Backreaction of mass and angular momentum accretion on black holes: General formulation of metric perturbations and application to the Blandford–Znajek process

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We study the metric backreaction of mass and angular momentum accretion on black holes. We first develop the formalism of monopole and dipole linear gravitational perturbations around Schwarzschild black holes in Eddington–Finkelstein coordinates against generic time-dependent matter. We derive the relation between the time dependence of the mass and angular momentum of the black hole and the energy–momentum tensors of accreting matter. As a concrete example, we apply our formalism to the Blandford–Znajek process around slowly rotating black holes. We find that the time dependence of the monopole and dipole perturbations can be interpreted as a slowly rotating Kerr metric with time-dependent mass and spin parameters, which are determined from the energy and angular momentum extraction rates of the Blandford–Znajek process. We also show that the Komar angular momentum and the area of the apparent horizon are decreasing and increasing in time, respectively, while they are consistent with the Blandford–Znajek argument of energy extraction in terms of black hole mechanics if we regard the time-dependent mass parameter as the energy of the black hole.
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1. Introduction

Black holes in astrophysical situations are usually assumed to be Kerr black holes, and the matter fields are treated as test fields. This is because the effects of matter distribution on the curvature are typically small, and then the spacetime is determined from the vacuum Einstein equations which only admit Kerr black holes as stationary regular black holes due to the uniqueness theorem in general relativity [1–3]. Nevertheless, if we take into account the effect of matter distribution on the spacetime, we can discuss the effect of the energy–momentum tensor on the metric by gravitational perturbations around the background black holes. In particular, if matter accretion on black holes exists, we expect that the mass and angular momentum of black holes secularly change. In this paper we would like to clarify this issue by explicitly studying the gravitational perturbations around black holes. As a first step, we focus on the case of the Schwarzschild black hole background.

The linear gravitational perturbations around Schwarzschild black holes were studied by Regge and Wheeler [4] and Zerilli [5,6]. For higher-order multipole perturbations, where the degrees of freedom of gravitational waves exist, the linearized Einstein equations reduce to second-order wave equations

called the Regge and Wheeler, and Zerilli, equations with the source terms [4–7]. Because we are now interested in the evolution of the black hole mass and angular momentum by matter accretion, we need to study monopole and dipole perturbations. In Refs. [8,9], monopole perturbations against generic stationary accreting matter around Schwarzschild black holes were studied. Recently, in Ref. [10], it was shown that both monopole and dipole perturbations for generic time-dependent matter around Schwarzschild black holes can be solved in a static coordinate system. In this paper we extend the formalism in Ref. [9], where Eddington–Finkelstein coordinates are used, to the case of monopole and dipole perturbations for generic time-dependent accreting matter. To study the evolution of black hole mass and angular momentum, regularity of the accreting matter on the black hole horizon is required and Eddington–Finkelstein coordinates are suitable for checking the regularity. As shown in Sect. 2, we derive the relation between the time dependence of the mass and angular momentum of the black hole and the energy–momentum tensors.

As an interesting phenomenon around rotating black holes we can consider energy extraction, not just increasing the black hole mass. Energy extraction by test particles is known as the Penrose process [11,12], and that by scattering waves is superradiance [13–18]. The energy extraction process by force-free electromagnetic fields is the Blandford–Znajek process [19], which is a candidate for the central engine for gamma-ray bursts and active galactic nucleus jets. The various aspects of the Blandford–Znajek process have been studied in Refs. [20–30]. In this paper we discuss the metric backreaction of energy extraction from rotating black holes by the Blandford–Znajek process. Because the discussions in Ref. [19] are based on the slow rotation approximation of Kerr black holes, we discuss the backreaction using non-linear gravitational perturbations around Schwarzschild black holes, where both the effects of the slow rotation and the backreaction of the Blandford–Znajek process are taken into account. In studying the non-linear gravitational perturbations, at each order, we need to solve equations whose forms are the same as those of linear order but the non-linear effects appear in the source terms. For this reason, our formalism can be applied to this problem.

This paper is organized as follows. In Sect. 2 we develop the formalism by extending the discussion in Ref. [9]. In Sect. 3, we briefly review the force-free electromagnetic fields considered in Ref. [19]. Applying the formalism in Sect. 2 to electromagnetic fields in Sect. 3, we study the metric backreaction of the Blandford–Znajek process in Sect. 4. The black hole mechanics in this situation are discussed in Sect. 5. Section 6 presents a summary and discussions. We use units in which $c = G = 1$.

2. Backreaction of mass and angular momentum accretion on Schwarzschild black holes

Let us consider the situation where the effect of matter distribution on curvature is weak. Then, we need to solve the Einstein equations

$$G_{\mu\nu} = 8\pi\epsilon T_{\mu\nu}, \tag{1}$$

with the small parameter ϵ. At the lowest order, $\mathcal{O}(\epsilon^0)$, the metric is given by a vacuum solution of the Einstein equations. For later convenience, in this section we choose the lowest-order vacuum solution as the Schwarzschild metric $g_{\mu\nu}^{\text{Sch}}$:

$$g_{\mu\nu}^{\text{Sch}} dx^\mu dx^\nu = -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \tag{2}$$

with $f = 1 - r_0/r$ and $r_0 = 2M$, where M denotes the background black hole mass. When we consider the effect of $\epsilon T_{\mu\nu}$, the spacetime will be described by the metric with a small deviation

from the Schwarzschild metric,

$$g_{\mu\nu} = g_{\mu\nu}^{\text{Sch}} + \epsilon h_{\mu\nu}. \tag{3}$$

The Einstein tensor of this metric becomes

$$
\begin{aligned}
G_{\mu\nu} &= \epsilon \delta G_{\mu\nu} \\
&= \epsilon \left[-\frac{1}{2} \nabla_\mu \nabla_\nu h^\alpha{}_\alpha - \frac{1}{2} \nabla^\alpha \nabla_\alpha h_{\mu\nu} + \nabla^\alpha \nabla_{(\mu} h_{\nu)\alpha} + \frac{1}{2} g_{\mu\nu} (\nabla^\alpha \nabla_\alpha h^\beta{}_\beta - \nabla^\alpha \nabla^\beta h_{\alpha\beta}) \right] + \mathcal{O}(\epsilon^2) \\
&=: \epsilon \mathcal{L}^{\text{Sch}}[h_{\alpha\beta}]_{\mu\nu} + \mathcal{O}(\epsilon^2),
\end{aligned}
\tag{4}
$$

where ∇_μ denotes the the covariant derivative of the Schwarzschild metric $g_{\mu\nu}^{\text{Sch}}$, and we raise or lower indices by $g_{\mu\nu}^{\text{Sch}}$. At $\mathcal{O}(\epsilon)$, we need to solve the equations

$$\epsilon \mathcal{L}^{\text{Sch}}[h_{\alpha\beta}]_{\mu\nu} = 8\pi \epsilon T_{\mu\nu}. \tag{5}$$

The energy–momentum tensor satisfies

$$\nabla^\mu T_{\mu\nu} = 0, \tag{6}$$

due to the Bianchi identity. We should note that the following discussion holds if only the basic equations formally take the form of Eq. (5). In particular, when we discuss the effect of the backreaction for the Blandford–Znajek process in Sect. 4, we will solve Eq. (5) with the effective energy–momentum tensor.

For a spherically symmetric spacetime background, we can decompose tensor quantities by the tensor spherical harmonics characterized by ℓ, m ($\ell = 0, 1, 2 \ldots, m = 0, \pm 1, \ldots \pm \ell$), and we can separately discuss even and odd parities and different ℓ, m modes when we solve Eq. (5) [4–6]. In this section we study $\ell = 0$ and $\ell = 1$ odd-parity time-dependent gravitational perturbations for a generic time-dependent matter distribution because those modes are important for the study of the backreaction of accreting matter on the mass and angular momentum of black holes. In Ref. [9], the case of a stationary energy–momentum tensor was discussed, and recently, in Ref. [10], the generic time-dependent case was discussed in the static coordinate system. In this paper we work in Eddington–Finkelstein coordinates (V, r, θ, Φ) with $dV = dt + f^{-1} dr$, $d\Phi = d\phi$, and the line element becomes

$$g_{\mu\nu}^{\text{Sch}} dx^\mu dx^\nu = -f dV^2 + 2 dV dr + r^2 (d\theta^2 + \sin^2\theta d\Phi^2). \tag{7}$$

In this coordinate system it is easy to discuss the regularity of tensor quantities at $r = r_0$ because the finiteness of the tensor components at $r = r_0$ coincides with the regularity condition at the horizon.

2.1. Monopole perturbations

The perturbed metric for $\ell = 0$ is given by

$$h_{\mu\nu}^{(+)}\big|_{\ell=0} dx^\mu dx^\nu = H_0(V, r) dV^2 + 2 H_1(V, r) dV dr, \tag{8}$$

where we choose the gauge condition $h_{rr} = h_{\theta\theta} (= h_{\Phi\Phi}/\sin^2\theta) = 0$ (see Appendix A). In this gauge choice there is a residual gauge mode $H_0 \to H_0 - 2f\eta(V)$, $H_1 \to H_1 + \eta(V)$, where $\eta(V)$

is an arbitrary function of V. We note that this residual gauge mode corresponds to the rescaling of the coordinate V. The generic energy–momentum tensor for $\ell = 0$ becomes

$$
\begin{aligned}
T_{\mu\nu}^{(+)}\big|_{\ell=0}&dx^\mu dx^\nu \\
&= T_{VV}(V,r)dV^2 + 2T_{Vr}(V,r)dVdr + T_{rr}(V,r)dr^2 + T_\Omega(V,r)r^2(d\theta^2 + \sin^2\theta d\Phi^2).
\end{aligned} \quad (9)
$$

The equation $\nabla^\mu T_{\mu V} = 0$ shows that the quantity

$$
\mathcal{A} = 4\pi r^2(fT_{Vr} + T_{VV}) \quad (10)
$$

satisfies

$$
\partial_r \mathcal{A} = -4\pi r^2 \partial_V T_{Vr}. \quad (11)
$$

The quantity $\mathcal{A}$ is interpreted as the accretion rate of the energy,

$$
\mathcal{A} = \int_0^{2\pi} \int_0^\pi T_\mu{}^\nu (\partial_V)^\mu (dr)_\nu r^2 \sin\theta d\theta d\Phi, \quad (12)
$$

which is related to the flux associated with the conservation law $\nabla^\mu(T_{\mu\nu}(\partial_V)^\nu) = 0$. We note that $(\partial_V)^\mu := \partial x^\mu / \partial V$ and $(dr)_\nu := \partial r / \partial x^\mu$. If the energy–momentum tensor is stationary, $\mathcal{A}$ becomes constant. Introducing a quantity $\mathcal{E}$ as

$$
\mathcal{E} = \int_0^{2\pi} \int_0^\pi T_{\mu\nu}(\partial_V)^\mu (\partial_V)^\nu r^2 \sin\theta d\theta d\Phi, \quad (13)
$$

we can write Eq. (11) as

$$
(f\partial_r + \partial_V)\mathcal{A} = \partial_V \mathcal{E}. \quad (14)
$$

In the static coordinates (t, r, θ, ϕ), Eq. (14) becomes $f\partial_r\mathcal{A} = \partial_t\mathcal{E}$, and we can easily see that this corresponds to the local energy conservation law.[1] The other components of the equations $\nabla^\mu T_{\mu\nu} = 0$ show the relation among T_{Vr}, T_{rr}, and T_Ω:

$$
4rT_\Omega - 2\partial_r(r^2(T_{Vr} + fT_{rr})) - r^2 T_{rr}\partial_r f - 2r^2 \partial_V T_{rr} = 0. \quad (15)
$$

In the same manner as Ref. [9], introducing new variables $\delta M(V,r)$ and $\lambda(V,r)$ as

$$
H_0(V,r) = \frac{2\delta M(V,r)}{r} + 2f\lambda(V,r), \qquad H_1(V,r) = -\lambda(V,r), \quad (16)
$$

the (V,V), (V,r), and (r,r) components of the Einstein equations give

$$
\partial_V \delta M = \mathcal{A}, \qquad \partial_r \delta M = -4\pi r^2 T_{Vr}, \qquad \partial_r \lambda = -4\pi r T_{rr}. \quad (17)
$$

These equations can be solved as

$$
\delta M = \delta m + \int_{V_0}^V \mathcal{A}(\bar{V},r)d\bar{V} - 4\pi \int_{r_0}^r \bar{r}^2 T_{Vr}(V_0,\bar{r})d\bar{r}, \quad (18)
$$

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[1] In our definition, $\mathcal{A}$ is positive when positive accretion into the black hole exists. The equation can be written in the conventional conservation form $\partial_t \mathcal{E} + f\partial_r(-\mathcal{A}) = 0$.

$$\lambda = -4\pi \int_{r_0}^{r} \bar{r} T_{rr}(V, \bar{r}) d\bar{r} + \chi(V), \tag{19}$$

where δm and V_0 are constants and $\chi(V)$ is an arbitrary function of V. The function $\chi(V)$ corresponds to the residual gauge mode, i.e. the rescaling of V. We note that the other components of the Einstein equations are automatically satisfied. To summarize, the perturbed metric for $\ell = 0$ is described by

$$h_{\mu\nu}^{(+)}\big|_{\ell=0} dx^\mu dx^\nu = \left(\frac{2\delta M}{r} + 2f\lambda \right) dV^2 - 2\lambda dV dr, \tag{20}$$

where δM and λ are given by Eqs. (18) and (19).

If there do not exist $\ell \geq 1$ perturbations, due to the spherical symmetry of the spacetime, we can calculate the Misner–Sharp mass[2] for the metric $g_{\mu\nu}^{\text{Sch}} + \epsilon h_{\mu\nu}$ at (V, r) as

$$M_{\text{MS}} = M + \epsilon \delta M, \tag{21}$$

where δM is given by Eq. (18). We can see that the constant δm denotes the deviation of the Misner–Sharp mass from the background mass parameter M at $V = V_0$ and $r = r_0$. We note that the degrees of freedom in choosing δm and V_0 are degenerate because if we change V_0, δm is shifted. Also, the quantity $\mathcal{A}$ determines the time dependence of the mass:

$$\partial_V M_{\text{MS}} = \epsilon \mathcal{A}. \tag{22}$$

2.2. Odd-parity dipole perturbations

The perturbed metric for the $\ell = 1$ odd-parity modes is given by

$$h_{\mu\nu}^{(-)}\big|_{\ell=1} dx^\mu dx^\nu = 4\sqrt{\pi/3}\, h_0(V, r) \sin\theta (\partial_\theta Y_{1,0}) dV d\Phi$$

$$= -2h_0(V, r) \sin^2\theta dV d\Phi, \tag{23}$$

where $Y_{1,0} = 2^{-1}\sqrt{3/\pi} \cos\theta$,[3] and we choose the gauge condition $h_{r\Phi} = 0$ (see Appendix A). In this gauge choice there is a residual gauge mode, $h_0 \to h_0 + r^2 \zeta(V)$, where $\zeta(V)$ is an arbitrary function of V. Note that this residual gauge mode corresponds to the shift of the coordinate Φ by the function of V. The generic energy–momentum tensor for the $\ell = 1$ odd-parity modes becomes

$$T_{\mu\nu}^{(-)}\big|_{\ell=1} dx^\mu dx^\nu = -2\sin^2\theta d\Phi[t_{V\Phi}(V, r)dV + t_{r\Phi}(V, r)dr]. \tag{24}$$

The non-trivial component of $\nabla^\mu T_{\mu\nu} = 0$ shows that the quantity

$$\mathcal{B} = \frac{16\pi r^2}{3r_0}(t_{V\Phi} + f t_{r\Phi}) \tag{25}$$

satisfies

$$\partial_r \mathcal{B} = -\frac{16\pi r^2}{3r_0} \partial_V t_{r\Phi}. \tag{26}$$

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[2] The Misner–Sharp mass for spherically symmetric spacetime is given by $M_{\text{MS}} = (1 - |dr|^2)r/2$, where r is the area radius and $|dr|^2 = g^{\mu\nu}(dr)_\mu(dr)_\nu$.

[3] The other cases $Y_{1,\pm1}$ can be obtained by acting the ladder operators of the spherical harmonics on the perturbed metric in Eq. (23) if needed.

The quantity $\mathcal{B}$ is interpreted as the accretion rate of the angular momentum,

$$\mathcal{B} = -\frac{1}{M} \int_0^{2\pi} \int_0^{\pi} T_\mu{}^\nu (\partial_\Phi)^\mu (dr)_\nu r^2 \sin\theta d\theta d\Phi, \qquad (27)$$

where $M = r_0/2$ and $(\partial_\Phi)^\mu := \partial x^\mu / \partial \Phi$. This is related to the flux associated with the conservation law $\nabla^\mu (T_{\mu\nu} (\partial_\Phi)^\nu) = 0$. When the energy–momentum tensor is stationary, $\mathcal{B}$ becomes constant. Introducing a quantity $\mathcal{J}$ as

$$\mathcal{J} = -\frac{1}{M} \int_0^{2\pi} \int_0^{\pi} T_{\mu\nu} (\partial_V)^\mu (\partial_\Phi)^\nu r^2 \sin\theta d\theta d\Phi, \qquad (28)$$

we can write Eq. (26) as

$$(f\partial_r + \partial_V)\mathcal{B} = \partial_V \mathcal{J}. \qquad (29)$$

In the static coordinates (t, r, θ, ϕ), Eq. (29) becomes $f\partial_r \mathcal{B} = \partial_t \mathcal{J}$, and we can easily see that this corresponds to the local angular momentum conservation law.[4] The (r, Φ) component of the Einstein equations becomes

$$\partial_r^2 h_0 - \frac{2}{r^2} h_0 = 16\pi t_{r\Phi}. \qquad (30)$$

The general solutions of this equation are given by

$$h_0(v, r) = \frac{C_1(V)}{r} + r^2 C_2(V) + h_0^{\mathrm{IH}}(V, r), \qquad (31)$$

where C_1 and C_2 are arbitrary functions of V, and h_0^{IH} is an inhomogeneous solution,

$$h_0^{\mathrm{IH}}(V, r) = 16\pi r^2 \int_{r_0}^r \frac{1}{\bar{r}^4} \left[\int_{r_0}^{\bar{r}} \tilde{r}^2 t_{r\Phi}(V, \tilde{r}) d\tilde{r} \right] d\bar{r}. \qquad (32)$$

The other components of the Einstein equations give

$$\partial_V C_1 = r_0 \mathcal{B}|_{r=r_0}. \qquad (33)$$

The general solution of this equation is given by

$$C_1 = r_0 \delta a + r_0 \int_{V_0}^V \mathcal{B}(\bar{V}, r_0) d\bar{V}, \qquad (34)$$

where δa is a constant. To summarize, the perturbed metric becomes

$$h_{\mu\nu}^{(-)}\big|_{\ell=1} dx^\mu dx^\nu = -\frac{2r_0 \sin^2\theta}{r} d\Phi dV \left[\delta a + \int_{V_0}^V \mathcal{B}(\bar{V}, r_0) d\bar{V} + \frac{r}{r_0} \left(h_0^{\mathrm{IH}} + r^2 C_2(V) \right) \right], \qquad (35)$$

where the function $C_2(V)$ corresponds to the residual gauge mode.

If there do not exist $m \neq 0$ perturbations, we can calculate the Komar angular momentum associated with the Killing vector ∂_Φ for the metric $g_{\mu\nu}^{\mathrm{Sch}} + \epsilon h_{\mu\nu}$ at the radius r as

$$J_{\mathrm{Komar}} = \epsilon M \left[\delta a + \int_{V_0}^V \mathcal{B}(\bar{V}, r_0) d\bar{V} + \frac{r}{6M} \left(2h_0^{\mathrm{IH}} - r\partial_r h_0^{\mathrm{IH}} \right) \right]. \qquad (36)$$

---

[4] In our definition, $\mathcal{B}$ is positive when positive angular momentum accretion onto the black hole exists. The equation can be written in the conventional conservation form $\partial_t \mathcal{J} + f\partial_r(-\mathcal{B}) = 0$.

We note that $h_0^{\mathrm{IH}} = \partial_r h_0^{\mathrm{IH}} = 0$ at $r = r_0$. The time dependence of J_{Komar} at the radius r becomes

$$\partial_V J_{\mathrm{Komar}} = \epsilon M \mathcal{B}. \tag{37}$$

We can see that δa corresponds to a constant shift in the Kerr parameter for slowly rotating cases, and $\mathcal{B}$ determines the time dependence of the angular momentum at the radius r.

2.3. Remarks

2.3.1. Uniqueness of the Kerr metric if $T_{\mu\nu} = 0$ in the exterior regions for $V \geq V_1$

Let us assume that the energy–momentum tensor $T_{\mu\nu}$ for $r \geq r_0$ vanishes for $V \geq V_1$ ($> V_0$). This corresponds to the situation that the matter fields are electrically neutral and they completely fall into the black hole at $V = V_1$. In that case, according to our formalism, the perturbed metric for $\ell = 0$ and odd-parity $\ell = 1$ modes becomes that for slowly rotating Kerr black holes for $r \geq r_0$ and $V \geq V_1$:

$$(g_{\mu\nu}^{\mathrm{Sch}} + \epsilon h_{\mu\nu}^{(+)}\big|_{\ell=0} + \epsilon h_{\mu\nu}^{(-)}\big|_{\ell=1})dx^\mu dx^\nu$$

$$= -\left[1 - \frac{2M_{[\mathrm{final}]}}{r}\right]dV^2 + 2dVdr + r^2(d\theta^2 + \sin^2\theta d\Phi^2) - \frac{4Ma_{[\mathrm{final}]}\sin^2\theta}{r}d\Phi dV, \tag{38}$$

with

$$M_{[\mathrm{final}]} = M + \epsilon\delta m + \epsilon\int_{V_0}^{V_1}\mathcal{A}(V,r_0)dV, \qquad a_{[\mathrm{final}]} = \epsilon\delta a + \epsilon\int_{V_0}^{V_1}\mathcal{B}(V,r_0)dV. \tag{39}$$

Note that we can evaluate δM in Eq. (18) at $r = r_0$ for $V \geq V_1$ because of the relation $\partial_r\delta M = -4\pi r^2\partial_V T_{Vr} = 0$ for $V \geq V_1$. Thus, the integrals of $\mathcal{A}$ and $\mathcal{B}$ at $r = r_0$ give the changes of the mass and the angular momentum of the black hole, respectively.

2.3.2. Vaidya metric

The Vaidya metric [31] is the exact spherically symmetric solution with radiating matter,

$$T_{\mu\nu}dx^\mu dx^\nu = \frac{d\mathcal{M}(V)/dV}{4\pi r^2}dV^2. \tag{40}$$

On the other hand, using our formalism with Eq. (40), we obtain

$$(g_{\mu\nu}^{\mathrm{Sch}} + \epsilon h_{\mu\nu}^{(+)}\big|_{\ell=0})dx^\mu dx^\nu = -\left(1 - \frac{2(M + \epsilon\mathcal{M}(V))}{r}\right)dV^2 + 2dVdr + r^2(d\theta^2 + \sin^2\theta d\Phi^2). \tag{41}$$

Thus, we can see that our linear perturbation formalism reproduces the exact Vaidya metric [31].[5]

2.3.3. The conservation laws and fluxes

The quantities $\mathcal{A}$ in Eq. (11) and $\mathcal{B}$ in Eq. (26) are related to the energy and angular momentum fluxes associated with the conservation laws $\nabla^\mu(T_{\mu\nu}(\partial_V)^\nu) = 0$ and $\nabla^\mu(T_{\mu\nu}(\partial_\Phi)^\nu) = 0$, respectively. We

---

[5] The metric in Eq. (41) takes the Kerr–Schild form [32,33]. It is known that the Einstein tensor of the Kerr–Schild form is linear to the unknown function (see, e.g., Refs. [34,35]). This is the reason why linear perturbation analysis can derive the exact solution.

should note that this discussion holds only if the basic equations formally take the form of Eq. (5). In particular, as discussed later, equations whose forms are the same as Eq. (5) but with $T_{\mu\nu}$ replaced by the effective energy–momentum tensors $T_{\mu\nu}^{\text{eff}}$ appear in the context of non-linear perturbations around the Schwarzschild metric. In that case, the equations $\nabla^{\mu}(T_{\mu\nu}^{\text{eff}}(\partial_V)^{\nu}) = 0$ and $\nabla^{\mu}(T_{\mu\nu}^{\text{eff}}(\partial_\Phi)^{\nu}) = 0$ hold for the covariant derivative with respect to the Schwarzschild metric, and the corresponding global conservation laws exist.

3. The energy–momentum tensor of the Blandford–Znajek process

3.1. The force-free electromagnetic fields around the Kerr spacetime

We consider the test electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with the electric current density vector j^μ on the Kerr spacetime. In this section, $g_{\mu\nu}$ denotes the Kerr metric and ∇_μ denotes the corresponding covariant derivative. The metric of the Kerr spacetime in the Boyer–Lindquist coordinates (t, r, θ, ϕ) is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma}dt^2 - \frac{2a \sin^2\theta (r^2 + a^2 - \Delta)}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$
$$+ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\sin^2\theta d\phi^2, \tag{42}$$

with

$$\Sigma = r^2 + a^2 \cos^2\theta, \qquad \Delta = r^2 + a^2 - 2Mr. \tag{43}$$

The constants M and a denote the mass and spin parameters. The black hole horizon is located at $r = r_+ = M + \sqrt{M^2 - a^2}$. The Maxwell equations on this spacetime are given by

$$\nabla^\mu F_{\mu\nu} = 4\pi j_\nu. \tag{44}$$

We note that the equations

$$\nabla_{[\mu}F_{\nu\rho]} = 0 \tag{45}$$

are automatically satisfied from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The energy–momentum tensor of the electromagnetic field,

$$T_{\mu\nu}^{\text{EM}} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{46}$$

satisfies

$$\nabla^\mu T_{\mu\nu}^{\text{EM}} = -4\pi F_{\nu\mu}j^\mu. \tag{47}$$

If the right-hand side of Eq. (47), i.e. the Lorentz force term, is neglected, the force-free condition

$$F_{\nu\mu}j^\mu = 0 \tag{48}$$

is satisfied. Then, the energy–momentum tensor of the electromagnetic field satisfies

$$\nabla^\mu T_{\mu\nu}^{\text{EM}} = 0. \tag{49}$$

We should note that under the condition $T_{\mu\nu}^{\text{particle}} \ll T_{\mu\nu}^{\text{EM}}$, where $T_{\mu\nu}^{\text{particle}}$ is the particle energy density, the total energy–momentum conservation equations $\nabla^\mu(T_{\mu\nu}^{\text{particle}} + T_{\mu\nu}^{\text{EM}}) = 0$ reduce to Eq. (49). This implies that $T_{\mu\nu}^{\text{particle}} \ll T_{\mu\nu}^{\text{EM}}$ is a sufficient condition for Eq. (48). To summarize, the force-free electromagnetic fields $F_{\mu\nu}$ can be obtained by solving Eq. (49) with Eq. (46), and the electric current density vector j^μ can be calculated from Eq. (44).

Because the Boyer–Lindquist coordinates do not cover the black hole horizon, the location $r = r_+$ becomes a coordinate singularity and tensors have apparently singular behavior there. In order to solve this problem, we introduce the Kerr–Schild coordinates (T, r, θ, Φ) by $dT = dt + 2Mrdr/\Delta$, $d\Phi = d\phi + adr/\Delta$. The Kerr metric in the Kerr–Schild coordinates becomes

$$
g_{\mu\nu}^{\text{KS}}dx^\mu dx^\nu = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dT^2 + \frac{4Mr}{\Sigma}dTdr - \frac{4Mr}{\Sigma}a\sin^2\theta dTd\Phi + \left(1 + \frac{2Mr}{\Sigma}\right)dr^2
$$

$$
+ \Sigma d\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\Phi^2 - 2a\sin^2\theta\left(1 + \frac{2Mr}{\Sigma}\right)drd\Phi. \quad (50)
$$

3.2.   The Blandford–Znajek solutions in the the Kerr–Schild coordinates

In Ref. [19], Blandford and Znajek studied stationary and axisymmetric force-free electromagnetic fields around a slowly rotating Kerr metric, and the energy and angular momentum extraction though the magnetic fields, called the Blandford–Znajek process. In this paper we focus on the so-called split-monopole solution, and the solution in the Kerr–Schild coordinates is given by [19,20]

$$
T_{\mu\nu}^{\text{BZ}} = \left(F_{\mu\alpha}F_\nu^{\ \alpha} - \frac{1}{4}g_{\mu\nu}^{\text{KS}}F_{\alpha\beta}F^{\alpha\beta}\right), \quad (51)
$$

where the explicit forms of $F_{\mu\nu}$ are

$$
F_{Tr} = \omega\partial_r A_\Phi, \quad (52)
$$

$$
F_{T\theta} = \omega\partial_\theta A_\Phi, \quad (53)
$$

$$
F_{T\Phi} = 0, \quad (54)
$$

$$
F_{r\theta} = \sqrt{|\det(g_{\mu\nu}^{\text{KS}})|}B_\Phi, \quad (55)
$$

$$
F_{r\Phi} = \partial_r A_\Phi, \quad (56)
$$

$$
F_{\theta\Phi} = \partial_\theta A_\Phi, \quad (57)
$$

and

$$
\omega = \omega_1\frac{a}{M} + \mathcal{O}(a^3), \quad (58)
$$

$$
B_\Phi = B_{\Phi 1}\frac{a}{M} + \mathcal{O}(a^3), \quad (59)
$$

$$
A_\Phi = A_{\Phi 0} + A_{\Phi 2}\left(\frac{a}{M}\right)^2 + \mathcal{O}(a^4). \quad (60)
$$

The functions $A_{\Phi 0}, A_{\Phi 2}, \omega_1$, and $B_{\Phi 1}$ are given by

$$
A_{\Phi 0} = -C\cos\theta, \quad (61)
$$

$$
A_{\Phi 2} = C\mathcal{F}(r)\cos\theta\sin^2\theta, \quad (62)
$$

$$\omega_1 = \frac{1}{8M}, \tag{63}$$

$$B_{\Phi 1} = \frac{-C}{8Mr^2}\left(1 + \frac{4M}{r}\right), \tag{64}$$

where C is implicitly assumed to be of different signs for different signs of $\cos\theta$, and the function $\mathcal{F}(r)$ is a regular solution of the differential equation

$$\frac{d^2\mathcal{F}}{dr^2} + \frac{2M}{r(r-2M)}\frac{d\mathcal{F}}{dr} - \frac{6\mathcal{F}}{r(r-2M)} + \frac{M(r+2M)}{r^3(r-2M)} = 0. \tag{65}$$

The explicit form of $\mathcal{F}(r)$ is

$$\mathcal{F}(r) = \left[\mathrm{Li}_2\left(\frac{2M}{r}\right) - \ln\left(1 - \frac{2M}{r}\right)\ln\frac{r}{2M}\right]\frac{r^2(2r-3M)}{8M^3}$$
$$+ \frac{M^2 + 3Mr - 6r^2}{12M^2}\ln\frac{r}{2M} + \frac{11}{72} + \frac{M}{3r} + \frac{r}{2M} - \frac{r^2}{2M^2}, \tag{66}$$

where Li_2 is the second polylogarithm function,

$$\mathrm{Li}_2(x) = -\int_0^1 dt\,\frac{\ln(1-tx)}{t}. \tag{67}$$

The asymptotic behaviors of $\mathcal{F}$ at $r = 2M$ and $r = \infty$ are

$$\mathcal{F} = \frac{6\pi^2 - 49}{72} + \frac{6\pi^2 - 61}{24M}(r - 2M) + \mathcal{O}((r-2M)^2), \tag{68}$$

$$\mathcal{F} = \frac{M}{4r} + \frac{M^2\ln(r/M)}{10r^2} - \frac{M^2(11 + 20\ln 2)}{200r^2} + \mathcal{O}((\ln r)/r^3), \tag{69}$$

respectively. We can confirm that $T_{\mu\nu}^{\mathrm{BZ}}$ satisfies the equations for force-free electromagnetic fields, Eq. (49).[6] Also, $F_{\mu\nu}$ satisfies the degenerate condition

$$\star F^{\mu\nu}F_{\mu\nu} = 0, \tag{70}$$

where $\star F_{\mu\nu} = F^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu}/2$ and $\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita tensor.[7] As shown in Refs. [19,20], the energy and angular momentum extraction rates are given by

$$\dot{E}_{\mathrm{BZ}} := -\int_0^{2\pi}\int_0^\pi \sqrt{|\det(g_{\mu\nu}^{\mathrm{KS}})|}\, T_\mu^{\mathrm{BZ}\nu}(\partial_T)^\mu (dr)_\nu d\theta d\Phi = \frac{\pi}{24}\frac{a^2C^2}{M^4} + \mathcal{O}(a^4), \tag{71}$$

$$\dot{J}_{\mathrm{BZ}} := \int_0^{2\pi}\int_0^\pi \sqrt{|\det(g_{\mu\nu}^{\mathrm{KS}})|}\, T_\mu^{\mathrm{BZ}\nu}(\partial_\Phi)^\mu (dr)_\nu d\theta d\Phi = \frac{\pi}{3}\frac{aC^2}{M^2} + \mathcal{O}(a^3). \tag{72}$$

We can see that the relation $\dot{E}_{\mathrm{BZ}} = \omega\dot{J}_{\mathrm{BZ}}$ holds at this order.

---

[6] We note that $F_{\mu\nu}$ satisfies Eq. (45), and this implies that A_μ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ exists.

[7] The degenerate condition in Eq. (70) can be derived from the force-free condition in Eq. (48) for non-zero j^μ (see, e.g., Ref. [28]). We also note that Eq. (70) is compatible with the ideal magnetohydrodynamic condition [20].

4.   The backreaction of the Blandford–Znajek process

4.1.   Perturbation scheme

The discussions in Sect. 3 are based on the test field approximation around the Kerr black holes. When we consider the backreaction of the Blandford–Znajek process on the spacetime, we regard the parameter C^2 as a small parameter so that the effect of the energy–momentum tensor $T^{\mathrm{BZ}}_{\mu\nu}(\propto C^2)$ on the spacetime is weak. Introducing dimensionless small parameters α, β as[8]

$$\alpha := \frac{a}{M}, \qquad \beta := \frac{C^2}{M^2}, \tag{73}$$

the energy–momentum tensor $T^{\mathrm{BZ}}_{\mu\nu}$ can be written by the Taylor series around $(\alpha, \beta) = (0,0)$ as

$$T^{\mathrm{BZ}}_{\mu\nu} = \beta T^{(0,1)}_{\mu\nu} + \alpha\beta T^{(1,1)}_{\mu\nu} + \alpha^2\beta T^{(2,1)}_{\mu\nu} + \mathcal{O}(\alpha^3\beta). \tag{74}$$

To discuss the backreaction of the Blandford–Znajek process, we need to solve the Einstein equations

$$G_{\mu\nu} = 8\pi T^{\mathrm{BZ}}_{\mu\nu} = 8\pi\beta T^{(0,1)}_{\mu\nu} + 8\pi\alpha\beta T^{(1,1)}_{\mu\nu} + 8\pi\alpha^2\beta T^{(2,1)}_{\mu\nu} + \mathcal{O}(\alpha^3\beta). \tag{75}$$

We expand the metric tensor as

$$g_{\mu\nu} = g^{\mathrm{Kerr}}_{\mu\nu} + g^{\mathrm{BZ}}_{\mu\nu}, \tag{76}$$

with

$$g^{\mathrm{Kerr}}_{\mu\nu} = g^{\mathrm{Sch}}_{\mu\nu} + \alpha h^{(1,0)}_{\mu\nu} + \alpha^2 h^{(2,0)}_{\mu\nu} + \mathcal{O}(\alpha^3), \tag{77}$$

$$g^{\mathrm{BZ}}_{\mu\nu} = \beta h^{(0,1)}_{\mu\nu} + \alpha\beta h^{(1,1)}_{\mu\nu} + \alpha^2\beta h^{(2,1)}_{\mu\nu} + \mathcal{O}(\alpha^3\beta). \tag{78}$$

The Einstein tensor becomes

$$G_{\mu\nu} = \beta G^{(0,1)}_{\mu\nu} + \alpha\beta G^{(1,1)}_{\mu\nu} + \alpha^2\beta G^{(2,1)}_{\mu\nu} + \mathcal{O}(\alpha^3\beta) + \mathcal{O}(\beta^2). \tag{79}$$

We note that the Einstein tensor at $\mathcal{O}(\beta^0)$ vanishes because the $\mathcal{O}(\beta^0)$ metric is the Kerr metric. At each order, we need to solve the following equations:

$$\beta G^{(0,1)}_{\mu\nu} = 8\pi\beta T^{(0,1)}_{\mu\nu}, \tag{80}$$

$$\alpha\beta G^{(1,1)}_{\mu\nu} = 8\pi\alpha\beta T^{(1,1)}_{\mu\nu}, \tag{81}$$

$$\alpha^2\beta G^{(2,1)}_{\mu\nu} = 8\pi\alpha^2\beta T^{(2,1)}_{\mu\nu}. \tag{82}$$

Schematically, we can write $G^{(0,1)}_{\mu\nu}$, $G^{(1,1)}_{\mu\nu}$, and $G^{(2,1)}_{\mu\nu}$ as

$$\beta G^{(0,1)}_{\mu\nu} = \beta\mathcal{L}^{\mathrm{Sch}}[h^{(0,1)}_{\alpha\beta}]_{\mu\nu}, \tag{83}$$

$$\alpha\beta G^{(1,1)}_{\mu\nu} = \alpha\beta\mathcal{L}^{\mathrm{Sch}}[h^{(1,1)}_{\alpha\beta}]_{\mu\nu} - 8\pi\alpha\beta\tilde{T}^{(1,1)}_{\mu\nu}, \tag{84}$$

$$\alpha^2\beta G^{(2,1)}_{\mu\nu} = \alpha^2\beta\mathcal{L}^{\mathrm{Sch}}[h^{(2,1)}_{\alpha\beta}]_{\mu\nu} - 8\pi\alpha^2\beta\tilde{T}^{(2,1)}_{\mu\nu}, \tag{85}$$

---

[8] The shape of the letter α is similar to a, and β reminds us of the magnetic fields B.

where $\tilde{T}_{\mu\nu}^{(1,1)}$ and $\tilde{T}_{\mu\nu}^{(2,1)}$ denote the effects of the non-linear perturbation, e.g. $\tilde{T}_{\mu\nu}^{(1,1)}$ is constructed from $h_{\mu\nu}^{(1,0)}$ and $h_{\mu\nu}^{(0,1)}$. Thus, at each order we solve the following equations, which can be seen as *linear perturbation around the Schwarzschild spacetime with the effective energy–momentum tensors*:

$$\beta\mathcal{L}^{\text{Sch}}[h_{\alpha\beta}^{(0,1)}]_{\mu\nu} = 8\pi\beta T_{\mu\nu}^{(0,1)}, \tag{86}$$

$$\alpha\beta\mathcal{L}^{\text{Sch}}[h_{\alpha\beta}^{(1,1)}]_{\mu\nu} = 8\pi\alpha\beta T_{\mu\nu}^{(1,1)} + 8\pi\alpha\beta\tilde{T}_{\mu\nu}^{(1,1)} =: 8\pi\alpha\beta T_{\mu\nu}^{\text{eff}(1,1)}, \tag{87}$$

$$\alpha^2\beta\mathcal{L}^{\text{Sch}}[h_{\alpha\beta}^{(2,1)}]_{\mu\nu} = 8\pi\alpha^2\beta T_{\mu\nu}^{(2,1)} + 8\pi\alpha^2\beta\tilde{T}_{\mu\nu}^{(2,1)} =: 8\pi\alpha^2\beta T_{\mu\nu}^{\text{eff}(2,1)}. \tag{88}$$

If we regard $\alpha^n\beta$ ($n = 0, 1, 2$) as small parameters, we can apply the formalism in Sect. 2 to these equations at each order.

4.2. Eddington–Finkelstein-like coordinates

We discuss the backreaction of the Blandford–Znajek process using the formalism developed in Sect. 2, which is written in Eddington–Finkelstein coordinates. It is convenient to introduce the Eddington–Finkelstein-like coordinates (V, r, θ, Φ) by $dV = dT + dr$. In this coordinate system, the Kerr metric becomes

$$g_{\mu\nu}^{\text{EF}}dx^\mu dx^\nu = -\left(1 - \frac{2Mr}{\Sigma}\right)dV^2 + 2dVdr + \Sigma d\theta^2 + \frac{(r^2 + a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta d\Phi^2$$
$$- 2a\sin^2\theta drd\Phi - \frac{4aMr}{\Sigma}\sin^2\theta dVd\Phi. \tag{89}$$

Then, $h_{\mu\nu}^{(n,0)}$ ($n = 0, 1, 2, \ldots$) in Eq. (77) can be obtained by taking a Taylor series around $a = 0$ for the metric in Eq. (89), i.e.

$$g_{\mu\nu}^{\text{EF}}dx^\mu dx^\nu = \left(g_{\mu\nu}^{\text{Sch}} + \alpha h_{\mu\nu}^{(1,0)} + \alpha^2 h_{\mu\nu}^{(2,0)} + \mathcal{O}(\alpha^3)\right)dx^\mu dx^\nu, \tag{90}$$

with

$$g_{\mu\nu}^{\text{Sch}}dx^\mu dx^\nu = -fdV^2 + 2dVdr + r^2(d\theta^2 + \sin^2\theta d\Phi^2), \tag{91}$$

$$\alpha h_{\mu\nu}^{(1,0)}dx^\mu dx^\nu = -2\alpha M\sin^2\theta d\Phi\left(\frac{2M}{r}dV + dr\right), \tag{92}$$

$$\alpha^2 h_{\mu\nu}^{(2,0)}dx^\mu dx^\nu = \alpha^2 M^2\left[-\frac{2M}{r^3}\cos^2\theta dV^2 + \cos^2\theta d\theta^2 + \frac{(r + M - M\cos(2\theta))\sin^2\theta}{r}d\Phi^2\right]. \tag{93}$$

In the Eddington–Finkelstein-like coordinates (V, r, θ, Φ), we obtain the following equations for the energy–momentum tensors of the Blandford–Znajek process discussed in Sect. 3:

$$\beta T_{\mu\nu}^{(0,1)}dx^\mu dx^\nu = \frac{M^2\beta}{2r^4}\left[\left(1 - \frac{2M}{r}\right)dV^2 - 2dVdr + r^2(d\theta^2 + \sin^2\theta d\Phi^2)\right], \tag{94}$$

$$\alpha\beta T_{\mu\nu}^{(1,1)}dx^\mu dx^\nu = -2\alpha\beta M\sin^2\theta d\Phi\left[\frac{r^3 - 8M^3}{8r^5}dV - \frac{2M^2 + r(r + 2M)}{4r^4}dr\right], \tag{95}$$

$$\alpha^2\beta T_{\mu\nu}^{(2,1)}dx^\mu dx^\nu = \alpha^2\beta T_{\mu\nu}^{(2,1)}\big|_{\ell=0}dx^\mu dx^\nu + \alpha^2\beta T_{\mu\nu}^{(2,1)}\big|_{\ell=2}dx^\mu dx^\nu, \tag{96}$$

with

$$\alpha^2\beta T_{\mu\nu}^{(2,1)}\big|_{\ell=0}dx^\mu dx^\nu$$

$$
= \frac{\alpha^2 \beta}{96 r^7} \Bigg[(96 M^5 - 64 M^4 r + 16 M^3 r^2 + r^5) dV^2
$$

$$
- 4r(4M^2 + r^2)(-4M^2 + 2Mr + r^2) dV dr
$$

$$
+ 4r^3(r + 2M)^2 dr^2 - 32 M^4 (M + r) r^2 (d\theta^2 + \sin^2\theta \, d\Phi^2) \Bigg], \tag{97}
$$

$$
\alpha^2 \beta T^{(2,1)}_{\mu\nu}\big|_{\ell=2} dx^\mu dx^\nu
$$

$$
= \frac{\alpha^2 \beta}{48 r^7} \sqrt{\frac{\pi}{5}} Y_{2,0} \Bigg[\Big(192 M^5 - r(32 M^4 + 16 M^3 r + r^4) + 192 M^2 (-2M + r) r^2 \mathcal{F} \Big) dV^2
$$

$$
+ 4r(32 M^4 + 8 M^3 r + 2 M r^3 + r^4 - 96 M^2 r^2 \mathcal{F}) dV dr - 4 r^3 (r + 2M)^2 dr^2
$$

$$
+ 16(2 M^3 - r M^2 + 12 r^3 \mathcal{F}) r^2 (d\theta^2 + \sin^2\theta \, d\Phi^2) \Bigg] - \sqrt{\frac{\pi}{5}} \frac{4\alpha^2 \beta M^2 \partial_r \mathcal{F}}{3 r^2} (\partial_\theta Y_{2,0}) dr d\theta
$$

$$
+ \sqrt{\frac{\pi}{5}} \frac{\alpha^2 \beta M^3 (r^2 + rM + 2M^2)}{3 r^5} \sum_{i,j=\theta,\Phi} \left(\hat{\nabla}_i \hat{\nabla}_j Y_{2,0} - \frac{1}{2} \gamma_{ij} \hat{\triangle} Y_{2,0} \right) dx^i dx^j, \tag{98}
$$

where $Y_{2,0} = 4^{-1}\sqrt{5/\pi}(-1 + 3\cos^2\theta)$, γ_{ij} is the metric of the unit sphere, i.e. $\sum_{i,j=\theta,\Phi} \gamma_{ij} dx^i dx^j = d\theta^2 + \sin^2\theta \, d\Phi^2$, and the operators $\hat{\nabla}_i$ and $\hat{\triangle}$ denote the covariant derivative and the Laplacian on γ_{ij}, respectively.

4.3. $\mathcal{O}(\beta)$ corrections

We can read $\mathcal{A}^{(0,1)}$, $T^{(0,1)}_{Vr}$, and $T^{(0,1)}_{rr}$ from the $\mathcal{O}(\beta^2)$ energy–momentum tensor in Eq. (94) as

$$
\mathcal{A}^{(0,1)} = 0, \qquad \beta T^{(0,1)}_{Vr} = -\frac{\beta M^2}{2 r^4}, \qquad T^{(0,1)}_{rr} = 0. \tag{99}
$$

From Eqs. (16), (18), and (19), we obtain the perturbed metric as

$$
\beta h^{(0,1)}_{\mu\nu} dx^\mu dx^\nu = \beta \left[\frac{2\delta m^{(0,1)}}{r} + \frac{2\pi M(r - 2M)}{r^2} \right] dV^2, \tag{100}
$$

where we set the residual gauge mode as $\chi^{(0,1)}(V) = 0$. For later convenience we choose $\delta m^{(0,1)} = -\pi M$; then, the total metric at this order is

$$
\left(g^{\text{Sch}}_{\mu\nu} + \beta h^{(0,1)}_{\mu\nu} \right) dx^\mu dx^\nu = -\left(1 - \frac{2M}{r} + \frac{4\pi M^2 \beta}{r^2} \right) dV^2 + 2 dV dr + r^2 (d\theta^2 + \sin^2\theta \, d\Phi^2). \tag{101}
$$

This is the Reissner–Nordström metric with a magnetic charge parameter $Q = 2\sqrt{\pi\beta} M = 2\sqrt{\pi} C$. We can see that the mass of the spacetime is M and the location of the event horizon is $r = r_H$, with

$$
r_H = M + \sqrt{M^2 - 4\pi M^2 \beta^2} = 2M - 2\pi M \beta + \mathcal{O}(\beta^2). \tag{102}
$$

One may think it strange for the spacetime to be the magnetic Reissner–Nordström metric, because the Blandford–Znajek solution is globally different from the magnetic monopole, but it describes the split monopole [19]. The reason is because Birkhoff's theorem for a specially symmetric spacetime holds *locally*, and the energy–momentum tensor of the Blandford–Znajek solution at $\mathcal{O}(\beta)$ is locally the same as that for the global magnetic monopole.

4.4. $\mathcal{O}(\alpha\beta)$ corrections: time dependence of the angular momentum

After some calculations, we obtain

$$8\pi\alpha\beta\tilde{T}_{\mu\nu}^{(1,1)}dx^\mu dx^\nu = 16\pi\alpha\beta\sin^2\theta d\Phi\left(\frac{M^4}{r^5}dV + \frac{M^3}{2r^4}dr\right). \qquad (103)$$

Thus, the effective energy–momentum tensor becomes

$$\begin{aligned}
8\pi\alpha\beta T_{\mu\nu}^{\mathrm{eff}(1,1)}dx^\mu dx^\nu &= 8\pi\alpha\beta T_{\mu\nu}^{(1,1)}dx^\mu dx^\nu + 8\pi\alpha\beta\tilde{T}_{\mu\nu}^{(1,1)}dx^\mu dx^\nu \\
&= -16\pi\alpha\beta\sin^2\theta d\Phi\left[\frac{M(r^3 - 16M^3)}{8r^5}dV - \frac{M(4M^2 + 2Mr + r^2)}{4r^4}dr\right].
\end{aligned} \qquad (104)$$

We can read

$$t_{V\Phi}^{\mathrm{eff}(1,1)} = \frac{M(r^3 - 16M^3)}{8r^5}, \qquad t_{r\Phi}^{\mathrm{eff}(1,1)} = -\frac{M(4M^2 + 2Mr + r^2)}{4r^4}, \qquad (105)$$

so $\mathcal{B}^{\mathrm{eff}(1,1)}$ in Eq. (25) and $h_0^{\mathrm{IH}(1,1)}$ in Eq. (32) become

$$\mathcal{B}^{\mathrm{eff}(1,1)} = -\frac{\pi}{3}, \qquad (106)$$

$$h_0^{\mathrm{IH}(1,1)} = \frac{\pi r^2}{36M}\left[-13 + \frac{8M^2\left(-18M^2 + 4Mr + 9r^2 - 12Mr\ln(2M/r)\right)}{r^4}\right]. \qquad (107)$$

We note that $\mathcal{B}^{\mathrm{eff}(1,1)}$ is constant because $T_{\mu\nu}^{\mathrm{eff}(1,1)}$ is not time dependent (see Eq. (26)). From Eq. (35), the perturbed metric becomes

$$\alpha\beta h_{\mu\nu}^{(1,1)}dx^\mu dx^\nu =$$
$$-\frac{4M\alpha\beta\sin^2\theta}{r}d\Phi dV\left[\delta a^{(1,1)} + \mathcal{B}^{\mathrm{eff}(1,1)}(V - V_0) + \frac{r}{2M}\left(h_0^{\mathrm{IH}(1,1)} + r^2 C_2^{(1,1)}(V)\right)\right]. \quad (108)$$

At this order, the Komar angular momentum at the radius r is

$$J_{\mathrm{Komar}} = \alpha M^2 + \alpha\beta M\left[\delta a^{(1,1)} + \mathcal{B}^{\mathrm{eff}(1,1)}(V - V_0) + \frac{r}{6M}\left(2h_0^{\mathrm{IH}(1,1)} - r\partial_r h_0^{\mathrm{IH}(1,1)}\right)\right]. \qquad (109)$$

We find that the time dependence of J_{Komar} coincides with the prediction from the angular momentum extraction rate of the Blandford–Znajek process in Eq. (72),

$$\partial_V J_{\mathrm{Komar}} = \alpha\beta M\mathcal{B}^{\mathrm{eff}(1,1)} = -\frac{\alpha\beta\pi M}{3}. \qquad (110)$$

We set $\delta a^{(1,1)} = 0$; then, $J_{\mathrm{Komar}} = \alpha M^2$ at $V = V_0$ and $r = r_0$. We also choose the gauge mode as $C_2^{(1,1)}(V) = 13\pi/(36M)$ so that the divergent behavior $h_0^{\mathrm{IH}(1,1)}$ at $r \to \infty$ is canceled in Eq. (108).

4.5. $\mathcal{O}(\alpha^2\beta)$ corrections: time dependence of the mass

In a similar way, we obtain $\tilde{T}_{\mu\nu}^{(2,1)}$ as

$$8\pi\alpha^2\beta\tilde{T}_{\mu\nu}^{(2,1)} = 8\pi\alpha^2\beta\tilde{T}_{\mu\nu}^{(2,1)}\big|_{\ell=0} + 8\pi\alpha^2\beta\tilde{T}_{\mu\nu}^{(2,1)}\big|_{\ell=2}, \qquad (111)$$

with

$$
8\pi\alpha^2\beta\tilde{T}^{(2,1)}_{\mu\nu}\big|_{\ell=0}dx^\mu dx^\nu
$$

$$
= \frac{\alpha^2\beta}{27r^8}\Bigg[\Big(-\pi(288M^6 - 936M^5 r + 288M^4 r^2 + 144M^3 r^3 + 26Mr^5 - 13r^6)
$$

$$
+ \mathcal{B}^{\mathrm{eff}(1,1)}(72M^2 r^3(-3M+r) - 36M^2(9M-4r)(2M-r)r(V-V_0))
$$

$$
+ 18M(2M-r)r^2(2(-3M+r)h_0^{\mathrm{IH}(1,1)} + (3M-2r)r\partial_r h_0^{\mathrm{IH}(1,1)})\Big)dV^2
$$

$$
+ 2r\Big(\pi(-144M^5 + 432M^4 r + 72M^3 r^2 - 36M^2 r^3 - 13r^5)
$$

$$
+ \mathcal{B}^{\mathrm{eff}(1,1)}(-54M^2 r^3 + 36M^2 r(-9M+4r)(V-V_0))
$$

$$
+ 18Mr^2(2(-3M+r)h_0^{\mathrm{IH}(1,1)} + (3M-2r)r\partial_r h_0^{\mathrm{IH}(1,1)})\Big)dVdr
$$

$$
+ 36r^2\Big(-6\mathcal{B}^{\mathrm{eff}(1,1)}M^2 r(V-V_0) + M(2M\pi(-2M^2 + 2Mr + r^2)
$$

$$
- 2r^2 h_0^{\mathrm{IH}(1,1)} + r^3\partial_r h_0^{\mathrm{IH}(1,1)})\Big)dr^2
$$

$$
+ 18Mr^3\Big(4M^2\pi(-3M^2 + Mr + 2r^2) + \mathcal{B}^{\mathrm{eff}(1,1)}(-6Mr^3 + 6Mr(-3M+r)(V-V_0))
$$

$$
+ (3M-r)r^2(-2h_0^{\mathrm{IH}(1,1)} + r\partial_r h_0^{\mathrm{IH}(1,1)})\Big)(d\theta^2 + \sin^2\theta d\Phi^2)\Bigg], \tag{112}
$$

and

$$
8\pi a^2 C^2 \tilde{T}^{(2,1)}_{\mu\nu}\big|_{\ell=2}dx^\mu dx^\nu
$$

$$
= \frac{4\alpha^2\beta}{27r^8}\sqrt{\frac{\pi}{5}}Y_{2,0}\Bigg[\Big(\pi(144M^6 - 792M^5 r + 144M^4 r^2 + 72M^3 r^3 + 13Mr^5 - 26r^6)
$$

$$
+ \mathcal{B}^{\mathrm{eff}(1,1)}(36M^2(3M-r)r^3 + 18M^2 r(18M^2 - 17Mr - 2r^2)(V-V_0))
$$

$$
+ 9Mr^2(2(6M^2 - 5Mr - 2r^2)h_0^{\mathrm{IH}(1,1)} - r(-2M+r)(-3M+2r)\partial_r h_0^{\mathrm{IH}(1,1)})\Big)dV^2
$$

$$
+ \frac{r}{2}\Big(\pi(288M^5 - 864M^4 r - 144M^3 r^2 + 72M^2 r^3 + 65r^5)
$$

$$
+ \mathcal{B}^{\mathrm{eff}(1,1)}(108M^2 r^3 + 72M^2(9M-r)r(V-V_0))
$$

$$
+ 36Mr^2((6M+r)h_0^{\mathrm{IH}(1,1)} + r(-3M+2r)\partial_r h_0^{\mathrm{IH}(1,1)})\Big)dVdr
$$

$$
+ 18Mr^2\Big(2M\pi(2M^2 - 2Mr - r^2) + 6\mathcal{B}^{\mathrm{eff}(1,1)}Mr(V-V_0)
$$

$$
+ 2r^2 h_0^{\mathrm{IH}(1,1)} - r^3\partial_r h_0^{\mathrm{IH}(1,1)}\Big)dr^2
$$

$$
+ \frac{3r^3}{4}\Big(\pi(144M^5 - 264M^4 r - 96M^3 r^2 + 13r^5) + \mathcal{B}^{\mathrm{eff}(1,1)}(72M^2 r^3 + 216M^3 r(V-V_0))
$$

$$
+ 12Mr^2((6M+r)h_0^{\mathrm{IH}(1,1)} + r(-3M+r)\partial_r h_0^{\mathrm{IH}(1,1)})\Big)(d\theta^2 + \sin^2\theta d\Phi^2)\Bigg]
$$

$$
+ 2\sqrt{\frac{\pi}{5}}\frac{\alpha^2\beta M}{27r^6}\Big(720M^4\pi + 13\pi r^4 + \mathcal{B}^{\mathrm{eff}(1,1)}(36Mr^3 + 36Mr(8M+3r)(V-V_0))
$$

$$+ 36r^2(3M + r)h_0^{\mathrm{IH}(1,1)} - 18r^3(2M + r)\partial_r h_0^{\mathrm{IH}(1,1)}\Big)(\partial_\theta Y_{2,0})dVd\theta$$

$$+ 2\sqrt{\frac{\pi}{5}}\frac{\alpha^2\beta}{27r^4}\Big(-180\mathcal{B}^{\mathrm{eff}(1,1)}M^2(V - V_0)$$

$$+ r(-72Mh_0^{\mathrm{IH}(1,1)} + r(-13\pi r + 18M\partial_r h_0^{\mathrm{IH}(1,1)}))\Big)(\partial_\theta Y_{2,0})drd\theta$$

$$+ \sqrt{\frac{\pi}{5}}\frac{\alpha^2\beta}{27r^5}\Big(\pi(-720M^5 - 72M^4r + 13r^5) + 72\mathcal{B}^{\mathrm{eff}(1,1)}M^2r(-9M + r)(V - V_0)$$

$$+ 36Mr^2((-6M + r)h_0^{\mathrm{IH}(1,1)} + 3Mr\partial_r h_0^{\mathrm{IH}(1,1)})\Big)\sum_{i,j=\theta,\Phi}\left(\hat{\nabla}_i\hat{\nabla}_j Y_{2,0} - \frac{1}{2}\gamma_{ij}\hat{\triangle}Y_{2,0}\right)dx^i dx^j. \quad (113)$$

The effective energy–momentum tensor $T_{\mu\nu}^{\mathrm{eff}(2,1)}$ is given by

$$8\pi\alpha^2\beta T_{\mu\nu}^{\mathrm{eff}(2,1)} = 8\pi\alpha^2\beta T_{\mu\nu}^{(2,1)} + 8\pi\alpha^2\beta\tilde{T}_{\mu\nu}^{(2,1)}. \quad (114)$$

From the expression for $T_{\mu\nu}^{\mathrm{eff}(2,1)}\big|_{\ell=0}$ we can read $\mathcal{A}^{\mathrm{eff}(2,1)}$, $T_{Vr}^{\mathrm{eff}(2,1)}$, and $T_{rr}^{\mathrm{eff}(2,1)}$ as

$$\mathcal{A}^{\mathrm{eff}(2,1)} = \frac{1}{24r^3}\Big[8M^2\mathcal{B}^{\mathrm{eff}(1,1)}(r - 6M) - \pi r(8M^2 + r^2)\Big], \quad (115)$$

$$\alpha^2\beta T_{Vr}^{\mathrm{eff}(2,1)} = \frac{\alpha^2\beta}{432\pi r^7}\Big[-\pi(288M^5 - 1008M^4r - 72M^3r^2 + 72M^2r^3 + 18Mr^4 + 35r^5)$$

$$+ \mathcal{B}^{\mathrm{eff}(1,1)}(-108M^2r^3 + 72M^2r(-9M + 4r)(V - V_0))$$

$$+ 36Mr^2(2(-3M + r)h_0^{\mathrm{IH}(1,1)} + (3M - 2r)r\partial_r h_0^{\mathrm{IH}(1,1)})\Big], \quad (116)$$

$$\alpha^2\beta T_{rr}^{\mathrm{eff}(2,1)} = \frac{\alpha^2\beta}{24\pi r^6}\Big[\pi(-16M^4 + 16M^3r + 12M^2r^2 + 4Mr^3 + r^4)$$

$$- 24\mathcal{B}^{\mathrm{eff}(1,1)}M^2r(V - V_0) + 4Mr^2(-2h_0^{\mathrm{IH}(1,1)} + r\partial_r h_0^{\mathrm{IH}(1,1)})\Big]. \quad (117)$$

We can see that $\mathcal{A}^{\mathrm{eff}(2,1)}$ is not a constant but does not depend on V. The value of $\mathcal{A}^{\mathrm{eff}(2,1)}$ at $r = r_0$ is

$$\mathcal{A}^{\mathrm{eff}(2,1)}\big|_{r=r_0} = -\frac{5\pi}{72}. \quad (118)$$

From Eqs. (16), (18), and (19), we obtain the perturbed metric as

$$\alpha^2\beta h_{\mu\nu}^{(2,1)}\big|_{\ell=0}dx^\mu dx^\nu = \alpha^2\beta\left[\left(\frac{2\delta M^{(2,1)}}{r} + 2f\lambda^{(2,1)}\right)dV^2 - 2\lambda^{(2,1)}dVdr\right], \quad (119)$$

with

$$\delta M^{(2,1)} = \delta m^{(2,1)} + \mathcal{A}^{\mathrm{eff}(2,1)}(V - V_0) - 4\pi\int_{r_0}^r \bar{r}^2 T_{Vr}^{\mathrm{eff}(2,1)}(V_0,\bar{r})d\bar{r}, \quad (120)$$

$$\lambda^{(2,1)} = -4\pi\int_{r_0}^r \bar{r}T_{rr}^{\mathrm{eff}(2,1)}(V,\bar{r})d\bar{r} + \chi^{(2,1)}(V), \quad (121)$$

where $\delta m^{(2,1)}$ is a constant and the function $\chi^{(2,1)}(V)$ corresponds to the residual gauge mode. We can see that the "mass term" $\delta M^{(2,1)}$ depends on time. However, because the spacetime is not spherically

symmetric at this order, the appropriate definition of the mass is not clear. We discuss this topic in the next section.

We should note that $\ell = 2$ even-parity metric perturbations also exist at $\mathcal{O}(\alpha^2\beta)$:

$$\alpha^2\beta h_{\mu\nu}^{(+)(2,1)}\big|_{\ell=2}dx^\mu dx^\nu = 4\alpha^2\beta\sqrt{\frac{\pi}{5}}Y_{2,0}\Big[H_{0,\ell=2}^{(2,1)}dV^2 + 2H_{1,\ell=2}^{(2,1)}dVdr + H_{2,\ell=2}^{(2,1)}dr^2$$
$$+ 2K_{\ell=2}^{(2,1)}r^2(d\theta^2 + \sin^2\theta d\Phi^2)\Big]. \tag{122}$$

The perturbed metric can be obtained by solving the Zerilli equation [5–7]. As shown in the next section, $\ell = 2$ metric perturbations do not affect the area of the apparent horizon, and thus these modes are not relevant for the discussion of black hole mechanics.

5. Black hole mechanics

In this section we discuss the relations among the area, mass, and angular momentum of a black hole. In the standard derivation of black hole mechanics [36,37], assuming time-translational and rotational Killing vectors in a vacuum spacetime before and after the dynamical process, the differences in the Bondi–Sachs energy and angular momentum, and therefore the energy and angular momentum of the whole system, are discussed. In the present case, however, we would like to determine the energy and angular momentum extraction in the presence of force-free electromagnetic fields without a time-translational Killing vector. We do not assume the stationary stages before and after the energy extraction. Moreover, to isolate the energy and angular momentum of the black hole from the ambient electromagnetic fields, we need to discuss them in terms of quasi-local quantities. In the present situation we show that the apparent horizon is a good candidate for the black hole horizon for this purpose, and that the first law of black hole mechanics holds if we take the appropriate time-dependent mass parameter of the apparent horizon.

5.1. Apparent horizon

In this subsection we discuss the apparent horizon for the metric $g_{\mu\nu} = g_{\mu\nu}^{\mathrm{Kerr}} + g_{\mu\nu}^{\mathrm{BZ}}$. Because the $V = \mathrm{const.}$ surface of this spacetime is timelike at $\mathcal{O}(\alpha^2\beta)$, we work in the Kerr–Schild coordinates (T, r, θ, Φ). We set the relation between T and V as $V = T + r - 2M$. The unit normal to the $T = \mathrm{const.}$ surface is given by

$$n_\mu dx^\mu = F_n dT, \tag{123}$$

where the function F_n is chosen so that $g^{\mu\nu}n_\mu n_\nu = -1$ and n^μ is future directed. The induced metric on the $T = \mathrm{const.}$ surface is given by

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{124}$$

and the projection operator $\gamma_\mu{}^\nu$ becomes

$$\gamma_\mu{}^\nu = \gamma_{\mu\alpha}g^{\alpha\nu}. \tag{125}$$

Because

- $Y_{0,0}$ perturbations come from $\mathcal{O}(\beta)$ and $\mathcal{O}(\alpha^2\beta)$,
- $Y_{1,0}$ perturbations come from $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\beta)$,
- $Y_{2,0}$ perturbations come from $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\alpha^2\beta)$

in the metric $g_{\mu\nu} = g_{\mu\nu}^{\text{Kerr}} + g_{\mu\nu}^{\text{BZ}}$, we can assume that the location of the apparent horizon at each hypersurface $T = \text{const.}$ is

$$r = \mathcal{R}(\theta; T)$$

$$= (\mathcal{R}^{(0,0)} + \beta\mathcal{R}^{(0,1)} + \alpha^2\mathcal{R}_{\ell=0}^{(2,0)} + \alpha^2\beta\mathcal{R}_{\ell=0}^{(2,1)})$$

$$+ \alpha(\mathcal{R}^{(1,0)} + \beta\mathcal{R}^{(1,1)})2\sqrt{\frac{\pi}{3}}Y_{1,0} + \alpha^2(\mathcal{R}_{\ell=2}^{(2,0)} + \beta\mathcal{R}_{\ell=2}^{(2,1)})4\sqrt{\frac{\pi}{5}}Y_{2,0}, \tag{126}$$

where the coefficients only depend on T [38]. From the results for the Kerr metric and $\mathcal{O}(\beta)$ perturbations, where the location of the apparent horizon coincides with that of the event horizon at this order, discussed in Sect. 4.3, we obtain

$$\mathcal{R}^{(0,0)} = 2M, \tag{127}$$

$$\alpha\mathcal{R}^{(1,0)} = 0, \tag{128}$$

$$\alpha^2\mathcal{R}_{\ell=0}^{(2,0)} = -\frac{\alpha^2 M}{2}, \tag{129}$$

$$\alpha^2\mathcal{R}_{\ell=2}^{(2,0)} = 0, \tag{130}$$

$$\beta\mathcal{R}^{(0,1)} = -2\pi\beta M. \tag{131}$$

Thus, we need to fix $\mathcal{R}_{\ell=0}^{(2,1)}$, $\mathcal{R}^{(1,1)}$, and $\mathcal{R}_{\ell=2}^{(2,1)}$. The unit normal to $r = \mathcal{R}(\theta; T)$ at each $T = \text{const.}$ surface is

$$s_\mu = F_s \gamma_\mu{}^\nu \bar{s}_\nu, \tag{132}$$

with $\bar{s}_\mu dx^\mu = dr - (\partial_\theta \mathcal{R})d\theta$, where the function F_s is chosen so that $g^{\mu\nu}s_\mu s_\nu = 1$ and s^μ is an outward vector. The induced metric on the $T = \text{const.}$ and $r = \mathcal{R}(\theta; T)$ surface is

$$q_{\mu\nu} = \gamma_{\mu\nu} - s_\mu s_\nu = g_{\mu\nu} + n_\mu n_\nu - s_\mu s_\nu. \tag{133}$$

The location of the apparent horizon is determined by

$$\theta_+ = q^{\mu\nu}\nabla_\mu(n_\nu + s_\nu) = 0. \tag{134}$$

After some calculations, we obtain

$$\mathcal{R}^{(1,1)} = 0, \tag{135}$$

$$\mathcal{R}_{\ell=0}^{(2,1)} = \left(2\mathcal{A}^{\text{eff}(2,1)}\Big|_{r=r_0} - \frac{2}{3}\mathcal{B}^{\text{eff}(1,1)}\right)(T - T_0) - \frac{151M\pi}{54} + 2\delta m^{(2,1)}, \tag{136}$$

$$\mathcal{R}_{\ell=2}^{(2,1)} = \frac{\mathcal{B}^{\text{eff}(1,1)}}{21}(T - T_0) + \frac{5\pi M}{54} + \frac{2M}{7}H_{0,\ell=2}^{(2,1)}\Big|_{r=r_0} - \frac{8M^2}{7}\partial_T K_{\ell=2}^{(2,1)}\Big|_{r=r_0} \tag{137}$$

as solutions of Eq. (134), where $T_0 = V_0$. Using our results in the previous section, we have the relation

$$2\mathcal{A}^{\text{eff}(2,1)}\Big|_{r=r_0} - \frac{2}{3}\mathcal{B}^{\text{eff}(1,1)} = \frac{\pi}{12}. \tag{138}$$

The area of the apparent horizon is given by

$$A_{\mathrm{AH}} = 16\pi M^2 - 32\pi M^2\beta - 4\pi M^2\alpha^2 + \frac{4\pi M\alpha^2\beta}{3}\left[7\pi M + 12\mathcal{R}^{(2,1)}_{\ell=0}\right] + \mathcal{O}(\alpha^3) + \mathcal{O}(\beta^2).$$

(139)

We should note that $\ell = 2$ terms in Eq. (126) do not affect the area because of the orthogonality of the spherical harmonics. Thus, the time dependence of the apparent horizon area is

$$\partial_T A_{\mathrm{AH}} = 16\pi M\alpha^2\beta\,\partial_T\mathcal{R}^{(2,1)}_{\ell=0} = \frac{4\pi^2 M\alpha^2\beta}{3}.$$

(140)

5.2.   Angular momentum

The Komar angular momentum at the apparent horizon is

$$J_{\mathrm{Komar}}|_{\mathrm{AH}} = \alpha M^2 + \alpha\beta M\mathcal{B}^{\mathrm{eff}(1,1)}(T - T_0).$$

(141)

The time dependence of $J_{\mathrm{Komar}}|_{\mathrm{AH}}$ is

$$\partial_T J_{\mathrm{Komar}}|_{\mathrm{AH}} = \alpha\beta M\mathcal{B}^{\mathrm{eff}(1,1)}$$

$$= -\frac{\alpha\beta\pi M}{3}$$

$$= -\dot{J}_{\mathrm{BZ}},$$

(142)

where $\dot{J}_{\mathrm{BZ}}$ is given by Eq. (72). Thus, this reproduces the angular momentum extraction rate of the Blandford–Znajek process in Eq. (72). This explicitly shows that the total angular momentum conservation law holds, i.e. the decreasing rate of the angular momentum of the black hole is balanced with the angular momentum extraction rate of the Blandford–Znajek process.

5.3.   Implications of the black hole mechanics

If we assume the relation of the first law of black hole mechanics [36],

$$dM = \frac{\kappa}{8\pi}dA + \Omega_H dJ,$$

(143)

we can obtain the implication of the time dependence of the black hole mass. Setting dA and dJ as $\partial_T A_{\mathrm{AH}}$ and $\partial_T J_{\mathrm{Komar}}|_{\mathrm{AH}}$ in Eqs. (140) and (142), the time dependence of the mass is suggested by

$$\partial_T M = \frac{\kappa}{8\pi}\frac{4\pi^2 M\alpha^2\beta}{3} + \Omega_H\left(-\frac{\alpha\beta\pi M}{3}\right).$$

(144)

If we assume

$$\kappa = \frac{1}{4M} + \mathcal{O}(\alpha) + \mathcal{O}(\beta), \qquad \Omega_H = \frac{\alpha}{4M} + \mathcal{O}(\alpha\beta) + \mathcal{O}(\alpha^2) + \mathcal{O}(\beta),$$

(145)

we obtain

$$\partial_T M = -\frac{\alpha^2\beta\pi}{24} \qquad = -\dot{E}_{\mathrm{BZ}},$$

(146)

where $\dot{E}_{\mathrm{BZ}}$ is given by Eq. (71). This reproduces the energy extraction rate of the Blandford–Znajek process in Eq. (72), although the quantity M in the first law is as yet undefined as a quasi-local quantity of the apparent horizon.

5.4. The Hawking mass

Because the spacetime at $\mathcal{O}(\alpha^2 \beta)$ is neither stationary nor spherically symmetric, it is not obvious how to define the mass of the black hole. In that case, a possible choice of quasi-local mass is the Hawking mass. The Hawking mass at the apparent horizon is given by [39,41]

$$M_{\text{Hawking}}|_{\text{AH}} = \sqrt{\frac{A_{\text{AH}}}{16\pi}}$$

$$= M - \pi \beta M - \frac{\alpha^2 M}{8} + \frac{\alpha^2 \beta}{6}\left(\pi M + 3\mathcal{R}_{\ell=0}^{(2,1)}\right) + \mathcal{O}(\alpha^3) + \mathcal{O}(\beta^2). \qquad (147)$$

The time dependence of $M_{\text{Hawking}}|_{\text{AH}}$ is

$$\partial_T M_{\text{Hawking}}|_{\text{AH}} = \frac{\alpha^2 \beta}{2}\partial_T \mathcal{R}_{\ell=0}^{(2,1)} = \frac{\pi \alpha^2 \beta}{24} > 0. \qquad (148)$$

While the absolute value is the desired value, this is positive. This is because the Hawking mass at the apparent horizon is the square root of the apparent horizon area, which is increasing in time. Thus, we consider that the Hawking mass is not suitable for the description of energy extraction by the Blandford–Znajek process. We note that even if we use the Hayward mass [40,41], the mass is increasing in time.

5.5. Comparison with the Kerr metric with time-dependent parameters

As shown in Appendix B, the Kerr metric with small parameter shifts of the mass and angular momentum takes the form of Eq. (B.12). In this subsection we show that the time dependence of $g_{\mu\nu} = g_{\mu\nu}^{\text{Kerr}} + g_{\mu\nu}^{\text{BZ}}$ can be understood in terms of the Kerr metric of Eq. (B.12) but with time-decreasing mass and angular momentum.

Let us consider the Kerr metric in the form of Eq. (B.12), but we replace the constants $\delta M^{(\text{phys})}$ and $\delta J^{(\text{phys})}$ by $\delta\bar{M}(V)$ and $\delta\bar{J}(V)$, which are functions of V. We denote this metric by $g_{\mu\nu}^{\text{Kerr}+(\delta\bar{M},\delta\bar{J})}$. We would like to compare $g_{\mu\nu} = g_{\mu\nu}^{\text{Kerr}} + g_{\mu\nu}^{\text{BZ}}$ with $g_{\mu\nu}^{\text{Kerr}+(\delta\bar{M},\delta\bar{J})}$. We set $\delta\bar{M}$ and $\delta\bar{J}$ as

$$\alpha^2 \beta \delta\bar{M} = -\frac{\alpha^2 \beta \pi}{24}(V - V_0) = -\dot{E}_{\text{BZ}}(V - V_0), \qquad (149)$$

$$\alpha\beta\delta\bar{J} = -\frac{\alpha\beta\pi M}{3}(V - V_0) = -\dot{J}_{\text{BZ}}(V - V_0). \qquad (150)$$

We also choose $\chi(V)$ in $g_{\mu\nu}^{\text{Kerr}+(\delta\bar{M},\delta\bar{J})}$ as (see Eq. (B.14))

$$\chi = \chi^{(2,1)} + \frac{1}{6M}\mathcal{B}^{\text{eff}(1,1)}(V - V_0). \qquad (151)$$

Then, after some calculations, we obtain

$$g_{\mu\nu}^{\text{Kerr}} + g_{\mu\nu}^{\text{BZ}} = g_{\mu\nu}^{\text{Kerr}+(\delta\bar{M},\delta\bar{J})} + g_{\mu\nu}^{\text{other}} + [\ell = 2 \text{ terms}] + \mathcal{O}(\alpha^3) + \mathcal{O}(\beta^2), \qquad (152)$$

where $g_{\mu\nu}^{\text{other}}$ does not depend on time. We note that $g_{\mu\nu}^{\text{other}}$ contains the $\mathcal{O}(\beta)$ effect, i.e. the perturbation corresponding to the magnetic Reissner–Nordström metric discussed in Sect. 4.3. Equation (152) shows that the $\ell = 0, 1$ time-dependent terms of $g_{\mu\nu} = g_{\mu\nu}^{\text{Kerr}} + g_{\mu\nu}^{\text{BZ}}$ can be expressed as the Kerr metric with time-dependent parameters, $g_{\mu\nu}^{\text{Kerr}+(\delta\bar{M},\delta\bar{J})}$, whose time dependence is determined from the energy and angular momentum extraction rates of the Blandford–Znajek process. If we regard

$M + \delta \bar{M}$ as a black hole mass, its time dependence coincides with Eq. (146). Therefore, this gives an appropriate time-dependent mass for the energy extraction in the present setting.

We have seen that the $\ell = 0, 1$ time-dependent terms of our results can be fitted by the Kerr metric with time-dependent parameters in Eddington–Finkelstein-like coordinates. We should note that it is essential which time coordinate we let the mass and angular momentum parameters depend on. For example, if we let them depend on time in the Boyer–Lindquist coordinates, our results cannot be fitted by the corresponding spacetime. Finding an appropriate coordinate system is not such a trivial problem, and what we have shown is that Eddington–Finkelstein-like coordinates are the appropriate choice.[9]

Finally, we comment on $\mathcal{A}^{\mathrm{eff}(2,1)}$ in Eq. (115). While $\mathcal{A}^{\mathrm{eff}(2,1)}$ is related to the flux associated with $\nabla^\mu (T^{\mathrm{eff}}_{\mu\nu} (\partial_V)^\nu) = 0$ (see Sect. 2.3.3), at this stage the physical meaning of $\mathcal{A}^{\mathrm{eff}(2,1)}$ is not clear. Because $\delta M^{(2,1)}$ in Eq. (120) is written by $\mathcal{A}^{\mathrm{eff}(2,1)}$, it is useful to consider the meaning of the "mass term" $\delta M^{(2,1)}$. If we compare the situation with the Kerr black hole case, $\delta M^{(2,1)}$ corresponds to $\delta M^{(2,1)}_{\mathrm{Kerr}}$ in Eq. (B.13). As shown in Eq. (B.20), the "mass term" $\delta M^{(2,1)}_{\mathrm{Kerr}}$ in the Kerr black hole case does not directly denote the variation of the mass of the black hole, and the variation of the physical mass is obtained by subtracting the effect of the spin from $\delta M^{(2,1)}_{\mathrm{Kerr}}$, the second term on the right-hand side of Eq. (B.20). This suggests that $\delta M^{(2,1)}$ also contains information on both the mass and the angular momentum of a black hole, and this is why $\mathcal{A}^{\mathrm{eff}(2,1)}\big|_{r=r_0}$ in Eq. (118) does not coincide with $-\dot{E}_{\mathrm{BZ}}$. In fact, $\mathcal{A}^{\mathrm{eff}(2,1)}$ can be written in terms of $\dot{E}_{\mathrm{BZ}}$ and $\dot{J}_{\mathrm{BZ}}$ as

$$\alpha^2 \beta \mathcal{A}^{\mathrm{eff}(2,1)} = -\dot{E}_{\mathrm{BZ}} - \frac{4Mr - 6M^2}{3r^3} \alpha \dot{J}_{\mathrm{BZ}}. \tag{153}$$

6.   Summary and discussion

We have developed the formalism of monopole and dipole linear gravitational perturbations around Schwarzschild black holes in Eddington–Finkelstein coordinates against generic time-dependent accreting matter. We derived the mass and angular momentum of black holes in terms of the energy–momentum tensor of accreting matter at the linear order. The time dependence of the mass and angular momentum are determined by the the accretion rates of the energy and angular momentum. In particular, after the accreting matter completely falls into the black hole at some finite time, $\ell = 0$ and $\ell = 1$ perturbations represent slowly rotating Kerr black holes, and the final mass and angular momentum are expressed by the total time integral of the accretion rates at $r = 2M$. We also showed that our formalism can reproduce the exact Vaidya solution [31].

Applying our formalism to the Blandford–Znajek process [19], we studied the metric backreaction. While we need to study the non-linear gravitational perturbations to discuss the backreaction of the Blandford–Znajek process, our formalism can be applied to this problem because the forms of equations at each order are the same as those of linear order with the source terms which contain the non-linear effects. We calculated the time-dependent Komar angular momentum and the area of the apparent horizon. The decreasing rate of the former coincides with the angular momentum loss rate estimated in terms of the stress–energy tensor of the force-free electromagnetic fields at infinity.

According to the test-field calculation of the energy and angular momentum extraction rates of the Blandford–Znajek process [19], there is no doubt that energy and angular momentum are transfered

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[9] In Ref. [42], as an extension of the Vaidya metric [31], the Kerr metric but with time-dependent mass and angular momentum parameters are discussed in a different coordinate system from this paper. It will be interesting to discuss the relation with our perturbative solution, but we leave this problem for future work.

to asymptotic regions. However, it is not clear how to describe the local metric behavior of the backreaction. We showed that the time dependence of $\ell = 0, 1$ modes are expressed by the Kerr metric but with time-decreasing mass and angular momentum parameters, which depend only on the ingoing null coordinate V. This suggests that the corresponding outgoing fluxes come directly from the vicinity of the event horizon. If we regard the corresponding mass parameter as the black hole mass, we saw that its decreasing rate coincides with the energy extraction rate of the Blandford–Znajek process, and that the first law of black hole mechanics holds for the apparent horizon in terms of this mass parameter but not the Hawking mass.

Finally, we comment on future work. It will be interesting to extend our analysis to higher-order solutions of the Blandford–Znajek process [21–23]. Applications to other situations, e.g. the Penrose process or the superradiance phenomenon, is possible. It will also be interesting to consider applications to modified gravity theories. If we consider some modified gravity theories and they admit solutions close to the Schwarzschild black holes, we expect that the field equations for the monopole and dipole gravitational perturbations take the same form as Eq. (5); then, our formalism can be applied.

Appendix A.   The gauge transformation of monopole and odd-parity dipole perturbations

A.1. $\ell = 0$ perturbations

The general perturbed metric for $\ell = 0$ linear perturbations in Eddington–Finkelstein coordinates is given by

$$h_{\mu\nu}^{(+)}dx^\mu dx^\nu = H_0(V,r)dV^2 + 2H_1(V,r)dVdr + H_2(V,r)dr^2 + 2K(V,r)r^2(d\theta^2 + \sin^2\theta\, d\Phi^2). \tag{A.1}$$

The general gauge transformation for this perturbed metric becomes

$$h_{\mu\nu}^{(+)} \to h_{\mu\nu}^{(+)} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{A.2}$$

with

$$\xi_\nu dx^\mu = \xi_V(V,r)dV + \xi_r(V,r)dr. \tag{A.3}$$

Under this gauge transformation, the components of the perturbed metric change as

$$H_0 \to H_0 - \frac{r_0}{r^2}(\xi_V + f\xi_r) + 2\partial_V \xi_V, \tag{A.4}$$

$$H_1 \to H_1 + \partial_r \xi_V + \frac{r_0}{r^2}\xi_r + \partial_V \xi_r, \tag{A.5}$$

$$H_2 \to H_2 + 2\partial_r \xi_r, \tag{A.6}$$

$$K \to K + \frac{1}{r}(\xi_V + f\xi_r). \tag{A.7}$$

If we choose the gauge with $H_2 = K = 0$, the residual gauge modes become $\xi_V = -f\tilde{\eta}(V)$, $\xi_r = \tilde{\eta}(V)$, where $\tilde{\eta}(V)$ is an arbitrary function of V, and the components of the perturbed metric transform as $H_0 \to H_0 - 2f \partial_V \tilde{\eta}$ and $H_1 \to H_1 + \partial_V \tilde{\eta}$.

A.2. $\ell = 1$ odd-parity perturbations

The general perturbed metric for the $\ell = 1$, $m = 0$ odd-parity linear perturbations in Eddington–Finkelstein coordinates is given by

$$h^{(-)}_{\mu\nu} dx^\mu dx^\nu = 4\sqrt{\pi/3} \sin\theta (\partial_\theta Y_{1,0}) d\Phi (h_0(V,r)dV + h_1(V,r)dr)$$
$$= -2\sin^2\theta d\Phi (h_0(V,r)dV + h_1(V,r)dr). \tag{A.8}$$

The general gauge transformation for this perturbed metric becomes

$$h^{(-)}_{\mu\nu} \to h^{(-)}_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{A.9}$$

with

$$\xi_\nu dx^\mu = -\sin^2\theta \xi^{(-)}(V,r)d\Phi. \tag{A.10}$$

Under this gauge transformation, the components of the perturbed metric change as

$$h_0 \to h_0 + \partial_V \xi^{(-)}, \qquad h_1 \to h_1 - \frac{2}{r}\xi^{(-)} + \partial_r \xi^{(-)}. \tag{A.11}$$

If we choose the gauge with $h_1 = 0$, the residual gauge modes become $\xi^{(-)} = r^2 \tilde{\zeta}(V)$, where $\tilde{\zeta}(V)$ is an arbitrary function of V, and h_0 transforms as $h_0 \to h_0 + r^2 \partial_V \tilde{\zeta}$.

Appendix B. The Kerr metric with small parameter shifts

The Kerr metric has two parameters: the mass M and the spin a. Let us consider shifts of those parameters in Eq. (89) as

$$M \to M + \alpha^2 \beta \delta M^{(\text{phys})}, \qquad a \to a + \alpha\beta \frac{\delta J^{(\text{phys})}}{M}, \tag{B.1}$$

where $\alpha := a/M$ and β are small parameters, and $\delta M^{(\text{phys})}$ and $\delta J^{(\text{phys})}$ are constants. Introducing the coordinate transformation

$$d\Phi \to d\Phi - \alpha\beta \frac{\delta J^{(\text{phys})}}{M} \frac{dr}{r^2} \tag{B.2}$$

and the gauge transformation at $\mathcal{O}(\alpha^2\beta)$ as

$$g^{\text{EF}}_{\mu\nu}[M + \alpha^2\beta \delta M^{(\text{phys})}, a + \alpha\beta\delta J^{(\text{phys})}/M]$$
$$\to g^{\text{EF}}_{\mu\nu}[M + \alpha^2\beta \delta M^{(\text{phys})}, a + \alpha\beta\delta J^{(\text{phys})}/M] + \alpha^2\beta(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu), \tag{B.3}$$

with

$$\xi_\mu = \xi_\mu|_{\ell=0} + \xi_\mu|_{\ell=2}, \tag{B.4}$$

$$\xi_\mu|_{\ell=0}dx^\mu = \xi^{\ell=0}_V dV + \xi^{\ell=0}_r dr, \tag{B.5}$$

$$\xi_\mu|_{\ell=2}dx^\mu = \xi_V^{\ell=2}Y_{2,0}dV + \xi_r^{\ell=2}Y_{2,0}dr - \frac{\xi_S^{\ell=2}}{\sqrt{6}}(\partial_\theta Y_{2,0})d\theta, \tag{B.6}$$

and

$$\xi_V^{\ell=0} = -\frac{2M\delta J^{(\mathrm{phys})}}{r^2} + \left(1 - \frac{2M}{r}\right)\tilde\chi(V), \tag{B.7}$$

$$\xi_r^{\ell=0} = -\frac{2\delta J^{(\mathrm{phys})}}{3r} - \tilde\chi(V), \tag{B.8}$$

$$\xi_V^{\ell=2} = 0, \tag{B.9}$$

$$\xi_r^{\ell=2} = \sqrt{\frac{\pi}{5}}\frac{4(3M+r)}{3r^2}\delta J^{(\mathrm{phys})}, \tag{B.10}$$

$$\xi_S^{\ell=2} = -\sqrt{\frac{2\pi}{15}}\frac{2(2M+r)}{r}\delta J^{(\mathrm{phys})}, \tag{B.11}$$

the metric becomes $g_{\mu\nu}^{\mathrm{Kerr}+(\delta M,\delta J)}$ with

$$
\begin{aligned}
g_{\mu\nu}^{\mathrm{Kerr}+(\delta M,\delta J)}dx^\mu dx^\nu &= \left(g_{\mu\nu}^{\mathrm{Sch}} + \alpha h_{\mu\nu}^{(1,0)} + \alpha^2 h_{\mu\nu}^{(2,0)}\right)dx^\mu dx^\nu - \frac{4\alpha\beta\delta J^{(\mathrm{phys})}\sin^2\theta}{r}dVd\Phi \\
&\quad + \alpha^2\beta\left[\frac{2\delta M_{\mathrm{Kerr}}^{(2,1)}}{r} + 2\left(1-\frac{2M}{r}\right)\lambda_{\mathrm{Kerr}}^{(2,1)}\right]dV^2 - 2\alpha^2\beta\lambda_{\mathrm{Kerr}}^{(2,1)}dVdr \\
&\quad + \alpha^2\beta\sqrt{\frac{\pi}{5}}Y_{2,0}\left[H_{0,\ell=2}^{\mathrm{Kerr}}dV^2 + 2H_{1,\ell=2}^{\mathrm{Kerr}}dVdr + H_{2,\ell=2}^{\mathrm{Kerr}}dr^2\right. \\
&\quad \left. + 2K_{\ell=2}^{\mathrm{Kerr}}r^2(d\theta^2 + \sin^2\theta d\Phi^2)\right] + \mathcal{O}(\alpha^3) + \mathcal{O}(\beta^2), \tag{B.12}
\end{aligned}
$$

where

$$\delta M_{\mathrm{Kerr}}^{(2,1)} = \delta M^{(\mathrm{phys})} + \frac{4Mr - 6M^2}{3r^3}\delta J^{(\mathrm{phys})}, \tag{B.13}$$

$$\lambda_{\mathrm{Kerr}}^{(2,1)} = -\frac{4M\delta J^{(\mathrm{phys})}}{3r^3} + \chi(V), \tag{B.14}$$

$$\chi(V) := \partial_V\tilde\chi(V), \tag{B.15}$$

and

$$H_{0,\ell=2}^{\mathrm{Kerr}} = \frac{8M(6M^2 - Mr - 3r^2)J^{(\mathrm{phys})}}{3r^5}, \tag{B.16}$$

$$H_{1,\ell=2}^{\mathrm{Kerr}} = \frac{8M(3M+2r)\delta J^{(\mathrm{phys})}}{3r^4}, \tag{B.17}$$

$$H_{2,\ell=2}^{\mathrm{Kerr}} = -\frac{16M\delta J^{(\mathrm{phys})}}{r^3}, \tag{B.18}$$

$$K_{\ell=2}^{\mathrm{Kerr}} = -\frac{4M(2M+r)\delta J^{(\mathrm{phys})}}{r^4}. \tag{B.19}$$

Here, we choose the coordinate system and the gauge so that $\mathcal{O}(\alpha\beta)$ and $\mathcal{O}(\alpha^2\beta)$ terms take a similar form to Sect. 2 for $\ell = 0, 1$, and the Regge–Wheeler gauge for $\ell = 2$. We obtain the relation

$$\delta M^{(\text{phys})} = \delta M^{(2,1)}_{\text{Kerr}} - \frac{4Mr - 6M^2}{3r^3}\delta J^{(\text{phys})}. \tag{B.20}$$

This implies that the "mass term" $\delta M^{(2,1)}_{\text{Kerr}}$ in the $\mathcal{O}(\alpha^2\beta)$ perturbations does not directly denote the variation of the mass of the black hole, and the variation of the physical mass $\delta M^{(\text{phys})}$ is obtained by subtracting the effect of the spin from $\delta M^{(2,1)}_{\text{Kerr}}$.

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