288	1.61	0.762	947.6	102.7	2.50
{3,1,4}	−0.163	1.15	0.00106	0.0373	0.302	4385.	78.2	11.2
{3,3,3}	0.277	1.16	0.0584	0.000678	1.63	4026.	15.6	2.85
{3,3,4}	−0.452	1.07	879.7	73.1	0.0923	3.66	12.4	31.3
{4,1,2}	0.0451	1.10	141.2	2.48	0.936	44.3	136.8	1.43
{4,1,3}	0.232	1.10	446.9	8.51	1.35	14.6	163.3	23.4
{4,2,4}	−0.00938	1.29	0.104	5.78	0.444	301.1	34.1	18.4
{4,3,1}	−0.443	1.04	648.1	9.61	2.51	13.5	93.6	31.1
{4,3,3}	0.112	1.18	259.3	7.11	0.933	20.6	44.5	8.86
{5,1,1}	0.0597	1.15	0.110	1.61	0.762	944.7	102.7	0.0156
{5,1,2}	0.0493	1.12	18.4	16.3	1.08	88.8	159.3	2.83
{5,1,3}	−0.458	1.12	2035.	41.3	2.14	3.20	101.1	19.9
{5,1,4}	−0.163	1.15	0.00424	0.0373	0.302	4386.	78.3	11.2
{5,3,1}	0.0595	1.28	39.4	65.3	0.680	25.2	49.1	15.1
{5,3,3}	0.413	1.02	274.6	436.1	−0.705	2.66	42.1	11.4
{6,1,1}	0.0597	1.15	0.00283	1.61	0.762	948.1	102.7	0.0166
{6,1,2}	0.0515	1.11	3.45	50.0	0.956	29.8	140.0	23.3
{6,1,3}	−0.330	1.10	169.1	9.07	1.41	9.78	179.7	0.771
{6,1,4}	−0.163	1.15	0.000673	0.0372	0.301	4393.	78.4	11.2
{6,2,3}	−0.249	0.969	5.84	72.3	1.19	15.3	199.6	6.31
{6,2,4}	−0.00937	1.29	5.78	0.0243	0.444	301.3	34.1	18.4
{6,3,3}	0.277	1.09	0.000598	0.0191	1.74	5105.	15.6	19.9
{7,1,2}	0.0451	1.10	1830.	32.2	0.936	3.42	136.8	1.43
{7,1,3}	0.232	1.10	2558.	48.7	1.35	2.56	163.2	23.2
{7,1,4}	−0.328	1.05	31.1	0.00369	1.66	250.8	746.2	23.4
{7,2,4}	−0.0133	1.26	144.8	506.6	0.447	3.41	41.9	23.9
{7,3,1}	−0.185	1.21	18.3	0.00671	1.02	272.3	40.7	0.623
{7,3,3}	0.112	1.18	1680.	46.1	0.933	3.18	44.5	8.98
{8,1,1}	0.0526	1.15	451.9	496.9	0.761	3.03	104.3	0.215
{8,1,2}	0.0497	1.12	535.1	474.5	1.08	3.05	159.0	2.79
{8,1,3}	−0.458	1.12	1786.	36.2	2.14	3.65	101.0	19.9
{8,2,4}	0.221	0.977	2282.	35.9	0.0816	3.51	5.75	1.15
{8,3,1}	−0.185	1.21	18.3	0.0107	1.02	272.3	40.7	0.623
{8,3,3}	0.413	1.02	166.3	264.1	−0.705	4.40	42.1	11.4
{9,1,1}	0.490	0.905	33.0	1.20	0.160	44.1	40.6	0.814
{9,1,2}	0.0515	1.11	29.4	425.3	0.956	3.50	139.9	23.1
{9,1,3}	−0.330	1.10	281.8	15.1	1.41	5.87	179.7	0.782
{9,1,4}	−0.311	0.998	14.5	177.9	0.585	6.45	406.1	37.2
{9,2,3}	−0.250	0.970	16.3	201.8	1.19	5.49	199.2	6.27
{9,2,4}	−0.0539	0.999	393.3	19.5	0.0637	3.78	22.2	2.80
{10,1,1}	−0.490	0.905	353.9	13.1	0.159	4.11	40.6	1.55
{10,1,2}	0.0451	1.10	0.0176	0.00118	0.936	6262.	136.8	1.43
{10,1,3}	−0.330	1.10	290.4	15.6	1.41	5.70	179.7	0.770
{10,1,4}	−0.311	0.998	17.8	218.7	0.585	5.24	406.4	37.1
{10,2,3}	−0.249	0.969	14.9	184.3	1.19	6.00	199.7	6.25
{10,2,4}	−0.0137	1.26	16.8	1.29	0.447	103.2	43.2	23.6

Table 10. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C'_i - D'_j - N_k$ with gCP symmetry (*continues. . . .*)

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$\{i,j,k\}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\delta_{\rm CP}/\pi$	α_{21}/π	α_{31}/π	m_1/meV	m_2/meV	m_3/meV	$m_{\beta\beta}$/meV	m_β/meV
{1,1,2}	0.02191	0.307	0.499	1.37	1.29	0.480	9.45	12.8	50.7	6.12	12.9
{1,1,4}	0.02184	0.284	0.489	1.36	1.03	1.53	1.10	8.67	50.3	2.59	8.77
{1,3,1}	0.02210	0.306	0.480	1.08	1.82	1.91	8.97	12.4	50.4	10.4	12.5
{1,3,3}	0.02164	0.308	0.453	1.05	1.19	1.69	1.00	8.66	49.8	2.74	8.77
{2,1,1}	0.02180	0.297	0.499	1.07	1.02	0.908	6.57	10.8	50.4	0.782	10.9
{2,2,4}	0.02205	0.308	0.476	1.79	1.36	1.08	0.468	8.61	49.7	3.53	8.77
{3,1,1}	0.02214	0.309	0.473	0.820	1.32	1.80	12.0	14.7	51.2	7.35	14.8
{3,1,4}	0.02227	0.283	0.466	1.53	1.24	0.851	3.88	9.43	50.0	2.98	9.53
{3,3,3}	0.02164	0.307	0.453	0.953	0.815	0.313	1.00	8.65	49.8	2.74	8.77
{3,3,4}	0.02167	0.300	0.573	1.10	1.18	1.74	10.6	13.7	50.3	5.22	13.7
{4,1,2}	0.02211	0.305	0.474	1.34	1.15	0.596	6.54	10.8	50.3	3.16	10.9
{4,1,3}	0.02340	0.298	0.507	1.58	0.408	0.0354	7.85	11.6	52.1	6.43	12.1
{4,2,4}	0.02205	0.308	0.476	1.79	1.36	1.08	0.468	8.61	49.7	3.53	8.77
{4,3,1}	0.02320	0.292	0.484	1.71	0.306	0.108	12.8	15.4	53.7	11.6	15.7
{4,3,3}	0.02108	0.310	0.477	1.50	0.205	0.142	4.26	9.60	49.7	4.45	9.60
{5,1,1}	0.02215	0.308	0.472	1.18	0.685	0.199	11.9	14.7	51.2	7.34	14.8
{5,1,2}	0.02205	0.312	0.480	1.40	1.31	0.229	5.40	10.1	49.9	3.96	10.3
{5,1,3}	0.02223	0.306	0.454	1.81	1.98	1.01	0.951	8.65	49.8	3.15	8.84
{5,1,4}	0.02227	0.283	0.466	1.53	1.24	0.851	3.88	9.43	50.0	2.99	9.53
{5,3,1}	0.02381	0.312	0.438	1.27	0.252	0.0796	10.0	13.2	51.9	9.87	13.6
{5,3,3}	0.02212	0.311	0.467	1.66	0.794	1.70	3.63	9.33	49.9	2.80	9.51
{6,1,1}	0.02215	0.308	0.472	1.18	0.686	0.200	12.0	14.7	51.2	7.33	14.8
{6,1,2}	0.02129	0.333	0.546	1.14	1.13	0.304	6.26	10.6	49.6	2.34	10.7
{6,1,3}	0.02223	0.307	0.460	1.26	1.64	1.20	10.7	13.7	50.9	8.78	13.8
{6,1,4}	0.02227	0.283	0.467	1.53	1.24	0.849	3.88	9.43	50.0	2.98	9.53
{6,2,3}	0.02160	0.287	0.487	1.00	0.999	1.00	4.82	9.86	50.4	0.493	9.92
{6,2,4}	0.02205	0.308	0.476	1.79	1.36	1.08	0.468	8.61	49.7	3.53	8.77
{6,3,3}	0.02257	0.298	0.551	1.29	1.02	0.578	0.877	8.64	49.8	0.801	8.85
{7,1,2}	0.02212	0.305	0.474	1.34	1.15	0.596	6.54	10.8	50.3	3.16	10.9
{7,1,3}	0.02339	0.298	0.507	1.58	0.408	0.0353	7.85	11.6	52.1	6.43	12.1
{7,1,4}	0.02234	0.300	0.473	1.78	0.226	0.105	2.62	8.99	51.4	4.65	9.35
{7,2,4}	0.02252	0.284	0.441	1.74	1.56	1.13	0.454	8.61	50.3	3.59	8.81
{7,3,1}	0.02209	0.306	0.480	1.08	1.82	1.91	8.96	12.4	50.4	10.4	12.5
{7,3,3}	0.02107	0.310	0.478	1.50	0.205	0.142	4.26	9.60	49.7	4.45	9.60
{8,1,1}	0.02216	0.308	0.476	1.10	0.620	0.170	12.5	15.1	51.4	8.92	15.3
{8,1,2}	0.02205	0.312	0.479	1.40	1.31	0.229	5.39	10.1	49.9	3.97	10.3
{8,1,3}	0.02223	0.306	0.454	1.81	1.98	1.01	0.951	8.65	49.8	3.15	8.84
{8,2,4}	0.02221	0.305	0.472	0.946	1.02	1.87	0.214	8.60	49.8	1.33	8.78
{8,3,1}	0.02209	0.306	0.480	1.08	1.82	1.91	8.96	12.4	50.4	10.4	12.5
{8,3,3}	0.02212	0.311	0.467	1.66	0.794	1.70	3.63	9.33	49.9	2.80	9.51
{9,1,1}	0.02209	0.302	0.479	1.06	1.01	0.957	7.64	11.5	50.4	0.911	11.6
{9,1,2}	0.02129	0.333	0.545	1.14	1.13	0.304	6.26	10.6	49.6	2.34	10.7
{9,1,3}	0.02223	0.307	0.460	1.26	1.64	1.20	10.7	13.7	50.9	8.78	13.8
{9,1,4}	0.02136	0.291	0.538	1.75	1.10	1.83	4.58	9.74	51.4	1.12	9.90
{9,2,3}	0.02160	0.287	0.487	1.00	0.997	0.999	4.82	9.86	50.4	0.493	9.92
{9,2,4}	0.02193	0.297	0.454	1.01	0.989	1.01	1.36	8.70	50.0	2.69	8.84
{10,1,1}	0.02208	0.302	0.479	0.946	0.986	1.04	7.68	11.5	50.4	0.913	11.6

Table 10. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C'_i - D'_j - N_k$ with gCP symmetry (*continues. . . .*)

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{10,1,2}	0.02212	0.305	0.474	1.34	1.15	0.596	6.54	10.8	50.3	3.16	10.9
{10,1,3}	0.02223	0.307	0.460	1.26	1.64	1.20	10.7	13.7	50.9	8.78	13.8
{10,1,4}	0.02136	0.291	0.538	1.75	1.10	1.83	4.58	9.74	51.4	1.12	9.90
{10,2,3}	0.02161	0.287	0.487	1.00	0.999	1.00	4.82	9.86	50.4	0.492	9.92
{10,2,4}	0.02248	0.285	0.443	1.75	1.57	1.12	0.445	8.61	50.2	3.62	8.80

Table 10. The best fit values of the input parameters, neutrino masses and mixing parameters at the minimum values of χ^2 for viable models $C'_i - D'_j - N_k$ with gCP symmetry.

A Finite modular group $\Gamma_5 \cong A_5$

The finite modular group $\Gamma_5 \cong A_5$ can be generated by the modular generators S and T which satisfy the multiplication rules:

$$S^2 = T^5 = (ST)^3 = 1\,. \tag{A.1}$$

The A_5 group has five irreducible representations: one singlet representation $\mathbf{1}$, two three-dimensional representations $\mathbf{3}$ and $\mathbf{3'}$, one four-dimensional representation $\mathbf{4}$ and one five-dimensional representation $\mathbf{5}$. In the present work, we shall follow the conventions of refs. [66–69] and the explicit forms of the generators S and T in the five irreducible representations are as follows:

$$\begin{aligned}
&\mathbf{1}: \quad S = 1\,, && T = 1\,,\\
&\mathbf{3}: \quad S = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\phi & \phi-1 \\ -\sqrt{2} & \phi-1 & -\phi \end{pmatrix}, && T = \mathrm{diag}(1,\omega_5,\omega_5^4)\,,\\
&\mathbf{3'}: \quad S = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1-\phi & \phi \\ \sqrt{2} & \phi & 1-\phi \end{pmatrix}, && T = \mathrm{diag}(1,\omega_5^2,\omega_5^3)\,,\\
&\mathbf{4}: \quad S = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & \phi-1 & \phi & -1 \\ \phi-1 & -1 & 1 & \phi \\ \phi & 1 & -1 & \phi-1 \\ -1 & \phi & \phi-1 & 1 \end{pmatrix}, && T = \mathrm{diag}(\omega_5,\omega_5^2,\omega_5^3,\omega_5^4)\,,\\
&\mathbf{5}: \quad S = \frac{1}{5}\begin{pmatrix} -1 & \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{6} \\ \sqrt{6} & (\phi-1)^2 & -2\phi & 2(\phi-1) & \phi^2 \\ \sqrt{6} & -2\phi & \phi^2 & (\phi-1)^2 & 2(\phi-1) \\ \sqrt{6} & 2(\phi-1) & (\phi-1)^2 & \phi^2 & -2\phi \\ \sqrt{6} & \phi^2 & 2(\phi-1) & -2\phi & (\phi-1)^2 \end{pmatrix}, && T = \mathrm{diag}(1,\omega_5,\omega_5^2,\omega_5^3,\omega_5^4)\,,
\end{aligned} \tag{A.2}$$

where $\omega_5 = e^{2\pi i/5}$ denotes the quintic root of unit and $\phi = (1+\sqrt{5})/2$ is the golden ratio. Then the Kronecker products between various irreducible representations follow immediately

$$\begin{aligned}
&\mathbf{1}\otimes\boldsymbol{R}=\boldsymbol{R}\otimes\mathbf{1}=\boldsymbol{R}, \quad \mathbf{3}\otimes\mathbf{3}=\mathbf{1_S}\oplus\mathbf{3_A}\oplus\mathbf{5_S}, \quad \mathbf{3'}\otimes\mathbf{3'}=\mathbf{1_S}\oplus\mathbf{3'_A}\oplus\mathbf{5_S}, \quad \mathbf{3}\times\mathbf{3'}=\mathbf{4}\oplus\mathbf{5},\\
&\mathbf{3}\otimes\mathbf{4}=\mathbf{3'}\oplus\mathbf{4}\oplus\mathbf{5}, \quad \mathbf{3'}\otimes\mathbf{4}=\mathbf{3}\oplus\mathbf{4}\oplus\mathbf{5}, \quad \mathbf{3}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3'}\oplus\mathbf{4}\oplus\mathbf{5},\\
&\mathbf{3'}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3'}\oplus\mathbf{4}\oplus\mathbf{5}, \quad \mathbf{4}\otimes\mathbf{4}=\mathbf{1_S}\oplus\mathbf{3_A}\oplus\mathbf{3'_A}\oplus\mathbf{4_S}\oplus\mathbf{5_S}, \quad \mathbf{4}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3'}\oplus\mathbf{4}\oplus\mathbf{5_1}\oplus\mathbf{5_2},\\
&\mathbf{5}\otimes\mathbf{5}=\mathbf{1_S}\oplus\mathbf{3_A}\oplus\mathbf{3'_A}\oplus\mathbf{4_S}\oplus\mathbf{4_A}\oplus\mathbf{5_{S,1}}\oplus\mathbf{5_{S,2}},
\end{aligned} \tag{A.3}$$

where $\boldsymbol{R}$ denotes any irreducible representation of A_5, and $\mathbf{4_S}$, $\mathbf{4_A}$, $\mathbf{5_1}$, $\mathbf{5_2}$, $\mathbf{5_{S,1}}$ and $\mathbf{5_{S,2}}$ stand for the two $\mathbf{4}$ and $\mathbf{5}$ which appear in the Kronecker products, where the subscript "$\boldsymbol{S}$" ("$\boldsymbol{A}$") refers to symmetric (antisymmetric) combinations. We now list the CG coefficients in our basis, and the results are summarized in table 11.

$\mathbf{3}\otimes\mathbf{3}=\mathbf{1_S}\oplus\mathbf{3_A}\oplus\mathbf{5_S}$	$\mathbf{3'}\otimes\mathbf{3'}=\mathbf{1_S}\oplus\mathbf{3'_A}\oplus\mathbf{5_S}$	$\mathbf{3}\otimes\mathbf{3'}=\mathbf{4}\oplus\mathbf{5}$
$\mathbf{1_S}: x_1y_1+x_2y_3+x_3y_2$ $\mathbf{3_A}: \begin{pmatrix} x_2y_3-x_3y_2 \\ x_1y_2-x_2y_1 \\ x_3y_1-x_1y_3 \end{pmatrix}$ $\mathbf{5_S}: \begin{pmatrix} 2x_1y_1-x_2y_3-x_3y_2 \\ -\sqrt{3}(x_1y_2+x_2y_1) \\ \sqrt{6}x_2y_2 \\ \sqrt{6}x_3y_3 \\ -\sqrt{3}(x_1y_3+x_3y_1) \end{pmatrix}$	$\mathbf{1_S}: x_1y_1+x_2y_3+x_3y_2$ $\mathbf{3'_A}: \begin{pmatrix} x_2y_3-x_3y_2 \\ x_1y_2-x_2y_1 \\ x_3y_1-x_1y_3 \end{pmatrix}$ $\mathbf{5_S}: \begin{pmatrix} 2x_1y_1-x_2y_3-x_3y_2 \\ \sqrt{6}x_3y_3 \\ -\sqrt{3}(x_1y_2+x_2y_1) \\ -\sqrt{3}(x_1y_3+x_3y_1) \\ \sqrt{6}x_2y_2 \end{pmatrix}$	$\mathbf{4}: \begin{pmatrix} \sqrt{2}x_2y_1+x_3y_2 \\ -\sqrt{2}x_1y_2-x_3y_3 \\ -\sqrt{2}x_1y_3-x_2y_2 \\ \sqrt{2}x_3y_1+x_2y_3 \end{pmatrix}$ $\mathbf{5}: \begin{pmatrix} \sqrt{3}x_1y_1 \\ x_2y_1-\sqrt{2}x_3y_2 \\ x_1y_2-\sqrt{2}x_3y_3 \\ x_1y_3-\sqrt{2}x_2y_2 \\ x_3y_1-\sqrt{2}x_2y_3 \end{pmatrix}$

$\mathbf{3}\otimes\mathbf{4}=\mathbf{3'}\oplus\mathbf{4}\oplus\mathbf{5}$	$\mathbf{3'}\otimes\mathbf{4}=\mathbf{3}\oplus\mathbf{4}\oplus\mathbf{5}$
$\mathbf{3'}: \begin{pmatrix} -\sqrt{2}(x_2y_4+x_3y_1) \\ \sqrt{2}x_1y_2-x_2y_1+x_3y_3 \\ \sqrt{2}x_1y_3+x_2y_2-x_3y_4 \end{pmatrix}$ $\mathbf{4}: \begin{pmatrix} x_1y_1-\sqrt{2}x_3y_2 \\ -x_1y_2-\sqrt{2}x_2y_1 \\ x_1y_3+\sqrt{2}x_3y_4 \\ -x_1y_4+\sqrt{2}x_2y_3 \end{pmatrix}$ $\mathbf{5}: \begin{pmatrix} \sqrt{6}(x_2y_4-x_3y_1) \\ 2\sqrt{2}x_1y_1+2x_3y_2 \\ -\sqrt{2}x_1y_2+x_2y_1+3x_3y_3 \\ \sqrt{2}x_1y_3-3x_2y_2-x_3y_4 \\ -2\sqrt{2}x_1y_4-2x_2y_3 \end{pmatrix}$	$\mathbf{3}: \begin{pmatrix} -\sqrt{2}(x_2y_3+x_3y_2) \\ \sqrt{2}x_1y_1+x_2y_4-x_3y_3 \\ \sqrt{2}x_1y_4-x_2y_2+x_3y_1 \end{pmatrix}$ $\mathbf{4}: \begin{pmatrix} x_1y_1+\sqrt{2}x_3y_3 \\ x_1y_2-\sqrt{2}x_3y_4 \\ -x_1y_3+\sqrt{2}x_2y_1 \\ -x_1y_4-\sqrt{2}x_2y_2 \end{pmatrix}$ $\mathbf{5}: \begin{pmatrix} \sqrt{6}(x_2y_3-x_3y_2) \\ \sqrt{2}x_1y_1-3x_2y_4-x_3y_3 \\ 2\sqrt{2}x_1y_2+2x_3y_4 \\ -2\sqrt{2}x_1y_3-2x_2y_1 \\ -\sqrt{2}x_1y_4+x_2y_2+3x_3y_1 \end{pmatrix}$

$\mathbf{3}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3'}\oplus\mathbf{4}\oplus\mathbf{5}$	$\mathbf{3'}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3'}\oplus\mathbf{4}\oplus\mathbf{5}$
$\mathbf{3}: \begin{pmatrix} -2x_1y_1+\sqrt{3}x_2y_5+\sqrt{3}x_3y_2 \\ \sqrt{3}x_1y_2+x_2y_1-\sqrt{6}x_3y_3 \\ \sqrt{3}x_1y_5-\sqrt{6}x_2y_4+x_3y_1 \end{pmatrix}$ $\mathbf{3'}: \begin{pmatrix} \sqrt{3}x_1y_1+x_2y_5+x_3y_2 \\ x_1y_3-\sqrt{2}x_2y_2-\sqrt{2}x_3y_4 \\ x_1y_4-\sqrt{2}x_2y_3-\sqrt{2}x_3y_5 \end{pmatrix}$ $\mathbf{4}: \begin{pmatrix} 2\sqrt{2}x_1y_2-\sqrt{6}x_2y_1+x_3y_3 \\ -\sqrt{2}x_1y_3+2x_2y_2-3x_3y_4 \\ \sqrt{2}x_1y_4+3x_2y_3-2x_3y_5 \\ -2\sqrt{2}x_1y_5-x_2y_4+\sqrt{6}x_3y_1 \end{pmatrix}$	$\mathbf{3}: \begin{pmatrix} \sqrt{3}x_1y_1+x_2y_4+x_3y_3 \\ x_1y_2-\sqrt{2}x_2y_5-\sqrt{2}x_3y_4 \\ x_1y_5-\sqrt{2}x_2y_3-\sqrt{2}x_3y_2 \end{pmatrix}$ $\mathbf{3'}: \begin{pmatrix} -2x_1y_1+\sqrt{3}x_2y_4+\sqrt{3}x_3y_3 \\ \sqrt{3}x_1y_3+x_2y_1-\sqrt{6}x_3y_5 \\ \sqrt{3}x_1y_4-\sqrt{6}x_2y_2+x_3y_1 \end{pmatrix}$ $\mathbf{4}: \begin{pmatrix} \sqrt{2}x_1y_2+3x_2y_5-2x_3y_4 \\ 2\sqrt{2}x_1y_3-\sqrt{6}x_2y_1+x_3y_5 \\ -2\sqrt{2}x_1y_4-x_2y_2+\sqrt{6}x_3y_1 \\ -\sqrt{2}x_1y_5+2x_2y_3-3x_3y_2 \end{pmatrix}$

Table 11. Tensor products and the corresponding CG coefficients for the finite modular symmetry A_5. Here all CG coefficients are presented in the form $x\otimes y$, where x_i denotes the elements of the left base vector x and y_j stands for the elements of the right base vector y (*continues...*).

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$\mathbf{3} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5}$	$\mathbf{3'} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{5}: \begin{pmatrix} \sqrt{3}(x_2y_5 - x_3y_2) \\ -x_1y_2 - \sqrt{3}x_2y_1 - \sqrt{2}x_3y_3 \\ -2x_1y_3 - \sqrt{2}x_2y_2 \\ 2x_1y_4 + \sqrt{2}x_3y_5 \\ x_1y_5 + \sqrt{2}x_2y_4 + \sqrt{3}x_3y_1 \end{pmatrix}$	$\mathbf{5}: \begin{pmatrix} \sqrt{3}(x_2y_4 - x_3y_3) \\ 2x_1y_2 + \sqrt{2}x_3y_4 \\ -x_1y_3 - \sqrt{3}x_2y_1 - \sqrt{2}x_3y_5 \\ x_1y_4 + \sqrt{2}x_2y_2 + \sqrt{3}x_3y_1 \\ -2x_1y_5 - \sqrt{2}x_2y_3 \end{pmatrix}$

$\mathbf{5} \otimes \mathbf{5} = \mathbf{1}_S \oplus \mathbf{3}_A \oplus \mathbf{3}'_A \oplus \mathbf{4}_S \oplus \mathbf{4}_A \oplus \mathbf{5}_{S,1} \oplus \mathbf{5}_{S,2}$
$\mathbf{1}_S : x_1y_1 + x_2y_5 + x_3y_4 + x_4y_3 + x_5y_2$
$\mathbf{3}_A : \begin{pmatrix} x_2y_5 + 2x_3y_4 - 2x_4y_3 - x_5y_2 \\ -\sqrt{3}x_1y_2 + \sqrt{3}x_2y_1 + \sqrt{2}x_3y_5 - \sqrt{2}x_5y_3 \\ \sqrt{3}x_1y_5 + \sqrt{2}x_2y_4 - \sqrt{2}x_4y_2 - \sqrt{3}x_5y_1 \end{pmatrix}$
$\mathbf{3}'_A : \begin{pmatrix} 2x_2y_5 - x_3y_4 + x_4y_3 - 2x_5y_2 \\ \sqrt{3}x_1y_3 - \sqrt{3}x_3y_1 + \sqrt{2}x_4y_5 - \sqrt{2}x_5y_4 \\ -\sqrt{3}x_1y_4 + \sqrt{2}x_2y_3 - \sqrt{2}x_3y_2 + \sqrt{3}x_4y_1 \end{pmatrix}$
$\mathbf{4}_S : \begin{pmatrix} 3\sqrt{2}x_1y_2 + 3\sqrt{2}x_2y_1 - \sqrt{3}x_3y_5 + 4\sqrt{3}x_4y_4 - \sqrt{3}x_5y_3 \\ 3\sqrt{2}x_1y_3 + 4\sqrt{3}x_2y_2 + 3\sqrt{2}x_3y_1 - \sqrt{3}x_4y_5 - \sqrt{3}x_5y_4 \\ 3\sqrt{2}x_1y_4 - \sqrt{3}x_2y_3 - \sqrt{3}x_3y_2 + 3\sqrt{2}x_4y_1 + 4\sqrt{3}x_5y_5 \\ 3\sqrt{2}x_1y_5 - \sqrt{3}x_2y_4 + 4\sqrt{3}x_3y_3 - \sqrt{3}x_4y_2 + 3\sqrt{2}x_5y_1 \end{pmatrix}$
$\mathbf{4}_A : \begin{pmatrix} \sqrt{2}x_1y_2 - \sqrt{2}x_2y_1 + \sqrt{3}x_3y_5 - \sqrt{3}x_5y_3 \\ -\sqrt{2}x_1y_3 + \sqrt{2}x_3y_1 + \sqrt{3}x_4y_5 - \sqrt{3}x_5y_4 \\ -\sqrt{2}x_1y_4 - \sqrt{3}x_2y_3 + \sqrt{3}x_3y_2 + \sqrt{2}x_4y_1 \\ \sqrt{2}x_1y_5 - \sqrt{3}x_2y_4 + \sqrt{3}x_4y_2 - \sqrt{2}x_5y_1 \end{pmatrix}$
$\mathbf{5}_{S,1} : \begin{pmatrix} 2x_1y_1 + x_2y_5 - 2x_3y_4 - 2x_4y_3 + x_5y_2 \\ x_1y_2 + x_2y_1 + \sqrt{6}x_3y_5 + \sqrt{6}x_5y_3 \\ -2x_1y_3 + \sqrt{6}x_2y_2 - 2x_3y_1 \\ -2x_1y_4 - 2x_4y_1 + \sqrt{6}x_5y_5 \\ x_1y_5 + \sqrt{6}x_2y_4 + \sqrt{6}x_4y_2 + x_5y_1 \end{pmatrix}$
$\mathbf{5}_{S,2} : \begin{pmatrix} 2x_1y_1 - 2x_2y_5 + x_3y_4 + x_4y_3 - 2x_5y_2 \\ -2x_1y_2 - 2x_2y_1 + \sqrt{6}x_4y_4 \\ x_1y_3 + x_3y_1 + \sqrt{6}x_4y_5 + \sqrt{6}x_5y_4 \\ x_1y_4 + \sqrt{6}x_2y_3 + \sqrt{6}x_3y_2 + x_4y_1 \\ -2x_1y_5 + \sqrt{6}x_3y_3 - 2x_5y_1 \end{pmatrix}$

Table 11. Tensor products and the corresponding CG coefficients for the finite modular symmetry A_5. Here all CG coefficients are presented in the form $x \otimes y$, where x_i denotes the elements of the left base vector x and y_j stands for the elements of the right base vector y.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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