String Field Theory in One Dimension and Matrix Models

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ABSTRACT

We discuss the singlet sector of the $d = 1$ matrix model in terms of a Dirac fermion formalism. The leading order two- and three- point functions of the density fluctuations are obtained by this method. This allows us to construct the effective action to that order and hence provide the equation of motion. This equation is compared with the one obtained from the continuum approach.

In recent works[1][2][3] we have studied the problem of two-dimensional quantum field theories coupled to gravity. Our original motivation to do this was to arrive at a natural setting for the theory space formulation[4][5] of string theory, where, (1) there is no restriction on the central charge of the matter sector, and, (2) the theory has, within it, the ingredients to describe trajectories which join special points in the theory space, namely the classical vacua which correspond to conformally invariant theories. One of our main results has been that the matter + gravity system can be regarded as a field theory of the Liouville mode and matter fields in the background of the fiducial metric. Generic couplings or backgrounds now depend both on the Liouville mode and on the matter degrees of freedom and satisfy equations of motion in $d+1$ variables (d = matter, 1 = Liouville). This is because reparametrization invariance of the theory implies that all objects, in which the conformal mode has been integrated, should be Weyl-invariant in its dependence upon the fiducial metric. This condition, stated as the vanishing of the 'B function', gives rise to the equations of motion[3]. Other related works are due to J. Polchinski[6], and, T. Banks and J. Lykken[7].

These ideas were illustrated in various situations:

(*a*) For d-scalar fields interacting with 2-dim. gravity, we proved that this system quantized in the light cone gauge is exactly mapped into the conformally invariant field theory of d + 1-scalar fields, with background charge. At $d = 25$ we obtained the exact tree level S-matrix and spectrum of the "$d = 26$ critical string".

(*b*) In the case of $d < 1$, we considered the $(m, m+1)$ minimal models coupled to gravity, and could effectively describe the interpolation between two minimal models, for m very large, by means of a 'string field' that depends only on the Liouville mode, a function $\kappa(\eta)$ which satisfies the field equation[3]

$$\left(\partial_\eta{}^2 + Q\partial_\eta + h\right)\kappa(\eta) = b\kappa^2(\eta) + o(\kappa^3) \quad (1)$$

(*c*) In the case of d-scalar fields coupled to gravity, perturbed by a 'tachyon' background, the tachyon coupling T, which depends on d coordinates ϕ_i and the Liouville mode η satisfies the $d+1$ dimensional field equation[1][3]

$$\left(\partial_\eta{}^2 + Q\partial_\eta + \partial_i\partial_i + 2\right)T(\phi, \eta) + T^2(\phi, \eta) + \cdots = 0 \quad (2)$$

where

$$Q = \sqrt{\frac{25-d}{3}} \quad (3)$$

The coupling of other backgrounds like the metric, antisymmetric tensor and dilaton can be discussed similarly. To see the spectrum from equation (2), one eliminates the linear derivative piece by defining $\tilde{T} = e^{-\frac{1}{2}Q\eta}T$

$$\left(\partial_\eta{}^2 + \partial_i\partial_i + \frac{1}{4}(8 - Q^2))\right)\tilde{T}(\phi,\eta) + e^{-\frac{1}{2}Q\eta}\tilde{T}^2(\phi,\eta) + \cdots = 0 \tag{4}$$

This equation tells us that the spectrum at $d = 1$ (i.e., $Q^2 = 8$) is that of a massless particle. For $d > 1$, there is a tachyon in the spectrum and hence for much the same reasons as in 26-dimensional critical string theories, where it ruins the perturbation expansion, these theories may not exist. It is likely that the tachyon perturbation drives $d > 1$ theories to a stable point which is $d = 1$. It would also be interesting to understand how one can reach models with $d < 1$ by appropriate perturbations of the $d = 1$ model.

Our main purpose here is to discuss the cut-off string field theory at $d = 1$[8][9][10], formulated as the quantum mechanics of the matrix hamiltonian which was originally discussed by Brézin, Itzykson, Parisi and Zuber[11]

$$H = -\frac{1}{2N}\nabla_M^2 + N\, tr\, V(M) \tag{5}$$

where ∇_H^2 is the laplacian in the space of hermetian matrices and $V(M)$ is a polynomial. We can expect the results of the continuum theory and that from the matrix model approach to agree in the low momentum region only.

Since this hamiltonian is invariant under $U(N)$ transformations, $M \to UMU^\dagger$, there would be wavefunctions transforming according to various different representations of $U(N)$. (To be more precise, these consist of the trivial representation and the representations that can be generated by taking products of the adjoint.) It is not yet clear whether states which transform nontrivially under $U(N)$ are related to the string degrees of freedom. Presently we will analyse the singlet sector of model. We use the fermionic representation of this sector as explained below. This representation has two major advantages.

(a) The model is well defined even for finite N and for noncritical values of the coupling. Hence the nature of the various regularizations are most clearly recognized in this picture.

(b) It is easier to see various approximate and exact symmetries of the system from this point of view.

As in well known, ∇_M^2 acting on the singlet sector wave function $\psi(\Lambda)$ has the form

$$\nabla_M^2\psi(\Lambda) = \frac{1}{\Delta(\Lambda)}\Big(\sum_i \frac{\partial^2}{\partial\lambda_i^2}\Big)\Delta(\Lambda)\psi(\Lambda) \tag{6}$$

where $\Lambda = (\lambda_1, \cdots, \lambda_N)$, λ_i being the eigenvalues of M. $\Delta(\Lambda)$ is the Vandermonde determinant

$$\Delta(\Lambda) = \prod_{i<j}(\lambda_i - \lambda_j) \tag{7}$$

If we use $\chi(\Lambda) = \Delta(\Lambda)\psi(\Lambda)$ instead of $\psi(\Lambda)$ as the wave function, the effective hamiltonian becomes

$$H_f = -\frac{1}{2N}\sum_i \frac{\partial^2}{\partial\lambda_i^2} + N\sum_i V(\lambda_i) \tag{8}$$

This is the hamiltonian for non-interacting particles. However, since $\psi(\Lambda)$ is symmetric, $\chi(\Lambda)$ is antisymmetric. Hence the problem reduces to that of noninteracting fermions moving in an external potential.

The ground state is obtained by filling the N states which are lowermost in energy. The corresponding total energy becomes singular when the Fermi energy of the system approaches the value of the potential at a stationary point. This is related to the fact that the time period of the classical orbits corresponding to the states near the fermi surface starts diverging when they can approach the stationary point. In the semiclassical analysis one can obtain the nature of the singularities. Also one can take the double-scaling limit by keeping fixed the energy difference between the value of the potential at the stationary point and that of Fermi energy as N goes to infinity. Inverse of this energy difference can be identified as the string coupling constant, g_{str}[9].

We have developed a Dirac fermion formalism for the states near the Fermi surface which works well for the leading contributions and lends valuable insight into many results.

To obtain the equations of motion we write this theory as a second quantized theory with the action

$$S = \int dt\, d\lambda\, \chi^\dagger(\lambda,t)(i\partial_t + \frac{1}{2N}\partial_\lambda^2 - NV(\lambda))\chi(\lambda,t) + \int dt\, d\lambda\, N a_0(\lambda,t)\chi^\dagger(\lambda,t)\chi(\lambda,t) \tag{9}$$

where $\chi(\lambda,t)$ is a second-quantized fermion field in two dimensions and $a_0(\lambda,t)$ is the source function conjugate to density. It corresponds a coupling of the form $\sum_n tr\ M^n(t)a_n(t)$ in the original matrix model, where $a_n(t) = \int d\lambda\ \lambda^n\ a_0^n(\lambda,t)$. The corresponding vacuum to vacuum amplitude $Z[a_0]$ contains the information of all correlation function of density. Let

$$F[a_0] = \ln Z[a_0] \tag{10}$$

By taking the Legendre transformation of $F[a_0]$ we obtain the effective action $\Gamma[\rho]$ where

$$\rho(\lambda,t) = \frac{\delta F[a_0]}{\delta a_0(\lambda,t)} \tag{11}$$

and

$$\Gamma[\rho] = \int d\lambda\ dt\ \rho(\lambda,t)a_0(\lambda,t) - F[a_0] \tag{12}$$

When $a_0 = 0$ we have

$$\frac{\delta\Gamma[\rho]}{\delta\rho(\lambda,t)} = 0 \tag{13}$$

which is the quantum equation of motion.

Since we are looking at an effective bosonic theory we define the field variable that is used for bosonizing relativistic fermion theories,

$$\phi(\lambda,t) = \int^\lambda (\rho(\lambda,t) - <\rho(\lambda,t)>) \tag{14}$$

We will find, perturbatively, the leading order terms in the equation of motion in terms of this variable.

Dirac fermion representation

In the following we briefly indicate how this result was obtained using a Dirac fermion representation. This is essentially by a change of variables from $\lambda \to \tau = \int^\lambda \rho_o(\lambda')d\lambda'$, and scaling the hamiltonian by the fermi level wave function,

$$\hat{H} = \left(e^{\mp iN\Theta_f}\rho_f^{-1/2}\right)H\left(\rho_f^{1/2}e^{\pm iN\Theta_f}\right) = \mp i\frac{d}{d\tau} - \frac{1}{2N}\frac{d}{d\tau}\rho_o^2\frac{d}{d\tau}. \tag{15}$$

where $H(\rho_f^{1/2}e^{\pm iN\Theta_f}) = E_f(\rho_f^{1/2}e^{\pm iN\Theta_f})$.

To get the scales right, let us make some estimates. The leading order large N solution of the equation

$$H\phi = E_f\phi \tag{16}$$

is

$$\phi = \text{constant} \times \frac{1}{\sqrt[4]{2(\epsilon_f - V(\lambda))}} \times e^{\pm iN\int^\lambda d\lambda'\sqrt{2(\epsilon_f - V(\lambda'))}} \tag{17}$$

where

$$\epsilon_f = \frac{E_f}{N} \tag{18}$$

If we choose the constant to be 1, we have

$$\rho_f(\lambda) = \frac{1}{\sqrt{2(\epsilon_f - V(\lambda))}} \qquad \Theta_f(\lambda) = \int^\lambda d\lambda'\sqrt{2(\epsilon_0 - V(\lambda))} \tag{19}$$

and

$$\rho_f\Theta_f' = 1 \tag{20}$$

Let the potential have a maximum at λ_0 with $V''(\lambda_0) \neq 0$. If we take a solution for ϵ_0 very near $V(\lambda_0)$ then most of the probability is concentrated near that tip. Classically this is manifested by the particle spending a lot of time near the turning point, which is very close to the rather flat region around the potential maximum.

In this region $V(\lambda) \approx V(\lambda_0) - \frac{1}{2}|V''(\lambda_0)|(\lambda - \lambda_0)^2$

$$\tau - a \sim \int_{\lambda_1}^{\lambda} d\lambda' \frac{1}{\sqrt{|V''(\lambda_0)|(\lambda' - \lambda_0)^2 - 2(V(\lambda_0) - \epsilon_0)}} \sim \frac{1}{(|V''(\lambda_0)|)^{1/2}} \int_{\lambda_0+\sqrt{2\mu}}^{\lambda} d\lambda' \frac{1}{\left((\lambda' - \lambda_0)^2 - 2\mu\right)^{1/2}} \tag{21}$$

By convention we make $V''(\lambda_0) = -1$ and define μ to be $V(\lambda_0) - \epsilon_0$.

Upon integration we get

$$\tau - a = \cosh^{-1}\left(\frac{\lambda - \lambda_0}{\sqrt{2\mu}}\right) \qquad (22)$$

or

$$\lambda = \lambda_0 + \sqrt{2\mu}\cosh(\tau - a) \qquad (23)$$

where a is the value of τ at the turning point.

Now

$$\rho_f^2 \sim \frac{1}{4\mu \sinh^2(\tau - a)} \qquad (24)$$

$$\hat{H} \sim \mp i\frac{\partial}{\partial\tau} - \frac{1}{8N\mu}\frac{\partial}{\partial\tau}\frac{1}{\sinh^2(\tau - a)}\frac{\partial}{\partial\tau} \qquad (25)$$

This estimate can be trusted, when τ is not too near a.

To recover an approximate relativistic fermion picture from a nonrelativistic one, the most natural thing to do is to take the reference energy level E_0 to be the Fermi level E_f. If we now want the expression of $\hat{H}$ in terms of τ to be a scaled expression, that is, if we want to keep $\tau - a$ as a scaled variable, we have to have $N\mu$ = fixed. (This is true irrespective of the semiclassical approximation that we made to reach this expression of $\hat{H}$.)

Strictly speaking, for this problem, the wave functions are not exactly like $\rho^{1/2}e^{+iN\Theta}$ and $\rho^{1/2}e^{-iN\Theta}$, but a specific linear combination which depends upon the energy and the the boundary conditions. In terms of τ variables, $\rho^{1/2}e^{\pm iN\Theta}$, after the relevant transformation looks like a plane wave in the leading order. The extent of classically allowed $\lambda - \lambda_0$ is roughly from 0 to say 1. Corresponding range of $\tau - a$ is from 0 to $\ln\frac{1}{\sqrt{\mu}}$. The level spacing goes as inverse of this range. Hence the boundary condition can give rise to mixing of left moving and right moving plane waves which can change the energy atmost by $\frac{1}{\ln\frac{1}{\sqrt{\mu}}}$. This vanishes in the scaling limit.

Thus we are allowed, in the scaling limit to deal with chiral states which are almost exact eigenstates. The hamiltonian which makes the right moving states near the Fermi surface look like plane waves is

$$\hat{H}_R = -i\partial_\tau - \frac{1}{2N}\partial_\tau\rho_f^2\partial_\tau \qquad (26)$$

The hamiltonian which does the same for the left moving states is

$$\hat{H}_L = i\partial_\tau - \frac{1}{2N}\partial_\tau\rho_f^2\partial_\tau \qquad (27)$$

Both the hamiltonians have the information about all the states. However, for the left moving states the second term in $\hat{H}_R$ cannot be considered as a small perturbation. Similar problem arise for right moving states and $\hat{H}_L$.

Thus, for the calculations where only states near the Fermi surface matter, one can describe the left moving states by $\hat{H}_L$ and the right moving by $\hat{H}_R$. This gives a Dirac hamiltonian. In the second-quantized notation the hamiltonian is

$$H = \int d\tau\left(\psi_+^\dagger \hat{H}_L\psi_+ + \psi_-^\dagger \hat{H}_R\psi_-\right) \qquad (28)$$

To be honest one should discard half the solutions of each of the hamiltonians, not to over count the states. This would be some ultraviolet cut off in the theory. This cutoff would refer to the value of the momenta where the second term starts dominating over the first. For calculations involving processes near the Fermi surface, this cutoff is not important.

In many of the leading order calculations, this problem does not show up. This effective ultraviolet cutoff parameter, in a certain region of τ, is finite in the scaled picture (as opposed to the semiclassical case). Hence one has to be careful about it.

We now briefly indicate the results of the calculation of the 2 and 3 point functions of the density which corroborate our guess of the effective action to leading order.

The two-point function of density fluctuations

$$G^{(2)}(1,2) = < 0|T\rho(\lambda_1,t_1)\rho(\lambda_2,t_2)|0>_c \qquad (29)$$

$\rho(\lambda,t)$ is the eigenvalue/fermion density. If we look only at the connected part, we would see the correlation of density fluctuation, $\rho(\lambda,t) - <\rho(\lambda,t)>$, only. This density fluctuation can be represented also by $\chi^\dagger\chi$ normal ordered with respect to the Fermi sea.

If we change over to τ variables we have

$$\begin{aligned}&\tilde{G}^{(2)}(1,2)\\&=<0|T\tilde{\rho}(\tau_1,t_1)\tilde{\rho}(\tau_2,t_2)|0>_c\\&=\frac{d\lambda_1}{d\tau_1}\frac{d\lambda_2}{d\tau_2}<0|T\rho(\lambda_1,t_1)\rho(\lambda_2,t_2)|0>\end{aligned}\tag{30}$$

We call $\tilde{\chi}_L=\psi_+$ and $\tilde{\chi}_R=\psi_-$

$$:\chi^\dagger\chi:\rightarrow:\psi_+^\dagger\psi_+:+:\psi_-^\dagger\psi_-:\tag{31}$$

$$\begin{aligned}&\tilde{G}^{(2)}(1,2)\\&=<0|T:\psi_+^\dagger(1)\psi_+(1)::\psi_+^\dagger(2)\psi_+(2):|0>\\&+(+\rightarrow-)\end{aligned}\tag{32}$$

Take $t_1>t_2$ and consider

$$\begin{aligned}&<0|:\psi_-^\dagger(1)\psi_-(1)::\psi_-^\dagger(2)\psi_-(2)::|0>\\&=S_p^{(-)}(1,2)S_h^{(-)}(1,2)\end{aligned}\tag{33}$$

In the leading order the particle and hole propagators are identical and charge conjugation symmetry is explicit,

$$S_p^{(-)}=S_h^{(-)}=\frac{\omega}{2\pi}\sum_{n=0}^{\infty}e^{-i(n+\frac{1}{2})\omega t_{12}^-}\tag{34}$$

Using these formulae the 2-point function is calculated to be

$$\begin{aligned}&\tilde{G}^{(2)}(1,2)=\\&\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}\Big[\sum_{j=-\infty}^{\infty}\frac{i\omega}{4\pi^3}\int_{-\infty}^{\infty}dE\frac{e^{-i(Et_{12}-j\omega\tau_{12})}}{E^2-(j\omega)^2+i\epsilon}\Big]\end{aligned}\tag{35}$$

The expression inside the square bracket is the correlator of a free bose field. This is not surprising since what we have done is to bosonize the noninteracting fermions in a finite volume. We identify the free Bose field through the well known relation

$$:\psi^\dagger\psi:=\partial_\tau\phi\tag{36}$$

One can then see equation (37) coming out immediately from the Bose field correlator.

The three-point function of density fluctuations

For fermions satisfying the Dirac equation, the three point function of density is zero. This is a consequence of the charge conjugation symmetry of the Dirac hamiltonian. In other words, it is a consequence of the symmetry of the problem under reflection about the Fermi level. However, we know that this symmetry is broken in the nonrelativistic model and this is caused by the second term in the hamiltonian. This term, treated as a perturbation, should provide systematic order by order contributions to the three point function.

The lowest order contribution to the three-point function

$$\begin{aligned}\tilde{G}^{(3)}(1,2,3)=<0|:\psi_+^\dagger(1)\psi_+(1)::\psi_+^\dagger(2)\psi_+(2):\\:\psi_+^\dagger(3)\psi_+(3):|0>+(+\rightarrow-)\end{aligned}\tag{37}$$

($t_1>t_2>t_3$) turns out to be the following lengthy expression after a long calculation.

$$\begin{aligned}&\tilde{G}^{(3)}(1,2,3)=\sum_{j_1,j_2=1}^{\infty}\frac{\omega^4}{8\pi^3}e^{-ij_1\omega t_1^- -i(j_2-j_1)t_2^- +ij_2t_3^-}\\&\Big[\omega^{-1}\frac{\partial\omega}{\partial\epsilon}\Big\{j_2(j_1-j_2)\Theta(j_1-j_2)(1-ij_1\omega t_1^-)\\&+j_2j_1(1-i(j_2-j_1)\omega t_2^-)\\&+j_1(j_2-j_1)\Theta(j_2-j_1)(1+ij_2\omega t_3^-)\Big\}\\&-j_2(j_1-j_2)\Theta(j_1-j_2)\Big(\frac{1}{2(\epsilon-V(1))}+\\&ij_1\omega\int^{\tau_1}\frac{d\tau}{2(\epsilon-V)}\Big)\\&-j_2j_1\Big(\frac{1}{2(\epsilon-V(2))}\\&+i(j_2-j_1)\omega\int^{\tau_2}\frac{d\tau}{2(\epsilon-V)}\Big)\\&-j_1(j_2-j_1)\Theta(j_2-j_1)\Big(\frac{1}{2(\epsilon-V(3))}-\\&ij_2\omega\int^{\tau_3}\frac{d\tau}{2(\epsilon-V)}\Big)\Big]\\&+\Big(t^-\rightarrow t^+\text{ and }\int^{\tau}\frac{d\tau'}{2(\epsilon-V)}\rightarrow-\\&\int^{\tau}\frac{d\tau'}{2(\epsilon-V)}\Big)\end{aligned}\tag{38}$$

$\Theta(x)$ being the Heaviside function.

The structure of the effective action

We want to keep only terms upto the order of $\frac{1}{N}$ in the equation of motion. It can be easily seen from N counting that if we normalize the two-point connected Green's function to be order 1, then the order of the n-point connected function is N^{2-n}. Hence we need to consider only the two-point and the three-point function. The leading contribution to the three-point function is of the order $1/N$. The $1/N$ contribution to the two-point function cancells off. This is because the two-point function of the density is $S_k(1,2)S_p(1,2)$ which is $S(1,2)^2$ in the lowest order. The next order is $\delta(S_k(1,2)S_p(1,2))$. Since, in the lowest order $\Delta S_k = -\Delta S_p$, the first correction to the two-point funcion is zero. The correction is therefore $\sim O(\frac{1}{N^2})$.

Hence, from what we have done till now, we can reconstruct in the lowest order quadratic and cubic pieces of the effective action. The quadratic piece is going to be that of a free boson field which is $2\pi \int dtd\tau \partial_+\phi\partial_-\phi$. We need to choose a three-vertex which gives the correct three point function. This three point function has two pieces. One is proportional to $\omega^{-1}\frac{\partial\omega}{\partial\epsilon}$, the other is not. This first term is the dominant one for fixed λ_i, if we calculate $< \prod_i \rho(\lambda_i, t_i) >$. However if we change over to scaled variable like $\tau - a$ then since

$$\begin{aligned} &\omega \sim |\ln \Delta\epsilon|^{-1}, \quad \Delta\epsilon = V(\lambda_0) - \epsilon \\ &\frac{1}{N\omega}\frac{\partial\omega}{\partial\epsilon} \sim \frac{1}{|\ln \Delta\epsilon|}\frac{1}{N\Delta\epsilon} \to 0 \end{aligned} \tag{39}$$

if $N\Delta\epsilon$ is held fixed when $N \to \infty$. On the other hand quantities like

$$\frac{1}{N}\frac{1}{2(\epsilon - V)} \sim \frac{1}{N\Delta\epsilon}\frac{1}{\sinh^2(\tau - a)} \tag{40}$$

remain finite. Hence we pay less attention to the piece proportional to $\frac{\partial\omega}{\partial\epsilon}$. The other piece is a sum of two chiral contributions. This indicates that the vertex is made of $\partial_+\phi$ and $\partial_-\phi$. $\partial_\pm = \frac{\partial}{\partial t\pm}$. In fact one can show that the required interaction piece of the effective action is of the form

$$\Gamma_{\rm int} = \frac{-2\pi^2}{3N}\int dtd\tau \rho_f^2(\tau)\{(\partial_+\phi)^3 - (\partial_-\phi)^3\}. \tag{41}$$

It is remarkable that some very similar action can be obtained if one tries to bosonize the fermion theory naively by using Mandelstam formulae[12],

$$\begin{aligned} &\psi_\pm^\dagger(\tau_1)\psi_\pm(\tau_2) = \mp\frac{i}{2\pi(\tau_1 - \tau_2)} \\ &\exp\Big(-\pi i \int_{\tau_1}^{\tau_2} d\tau(\dot\phi \pm \phi') + O(\tau_1 - \tau_2)^2\Big) \end{aligned} \tag{42}$$

(Note that our normalization of ϕ is different from Mandelstam's.) Now, one can separately differentiate in τ_1 and τ_2 and then take the limit $\tau_1 \to \tau_2$ and use the result in equation ( ) to obtain the bosonic expression for the perturbation.

We know that Mandelstam formulae depend crucially on the short distance properties of the Green's function, which can be modified if the perturbation is singular. This is precisely the case here. Yet this procedure gives the same leading order effective action,except for a $\frac{1}{N}\int dtd\tau\rho_f^2\partial_\tau^3\phi$ term (which, if genuinely present, should shifts the background ϕ from zero to a value $\sim 0(\frac{1}{N})$ and in that process give $0(\frac{1}{N^2})$ correction to the two point function which no longer remain translation-invariant). It is possible that there is a generalization of the Mandelstam formulea in our case, where terms more singular than $\frac{1}{\tau_1-\tau_2}$ appear, but they are always multiplied by higher powers of $1/N$ (or $g_{\rm str}$).

The equation of motion in the lowest order looks like

$$\partial_+\partial_-\phi = \frac{\pi}{2N}\Big[\partial_+\{\rho_f^2(\partial_+\phi)^2\} - \partial_-\{\rho_f^2(\partial_-\phi)^2\}\Big] \tag{43}$$

since

$$\rho_f^2(\tau) \sim \frac{1}{4\mu\sinh^2(\tau - a)} \tag{44}$$

for large $\tau - a$, i.e. for points far away from the turning point,

$$\rho_f^2(\tau) \sim \frac{e^{-2(\tau-a)}}{\mu} \tag{45}$$

Then

$$\begin{aligned} \partial_+\partial_-\phi = &\frac{\pi}{N\mu}e^{-2(\tau-a)}\Big[-(\partial_+\phi)^2 + (\partial_-\phi)^2 \\ &+ \partial_+\phi\partial_+^2\phi - \partial_-\phi\partial_-^2\phi\Big] \end{aligned} \tag{46}$$

This is very similar to the tachyon equation. Note, however, that the interaction terms consist solely of derivatives of ϕ and not ϕ itself. Also it can be written entirely in terms of the currents $j_\pm = \partial_\pm \phi +$ higher order terms.

Note added

While this work was in progress we became aware of similar works by S.R. Das and A. Jevicki[13] and[14] J. Polchinski.

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