The final fate of instability of Reissner-Nordström-AdS black holes

Jun-ichirou Koga[1](a), Kengo Maeda[2](b) and Shunsuke Fujii[3](c)

(a) *Research Institute for Science and Engineering, Waseda University, Shinjuku, Tokyo 169-8555*
(b) *Faculty of Engineering, Shibaura Institute of Technology, Saitama, 330-8570*
(c) *Iwate Prefectural Morioka First High School, Iwate, 020-8515*

Abstract

We investigate dynamical and thermodynamical instability of the 4-d Reissner–Nordström–anti de–Sitter (RN-AdS) black holes. In accordance with the holographic model of superconductor, we find, by computing the quasi-normal mode, that the RN-AdS black holes are unstable below the critical temperature. We also calculate the difference of the entropy between the RN-AdS black hole and the hairy black hole that emerges below the critical temperature. It is shown that the hairy black hole carries larger entropy than the RN-AdS black hole in the micro-canonical ensemble, which implies that the RN-AdS black holes evolve towards the hairy black holes, due to the instability.

1 Introduction

Based on AdS/CFT correspondence, the simplest model of holographic superconductor [1] has been constructed in the theory described by the bulk action

$$S = \int \sqrt{-g}\, d^4x \left[R + \frac{6}{L^2} - \frac{F^{ab}F_{ab}}{4} - m^2|\psi|^2 - (\bar{D}^a\bar{\psi})(D_a\psi) \right], \tag{1}$$

where L is the scale of the cosmological constant, $D_a = \nabla_a - iqA_a$, and m and q are the mass and the charge of the scalar field ψ, respectively. (See for reviews [2–4]). In this model, we have the RN-AdS black hole as the unique solution in the bulk when the temperature of the black hole is high enough, which corresponds to the normal state in the boundary field theory. However, at some critical temperature T_c, there emerges the so-called marginally stable solution [5], which is a black hole solution with a tiny scalar hair, and it is argued that the emergence of this solution signals instability of the RN-AdS black holes below T_c, implying phase transition at T_c. Although this argument of instability seems natural, it has not been explicitly shown so far whether the RN-AdS black holes are really unstable below T_c, and whether they evolve, due to this instability, towards black holes with the scalar hair, which correspond to the superconducting state in the holographic superconductor.

In order to clarify the dynamical and thermodynamical aspects of this instability of the RN-AdS black hole, we computed [6] the quasi-normal mode on the background of the RN-AdS black hole, and the difference of the entropy between the RN-AdS black hole and the hairy black hole in the micro-canonical ensemble near T_c. We will describe here the primary analyses and results presented in [6].

2 Quasi-normal modes

We first analyze the quasi-normal mode of the scalar filed ψ on the 4-d RN-AdS black hole background near T_c. The metric and the gauge potential of this black hole are given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad A_\mu dx^\mu = \Phi(r)dt = \rho\left(\frac{1}{r_+} - \frac{1}{r}\right)dt, \tag{2}$$

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[1] Email address: koga@waseda.jp

[2] Email address: maeda302@sic.shibaura-it.ac.jp

[3] Email address: fujii@th.phys.titech.ac.jp

where $f(r) = k - 2M/r + \rho^2/4r^2 + r^2/L^2$, k is the curvature of the 2-d Einstein subspace, and M, ρ, and r_+ are the mass, the charge, and the horizon radius of the black hole, respectively. The holographic model of superconductor has a planar horizon, and hence $k = 0$ in this case. Although we considered in [6] all the cases of k, we thus focus here on the case of $k = 0$.

The boundary condition for the quasi-normal mode of the scalar field ψ is imposed such that ψ is purely in-going near the horizon. By writing as

$$\psi = e^{-i\omega t}\,(1-u)^{-i\omega/2\kappa}\,\Psi(u), \tag{3}$$

where u is the new radial coordinate defined by $u \equiv r_+/r$ and κ is the surface gravity, this boundary condition is found to require that the new variable $\Psi(u)$ is regular on the horizon. On the other hand, we impose the Dirichlet boundary condition $\psi \to 0$ at infinity. Then, the scalar field equation derived from Eq. (1) yields the asymptotic form of $\Psi(u)$ as $\Psi(u) \to u^{\Delta_+}$, where $\Delta_+ \equiv 3/2 + \sqrt{9/4 + m^2L^2}$.

We now consider the perturbation *of* the RN-AdS black hole near T_c. We perturb the horizon radius r_+, the surface gravity κ, the metric function $f(u)$, and the gauge field $\Phi(u)$ (the latter two now being considered as functions of u), as

$$r_+ = r_c + \delta r_+, \qquad \kappa = \kappa_c + \delta\kappa, \qquad f(u) = f_c(u) + \delta f(u), \qquad \Phi(u) = \Phi_c(u) + \delta\Phi(u), \tag{4}$$

where the subscript c stands for quantities at T_c. We also note that the quasi-normal frequency ω approaches zero when $T = \kappa/2\pi \to T_c$, according to the critical slowing down [7]. Since we have the marginally stable solution $\Psi_0(u)$ at T_c, we can write $\Psi(u)$ as

$$\Psi(u) = \Psi_0(u) + \omega\Psi_1(u), \tag{5}$$

where it is understood that ω is also small near T_c. Then, the leading and the next-to-leading order terms of the scalar field equation derived from Eq. (1) are written as

$$\mathcal{D}\,\Psi_0(u) = 0, \qquad \omega\,\mathcal{D}\,\Psi_1(u) = j_1(u) + \omega\, j_2(u), \tag{6}$$

where $\mathcal{D}$ is a Sturm-Liouville differential operator, and the source terms (the right-hand side) in the second equation are divided into those which do not depend on ω (the first term) and those which are linear in ω (the second term). Under the boundary condition we imposed above, the differential operator $\mathcal{D}$ is shown to be Hermitian. Thus, by considering the inner product of $\omega\,\mathcal{D}\,\Psi_1(u)$ and $\Psi_0(u)$, and employing Eq. (6), we obtain

$$\omega = -\left(\int_0^1 j_1(u)\,\Psi_0(u)\,du\right) \Bigg/ \left(\int_0^1 j_2(u)\,\Psi_0(u)\,du\right). \tag{7}$$

In order to evaluate the frequency of the quasi-normal mode, we substitute the numerical solution of $\Psi_0(u)$ into Eq. (7), and perform the integration in Eq. (7) numerically. The imaginary part ω_I of (7) for $\rho L = 60$, normalized by $-r_c^2/\delta r_+$, is shown in the left panel of Figure 1. We see that $\omega_I < 0$ for $\delta r_+ > 0$, and $\omega_I > 0$ for $\delta r_+ < 0$. Since we have $\delta r_+(T - T_c) > 0$ for ρ fixed, we see that the RN-AdS black holes with $T > T_c$ are dynamically stable, while those with $T < T_c$ are dynamically unstable.

3 Entropy

Now we consider the perturbative solution of the hairy black hole near T_c, and compare its entropy with that of the RN-AdS black hole in the micro-canonical ensemble. To do so, we take the ansatz

$$ds^2 = -g(y)d\tau^2 + \frac{dy^2}{g(y)} + R^2(y)d\Omega_2^2, \qquad A_\mu dx^\mu = \phi(y)d\tau, \qquad \psi = \psi(y), \tag{8}$$

where the radial coordinate y is related to r as $y = r/r_c$ for the background RN-AdS solution at T_c, and we consider the perturbation of these field variables near T_c as

$$\psi(y) = \epsilon^{1/2}\psi_1(y) + \cdots, \; g(y) = g_c(y) + \epsilon g_1(y) + \cdots, \; R(y) = r_c y + \epsilon R_1(y) + \cdots, \; \phi(y) = \phi_c(y) + \epsilon\phi_1(y) + \cdots,$$

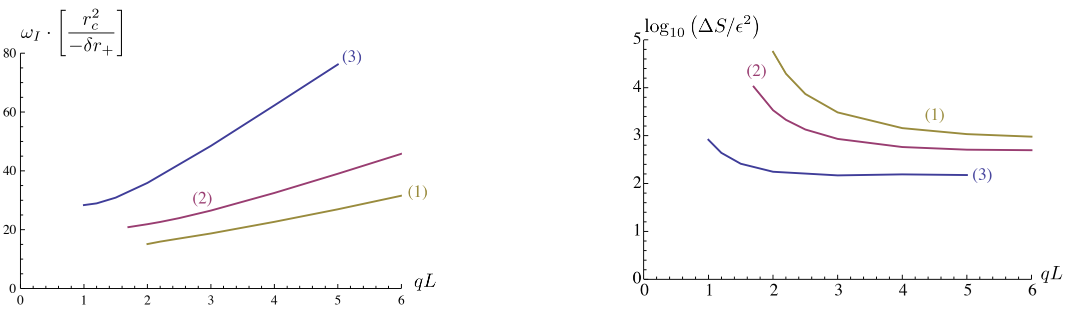

Figure 1: The imaginary part ω_I of the quasi-normal frequency is shown in the left panel, and the difference of the entropy ΔS is shown in the right panel, for $k = 0$ and $\rho L = 60$. The line (1) represents the result for $m^2L^2 = 7/4$, the line (2) for $m^2L^2 = 0$, and the line (3) for $m^2L^2 = -2$.

as well as the perturbation of the temperature T as

$$T = T_c + \epsilon T_1 + \cdots , \tag{9}$$

where the subscript c stands for quantities at T_c. Thus, the small parameter ϵ measures the deviation from the phase transition point ($T = T_c$). The field equations derived from Eq. (1) are then written in terms of the perturbed variables as

$$\frac{\left(y^2 g_c \psi_1'\right)'}{y^2} = \left(m^2 - \frac{q^2\phi_c^2}{g_c}\right)\psi_1,\ \frac{\left(y^2 \phi_1'\right)'}{y^2} = \frac{2q^2\phi_c\psi_1^2}{g_c} - \frac{2}{r_c}\left(\frac{R_1'}{y} - \frac{R_1}{y^2}\right)\phi_c',\ R_1'' = -\frac{r_c y}{2}\left(\frac{q^2\phi_c^2\psi_1^2}{g_c^2} + \psi_1'^2\right),$$
$$\frac{(yg_1)'}{y^2} = -\frac{1}{2}\phi_c'\phi_1' - \frac{m^2}{2}\psi_1^2 + \frac{1}{2}g_c\psi_1'^2 + \frac{q^2}{2g_c}\phi_c^2\psi_1^2 - \frac{1}{r_c}\left(g_c' + \frac{2g_c}{y}\right)\left(\frac{R_1'}{y} - \frac{R_1}{y^2}\right)g_c' - \frac{2kR_1}{r_c^3 y^3}.$$

From these forms of the field equations, we see that once we have the solution of ψ_1 at hand, other perturbed field variables are given by quadratures, and hence integration constants in those quadratures are determined from the asymptotic behavior of these variables. Since we are concerned here with the micro-canonical ensemble, we require that these variables behave asymptotically such that the mass M and the charge ρ of the hairy black hole ($\psi_1 \neq 0$) are the same as those of the RN-AdS black hole ($\psi_1 = 0$) not only at T_c but also away from T_c. In what follows, we shall fix the charge ρ of the black hole, and set ϵ in terms of the mass M as

$$\epsilon = \frac{M_c - M}{M_c}, \tag{10}$$

where M_c is the mass at T_c.

Then, we apply the first law of black hole thermodynamics with $\rho =$ const., i.e., $\delta S = \delta M/T$, in order to obtain the entropies of the hairy black hole and the RN-AdS black hole. By using Eqs. (9) and (10), and integrating the first law, we obtain

$$S(\epsilon) = S_c - \int_0^\epsilon \frac{M_c}{T} d\epsilon = S_c - \frac{M_c}{T_c}\epsilon + \frac{1}{2}\frac{M_c T_1}{T_c^2}\epsilon^2 + \cdots , \tag{11}$$

where $S(\epsilon)$ is the entropy of the black hole with the mass M defined by Eq. (10) in terms of ϵ, and S_c is the entropy at T_c. Note that the hairy black hole and the RN-AdS black hole with the same M have the same value of ϵ. Therefore, the difference $\Delta S \equiv S_{\text{hairy}} - S_{\text{RNAdS}}$ between the entropy S_{hairy} of the hairy black hole and S_{RNAdS} of the RN-AdS black hole arises at order of ϵ^2 as

$$\Delta S = \frac{1}{2}\frac{M_c \Delta T_1}{T_c^2}\epsilon^2, \tag{12}$$

where ΔT_1 is the difference of the temperature perturbation T_1 between the hairy and the RN-AdS black holes, which is expressed as

$$\frac{\Delta T_1}{T_c} = \frac{\psi_1^2(1)}{4} + \frac{1}{2}\int_1^\infty \left(\frac{y g_c'\psi_1'}{g_c} - \frac{m^2\psi_1}{g_c}\right)\psi_1 dy + \frac{1}{L^2 g_c'(1)}\int_1^\infty \frac{y(y-1)}{g_c}\left(\frac{kL^2}{r_c^2} + 3 + y(2+y)\right)\frac{q^2\phi_c^2\psi_1^2}{g_c}dy.$$

The behavior of ΔS for $\rho L = 60$, normalized by ϵ^2, is shown in the right panel of Figure 1. We see that the entropy of the hairy black hole is larger than that of the RN-AdS black hole.

4 Conclusion and discussion

We first investigated the quasi-normal mode of the scalar field on the background of the RN-AdS black hole. We found that the imaginary part of the quasi-normal mode frequency changes its sign at T_c, and it is positive when the temperature of the black hole is smaller than T_c, which indicates that the RN-AdS black holes are dynamically unstable below T_c. We also found that the hairy black hole with the same mass and charge as the RN-AdS black hole possesses larger entropy than the RN-AdS black hole. These results support the scenario underlying the holographic model of superconductor, where the phase transition is expected to occur at T_c, and the RN-AdS black holes below T_c evolve towards the hairy black holes. Although we presented here the numerical results only for the case where the 2-d Einstein subspace has planar symmetry ($k = 0$), the same instability has been found [6] to occur also in the cases of $k = 1$ and $k = -1$, i.e., without depending on the topology of the 2-d Einstein subspace.

It is worthwhile mentioning the aspect of this instability from the viewpoint of the bulk spacetime. The electric charge flux $\mathcal{F}$ through the horizon due to the instability and the charge density σ outside the hairy black hole are computed near T_c as

$$\mathcal{F} \equiv j_a k^a|_{r=r_c} = 4\,\omega_I \rho\, q^2 e^{2\omega_I t} \int_0^1 \frac{(1-u)\Psi_0^2(u)}{u^4 f_c(u)} du, \qquad \sigma \equiv j_a n^a = 2\,\epsilon\,\rho\,q^2 \frac{(1-y^{-1})}{r_c^2\sqrt{g_c(y)}} \psi_1^2(y), \tag{13}$$

where j_a is the electric current density carried by the scalar field, k^a is the future-directed null Killing vector tangent to the horizon, and n^a is the past-directed unit normal to the $\tau =$ const. hypersurface. We thus see that both $\mathcal{F}$ and σ have the same sign as ρ below T_c, which implies that the charge of the same sign as the black hole charge ρ has been extracted from the black hole. We also note that the energy-momentum tensor of the scalar field alone is *not* conserved, because it is interacting with the gauge field, as well as gravity. Although we fixed the gauge field as the background in deriving the quasi-normal mode, the back reaction of the scalar field onto the gauge field, which is at higher order in the present perturbation analysis, transfers energy of the gauge field to the scalar field. Thus, the scalar field gains energy from the gauge field, and drops into the black hole the electric charge of the sign opposite to that of the black hole.

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