



Momentum transport properties of a hot and dense QCD matter in a weak magnetic field

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Abstract We have studied the momentum transport properties of a hot and dense QCD matter in the presence of weak magnetic field by determining the shear (η) and bulk (ζ) viscosities in the relaxation time approximation of kinetic theory. The dependence of η and ζ on the temperature has been explored in the presence of weak magnetic field (B -field) and finite chemical potential (μ). It is observed that both shear and bulk viscosities get decreased in the presence of a weak magnetic field, whereas the finite chemical potential increases these viscosities, specifically at low temperatures. This study is important to understand the sound attenuation through the Prandtl number (Pr), the nature of the flow through the Reynolds number (Re), the fluidity and location of transition point of the matter through the ratios η/s and ζ/s (s is the entropy density), respectively. The Prandtl number is observed to increase in the weak magnetic field, whereas the presence of a finite chemical potential reduces its magnitude as compared to the scenario of absence of B -field and μ . However, Pr still remains larger than unity, indicating that the energy dissipation due to the sound attenuation is mostly governed by the momentum diffusion. It is noticed that the weak magnetic field makes the Reynolds number larger, whereas the chemical potential makes it smaller than that in the absence of B -field and μ . We have observed that the ratio η/s decreases in the weak magnetic field regime, whereas the finite chemical potential increases its value, but the ratio ζ/s is found to decrease in the presence of weak magnetic field as well as finite chemical potential.

1 Introduction

The ultrarelativistic heavy ion collisions (URHICs) at Relativistic Heavy Ion Collider (RHIC) and Large Hadron Col-

lider (LHC) have provided strong evidence of the formation of a strongly interacting matter, known as the quark-gluon plasma (QGP). One of the salient goals of these experiments is to divulge the transport properties of the QGP. The transport coefficients are sensitive to the relevant degrees of freedom and their respective interactions within the QGP medium. For example, the shear viscosity (η) gives information about the momentum transfer in the presence of inhomogeneity of fluid velocity and the bulk viscosity (ζ) delineates the change of local pressure due to either contraction or expansion of fluid. Shear viscosity is related to a change in shape at constant volume, whereas bulk viscosity is related to a change in volume at constant shape. The dimensionless ratios of shear and bulk viscosities (η/s and ζ/s) to entropy density (s) characterize the intrinsic ability of a system to relax towards equilibrium. The exploration of shear and bulk viscosities reveal about the fluid dynamical behavior of the medium. The shear and bulk viscosity calculations were performed through various approaches, such as the relativistic Boltzmann transport equation in the relaxation time approximation [1–3], the Green–Kubo formula [4–7], the lattice simulations [8,9], the molecular dynamics simulation [10], the perturbation theory [11–14] etc. The $N = 4$ supersymmetric $SU(N_c)$ Yang–Mills theory has estimated the lower bound of the ratio η/s as close to $1/(4\pi)$, which is also known as the Kovtun–Son–Starinets (KSS) lower bound [15]. This estimated lower bound has been conjectured to be the lower bound of η/s for different physical systems, such as helium, nitrogen and water at pressures 0.1 MPa, 10 MPa, and 100 MPa, respectively. The transition from hadrons to quark-gluon plasma has a similar behavior in the ratio η/s [16].

In addition, the Au–Au collision at RHIC has also reported very low value of $\eta/s \sim 1/(4\pi)$ for the QGP medium formed indicating that the hot and dense matter produced in URHICs behaves like a perfect fluid. Using lattice gauge theory principles, η/s has been studied for few values

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of temperature in pure Yang–Mills theory [8, 17, 18]. According to the Yang–Mills theory and perturbative QCD [19], an increase in η/s is observed in the presence of dynamical quarks [8, 20]. Through the functional diagrammatic approach to QCD, references [21, 22] have evaluated η in Yang–Mills theory and the results are in good agreement with the lattice results for QCD with (2+1)-quark flavors. Both the approaches have observed a minimal η/s of about 0.2 near the phase transition temperature, which is slightly larger than $1/(4\pi)$. Similar results were also obtained in perturbation theory [22, 23]. For massless QGP, the bulk viscosity is very small compared to the shear viscosity for which it was neglected by some early viscous hydrodynamic simulations in the dissipative part of the energy–momentum tensor [24, 25]. The vanishing of the ratio ζ/s explains the restoration of chiral symmetry of the matter. On the other hand, a sharp rise of the ratio ζ/s in the vicinity of the phase transition temperature of matter is reported in Ref. [4]. Large value of the bulk viscosity signifies large fluctuations in the pressure. Although ζ vanishes for QGP with massless flavors at the classical level, but the quantum effects break the conformal symmetry of QCD, thus generating a nonzero bulk viscosity, which is described by the lattice calculation in the $SU(3)$ gauge theory [26]. As a result, the ratio ζ/s acts as a measure of the deviation of the strongly interacting matter from conformality.

The aforementioned estimations were made for most central collisions. However, in noncentral events of heavy ion collisions, when two nuclei travelling with ultrarelativistic speeds collide with each other, an intensely strong magnetic field perpendicular to the collision plane is expected to be produced at very early stages. Depending on the centrality, the strength of the magnetic field may vary between m_π^2 (10^{18} Gauss) at RHIC and $15 m_\pi^2$ at LHC [27] and at extreme cases it may reach $50 m_\pi^2$. The magnetic field is very strong for very short duration and becomes weak. So, there are two limits: strong magnetic field and weak magnetic field. In the strong magnetic field limit, the energy scale associated with the magnetic field is greater than the energy scale related to the temperature ($|q_f B| \gg T^2$, where $|q_f|$ is the absolute electronic charge of quark with flavor f). On the other hand, in the weak magnetic field limit, the energy scale associated with the magnetic field is smaller than the energy scale related to the temperature ($|q_f B| \ll T^2$). According to some observations [28, 29], the lifetime of such magnetic field gets significantly extended in an electrically conducting medium and is comparable with the lifetime of the partonic medium. In addition, high baryon densities are expected to be evidenced in Compressed Baryonic Matter (CBM) experiment at Facility for Antiproton and Ion Research (FAIR) and Nuclotron-based Ion Collider fAcility (NICA) at Joint Institute for Nuclear Research (JINR) in fixed target experiments. Thus, the shear viscosity, the bulk viscosity and the

associated transport properties of the medium are prone to be altered by the presence of both magnetic field and chemical potential. Previously, the effects of magnetic field on the QCD thermodynamics [30–33], the heavy quark diffusion [34], the conductive properties [29, 35–41], the magnetohydrodynamics [42, 43], the photon and dilepton productions from QGP [44–47], etc. have been explored. Recently, the collective effects of weak magnetic field and finite chemical potential on the charge transport, the heat transport and some related transport coefficients were explored in Ref. [48]. Viscous properties were also studied previously by using different models and approximations at finite magnetic field. For example, in Ref. [49] authors had employed the diluted instanton liquid model and the Green–Kubo formula to study the shear viscosity of the $SU(2)$ light-flavor quark matter at finite temperature under the strong magnetic field limit. In Ref. [50] authors had investigated the viscosities of the quark-gluon plasma in the presence of the strong magnetic field with the leading-log and lowest Landau level (LLL) approximations. Authors in Ref. [51] had computed the shear viscosity of two-flavor QCD plasma in a magnetic field by using the perturbative QCD at leading log order. In Ref. [52] authors had investigated the viscosities using the nonresistive dissipative magnetohydrodynamics from the Boltzmann equation in the 14-moment approximation at finite magnetic field. Authors in Ref. [53] had estimated viscosities using the relativistic Boltzmann transport equation in the relaxation time approximation, but for a hot and dense hadronic matter. In Ref. [54] authors had investigated the effects of the strong magnetic field-induced and asymptotic expansion-induced anisotropies on viscosities for a hot QCD matter using the kinetic theory approach, while in Ref. [55], the effects of the strong magnetic field and density on viscosities had been explored. In the present work, (i) we have studied shear and bulk viscosities for a hot QCD matter in the presence of both magnetic field and finite chemical potential. We have estimated the viscosities by solving the relativistic Boltzmann transport equation in the kinetic theory approach and used the weak magnetic field limit, where the energy scale associated with the temperature is larger than the energy scale related to the magnetic field, i.e. $T^2 \gg |q_f B|$. So, we have used the ansatz method in the weak magnetic field limit to calculate viscosities in the first part of Sect. 2, where the terms containing ω_c (cyclotron frequency) and its higher orders have been neglected. (ii) In the second part of Sect. 2, we have revisited the viscosity coefficients in the general configuration of magnetic field (no weak or strong magnetic field limit) and observed how they are related to the viscosities calculated using the ansatz method. (iii) We have extended our study to know the collective effects of weak magnetic field and density on some applications of viscosities, such as the Prandtl number (Pr), the Reynolds number (Re), specific shear viscosity (η/s) and specific bulk viscosity (ζ/s). (iv) We have

used the quasiparticle model, wherein the interactions among the medium constituents have been incorporated through the thermal masses of particles.

The present work is organized as follows. In Sect. 2, the momentum transport properties have been studied by deriving the response functions, *viz.* the shear viscosity and the bulk viscosity in the kinetic theory approach with a short description of the quasiparticle model. The results are presented in Sect. 3 while Sect. 4 discusses some applications of both the viscosities in terms of the Prandtl number, the Reynolds number and the ratios η/s and ζ/s . The work is summarized in Sect. 5.

2 Momentum transport properties

A fluid system slightly shifted from its equilibrium state due to the nonuniformity of its constituent flow with respect to the macroscopic velocity, can possess finite shear and bulk viscosities. We calculate the viscosities by assuming a local temperature $T(x)$ and flow velocity $u^\mu(x)$. For a nonequilibrium system, the dissipative part of the energy-momentum tensor $\Delta T^{\mu\nu}$ is written in terms of the equilibrium energy-momentum tensor $T_{(0)}^{\mu\nu}$ as

$$\Delta T^{\mu\nu} = T^{\mu\nu} - T_{(0)}^{\mu\nu}. \quad (1)$$

For the partonic system, $\Delta T^{\mu\nu}$ can also be written in terms of the infinitesimal changes of the quark, antiquark and gluon distribution functions as

$$\Delta T^{\mu\nu} = \int \frac{d^3 p}{(2\pi)^3} p^\mu p^\nu \left[\sum_f g_f \frac{(\delta f_f + \delta \bar{f}_f)}{\omega_f} + g_g \frac{\delta f_g}{\omega_g} \right], \quad (2)$$

where ‘ f ’ represents the flavor index for three flavors u, d and s . In Eq. (2), g_f and δf_f ($\delta \bar{f}_f$) denote the degeneracy factor and the infinitesimal change in the quark (antiquark) distribution function of f th flavor, respectively. For the gluon, g_g and δf_g denote the degeneracy factor and the infinitesimal change in its distribution function, respectively. The infinitesimal changes in quark, antiquark and gluon distribution functions are defined as $\delta f_f = f_f - f_f^0$, $\delta \bar{f}_f = \bar{f}_f - \bar{f}_f^0$ and $\delta f_g = f_g - f_g^0$, respectively. Here, f_f^0 , \bar{f}_f^0 and f_g^0 are the equilibrium distribution functions for quark, antiquark and gluon, respectively, which have the following forms,

$$f_f^0 = \frac{1}{e^{\beta(u^\alpha p_\alpha - \mu_f)} + 1}, \quad (3)$$

$$\bar{f}_f^0 = \frac{1}{e^{\beta(u^\alpha p_\alpha + \mu_f)} + 1}, \quad (4)$$

$$f_g^0 = \frac{1}{e^{\beta u^\alpha p_\alpha} - 1}, \quad (5)$$

where $T = \beta^{-1}$, u^α denotes the four-velocity of fluid and μ_f represents the chemical potential of f th flavor of quark. In above equations, for quark and antiquark, $p_\alpha \equiv (\omega_f, \mathbf{p})$ with $\omega_f = \sqrt{\mathbf{p}^2 + m_f^2}$ and for gluon, $p_\alpha \equiv (\omega_g, \mathbf{p})$. In order to determine the infinitesimal change in the particle distribution function, we are going to solve the relativistic Boltzmann transport equation in the relaxation time approximation for finite magnetic field and chemical potential,

$$p^\mu \frac{\partial f_f(x, p)}{\partial x^\mu} + \mathcal{F}^\mu \frac{\partial f_f(x, p)}{\partial p^\mu} = -\frac{p_\nu u^\nu}{\tau_f} \delta f_f(x, p), \quad (6)$$

where $f_f = \delta f_f + f_f^0$. The external force $\mathcal{F}^\mu = q F^{\mu\nu} p_\nu = (p^0 \mathbf{v} \cdot \mathbf{F}, p^0 \mathbf{F})$, where $F^{\mu\nu}$ represents the electromagnetic field strength tensor and \mathbf{F} denotes the Lorentz force, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. The relations between the components of $F^{\mu\nu}$ and the components of electric and magnetic fields are given by $F^{0i} = E^i$, $F^{i0} = -E^i$ and $F^{ij} = \frac{1}{2} \epsilon^{ijk} B_k$. The relaxation times for quarks (antiquarks), τ_f ($\tau_{\bar{f}}$) and for gluons, τ_g are respectively written [56] as

$$\tau_{f(\bar{f})} = \frac{1}{5.1 T \alpha_s^2 \log(1/\alpha_s) [1 + 0.12(2N_f + 1)]}, \quad (7)$$

$$\tau_g = \frac{1}{22.5 T \alpha_s^2 \log(1/\alpha_s) [1 + 0.06N_f]}. \quad (8)$$

To solve Eq. (6), we take the following ansatz which was first suggested by Ref. [36],

$$f_f = f_f^0 - \tau_f q \mathbf{E} \cdot \frac{\partial f_f^0}{\partial \mathbf{p}} - \mathbf{\Gamma} \cdot \frac{\partial f_f^0}{\partial \mathbf{p}}, \quad (9)$$

where $\mathbf{\Gamma}$ is associated with the magnetic field. The partial derivatives in the above ansatz are evaluated as

$$\begin{aligned} \frac{\partial f_f^0}{\partial p_x} &= -\beta v_x f_f^0 (1 - f_f^0), & \frac{\partial f_f^0}{\partial p_y} &= -\beta v_y f_f^0 (1 - f_f^0), \\ \frac{\partial f_f^0}{\partial p_z} &= -\beta v_z f_f^0 (1 - f_f^0). \end{aligned}$$

Assuming that the electric field is along x-direction ($\mathbf{E} = E \hat{x}$) and the magnetic field is along z-direction ($\mathbf{B} = B \hat{z}$), the relativistic Boltzmann transport equation (6) using the ansatz (9) can be rewritten as

$$\begin{aligned} \frac{\tau_f}{p_0} p^\mu \frac{\partial f_f^0}{\partial x^\mu} + \beta f_f^0 (1 - f_f^0) (\Gamma_x v_x + \Gamma_y v_y + \Gamma_z v_z) \\ + \tau_f q E v_x \frac{\partial f_f}{\partial p_0} - q B \tau_f \left(v_x \frac{\partial f_f}{\partial p_y} - v_y \frac{\partial f_f}{\partial p_x} \right) = 0. \end{aligned} \quad (10)$$

The partial derivatives in the above equation are calculated as

$$\begin{aligned} v_x \frac{\partial f_f}{\partial p_0} &= -\beta v_x f_f^0 (1 - f_f^0) - q E \tau_f \beta f_f^0 \\ &\quad \times (1 - f_f^0) v_x^2 \left(\frac{1}{\omega_f} + \beta \right) \end{aligned}$$

$$\begin{aligned}
& -\beta f_f^0 (1 - f_f^0) \Gamma_x v_x^2 \left(\frac{1}{\omega_f} + \beta \right) \\
& -\beta f_f^0 (1 - f_f^0) \Gamma_y v_x v_y \left(\frac{1}{\omega_f} + \beta \right) \\
& -\beta f_f^0 (1 - f_f^0) \Gamma_z v_x v_z \left(\frac{1}{\omega_f} + \beta \right), \quad (11)
\end{aligned}$$

$$\begin{aligned}
v_x \frac{\partial f_f}{\partial p_y} = & -\beta v_x v_y f_f^0 (1 - f_f^0) - q E \tau_f \beta f_f^0 \\
& \times (1 - f_f^0) v_x^2 v_y \left(\frac{1}{\omega_f} + \beta \right) \\
& -\beta f_f^0 (1 - f_f^0) \Gamma_x v_x^2 v_y \left(\frac{1}{\omega_f} + \beta \right) \\
& -\beta f_f^0 (1 - f_f^0) \Gamma_y v_x v_y^2 \left(\frac{1}{\omega_f} + \beta \right) \\
& + \frac{v_x \Gamma_y \beta f_f^0 (1 - f_f^0)}{\omega_f} - \beta f_f^0 (1 - f_f^0) \\
& \times \Gamma_z v_x v_y v_z \left(\frac{1}{\omega_f} + \beta \right), \quad (12)
\end{aligned}$$

$$\begin{aligned}
v_y \frac{\partial f_f}{\partial p_x} = & -\beta v_y v_x f_f^0 (1 - f_f^0) - q E \tau_f \beta f_f^0 \\
& \times (1 - f_f^0) v_y v_x^2 \left(\frac{1}{\omega_f} + \beta \right) \\
& + \frac{q E \tau_f \beta f_f^0 (1 - f_f^0) v_y}{\omega_f} \\
& -\beta f_f^0 (1 - f_f^0) \Gamma_x v_y v_x^2 \left(\frac{1}{\omega_f} + \beta \right) \\
& + \frac{\Gamma_x \beta f_f^0 (1 - f_f^0) v_y}{\omega_f} \\
& -\beta f_f^0 (1 - f_f^0) \Gamma_y v_y^2 v_x \left(\frac{1}{\omega_f} + \beta \right) \\
& -\beta f_f^0 (1 - f_f^0) \Gamma_z v_y v_z v_x \left(\frac{1}{\omega_f} + \beta \right). \quad (13)
\end{aligned}$$

Substituting the values of partial derivatives in Eq. (10) and then dropping higher order velocity terms, we obtain

$$\begin{aligned}
J - \beta f_f^0 (1 - f_f^0) \tau_f q E v_x \\
+ \beta f_f^0 (1 - f_f^0) (\Gamma_x v_x + \Gamma_y v_y + \Gamma_z v_z) \\
- \frac{q B \tau_f \beta f_f^0 (1 - f_f^0)}{\omega_f} (v_x \Gamma_y - v_y \Gamma_x) \\
+ \frac{\tau_f^2 q B q E v_y \beta f_f^0 (1 - f_f^0)}{\omega_f} = 0. \quad (14)
\end{aligned}$$

In getting the above equation, we have also replaced $J = \frac{\tau_f}{p_0} p^\mu \frac{\partial f_f^0}{\partial x^\mu}$. For quark distribution function, we have calculated J as

$$\begin{aligned}
J = & -\beta \tau_f f_f^0 (1 - f_f^0) \left[\left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \\
& \left. + p^l \left(\frac{\partial_l P}{\varepsilon + P} - \frac{\partial_l T}{T} \right) - \frac{T p^l}{\omega_f} \partial_l \left(\frac{\mu_f}{T} \right) - \frac{p^k p^l}{2\omega_f} W_{kl} \right], \quad (15)
\end{aligned}$$

where $W_{kl} = \partial_k u_l + \partial_l u_k - \frac{2}{3} \delta_{kl} \partial_j u^j$. Similarly for antiquark and gluon distribution functions, we get

$$\begin{aligned}
\bar{J} = & -\beta \tau_{\bar{f}} \bar{f}_f^0 (1 - \bar{f}_f^0) \left[\left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \\
& \left. + p^l \left(\frac{\partial_l P}{\varepsilon + P} - \frac{\partial_l T}{T} \right) + \frac{T p^l}{\omega_f} \partial_l \left(\frac{\mu_f}{T} \right) - \frac{p^k p^l}{2\omega_f} W_{kl} \right], \quad (16)
\end{aligned}$$

$$\begin{aligned}
J_g = & -\beta \tau_g f_g^0 (1 + f_g^0) \left[\left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_g} \right\} \partial_l u^l \right. \\
& \left. + p^l \left(\frac{\partial_l P}{\varepsilon + P} - \frac{\partial_l T}{T} \right) - \frac{p^k p^l}{2\omega_g} W_{kl} \right], \quad (17)
\end{aligned}$$

respectively. With the help of Eqs. (14), (15) and (9), we get the nonequilibrium part of the quark distribution function (determined in Appendix A) as

$$\begin{aligned}
\delta f_f = & q E \tau_f v_x \beta f_f^0 (1 - f_f^0) + v_x \beta f_f^0 (1 - f_f^0) \\
& \times \left[\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \\
& + \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\
& - \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{kx}}{2} - \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{ky}}{2} \\
& + \frac{(\tau_f - \omega_c^2 \tau_f^3) q E}{1 + \omega_c^2 \tau_f^2} \left. \right] + v_y \beta f_f^0 (1 - f_f^0) \\
& \times \left[\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \\
& - \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\
& - \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{ky}}{2} \\
& + \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{kx}}{2} - \frac{2\omega_c \tau_f^2 q E}{1 + \omega_c^2 \tau_f^2} \left. \right], \quad (18)
\end{aligned}$$

where the cyclotron frequency, ω_c is defined as $\omega_c = \frac{qB}{\omega_f}$. In a first order theory, for infinitesimal deviation of the system from its equilibrium, the spatial component of the dissipative part of the energy-momentum tensor is defined [56–58] as

$$\Delta T^{ij} = -\eta W^{ij} - \zeta \delta^{ij} \partial_l u^l. \quad (19)$$

Here the shear viscosity and the bulk viscosity are described as the coefficients of the traceless part and the trace part of ΔT^{ij} , respectively.

In the weak magnetic field regime, 3-dimensional dynamics is retained, unlike the strong magnetic field regime, where 3-dimensional dynamics for charged particles gets reduced to 1-dimensional dynamics and only longitudinal (along the direction of magnetic field) component of ΔT^{ij} exists. It is very important to note that, at least in the weak magnetic field limit, we do not split ΔT^{ij} into different components, rather, the effect of magnetic field enters mainly through the cyclotron frequency (ω_c). According to this specific limit, we neglect the terms containing ω_c and its higher orders in the numerator. Thus, Hall-type shear and bulk viscosities are not obtained in this part of this section. In the general configuration of magnetic field, different components of aforesaid viscosities are obtained in the next part of this section. We get the spatial component of Eq. (2) (determined in Appendix B) as

$$\begin{aligned} \Delta T^{ij} &= \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \frac{\beta p^i p^j}{\omega_f} \\ &\times \left[2qEv_x \frac{\tau_f f_f^0 (1 - f_f^0)}{1 + \omega_c^2 \tau_f^2} + 2\bar{q}Ev_x \frac{\tau_{\bar{f}} \bar{f}_f^0 (1 - \bar{f}_f^0)}{1 + \omega_c^2 \tau_{\bar{f}}^2} \right. \\ &+ \left(\frac{\tau_f f_f^0 (1 - f_f^0)}{1 + \omega_c^2 \tau_f^2} + \frac{\tau_{\bar{f}} \bar{f}_f^0 (1 - \bar{f}_f^0)}{1 + \omega_c^2 \tau_{\bar{f}}^2} \right) \\ &\times \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &- \left(\frac{\tau_f f_f^0 (1 - f_f^0)}{1 + \omega_c^2 \tau_f^2} + \frac{\tau_{\bar{f}} \bar{f}_f^0 (1 - \bar{f}_f^0)}{1 + \omega_c^2 \tau_{\bar{f}}^2} \right) \frac{p_k p_l}{2p_0} W_{kl} \Big] \\ &+ g_g \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j \tau_g}{\omega_g} \beta f_g^0 (1 + f_g^0) \\ &\times \left[\left\{ \omega_g \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_g} \right\} \partial_l u^l \right. \\ &\left. - \frac{p^k p^l}{2\omega_g} W_{kl} + p^l \left(\frac{\partial_l P}{\varepsilon + P} - \frac{\partial_l T}{T} \right) \right]. \end{aligned} \quad (20)$$

Comparing Eqs. (19) and (20), we get the shear viscosity of a weakly magnetized hot and dense QCD matter as

$$\begin{aligned} \eta &= \frac{\beta}{30\pi^2} \sum_f g_f \int dp \frac{p^6}{\omega_f^2} \\ &\times \left[\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} f_f^0 (1 - f_f^0) + \frac{\tau_{\bar{f}}}{1 + \omega_c^2 \tau_{\bar{f}}^2} \bar{f}_f^0 (1 - \bar{f}_f^0) \right] \\ &+ \frac{\beta}{30\pi^2} g_g \int dp \frac{p^6}{\omega_g^2} \tau_g f_g^0 (1 + f_g^0). \end{aligned} \quad (21)$$

Similarly, the comparison between Eqs. (19) and (20) gives the bulk viscosity as

$$\begin{aligned} \zeta &= \frac{1}{3} \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{\omega_f} \\ &\times \left[f_f^0 (1 - f_f^0) A_f + \bar{f}_f^0 (1 - \bar{f}_f^0) \bar{A}_f \right] \\ &+ \frac{1}{3} g_g \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{\omega_g} f_g^0 (1 + f_g^0) A_g. \end{aligned} \quad (22)$$

The factors A_f , \bar{A}_f and A_g in Eq. (22) are respectively written as

$$A_f = \frac{\tau_f \beta}{3(1 + \omega_c^2 \tau_f^2)} \left[\frac{p^2}{\omega_f} - 3 \left(\frac{\partial P}{\partial \varepsilon} \right) \omega_f \right], \quad (23)$$

$$\bar{A}_f = \frac{\tau_{\bar{f}} \beta}{3(1 + \omega_c^2 \tau_{\bar{f}}^2)} \left[\frac{p^2}{\omega_f} - 3 \left(\frac{\partial P}{\partial \varepsilon} \right) \omega_f \right], \quad (24)$$

$$A_g = \frac{\tau_g \beta}{3} \left[\frac{p^2}{\omega_g} - 3 \left(\frac{\partial P}{\partial \varepsilon} \right) \omega_g \right]. \quad (25)$$

The calculation of viscosity requires nonzero velocity gradient. But there exist different frames to define velocity u^μ , for example, u^μ denotes the velocity of baryon number flow in the Eckart frame, whereas it denotes the velocity of energy flow in the Landau–Lifshitz frame. Therefore the freedom to choose a specific frame creates arbitrariness. To avoid this arbitrariness, one needs the “condition of fit”, i.e. if one chooses the Landau–Lifshitz frame, then the condition of fit in the local rest frame demands the “00” component of the dissipative part of the energy-momentum tensor to be zero ($\Delta T^{00} = 0$). In order to satisfy this Landau–Lifshitz condition, the factors A_f , \bar{A}_f and A_g should be replaced as $A_f \rightarrow A'_f = A_f - b_f \omega_f$, $\bar{A}_f \rightarrow \bar{A}'_f = \bar{A}_f - \bar{b}_f \omega_f$ and $A_g \rightarrow A'_g = A_g - b_g \omega_g$. The Landau–Lifshitz conditions for A_f , \bar{A}_f and A_g are respectively given by

$$\sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \omega_f f_f^0 (1 - f_f^0) (A_f - b_f \omega_f) = 0, \quad (26)$$

$$\sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \omega_f \bar{f}_f^0 (1 - \bar{f}_f^0) (\bar{A}_f - \bar{b}_f \omega_f) = 0, \quad (27)$$

$$g_g \int \frac{d^3 p}{(2\pi)^3} \omega_g f_g^0 (1 + f_g^0) (A_g - b_g \omega_g) = 0. \quad (28)$$

The quantities b_f , \bar{b}_f and b_g are arbitrary constants and are associated with the particle and energy conservations for a thermal medium having asymmetry between the numbers of particles and antiparticles [59]. These quantities can be obtained by solving Eqs. (26), (27) and (28). After substituting $A_f \rightarrow A'_f$, $\bar{A}_f \rightarrow \bar{A}'_f$ and $A_g \rightarrow A'_g$ in Eq. (22) and simplifying, we get the bulk viscosity of a weakly magnetized hot and dense QCD matter as

$$\begin{aligned} \zeta = & \frac{\beta}{18\pi^2} \sum_f g_f \int d\mathbf{p} \, \mathbf{p}^2 \left[\frac{\mathbf{p}^2}{\omega_f} - 3 \left(\frac{\partial P}{\partial \varepsilon} \right) \omega_f \right]^2 \\ & \times \left[\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} f_f^0 (1 - f_f^0) + \frac{\tau_{\bar{f}}}{1 + \omega_c^2 \tau_{\bar{f}}^2} \bar{f}_f^0 (1 - \bar{f}_f^0) \right] \\ & + \frac{\beta}{18\pi^2} g_g \int d\mathbf{p} \, \mathbf{p}^2 \left[\frac{\mathbf{p}^2}{\omega_g} - 3 \left(\frac{\partial P}{\partial \varepsilon} \right) \omega_g \right]^2 \tau_g f_g^0 (1 + f_g^0). \end{aligned} \quad (29)$$

In this part, we have obtained the shear and bulk viscosities using the ansatz method in the weak magnetic field limit. In the next part, we are going to determine different components of shear and bulk viscosities in the general configuration of magnetic field.

Momentum transport coefficients in the general configuration of magnetic field: In the presence of an arbitrary magnetic field, the infinitesimal change in the distribution function of charged particles (quarks and antiquarks) is written as

$$\delta f = \sum_{l=0}^4 C_l Y_{mn}^l v_m v_n. \quad (30)$$

The spatial component of the nonequilibrium part of the energy-momentum tensor is written as

$$\Delta T_{ij} = \sum_{l=0}^4 \eta_l Y_{ij}^l, \quad (31)$$

where η_0 , η_1 , η_2 , η_3 and η_4 denote five shear viscosity coefficients. For the calculation of the viscosities, it is sufficient to take only spatial component of the nonequilibrium part of the energy-momentum tensor. In the above tensor, we have excluded the bulk viscosity part to determine the shear viscosity coefficients. In terms of the infinitesimal change in the particle distribution function, ΔT_{ij} has the following form,

$$\Delta T_{ij} = \sum_f g_f \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \omega_f v_i v_j \delta f. \quad (32)$$

Substituting the value of δf (30) in Eq. (32) and then simplifying, we get

$$\begin{aligned} \Delta T_{ij} = & \frac{1}{15} \sum_f g_f \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \omega_f v^4 \\ & \times (\delta_{ij} \delta_{mn} + \delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \sum_{l=0}^4 C_l Y_{mn}^l. \end{aligned} \quad (33)$$

In the above equations, Y_{ij}^0 , Y_{ij}^1 , Y_{ij}^2 , Y_{ij}^3 and Y_{ij}^4 are respectively expressed [57, 60] as

$$Y_{ij}^0 = (3b_i b_j - \delta_{ij}) \left(b_k b_l V_{kl} - \frac{1}{3} \nabla \cdot \mathbf{V} \right), \quad (34)$$

$$\begin{aligned} Y_{ij}^1 = & 2V_{ij} + (b_i b_j - \delta_{ij}) \nabla \cdot \mathbf{V} + \delta_{ij} V_{kl} b_k b_l \\ & - 2V_{ik} b_k b_j - 2V_{jk} b_k b_i + b_i b_j V_{kl} b_k b_l, \end{aligned} \quad (35)$$

$$Y_{ij}^2 = 2V_{ik} b_k b_j + 2V_{jk} b_k b_i - 4b_i b_j V_{kl} b_k b_l, \quad (36)$$

$$Y_{ij}^3 = V_{ik} b_{jk} + V_{jk} b_{ik} - V_{kl} b_{ik} b_j b_l - V_{kl} b_{jk} b_i b_l, \quad (37)$$

$$Y_{ij}^4 = 2V_{kl} b_{ik} b_j b_l + 2V_{kl} b_{jk} b_i b_l, \quad (38)$$

where $b_{ij} = \epsilon_{ijk} b_k$ and $V_{ij} = \frac{1}{2} \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right)$, with V_i and $b_i = \frac{\mathbf{B}}{B}$ denoting the fluid velocity and the unit vector along the direction of magnetic field, respectively. Imposing the condition, $\nabla \cdot \mathbf{V} = 0$, and using the relations, such as $V_{ij} b_i b_j = 0$, $b_{ij} v_i v_j = 0$, $b_i b_i = 1$, $b_i b_i = 0$ and $b_{ij} b_j = 0$, we determine η_1 , η_2 , η_3 and η_4 . On the other hand, η_0 remains the same as in the absence of magnetic field and is given by

$$\begin{aligned} \eta_0 = & \frac{\beta}{30\pi^2} \sum_f g_f \int d\mathbf{p} \, \frac{\mathbf{p}^6}{\omega_f^2} \left[\tau_f f_f^0 (1 - f_f^0) \right. \\ & \left. + \tau_{\bar{f}} \bar{f}_f^0 (1 - \bar{f}_f^0) \right]. \end{aligned} \quad (39)$$

Comparing Eqs. (31) and (33), and requiring the consistency of both these equations, we have

$$\eta_1 = \frac{2}{15} \sum_f g_f \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \omega_f v^4 C_1, \quad (40)$$

$$\eta_2 = \frac{2}{15} \sum_f g_f \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \omega_f v^4 C_2, \quad (41)$$

$$\eta_3 = -\frac{2}{15} \sum_f g_f \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \omega_f v^4 C_3, \quad (42)$$

$$\eta_4 = -\frac{2}{15} \sum_f g_f \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \omega_f v^4 C_4. \quad (43)$$

The factors C_1 , C_2 , C_3 and C_4 can be calculated by using the relativistic Boltzmann transport equation in the relaxation time approximation at finite magnetic field and chemical potential (6). To proceed for the calculation, we take only the spatial components in Eq. (6) and keep only the magnetic field part in the Lorentz force. Then, we split f as $f = f_0 + \delta f$ in the left hand side of Eq. (6) and

keep only f_0 . In doing so, the second term will vanish due to the appearance of the expression $q(\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_0}{\partial \mathbf{p}} = -q\beta [(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}] f_0 (1 - f_0) = 0$. So, in order to keep the magnetic field dependence, we need to keep δf in the second term. Thus, Eq. (6) gets simplified into

$$\frac{p_i}{\omega_f} \frac{\partial f_0}{\partial x_i} - \frac{qB}{\omega_f} b_{ij} v_j \frac{\partial (\delta f)}{\partial v_i} = -\frac{\delta f}{\tau_f}, \quad (44)$$

where $\frac{p_i}{\omega_f} \frac{\partial f_0}{\partial x_i} = -\beta \omega_f v_i v_j V_{ij} f_0 (1 - f_0)$ and the value of δf is given in Eq. (30). Now, Eq. (44) becomes

$$\begin{aligned} \beta \omega_f V_{ij} v_i v_j f_0 (1 - f_0) &= -\omega_c b_{ij} v_j \frac{\partial}{\partial v_i} \left(\sum_{l=0}^4 C_l Y_{mn}^l v_m v_n \right) \\ &\quad + \frac{\sum_{l=0}^4 C_l Y_{mn}^l v_m v_n}{\tau_f} \\ &= -2\omega_c b_{ij} v_j \left(\sum_{l=0}^4 C_l Y_{im}^l v_m \right) \\ &\quad + \frac{\sum_{l=0}^4 C_l Y_{mn}^l v_m v_n}{\tau_f}. \end{aligned} \quad (45)$$

Using the relations $\nabla \cdot \mathbf{V} = 0$, $V_{ij} b_i b_j = 0$, $b_{ij} v_i v_j = 0$, $b_i b_i = 1$, $b_{ij} b_i = 0$ and $b_{ij} b_j = 0$ in above equation, and then comparing the same tensor structures on both sides of Eq. (45), C_1 , C_2 , C_3 and C_4 can be obtained (in Appendix C) as

$$C_1 = \frac{\beta \omega_f \tau_f f_0 (1 - f_0)}{2(1 + 4\omega_c^2 \tau_f^2)}, \quad (46)$$

$$C_2 = \frac{\beta \omega_f \tau_f f_0 (1 - f_0)}{2(1 + \omega_c^2 \tau_f^2)}, \quad (47)$$

$$C_3 = -\frac{\beta \omega_f \omega_c \tau_f^2 f_0 (1 - f_0)}{(1 + 4\omega_c^2 \tau_f^2)}, \quad (48)$$

$$C_4 = -\frac{\beta \omega_f \omega_c \tau_f^2 f_0 (1 - f_0)}{2(1 + \omega_c^2 \tau_f^2)}. \quad (49)$$

Substituting the values of C_1 , C_2 , C_3 and C_4 in Eqs. (40), (41), (42) and (43) and then simplifying, we get η_1 , η_2 , η_3 and η_4 respectively as

$$\begin{aligned} \eta_1 &= \frac{\beta}{30\pi^2} \sum_f g_f \int dp \frac{p^6}{\omega_f^2} \left[\frac{\tau_f}{1 + 4\omega_c^2 \tau_f^2} f_f^0 (1 - f_f^0) \right. \\ &\quad \left. + \frac{\tau_{\bar{f}}}{1 + 4\omega_c^2 \tau_{\bar{f}}^2} \bar{f}_f^0 (1 - \bar{f}_f^0) \right], \end{aligned} \quad (50)$$

$$\eta_2 = \frac{\beta}{30\pi^2} \sum_f g_f \int dp \frac{p^6}{\omega_f^2} \left[\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} f_f^0 (1 - f_f^0) \right]$$

$$+ \frac{\tau_{\bar{f}}}{1 + \omega_c^2 \tau_{\bar{f}}^2} \bar{f}_f^0 (1 - \bar{f}_f^0) \right], \quad (51)$$

$$\begin{aligned} \eta_3 &= \frac{\beta}{30\pi^2} \sum_f g_f \int dp \frac{p^6}{\omega_f^2} \left[\frac{2\omega_c \tau_f^2}{1 + 4\omega_c^2 \tau_f^2} f_f^0 (1 - f_f^0) \right. \\ &\quad \left. + \frac{2\omega_c \tau_{\bar{f}}^2}{1 + 4\omega_c^2 \tau_{\bar{f}}^2} \bar{f}_f^0 (1 - \bar{f}_f^0) \right], \end{aligned} \quad (52)$$

$$\begin{aligned} \eta_4 &= \frac{\beta}{30\pi^2} \sum_f g_f \int dp \frac{p^6}{\omega_f^2} \left[\frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} f_f^0 (1 - f_f^0) \right. \\ &\quad \left. + \frac{\omega_c \tau_{\bar{f}}^2}{1 + \omega_c^2 \tau_{\bar{f}}^2} \bar{f}_f^0 (1 - \bar{f}_f^0) \right]. \end{aligned} \quad (53)$$

Neglecting the factor 4 in the denominator of Eq. (50), one can find that $\eta_1 = \eta_2 = \eta$ (charged particle part), where η is given in Eq. (21). In the above description of different shear viscosity coefficients, the gluon part of the shear viscosity has been excluded, because magnetic field has almost no effect on the electrically neutral gluons, thus, this part of the viscosity does not split into different components in the presence of magnetic field. So, one can add the gluon part to the charged particle part to get the total shear viscosity of the hot medium of quarks, antiquarks and gluons like in Eq. (21). Now, excluding the shear viscosity part and including only the bulk viscosity part, ΔT_{ij} is expressed [57] as

$$\Delta T_{ij} = \xi_0 \delta_{ij} \nabla \cdot \mathbf{V} + \xi_1 (\delta_{ij} V_{kl} b_k b_l + b_i b_j \nabla \cdot \mathbf{V}). \quad (54)$$

Thus, there also exist two different bulk viscosity coefficients in the presence of an arbitrary magnetic field, such as ξ_0 and ξ_1 . The volume or bulk viscosity coefficient ξ_0 remains the same as in the absence of magnetic field and is given by

$$\begin{aligned} \xi_0 &= \frac{\beta}{18\pi^2} \sum_f g_f \int dp p^2 \left[\frac{p^2}{\omega_f} - 3 \left(\frac{\partial P}{\partial \varepsilon} \right) \omega_f \right]^2 \\ &\quad \times \left[\tau_f f_f^0 (1 - f_f^0) + \tau_{\bar{f}} \bar{f}_f^0 (1 - \bar{f}_f^0) \right]. \end{aligned} \quad (55)$$

On the other hand, ξ_1 which is the cross effect between the ordinary and volume viscosities vanishes for a plasma (for details, please see Ref. [57]). Thus, to see the magnetic field-dependence, we use the bulk viscosity obtained through the ansatz method at weak magnetic field limit in the first part of this section (29).

The aforementioned transport properties are studied considering the quasiparticle model (QPM) of QGP medium. In quasiparticle models [61–64], QGP is described as a system of massive noninteracting quasiparticles and the mass of the quasiparticle arises due to the interactions of quarks and gluons with the thermal medium. In the kinetic theory approach with the quasiparticle model description, the interactions among partons have been considered to be contained only in

their quasiparticle masses. Quasiparticle masses of particles have been derived from the hard thermal loop (HTL) perturbation theory at high temperatures [65, 66]. We note that the estimation of the quasiparticle model depends on the requirement of thermodynamic consistency, which has already been tested in notable works like [61, 62, 66]. It assumes that the deconfined quarks and gluons remain the relevant degrees of freedom even in the quasiparticle model, which is a justified assumption for high temperatures $T > T_c$ and for small chemical potentials $\mu < 2T_c$, because this model reproduces the leading-order perturbative results and in addition, it represents a thermodynamically consistent effective resummation of the leading-order thermal contributions. For the thermodynamic consistency, there are some conditions which need to be satisfied, for example, the derivative of pressure with respect to the square of quasiparticle mass requires to vanish. At high temperatures, the thermodynamic consistency can be fulfilled, because the thermodynamic quantities can be perturbatively expanded in powers of coupling g and the full expressions represent a thermodynamically consistent resummation of terms of all orders in coupling g [66]. Thus, the coupling must be very small and this is unambiguously satisfied in high temperature QGP phase [61, 66].

The quasiparticle model was successfully used to study the equation of state for the partonic medium [67, 68]. This model had also been studied in different approaches, such as the Nambu–Jona–Lasinio (NJL) and Polyakov NJL based quasiparticle models [69–71], quasiparticle model in a strong magnetic field [54, 55], quasiparticle model with Gribov–Zwanziger quantization [72, 73], thermodynamically consistent quasiparticle model [74, 75] etc. In a hot and dense medium, the thermal mass (squared) of quark is given [65, 66] by

$$m_{fT}^2 = \frac{g^2 T^2}{6} \left(1 + \frac{\mu_f^2}{\pi^2 T^2} \right). \quad (56)$$

In the similar environment, the thermal mass (squared) of gluon is given [64, 66, 76] by

$$m_{gT}^2 = \frac{g^2 T^2}{6} \left(N_c + \frac{N_f}{2} + \frac{3}{2\pi^2 T^2} \sum_f \mu_f^2 \right). \quad (57)$$

In the above equations, $g^2 = 4\pi\alpha_s$, where α_s denotes the one-loop strong running coupling at finite temperature, chemical potential and weak magnetic field, and is expressed [77] as

$$\alpha_s(\Lambda^2, eB) = \frac{\alpha_s(\Lambda^2)}{1 + b_1 \alpha_s(\Lambda^2) \ln \left(\frac{\Lambda^2}{\Lambda^2 + eB} \right)}. \quad (58)$$

Here $\alpha_s(\Lambda^2)$ is the one-loop strong running coupling in the absence of magnetic field, which is given by

$$\alpha_s(\Lambda^2) = \frac{1}{b_1 \ln \left(\frac{\Lambda^2}{\Lambda_{\overline{MS}}^2} \right)}, \quad (59)$$

with $b_1 = \frac{11N_c - 2N_f}{12\pi}$, $\Lambda_{\overline{MS}} = 0.176$ GeV and $\Lambda = 2\pi\sqrt{T^2 + \mu_f^2/\pi^2}$ for electrically charged particles (quarks and antiquarks) and $\Lambda = 2\pi T$ for gluons. The chemical potentials for all flavors have been kept the same, i.e. $\mu_f = \mu$.

3 Results and discussions

Figures 1 and 2 show the temperature dependence of shear (η) and bulk (ζ) viscosities in the presence of weak magnetic field and finite chemical potential, respectively. It can be seen from these figures that the influence of weak magnetic field on η and ζ is less pronounced than the influence of chemical potential. Compared to the thermal medium at $eB = 0$, $\mu = 0$, the decrease of η and ζ due to the weak magnetic field is meagre, contrary to their discernible increase due to the finite chemical potential. These effects of weak magnetic field and chemical potential on shear and bulk viscosities are more conspicuous at low temperatures. Thus, the reduction in η leads to a decrease in the momentum transport in the presence of weak magnetic field, whereas the finite chemical potential creates favorable condition for the momentum transport in hot QCD matter and it becomes easy for a particle to carry momentum over great distances. It can also be inferred that their effects on the momentum transport get suppressed at higher temperatures. Further, the reduction in ζ in weak magnetic field regime explains small fluctuations in the pressure, contrary to large fluctuations at finite chemical potential. At finite magnetic field, the magnetic catalysis phenomenon enhances the dynamical symmetry breaking, thus triggering the binding of oppositely charged particles. It results in a stronger interaction between the constituents of the medium, which thus reduces the viscosities. In addition, with the magnetic field, cyclotron frequency increases and particle distributions decrease, which also give a decreasing effect to the viscosities. But, in the weak magnetic field limit this decrease is meagre, which can be understood from the fact that, in this limit, the magnetic field is the weak energy scale and the temperature is the strong energy scale. So, at high temperature phase, the effects of weak magnetic field on the abovementioned quantities and phenomenon are less pronounced. Thus, the viscosities have a negligible dependence on the magnetic field. Throughout the temperature range, the shear viscosity remains nearly two orders of magnitude larger than the bulk viscosity. Thus, the momentum transfer across the layer exceeds the momentum transfer along the

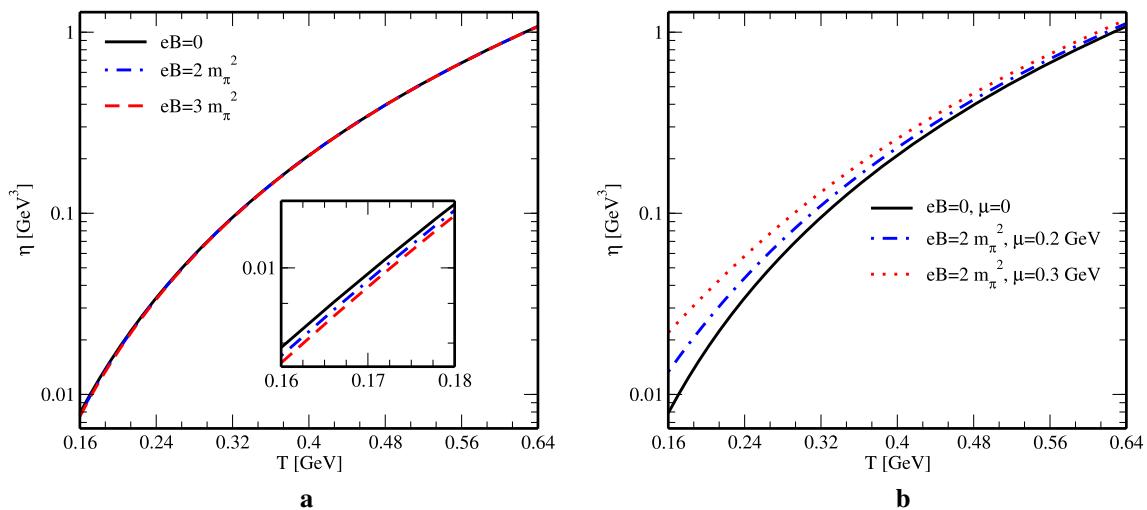


Fig. 1 The variation of the shear viscosity with temperature **(a)** in the presence of weak magnetic field and **(b)** in the presence of finite chemical potential

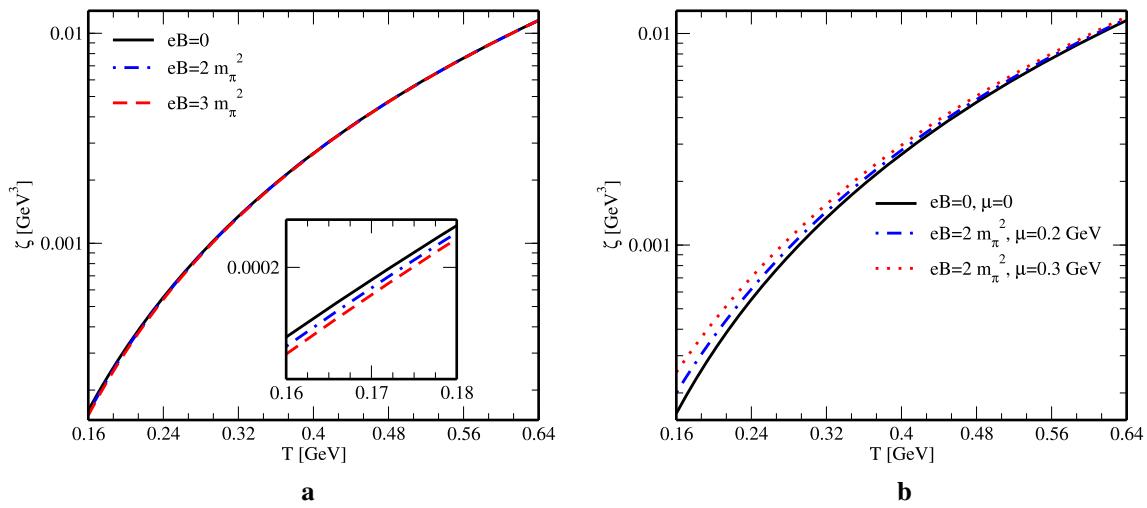


Fig. 2 The variation of the bulk viscosity with temperature **(a)** in the presence of weak magnetic field and **(b)** in the presence of finite chemical potential

layer. The dominance of shear viscosity over bulk viscosity also describes that the change in shape at constant volume is dominant as compared to the change in volume at constant shape.

The enhancement of shear viscosity at finite chemical potential also supports the reduction of elliptic flow in the similar regime, which can be understood as follows. We know that v_2 measures the flow anisotropy in the azimuthal plane. The shear viscosity being a result of frictional force and the frictional force being proportional to the flow velocity have noticeably large effects on the fast-moving particles in the collision plane. Thus anisotropy gets reduced, resulting a decrease in v_2 at finite chemical potential. Although the bulk viscosity is very small, but the emergence of finite chemical potential tends to enhance its magnitude. It thus explains that

the chemical potential supports the deviation of the strongly interacting matter from conformality.

For the comparison, we have plotted four shear viscosity coefficients, η_1 , η_2 , η_3 and η_4 as functions of temperature at weak magnetic field and finite chemical potential in Fig. 3. One can see that, η_1 and η_2 are almost indistinguishable, whereas η_3 and η_4 are distinguishable. This is expected, because the appearance of factor 4 in the denominator does not affect much, so η_1 (50) is almost equal to η_2 (51), whereas the appearance of factor 2 in the numerator does affect noticeably, so the difference between η_3 (52) and η_4 (53) is conspicuous. Both η_3 and η_4 directly depend on magnetic field through the cyclotron frequency ω_c (can be seen in Eqs. (52) and (53)) and hence called Hall-type shear viscosity coefficients. These Hall-type shear viscosity

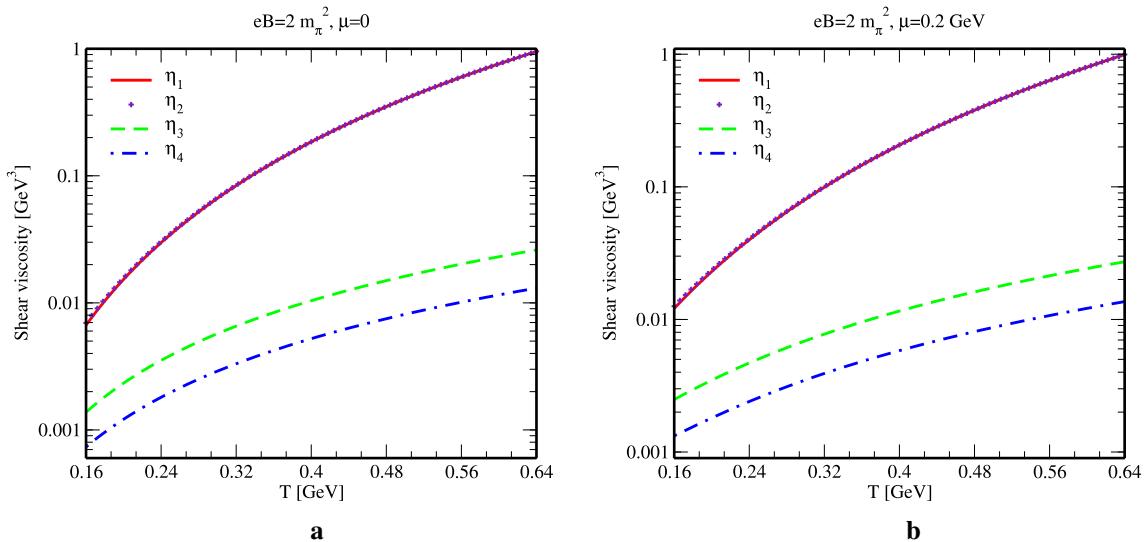


Fig. 3 Variations of the four shear viscosity coefficients with temperature **(a)** in the presence of weak magnetic field and **(b)** in the presence of finite chemical potential

coefficients η_3 and η_4 are found to be much smaller than the shear viscosity coefficients η_1 and η_2 , which explains that, η_2 ($\approx \eta_1$) is the dominant shear viscosity coefficient. One can also notice that, η_2 ($\approx \eta_1$) is exactly equal to the charged particle part of η obtained using the ansatz method in the first part of Sect. 2, i.e. η_2 ($\approx \eta_1$) = η (charged particle part). As compared to the $\mu = 0$ case (Fig. 3a), these coefficients get increased at finite chemical potential (Fig. 3b).

4 Applications

In this section, we are going to study the effects of weak magnetic field and finite chemical potential on the Prandtl number, the Reynolds number, the ratio of shear viscosity to entropy density, η/s and the ratio of bulk viscosity to entropy density, ζ/s .

4.1 Prandtl number

The momentum diffusion and the thermal diffusion are not completely independent, rather, they are related through the Prandtl number (Pr) as

$$\text{Pr} = \frac{\eta/\rho}{\kappa/C_p}, \quad (60)$$

where C_p represents the specific heat at constant pressure, ρ is the mass density and κ denotes the thermal conductivity. The Prandtl number is important to understand the effects of momentum diffusion and thermal diffusion on the sound attenuation in a medium. For $\text{Pr} < 1$, the dominance of thermal diffusion over momentum diffusion in the sound

attenuation is implied, unlike the case where $\text{Pr} > 1$. The estimation of the Prandtl number is carried out in a weak magnetic field, using the expression of the thermal conductivity in the similar environment (written in Appendix D) from our recent work [48]. C_p and ρ are calculated from the energy-momentum tensor ($C_p = \partial(u_\mu T^{\mu\nu} u_\nu - \Delta_{\mu\nu} T^{\mu\nu}/3)/\partial T$, with $\Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$) and the particle flow four-vector ($\rho = \sum_{f,\bar{f},g} m_{f,\bar{f},g} u_\mu u_\nu N^\mu$, with $m_{f,\bar{f},g}$ denoting the quasi-particle mass), respectively.

Figure 4 shows the variation of the Prandtl number as a function of temperature for different values of magnetic field and chemical potential. It can be observed that $\text{Pr} > 1$ and it increases with temperature. The presence of weak magnetic field increases Pr (Fig. 4a), whereas the finite chemical potential decreases its magnitude (Fig. 4b). The changes of Pr are higher for lower temperatures. The values of the Prandtl number imply that the sound attenuation is mostly governed by the momentum diffusion for the hot QCD matter and it is more pronounced in the presence of weak magnetic field than at finite chemical potential. Here, one can notice that the effect of magnetic field on the Prandtl number is measurable, unlike the effect on η and ζ , which can be comprehended as follows. The Prandtl number is the ratio of the momentum diffusion to the thermal diffusion. Both the thermal and momentum diffusions get noticeably affected by the presence of weak magnetic field and the effect on the momentum diffusion (Fig. 5a) is found to be larger than the effect on the thermal diffusion (Fig. 5b), so their ratio, i.e. the Prandtl number is noticeably affected by the magnetic field.

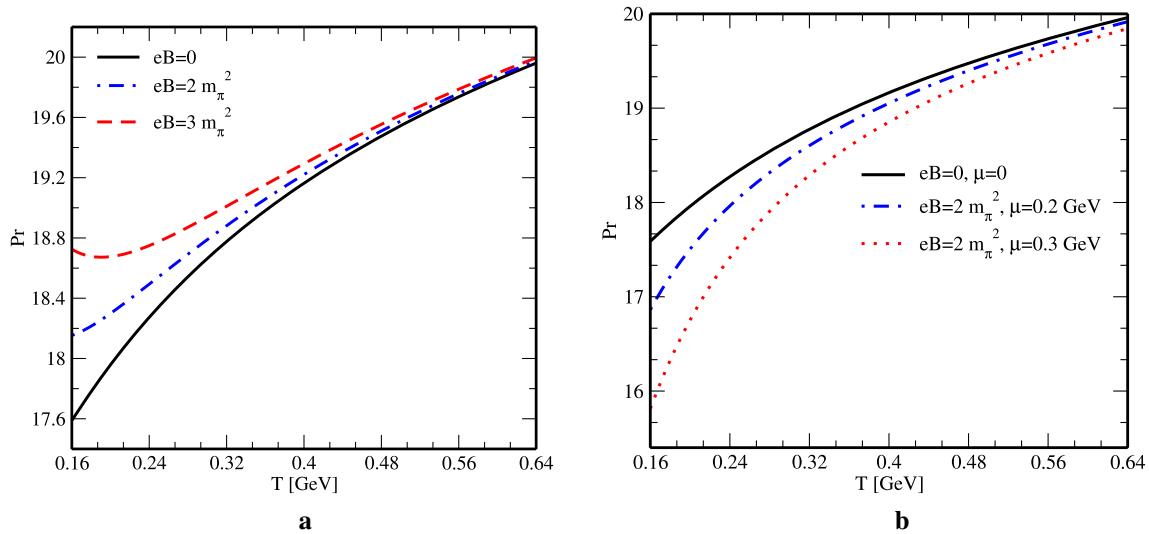


Fig. 4 The variation of the Prandtl number with temperature (a) in the presence of weak magnetic field and (b) in the presence of finite chemical potential

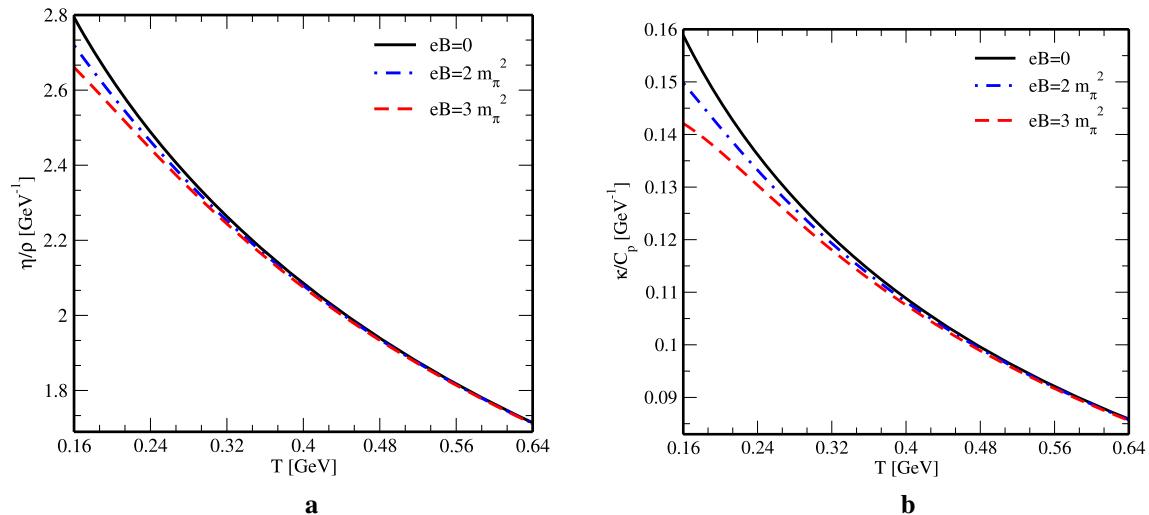


Fig. 5 Variations of (a) η/ρ and (b) κ/C_p with temperature at different values of magnetic field

4.2 Reynolds number

The viscous behavior of a medium can be understood by studying the Reynolds number,

$$Re = \frac{Lv}{\eta/\rho}, \quad (61)$$

where η/ρ represents the kinematic viscosity, and L and v are the characteristic length and velocity of the flow, respectively. Laminar or turbulent nature of the flow is specified by the Reynolds number, i.e. Re requires to be much larger than 1 for a turbulent flow while lower values correspond to a laminar flow, describing a more viscous fluid [78]. The proper time evolution of the thermodynamic quantities in the second-order dissipative relativistic fluid dynamics and their

dependence on the Reynolds number have been studied in Ref. [79]. The Reynolds number of quark matter has been estimated to be around 10 using the Kubo formula and NJL model [80]. For initial QGP, (3+1)-dimensional fluid dynamical model reports the range of Re to be 3–10 [81], whereas its upper bound is estimated to be approximately 20 in the holographic model [78]. In the present work, the Reynolds number for a weakly magnetized hot and dense QCD matter is estimated with $v \simeq 1$ and $L = 4 \text{ fm}$.

Figure 6 depicts the variation of the Reynolds number with the temperature in the presence of weak magnetic field and finite chemical potential. The Reynolds number is found to increase with the temperature. A small increase in the magnitude of Re is noticed due to the weak magnetic field (Fig. 6a), contrary to a large decrease due to the finite chemical poten-

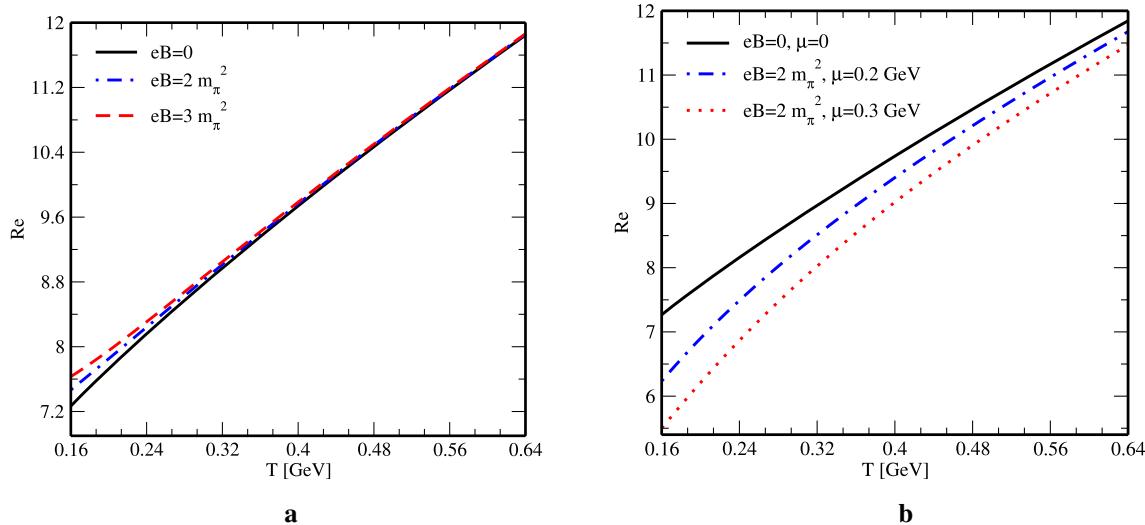


Fig. 6 The variation of the Reynolds number with temperature (a) in the presence of weak magnetic field and (b) in the presence of finite chemical potential

tial (Fig. 6b). The range of Re is found to be 5.49–11.86 in the temperature range, 160–640 MeV, indicating that the characteristic length scale of the hot QCD system prevails over its kinematic viscosity with the flow remaining laminar. It can be seen that the effect of magnetic field on the Reynolds number is measurable, unlike the effect on η and ζ , which can be understood as follows. The Reynolds number is the ratio of the product of characteristic length and velocity of the flow (Lv) to the kinematic viscosity. Since Lv has been taken to be constant, the magnitude of effect due to the weak magnetic field is decided by the kinematic viscosity, which is the ratio of the shear viscosity to the mass density. Since the influence of magnetic field on this ratio is noticeable, a measurable effect of magnetic field on the Reynolds number is observed.

4.3 Ratios η/s and ζ/s

In order to determine the ratios η/s and ζ/s , entropy density (s) is first evaluated from the energy–momentum tensor and baryon density (n_B) using the following equation:

$$S = \frac{u_\mu T^{\mu\nu} u_\nu - \sum_f \mu_f n_B - \Delta_{\mu\nu} T^{\mu\nu}/3}{T}, \quad (62)$$

where n_B is defined as

$$n_B = \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \left(f_f^0 - \bar{f}_f^0 \right). \quad (63)$$

The entropy density is observed to decrease with an increase of magnetic field at a fixed temperature (Fig. 7a). On the other hand, an increase in the value of entropy density is observed at finite chemical potential (Fig. 7b). Thus, the presence of magnetic field makes the system less disordered, whereas the

disorder is larger at finite chemical potential. The observations on entropy density, shear and bulk viscosities facilitate the exploration of ratios η/s and ζ/s .

Figure 8a and 8b display the effects of weak magnetic field and finite chemical potential on the variations of η/s and ζ/s with temperature, respectively. In a weak magnetic field at zero chemical potential, the ratio η/s gets slightly decreased and becomes nearer to the conjectured lower bound $1/(4\pi)$, specifically at low temperatures (Fig. 8a). It can be understood from the fact that, both η and s get reduced due to the weak magnetic field, with the reduction of η being more than that of s , thus resulting in an overall decrease of η/s in the said regime. However, in the additional presence of chemical potential, η/s becomes slightly greater than that at $\mu = 0$, $eB = 0$, but still not very far from the lower bound. Thus, the hot QCD matter shows the characteristic of a nearly perfect fluid in the said regime. The ratio ζ/s is found to be very small as compared to the ratio η/s and it exhibits a non-monotonic behavior at low temperatures (Fig. 8b). Above the phase transition temperature $T_c = 0.16$ GeV, there is a broad smooth minimum in the ratio ζ/s and then, this ratio gradually increases at higher temperatures. The presence of weak magnetic field slightly decreases the magnitude of ζ/s , which corroborates the observations on ζ and s in the similar environment, whereas a comparatively large decrease is observed in the additional presence of chemical potential. Unlike $eB = 0$, $\mu = 0$ case, no nonmonotonic behavior of ζ/s near T_c is found in the presence of both weak magnetic field and finite chemical potential.

The inclusion of magnetic field in the lattice QCD calculations is an emerging area of research. To the best of our knowledge, no lattice QCD results on viscosities are available at finite magnetic field, so it may not be plausible to compare

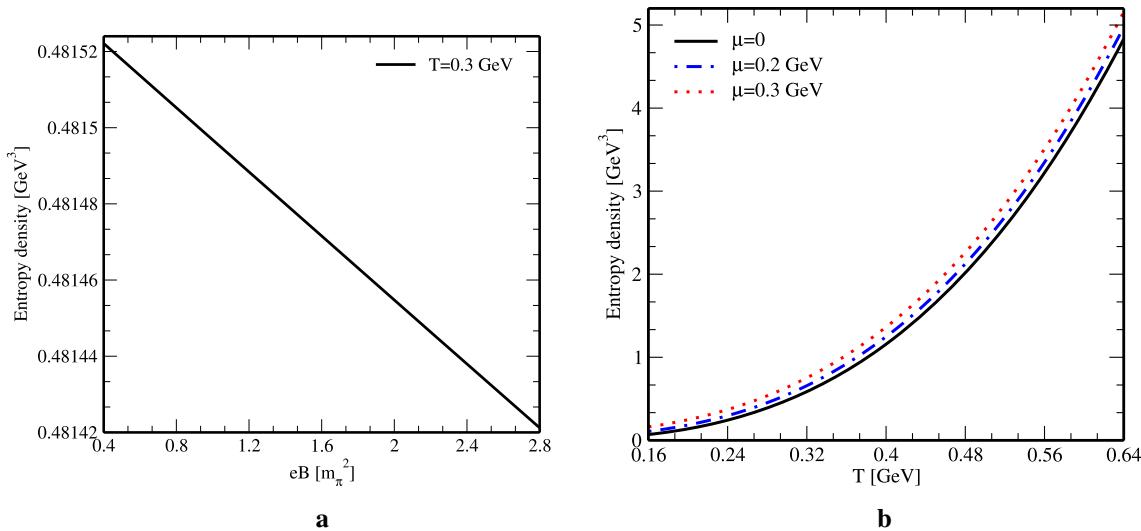


Fig. 7 The variation of the entropy density **(a)** with magnetic field at a fixed temperature and **(b)** with temperature at different values of chemical potential

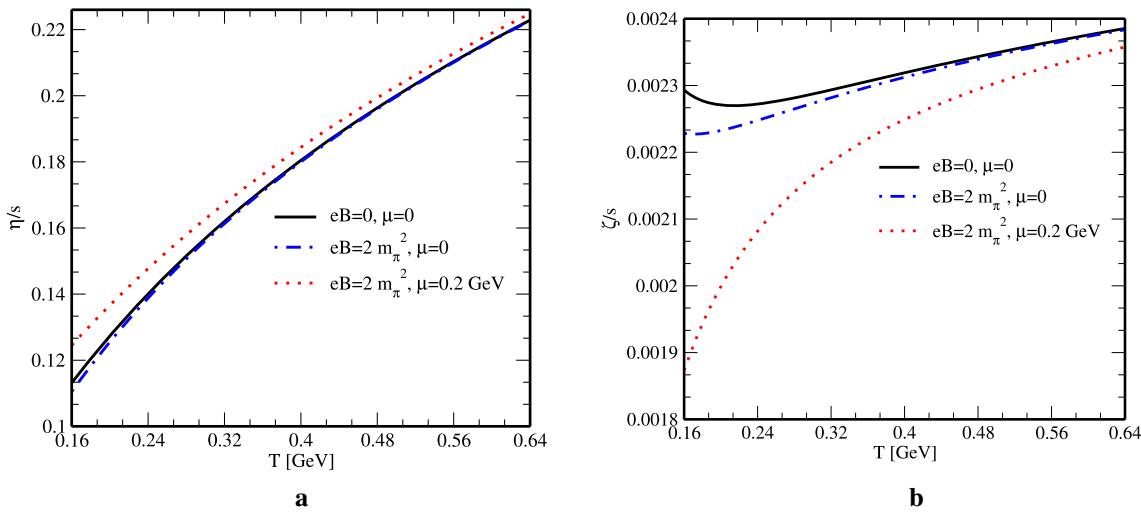


Fig. 8 Variations of **(a)** η/s and **(b)** ζ/s with temperature in the presence of weak magnetic field and finite chemical potential

our results on viscosities with the lattice QCD calculations at the equal base. We may however update the lattice QCD results at zero magnetic field. According to the lattice results [17, 82], η/s becomes minimum and ζ/s becomes maximum near the phase transition temperature. Compared to the lattice result of ζ/s in Ref. [82], our result in the presence of weak magnetic field is smaller. Lattice calculation for an SU(3) pure gauge model in Ref. [17] reports the upper bound for η/s of QGP to be 1, and for the temperature range 0.16–0.64 GeV, our result on η/s in weak magnetic field is slightly less than the lattice result. Another lattice work in Ref. [18] estimates η/s to be nearly 0.102 at $T = 1.24T_c$ and 0.134 at $T = 1.65T_c$, whereas our weak magnetic field calculation observes slight larger values of η/s at these temperatures. Lattice calculation in Ref. [26] reports a very small value

of ζ/s (<0.15) except near T_c and even becomes extremely small away from T_c , whereas our weak magnetic field result lies below the lattice result on the ratio ζ/s . The Ref. [8] has studied the SU(3)-gluodynamics shear viscosity temperature dependence on the lattice and found that for a temperature range $T_c - 1.5T_c$, η/s ranges 0.24–0.27 approximately. For this temperature range, our calculation estimates η/s in the ranges 0.113–0.14 at $eB = 0$ and 0.11–0.139 at $eB \neq 0$. Thus, lattice results lie above our results, and the presence of weak magnetic field shifts η/s more towards the lower bound ($1/(4\pi)$), thus making the medium to show nearly perfect fluid characteristics. On the other hand, Ref. [9] has studied the SU(3)-gluodynamics bulk viscosity temperature dependence on the lattice and found a very small value of ζ/s for $T \geq 1.1T_c$ and in fact this matches with our result

at $eB = 0$, whereas the result at weak magnetic field is less than the aforesaid lattice estimation in the same temperature range. Since ζ/s vanishes for a conformal QGP, the decrease of ζ/s in weak magnetic field drives the medium towards the conformal symmetry of QGP phase.

5 Summary

The momentum transport properties of a hot and dense QCD matter have been studied in terms of the shear and bulk viscosities in the presence of weak magnetic field and finite chemical potential using the kinetic theory approach. In general, the emergence of magnetic field breaks the isotropy of the medium and splits the shear viscosity into five components ($\eta_0, \eta_1, \eta_2, \eta_3$ and η_4) and the bulk viscosity into two components (ζ_0 and ζ_1). Out of these seven components, η_0 and ζ_0 retain their forms same as in the absence of magnetic field. On the other hand, ζ_1 vanishes, whereas η_1, η_2, η_3 and η_4 are magnetic field-dependent, out of which η_1 and η_2 are dominant shear viscosity coefficients, and η_3 and η_4 are called as Hall-type shear viscosity coefficients. In addition, we also calculated the shear (η) and bulk (ζ) viscosities using the ansatz method in weak magnetic field limit. In this method, the magnetic field-dependence of the bulk viscosity could be seen. This method under the weak magnetic field limit gives only the dominant contribution of the shear viscosity, because we found that $\eta = \eta_2 (\approx \eta_1)$. We observed that the presence of weak magnetic field decreases both η and ζ , thus reducing the transport of momentum across and along the layer as compared to the zero magnetic field case. The presence of finite chemical potential increases both η and ζ . The presence of weak magnetic field makes the Prandtl number (Pr) larger than its value in the absence of both magnetic field and chemical potential, however at finite chemical potential, Pr becomes smaller and in all cases, Pr is found to be greater than unity. Thus, the sound attenuation is mostly governed by the momentum diffusion and the weak magnetic field makes the dominance of momentum diffusion over thermal diffusion stronger, whereas the chemical potential makes this dominance weaker. The Reynolds number (Re) is found to be increased in an ambience of weak magnetic field, but it gets decreased at finite chemical potential and the flow remains laminar. A meagre decrease due to the weak magnetic field and a noticeable increase due to the finite chemical potential in the magnitude of η/s are observed. The magnitude of ζ/s and the nonmonotonicity in its variation with temperature get waned in the presence of weak magnetic field and finite chemical potential.

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A Derivation of Eq. (18)

Since magnetic field is taken along z -direction, no explicit dependence of magnetic field on spatial velocity gradient along z -direction can be observed. Now, J is calculated as

$$J = -\beta\tau_f f_f^0 \left(1 - f_f^0\right) \left[\left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l + p^l \left(\frac{\partial_l P}{\varepsilon + P} - \frac{\partial_l T}{T} \right) - \frac{T p^l}{\omega_f} \partial_l \left(\frac{\mu_f}{T} \right) - \frac{p^k p^l}{2\omega_f} W_{kl} \right], \quad (\text{A.1})$$

where $W_{kl} = \partial_k u_l + \partial_l u_k - \frac{2}{3} \delta_{kl} \partial_j u^j$. After substituting the value of J (A.1) in Eq. (14) and then simplifying, we obtain

$$\begin{aligned} & -\frac{p_0 p_x v_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l + \frac{p_k v_x W_{kx}}{2} \\ & + \frac{\Gamma_x v_x}{\tau_f} - \omega_c \Gamma_y v_x - q E v_x \\ & - \frac{p_0 p_y v_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l + \frac{p_k v_y W_{ky}}{2} \\ & + \frac{\Gamma_y v_y}{\tau_f} + \omega_c \Gamma_x v_y + \omega_c \tau_f q E v_y \\ & + \frac{\Gamma_z v_z}{\tau_f} + \frac{T p^l}{\omega_f} \partial_l \left(\frac{\mu_f}{T} \right) - p^l \left(\frac{\partial_l P}{\varepsilon + P} - \frac{\partial_l T}{T} \right) = 0, \end{aligned} \quad (\text{A.2})$$

where $|\mathbf{p}| = p$ and ω_c is the cyclotron frequency, $\omega_c = \frac{qB}{\omega_f}$. Equating the coefficients of v_x, v_y and v_z on both sides of the above equation, we get

$$\begin{aligned} & -\frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l + \frac{p_k W_{kx}}{2} \\ & + \frac{\Gamma_x}{\tau_f} - \omega_c \Gamma_y - q E = 0, \end{aligned} \quad (\text{A.3})$$

$$-\frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l + \frac{p_k W_{ky}}{2} + \frac{\Gamma_y}{\tau_f} + \omega_c \Gamma_x + \omega_c \tau_f q E = 0, \quad (\text{A.4})$$

$$\frac{\Gamma_z}{\tau_f} = 0. \quad (\text{A.5})$$

Now, Γ_x , Γ_y and Γ_z can be obtained by solving Eqs. (A.3), (A.4) and (A.5) as

$$\begin{aligned} \Gamma_x &= \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &+ \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &- \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{kx}}{2} - \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{ky}}{2} \\ &+ \frac{(\tau_f - \omega_c^2 \tau_f^3) q E}{1 + \omega_c^2 \tau_f^2}, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} \Gamma_y &= \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &- \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &- \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{ky}}{2} + \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{kx}}{2} \\ &- \frac{2\omega_c \tau_f^2 q E}{1 + \omega_c^2 \tau_f^2}, \end{aligned} \quad (\text{A.7})$$

$$\Gamma_z = 0. \quad (\text{A.8})$$

Substituting the values of Γ_x , Γ_y and Γ_z in Eq. (9) and then simplifying, we get the nonequilibrium part of the quark distribution function as follows,

$$\begin{aligned} \delta f_f &= q E \tau_f v_x \beta f_f^0 (1 - f_f^0) + v_x \beta f_f^0 (1 - f_f^0) \\ &\times \left[\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \\ &+ \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &- \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{kx}}{2} - \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{ky}}{2} \\ &\left. + \frac{(\tau_f - \omega_c^2 \tau_f^3) q E}{1 + \omega_c^2 \tau_f^2} \right] + v_y \beta f_f^0 (1 - f_f^0) \\ &\times \left[\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \end{aligned}$$

$$\begin{aligned} &- \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &- \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{ky}}{2} \\ &\left. + \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{kx}}{2} - \frac{2\omega_c \tau_f^2 q E}{1 + \omega_c^2 \tau_f^2} \right]. \end{aligned} \quad (\text{A.9})$$

B Derivation of Eq. (20)

The spatial component of Eq. (2) can be written as

$$\begin{aligned} \Delta T^{ij} &= \int \frac{d^3 p}{(2\pi)^3} p^i p^j \left[\sum_f g_f \frac{(\delta f_f + \delta \bar{f}_f)}{\omega_f} + g_g \frac{\delta f_g}{\omega_g} \right] \\ &= \Delta T_q^{ij} + \Delta T_{\bar{q}}^{ij} + \Delta T_g^{ij}. \end{aligned} \quad (\text{B.1})$$

Using the expression of δf_f (18), the quark part ΔT_q^{ij} in Eq. (B.1) is determined as follows,

$$\begin{aligned} \Delta T_q^{ij} &= \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j}{\omega_f} \delta f_f \\ &= \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j}{\omega_f} \beta f_f^0 (1 - f_f^0) \left[q E \tau_f v_x \right. \\ &+ v_x \left(\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \\ &+ \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &- \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{kx}}{2} - \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{ky}}{2} \\ &\left. \left. + \frac{(\tau_f - \omega_c^2 \tau_f^3) q E}{1 + \omega_c^2 \tau_f^2} \right) \right] \\ &+ v_y \left(\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \\ &- \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &- \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{ky}}{2} \\ &\left. \left. + \frac{\omega_c \tau_f^2}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{kx}}{2} - \frac{2\omega_c \tau_f^2 q E}{1 + \omega_c^2 \tau_f^2} \right) \right]. \end{aligned} \quad (\text{B.2})$$

In the weak magnetic field limit, the terms containing ω_c and its higher powers in the numerator can be dropped. Thus,

Eq. (B.2) becomes

$$\begin{aligned} \Delta T_q^{ij} &= \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j}{\omega_f} \beta f_f^0 (1 - f_f^0) \left[q E \tau_f v_x \right. \\ &\quad + v_x \left(\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_x}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \\ &\quad \left. - \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{kx}}{2} + \frac{\tau_f q E}{1 + \omega_c^2 \tau_f^2} \right) \\ &\quad + v_y \left(\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_0 p_y}{p^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right. \\ &\quad \left. - \frac{\tau_f}{1 + \omega_c^2 \tau_f^2} \frac{p_k W_{ky}}{2} \right) \left. \right] \\ &= \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j \tau_f}{\omega_f} \beta f_f^0 (1 - f_f^0) \\ &\quad \times \left[\frac{2q E v_x}{1 + \omega_c^2 \tau_f^2} - \frac{1}{1 + \omega_c^2 \tau_f^2} \frac{p_k p_l}{2\omega_f} W_{kl} \right. \\ &\quad \left. + \frac{1}{1 + \omega_c^2 \tau_f^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right]. \quad (\text{B.3}) \end{aligned}$$

Similarly, the antiquark part $\Delta T_{\bar{q}}^{ij}$ is written as

$$\begin{aligned} \Delta T_{\bar{q}}^{ij} &= \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j \tau_{\bar{f}}}{\omega_f} \beta \bar{f}_f^0 (1 - \bar{f}_f^0) \\ &\quad \times \left[\frac{2\bar{q} E v_x}{1 + \omega_c^2 \tau_{\bar{f}}^2} - \frac{1}{1 + \omega_c^2 \tau_{\bar{f}}^2} \frac{p_k p_l}{2\omega_f} W_{kl} \right. \\ &\quad \left. + \frac{1}{1 + \omega_c^2 \tau_{\bar{f}}^2} \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \right]. \quad (\text{B.4}) \end{aligned}$$

The gluon part ΔT_g^{ij} retains its form same as that in the absence of magnetic field,

$$\begin{aligned} \Delta T_g^{ij} &= g_g \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j \tau_g}{\omega_g} \beta f_g^0 (1 + f_g^0) \\ &\quad \times \left[\left\{ \omega_g \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_g} \right\} \partial_l u^l \right. \\ &\quad \left. - \frac{p^k p^l}{2\omega_g} W_{kl} + p^l \left(\frac{\partial_l P}{\varepsilon + P} - \frac{\partial_l T}{T} \right) \right]. \quad (\text{B.5}) \end{aligned}$$

Adding Eqs. (B.3), (B.4) and (B.5) and then simplifying, we get

$$\begin{aligned} \Delta T^{ij} &= \sum_f g_f \int \frac{d^3 p}{(2\pi)^3} \frac{\beta p^i p^j}{\omega_f} \left[2q E v_x \frac{\tau_f f_f^0 (1 - f_f^0)}{1 + \omega_c^2 \tau_f^2} \right. \\ &\quad \left. + 2\bar{q} E v_x \frac{\tau_{\bar{f}} \bar{f}_f^0 (1 - \bar{f}_f^0)}{1 + \omega_c^2 \tau_{\bar{f}}^2} \right. \\ &\quad \left. + \left(\frac{\tau_f f_f^0 (1 - f_f^0)}{1 + \omega_c^2 \tau_f^2} + \frac{\tau_{\bar{f}} \bar{f}_f^0 (1 - \bar{f}_f^0)}{1 + \omega_c^2 \tau_{\bar{f}}^2} \right) \right. \\ &\quad \times \left\{ \omega_f \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_f} \right\} \partial_l u^l \\ &\quad - \left(\frac{\tau_f f_f^0 (1 - f_f^0)}{1 + \omega_c^2 \tau_f^2} + \frac{\tau_{\bar{f}} \bar{f}_f^0 (1 - \bar{f}_f^0)}{1 + \omega_c^2 \tau_{\bar{f}}^2} \right) \frac{p_k p_l}{2p_0} W_{kl} \right] \\ &\quad + g_g \int \frac{d^3 p}{(2\pi)^3} \frac{p^i p^j \tau_g}{\omega_g} \beta f_g^0 (1 + f_g^0) \\ &\quad \times \left[\left\{ \omega_g \left(\frac{\partial P}{\partial \varepsilon} \right) - \frac{p^2}{3\omega_g} \right\} \partial_l u^l \right. \\ &\quad \left. - \frac{p^k p^l}{2\omega_g} W_{kl} + p^l \left(\frac{\partial_l P}{\varepsilon + P} - \frac{\partial_l T}{T} \right) \right]. \quad (\text{B.6}) \end{aligned}$$

C Derivation of $C_l, l = 1, 2, 3, 4$

Making use of the relations $\nabla \cdot \mathbf{V} = 0$, $V_{ij} b_i b_j = 0$, $b_{ij} v_i v_j = 0$, $b_i b_i = 1$, $b_{ij} b_i = 0$ and $b_{ij} b_j = 0$ in Eq. (45), we get

$$\begin{aligned} &\beta \omega_f V_{ij} v_i v_j f_0 (1 - f_0) \\ &= -2\omega_c \left[C_1 b_{ij} v_j Y_{im}^1 v_m + C_2 b_{ij} v_j Y_{im}^2 v_m \right. \\ &\quad + C_3 b_{ij} v_j Y_{im}^3 v_m + C_4 b_{ij} v_j Y_{im}^4 v_m \left. \right] \\ &\quad + \frac{1}{\tau_f} \left[C_1 Y_{ij}^1 v_i v_j + C_2 Y_{ij}^2 v_i v_j \right. \\ &\quad \left. + C_3 Y_{ij}^3 v_i v_j + C_4 Y_{ij}^4 v_i v_j \right] \\ &= -2\omega_c \left[C_1 b_{ij} v_j (2V_{im} - 2V_{ik} b_k b_m - 2V_{mk} b_k b_i) v_m \right. \\ &\quad + C_2 b_{ij} v_j (2V_{ik} b_k b_m + 2V_{mk} b_k b_i) v_m \\ &\quad + C_3 b_{ij} v_j (V_{ik} b_{mk} + V_{mk} b_{ik} - V_{kl} b_{ik} b_{ml}) \\ &\quad - V_{kl} b_{mk} b_l b_l) v_m \\ &\quad + C_4 b_{ij} v_j (2V_{kl} b_{ik} b_m b_l + 2V_{kl} b_{mk} b_l b_l) v_m \left. \right] \\ &\quad + \frac{1}{\tau_f} \left[C_1 (2V_{ij} - 2V_{ik} b_k b_j - 2V_{jk} b_k b_i) v_i v_j \right. \\ &\quad + C_2 (2V_{ik} b_k b_j + 2V_{jk} b_k b_i) v_i v_j \\ &\quad + C_3 (V_{ik} b_{jk} + V_{jk} b_{ik}) \\ &\quad - V_{kl} b_{ik} b_j b_l - V_{kl} b_{jk} b_i b_l) v_i v_j \\ &\quad + C_4 (2V_{kl} b_{ik} b_j b_l + 2V_{kl} b_{jk} b_i b_l) v_i v_j \left. \right]. \quad (\text{C.1}) \end{aligned}$$

Now, Eq. (C.1) gets further simplified into

$$\begin{aligned} &\beta \omega_f V_{ij} v_i v_j f_0 (1 - f_0) \\ &= -2\omega_c \left[2C_1 V_{ik} b_{ij} v_j v_k - 2C_1 V_{ik} b_{ij} b_k v_j (\mathbf{b} \cdot \mathbf{v}) \right. \\ &\quad \left. - 2C_2 V_{ik} b_{ij} v_j v_k + 2C_2 V_{ik} b_{ij} b_k v_j (\mathbf{b} \cdot \mathbf{v}) \right. \\ &\quad \left. - 2C_3 V_{ik} b_{jk} v_i v_k + 2C_3 V_{ik} b_{jk} b_i v_i v_k \right. \\ &\quad \left. - 2C_4 V_{ik} b_{jk} b_i v_i v_k + 2C_4 V_{ik} b_{jk} b_i b_k v_i v_k \right]. \end{aligned}$$

$$\begin{aligned}
& +2C_2 V_{ik} b_{ij} b_k v_j (\mathbf{b} \cdot \mathbf{v}) + 2C_3 V_{ij} v_i v_j \\
& -4C_3 V_{ij} b_i v_j (\mathbf{b} \cdot \mathbf{v}) + 2C_4 V_{ij} b_i v_j (\mathbf{b} \cdot \mathbf{v}) \\
& +\frac{1}{\tau_f} [2C_1 V_{ij} v_i v_j - 4C_1 V_{ij} b_i v_j (\mathbf{b} \cdot \mathbf{v}) \\
& +4C_2 V_{ij} b_i v_j (\mathbf{b} \cdot \mathbf{v}) + C_3 V_{ik} b_{jk} v_i v_j + C_3 V_{jk} b_{ik} v_i v_j \\
& -2C_3 V_{kl} b_{ik} b_l v_i (\mathbf{b} \cdot \mathbf{v}) + 4C_4 V_{kl} b_{ik} b_l v_i (\mathbf{b} \cdot \mathbf{v})]. \tag{C.2}
\end{aligned}$$

Comparing the same tensor structures on both sides of Eq. (C.2), we have

$$-4\omega_c C_3 + \frac{2C_1}{\tau_f} = \beta \omega_f f_0 (1 - f_0), \tag{C.3}$$

$$-2\omega_c (2C_4 - 4C_3) + \frac{1}{\tau_f} (-4C_1 + 4C_2) = 0, \tag{C.4}$$

$$-2\omega_c (-2C_1 + 2C_2) - \frac{1}{\tau_f} (-2C_3 + 4C_4) = 0, \tag{C.5}$$

$$-4\omega_c C_1 - \frac{2C_3}{\tau_f} = 0. \tag{C.6}$$

After solving Eqs. (C.3), (C.4), (C.5) and (C.6), we get

$$C_1 = \frac{\beta \omega_f \tau_f f_0 (1 - f_0)}{2 (1 + 4\omega_c^2 \tau_f^2)}, \tag{C.7}$$

$$C_2 = \frac{\beta \omega_f \tau_f f_0 (1 - f_0)}{2 (1 + \omega_c^2 \tau_f^2)}, \tag{C.8}$$

$$C_3 = -\frac{\beta \omega_f \omega_c \tau_f^2 f_0 (1 - f_0)}{(1 + 4\omega_c^2 \tau_f^2)}, \tag{C.9}$$

$$C_4 = -\frac{\beta \omega_f \omega_c \tau_f^2 f_0 (1 - f_0)}{2 (1 + \omega_c^2 \tau_f^2)}. \tag{C.10}$$

D Thermal conductivity

In relativistic hydrodynamics there exist different frames. The freedom to choose a specific frame creates arbitrariness. To avoid arbitrariness, one needs the “condition of fit”, i.e. if one chooses the Landau frame, then the condition of fit in the local rest frame requires the “0” component of the heat flow four-vector to be zero, i.e. $Q^0 = 0$, which can be understood from the fact that in the rest frame of the heat bath or fluid, heat flow four-vector is orthogonal to the fluid four-velocity, i.e. $Q^\mu u_\mu = 0$, where $u_\mu = (1, 0, 0, 0)$ in the local rest frame. Thus in the rest frame of the fluid, the heat flow is purely spatial. This concept has been used in the study of the thermal conductivity. In this way, the results also remain independent of the choice of frame in relativistic hydrodynamics [83, 84].

For a weakly magnetized hot and dense QCD matter, the thermal conductivity is given [48] by

$$\begin{aligned}
\kappa = & \frac{\beta^2}{6\pi^2} \sum_f g_f \int dp \frac{p^4}{\omega_f^2} \\
& \times \left[\frac{\tau_f}{1 + \omega_c^2 \tau_f^2} (\omega_f - h_f)^2 f_f^0 (1 - f_f^0) \right. \\
& \left. + \frac{\tau_{\bar{f}}}{1 + \omega_c^2 \tau_{\bar{f}}^2} (\omega_f - \bar{h}_f)^2 \bar{f}_f^0 (1 - \bar{f}_f^0) \right]. \tag{D.1}
\end{aligned}$$

References

1. P. Danielewicz, M. Gyulassy, Phys. Rev. D **31**, 53 (1985)
2. K. Heckmann, M. Buballa, J. Wambach, Eur. Phys. J. A **48**, 142 (2012)
3. S. Yasui, S. Ozaki, Phys. Rev. D **96**, 114027 (2017)
4. D. Kharzeev, K. Tuchin, J. High Energy Phys. **0809**, 093 (2008)
5. M.A.V. Basagoiti, Phys. Rev. D **66**, 045005 (2002)
6. G.D. Moore, O. Saremi, J. High Energy Phys. **0809**, 015 (2008)
7. S. Plumari, A. Puglisi, F. Scardina, V. Greco, Phys. Rev. C **86**, 054902 (2012)
8. N.Y. Astrakhantsev, V.V. Braguta, A.Y. Kotov, J. High Energy Phys. **1704**, 101 (2017)
9. N.Y. Astrakhantsev, V.V. Braguta, A.Y. Kotov, Phys. Rev. D **98**, 054515 (2018)
10. B.A. Gelman, E.V. Shuryak, I. Zahed, Phys. Rev. C **74**, 044908 (2006)
11. P.B. Arnold, G.D. Moore, L.G. Yaffe, J. High Energy Phys. **11**, 001 (2000)
12. P.B. Arnold, G.D. Moore, L.G. Yaffe, J. High Energy Phys. **05**, 051 (2003)
13. P.B. Arnold, C. Dogan, G.D. Moore, Phys. Rev. D **74**, 085021 (2006)
14. Y. Hidaka, R.D. Pisarski, Phys. Rev. D **78**, 071501(R) (2008)
15. P.K. Kovtun, D.T. Son, A.O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005)
16. L.P. Csernai, J.I. Kapusta, L.D. McLerran, Phys. Rev. Lett. **97**, 152303 (2006)
17. A. Nakamura, S. Sakai, Phys. Rev. Lett. **94**, 072305 (2005)
18. H.B. Meyer, Phys. Rev. D **76**, 101701(R) (2007)
19. P.B. Arnold, G.D. Moore, L.G. Yaffe, J. High Energy Phys. **0305**, 051 (2003)
20. H.B. Meyer, Nucl. Phys. A **830**, 641C (2009)
21. M. Haas, L. Fister, J.M. Pawłowski, Phys. Rev. D **90**, 091501 (2014)
22. N. Christiansen, M. Haas, J.M. Pawłowski, N. Strodthoff, Phys. Rev. Lett. **115**, 112002 (2015)
23. G. Jackson, A. Peshier, J. Phys. G **45**, 095001 (2018)
24. A. Muronga, Phys. Rev. Lett. **88**, 062302 (2002)
25. H. Song, U. Heinz, J. Phys. G **36**, 064033 (2009)
26. H.B. Meyer, Phys. Rev. Lett. **100**, 162001 (2008)
27. V. Skokov, A. Illarionov, V. Toneev, Int. J. Mod. Phys. A **24**, 5925 (2009)
28. K. Tuchin, Phys. Rev. C **82**, 034904 (2010)
29. S. Rath, B.K. Patra, Phys. Rev. D **100**, 016009 (2019)
30. S. Rath, B.K. Patra, J. High Energy Phys. **1712**, 098 (2017)
31. A. Bandyopadhyay, B. Karmakar, N. Haque, M.G. Mustafa, Phys. Rev. D **100**, 034031 (2019)
32. S. Rath, B.K. Patra, Eur. Phys. J. A **55**, 220 (2019)

33. B. Karmakar, R. Ghosh, A. Bandyopadhyay, N. Haque, M.G. Mustafa, Phys. Rev. D **99**, 094002 (2019)

34. K. Fukushima, K. Hattori, H.-U. Yee, Y. Yin, Phys. Rev. D **93**, 074028 (2016)

35. K. Hattori, D. Satow, Phys. Rev. D **94**, 114032 (2016)

36. B. Feng, Phys. Rev. D **96**, 036009 (2017)

37. M. Kurian, V. Chandra, Phys. Rev. D **96**, 114026 (2017)

38. K. Fukushima, Y. Hidaka, Phys. Rev. Lett. **120**, 162301 (2018)

39. L. Thakur, P.K. Srivastava, Phys. Rev. D **100**, 076016 (2019)

40. M. Kurian, S. Mitra, S. Ghosh, V. Chandra, Eur. Phys. J. C **79**, 134 (2019)

41. S. Rath, B.K. Patra, Eur. Phys. J. C **80**, 747 (2020)

42. V. Roy, S. Pu, L. Rezzolla, D. Rischke, Phys. Lett. B **750**, 45 (2015)

43. G. Inghirami, L. Del Zanna, A. Beraudo, M.H. Moghaddam, F. Becattini, M. Bleicher, Eur. Phys. J. C **76**, 659 (2016)

44. H. van Hees, C. Gale, R. Rapp, Phys. Rev. C **84**, 054906 (2011)

45. K. Tuchin, Phys. Rev. C **88**, 024910 (2013)

46. K.A. Mamo, J. High Energy Phys. **1308**, 083 (2013)

47. C. Shen, U.W. Heinz, J.-F. Paquet, C. Gale, Phys. Rev. C **89**, 044910 (2014)

48. S. Rath, S. Dash, [arXiv:2112.11802](https://arxiv.org/abs/2112.11802) [hep-ph]

49. Seung-i Nam, Chung-W. Kao, Phys. Rev. D **87**, 114003 (2013)

50. K. Hattori, Xu.-G. Huang, D.H. Rischke, D. Satow, Phys. Rev. D **96**, 094009 (2017)

51. S. Li, Ho.-U. Yee, Phys. Rev. D **97**, 056024 (2018)

52. G.S. Denicol et al., Phys. Rev. D **98**, 076009 (2018)

53. A. Das, H. Mishra, R.K. Mohapatra, Phys. Rev. D **100**, 114004 (2019)

54. S. Rath, B.K. Patra, Phys. Rev. D **102**, 036011 (2020)

55. S. Rath, B.K. Patra, Eur. Phys. J. C **81**, 139 (2021)

56. A. Hosoya, K. Kajantie, Nucl. Phys. B **250**, 666 (1985)

57. E.M. Lifshitz, L.P. Pitaevskii, *Physical Kinetics* (Pergamon Press, Oxford, 1981)

58. L.D. Landau, E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1987)

59. P. Chakraborty, J.I. Kapusta, Phys. Rev. C **83**, 014906 (2011)

60. K. Tuchin, J. Phys. G **39**, 025010 (2012)

61. A. Peshier, B. Kämpfer, O.P. Pavlenko, G. Soff, Phys. Lett. B **337**, 235 (1994)

62. M.I. Gorenstein, S.N. Yang, Phys. Rev. D **52**, 5206 (1995)

63. M. Bluhm, B. Kämpfer, R. Schulze, D. Seipt, Eur. Phys. J. C **49**, 205 (2007)

64. H. Berrehrah, E. Bratkovskaya, W. Cassing, P.B. Gossiaux, J. Aichelin, M. Bleicher, Phys. Rev. C **89**, 054901 (2014)

65. E. Braaten, R.D. Pisarski, Phys. Rev. D **45**, R1827 (1992)

66. A. Peshier, B. Kämpfer, G. Soff, Phys. Rev. D **66**, 094003 (2002)

67. V. Goloviznin, H. Satz, Z. Phys. C **57**, 671 (1993)

68. A. Peshier, B. Kämpfer, O.P. Pavlenko, G. Soff, Phys. Rev. D **54**, 2399 (1996)

69. K. Fukushima, Phys. Lett. B **591**, 277 (2004)

70. S.K. Ghosh, T.K. Mukherjee, M.G. Mustafa, R. Ray, Phys. Rev. D **73**, 114007 (2006)

71. H. Abuki, K. Fukushima, Phys. Lett. B **676**, 57 (2009)

72. N. Su, K. Tywoniuk, Phys. Rev. Lett. **114**, 161601 (2015)

73. W. Florkowski, R. Ryblewski, N. Su, K. Tywoniuk, Phys. Rev. C **94**, 044904 (2016)

74. V.M. Bannur, J. High Energy Phys. **0709**, 046 (2007)

75. V.M. Bannur, Phys. Rev. C **75**, 044905 (2007)

76. J.-P. Blaizot, A. Ipp, A. Rebhan, U. Reinoso, Phys. Rev. D **72**, 125005 (2005)

77. A. Ayala et al., Phys. Rev. D **98**, 031501 (2018)

78. B. McInnes, Nucl. Phys. B **921**, 39 (2017)

79. A. Muronga, Phys. Rev. C **69**, 034903 (2004)

80. T. Fukutome, M. Iwasaki, Prog. Theor. Phys. **119**, 991 (2008)

81. L.P. Csernai, D.D. Strottman, Cs. Anderlik, Phys. Rev. C **85**, 054901 (2012)

82. F. Karsch, D. Kharzeev, K. Tuchin, Phys. Lett. B **663**, 217 (2008)

83. S. Gavin, Nucl. Phys. A **435**, 826 (1985)

84. J.I. Kapusta, B. Müller, M. Stephanov, Phys. Rev. C **85**, 054906 (2012)