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Asymptotically Safe Quantum Field Theories in Perturbative Regime and Beyond

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Abstract

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by Vedran SKRINJAR

The range of topics covered in this thesis reflects the extent to which the tree of Asymptotic Safety research has branched out in recent years. The thesis consists of two parts, one dealing with pure particle models and one adding gravity to the mix. Having roots in a 1996 paper by Reuter, Asymptotic Safety was for a long time thought of as "only" a promising approach to quantum gravity. In 2014 an important new branch was grafted to the tree when Litim and Sannino discovered first examples of asymptotically safe quantum field theories structurally similar to the Standard Model. In the first part of the thesis we work in perturbation theory and with instanton calculus to extend the work of Litim and Sannino. The extension goes in two directions. A more formal one studies the instanton effects in their model. The other, more pragmatic one, seeks asymptotically safe minimal extensions of the Standard Model. Despite being historically first, asymptotic safety in quantum gravity context is studied after pure particle models here. The reasons are that quantum gravity is a logical extension of "beyond the Standard Model" particle physics, and that it relies heavily on the less-known functional renormalization group. We will show that a generic matter-gravity fixed point of the renormalization group flow always contains gravity-induced non-minimal matter-gravity interactions. This computation will also demonstrate the practical necessity of dealing with background independence in Asymptotic Safety, which is the final topic that we address.

To Antonela,

for love, support and infinite patience

To mom,

for teaching me about strength and independence

To grandma,

for trying to understand quantum mechanics
despite having only five years of formal education,

gods know I didn't do much better....

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Contents

List of Abbreviations

BFA	background field approximation
BFM	background field method
BSM	beyond Standard Model *(referring to UV completions of SM)*
CORE	conformally reduced *(gravity model)*
EAA	effective average action
EFT	effective field theory
ERGE	exact renormalization group equation
FRG	functional renormalization group
GR	General (Theory of) Relativity
IR	infrared
LHS	left hand side
LO	leading order
msWI	modified split Ward identities
msWWI	modified split-Weyl Ward identities
NLO	next-to-leading order *(analogously for NNLO)*
QFT	quantum field theory
RG	renormalization group
RHS	right hand side
SM	Standard Model *(i.e. Glashow-Salam-Weinberg model)*
s.t.	such that
UV	ultraviolet
VEV	vacuum expectation value
wrt	with respect to

$\int_x$	stands for 4d integral $\int d^4x$ in either Lorentzian or Euclidean spacetime
$F^2 \equiv F^i_{\mu\nu} F_i^{\mu\nu}$	trace of square of $SU(N)$ stress-energy tensor $F = dA + A \wedge A$
$G^2 \equiv G^i_{\mu\nu} G_i^{\mu\nu}$	trace of square of $SU(N)$ stress-energy tensor $G = dA + gA \wedge A$
$\frac{d}{dt}$	Derivative wrt logarithm of the RG scale ("RG time"). Also denoted by a dot.

Chapter 1

Introduction

Asymptotic Safety

The central idea and a leitmotif of this thesis is a property of some quantum field theories (QFTs) called "asymptotic safety". A QFT is "asymptotically safe" if it is complete in the Wilsonian sense, meaning that its renormalization group flow reaches an ultraviolet fixed point. The more familiar notion of asymptotic freedom corresponds to one possible realization of the asymptotic safety, namely one in which a non-interacting fixed point is reached. The notion of interacting fixed points, or conformal field theories, is well known both in statistical physics (for example Wilson-Fisher fixed point) and in high energy particle physics (for example Banks-Zaks fixed point). The main novel ingredient that the asymptotic safety introduces is the idea that the interacting fixed point is reached in the high energy or ultraviolet (UV) regime, and not in the low energy or infrared (IR) regime.

With the advent of String Theory, and to a lesser extent other quantum gravity theories such as Loop Quantum Gravity, it has become a rather common opinion that fundamental physics may not be described within a QFT formalism[1]. This line of reasoning often argues that some kind of fundamental non-locality is necessary to tame the UV divergences in fundamental physical theories. Such non-locality is then presumed to be achieved by adopting extended fundamental degrees of freedom, as opposed to "pointlike" quantum fields. The fact that the QCD is perfectly well defined at all energy scales is usually disregarded; possibly because it is fundamental on its own, but it has to be interpreted as an EFT when included in the full Standard Model.

The asymptotic Safety program may be defined as a research program that is trying to formulate fundamental particle physics, including gravity, as an asymptotically safe quantum field theory. The main conjecture or premise of the Asymptotic Safety program is that the Standard Model[2] coupled to gravity reaches a UV fixed point and in this way avoids the Landau poles. Consequently, it is usually presumed

[1] The sentiment changed with the development of AdS/CFT, but this discussion is complex and beyond our scope.

[2] Appropriately extended to take care of matter-antimatter asymmetry, bariogenesys, neutrino masses, dark matter, etc.

that quantum fields may be considered fundamental degrees of freedom and "more fundamental" degrees of freedom are usually not invoked[3].

Until a couple of years ago the research within the Asymptotic Safety program was strongly focused on the problem of quantum gravity. The reason for this was that there were no known particle physics models at the time which featured interacting UV fixed points. This all changed in 2014 with a paper by Litim and Sannino in which they introduced for the first time an asymptotically safe gauge theory which we will refer to as LISA in this thesis. LISA has an advantage over the rest of the Asymptotic Safety program which is that the fixed point found in this model is rigorously established. This level of confidence is achieved because LISA is formulated in the controllable Veneziano limit of the standard perturbation theory. It has therefore set up a paradigm for many subsequent high energy theory and phenomenology developments. The first part of this thesis will therefore be centered on this new, pure-matter approach to the notion of asymptotic safety. We will make use of standard tools, s.a. perturbation theory and instanton calculus, to try to pave the way for future "beyond Standard Model" (BSM) phenomenology and model building within the asymptotic safety paradigm.

The second part of the thesis will focus on the inclusion of gravity in the asymptotic safety paradigm. Here we will be dealing with pure gravity, but also with some steps towards building a unified view of high energy physics by putting matter and gravity into one consistent formalism. Both main sections in this chapter are dealing with some of the open questions in contemporary Asymptotic Safety research. An important difference with respect to the first part of the thesis is the fact that once one includes gravity in this formulation, one has to leave perturbation theory behind. In fact, both main sections of the second part of the thesis rely heavily on a non-perturbative technique known as the functional renormalization group (FRG). Since this set of techniques is not familiar to a wide range of particle physicists we will dedicate a whole section to introducing the FRG in detail.

Organization of the Thesis

This thesis is structured around four research papers [1]–[4] which are, on first sight, studying somewhat disjoint topics. The sections in chapter 2 are based around research papers exploring instanton effects in LISA and studying some possible Standard Model extensions within the asymptotic safety paradigm. The research papers at the center of chapter 3 are exploring the implications of gravity-matter interactions for the existence of a combined UV fixed point and potential solutions of Ward identities related to background independence in quantum gravity. While the main underlying idea connecting all these sections is the asymptotic safety conjecture there are deeper connections with practical consequences as we now explain.

[3]It is clear that asymptotic safety is a property of the renormalization group flow, and as such it is compatible with other degrees of freedom besides quantum fields. (e.g. lattice, tensor models, etc.)

We begin in section 2.1 with a review of the Litim-Sannino model. In section 2.2 we show that instantons are infrared objects in LISA, just as they are in QCD. In particular, we study mass-deformed LISA because it runs to (large N_c) pure YM theory in the IR instead of running to a Gaussian fixed point like massless LISA. Our computations show that the vacuum of the theory is well described by the standard instanton fluid model and whereas the expectation value of the instanton scale is lower than common fermion mass, the coupling value at instanton scale is very perturbative due to matching conditions to LISA. First implication of this statement is that mass-deformed LISA is a very good framework for studying instanton physics because it allows for unusually good perturbative control. Second implication is that, since instantons are IR objects, the fixed point originally found in standard perturbation theory is stable under the inclusion of instanton effects.

Having established with confidence that asymptotic safety may be realized in a complex gauge theory structurally similar to Standard Model and consisting of all known kinds of matter (to wit scalars, vectors and spinors), we proceed in section 2.3 to relax the assumptions made in LISA. To be as conservative and as simple as possible we have decided to work with well-known beta functions for the Standard Model augmented by a minimalistic BSM sector. The BSM sector is modeled on LISA and it consists of a new family of vectorlike fermions minimally coupled to the Standard Model and a set of non-gauged scalars required to write Yukawa interactions between the BSM fermions. A subset of these models is also interesting because it contains a dark matter candidate. Some of the lessons of this section are that the physical fixed points may not appear in the regime of validity of perturbation theory and that the hypercharge may be a source of problem for successful model building within the asymptotic safety scenario.

Both of these lessons are relevant for the second part of the thesis, contained in chapter 3. First of all, there is a recent finding that gravity may have beneficial effects on the renormalization group flow of the hypercharge. We do not pretend that this is the only, or even the most important reason for quantizing gravity, but consider it a small example of how lack of a good quantum theory of gravity is still a pressing issue in fundamental particle physics. If we study the RG flow of Newton's constant in perturbation theory we find that it grows with energy scale and extrapolation suggests that it becomes non-perturbative. This reinforces the point made by the second lesson above. It is not improbable that the physical fixed point lies just outside the scope of the standard perturbation theory. We should, therefore, include non-perturbative techniques in our toolkit. Section 3.1 describes a technique, called functional renormalization group (FRG), which is particularly suitable for studying such problems. In fact, FRG is an analytic non-perturbative technique which is perfectly suited for analysis of the semi-perturbative fixed points of the sort that we often find in section 2.3. FRG is introduced in chapter 3 primarily for the purposes of studying pure-gravity and matter-gravity interactions in the high-energy regime.

In section 3.2 we describe a common gravity-matter fixed point in a particular truncation of the FRG and we show that a general structure of such fixed points is one at which both matter and gravity are interacting. In particular, it is clear from this analysis that the question of background independence is an important one even for practical computations in quantum gravity, and this question is then further addressed in section 3.3.

We hope it is evident from this brief overview that the current Asymptotic Safety research has branched out in many different directions and that they are all different facets of the same underlying idea - that fundamental particle physics (including gravity) may be described by a consistent quantum field theory. This thesis accepts the multifaceted state of the body of research on Asymptotic Safety and it tries to bring together a number of topics which may be superficially considered disjoint but which at the same time share deep connections.

List of Publications

The work presented in this thesis is based on the following publications:

- C. M. Nieto, R. Percacci, V. Skrinjar, "Split Weyl Transformations in Quantum Gravity", Phys. Rev., vol. D96, no. 10, p. 106 019, 2017., arXiv: 1708.09760[gr-qc]

- A. Eichhorn, S. Lippoldt, V. Skrinjar, "Nonminimal Hints for Asymptotic Safety", Phys. Rev., vol. D97, no. 2, p. 026 002, 2018., arXiv: 1710.03005[hep-th]

- F. Sannino, V. Skrinjar, "Safe and Free Instantons", arXiv: 1802.10372[hep-th]

- D. Barducci, M. Fabbrichesi, C. M. Nieto, R. Percacci, V. Skrinjar,"In search of a UV completion of the standard model - 378.000 models that don't work ", arXiv:1807.05584[hep-ph]

Chapter 2

Asymptotic Safety in Particle Physics

We have mentioned in the Introduction that asymptotic safety is a concept intimately related to the notion of asymptotic freedom [5], [6] and that one could think of asymptotic freedom as one particular realization of asymptotic safety. An asymptotically safe quantum field theory (QFT) is a QFT that reaches a fixed point of the renormalization group (RG) flow in the deep UV. Such a theory is fundamental according to Wilson [7], [8] because it is well defined at all energy scales, even as the energy of the processes is taken to infinity. The reason for this is that, by definition, the couplings in the action go to finite, "fixed point" values in the deep UV. This is precisely the kind of behavior that one finds in QCD. In particular, the gauge coupling in QCD vanishes logarithmically as energies are taken to infinity (provided the theory does not contain too many fermions). All observables in QCD are thus well defined and the theory does not suffer from a Landau pole. The notion of asymptotic safety is a simple generalization of this idea, and it merely assumes that the fixed point that one reaches in the deep UV is not a Gaussian (non-interacting) one, but a fully interacting quantum field theory with non-trivial interactions at all scales.

Surprisingly enough, the concept of asymptotic safety first appeared in high energy physics in the context of UV completions of Einstein's gravity [9], the idea being that Newton's coupling (and possibly the comoslogical constant) flow to finite values in the deep UV. Such behavior would allow gravity to be quantized as a quantum field theory. Quantization of gravity and its interactions with matter along these lines will be discussed in much more detail in Chapter 3. The focus of this chapter is on the scope of the asymptotic safety paradigm for building phenomenologically viable UV completions of the Standard Model. We will begin by introducing a model, dubbed LISA, which introduced the paradigm to the particle physics community in 2014.

2.1 Introduction

LISA

In a 2014 paper [10] Litim and Sannino introduced a gauge theory that they showed to be asymptotically safe. The Lagrangian of the theory consists of an $SU(N_c)$ gauge field coupled to N_f vector-like fermions and a (non-gauged) scalar field. All three kinds of fields are shown to be required for the existence of a UV fixed point. The reasoning goes as follows. One loop beta function of the gauge coupling in QCD with N_c colors and N_f vectorlike fermions takes form,

$$\beta_g = -B\alpha_g^2, \tag{2.1}$$

where α_g is related to the gauge coupling g by,

$$\alpha_g = \frac{g^2 N_c}{(4\pi)^2}. \tag{2.2}$$

The model is asymptotically free provided the one loop coefficient, B, is positive. If B is negative then the theory has a Gaussian fixed point in the IR, and the coupling grows without bound in the UV. Let us now consider the two loop beta function β_g. It takes form

$$\beta_g = -B\alpha_g^2 + C\alpha_g^3. \tag{2.3}$$

The right hand side of the two loop beta function clearly vanishes for

$$\alpha_g^* = B/C\,, \tag{2.4}$$

in addition to the Gaussian FP at $\alpha_g = 0$. If $B > 0$ and $C > 0$ the non-Gaussian fixed point α_g^* of the RG flow is reached in the IR and is known as the Banks-Zaks fixed point. It describes a theory which is asymptotically free in the UV, but which flows to an interacting conformal field theory in the IR. The domain of parameters N_f and N_c for which this dynamics exists is known as QCD conformal window. For N_f and N_c such that

$$\epsilon = N_f/N_c - 11/2 \tag{2.5}$$

satisfies $|\epsilon| << 1$ and $sign(\epsilon) = -1$ the boundary of the conformal window can be rigorously established because

$$B = -4\epsilon/3 \;\;\&\;\; C = 25 + 26\epsilon/23, \tag{2.6}$$

so the fixed point is completely perturbative, $\alpha_g^* << 1$. The non-Gaussian fixed point $\alpha_g^* = B/C$ would be a UV fixed point if there was a way to flip the sign of both B and C coefficients. The sign of B may be changed by requiring ϵ to have a positive sign, which is achievable by taking N_f sufficiently larger than N_c so as to barely break asymptotic freedom. The problem comes from C whose sign cannot be changed in QCD alone.

Litim and Sannino have shown in [10] that one can effectively change the sign of C by coupling the theory to scalars. In fact, Yukawa couplings enter the two loop beta function for the gauge field with opposite sign of the cubic term which is precisely the kind of contribution needed to effectively render C negative. The system at this next-to-leading order (NLO) is described by the two loop gauge beta function, and one loop Yukawa beta function which take the following form [10],

$$\beta_g = \alpha_g^2 \left[\frac{4}{3}\epsilon + (25 + \frac{26}{3}\epsilon)\alpha_g - 2(\frac{11}{2} + \epsilon)^2 \alpha_y\right], \tag{2.7}$$

$$\beta_y = \alpha_y \left[(13 + 2\epsilon)\alpha_y - 6\alpha_g\right], \tag{2.8}$$

where

$$\alpha_y = y^2 N_c / (4\pi)^2. \tag{2.9}$$

Besides the Gaussian fixed point this model also admits a non-Gaussian (i.e. interacting) UV fixed point at,

$$(\alpha_g^*, \alpha_y^*) = (\frac{26}{57}\epsilon, \frac{12}{57}\epsilon), \tag{2.10}$$

to order $\mathcal{O}(\epsilon^2)$. This is the central result of the Litim-Sannino paper.

This result shows that one could imagine a UV completion of the Standard Model to be a non-trivial quantum field theory based on scalars, spinors and gauge fields. It is a counter-example to two widely held beliefs which we already mentioned in 1: 1) the UV completion of the Standard Model requires new degrees of freedom beyond quantum fields, e.g. strings; and 2) the QFTs which are sufficiently complex to be phenomenologically relevant always fall into the category of EFTs.

Schemes

Let us discuss the nomenclature and the motivation for the perturbation theory ordering in (2.7)-(2.8). Leading order (LO) was defined as having one-loop gauge beta function, and no Yukawa or scalar beta functions ("zero-loop"). We call this 1-0-0 scheme, where the nomenclature is s.t. the n-l-m scheme would consist of n-loop gauge, l-loop Yukawa, and m-loop scalar beta functions. Next-to-leading order (NLO) corresponds to the 2-1-0 scheme which refers to using two-loop gauge running, one-loop Yukawa running, and ignoring the running of the scalar couplings. This scheme will be used in both of the following sections, 2.2 and 2.3. Finally, [10] contains a check of the stability of the UV fixed point in the NNLO or 3-2-1 scheme (with naming convention being clear by now). In fact, they find a UV fixed point in the space of gauge, Yukawa and two scalar self-coupling operators. Its location in the gauge-Yukawa subsector is a small perturbation of the NLO result. We will be using this scheme in section 2.3 only.

The reason for working in these schemes, instead of doing naive perturbation theory ordering where all beta functions are taken to the same order is discussed in detail in [10] in the context of LISA, and in [11] in the context of the Standard Model. The core of the argument is that conformal symmetry of the bare action implies so-called Weyl consistency conditions for the quantum theory which relate the coefficients of the beta function at different loop orders [12]–[15]. The schemes proposed in [10], [11], which we also use here, are explicitly constructed to respect the Weyl consistency conditions (and in this sense also the conformal symmetry).

For the sake of completeness we mention that the Weyl consistency conditions read,

$$\frac{\partial \beta^j}{\partial g_i} = \frac{\partial \beta^i}{\partial g_j}\,, \tag{2.11}$$

where $\{g_i\}$ is the set of all couplings in the theory, $\beta_i = dg_i/dlog\mu$ is the beta function of g_i, and most importantly $\beta^i = \chi^{ij}\beta_j$. χ can be thought of as a metric in the coupling space. Its definition comes from a result describing the relation between the derivative of the a-function w.r.t. the couplings and the beta function of the couplings, and in two dimensions it is directly related to the Zamolodchikov metric. The reason why Weyl consistency conditions relate coefficients at different perturbation-theory orders of the beta function is that χ depends on the couplings.

We do not discuss this matter any further in this thesis and refer the reader to the aforementioned references for more details on the topic. In particular the reference [16] contains the explicit form of χ valid for LISA; see also [17].

In the following section we study the role of instantons in LISA. Some of the main motivations for doing so are to check that instantons do not affect the existence of the fixed point and to describe the vacuum structure of LISA. We begin with a quick review of mean field theory approach to instantons in QCD.

2.2 Controllable Instantons

Before discussing the role of instantons in LISA, we start by briefly reviewing the salient points about the instanton calculus for pure Yang Mills (YM) and QCD. Following this introduction we will study mass-deformed LISA in Diakonov and Petrov's famous mean field theory approach to instanton fluid description of the QCD vacuum to determine the density of instantons per unit volume as function of the fermion mass. This quantity is of fundamental importance due to its direct relation with nuclear physics observables. This section is based on [1] which extended the instanton calculus to asymptotically safe theories for the first time.

2.2.1 Instanton calculus review

Steepest Descent Method

In quantum field theory (QFT) one aims at computing the partition function,

$$Z[\mathcal{J}] = \int \mathcal{D}\phi \, e^{\mathrm{i}S[\phi;\lambda]+\mathcal{J}\phi} \,, \tag{2.12}$$

where $S[\phi;\lambda]$ is the sum of a classical action, a gauge-fixing action and a ghost action, depending on the fields ϕ and the couplings λ, and $\mathcal{J}$ is a source for ϕ. If the action is non-integrable one usually attempts to solve the problem through perturbation theory which amounts to expanding the action in powers of small coupling constants λ. Solutions of the classical theory corresponding to $S[\phi;\lambda]$ are specific classical field configurations $\bar{\phi}$. Since the first variation of the action vanishes on these configurations they represent stationary points, or extrema, of the action. The integrand on the right hand side (RHS) of (2.12) is clearly an oscillating function, and thus one may attempt to evaluate the integral by performing an expansion around the classical solution $\bar{\phi}$. Symbolically, we have

$$Z[\mathcal{J}] = \int \mathcal{D}\phi \, e^{\mathrm{i}\left[S[\bar{\phi}]+\frac{1}{2}\phi S^{(2)}[\bar{\phi}]\phi+\mathcal{O}(\phi^3)\right]+\mathcal{J}\phi} \,. \tag{2.13}$$

This is the core of the steepest descent method for approximating the value of complex integrals. The vacuum solution is defined as the classical configuration that minimizes the energy functional - the Hamiltonian. In the case of (comparatively) simple QFTs there is just one vacuum state and thus there is but a single field configuration $\bar{\phi}$ around which one should expand the partition function. This is precisely the situation described by equation (2.13).

For Yang-Mills (YM) theories, often coupled to scalars or fermions, and occasionally coupled to gravity, the vacuum structure is more complicated and if one would naively apply the above prescription several important phenomena would be unaccounted for, such as a deeper understanding of chiral symmetry breaking, the generation of the eta prime mass in QCD, etc.

Let us therefore briefly review the correct approach applicable to a generic QFT [18]–[22]. We begin by Euclideanizing the QFT by performing the Wick rotation. The Wick rotation is a complicated and deep subject, but for our purposes it simply reduces to "rotating" the time coordinate: $t \to \tau = -\mathrm{i}t$. Despite its conceptual simplicity in absence of gravity, there are still some technicalities to which one has to pay attention, such as treating gauge fields and fermions with care during the Euclideanization. The Euclidean action S_E is a functional of Euclidean fields $\phi_E(x)$ living on a 4D Euclidean space described by coordinates $x = (x_1, x_2, x_3, \tau)$. When solving the equations of motion one has to set up boundary conditions for $|x| \to \infty$ such that the action remains finite. Usually our conditions require $\phi \to$ const for $|x| \to \infty$. If the potential has only one extremum there is going to be a single vacuum solution (constant field configuration in all of the space) and therefore the naive perturbation theory described by (2.13) is valid. If, however, the potential has more than one degenerate vacuum, then there exist classical solutions interpolating between these Euclidean vacuua. These finite-action topologically-stable solutions to classical Euclidean equations of motion are called instantons or pseudoparticles [23], [24]. Instantons are topologically stable since going from one such field configuration to another would require bridging an infinite action barrier.

The correct application of the steepest descent method to the Euclideanized version of (2.13) involves a summation over all the instanton configurations. Even though one does not find instantons as classical solutions to Lorentzian equations of motion, it is clear that the Lorentzian partition function can be obtained by Wick rotating the Euclidean partition function, and thus instantons have to be incorporated in the Lorentzian computation. Being interpreted as fields that interpolate between different vacuua, instantons are crucial for understanding a rich vacuum structure in YM theories.

From $SU(2)$ to $SU(N)$

The $SU(2)$ color group plays a special role in instanton physics since $SU(N)$ instantons can be obtained from the $SU(2)$ instantons [18], [25]. Consider a Euclidean YM action,

$$S[A] = \frac{1}{4}\int_x G_a^{\mu\nu}(A)G_{a\mu\nu}(A) \tag{2.14}$$

where A_μ^a is the gauge field associated with the $SU(2)$ group. To "find" instantons we look for gauge configurations for which the action is bounded, but rather than asking for $A_\mu^a(x)$ to decay faster than $1/x$ for $|x| \to \infty$, we require it to become pure gauge,

$$A_\mu \xrightarrow{|x|\to\infty} \mathrm{i}S\partial_\mu S^\dagger \ , \tag{2.15}$$

where S are $SU(2)$ matrices (not to be confused with the action) that depend on angles only. $SU(2)$ instantons are related to maps from $SU(2)$ to itself. Such maps are classified by the third homotopy group and they fall into topologically distinct classes. In the case of $SU(2)$ these are labelled by integer numbers, and members from different classes cannot be continuously mapped into each other [1]. Continuous gauge transformations that are defined on all of $\mathbb{R}^4$ necessarily have winding number zero at infinity. Such transformations preserve the topological invariant of the instanton.

The integers labelling distinct topological classes of instantons can be thought of as topological charges. Furthermore, for a given instanton configuration the topological charge is given by,

[1] One can think of a class label as a winding number saying how many times a map winds around the target sphere.

$$n = \frac{g^2}{32\pi^2} \int_x G_a^{\mu\nu} \tilde{G}_{a\mu\nu}, \qquad n \in \mathbb{Z} \tag{2.16}$$

where g is the gauge coupling. One can complete the square in the action as follows (suppressing indices),

$$S = \frac{1}{4} \int_x GG = \int_x \frac{1}{4} G\tilde{G} + \frac{1}{8}(G - \tilde{G})^2 = n\frac{8\pi^2}{g^2} + \frac{1}{8} \int_x (G - \tilde{G})^2 \tag{2.17}$$

The value of the action for the instanton of topological charge n thus equals:

$$S|_{\text{n-instanton}} = n\frac{8\pi^2}{g^2}. \tag{2.18}$$

This value is achieved when the field satisfies the self-duality condition, $G = \tilde{G}$. Whereas the equation (2.18) holds true for positive n, negative values of n can be obtained via a parity transformation. Then $G\tilde{G} \to -G\tilde{G}$ and, following the same argument as above, the action reaches the minimum value of $|n|\frac{8\pi^2}{g^2}$ for the field configuration which is anti self-dual, $G = -\tilde{G}$. Such a field configuration is called an anti-instanton.

Using Bianchi identities, one can show that the field satisfying the (anti-)self-duality condition is on-shell, i.e. it automatically satisfies the equations of motion. Computing the value of the action on an instanton configuration constitutes the first important result of the instanton calculus. Starting from the asymptotics (2.15) and assuming the same directional dependence of the solution in all spacetime points one can write an ansatz for the instanton. Requiring absence of singularities at the origin of space and self-duality of the solution suffices to uniquely fix the instanton up to collective coordinates [18]. The result is the famous BPST instanton ($SU(2)$ instanton with charge n=1) [23],

$$A_\mu^a = \frac{2}{g}\eta_{a\mu\nu}\frac{(x - x_0)^\nu}{(x - x_0)^2 + \rho^2} . \tag{2.19}$$

The BPST instanton takes the above explicit form in the so-called regular gauge. The parameter ρ, known as the instanton size, is one of the instanton collective coordinates, the remaining ones being the instanton location x_0, and the orientation in color space. Finally, the symbols mixing the Lorentz and color indices, $\eta_{a\mu\nu}$, are known as 't Hooft symbols [24].

The generalisation to simple Lie algebras is obtained directly from the $SU(2)$ BPST instanton exploiting the fact that any $SU(N)$ group contains $SU(2)$ subgroups. To deduce the $SU(N)$ instantons one simply embeds the BPST solution (2.19) into $SU(N)$. The choice of embedding is free, with the most common choice being the so-called minimal embedding. It consists in taking the $SU(N)$ generators in the fundamental group, and taking the first three generators $T^1, ..., T^3$ to be block-diagonal with $SU(2)$ generators embedded in the upper-left corner. The $SU(N)$ BPST instanton is obtained by contracting the first three generators T^a, $a = 1, 2, 3$ with the

BPST solution (2.19). One can analogously obtain $SU(N)$ instantons with charge $n \neq 1$ from other $SU(2)$ solutions. This simple prescription works because the third homotopy group of $SU(N)$ is $\mathbb{Z}$ for all N, and with the minimal embedding each equivalence class of $SU(N)$ solutions contains a representative $SU(2)$ instanton.

Instanton Ensembles

Our next goal is to construct a partition function for an instanton ensemble. The reason for this is that instanton ensembles play an important role in determining the structure of the QCD vacuum. As a first step we review and discuss some one-instanton partition functions. We begin with the famous result for the one-instanton partition function which was given by 't Hooft in 1976 [24].

The vacuum-to-vacuum transition amplitude in presence of a single instanton is given by the following one-loop instanton calculus result for an $SU(N_c)$ pure Yang-Mills theory [24], [26],

$$\begin{aligned} W^{(1)} &= \frac{4}{\pi^2} \frac{\exp\left(-\alpha(1) - 2(N_c-2)\alpha(1/2)\right)}{(N_c-1)!(N_c-2)!} \times \\ &\times \int d^4x d\rho \rho^{-5} \left(\frac{4\pi^2}{g_0^2}\right)^{2N_c} \exp\left(-\frac{8\pi^2}{g_{1L}^2}\right) \qquad (2.20) \\ &\equiv C_c \int d^4x d\rho \rho^{-5} \left(\frac{8\pi^2}{g_0^2}\right)^{2N_c} \exp\left(-\frac{8\pi^2}{g_{1L}^2}\right) \qquad (2.21) \end{aligned}$$

The expressions on the right hand side follow from a leading order expansion around the BPST instanton followed by the computation of one-loop determinants (normalized to the perturbation theory vacuum). Besides integrating over the eigenvalues of the Laplacians one performs a summation over all collective coordinates. The remaining integral in the above expressions corresponds to the sum over the instanton size ρ and its integrand is referred to as the instanton density [2]. The integral over ρ is left implicit because it is IR divergent due to the running coupling in the exponent. Clearly one has to tame this behavior for the result to be meaningful, and we will elaborate on this shortly.

If the Yang-Mills theory is coupled to N_f Dirac fermions then, at one loop, they contribute to the above result via fermion determinant. Zero and non-zero modes of the fermion determinant should be considered separately. The non-zero modes contribute the following term to the partition function [24],

$$\exp\left[-\frac{2N_f}{3} log(\rho/\rho_0) + 2N_f\alpha(1/2)\right] , \qquad (2.22)$$

[2] Note that the numerical factor C_c depends only on the number of colors and it also contains the factor 2^{-2N_c}.

where the log term is the fermion contribution to the one-loop running of the gauge coupling and $\alpha(x)$ is a numerical function defined in [24][3]. Assuming that the fermions share a common mass m, the zero modes contribute a term

$$(m\rho)^{N_f}. \tag{2.23}$$

We can now generalise the result in (2.21) to include the fermions using the one-loop running of the QCD gauge coupling,

$$\frac{8\pi^2}{g_{1L}^2} = \frac{8\pi^2}{g_0^2} - b\,log(\rho/\rho_0)\,, \quad \text{with} \quad b = \frac{11}{3}N_c - \frac{2}{3}N_f\,, \tag{2.24}$$

and derive

$$W^{(1L)} = \frac{4}{\pi^2}\frac{\exp(-\alpha(1)+4\alpha(1/2))}{(N_c-1)!(N_c-2)!}\exp(2(N_f-N_c)\alpha(1/2))m^{N_f}\left(\frac{4\pi^2}{g_0^2}\right)^{2N_c}\times$$

$$\times \int d^4x d\rho \rho^{-5+N_f}\exp\left(-\frac{8\pi^2}{g_0^2}+\left(\frac{11}{3}N_c-\frac{2}{3}N_f\right)log(\rho/\rho_0)\right) \tag{2.25}$$

$$= C_{cf}\, m^{N_f}\int d^4x d\rho \rho^{-5+N_f}\left(\frac{8\pi^2}{g_0^2}\right)^{2N_c}\exp\left(-\frac{8\pi^2}{g_{1L}^2}\right)\,. \tag{2.26}$$

The zero mode contributions given in (2.23) imply that the one-instanton amplitude vanishes as the common fermion mass is taken to zero. This was noted and thoroughly discussed in [24], see also [27], [28].

Let us now briefly discuss the issue of IR divergences of the one-instanton amplitude. Conceptually, it is reasonable to expect that if QCD forms gluon condensates, then they should be described by a statistical ensemble of the instantons forming them. Since the description of the vacuum in terms of an instanton ensemble is more realistic, we may expect that instanton interactions would “remove” the IR divergence. The early attempts in this direction imagined the QCD vacuum to be described by an instanton gas [22], [29]. This was demonstrated to be a poor description of the physical vacuum, since instantons were much more strongly interacting. The solution came in the form of Shuryak’s instanton liquid model in 1982 [30]. He had shown that a simple model of the instanton medium as a liquid with only two free parameters can effectively explain a number of nuclear physics observables. His model assumes that all instantons have the same size, $\bar{\rho}$, and he obtained the instanton size and the density of the instanton liquid from the empirical value of the gluon condensate. The approach thus doesn’t explain why the instanton density is a delta-like peak around some $\bar{\rho}$, but such a description has predictive power and seems to explain nuclear physics data well.

The above ideas were developed more systematically within the mean field theory by Diakonov and Petrov in 1983 and 1985, aiming at a description of an ensemble of instantons from first principles. Failure of the instanton gas picture had

[3] $\alpha(1/2) = 0.145873$ and $\alpha(1) = 0.443307$

implied that the instanton interactions should be modeled even if the medium itself will turn out to be rather dilute. This was motivated, among other things, by the expectation that the instanton interactions would remove the IR divergence. Therefore, Diakonov and Petrov introduced a modified variational procedure [31] in an attempt to approximate the exact multi-instanton partition function. They applied their method to pure Yang-Mills theory and besides curing the IR problem they also successfully computed a number of physical observables. In a later work [28] the method was extended to include gauged fermions.

Before moving on let us comment on one more issue regarding the master equation (2.26). It follows from the one-loop computation that the coupling in the exponential term is renormalized, but the one in the pre-exponential factor is not. In the literature, this problem is often addressed by recognizing that at two loops the pre-exponential factor gets renormalized [31] and thus one replaces the bare coupling by the one-loop running coupling, and the one-loop coupling by the two-loop coupling. For completeness we also provide the standard result for the two loop running coupling [27], [32],

$$\frac{8\pi^2}{g_{2L}^2} = \frac{8\pi^2}{g_0^2} - b\log\rho/\rho_0 + \frac{b'}{b}\log(1-\frac{g_0^2}{8\pi^2}\log\rho/\rho_0)\,, \tag{2.27}$$

$$b' = \frac{51}{9}N_c^2 - \frac{19}{3}N_f\,. \tag{2.28}$$

Note that the behavior of the coupling given in (2.27) is not the exact two-loop one. In fact, this is only the leading UV contribution valid in the deep UV regime for the asymptotically free phase of QCD [4]. This will be further elaborated when we discuss the application of instanton calculus to LISA.

Mean Field Theory and Large N_c

Pure Yang-Mills theory at large N_c is an important step towards studying instantons in the conformal window as well as asymptotically safe instantons. In fact, many of the formulae derived in this subsection can be adapted to include the effects of fermions in theories such as Banks-Zaks and LISA. We will now briefly outline the variational approach to mean field theory due to Diakonov and Petrov [31] and present the main results. We particularly focus on the large-N_c limit [33].

In absence of the exact results for the vacuum structure of the YM theory one may assume that the pure YM vacuum is given by a background gauge field configuration which can be described as consisting of an ensamble of instantons. Such a background may then be approximated by a simple sum of localized one-instanton

[4]This is clear since the expression (2.27) is manifestly ignorant of the possible existence of a perturbative IR fixed point.

solutions. Starting from this ansatz the ground state can be derived by introducing a modification of Feynman's variational principle.

The modified variational principle is used to approximate a partition function Z,

$$Z = \int \mathrm{D}\phi e^{-S[\phi]} \, . \tag{2.29}$$

It consists in taking the action S, modifying it slightly to get an action S_1 so that S_1 has the minimum on the ansatz field configuration, and then using the inequality

$$\begin{aligned} Z &\equiv \left(\frac{1}{Z_1} \int D\phi e^{-(S[\phi]-S_1[\phi])} e^{-S_1[\phi]} \right) Z_1 \\ &= Z_1 \langle e^{-(S[\phi]-S_1[\phi])} \rangle \geq Z_1 e^{-\langle S-S_1 \rangle} \, , \end{aligned} \tag{2.30}$$

where he expectation values $\langle \, . \, \rangle$ are taken with respect to the measure $\exp(-S_1)$. The new partition function Z_1 is defined analogously to Z, with the action S_1 in place of S. By maximizing the RHS (within the parameter space of our ansatz, see below) we obtain the best estimate on Z.

Let the background field be given by $\bar{A} = \sum_I A_I + \sum_{\bar{I}} A_{\bar{I}}$, where I runs over the instantons and $\bar{I}$ over anti-instantons. The Lagrangian may be rewritten as follows,

$$-\frac{1}{4g^2} F^2(\bar{A}) = -\frac{1}{4g^2} \left(\sum_{i=I,\bar{I}} F^2(A_i) + F^2(\bar{A}) - \sum_{i=I,\bar{I}} F^2(A_i) \right) \tag{2.31}$$

$$\equiv -\frac{1}{4g^2} \left(\sum_{i=I,\bar{I}} F^2(A_i) + U_{int} \right) \, , \tag{2.32}$$

where the first term is the Lagrangian of a non-interacting instanton gas, and the second term describes all n-body interactions in the medium. From here on we use notation $1/4g^2 F^2 = 1/4 G^2$. Including the bosonic statistics factors $N_\pm$ in front of the partition function, normalizing both sides of (2.30) to the perturbation theory vacuum, and regularizing the determinants, at one loop order we obtain the following expression

$$\left. \frac{Z}{Z_{ptb}} \right|_{reg,1L} \geq \frac{1}{N_+! N_-!} \int \prod_i^{N_+ + N_-} d\gamma_i \, d(\rho_i) e^{-\beta(\bar{\rho}) U_{int}(\gamma_i)} \tag{2.33}$$

$$\equiv \frac{1}{N_+! N_-!} \int \prod_i^{N_+ + N_-} d\gamma_i \, e^{-E(\gamma_i)} \, . \tag{2.34}$$

In this expression γ_i represent the collective coordinates of the i-th pseudoparticle (see 2.2.1). $d(\rho)$ stands for the one-instanton density (2.21), and we use the standard notation,

$$\beta(\rho) \equiv 8\pi^2 / g^2(\rho) \, . \tag{2.35}$$

Please remember that in this section $\beta(\rho)$ is the above shorthand for the inverse running coupling, and not the beta function. In the expression (2.33) $\beta(\rho)$ is renormalized by one-loop determinants at a scale $\bar{\rho}$ corresponding to the average instanton size. In the second line, (2.34), we've introduced a compact notation,

$$E(\gamma_i) = \beta(\bar{\rho})U_{int}(\gamma_i) - \sum_i \log d(\rho_i) \, . \tag{2.36}$$

If the medium is sufficiently dilute one can consider only two-particle interactions in the interaction term, all the other ones being subdominant [5]. This is the key physical ingredient beyond the simple instanton gas model. The interaction potential has been determined in [31]. Integrating over the relative angle between two instantons in color space, and integrating over the instanton separation one obtains a remarkably simple expression,

$$U_{int}^{2-body}(\rho_1, \rho_2) = \gamma^2 \rho_1^2 \rho_2^2 \, , \qquad \gamma^2 = \frac{27\pi^2}{4} \frac{N_c}{N_c^2 - 1} \, , \tag{2.37}$$

where $\rho_{1,2}$ are the sizes of the two pseudoparticles, and the coupling γ^2 has the characteristic $1/N_c$ behavior.

We may now use the variational principle. Assuming that the effect of the two-body interactions can be well captured by a modification of the one-instanton densities $d(\rho)$, we write,

$$E_1(\gamma_i) = -\sum_I^{N_+} \log \mu_+(\rho_I) - \sum_{\bar{I}}^{N_-} \log \mu_-(\rho_{\bar{I}}) \, , \tag{2.38}$$

where $\mu_\pm$ are effective densities to be determined through maximization of the RHS of (2.30). To write explicitly the RHS of (2.30) as a function of $\mu_\pm$ we first of all need Z_1. This is obtained by substituting E_1 in place of E in (2.34), which gives us

$$Z_1 = \frac{1}{N_+! N_-!} V^{N_+ + N_-} (\mu_+^0)^{N_+} (\mu_-^0)^{N_-} \, , \tag{2.39}$$

where,

$$\mu_\pm^0 = \int_0^\infty d\rho \; \mu_\pm(\rho) \, . \tag{2.40}$$

Second piece that we need to evaluate (2.30) is $\langle E - E_1 \rangle$. We will express it in terms of

$$\overline{\rho_\pm^2} = \frac{1}{\mu_\pm^0} \int d\rho \; \rho^2 \mu_\pm(\rho) \, . \tag{2.41}$$

[5] In fact first corrections to this computation come not from considering higher order interactions but from considering 2-loop beta functions [31].

Now we can substitute Z_1 and $\langle E - E_1 \rangle$ in the RHS of (2.30) and minimize it wrt $\mu_\pm$. The following computation is not difficult, but it is a bit lengthy so we refer the reader to [31] for some additional steps. There's an arbitrary constant appearing in the optimal $\mu_\pm$, and if these are chosen equal then $\mu_+ = \mu_- \equiv \mu$. Writing $N_+ + N_- = N$, we find the optimal μ to be,

$$\mu(\rho) = d(\rho) \exp\left(-\frac{\beta\gamma^2 N}{V}\overline{\rho^2}\rho^2\right), \tag{2.42}$$

where $\beta \equiv \beta(\bar{\rho}) = 8\pi^2/g^2(\bar{\rho})$. This can be reinserted in (2.41) to give,

$$(\overline{\rho^2})^2 = \frac{\nu}{\beta\gamma^2 N/V}, \quad \nu = \frac{b-4}{2}. \tag{2.43}$$

This expression can be further inserted in the optimal μ, and μ^0 can be easily found using the explicit form of the optimal μ and of the one-instanton density. Finally we can determine the RHS of (2.39), see [31] for more details.

Instead of keeping the number of pseudoparticles N fixed we can work in the grand canonical ensemble. This allows us to find the average number of instantons in the medium by maximizing the RHS of (2.39) as a function of N. For the bosonic factors we set $N_\pm! = (N/2)!$ and use the Stirling approximation. This brings us to the following important expression for the average instanton number,

$$\langle N \rangle = V\Lambda_{YM}^4 \left(\Gamma(\nu) C_{cf} \tilde{\beta}^{2N_c} (\beta\gamma^2\nu)^{-\nu/2}\right)^{\frac{2}{\nu+2}}, \tag{2.44}$$

where $\tilde{\beta} = 8\pi^2/g_0^2$. Note that $\overline{\rho^2}$ enters this equation through $\beta = 8\pi^2/g_{1L}^2(\overline{\rho^2})$, so (2.43) and (2.44) should be solved simultaneously (consistently) for $\langle N \rangle$ and $\overline{\rho^2}$. The importance of the average number of instantons comes from the fact that it is related both to the gluon condensate (see (2.46)), vacuum energy (see (2.48)), topological susceptiblity (see (2.51)), and in a theory with fermions, to the $U(1)$ axial anomaly.

Substituting the optimal effective density μ on the RHS of (2.30), in terms of the number of instantons per unit volume the partition function takes the following simple form,

$$Z = \exp\left[\frac{1}{2}(\nu+1)\langle N \rangle\right]. \tag{2.45}$$

We can solve numerically for the expectation values of the instanton size and of the density of instantons in the vacuum. To do that we need to perform the aforementioned RG improvement by promoting $\tilde{\beta}$ to β and β to $8\pi^2/g_{(2L)}^2(\bar{\rho})$. Note that it is useful to introduce a free parameter a, called a fudge factor, in the log term of the one-loop running coupling (2.24). The fudge factor essentially parametrizes the uncertainty of the actual confining scale Λ_{YM}. The numerical results are shown in Figure 2.1. Even for modest values of N_c shown in the figure, it is evident that the

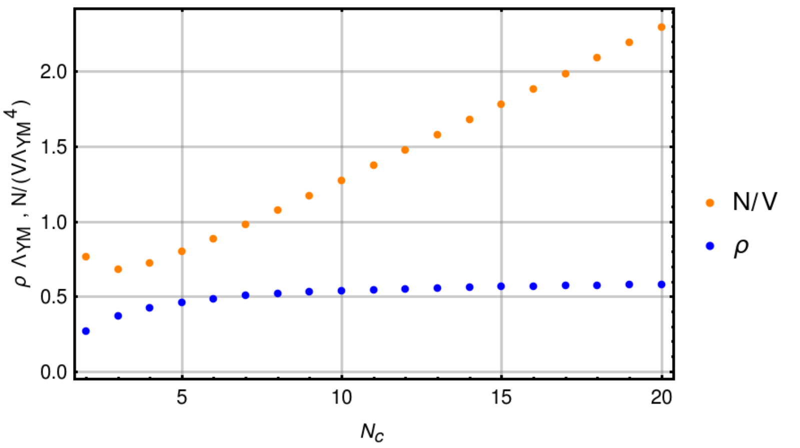

FIGURE 2.1: Instanton size and density of instantons as function of N_c

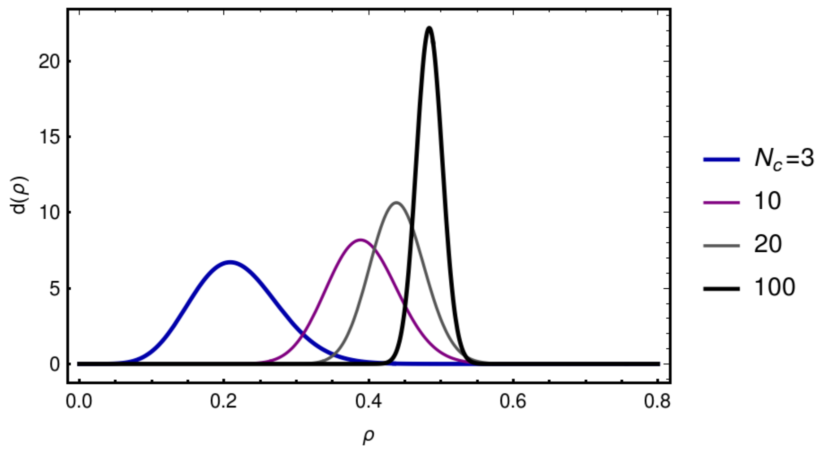

FIGURE 2.2: Effective instanton density profile as a function of N_c. (Normalized to unity.)

density of instantons increases as $\mathcal{O}(N_c)$, whereas the average instanton size is independent of N_c and is always of $\mathcal{O}(1)$.

We can also study the dependence of the effective instanton density $d(\rho)$ on the number of colors N_c. The results are shown in Figure 2.2. Already from (2.20) we know that the amplitude decreases rapidly with N_c, but what we consider here is the shape and the spread of the distribution. (To this end we normalize all distributions to $\mu^0 = 1$.) In particular, we notice that the distribution has a prominent peak centered about the average instanton size and in the large N_c limit becomes delta-like [33].

Recall the relation between the full Lagrangian and the instanton-gas Lagrangian, $F^2 = \sum_i F_i^2 + 32\pi^2 U_{int}$. Since we know the value of the action for the BPST instanton (see (2.18)), we know that

$$\langle \int \frac{d^4x}{32\pi^2} F^2 \rangle = \langle N \rangle + \langle U_{int} \rangle \,. \tag{2.46}$$

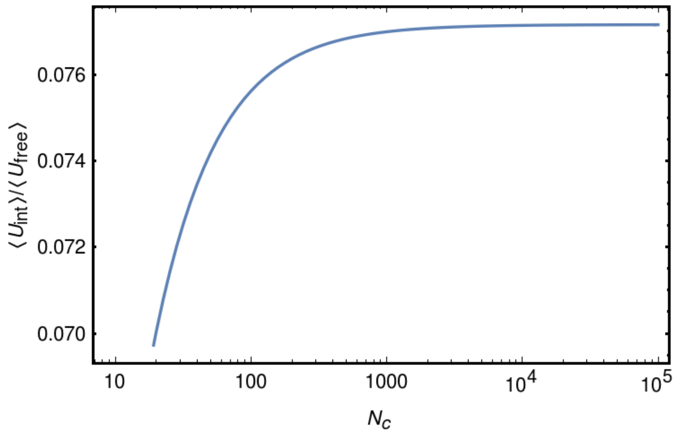

FIGURE 2.3: Ratio of interaction energy to free energy as a function of N_c. We've fixed a=1/10.

From (2.33) it follows that $\langle U_{int}\rangle = -\partial \log Z/\partial\beta$, which can be used in (2.45) to obtain,

$$\langle U_{int}\rangle = \frac{\nu}{2\beta}\langle N\rangle \,. \tag{2.47}$$

Figure 2.3 shows the ratio of the interaction energy to free energy. Since the free energy is larger than the interaction energy we can trust the simplified two-body interaction model. Further, because the gluon field VEV is related to the trace of the stress energy tensor (SET) by the scale anomaly relation, and since the trace of the SET is in a direct relation to the vacuum energy density, we obtain the following leading-order expression for the vacuum energy density,

$$\mathcal{E} = -\frac{b}{4}\frac{\langle N\rangle}{V} \,. \tag{2.48}$$

Notice that it grows quadratically with N_c, with an additional factor of N_c with respect to non-interacting instanton gas [33].

Let us now compute the topological susceptibility. This is of particular interest because it is an observable. We start by adding the topological theta-term, $\frac{i\theta}{32\pi^2}\int d^4x F\tilde{F}$, to the action. The topological susceptibility is defined by,

$$\chi_{top} = -\frac{\partial^2 \log Z}{\partial\theta^2}|_{\theta=0} = \langle\left(\int d^4x \frac{F\tilde{F}}{32\pi^2}\right)^2\rangle \,. \tag{2.49}$$

In particular, adding the θ-term to the partition function doesn't modify the computation of $\mu(\rho)$, or $\overline{\rho^2}$, and thus the only modification to (2.45) is in an additional term $+i\theta(N_+ - N_-)$. Self-consistently, by rewriting this as

$$Z = \exp\left[\frac{\nu+2}{2}\langle N\rangle(1 - \frac{\theta^2}{\nu+2} + \mathcal{O}(\theta^4))\right] \,, \tag{2.50}$$

and taking the derivative as in (2.49) we get [31],

$$\chi_{top} = \langle N \rangle \,. \tag{2.51}$$

We are now ready to investigate the role of instantons in controllable asymptotically safe quantum field theories.

2.2.2 Safe Instantons

Here we extend the instanton calculus to asymptotically safe quantum field theories, or more precisely to the first discovered controllable asymptotically safe four dimensional gauge theory, here dubbed LISA [10].

As already mentioned, 't Hooft's master formula is a one-loop result which requires an RG-improvement in order for us to be able to extract any numbers from it. We will do this RG improvement, working with two-loop gauge running couplings. In order to work in a consistent scheme, we therefore have to satisfy ourselves with the 2-1-0 scheme. It is clear that scalars drop out completely from the following computation because they are not gauged. For this reason we do not care that scalar self-interaction couplings do not run in 2-1-0 scheme. Yukawa couplings on the other hand do not appear directly in the master formula, but they do contribute to the running gauge coupling.

We will be studying the RG flow of LISA along the separatrix, a line connecting the IR Gaussian FP to the UV interacting FP in the coupling space. To simplify the discussion, we deal with the Yukawa coupling following [34], by expressing it as a function of α_g along the separatrix. This is achieved by finding the solution for the zero of the Yukawa beta function, to leading order in ϵ, of the form $\alpha_y = \alpha_y(\alpha_g)$. The running gauge coupling will be numerically correct to order ϵ. Crucially, the qualitative picture of having a running coupling interpolating between a Gaussian FP in the IR and a perturbative, non-Gaussian FP in the UV persists. In particular, substituting the flow of the Yukawa coupling, $\alpha_y(\alpha_g)$ to leading order in ϵ along the separatrix in the above expression for the cubic term leads exactly to the beta function (2.3), with $B = -4\epsilon/3$ and

$$C = -\frac{2}{3}\frac{57 - 46\epsilon - 8\epsilon^2}{13 + \epsilon} \,. \tag{2.52}$$

Both B and C being negative, the fixed point appears at the physical value $\alpha_* = B/C > 0$. Despite the fact there are no explicit scalar contributions in the following computations, note that the asymptotically safe dynamics could not have been possible without the inclusion of scalars in the theory, since the Yukawa contribution to the cubic term of the gauge beta function was crucial for shifting the fixed point to the domain of physical values.

The "exact" two-loop running is given by [34]

$$\alpha(\mu) = \frac{\alpha_*}{1 + W(z(\mu))} \,. \tag{2.53}$$

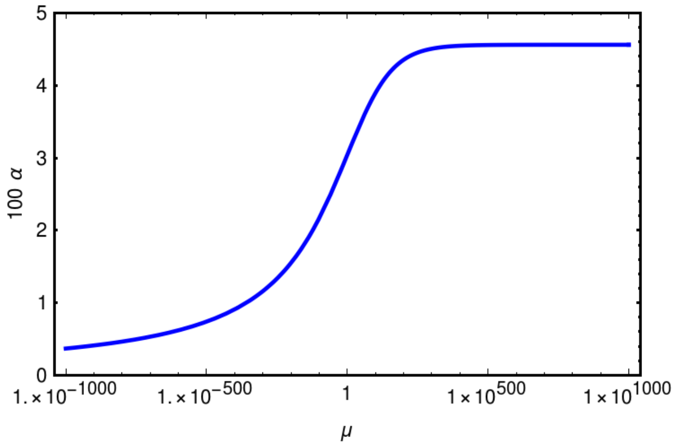

FIGURE 2.4: The figure shows the running gauge coupling in LISA for $\epsilon = 1/10$. The inflection corresponds to the scale $\Lambda_c = 1$.

W stands for the Lambert (or productlog) function, α_* is the fixed point, and

$$z(\rho) = e^{1/2-\log 2}(\rho\Lambda_c)^{-\alpha_* B}\,. \tag{2.54}$$

Expansion around $\mu \to \infty$ yields equation (2.27). The running stemming from (2.53) is manifestly bounded and it interpolates between $\alpha = 0$ in the IR and $\alpha = \alpha_*$ in the UV, as it can be seen from Figure 2.4.

Just as the one-loop running in the QCD introduces a natural length scale Λ_{QCD}, so does the "exact" two-loop running (2.53) between two fixed points introduce a natural length scale called Λ_c (where c stands for critical). Λ_c is the two-loop RG-invariant scale, and it has a direct analogue in the Banks-Zaks theory for example. We obtain it as follows. $\partial_\alpha \beta_\alpha(\alpha)$ vanishes for $\alpha = {}^2\!/_3\, \alpha_* \equiv \alpha_c$. The scale at which one reaches this value of the coupling is critical in the sense that at this scale the gauge coupling changes scaling from canonical to a non-Gaussian one. This scale,

$$\mu(\alpha_c) \equiv \Lambda_c = (2e^{-\frac{1}{2}})^{-1/\theta_*}(1 - \frac{\alpha}{\alpha_*})^{-1/\theta_*}\,\mu\,, \tag{2.55}$$

is the two-loop RG-invariant scale in the sense that $\mu\partial_\mu(\Lambda_c) = 0$ (to linear order). We have also introduced notation θ_* to mean the eigenvalue of the RG flow at the UV FP,

$$\theta_* = \frac{\partial \beta_\alpha}{\partial \alpha}|_{\alpha_*} = \alpha_* B\,. \tag{2.56}$$

Let us now compute the average instanton size and the density of instantons per unit volume following the large-N_c computation. Inserting the RG-improvements in the one-loop master equation (2.26) we have,

$$d_{2L}(\rho) = C_{cf} m^{N_f} \rho^{N_f-5} (b \log M\rho)^{2N_c} e^{-\frac{8\pi^2}{g_{2L}^2}} \tag{2.57}$$

$$= C_{cf} \exp(1/2 - \log 2)^{-\frac{8\pi^2}{g_*^2}} m^{N_f} \rho^{N_f-5} (\rho\Lambda_c)^{\frac{1}{2}BN_c} \times$$

$$\times \quad (b \log M\rho)^{2N_c} W(z(\rho))^{\frac{8\pi^2}{g_*^2}}. \tag{2.58}$$

In the above expression we used C_{cf} as defined in (2.25), Λ_c defined in (2.55), one-loop beta coefficient b given in (2.24), and one-loop RG-invariant scale M,

$$M = \frac{1}{\rho_c} \exp\left(-\frac{1}{b}\frac{8\pi^2}{g(\rho_c)^2}\right) = \Lambda_c \exp\left(-\frac{3}{2}\frac{C}{B^2}\right). \tag{2.59}$$

Setting $\rho^2 \to \overline{\rho^2}$ in second line of (2.58) the expression for the instanton density takes a form which is similar to what we had in the pure YM case. In fact, defining,

$$f(\bar{\rho}) = C_{cf} \exp(1/2 - \log 2)^{-\frac{8\pi^2}{g_*^2}} \left(\frac{b}{2} \log M^2\overline{\rho^2}\right)^{2N_c} W(\overline{\rho^2})^{\frac{8\pi^2}{g_*^2}}, \tag{2.60}$$

we can write the two-loop instanton density as,

$$d_{2L}(\rho) = f(\bar{\rho}) m^{N_f} \rho^{N_f-5} (\rho\Lambda_c)^{\frac{1}{2}BN_c}. \tag{2.61}$$

Note that $BN_c/2 = b$, so the dependence on ρ for $N_f = 0$ is exactly the same as in pure-YM theory. In other words, for $N_f = 0$, $d_{2L} \sim \rho^{b-5}$ which was causing the IR divergence problems in the one-instanton result. For $N_f > 0$ and $B > 0$ nothing changes, but LISA has $B < 0$. The qualitative dependence on ρ remains the same in (2.61) because N_f dominates over $1/2BN_c$ (B being of order ϵ, and $N_c < N_f$). We can thus proceed as before, and in fact one can obtain the effective two-loop density $\mu(\rho)$ from (2.61) in an analogous way to the large-N_c computation. $\mu(\rho)$ thus again takes form (2.42), with $d(\rho)$ being substituted by the above two-loop instanton density, so we still have a Gaussian suppression of the IR instantons. With the effective instanton density μ known, we may re-insert it into the definition (2.41) of $\overline{\rho^2}$ and compute the Gaussian integral. The expectation value of ρ that we get is again of the form (2.43), but now with ν given by,

$$\nu = \frac{1}{2}\left(\frac{1}{2}BN_c + N_f - 4\right). \tag{2.62}$$

As a side note, since β is positive, ν has to be positive too if $\bar{\rho}$ is to be positive. In fact, $1/2BN_c = -4/6\epsilon N_c$, whereas $N_f = (11/2 + \epsilon)N_c \simeq 11/2N_c$, and thus ν is clearly positive in LISA.

The minimization of the partition function can now be performed in complete analogy to the derivation of the average instanton number in the pure YM theory and we obtain

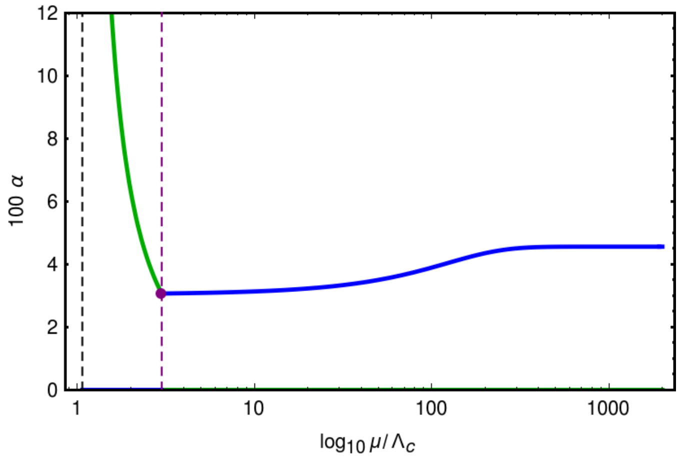

FIGURE 2.5: The blue line shows the LISA running for $\epsilon = 1/10$, and green line corresponds to the pure YM running. Purple dot shows the matching couplings at the fermion mass scale which is given by purple dashed line. Black dashed line is the scale Λ_{YM}.

$$\langle N \rangle = V\Lambda_c^4 \left[\Gamma(\nu) (\frac{m}{\Lambda_c})^{N_f} f(\bar{\rho}) (\beta\gamma^2\nu)^{-\frac{\nu}{2}} \right]^{\frac{2}{2+\nu}} . \tag{2.63}$$

Comparing to (2.44), the most notable difference is the appearance of the RG-invariant scale Λ_c instead of the IR-divergence scale $\Lambda \simeq \Lambda_{YM}$. Another important thing is that $\tilde{\beta}^{2N_c}$ is replaced by $(\frac{m}{\Lambda_c})^{N_f} f(\bar{\rho})$, which renormalizes the one-loop result (2.44). The partition function still has the same form (2.45) as in the pure YM case, but with new values for ν and $\langle N \rangle$.

Solving the equations for $\bar{\rho}$ and N/V with LISA running couplings, the way we did in the pure YM case, leads us to the results shown in the Figure 2.6. Crucially, the results are inconsistent with the hypotheses in the sense that $\bar{\rho}^{-1}$ that we find is always smaller than m, i.e. it is more IR than the scale m where we decouple fermions.

This leads us to look for the solution below the energy scale m, where running of the couplings is given by pure YM beta functions [6]. We thus proceed by considering again the equations (2.43) and (2.44).

We know from the subsection 2.2.1 that the solutions for instantons in the pure YM theory are internally consistent, meaning that $\bar{\rho}^{-1} \gg \Lambda_{YM}$. When solving the equations for LISA, since we didn't find any solutions for $\bar{\rho}^{-1} > m$, we additionally have to make sure that the consistency condition $\bar{\rho}^{-1} < m$ is met when using the pure YM running coupling.

Our results are shown in the top panel of Figure 2.7 which shows the ratio of m to $\bar{\rho}^{-1}$ as a function of m measured in units of Λ_c. Results for $\bar{\rho}$ are well within the required consistency range. Bottom panel shows the inverse instanton length as a

[6] Equivalently, we may look for solutions using the running coupling defined as a piecewise function, equal to LISA running coupling beyond energy m and equal to matching pure YM running coupling below m.

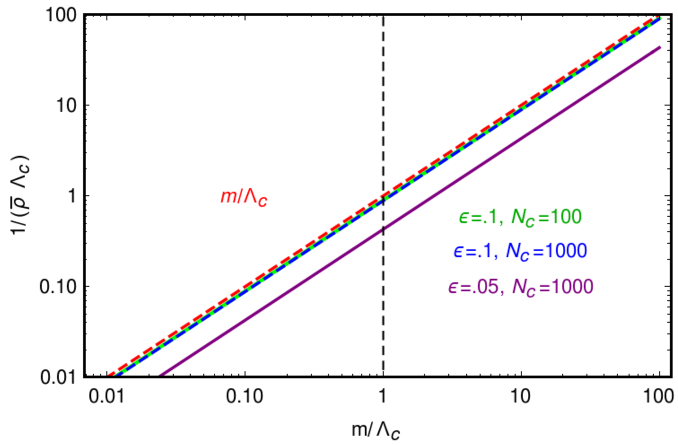

FIGURE 2.6: The figure shows solutions for $\bar{\rho}$, obtained using LISA beta functions, for various choices of ϵ and N_c (green, blue and purple). Red dashed line shows the fermion mass.

function of mass, and we can clearly see the power law decrease of the instanton scale as m is taken to zero.

As an additional consistency check one may study the behavior of Λ_{YM}. In fact, here it is not an arbitrary number but it is specified by the following one loop matching conditions

$$\frac{8\pi^2}{g_{YM}^2(m)} \equiv -\frac{11}{3} N_c \, log\left(a\frac{\Lambda_{YM}}{m}\right) = \frac{8\pi^2}{g_{LISA}^2(m)} = \frac{N_c}{2\alpha(m)}\,, \tag{2.64}$$

which yields,

$$\Lambda_{YM} = \frac{m}{a}\exp\left(-\frac{3}{22}\frac{1}{\alpha(m)}\right)\,. \tag{2.65}$$

For small enough ϵ the exponential term is flat as a function of m, so the dependence on mass here is essentially linear. Finally, we can plot $\bar{\rho}\Lambda_{YM}$ as a function of m/Λ_c and we find that it is exactly constant, taking value $\bar{\rho} = 0.390\Lambda_{YM}^{-1}$ for $a = 1/10$, $N_c = 1000$ and $\epsilon = -1/10$.

Let us now discuss the instanton energy and the topological susceptibility. Since the couplings are renormalized at the energy scale corresponding to the inverse of the average instanton size, and since the instanton size turns out to be such that they sit well within the pure YM regime, the analysis closely follows the pure YM case.

In particular, the partition function again takes the simple form (2.45) with $\langle N\rangle$ and ν given by (2.44) and (2.43) respectively. The total energy is given by a sum of the free energy term, $\langle N\rangle$, and the interaction term. The interaction term comes from the derivative of the partition function wrt $\beta = 8\pi^2/g_{2L}^2(\bar{\rho})$. This dependence is hidden in $\langle N\rangle$ where it appears in the same form as it did in the pure YM case, which means the interaction energy can again be written as $\langle U_{int}\rangle = \nu\langle N\rangle/(2\beta)$. The ratio of the interaction energy to free energy thus follows the curve shown in Figure 2.3. In fact the shape of that curve changes significantly if two-loop running

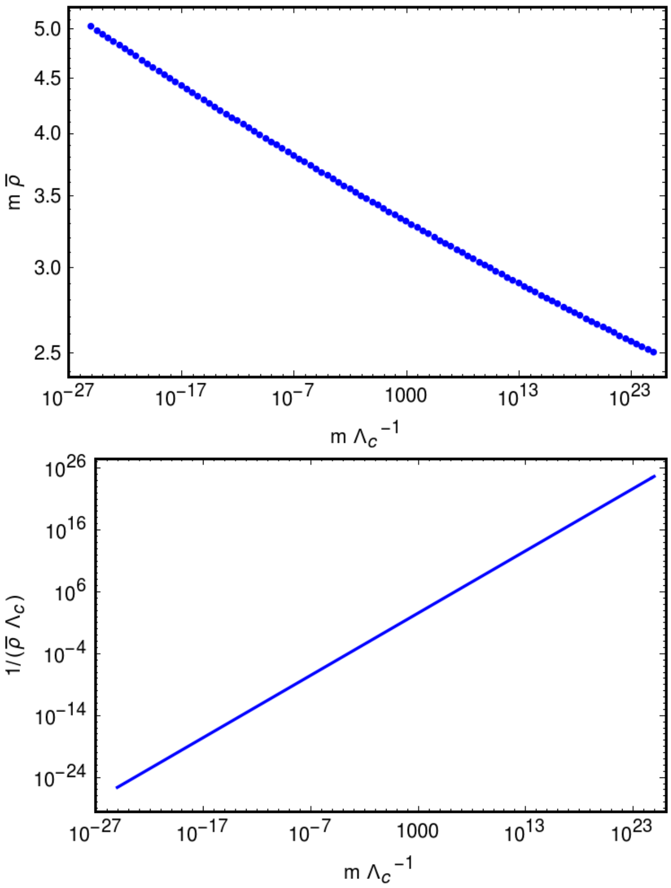

FIGURE 2.7: Top panel shows ratio $m/\bar{\rho}^{-1}$ and bottom panel shows $\bar{\rho}^{-1}$ as functions of m. In both figures $a = 1/10$, $N_c = 1000$ and $\epsilon = 1/10$.

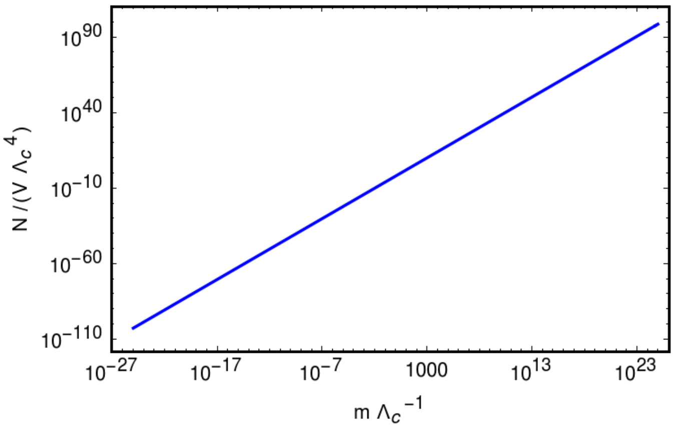

FIGURE 2.8: The figure shows density of the instantons per unit volume measured in units of Λ_c^{-4}.

is used instead of the one-loop running and the overall method is more stable when compared to the QCD case.

If one fixes m, N_c and a (e.g. $m = 1/10\ \Lambda_c$, $N_c = 1000$, $a = 1/10$) one can study $\bar{\rho}\Lambda_{YM}$ as a function of ϵ and find that it is constant (and in our example) equal to $\bar{\rho} = 0.390\Lambda_{YM}^{-1}$. The reason why $\bar{\rho}(\epsilon)$ is constant in units of Λ_{YM} is related to the fact that Λ_{YM} decreases rapidly with decreasing $|\epsilon|$, thus compensating for the rapidly growing $\bar{\rho}$ in units of Λ_c.

The determination of the topological susceptibility proceeds as described in the previous section, see equation (2.51). As we've discussed above, $\Lambda_{YM}\bar{\rho}$ is essentially m- and ϵ-independent. In this sense N/V depends only on the explicit factor $\bar{\rho}^{-4}$. It is then clear that N/V will rapidly decrease with decreasing m/Λ_c. This is confirmed in Figure 2.8. It is worth mentioning that the quantity $N/(V\Lambda_{YM}^4)$ is independent of the fermion mass, same as $\bar{\rho}\Lambda_{YM}$.

2.2.3 Discussion

Let us recapitulate the main points of this section and reiterate the connection to and the implications for the following section.

Instantons are non-trivial gauge configurations which are extrema of the bare Euclidean action. As such they have to be included in the computation of the Euclidean partition function, and thus they will directly contribute to the analytically continued Lorentzian partition function. Instantons were shown to be extremely important for the description of the vacuum of gauge theories, as exemplified particularly by the success of the instanton fluid model applied to QCD. In this section we have seen that the standard mean field techniques used for the pure Yang-Mills theory can be straightforwardly extended to a more complicated theory, namely mass-deformed LISA.

Being deep within the regime of perturbation theory at all energy scales, we do not have to deal with the complications usually introduced by chiral symmetry breaking. In fact, chiral condensate is a known order parameter for the χSB, and most of the complications related to going from pure Yang-Mills to QCD is related to the role of instantons in the formation of the condensate and the effect of the chiral condensate on the instanton fluid. In LISA the couplings stay arbitrarily weak for all energy scales, which means that chiral condensate never forms. Heavy fermions can then be dealt with in a straightforward way by including their contribution to the running gauge coupling.

The above analysis relies on a simplified two-body interaction model of the instanton interactions. We have argued for this approximation based on the fact that the interaction energy is subdominant to the free energy of the system. We have then demonstrated that the instantons appear in the deep IR regime with respect to the fermion mass scale. This begs the question whether the analysis remains stable, since the gauge coupling in pure YM can grow without bound in the IR. What we find is that the YM coupling g^2 at the instanton scale $\mu = \bar{\rho}^{-1}$ takes fully perturbative values. For example, for $N_c = 100$ (ϵ is irrelevant here since we're computing the value of the pure YM coupling) we find $g^2 = 0.0344$ or $\alpha = (g^2 N_c)/(4\pi)^2 = 0.0218$.

With this we conclude that LISA is an interesting proof-of-concept for the paradigm of asymptotically safe extensions of the Standard Model. Its main limitation, of course, is that the fixed point is reached in the Veneziano limit. Following "Occam's razor" argument we would prefer to find phenomenologically viable extensions of the Standard Model which contain much fewer degrees of freedom. In the following section we pave the way for doing exactly that.

2.3 Searching for UV completions of the Standard Model

Having established the existence of asymptotically safe and "Standard Model-like" quantum field theories in the Veneziano limit and having subsequently confirmed their stability under the inclusion of instanton effects, we will now attempt to construct phenomenologically viable extensions of the Standard Model (SM) based on the asymptotic safety paradigm. Finding such models is of paramount importance for the search for UV completions of the known particle physics in the form of asymptotically safe QFTs. Only if we show that such models can exist in principle do we move from the realm of formal, mathematical physics to the realm of phenomenologically interesting physics. We lay the track for this line of research in this section.

In particular, while asymptotically free theories have been studied in great detail and for a long time, work on asymptotically safe models for particle physics has only begun quite recently. For some early references studying toy models of subsectors of the Standard Model via functional renormalization group see [35]–[40]. As already discussed in detail in section 2.1, true breakthrough came with LISA [10]. That LISA was a "game changer" can be seen from the amount of model building it has inspired in a relatively short amount of time: [34], [41]–[48]. As we will explain below, most of the models considered thus far are not entirely satisfying from the phenomenological point of view due to either being dependent on large-N_f resummation techniques or due to not considering the full SM gauge group. Our work aims at being conservative in that it may be interpreted as looking for a minimum amount of vector-like fermions that one should add to the SM in order to produce a physical UV fixed point.

2.3.1 Bird's-eye view of the program

In LISA the fixed point arises from a cancellation between one- and two-loop terms in the β-functions, see section 2.1. Crucially, this dynamics occurs in the Veneziano limit which provides the small expansion parameter

$$\epsilon = \frac{N_f}{N_c} - \frac{11}{2} \,. \tag{2.66}$$

It is reasonable to expect that there exist asymptotically safe models also for finite values of ϵ, and for more general choices of matter content. General conditions for the existence of such fixed points have been discussed in [10], [49].

The Standard Model by itself doesn't seem to be asymptotically safe, because of the Landau pole in the $U(1)$ gauge coupling [50], [51], see Figure 2.9, and the uncertain fate of the Higgs quartic interaction [52]. The Landau pole can only be avoided by assuming that this gauge coupling is identically zero at all energies. This is known as the triviality problem.

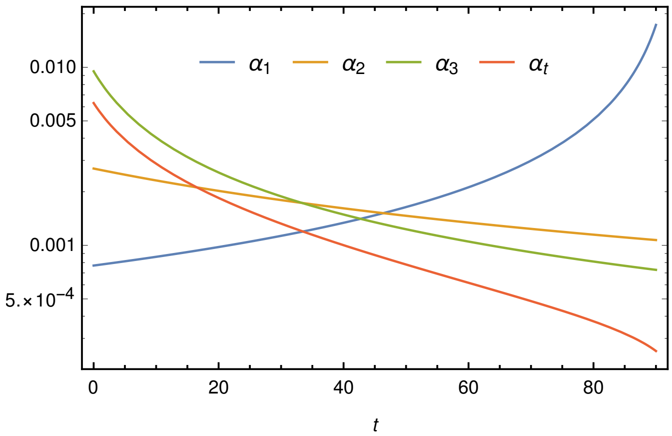

FIGURE 2.9: Running of the gauge couplings α_i and Yukawa α_t for the SM in the 3-2-1 scheme as a function of $t = \ln(\mu/M_Z)$. Around $t \simeq 45$ the three gauge couplings come close together. At larger values of t, α_1 begins its ascent towards the Landau pole.

Can the SM be turned into an asymptotically safe theory by extending its matter content? The simplest extensions consist of multiple generations of vector-like fermions carrying diverse representations of the SM gauge group. The choice of vector-like fermions is motivated by their not giving rise to gauge anomalies and their masses being technically natural. As mentioned above, such models have been studied in two independent ways so far.

The authors of [42] have studied the β-functions to two-loop order in the simplified case of $SU(3)_c \times SU(2)_L$ gauge interactions and a Yukawa-like interaction among the vector-like fermions. They find several UV fixed points, which they match to the low-energy SM in a number of benchmark cases. We will not discuss these toy models here any further, but the interested reader is instead referred to [4] where we have thoroughly analyzed them in both 2-1-0 and 3-2-1 schemes (see section 2.1 for definitions of the schemes).

In a parallel development, the authors of [43]–[46], [53] studied asymptotic safety of the full SM gauge sector, again extended by N_f vector-like fermions, by means of a large-N_f resummation of the perturbative series of the β-functions. They find several UV fixed points, which however cannot be matched to the low-energy SM in a consistent manner [46]. Since our focus here is on the smallest possible extensions of the SM, and since the large-N_f techniques are beyond the scope of this work we do not comment on this approach any further.

To move forward the quest for asymptotically safe extensions of the SM, we report our results for a large class of models based on the SM interactions: gauge group $SU_c(3) \times SU_L(2) \times U_Y(1)$, top Yukawa, and Higgs quartic self-interaction; the vector-like fermions are charged under these intercations and have in addition Yukawa interactions with new scalar fields. The restriction to the top Yukawa makes the form of the β-functions more manageable—and it is in line with earlier investigations. The models differ in the number of copies, N_f, of the vector-like fermions and in the representation of the gauge groups that they carry.

In contrast to [43]–[46], [53] we do not use resummed β-functions. However, we go beyond the two-loop results considered so far in the literature and work in both 2-1-0 and 3-2-1 schemes. By comparing the results at these two different orders in the perturbative expansion we are able to assess quantitatively the impact of radiative corrections and therefore decide whether a given fixed point is within the perturbative domain or not. This selection is supported by the use of other tests of perturbativity that the fixed points must satisfy, as discussed in detail in subsection 2.3.2.

The core of our work consists of a systematic search for reliable fixed points in a large grid in the parameter space spanned by the number of vector-like fermions, N_f, and their $SU(3)_c$, $SU(2)_L$, and $U(1)_Y$ quantum numbers. We first find all the zeroes of the β-functions for each model in the grid. We then test each prospective fixed point against two conditions:

- The fixed point must be in the domain of perturbation theory.
- The fixed point can be connected to the SM at low energy.

In particular, the first condition requires that the FP is stable under radiative corrections. We find, a posteriori, that this is the case only when the perturbativity tests defined in subsection 2.3.2 are satisfied. The second condition would generally require a delicate numerical analysis of the trajectories emanating from it. See "Other interesting results” in subsection 2.3.3 for example. In most cases, however, we find that a much simpler condition is adequate for our purposes: the FP must not have any coupling that is zero and irrelevant, because such couplings must be identically zero at all scales to avoid Landau poles and thus cannot match to SM in the IR. As we shall see, these two requirements taken together, while quite reasonable, are very restrictive.

2.3.2 Methodology

Let us describe the general procedure that will be followed in the rest of this section. This will allow us to better motivate the requirements introduced in subsection 2.3.1 that we impose on the fixed points in order for them to be considered physical.

Recall we have mentioned in section 2.1 that the perturbative β-functions of the SM and its extensions have a natural hierarchy originating from the Weyl consistency conditions [11]–[14], [54]. A consistent solution of (2.11) relates different orders in the perturbative expansion and indicates that the gauge couplings must have the highest order in the loop expansion, while the Yukawa coupling must be computed at one order less and the quartic interaction still one order less. This leaves us in practice with two perturbation theory approximation schemes:

- 2-1-0 scheme, in which the gauge couplings are renormalized at the two-loop order (NLO)
- 3-2-1 scheme, in which the gauge couplings are renormalized at the three-loop order (NNLO)

Our analysis confirms the expectations coming from LISA, that the fixed points arise from the cancellation of the two-loop gauge and one-loop Yukawa contributions. We thus find promising fixed points in the 2-1-0 scheme and by comparing the results to the 3-2-1 scheme it is possible to estimate their robustness under the radiative corrections and thereby to also judge reliability of the perturbative computation.

Other approximation schemes do exist, for example one may naively retain all beta functions to the same order or keep the gauge beta functions one order higher than the others. We refer the reader to [55] where the respective merits and shortcomings of these alternative schemes are discussed, but we do not use them in this thesis.

Perturbativity test 1: coupling constants

For each model that we consider we first solve the beta functions to list all the zeroes that they contain. Since we have a system of algebraic equations of a relatively high order there are generally very many spurious solutions. Our task is to eliminate all the spurios solutions so that we may focus on studying physically interesting ones. First step along this way is the study of the fixed point values.

Beyond perturbation theory, e.g. in stronlgy interacting regime of QCD, couplings α_i may take any non-negative real values. When we work in perturbation theory, however, we demand all the couplings to be small at all energy scales from the IR matching scale all the way up to the UV fixed point. This is a minimum requirement that has to be satisfied if perturbation theory is to be trusted.

In practice "small" means that in going to the next order of the expansion, the position of the fixed point, as well as its other properties, should be reasonably robust. We will see that this implies that, in addition to being non-negative, the numerical value of a FP must satisfy the condition[7]

$$\alpha_i^* \equiv \left(\frac{g_i^*}{4\pi}\right)^2 < 1\,. \tag{2.67}$$

The condition (2.67) suffices to keep the perturbative expansion within its domain of validity if the expansion coefficients are of the same order and not too large (see "Perturbativity test 3" in particular). If they are not, the condition in (2.67) should be strengthened and only smaller values allowed to prevent higher order terms from being dominating.

Perturbativity test 2: anomalous dimensions

Once we identify candidate fixed points whose couplings satisfy first perturbativity test, we can study the flow in fixed points' immediate neighborhood, i.e. we study the flow of the perturbation $y_i \equiv g_i - g_i^*$ around the FP $\{g_i^*\}_i$. To this end, we linearize the β-functions as

$$\frac{dy_i}{dt} = M_{ij} y_j\,, \tag{2.68}$$

where

$$M_{ij} \equiv \frac{\partial \beta_i}{\partial g_j} \tag{2.69}$$

is referred to as the stability matrix. Next, we diagonalize the linear system by going to the variables $z_i = (S^{-1})_{ij} y_j$, defined by the equation

$$(S^{-1})_{ij} M_{jl} S_{ln} = \delta_{in} \vartheta_n\,, \tag{2.70}$$

so that the β-functions and their solutions are in the simplified form

$$\frac{dz_i}{dt} = \vartheta_i z_i \quad \text{and} \quad z_i(t) = c_i\, e^{\vartheta_i t} = c_i \left(\frac{\mu}{\mu_0}\right)^{\vartheta_i}. \tag{2.71}$$

We refer to the eigenvalues θ_i of the stability matrix (2.69) as scaling exponents or critical exponents. From the expression of z_i's as functions of μ, we see that there are different situations depending on the sign of ϑ_i:

[7] Note additional factor of (4π) wrt standard definition; this convention is chosen for compatibility with section 2.1 and the original LISA paper [10].

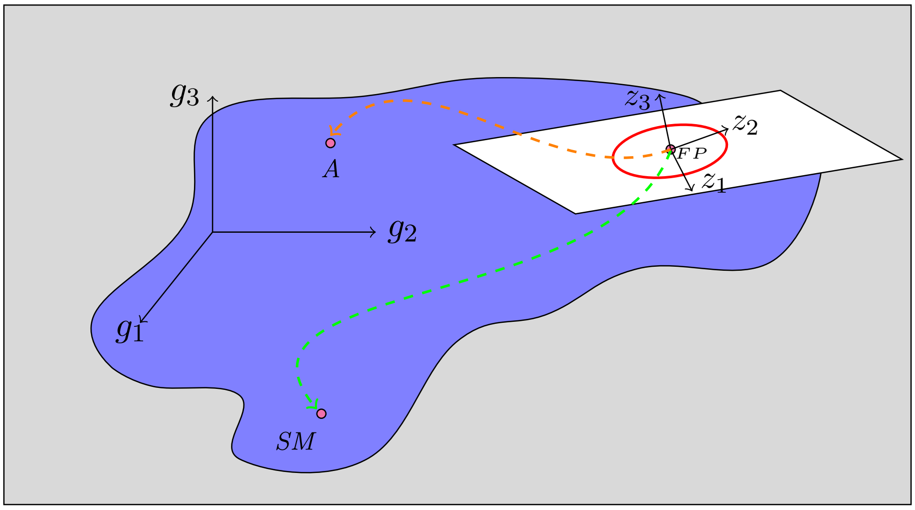

FIGURE 2.10: Theory space of couplings g_i where only 3 axes are shown for simplicity. For a given fixed point we show the UV safe surface (blue region), the linearized UV critical surface around the fixed point (white plane), the new set of coordinates z_i, a small region of possible initial points for the flow (red circle) and two UV safe trajectories ending at a given matching scale $\mathcal{M}$ (green and orange dashed lines, the former going to the SM, the latter going to a different IR physics A).

- For $\vartheta_i > 0$, as we increase μ we move away from the fixed point and z_i increases without control; the direction z_i is said to be irrelevant.
- If $\vartheta_i < 0$, as we increase μ we approach the fixed point; the direction z_i is said to be relevant.
- If $\vartheta_i = 0$, the fate of z_i is undetermined to linear order (see "Caveats" below); the direction z_i is said to be marginal.

Asymptotically safe theories correspond to trajectories lying on the surface whose tangent space at the fixed point is spanned by the relevant directions. This tangent space, known as the UV critical surface, is shown in Figure 2.10 as a white plane, whereas the full UV critical surface is shown in blue.

The critical exponents are universal quantities—meaning that they are invariant under a general transformation in the space of couplings [56]. Similarly to the coupling constants, there are restrictions on their size in perturbation theory. In general the beta function of g_i, the dimensionless version of a generic dimensionful coupling $\bar{g}_i = g_i \mu^{d_i}$, has the form

$$\beta_i = -d_i g_i + \beta_i^{quant}(g_j), \tag{2.72}$$

where β_i^{quant} encodes the quantum contributions to the β-functions. Therefore, the stability matrix is given by

$$M_{ij} = -d_i \delta_{ij} + \frac{\partial \beta_i^{quant}}{\partial g_j} \tag{2.73}$$

which shows that its eigenvalues equal the (negative) classical scaling dimensions plus quantum corrections, also known as anomalous dimensions. The quantity $-\vartheta_i$ thus represents the full scaling dimension of the coupling $\bar{g}_i$. If we want a perturbative computation to be trustworthy, we must demand that the anomalous scaling dimensions be small. In the case of (classically) marginal couplings that we consider $d_i = 0$, which means that

$$|\vartheta_i| < O(1)\,. \tag{2.74}$$

There is a degree of arbitrariness about where exactly one should set this bound. In our study, we look at the scaling dimensions for the models under examination and set the bound in the first gap in the distribution of their $O(1)$ values.

First two perturbativity tests are quite powerful and together they successfully characterize most fixed points as either perturbative or uninteresting (either because non-perturbative or non-physical). For some fixed points it happens that the couplings or critical exponents are in a domain where we cannot discern with complete confidence whether the fixed point should be considered as physically relevant or beyond the scope of perturbation theory. To this end, but also in order to confirm the classification of the remaining fixed points to the two groups, we apply two more perturbativity tests.

Perturbativity test 3: expansion coefficients

A simple perturbativity test consists in comparing the coefficients of different-loop contributions to the beta functions. Let us write the beta functions of the gauge couplings α_i in the schematic form

$$\beta_i = \left(A^{(i)} + B_r^{(i)}\alpha_r + C_{rs}^{(i)}\alpha_r\alpha_s\right)\alpha_i^2, \tag{2.75}$$

where A, B and C are the one-, two- and three-loop contributions. Even though the β-functions vanish at FPs, we can split each beta function in the following way

$$0 = \beta_i|_{FP} = A_*^{(i)} + B_*^{(i)} + C_*^{(i)}\,, \tag{2.76}$$

where $A_*^{(i)} = A^{(i)}\alpha_{i*}^2$, $B_*^{(i)} = B_r^{(i)}\alpha_{r*}\alpha_{i*}^2$ and $C_*^{(i)} = B_{rs}^{(i)}\alpha_{r*}\alpha_{s*}\alpha_{i*}^2$. When we insert the FP couplings calculated in the 3-2-1 approximation scheme, we expect the three-loop contribution $C_*^{(i)}$ to be smaller than the two-loop term $B_*^{(i)}$, and two-loop term smaller than one-loop term $A_*^{(i)}$ (in absolute value).

$$\rho_i < \sigma_i < 1\,, \quad \text{where} \quad \rho_i = |C_*^{(i)}/A_*^{(i)}| \quad \text{and} \quad \sigma_i = |B_*^{(i)}/A_*^{(i)}|\,. \tag{2.77}$$

In the following, when we report results in the 2-1-0 approximation scheme, we give the values of ρ_i defined at the 2-1-0 fixed point and when we report results in the

3-2-1 approximation scheme, we give the values of ρ_i and σ_i defined at the 3-2-1 approximation scheme fixed point.

Perturbativity test 4: central charges

A QFT at a FP is a conformal field theory (CFT), and a given CFT is characterized by two local functions: c and a. We refer to these functions collectively as central charges or CFT functions. They are defined as coefficients of the Weyl tensor W and the Euler density E_4 in the trace of the energy-momentum tensor of the theory,

$$\langle T^\mu_\mu \rangle = cW^2 - aE_4 + \dots \, . \tag{2.78}$$

The ellipses on the RHS denote operators constructed from the fields in the theory. A function related to the CFT function a, often denoted $\tilde{a}$, was proven to be monotonically decreasing following the RG flow from a UV fixed point to an IR one [12], [13]. In fact, the RG flow of the $\tilde{a}$-function is related to the dynamics by means of the β-functions of the theory through the relation

$$\mu \frac{\partial \tilde{a}}{\partial \mu} = -\chi_{ij} \beta^i \beta^j \, , \tag{2.79}$$

where the tensor χ_{ij} is known as the Zamolodchikov metric, which we have also seen enters the Weyl consistency conditions (2.11). Evaluated at a fixed point, $\tilde{a}$ reduces to the a-function.

In the models that we study below there is only a UV fixed point present, whereas dynamics in the IR is not conformal, but usual one with dynamically generated confinement length scale. Nevertheless, central charges of the UV fixed points can still be used to test whether the fixed points are reliable.

In any CFT, both a and c have to be positive, and their ratio has to satisfy the so-called collider bounds [57], namely

$$\frac{1}{3} \leq \frac{a}{c}\Big|_{FP} \leq \frac{31}{18} \, . \tag{2.80}$$

In perturbation theory central charges are expanded in series,

$$\tilde{a} = \tilde{a}_{free} + \frac{\tilde{a}^{(1)}}{(4\pi)^2} + \frac{\tilde{a}^{(2)}}{(4\pi)^4} + \dots \tag{2.81}$$

$$c = c_{free} + \frac{c^{(1)}}{(4\pi)^4} + \dots \tag{2.82}$$

and since free-field theory contributions are positive [58],

$$\tilde{a}_{free} = \frac{1}{(4\pi)^2} \frac{n_s + 11 n_w/2 + 62\, n_v}{360} \tag{2.83}$$

$$c_{free} = \frac{1}{(4\pi)^2} \frac{n_s/6 + n_w + 2\, n_v}{20} \, , \tag{2.84}$$

—n_s, n_w, and n_v referring to the number of scalar, Weyl and vector degrees of freedom, respectively—the positivity of the CFT functions is ensured in perturbation theory.

One can estimate the strength of quantum effects from the relative changes of the central charges to decide whether a FP is within the domain of validity of perturbation theory. If we write the a-function as $a = a_{free} + a_q$, a_q referring to the contribution of quantum corrections, and similarly for the c-function; the relative changes in central charges are defined by:

$$\delta a \equiv \frac{a - a_{free}}{a_{free}} = \frac{a_q}{a_{free}} \quad \text{and} \quad \delta c \equiv \frac{c - c_{free}}{c_{free}} = \frac{c_q}{c_{free}} \,. \tag{2.85}$$

If δa or δc become smaller than -1 the fixed point is unphysical because it cannot correspond to a CFT (since $c > 0$ and $a > 0$ are guaranteed for any CFT). A fixed point for which δc or δa is of order 1 should be discarded as well since quantum corrections are then comparable in size to the free-theory contribution, invalidating the expansion.

The central charges in the 2-1-0 approximation can be easily computed by embedding the models in the general gauge-Yukawa Lagrangian of [17]. Computation in the 3-2-1 scheme is significantly more complicated due to a major increase in complexity of the Zamolodchikov metric. We do not pursue the 3-2-1 computation both for this reason and because the results in the 2-1-0 scheme seem to confirm that the first two perturbativity criteria, namely constraints on coupling sizes and anomalous dimensions, are entirely compatible with the CFT tests.

RG Matching to Standard Model

Once we have an understanding of the fixed point structure and the perturbativity tests are satisfied, it remains to be checked if there exists a trajectory connecting a given FP to the Standard Model at some IR scale. This may be accomplished in the following manner. First all the SM couplings are run via SM beta functions to a common RG scale, which we take to be 1.83 TeV, where they take values:

$$\alpha_1 = 0.000795\,, \quad \alpha_2 = 0.00257\,, \quad \alpha_3 = 0.00673\,, \quad \alpha_t = 0.00478\,.$$

This defines the IR target for the RG flow from the UV fixed point. Next, the RG flow is started from a point numerically as close as possible to the FP and (approximately, to given precision) lying on the UV critical surface. Domain of such points is shown as a red circle in Figure 2.10. This guarantees that, to high precision, the flow towards the UV ends at the fixed point. The system is then allowed to flow by means of the full beta functions towards the IR. The initial point of the flow is varied until the trajectory hits the target SM values to a predetermined accuracy.

The scale at which one starts the flow, μ_0, is not known a priori. If one reaches the target values of the couplings after some RG time $t = \log(\mu/\mu_0) < 0$, the corresponding scale μ is identified with 1.83 TeV and the starting scale is identified as $\mu_0 = \mu e^t$.

Caveats

Finally, before moving on to the results of the analysis, let us discuss two caveats.

Despite it being a well known fact, it should be stressed that even though the β-function of a single coupling is independent of the gauge choice in dimensional regularization, and is regularization scheme-independent up to NLO; if there are several couplings running together, their β-functions depend on the scheme already at the NLO [59]. There is therefore a degree of ambiguity in the position of the fixed points we are going to discuss because their position could be moved by changing the regularization scheme. We assume that these changes are small if the fixed point is found within the perturbative regime. One should however bear in mind this problem of regularization-scheme dependence in all the discussions to follow.

Second comment is about marginal couplings; namely, if one of the eigenvalues is equal to zero, the linear approximation does not give us information about the RG behavior in the direction associated to it. In this case we have to go beyond the linear expansion. At second order in the couplings y_i, the β-functions take the form

$$\frac{dy_i}{dt} = M_{ij} y_j + P_{ijk} y_j y_k \,, \quad \text{where} \quad P_{ijk} = \frac{\partial^2 \beta_i}{\partial g_j \partial g_k}. \tag{2.86}$$

The structure of these quadratic flows is quite complicated to describe in full generality, with the fate of a specific trajectory depending strongly on the position of the initial point in the neighbourhood of the fixed point.

However, marginal couplings do not generally occur for a fully interacting fixed point — they can always be identified with some coupling that is itself zero at the fixed point. We had shown in an appendix of [4] that the structure of the β-functions is such that the flow of the marginal couplings near the fixed point is of the form

$$\frac{dy_i}{dt} = P_{iii} y_i^2 \,, \tag{2.87}$$

with no summation implied. Our flows will always be written in terms of the α_i, which are bound to be positive. Therefore, marginal directions $y_i = \alpha_i$ with $P_{iii} < 0$ are UV attractive and are called marginally relevant while those with $P_{iii} > 0$ are UV repulsive and are called marginally irrelevant. A well-known example of a marginally relevant direction is that of the $SU_c(3)$ gauge coupling of strong interactions. Altogether, the UV critical surface is thus spanned by the relevant and marginally relevant directions.

Methods summary

Given a model, we first look for all the fixed points of the β-functions. Since the β-functions are given in the form of a Taylor expansion, they will have several zeroes that are mere artifacts of the expansion, and we have to select those that may be physical. The criteria we apply are stability under radiative corrections and matching to the SM at low energy.

We begin by analyzing the fixed points of the 2-1-0 approximation scheme. In the first step, we retain only those fixed points that can be reasonably assumed to be within the perturbative regime, that is, those for which the couplings and the scaling exponents satisfy the bounds in (2.67) and (2.74), as well as the other perturbativity criteria discussed above. We check a posteriori that these bounds are indeed reasonable indicators of radiative stability.

We then compare with the results of the similar analysis in the 3-2-1 approximation scheme. We retain only those fixed points that can be reasonably identified in both approximations. Their number is quite small. We find that this identification is possible if the couplings and scaling exponents are sufficiently small whereas fixed points that have large couplings or scaling exponents cannot be recognized across the two approximations and must be discarded.

Finally, for the fixed points that are radiatively stable in the sense just described, we look for the possibility of matching to the SM at low energy.

If a fixed point satisfies all these conditions, we say that it should be considered physical; otherwise, the FP should be rejected as unphysical.

A question that one might naturally raise is whether the SM itself allows for physical fixed points? If it did have perturbative fixed points there would still be two logical possibilities, nature would decide whether the FP would be connected to the observed IR physics via RG flow or not. We have studied this question in [4] and we have found that the Standard Model on its own does not possess any perturbative fixed points in eiher 2-1-0 or 3-2-1 schemes. We therefore proceed directly to beyond-Standard-Model (BSM) physics and in the next subsection we study SM augmented by a BSM sector consisting of vector-like fermions and non-gauged scalars as described in 2.3.1.

2.3.3 Standard Model extensions

We consider a minimal extension of the SM by adding new matter content charged under the SM group $SU_c(3) \times SU_L(2) \times U_Y(1)$. The gauge sector is not modified. Following [10], [42], [49], [55], we take N_f families of vector-like fermions minimally coupled to the SM. The idea is to consider a new type of Yukawa interactions among the vector-like fermions such that their contribution generate new zeros in the gauge β-functions. Accordingly, new scalar fields must be included as well. These scalars are taken to be singlets of the SM group while the fermions will be described by the representations R_3 under $SU_c(3)$, R_2 under $SU_L(2)$, and have hypercharge Y of the gauge group $U_Y(1)$. Denoting by S_{ij} the matrix formed with N_f^2 complex scalar fields, the Lagrangian characterizing this minimal BSM extension is

$$\mathcal{L} = \mathcal{L}_{SM} + \mathrm{Tr}(\bar{\psi} i \not{D} \psi) + \mathrm{Tr}(\partial_\mu S^\dagger \partial_\mu S) - y \mathrm{Tr}(\bar{\psi}_L S \psi_R + \bar{\psi}_R S^\dagger \psi_L). \tag{2.88}$$

In (2.88), $\mathcal{L}_{SM}$ stands for the SM Lagrangian, y is the BSM Yukawa coupling, which we assume to be the same for all fermions, the trace sums over the SM representation indices as well as the flavor indices, and we have decomposed ψ as $\psi = \psi_L + \psi_R$ with $\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi$. We neglect the role of quartic self interactions of the scalars S_{ij} as well as portal couplings of the latter to the Higgs sector.

This extension of the SM is simple enough to allow explicit computations while giving rise to new features in the RG flow of the theory. The vector-like fermions serve as a proxy for more elaborate extensions; they do not introduce gauge anomalies and do not induce a large renormalization of the Higgs mass—they are technically natural.

The β-functions

Within the model defined by the Lagrangian (2.88), we look for fixed points satisfying the requirements discussed in subsection 2.3.2. We start the analysis in the 2-1-0 approximation scheme and write the β-functions of the system (2.88) in terms of the SM couplings,

$$\alpha_i = \frac{g_i^2}{(4\pi)^2} \text{ for } i = 1,2,3, \quad \text{and} \quad \alpha_t = \frac{y_t^2}{(4\pi)^2}, \tag{2.89}$$

augmented by the new coupling

$$\alpha_y = \frac{y^2}{(4\pi)^2}. \tag{2.90}$$

In the following, of all the SM Yukawa couplings we dynamically keep track of only the top-Yukawa. In doing so we are in line with previous investigations and we keep the system manageable. The β-functions will depend on the dimensions of the fermion representations d, the Casimir invariants C, and the Dynkin indices S, which are defined for general representation as

$$d_{R_2} = 2\ell + 1, \qquad d_{R_3} = \frac{1}{2}(p+1)(q+1)(p+q+2), \tag{2.91}$$

$$C_F^{(2)} = C_{R_2} = \ell(\ell+1), \qquad C_F^{(3)} = C_{R_3} = p + q + \frac{1}{3}(p^2 + q^2 + pq), \tag{2.92}$$

$$S_F^{(2)} = S_{R_2} = \frac{d_{R_2} C_{R_2}}{3}, \qquad S_F^{(3)} = S_{R_3} = \frac{d_{R_3} C_{R_3}}{8}. \tag{2.93}$$

Here integer or half-integer ℓ denotes the highest weight of the $SU_L(2)$ representation R_2, and (p,q), for integer p and q, denotes the weights of an irreducible R_3 representation of $SU_c(3)$.

In the 2-1-0 approximation scheme, the β-functions are given by [60]–[63]

$$\beta_1^{\rm NLO} = \left(B_1 + M_1\alpha_1 + H_1\alpha_2 + G_1\alpha_3 - D_1\alpha_y - \frac{17}{3}\alpha_t\right)\alpha_1^2, \tag{2.94}$$

$$\beta_2^{\rm NLO} = \Big(-B_2 + M_2\alpha_2 + H_2\alpha_1 + G_2\alpha_3 - D_2\alpha_y - 3\,\alpha_t\Big)\alpha_2^2, \tag{2.95}$$

$$\beta_3^{\rm NLO} = \Big(-B_3 + M_3\alpha_3 + H_3\alpha_1 + G_3\alpha_2 - D_3\alpha_y - 4\,\alpha_t\Big)\alpha_3^2, \tag{2.96}$$

$$\beta_t^{\rm LO} = \left(9\alpha_t - \frac{17}{6}\alpha_1 - \frac{9}{2}\alpha_2 - 16\,\alpha_3\right)\alpha_t, \tag{2.97}$$

$$\beta_y^{\rm LO} = \Big(T\alpha_y - F_1\alpha_1 - F_2\alpha_2 - F_3\alpha_3\Big)\alpha_y, \tag{2.98}$$

where we have included the gauge and matter contributions in the coefficients B_i, M_i, H_i, G_i and D_i, for $i = 1,2,3$. These coefficient are expressed in terms of d_{R_2}, d_{R_3}, C_{R_2}, C_{R_3}, S_{R_2}, S_{R_3}, Y and N_f as follows. For the diagonal and mixing gauge contributions to the gauge β-functions we have

$$B_1 = \frac{41}{3} + \frac{8}{3}N_f Y^2 d_{R_2} d_{R_3}, \qquad M_1 = \frac{199}{9} + 8Y^4 N_f d_{R_2} d_{R_3}, \tag{2.99}$$

$$H_1 = 9 + 8Y^2 N_f C_{R_2} d_{R_2} d_{R_3}, \qquad G_1 = \frac{88}{3} + 8N_f Y^2 C_{R_3} d_{R_2} d_{R_3}, \tag{2.100}$$

$$B_2 = \frac{19}{3} - \frac{8}{3}N_f S_{R_2} d_{R_3}, \qquad M_2 = \frac{35}{3} + 4N_f S_{R_2} d_{R_3}\left(2\,C_{R_2} + \frac{20}{3}\right), \tag{2.101}$$

$$H_2 = 3 + 8N_f Y^2 S_{R_2} d_{R_3}, \qquad G_2 = 24 + 8N_f S_{R_2} C_{R_3} d_{R_3}, \tag{2.102}$$

$$B_3 = 14 - \frac{8}{3}N_f S_{R_3} d_{R_2}, \qquad M_3 = -52 + 4N_f S_{R_3} d_{R_2}(2\,C_{R_3} + 10), \tag{2.103}$$

$$G_3 = 9 + 8N_f S_{R_3} C_{R_2} d_{R_2}, \qquad H_3 = \frac{11}{3} + 8N_f Y^2 S_{R_3} d_{R_2}. \tag{2.104}$$

For the Yukawa contribution to the gauge β-functions we have

$$D_1 = 4N_f^2 Y^2 d_{R_2} d_{R_3},\; D_2 = \frac{1}{3} 4N_f^2 C_{R_2} d_{R_2} d_{R_3},\; D_3 = \frac{1}{8} 4N_f^2 C_{R_3} d_{R_2} d_{R_3}, \tag{2.105}$$

whereas the running of the new coupling α_y is characterized by the coefficients

$$T = 2(N_f + d_{R_2} C_{R_3}),\;\; F_1 = 12\, Y^2,\; F_2 = 12\, C_{R_2},\; F_3 = 12\, C_{R_3}. \tag{2.106}$$

All the new contributions to the gauge couplings running are multiplied by N_f, meaning that we can go back to the SM by taking the $N_f \to 0$ limit.

Due to the simplicity of the β-functions to this order in perturbation theory, we can find analytic solutions of the equations

$$\beta_i^{\rm NLO} = \beta_t^{\rm LO} = \beta_y^{\rm LO} = 0 \tag{2.107}$$

as functions of Y, ℓ, p, q and N_f. All these solutions can be split in two categories according to whether they depend on the hypercharge Y or not. All the latter have $\alpha_1^* = 0$.

For the gauge couplings, the β-functions in the 3-2-1 approximation scheme, are given, using the variables in (2.89), as follows

$$\begin{aligned}
\beta_1^{\rm NNLO} \;=\;& \beta_1^{\rm NLO} + \Big[-M_{11}\alpha_1^2 + M_{12}\alpha_1\alpha_2 - M_{13}\alpha_1\alpha_3 - G_{23}\alpha_2\alpha_3 + H_{11}\alpha_2^2 + G_{11}\alpha_3^2 \\
& + \frac{315}{8}\alpha_t^2 + K_{y1}\alpha_y^2 - \frac{2827}{144}\alpha_1\alpha_t - \frac{785}{16}\alpha_2\alpha_t - \frac{58}{3}\alpha_3\alpha_t \\
& - (K_{11}\alpha_1 + K_{12}\alpha_2 + K_{13}\alpha_3)\,\alpha_y + \frac{3}{2}(\alpha_1 + \alpha_2 - \alpha_\lambda)\,\alpha_\lambda \Big]\, \alpha_1^2,
\end{aligned} \tag{2.108}$$

$$\begin{aligned}
\beta_2^{\rm NNLO} \;=\;& \beta_2^{\rm NLO} + \Big[- M_{22}\alpha_2^2 + M_{21}\alpha_2\alpha_1 - M_{23}\alpha_2\alpha_3 - G_{13}\alpha_1\alpha_3 - H_{22}\alpha_1^2 + G_{22}\alpha_3^2 \\
& + \frac{147}{8}\alpha_t^2 + K_{y2}\alpha_y^2 - \frac{729}{16}\alpha_2\alpha_t - \frac{593}{48}\alpha_1\alpha_t - 14\,\alpha_3\alpha_t \\
& - (K_{22}\alpha_2 + K_{21}\alpha_1 + K_{23}\alpha_3)\,\alpha_y + \frac{1}{2}(\alpha_1 + 3\,\alpha_2 - 3\,\alpha_\lambda)\,\alpha_\lambda \Big]\, \alpha_2^2,
\end{aligned} \tag{2.109}$$

$$\beta_3^{\rm NNLO} \;=\; \beta_3^{\rm NNLO} + \Big[-M_{33}\alpha_3^2 + M_{31}\alpha_3\alpha_1 - M_{32}\alpha_3\alpha_2 - G_{12}\alpha_1\alpha_2 - H_{33}\alpha_2^2 + G_{33}\alpha_2^2 \tag{2.110}$$

$$+30\,\alpha_t^2 + K_{3y}\alpha_y^2 - 80\,\alpha_3\alpha_t - \frac{101}{12}\alpha_1\alpha_t - \frac{93}{4}\alpha_2\alpha_t \tag{2.111}$$

$$- (K_{33}\alpha_3 + K_{31}\alpha_1 + K_{32}\alpha_2)\,\alpha_y \Big]\, \alpha_3^2\,. \tag{2.112}$$

For the Yukawa and quartic Higgs couplings, the β-functions are given by

$$\begin{aligned}
\beta_t^{\rm NLO} &= \beta_t^{\rm LO} + \Big[-24\,\alpha_t^2 + 3\,\alpha_\lambda^2 - 12\,\alpha_t\alpha_\lambda + \left(\frac{131}{8}\alpha_1 + \frac{225}{8}\alpha_2 + 72\,\alpha_3\right)\alpha_t \\
&\quad + \frac{1187}{108}\alpha_1^2 + \frac{3}{2}\alpha_1\alpha_2 - \frac{23}{2}\alpha_2^2 + \frac{38}{9}\alpha_1\alpha_3 + 18\,\alpha_2\alpha_3 - 216\,\alpha_3^2 \\
&\quad + \frac{58}{27}B_{t1}\alpha_1^2 + 2B_{t2}\alpha_2^2 + \frac{160}{9}B_{t3}\alpha_3^2\Big]\,\alpha_t \qquad (2.113) \\
\beta_y^{\rm NLO} &= \beta_y^{\rm LO} + \Big[(4-V)\alpha_y^2 + (V_1\alpha_1 + V_2\alpha_2 + V_3\alpha_3)\,\alpha_y \qquad (2.114) \\
&\quad + W_1\alpha_1^2 + W_2\alpha_2^2 + W_3\alpha_3^2 - W_{12}\alpha_1\alpha_2 - W_{13}\alpha_1\alpha_2 - W_{23}\alpha_2\alpha_3\Big]\alpha_y, \qquad (2.115) \\
\beta_\lambda^{\rm LO} &= 12\,\alpha_\lambda^2 - (3\,\alpha_1 + 9\,\alpha_2)\,\alpha_\lambda + \frac{9}{4}\left(\frac{1}{3}\alpha_1^2 + \frac{2}{3}\alpha_1\alpha_2 + \alpha_2^2\right) + 12\,\alpha_t\alpha_\lambda - 12\,\alpha_t^2, \qquad (2.116)
\end{aligned}$$

where we have introduced several coefficients containing the gauge and Yukawa contributions which depend on N_f and the group representations of the SM and new vector-like fermions. The interested reader can find the explicit form of the coefficients and original sources of various pieces of beta functions in [4].

It is not possible to find analytic solutions for the fixed points in the 3-2-1 approximation scheme. The system

$$\beta_i^{\rm NNLO} = \beta_t^{\rm NLO} = \beta_y^{\rm NLO} = \beta_\lambda^{\rm LO} = 0 \qquad (2.117)$$

must be solved numerically, separately for each given choice of (N_f, Y, p, q, ℓ). No separation between Y-independent and -dependent solutions can be established before solving the equations, but it can be confirmed a posteriori.

Results 1: colorless vector-like fermions

In order to find fixed points satisfying the conditions (2.67) and (2.74), we generate a grid in the space spanned by the quantum numbers (N_f, ℓ, Y) for three specific $SU_c(3)$ representations: colorless ($p = q = 0$), fundamental ($p = 1, q = 0$) and adjoint ($p = q = 1$). For each of these representations, we consider the following values for the number of vector-like fermions, their isospin and hypercharge:

$$\begin{aligned}
N_f &\in [1, 300], \;\; \textit{in steps of 1}\,, \qquad (2.118) \\
\ell &\in [1/2, 10], \;\; \textit{in steps of 1/2}\,, \qquad (2.119) \\
Y &\in [0, 10], \;\; \textit{in steps of 1/2}\,. \qquad (2.120)
\end{aligned}$$

This amounts to 126,000 points for each representation of $SU_c(3)$.

We start here by considering the colorless case and we discuss the other two representations below. Colorless vector-like fermions are the least phenomenologically restricted and therefore the most attractive candidates for a successful extension of the SM. In the 2-1-0 approximation scheme we find that only the Y-independent set of solutions contains fixed points fulfilling the required conditions ($\alpha < 1$, $|\vartheta| < O(1)$).

To set the precise bound on $|\vartheta|$, we plot in Figure 2.11 the largest eigenvalue of the stability matrix versus one representative coupling. For the Y-independent solutions there are solutions with eigenvalues between 0 and 2.21, and then there is a gap up to 62.6; for the Y-dependent solutions there are no eigenvalues less than 9.63. Accordingly, we decide to consider fixed points with $|\vartheta| < 3$ [8]. In this way we probably include some fixed points that are not in the scope of perturbation theory, but we prefer further checks of such FPs over failing to catch good FPs that are only apparently non-perturbative. We discard all the Y-dependent fixed points since there is always an eigenvalue which is of order 10 or greater.

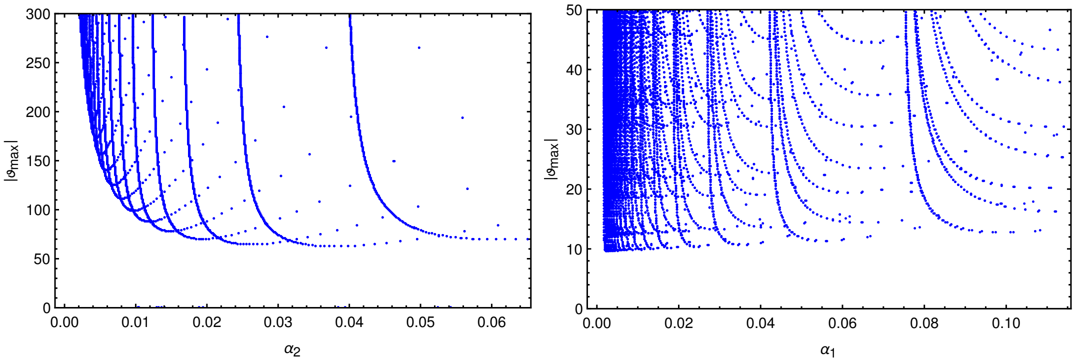

FIGURE 2.11: Distribution of the largest eigenvalue of the stability matrix for the colorless models. Left: Y-independent solutions. Right: Y-dependent solutions.

The above selection of the viable fixed points is confirmed by the study of their CFT central charges. There are 20 Y-independent fixed points with eigenvalues up to about ± 2, they are depicted in Figure 2.12 with the changes in the a- and c-functions, as well as their ratio. All of them have positive a and c. The fixed point with least variation in the central charges is that with $(N_f, \ell) = (1, 1)$, having $\delta a \simeq -0.0007$ and $\delta c \simeq 0.08$. The one with the largest change is that with $(N_f, \ell) = (1, 1/2)$, having $\delta a \simeq -0.2$ and $\delta c \simeq 0.8$. All these fixed points except for one pass the collider bound test (2.80). There are 69 Y-dependent fixed points with eigenvalues up to ± 10. None of them have positive a or c, and δa and δc are always of $O(1)$ so they should all be discarded. These results confirm our classification of the fixed points in Figure 2.13 according to the size of their eigenvalues and the ratio ρ.

Having applied all the criteria discussed in section 2.3.2, we find that for any value of the hypercharge Y the only group representations producing satisfactory

[8] Any threshold up to 62.2 would yield the same set of Y-independent fixed points.

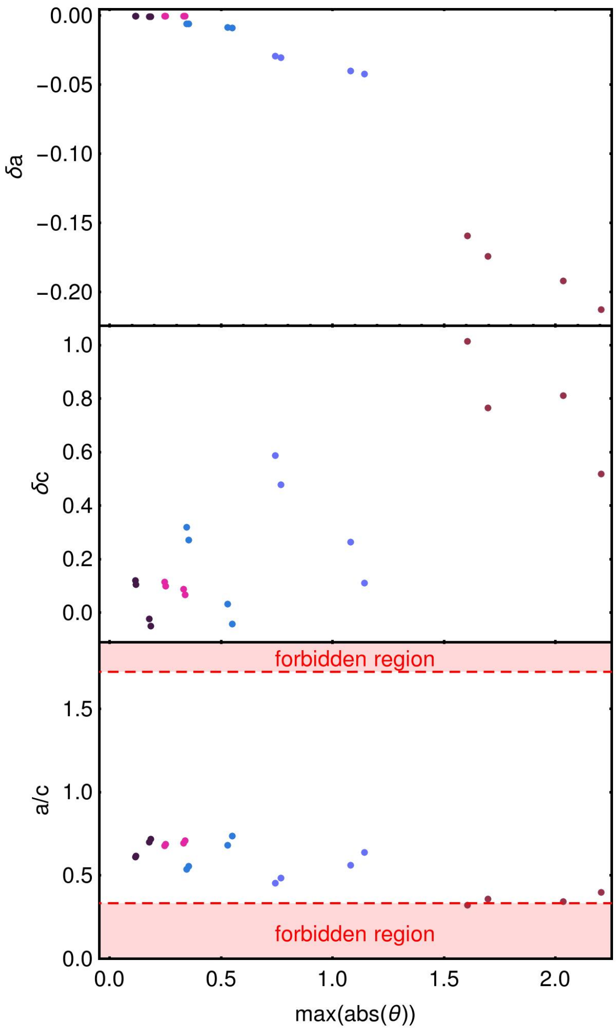

FIGURE 2.12: Variation δa, δc, and ratio a/c of the central charges as functions of the largest eigenvalue for the 20 colorless, Y-independent fixed points discussed in the text. The collider bounds (2.80) require a/c not to fall in the forbidden region in the bottom panel.

candidate fixed points are those collected, together with the corresponding eigenvalues, in Figure 2.13. The eigenvalues of the stability matrix turn out to be Y-independent as well.

(N_f, ℓ)	α_1^*	α_2^*	α_3^*	α_t^*	α_y^*	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5		ρ_2
$(1, \frac{1}{2})$	0	0.200	0	0	0.300	2.04	−0.900	0.884	0	0	P_{16}	3.97
	0	0.213	0	0.106	0.319	2.21	1.19	0.743	0	0	P_{17}	4.33
	0	0.179	0	0	0	−1.61	0.893	−0.804	0	0	P_{18}	3.28
	0	0.189	0	0.0943	0	−1.70	1.15	0.697	0	0	P_{19}	3.53
$(1, 1)$	0	0.0137	0	0	0.0411	0.333	−0.0616	0.0135	0	0	P_{16}	0.194
	0	0.0140	0	0.0070	0.0420	0.341	0.0633	0.0137	0	0	P_{17}	0.198
	0	0.0103	0	0	0	−0.247	−0.0464	0.0103	0	0	P_{18}	0.0963
	0	0.0105	0	0.0052	0	−0.251	0.0473	0.0104	0	0	P_{19}	0.0973
$(2, \frac{1}{2})$	0	0.104	0	0	0.117	1.0833	−0.467	0.328	0	0	P_{16}	1.71
	0	0.108	0	0.0542	0.122	1.14	0.525	0.315	0	0	P_{17}	1.81
	0	0.0827	0	0	0	−0.744	−0.372	0.303	0	0	P_{18}	1.19
	0	0.0856	0	0.0428	0	−0.770	0.427	0.283	0	0	P_{19}	1.23
$(3, \frac{1}{2})$	0	0.0525	0	0	0.0472	0.530	−0.236	0.109	0	0	P_{16}	0.763
	0	0.0543	0	0.0272	0.0489	0.552	0.251	0.109	0	0	P_{17}	0.794
	0	0.0385	0	0	0	−0.346	−0.173	0.0897	0	0	P_{18}	0.471
	0	0.0394	0	0.0197	0	−0.355	0.182	0.0896	0	0	P_{19}	0.483
$(4, \frac{1}{2})$	0	0.0189	0	0	0.0141	0.179	−0.0849	0.0179	0	0	P_{16}	0.246
	0	0.0194	0	0.0097	0.0146	0.185	0.0880	0.0182	0	0	P_{17}	0.253
	0	0.0130	0	0	0	−0.117	−0.0584	0.0130	0	0	P_{18}	0.141
	0	0.0132	0	0.0066	0	−0.119	0.0599	0.0132	0	0	P_{19}	0.143

FIGURE 2.13: Set of FPs and eigenvalues for colorless vector-like fermions in the 2-1-0 approximation scheme. We highlight in green the FPs that appear also in the 3-2-1 approximation. In the last column we show the ratio ρ_2 defined in Eq. (2.76).

The bounds on N_f and ℓ come from the behavior of the eigenvalues as functions of these parameters. If we plot one of the eigenvalues as a function of N_f for several values of l, we observe that it increases very fast. From Figure 2.14, we see that only models with small N_f produce sufficiently small eigenvalues.

It is important to realize that the large scaling dimensions of models with large N_f frustrate the apparently promising strategy of increasing the number of generations N_f of vector-like fermions in order to increase the NLO term in the gauge β-functions to cancel the (N_f-independent) LO term with smaller (and therefore more perturbative) values of the couplings α_i.

In Figure 2.13 we also show the ratio ρ_2. As discussed in subsection 2.3.2, under "Perturbativity test 3", this shows how large the three-loop contribution is with

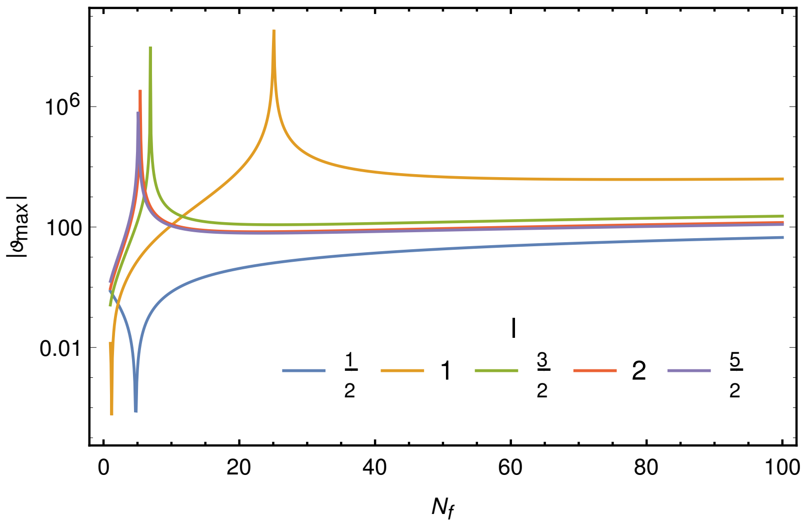

FIGURE 2.14: Behavior of a given eigenvalue $|\vartheta|$ as a function of N_f for several values of ℓ in the colorless case. The scaling dimension increases very fast with N_f, and only small values of N_f, ℓ produce $|\vartheta| < O(1)$.

respect to the two-loop contribution.

Now that we have isolated the candidates to study, we check whether these fixed points can be connected to the SM via the RG flow. We find that β_1 is proportional to α_1^2 and so, in order to avoid Landau poles, α_1 has to vanish at all energy scales. In conclusion, although we have perturbative fixed points, these cannot be matched to the SM because g_1 is different from zero at the TeV scale.

We then perform a similar analysis in the 3-2-1 approximation scheme. Since we see in Figure 2.13 that the fixed point with $|\vartheta| > 1$ produce a rather large ρ_2 ratio, we stick to solutions having $|\vartheta| < 1$. We find that the same combinations of N_f and ℓ that provide perturbative fixed points in the 2-1-0 case also give viable solutions here. Moreover, the solutions turn out to be Y-independent as well.

In Figure 2.15 we show the fixed point solutions satisfying the criteria in (2.67) and (2.74). Almost all the fixed points in Figure 2.15 can be traced back to fixed points that were already present in the 2-1-0 approximation scheme and listed in Figure 2.13. Notice that for a given pair (N_f, ℓ), not all the fixed points in 2-1-0 persist. For those that do, the values of α^* and ϑ change by relatively small amount. We can then claim that the solutions given in Figure 2.15 are radiatively stable fixed points.

Unfortunately, when we look at trajectories lying on the UV critical surface, we find that the coupling α_1 must be zero at all scales in all the models. The abelian interactions suffer from the triviality problem and no matching to the SM is possible if asymptotic safety is assumed. All these colorless models are therefore ruled out.

(N_f, l)	α_1^*	α_2^*	α_3^*	α_t^*	α_y^*	α_λ^*	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5	ϑ_6	σ_2	ρ_2
$(1, 1)$	0	0.0096	0	0.0048	0	0.0039	−0.244	0.0655	0.0430	0.0103	0	0	0.918	0.0821
	0	0.0119	0	0.0060	0.0343	0.0048	0.301	0.0813	0.0531	0.0134	0	0	0.8601	0.140
$(2, \frac{1}{2})$	0	0.0498	0	0.0259	0	0.0211	−0.592	0.382	0.282	0.200	0	0	0.581	0.418
	0	0.0567	0	0.0296	0.0734	0.0242	0.696	0.442	0.314	0.224	0	0	0.5012	0.499
$(3, \frac{1}{2})$	0	0.0291	0	0.0148	0	0.0120	−0.306	0.2080	0.132	0.0827	0	0	0.737	0.263
	0	0.0362	0	0.0184	0.0353	0.0150	0.403	0.262	0.165	0.100	0	0	0.645	0.354
$(4, \frac{1}{2})$	0	0.0117	0	0.0059	0	0.0048	−0.112	0.0804	0.052	0.0130	0	0	0.887	0.113
	0	0.0162	0	0.0081	0.0125	0.0066	0.161	0.112	0.0723	0.0179	0	0	0.823	0.177

FIGURE 2.15: Fixed points and eigenvalues for colorless vector-like fermions, in the 3-2-1 approximation scheme. The last two columns give the values of the ratios σ_2 and ρ_2 (see 2.77).

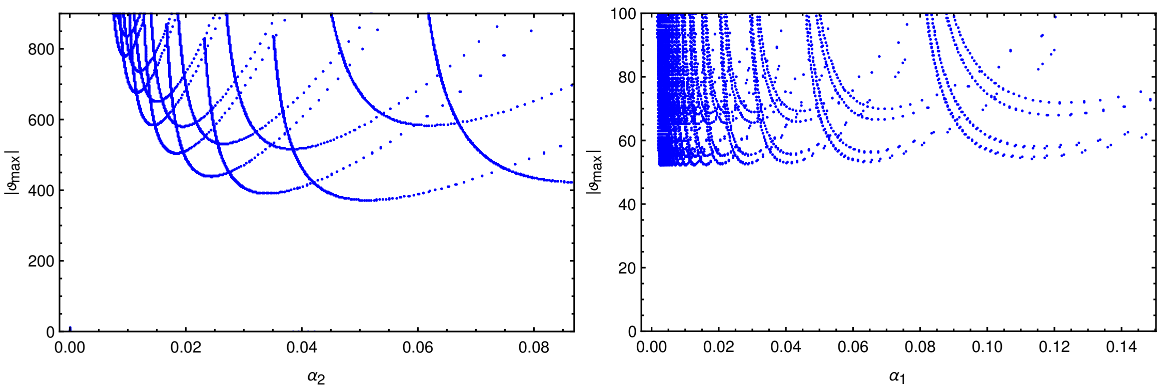

FIGURE 2.16: Distribution of the largest eigenvalue of the stability matrix for the $SU(3)$ fundamental representation. Left: Y-independent solutions. Right: Y-dependent solutions.

Results 2: vector-like fermions in fundamental of $SU_c(3)$

For the fundamental representation ($p = 1$ and $q = 0$) we follow the same procedure as before and generate 126,000 models by scanning the same grid in the (N_f, ℓ, Y) space. We again split the solutions into Y-independent and Y-dependent. The distribution of eigenvalues in Figure 2.16 shows that there are no fixed points with $|\vartheta| < 52.1$ for the Y-dependent solutions, whereas for the Y-independent solutions there is a gap between 10.8 and 372.

Accordingly, we eliminate all Y-dependent solutions and impose the bound $|\vartheta| < 11$ for those that are Y-independent. In this way, even more than in the preceding section, we include models that are probably unreliable, but these can be eliminated at a later stage. For the Y-independent solutions, we find the combinations of N_f and ℓ in Figures 2.17 and 2.18 that generate satisfactory candidate fixed points.

This selection is confirmed by the study of the central charges for these models. Among the 49 distinct Y-independent fixed points with eigenvalues up to ± 10, all have positive c-function, but 6 of them have a negative a-function (with one more being borderline acceptable). The CFT test seems to work well here: all fixed points with reasonable critical exponents pass it, whereas the ones with relatively large exponents do not. An unexpected fact is that the separation between "large" and "small" exponents seems to be around (maximum $|\theta|$ of) order 3.

For these perturbative and "semi-perturbative" fixed points, we also notice that the a-function is generically pushed toward 0 ($a_q < 0$) whereas the c function is generically shifted to larger values ($c_q > 0$). This is why the fixed points with negative a-function still seem to pass the c-function test. If one considers δc instead, then for most of these fixed points $\delta c > 1$, but apparently not for all. Finally, if one also studies the collider bounds one finds that ten more fixed points are excluded. We observe that if a fixed point barely satisfies one or both of the a and c tests, then it will most likely not satisfy the collider bound. The collider bound test thus seem to be the most stringent.

(N_f, l)	α_1^*	α_2^*	α_3^*	α_t^*	α_y^*	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5		ρ
$(1, \frac{1}{2})$	0	0.0411	0	0	0.0264	0.378	−0.185	0.0936	0	0	P_{16}	0.522
	0	0.0422	0	0.0211	0.0271	0.389	0.195	0.0936	0	0	P_{17}	0.537
	0	0.0385	0	0	0	−0.346	−0.173	0.0897	0	0	P_{18}	0.471
	0	0.0394	0	0.0197	0	−0.355	0.182	0.0896	0	0	P_{19}	0.483
$(1, 1)$	0	0	0.417	0	0	−6.67	−6.67	4.17	0	0	P_{11}	20.9
	0	0	0.521	0	0.417	10.8	−8.33	4.00	0	0	P_9	31.8
$(1, \frac{3}{2})$	0	0	0.176	0	0	−2.81	−2.81	1.52	0	0	P_{11}	5.45
	0	0	0.205	0.365	0	3.84	−3.28	1.52	0	0	P_{10}	7.21
	0	0	0.195	0	0.120	3.49	−3.12	1.51	0	0	P_9	6.60
	0	0	0.232	0.413	0.143	4.83	3.72	1.55	0	0	P_8	9.06
$(1, 2)$	0	0	0.0982	0	0	−1.57	−1.57	0.720	0	0	P_{11}	2.42
	0	0	0.108	0.193	0	1.88	−1.74	0.735	0	0	P_{10}	2.88
	0	0	0.105	0	0.0526	1.78	−1.68	0.730	0	0	P_9	2.73
	0	0	0.117	0.208	0.0586	2.15	1.88	0.749	0	0	P_8	3.30
$(1, \frac{5}{2})$	0	0	0.0600	0	0	−0.960	−0.960	0.360	0	0	P_{11}	1.27
	0	0	0.0646	0.115	0	1.08	−1.03	0.371	0	0	P_{10}	1.44
	0	0	0.0632	0	0.0266	1.04	−1.01	0.368	0	0	P_9	1.39
	0	0	0.0683	0.121	0.0288	1.18	1.09	0.380	0	0	P_8	1.59
$(1, 3)$	0	0	0.0412	0.0733	0.0150	0.689	0.660	0.184	0	0	P_8	0.839
	0	0	0.0388	0	0.0141	0.632	−0.621	0.178	0	0	P_9	0.758
	0	0	0.0395	0.0702	0	0.647	−0.632	0.180	0	0	P_{10}	0.778
	0	0	0.0372	0	0	−0.596	−0.596	0.174	0	0	P_{11}	0.707
$(1, \frac{7}{2})$	0	0	0.0221	0	0	−0.354	−0.354	0.0737	0	0	P_{11}	0.384
	0	0	0.0232	0.0413	0	0.376	−0.371	0.0764	0	0	P_{10}	0.415
	0	0	0.0229	0	0.0073	0.370	−0.366	0.0756	0	0	P_9	0.406
	0	0	0.0241	0.0428	0.0077	0.394	0.385	0.0784	0	0	P_8	0.441
$(1, 4)$	0	0	0.0114	0	0	−0.182	−0.182	0.0235	0	0	P_{11}	0.182
	0	0	0.0118	0.0210	0	0.191	−0.189	0.0235	0	0	P_{10}	0.195
	0	0	0.0117	0	0.0033	0.188	−0.187	0.0233	0	0	P_9	0.191
	0	0	0.0122	0.0217	0.0035	0.197	0.195	0.0242	0	0	P_8	0.205
$(1, \frac{9}{2})$	0	0	0.0033	0	0	−0.0530	−0.0530	0.0022	0	0	P_{11}	0.0495
	0	0	0.0034	0.0061	0	0.0550	−0.0549	0.0023	0	0	P_{10}	0.0523
	0	0	0.0034	0	0.0009	0.0544	−0.0544	0.0023	0	0	P_9	0.0516
	0	0	0.0035	0.0063	0.0009	0.0566	0.0564	0.0023	0	0	P_8	0.0547

FIGURE 2.17: Fixed points and eigenvalues for vector-like fermions in the fundamental representation of $SU_c(3)$, in the 2-1-0 approximation scheme, with $N_f = 1$. We highlight in green the fixed points that appear also in the 3-2-1 approximation scheme. The last column gives the values of the ratio ρ for α_2 or α_3 depending on the case (see 2.77).

(N_f, l)	α_1^*	α_2^*	α_3^*	α_t^*	α_y^*	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5		ρ
$(2, \frac{1}{2})$	0	0	0.176	0	0	−2.81	−2.81	1.52	0	0	P_{11}	5.45
	0	0	0.205	0.365	0	3.84	−3.28	1.52	0	0	P_{10}	7.21
	0	0	0.260	0	0.260	5.91	−4.16	1.59	0	0	P_9	11.1
	0	0	0.330	0.588	0.330	8.99	5.29	1.68	0	0	P_8	17.4
$(2, 1)$	0	0	0.0600	0	0	−0.960	−0.960	0.360	0	0	P_{11}	1.27
	0	0	0.0646	0.115	0	1.08	−1.03	0.371	0	0	P_{10}	1.44
	0	0	0.0727	0	0.0529	1.30	−1.16	0.390	0	0	P_9	1.77
	0	0	0.0795	0.141	0.0578	1.50	1.27	0.405	0	0	P_8	2.07
$(2, \frac{3}{2})$	0	0	0.0221	0	0	−0.354	−0.354	0.0737	0	0	P_{11}	0.384
	0	0	0.0232	0.0413	0	0.376	−0.371	0.0764	0	0	P_{10}	0.415
	0	0	0.0252	0	0.0144	0.417	−0.403	0.0810	0	0	P_9	0.475
	0	0	0.0266	0.0473	0.0152	0.448	0.426	0.0842	0	0	P_8	0.520
$(2, 2)$	0	0	0.0033	0	0	−0.0530	−0.0530	0.0022	0	0	P_{11}	0.0495
	0	0	0.0034	0.0061	0	0.0550	−0.0549	0.0023	0	0	P_{10}	0.0523
	0	0	0.0036	0	0.0017	0.0587	−0.0584	0.0024	0	0	P_9	0.0579
	0	0	0.0038	0.0068	0.0018	0.0612	0.0608	0.0025	0	0	P_8	0.0616
$(3, \frac{1}{2})$	0	0	0.0600	0	0	−0.960	−0.960	0.360	0	0	P_{11}	1.27
	0	0	0.0646	0.115	0	1.08	−1.03	0.371	0	0	P_{10}	1.44
	0	0	0.0882	0	0.0784	1.77	−1.41	0.423	0	0	P_9	2.47
	0	0	0.0985	0.175	0.0876	2.10	1.58	0.443	0	0	P_8	3.01
$(3, 1)$	0	0	0.0114	0	0	−0.182	−0.182	0.0227	0	0	P_{11}	0.182
	0	0	0.0118	0.0210	0	0.191	−0.189	0.0235	0	0	P_{10}	0.195
	0	0	0.0143	0	0.0095	0.237	−0.229	0.0276	0	0	P_9	0.264
	0	0	0.0150	0.0267	0.0100	0.252	0.241	0.0288	0	0	P_8	0.288
$(4, \frac{1}{2})$	0	0	0.0221	0	0	−0.354	−0.354	0.0737	0	0	P_{11}	0.384
	0	0	0.0232	0.0413	0	0.376	−0.371	0.0764	0	0	P_{10}	0.415
	0	0	0.0335	0	0.0268	0.607	−0536	0.0987	0	0	P_9	0.763
	0	0	0.0361	0.0642	0.0289	0.670	0.577	0.104	0	0	P_8	0.866
$(5, \frac{1}{2})$	0	0	0.0033	0	0	−0.0530	−0.530	0.0022	0	0	P_{11}	0.0495
	0	0	0.0343	0.0061	0	0.0550	−0.0549	0.0023	0	0	P_{10}	0.0523
	0	0	0.0052	0	0.0038	0.0850	−0.0829	0.0034	0	0	P_9	0.1010
	0	0	0.0055	0.0097	0.0040	0.0903	0.0878	0.035	0	0	P_8	0.111

FIGURE 2.18: Same as Table 2.17 but with $N_f > 1$.

When one tries to match the above UV FPs to the SM at low energies, it turns out that the abelian gauge coupling α_1 must again be zero at all scales. None of these fixed points is thus physically viable.

(N_f, l)	α_1^*	α_2^*	α_3^*	α_t^*	α_y^*	α_λ^*	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5	ϑ_6	σ	ρ
$(1, \frac{1}{2})$	0	0.0291	0	0.0148	0	0.0120	−0.306	0.208	0.132	0.0827	0	0	0.737	0.263
	0	0.0305	0	0.0155	0.0209	0.0126	0.322	0.219	0.139	0.0863	0	0	0.719	0.281
$(1, \frac{5}{2})$	0	0	0.0346	0	0	0	−0.748	−0.748	0.295	0	0	0	0.577	0.423
	0	0	0.0355	0	0.0167	0	−0.774	0.768	0.304	0	0	0	0.559	0.441
$(1, 3)$	0	0	0.0252	0	0	0	−0.501	−0.501	0.156	0	0	0	0.676	0.323
	0	0	0.0258	0	0.0101	0	−0.516	0.514	0.160	0	0	0	0.664	0.336
$(1, \frac{7}{2})$	0	0	0.0171	0	0	0	−0.315	−0.315	0.0670	0	0	0	0.771	0.228
	0	0	0.0177	0.0358	0	0.0221	0.969	−0.329	0.290	0.0723	0	0	0.758	0.242
	0	0	0.0175	0	0.0058	0	−0.324	0.324	0.0717	0	0	0	0.763	0.237
	0	0	0.0182	0.0368	0.0061	0.0227	0.998	0.334	0.298	0.0742	0	0	0.748	0.252
$(1, 4)$	0	0	0.098	0	0	0	−0.170	−0.170	0.0223	0	0	0	0.864	0.136
	0	0	0.0102	0.0193	0	0.0119	0.521	−0.177	0.165	0.0231	0	0	0.856	0.144
	0	0	0.0101	0	0.0029	0	−0.175	0.175	0.0229	0	0	0	0.859	0.141
	0	0	0.0104	0.0198	0.0030	0.0123	0.536	0.182	0.170	0.0237	0	0	0.8505	0.149
$(1, \frac{9}{2})$	0	0	0.0032	0	0	0	−0.0519	−0.0519	0.0022	0	0	0	0.955	0.0451
	0	0	0.0033	0.0059	0	0.0037	0.159	−0.0537	0.0526	0.0023	0	0	0.952	0.0476
	0	0	0.0032	0	0.0008	0	−0.0532	0.0532	0.0023	0	0	0	0.953	0.0469
	0	0	0.0033	0.0061	0.0009	0.00038	0.1635	0.0551	0.0540	0.0023	0	0	0.9505	0.0495
$(2, 1)$	0	0	0.346	0	0	0	−0.748	−0.748	0.295	0	0	0	0.577	0.423
	0	0	0.0381	0	0.0319	0	−0.846	0.824	0.326	0	0	0	0.5077	0.492
$(2, \frac{3}{2})$	0	0	0.0171	0	0	0	−0.315	−0.315	0.0699	0	0	0	0.771	0.228
	0	0	0.0177	0.0358	0	0.0221	0.969	−0.329	0.295	0.0723	0	0	0.758	0.242
	0	0	0.0187	0	0.0113	0	−0.350	0.349	0.0767	0	0	0	0.737	0.263
$(2, 2)$	0	0	0.0032	0	0	0	−0.0519	−0.0519	0.0022	0	0	0	0.955	0.0451
	0	0	0.0033	0.0059	0	0.0037	0.159	−0.0537	0.0526	0.0023	0	0	0.952	0.0476
	0	0	0.0035	0	0.0016	0	−0.0570	0.0570	0.0024	0	0	0	0.948	0.0521
	0	0	0.0036	0.0065	0.0017	0.0040	0.1756	0.0592	0.0579	0.0025	0	0	0.945	0.552
$(3, \frac{1}{2})$	0	0	0.0346	0	0	0	−0.748	−0.748	0.295	0	0	0	0.577	0.423
	0	0	0.0417	0	0.0440	0	−0.950	0.913	0.359	0	0	0	0.431	0.569
$(3, 1)$	0	0	0.0098	0	0	0	−0.170	−0.170	0.0223	0	0	0	0.864	0.136
	0	0	0.0102	0.0193	0	0.119	0.521	−0.177	0.165	0.0231	0	0	0.856	0.144
	0	0	0.0118	0	0.0081	0	0.208	−0.208	0.0270	0	0	0	0.819	0.181
	0	0	0.0123	0.0237	0.0085	0.0147	0.641	0.218	0.200	0.0281	0	0	0.8062	0.194
$(4, \frac{1}{2})$	0	0	0.0171	0	0	0	−0.315	−0.315	0.0699	0	0	0	0.771	0.228
	0	0	0.0177	0.0358	0	0.0221	0.969	−0.329	0.290	0.0723	0	0	0.758	0.242
	0	0	0.0226	0	0.0196	0	0.439	−0.437	0.0931	0	0	0	0.647	0.353
$(5, \frac{1}{2})$	0	0	0.0033	0	0	0	−0.0519	−0.0519	0.0022	0	0	0	0.955	0.0451
	0	0	0.0033	0.0059	0	0.0037	0.159	−0.0537	0.0526	0.0023	0	0	0.952	0.0476
	0	0	0.0048	0	0.0035	0	0.0798	−0.0793	0.0034	0	0	0	0.914	0.0859
	0	0	0.0050	0.0092	0.0037	0.0057	0.248	0.0843	0.0809	0.0035	0	0	0.9066	0.0934

FIGURE 2.19: Fixed points and eigenvalues for vector-like fermions in the fundamental representation of $SU_c(3)$, in the 3-2-1 approximation scheme. The last two columns give the values of the ratio σ and ρ for α_2 or α_3 depending on the case (see 2.77).

In the 3-2-1 approximation scheme, there exist fixed points that can be reasonably traced back to those in the 2-1-0 approximation scheme. These solutions are shown in Figure 2.19, where we have included only FP with $|\vartheta| < 1$ in order to get small ratios ρ_i and σ_i. However, they all have at least one coupling that has to be zero at all scales, thus preventing a proper matching to the SM. We conclude that also all the models with the vector-like fermions in the fundamental representation of $SU_c(3)$ cannot provide an asymptotically safe extension to the SM.

Results 3: vector-like fermions in adjoint of $SU_c(3)$

For the adjoint representation ($p = q = 1$), the search over the same grid of values of (N_f, ℓ, Y) (thereby covering 126,000 further models) does not produce any FPs within the perturbative domain. In addition, there are always large eigenvalues of

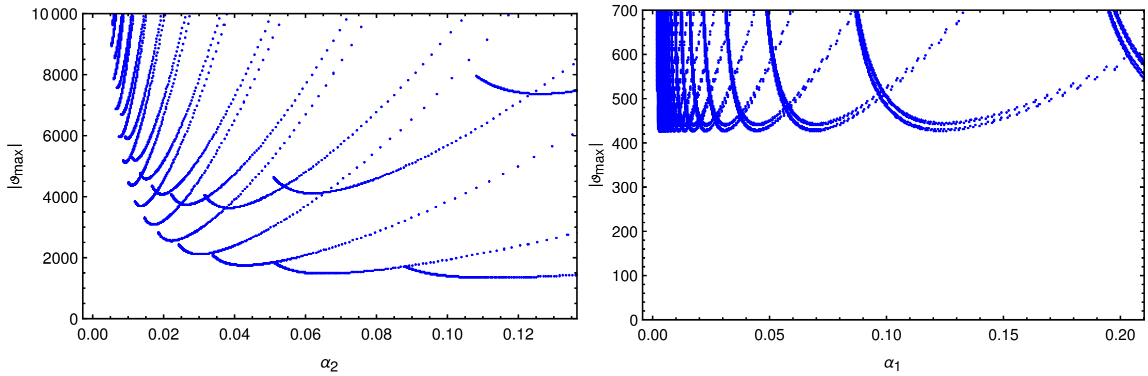

FIGURE 2.20: Distribution of the largest eigenvalue of the stability matrix for the $SU(3)$ adjoint representation. Left: Y-independent solutions. Right: Y-dependent solutions.

the stability matrix. That these models do not exhibit any viable fixed points is true both in the 2-1-0 and in the 3-2-1 approximation scheme.

In Figure 2.20 we show the distribution of eigenvalues in the 2-1-0 approximation scheme. We clearly see that the eigenvalues are rather large. In fact, the minimum eigenvalue in the Y-independent set of solutions is 1342, while in the Y-dependent set is 426.

This problem is confirmed by the study of the central charges. For the Y-independent fixed points we find for all fixed points δa of $O(1000)$. Similarly, the Y-dependent the fixed points have δa of $O(100)$. Tests of the c-function confirm these results, even though the a-function seems to be more sensitive, in the sense that it suffers greater relative change.

Again, we come up empty handed. The models with the vector-like fermions in the adjoint representation of $SU_c(3)$ do not provide a viable asymptotically safe extension of the SM. Higher $SU_c(3)$ representations are disfavored by experimental constraints because of the early onset of the modifications in the α_3 running. We may also conjecture from the previous analyses that they would produce even less reliable fixed point candidates.

Relaxing the criteria?

Having ruled out all possible candidates, one may wonder if the criteria in (2.67) and (2.74) might be too stringent and make us miss some potentially interesting models. Under relaxed assumptions we can indeed find fixed points that naively seem to be good candidates for an asymptotically safe extension of the SM. This is achieved if we allow for larger values of ϑ, i.e. if we allow a violation of the condition (2.74).

For example: consider the phenomenologically interesting case of $N_f = 3$ colorless vector-like fermions in the representation $\ell = 1/2$, $Y = 3/2$. Its FP coupling constants and eigenvalues are given in Figure 2.21. This example provides an interesting and non-trivial extension of the SM which includes non-Gaussian fixed point

(N_f, ℓ, Y)	α_1^*	α_2^*	α_3^*	α_t^*	α_y^*	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5	ρ_1
$(3, 1/2, 3/2)$	0.188	0	0	0	0.778	33.2	−3.36	−0.817	0	0	2.69

FIGURE 2.21: Values of the couplings at the fixed point, eigenvalues and ρ ratio for the model that almost works (2-1-0 approximation scheme).

value for the gauge coupling α_1 as well as for the Yukawa coupling α_y[9]. In particular, we see from Figure 2.21 that the first two scaling dimensions violate the criterion (2.74), but the fixed-point couplings satisfy the condition (2.67). We do not find any couplings frozen to zero and therefore a non-trivial RG flow towards the IR is possible. Let us therefore briefly suspend disbelief and proceed with the analysis of this candidate. Interestingly, we find a good matching to the SM couplings in the IR, with an error of the order of per mille, see Figure 2.22.

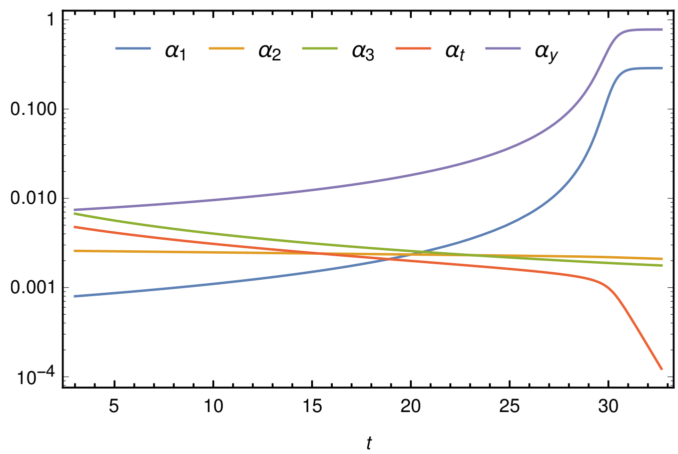

FIGURE 2.22: Evolution of the couplings with t in a logarithmic scale for the fixed point in Figure 2.21). This running provides a trajectory in the theory space connecting the fixed point and the physics at a matching scale around 2 TeV.

This model would seem to be a very promising candidate for an asymptotically safe extension of the SM, and yet it is not radiatively stable—a fact that vindicates the role of criteria in (2.74) as a filter for the physical fixed points . The 3-2-1 approximation scheme β-functions generate very different FPs, none of which can be traced back to the above one in the 2-1-0 approximation scheme. Moreover, all the 3-2-1 fixed points have a trivial coupling and cannot provide a viable extension of the SM.

Combining representations

Combining vector-like fermions in different representations, as done, for instance, in [45], [46], provides other examples of models that almost work. In the simplest scenario, we can try to construct a model with two types of vector-like fermions.

[9] Recall that the quartic scalar interaction does not renormalize in the 2-1-0 scheme.

In that case, we duplicate the last three terms in equation (2.88) for fermions $\tilde{\psi}$ and scalars $\tilde{S}$. We call the extra Yukawa coupling z with,

$$\alpha_z = \frac{z^2}{(4\pi)^2} \tag{2.121}$$

and assume no mixing between the two families.

α_1^*	α_2^*	α_3^*	α_t^*	α_y^*	α_z^*	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5	ϑ_6	ρ_2	ρ_1
0.226	0.193	0	0	0.778	0.534	241	24.2	−2.85	−2.28	−1.51	0	0	0

FIGURE 2.23: Values of the couplings at the fixed point of interest, eigenvalues and ρ ratios for the model combining 3 fields in the representation $(1,2,3/2)$ and 8 fields in the representation $(1,5,0)$ (2-1-0 approximation scheme).

Since many of the extensions beyond the SM attempt to describe dark matter, we take one of the possible minimal models discussed in [64] and identify some of the vector-like fermions with dark matter. This exercise makes clear the potential relevance of the asymptotic safety in selecting physics BSM.

In addition to the N_f vector-like fermions in the last-considered representation, we take N_{f_2} vector-like fermions with quantum numbers $p = q = 0$, $\ell = 2$ and $Y = 0$. That is, we take colorless quintuplets with no hypercharge. Within the 2-1-0 approximation scheme, for this combination $(1,2,3/2) \oplus (1,5,0)$, we realize that fixed points split in two categories: fixed points that depend on the number of quintuplets N_{f_2} and fixed points that do not. Clearly, the latter have $\alpha_y = 0$ so that the vector-like fermions enter only via loops in the gauge β-functions. Consequently, the conditions to lie on the critical surface of those fixed points imply that α_2 is identically equal to zero. This feature makes the corresponding fixed points uninteresting.

For the remaining fixed points, we find that in order to have $\alpha_i < 1$ for all couplings, the minimum number of quintuplets should be equal to eight. Taking the minimal case of $N_{f_2} = 8$, we find 6 fixed points, all of them having one large eigenvalue around 250. Thus, according to our perturbativity requirements, these fixed points are not reliable since there is always one ϑ which is much larger than 1. This is similar to what happens in the previous $SU(2)$-triplet model.

Despite non-perturbativity, we can find a matching with the SM. The only difference with respect to the previous model is that, in the present case, two matching scales are needed, the reason being that the large number of quintuplets makes α_2 decrease fast so that these fields must be decoupled at very high energies. In Figure 2.24 we show the logarithmic running of the couplings and the two different matching scales. The quintuplets decouple at an energy scale $O(10^{13})$ TeV (and must be considered "wimpzilla" dark matter [65]), and the doublets at an energy scale of the order 2 TeV. All the couplings go to the fixed point in Table 2.23

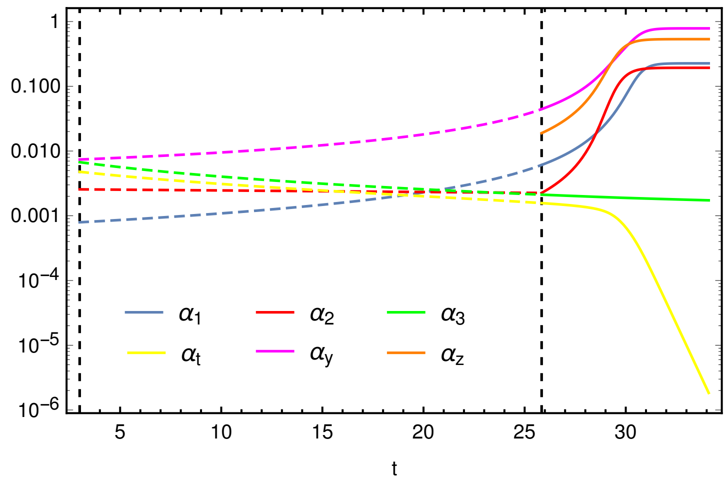

FIGURE 2.24: Evolution of the couplings with t in a logarithmic scale for the fixed point in Table 2.23 within the 210 approximation with 3 fields in $(1, 2, 3/2)$ and 8 fields in $(1, 5, 0)$. This running provides a trajectory in the theory space connecting the fixed point to a matching scale around 2 TeV passing through another matching (for the quintuplets) at about 10^{13} TeV.

Even though Figure 2.24 shows a nice flow of the coupling constants toward the SM, the size of the eigenvalues spells doom for the model and in fact the fixed point does not survive in the 3-2-1 approximation scheme.

2.3.4 Discussion

A systematic scan covering 378,000 possible extensions of the Standard Model based on vector-like fermions charged under the SM gauge group, carrying various representations and coming in several generations shows that there are no fixed points in the β-function of the models that satisfy both the perturbativity criteria and the RG flow compatible with the Standard Model in the IR. Non-perturbative FPs that appear in the 2-1-0 approximation scheme are absent when probed in the 3-2-1 approximation scheme. Fixed points that present themselves in both schemes or appear only at the higher order always contain a trivial solution in which at least one of the couplings is "frozen" to zero thus suggesting that the Landau problem of the LO theory persists at higher orders in the perturbative expansion. We conclude that it is not possible, with the models we have examined, to extend the SM up to arbitrarily high energies in perturbation theory.

This result can be interpreted in a number of ways. Within the context of asymptotic safety paradigm there is a multitude of scenarios that could be realized in nature. First possibility is, of course, that the asymptotic safety is a concept that is realized only in hypothetical models, but not in our physical reality. Second possibility is that the asymptotic safety is realized in our Universe. This possibility can be further divided as follows:

i) Above models may possess physical FPs that our computation has missed. We have observed that LISA-like fixed points arise from the cancellation of the gauge and Yukawa terms in the 2-1-0 scheme (analogous observation has been made about LISA [10]). In our computation, following standard practice, only the top Yukawa was tracked of all SM Yukawas, and all others were set to zero. This clearly underestimates the total Yukawa contribution to the beta functions, and it might be that the inclusion of the other Yukawa couplings is enough to generate physical fixed points that we've missed above. There is no certainty that this would happen, but it is clear that the computation would be more complicated and that the analysis would have to be performed in a much higher-dimensional (coupling) space. For this reason a simpler alternative computation could be done, parametrizing with a parameter n the effective number of Yukawa contributions. If reasonable values of n would yield interesting fixed points then this would make the full computation well worth pursuing, whereas if we would fail to identify interesting FPs even for large values of n it is likely that the full computation would not reveal any physical fixed points.

ii) Our results might indicate that the search must be enlarged to include models with more sophisticated BSM sectors. The gauge group may be extended, e.g. in the sense of Pati-Salam unification, which would potentially solve the $U(1)$-triviality problem and also be closer in spirit to the perturbative LISA fixed point. Work in this direction has already started [66]. In addition to the above, more complicated BSM matter sector could be studied. It is not clear at this stage how much such extensions would help since the vector-like fermions serve as a proxy for more general matter fields, and thus our results may be pointing to the possibility that the physical FPs exist only outside the scope of perturbation theory. Nevertheless, we think that this possibility is very well worth exploring, particularly because of its potential relevance for the ultra-high-energy model building. For example, notice that the model in Figure 2.22 reaches the non-Gaussian regime at around $10^{14} GeV$. Such non-standard dynamics in the early Universe may allow for rich inflation, leptogenesis, dark matter, and other relevant model building [41], [48], [67].

iii) Gravity might be an essential ingredient: in scenario ii) gravitational RG running kicks in around the Planck scale and it modifies the location and the properties of the FP, but it is not responsible for generating it. An alternative scenario that we should keep in mind is the possibility that rendering the Standard Model UV completions free of Landau poles may require the inclusion of gravity. That this might be achieved was shown for the first time in perturbation theory in [68]. This work inspired some controversy [69]–[72] which appears to be solved now, with latest answers [73]–[75] confirming the original work. Compatible results were also derived via functional renormalization group [76]–[81]. The main idea here is that gravity may feed into the running couplings of the particle model so as to induce asymptotic safety, taking on much more essential role than in case ii).

There is an important point about asymptotically safe QFTs which we didn't stress enough so far, and it is their strong predictivity. Consider, for example, perturbative fixed points in Figure 2.13 and Figure 2.15. It is never the case that all of the couplings are relevant, i.e. that all of them are free parameters. In fact, we find a couple of models which are maximally predictive; all of their (non-marginal) directions are irrelevant, meaning they do not have a single "tuneable" parameter. Generically what we expect from asymptotically safe theories from the above analysis, and from most analyses of the asymptotic safety of pure gravity or gravity-matter systems, are non-Gaussian UV fixed points which have a small number of relevant directions. This makes generic asymptotically safe models highly predictive, because it implies that the model has a small number of free parameters that should be input from the experiment, whereas all other parameters are then fixed by requiring the RG trajectory to sit on the UV critical surface. In principle this means that an asymptotically safe completion of the Standard Model may be able to explain the relations between many Standard Model parameters that have hitherto being considered as independent. This is particularly important in light of the negative LHC results.

In this section we have neglected gravity, and we have unsuccessfully searched for a viable fixed point within a certain class of BSM models. We have additionally discovered that the $U(1)$ coupling continues to be troublesome beyond Standard Model. One of the main tasks for the future models will therefore be to solve the problem of the $U(1)$ coupling running to the Landau pole. Motivated by the possibility that the UV completions of the SM possess non-perturbative FPs, and by the prospect of quantum gravity solving the triviality problem, in the next section we review functional renormalization group, a technique that promises us the ability to study non-perturbative RG flows.

Chapter 3

Asymptotic Safety in Gravity

This chapter is a logical extension of the previous one in two directions. Whereas the previous chapter was dealing with particle physics models in absence of gravity, using fairly standard techniques, this chapter goes beyond that in both ways.

Gravity is a fundamental force that has to be included in any description of fundamental physics that wants to claim any kind of completeness. It is evident from the previous chapter that pure particle models can be asymptotically safe, but at the same time it is difficult to construct realistic and phenomenologically viable asymptotically safe particle models. The Asymptotic Safety program has so far shown that pure gravity may be asymptotically safe on its own, and it may even impress its good RG qualities on matter models coupled to it. In this chapter we explore these questions in more detail.

Quantum Gravity

In the language of QFT we would say that Einstein's theory is an incredibly successful effective field theory (EFT). One reason for its validity over orders of magnitude of length scales is the smallness of Newton's coupling. Because of it, first quantum corrections to Newton's potential are so exceedingly small that they may never be directly measured [82]–[85]. The cutoff scale of the General Theory of Relativity (GR) is directly related to Newton's constant, and is known by the name of Planck energy.

Due to smallness of Newton's coupling the Planck scale is so high that any Earth-bound experiment that is supposed to perform direct measurements at the Planck scale is hardly concievable. This implies that for the purposes of explaining eventual future observations of radiative corrections to Newton's potential and similar observables the only quantum formulation of gravity that will ever be needed is given by the simple EFT of GR. Despite the fact that EFT of GR is a good description of quantum gravity all the way up to the Planck scale, it is clear that EFT is not a UV completion of GR but it is rather a good computational tool at "low energies".

What we care about in this chapter is a UV completion of the quantum GR which is supposed to provide us with insights into some fundamental questions regarding, for example, the nature of the microscopic structure of spacetime. The answers to these kinds of questions are what we primarily want from a quantum theory of gravity, and it is where EFT falls short of providing an answer. A consistent quantum theory of gravity should be a UV completion of GR so that it is well defined at all energy scales, and so that it consequently has some prospect of answering questions about the structure of spacetime, centers of the black holes, initial singularity of the cosmos, etc.

Asymptotically Safe Quantum Gravity

In sections 3.2 and 3.3 we will work in the context of an approach to quantum gravity commonly known as the Asymptotic Safety program. The main premise of the Asymptotic Safety program is that gravity is an interaction that may be quantized as a quantum field theory, where the fundamental field is (probably) the metric, and such that its renormalization group flow flows from GR in the IR towards an interacting QFT in the deep UV. [1] In this sense gravity is postulated to be fundamental according to Wilson.

Every theory of quantum gravity forces us to some trade-offs. String theory (at least as thought of in 1960s through 1980s) required us to abandon quantum fields, background independence and more. On the other hand string theory may be studied using perturbation theory, so that one has standard level of control in most computations. In Asymptotic Safety, instead, physics stays rather familiar, and we are required to trade away our standard mathematical tools. In fact, the fixed point, if it exists, should exist within a nonperturbative regime of the couplings (e.g. Newton's coupling). This implies that in principle we may continue to rely on the acquired physical intuition, but we have to acknowledge decreased reliability of the computations.

On the other hand Asymptotic Safety is a very promising approach to quantum gravity, not the least because it turns out to be very practical. By staying close to standard physics it is comparatively simple to extract physical predictions (and postdictions) from this theory than it is from most of its competitors. For example, we have found in section 2.3 that the $U(1)$ coupling may present a problem for finding a UV complete particle physics model, but there are already hints that these problems may be cured by coupling $U(1)$ to gravity because as gravity is flowing to a fixed point its backreaction modifies the RG flow of $U(1)$ to the point that it renders it asymptotically safe/free (instead of UV diverging) [80]. Even more importantly, in principle Asymptotic Safety may be a highly predictive (falsifiable) paradigm, and this seems to be realized in practice in many systems studied so far. For example it seems to be the case that one can predict the mass of the top quark [86], the mass of the Higgs particle [87], and much more within this framework.

It is difficult to stress too much the importance of the above statement. As we've said in the introduction to this chapter, it is hard to imagine an experiment directly measuring Newton's potential close to Planck's energy. It is therefore extremely

[1] This may be compared to mass-deformed LISA which was connecting pure Yang-Mills theory in the IR to an interacting theory of gauge fields, fermions and scalars in the deep UV.

interesting and important that Asymptotic Safety promises to be able to make predictions about experimentally accessible IR physics using the postulate of existence of a UV FP. Just as it is hard to imagine experiments at Planck scale, so it is hard to imagine a proof of existence of a gravitational FP with mathematical level of rigor. Crucial idea here is that such a thing is not necessary by any means because for practical purposes all that matters is that in some approximation there is a fixed point, and the existence of the fixed point in this approximation gives us non-trivial information about the experimentally accessible physics.

We begin the next section by giving more technical details about the techniques used to study Asymptotic Safety of gravity. This will be of direct use in the rest of the chapter.

3.1 Introduction

In the following two sections we will deal with non-perturbative QFT in the sense of non-perturbative couplings[2]. This kind of non-perturbativity can be very complicated. Consider QCD close to or below confinment scale Λ_{QCD}. There's a number of phenomena that occur there, from confinment, over chiral symmetry breaking to effectively changing the fundamental degrees of freedom from quarks and gluons to mesons. Perturbative QCD is of little help in this regime. Historically there were two routes that physicists followed in dealing with strong dynamics. First is to (partially) abandon analytic computations and proceed with lattice formulation of QFT (meaning numeric, Monte Carlo simulations). The second one was essentially guessing low-energy Lagrangians and then again combining perturbation theory with this low-energy effective QFT description (also known as chiral perturbation theory).

Strong Gravitational Dynamics

If gravity is non-perturbative in the deep UV, and we assume so from perturbation theory (and also from functional renormalization group that will be discussed in a moment), we can again try to follow the two aforementioned approaches.

First of them would be a lattice formulation of gravity [88]–[96]. Here one starts from an optimistic expectation that the gravitational UV dynamics can be obtained by discretizing GR. The caveat is that lattice formulations have to assume microscopic structure of the theory and thus we are placing faith in the idea that discretization of geometry is the right thing to do. This is opposite from what is done in QCD where one discretizes UV action in weak-dynamics regime and then computes low-energy states/observables in the strong-dynamics regime. In gravity one is guessing the microscopic structure in the strong-dynamics regime from the very

[2]This meaning of "non-perturbative" is distinct from the meaning used in chapter 2 where non-perturbativity referred to non-analiticity in the sense of instanton effects.

beginning so all modern approaches (e.g. Causal Dynamic Triangulations and Euclidean Dynamic Triangulations) essentially assume that metric is the fundamental degree of freedom and then discretize it.

The second approach is even more optimistic as it entails guessing the fundamental (UV) degrees of freedom and then trying to construct more appropriate Lagrangians, often hoping to maintain perturbativity in this new description. One could argue that String Theory, Loop Quantum Gravity, Group Field Theory/Tensor Networks, and some other approaches follow this strategy. This approach is notoriously difficult, not the least because it often requires understanding or developing a significant amount of non-standard mathematics, but even more so because such theories are often so different from the Standard Model and the rest of "standard" physics that it becomes incredibly difficult to extract solid physical predictions.

The approach that we will follow here is a third road which is optimistic in its own way. If gravity becomes non-perturbative in the UV, but not very non-perturbative, then classical intuition of having the metric as a fundamental degree of freedom may still be correct (or at least useful) one, and what's even more important it allows us another venue for attacking the problem of quantization of gravity. If non-perturbativity of the UV dynamics is small enough so that basic perturbative reasoning is still correct, for example regarding operator ordering, one may go far by using a technique called functional renormalization group or FRG.

The last scenario is what the bulk of research on Asymptotic Safety over the past two decades is pointing to. Additionally, it is maybe the most interesting option of all the aforementioned. [3]

Overview of the FRG

The functional renormalization group (FRG) is a collective name for a number of different techniques that all start from the partition function, then modify it in some way, and finally rewrite it as a differential RG flow equation. A defining feature in all of them is to not make any approximations when deriving the flow equation. In this sense the equations may be considered exact. The equation that is most widely used in the Asymptotic Safety program is called Wetterich equation [97], and it was applied to gravity for the first time by Reuter in 1996 [98].

Let us denote all the fields in a theory by Φ, and let the action $S[\Phi]$ also contain a gauge-fixing action and the ghost action corresponding to it. Wetterich equation follows from a modified partition function Z_k, obtained from the standard partition function,

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]} \,, \tag{3.1}$$

[3] Non-perturbative physics is much less understood and much less developed than standard perturbative one, yet we know that it contains a number of extremely interesting phenomena s.a. chiral symmetry breaking. If we are truly witnessing an "almost perturbative" non-perturbativity we are given a chance to study novel non-perturbative phenomena using both new analytic techniques and Monte Carlo simulations.

by adding to the action $S[\Phi]$ a cutoff action $\Delta S_k[\Phi]$. k stands for the RG scale[4] and $\Delta S_k[\Phi]$ provides an infrared regulator of the form

$$\Delta S_k[\Phi] = \frac{1}{2}\int_x \Phi R_k(\Delta)\Phi \, . \tag{3.2}$$

The cutoff action, which can also be thought of as a momentum-dependent mass term, will be discussed in more detail below; for the derivation of the Wetterich equation it is crucial only that it is quadratic in the fluctuating field so that it does not affect the vertices but only the propagators.

The k-dependent partition function reads,

$$Z_k[j] = \int \mathcal{D}\Phi \exp\left(-S[\Phi] - \Delta S_k[\Phi] + \int_x j\Phi\right) \, . \tag{3.3}$$

We now define $W_k[j] = \log Z_k[j]$ and then define its Legendre transform ($\phi = \langle\Phi\rangle$),

$$\tilde{\Gamma}_k[\phi] = -W_k[j_\phi] + \int_x j_\phi \phi \, . \tag{3.4}$$

The Wetterich equation is an exact RG flow equation for the effective average action (EAA) which is defined by subtracting the cutoff from $\tilde{\Gamma}_k$,

$$\Gamma_k[\phi] = \tilde{\Gamma}_k[\phi] - \Delta S_k[\phi] \, . \tag{3.5}$$

Let us now derive Wetterich equation. RG scale derivative of W_k reads,

$$k\frac{d}{dk}W_k \equiv \frac{d}{dt}W_k = -\frac{d}{dt}\langle\Delta S_k\rangle = -\frac{1}{2}Tr\langle\Phi\Phi\rangle\frac{dR_k}{dt} \, . \tag{3.6}$$

From the definition of the EAA and the W_k as the generator of Green's functions we have,

$$\frac{d}{dt}\Gamma_k = -\frac{d}{dt}W_k - \frac{d}{dt}\Delta S_k \tag{3.7}$$

$$= \frac{1}{2}Tr\left(\langle\Phi\Phi\rangle - \langle\Phi\rangle\langle\Phi\rangle\right) \tag{3.8}$$

$$= \frac{1}{2}Tr\frac{\delta^2 W_k}{\delta j \delta j}\frac{d}{dt}R_k \tag{3.9}$$

From $\frac{\delta\tilde{\Gamma}_k}{\delta\phi} = j$ we deduce that $\frac{\delta^2 W_k}{\delta j\delta j} = (\frac{\delta^2\tilde{\Gamma}_k}{\delta\phi\delta\phi})^{-1}$. This identity can be substituted in the RHS of the above RG flow, after which $\tilde{\Gamma}_k$ can be rewritten in terms of the EAA. These transformations bring us to the final form of the Wetterich equation,

$$k\partial_k\Gamma_k[\phi] \equiv \dot{\Gamma}_k[\phi] = \frac{1}{2}STr\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1}\dot{R}_k \, . \tag{3.10}$$

[4]We use k instead of μ for the RG scale because it is the standard notation in the Asymptotic Safety literature.

Let us explain some FRG technology by working through an example. We will consider a scalar theory with Lagrangian equal to

$$\mathcal{L} = \frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + \lambda\phi^4 \,. \tag{3.11}$$

First important notion is the notion of truncations. When dealing with non-perturbative physics infinitely many operators are generated by the RG flow, just as in the usual (perturbatively non-renormalizable) EFTs. We call the infinite dimensional space of all possible couplings/operators allowed by the symmetries of the theory a theory space. Before starting the computation one has to decide what subset of operators in the theory space one wants to work with, and this choice defines one's truncation (of the theory space). In the course of the computation only the couplings belonging to the truncation are considered and the rest is "projected out" (i.e. neglected).

In the case of Wetterich equation, just as with any other FRG equation, in practice one always works within a truncation, so they are not literally “Exact” RG Equations despite being commonly given this name. Nonetheless, they are much more powerful than standard perturbation theory since they do allow us to work with infinitely many couplings. In this sense they truly are functional RG equations.

In particular, instead of studying the Lagrangian shown above, we may decide right away to study a much more general Lagrangian with the same symmetries,

$$\mathcal{L} = \frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + V(\phi^2). \tag{3.12}$$

This kind of truncation would be called a (leading order) derivative expansion. The naming comes from the fact that if you think of expanding the potential V in power series, we're keeping all powers of ϕ in our truncation, but we are only keeping up to two derivatives. Next order in the derivative expansion would allow a generic function of ϕ in front of the kinetic term, the one after that would include generic four-derivative operators, and so on. This expansion may work very well if the physics in the UV is very local. If physics is not very local, then having exact dependence on ϕ at the cost of throwing away all derivatives may be a bad choice. In the case of much less local physics a better truncation would be,

$$\mathcal{L} = \frac{1}{2}\phi f(\eta_{\mu\nu}\partial^\mu\partial^\nu)\phi. \tag{3.13}$$

We call such an expansion a vertex expansion, or a field expansion, because it only keeps vertices with up to a certain number of fields in them (two in the above case), but it keeps the full momentum dependence of the vertices. In more complicated theories, e.g. Yang-Mills or gravity, this separation of truncations is not going to be as clear-cut, but there will still exist analogous systematic approaches.

In order to study (3.12) using the Wetterich equation (3.10) we need to construct the effective average action. We can choose it as,

$$\Gamma_k[\phi] = \int_x \frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + V_k(\phi^2)\ . \tag{3.14}$$

In particular, to write the Wetterich equation we will need the Hessian,

$$\Gamma_k[\phi]^{(2)} = -\partial^2 + \frac{\delta^2 V_k(\phi^2)}{\delta\phi\delta\phi} = -\partial^2 + 2V_k' + 4\phi^2 V_k''\ , \tag{3.15}$$

where primes generically stand for derivatives wrt the argument of the function (in this case ϕ^2), and the derivative of the EAA wrt the RG time,

$$\dot{\Gamma}_k = \int_x \dot{V}_k\ . \tag{3.16}$$

In addition to the EAA, Wetterich equation requires us to specify the cutoff action ΔS_k. From the derivation of the Wetterich equation we known that the cutoff action has to be of the form,

$$\Delta S_k = \int_x \frac{1}{2}\phi R_k(\eta_{\mu\nu}\partial^\mu\partial^\nu)\phi\ . \tag{3.17}$$

In order for this action to act as a good IR cutoff the regulator function R_k has to satisfy a couple of conditions. First condition is that $R_k(z) \to 0$ for $k \to 0$ (for every z), so that correct quantum effective action is reached in the IR. Second, $R_k(z)$ for fixed k is a monotonically decreasing function of z, and for fixed z is a monotonically increasing function of k. Next, for $z = 0$ the regulator equals k^2. Finally, for $z > k$ the regulator goes to zero sufficiently fast (so as to only suppress the IR modes). Besides these few conditions the choice of R_k is completely arbitrary.

One particular choice that proves to be very convenient for analytical computations is known as Litim's cutoff (or optimized cutoff) and it corresponds to the following choice:

$$R_k(z) = (k^2 - z)\theta(k^2 - z)\ . \tag{3.18}$$

Suppose we decided to leave wavefunction renormalization constant in the Lagrangian; at the very least it would be a function of the RG scale k.[5] Then one would have to decide whether to include the wavefunction renormalization constant in the R_k or whether to leave it out. Since R_k is acted upon by the $k\partial_k$ operator on the RHS of the Wetterich equation this affects the RG flow that one obtains. The choice of not including the wavefunction renormalization constant in the cutoff is usually referred to as a "pure cutoff". The other choice, which will be used in the following section, is often employed but does not have any particular name.

Let us proceed with the pure cutoff in which case the RG-time derivative of the R_k reads,

$$\dot{R}_k(z) = 2k^2\theta(k^2 - z)\ . \tag{3.19}$$

[5] More generally it would also be a field dependent function.

We have all the pieces we need to assemble Wetterich equation, but in the end we want to write it for dimensionless couplings. For this reason the final step that we have to do is to pass from dimensionful to dimensionless fields and couplings. In particular $\phi = k^{d/2-1}\varphi$, $V_k(\phi^2) = k^d v_k(\varphi^2)$, and eigenvalues of Δ are $z = k^2 y$. For example dimensionless form of the kinetic term is [6]

$$\int d^d x \phi \Delta \phi \to \int d^d x k^{-d} k^{2(d/2-1)} \varphi z k^2 \varphi = \int d^d x \varphi y \varphi \,.$$

After these transformations we have the following dimensionless expressions,

$$\Gamma_k \to \int_x \frac{1}{2} \eta_{\mu\nu} \partial^\mu \varphi \partial^\nu \varphi + v_k(\varphi^2)\,, \tag{3.20}$$

$$\dot{\Gamma}_k \to \left(d v_k + \dot{v}_k - (d-2)\varphi^2 v_k' \right)\,, \tag{3.21}$$

$$\Gamma_k^{(2)} \to y + (2v_k' + 4\varphi^2 v_k'')\,, \tag{3.22}$$

$$\tag{3.23}$$

where primes on v_k stand for derivatives wrt its argument, φ^2. Writing (3.18) as $R_k(z) = k^2 r_k(y)$ the dimensionless regulator takes form,

$$r_k = (1-y)\theta(1-y)\,. \tag{3.24}$$

Dimensionless form of the dR_k/dt can be read-off from (3.19), or reconstructed using (3.24) and

$$\dot{r}_k = -2y r_k'(y)\,, \tag{3.25}$$

which together imply[7],

$$\dot{R}_k = 2k^2(r_k - y r_k') = 2k^2\theta(1-y)\,. \tag{3.26}$$

We may now assemble the Wetterich equation. It reads,

[6] There are two ways to think of this transformation. Either metric is considered dimensionless and coordinates are thought of as dimensionful, in which case we've passed from dimensionful x on the LHS to dimensionless x on the RHS, or coordinates are thought of as dimensionless but metric is dimensionful ($[g_{..}] = 2$, $[g^{..}] = -2$, $[det(g)] = -2$), in which case we've passed from dimensionful Minkowski metric on the LHS to dimensionless on the RHS. The end result is the same, all fields and eigenvalues are dimensionless on the RHS.

[7] First equality in (3.26) is correct, but in the second one we are dropping some terms of the form $\sim x\delta(x)$ which are distributionally zero here, but should be kept in general.

$$
\begin{aligned}
\dot{\Gamma}_k =& \frac{1}{2} STr \left(\Gamma_k^{(2)} + R_k \right)^{-1} \dot{R}_k \\
=& \frac{1}{2} STr \left(y + (2v_k' + 4\varphi^2 v_k'') + r_k \right)^{-1} (2r_k - 2y r_k') \\
=& \frac{1}{2} STr \left(1 + (2v_k' + 4\varphi^2 v_k'') \right)^{-1} 2\theta(1-y) \, ,
\end{aligned}
$$

where first line is still given in terms of dimensionful quantities and we pass to dimensionless quantities in the following two lines. Substituting (3.21) in place of $\dot{\Gamma}_k$ on the LHS we write,

$$
\dot{v}_k = -d v_k + (d-2)\varphi^2 v_k' + STr \frac{\theta(1-y)}{1 + 2v' + 4\varphi^2 v''} \, . \tag{3.27}
$$

Trace is extremely simple to calculate in this case, and taking the eigenvalues to be $y \equiv p^2$ we've got,

$$
STr\, \theta(1-y) = \int \frac{d^d p}{(2\pi)^d} \theta(1-p^2) \tag{3.28}
$$

$$
= \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^1 |p|^{d-1} d|p| \tag{3.29}
$$

$$
= \frac{1}{d} \frac{\Omega_{d-1}}{(2\pi)^d} \, , \tag{3.30}
$$

where Ω_{d-1} comes from integration over the solid angle. Substituting the trace back into the flow equation we arrive at,

$$
\dot{v}_k = -d v_k + (d-2)\varphi^2 v_k' + \frac{1}{d} \frac{\Omega_{d-1}}{(2\pi)^d} \frac{1}{1 + 2v' + 4\varphi^2 v''} \, . \tag{3.31}
$$

Let us extract the beta function of the quartic coupling, assuming potential $v = \sum_{n=0}^{\infty} (n!)^{-1} \lambda_{2n} \varphi^{2n}$. We extract the required beta functions using a "projection technique". In this case the projection technique to extract $\dot{\lambda}_{2n}$ is to differentiate the flow equation n times wrt φ^2 followed by evaluating it at $\varphi = 0$. Doing this on both sides of the equation, we arrive at the following expression,

$$
\dot{\lambda}_4 = -d\lambda_4 + (d-2) 2\lambda_4 + \frac{1}{d} \frac{\Omega_{d-1}}{(2\pi)^d} \left(\frac{72\lambda_4^2}{(1+2\lambda_2)^3} - \frac{10\lambda_6}{(1+2\lambda_2)^2} \right) \, . \tag{3.32}
$$

We can guess from this equation that the beta function for each coupling λ_{2n} is going to depend on couplings $\{\lambda_{2n}, \lambda_{2(n-1)}, \lambda_{2(n+1)}\}$. We may follow two separate roads here. One is to go back to the full functional form of the Wetterich equation (3.31) and solve numerically for the full potential. Second one, which we follow here, is to truncate the infinite tower of beta functions. We do this by choosing $\lambda_6 = 0$, which is a simple but arbitrary choice. To keep things simple we will also continue in the massless limit, $\lambda_2 = 0$. Then the beta function of λ_4 reduces to

$$\dot{\lambda}_4 = (d-4)\lambda_4 + \frac{1}{d}\frac{\Omega_{d-1}}{(2\pi)^d}72\lambda_4^2\,. \tag{3.33}$$

It is clear from this expression that the system admits a non-Gaussian fixed point for $d < 4$ (which "deforms" with d to join the Gaussian fixed point for $d \geq 4$). We find the fixed points by setting the LHS to zero and solving the remaining algebraic equation for the fixed point λ_4^* (i.e. $\beta_4(\lambda_4^*) = 0$),

$$\lambda_4^* = \frac{(4-d)d(2\pi)^d}{72\Omega_{d-1}}\,. \tag{3.34}$$

Had we kept λ_2, λ_6 and higher order couplings we would have found the fixed point in an analogous manner, by solving a system of algebraic equations. To conclude, let us restrict ourselves to the 4d case ($d \to 4$, $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2) \to 2\pi^2$),

$$\dot{\lambda}_4 = \frac{1}{4}\frac{2\pi^2}{(2\pi)^4}72\lambda_4^2 = \frac{36}{16\pi^2}\lambda_4^2\,. \tag{3.35}$$

We can redefine the couplings to match the more familiar form of the potential, $\frac{\lambda_4}{4!}\varphi^4$, so that the one-loop coefficient takes on its usual value, $3/(16\pi^2)$. This shows that the Wetterich equation correctly reproduces the universal beta function for the quartic coupling!

Limitations of the FRG

Now that we have established the methodology of the functional renormalization group, let us briefly discuss its limitations.

First obvious limitation is the fact that the running functional here is not the effective action, but a related functional known as the effective average action (EAA). This problem can be bypassed to a certain extent by working with other flow equations besides the Wetterich equation. For example, one may write a proper-time-regularized FRG equation for Wilsonian effective action [99]–[101]. This is physically more transparent, but then one (probably) loses some nice mathematical properties of the Wetterich equation which make it computationally efficient (e.g. its one-loop structure). Most of the Asymptotic Safety literature is using the Wetterich equation, although there may definitely be some advantages to using proper-time FRG.

Second limitation has to do with choosing a truncation. Choosing relevant operators (in non-technical sense) and robust, trustworthy truncations strongly depends on our understanding of the physical system. When constructing chiral Lagrangians one can (and should) follow experimentally confirmed conservation laws, symmetry principles etc. We have no such help at our disposal when dealing with gravity, simply because we do not know the UV physics. This means that one relies exclusively on (semi-)classical thinking, and one trusts that metric may be used as the fundamental degree of freedom, and that canonical operator ordering is still valid in the deep UV. In this way one accumulates evidence for the existence of the fixed point by first finding it in a specific truncation, and then checks for its"stability" in the extended truncations. By stability we mean that the fixed point survives at the approximately same location with reasonably similar properties such as values of the critical exponents. In the best case scenario one may hope for convergence of gauge-independent quantities towards some final values in large, robust truncations. It should be clear, however, that this is by no means the kind of convergence that one expects in perturbation theory. Bottom line is that one should approach thinking about gravity in the Asymptotic Safety scenario from more of a lattice/engineering/computational scientist point of view than from a typical particle physicist viewpoint.

Third limitation is the arbitrariness involved in choosing the regulator function R_k. Just like the gauge choice is in principle arbitrary, but there may be such a thing as a very convenient choice, so too there are good choices of regulators. One such choice is the optimized cutoff used above, and the reason for its widespread use is mostly the fact that it often leads to simple analytic expressions. One should again be aware, however, that we are not working with perturbation series here, so it is not entirely clear how much the results may depend on the choice of regulator. Taking $k \to 0$ in the end of the computation removes the regulator and returns the quantum effective action. In practice we're always truncating the theory space so we're getting an approximation to the exact quantum effective action in the IR. Working beyond perturbation theory, it is fair to say that the only way to gain confidence about (reasonable) invariance of the results under the choice of R_k is to perform the same computation for multiple choices of R_k and verify explicitly stability of the results.

FRG for gravity

Let us discuss some peculiarities of the FRG applied to gravity. A good general reference for these topics is the recent book by Percacci[102]. First thing to note is that one cannot directly write Wetterich equation for the full quantum metric field g. The reason is that one has to gauge-fix and construct the argument of the cutoff profile R_k using a metric, but this clearly cannot be the full metric g. We thus introduce another metric, called background metric, $\overline{g}$ for these purposes. This directly implies that the EAA depends on two fields, the background metric and a fluctuation field h. We may also write it as a function of the full field g and the background field $\overline{g}$.

Having two metrics introduces additional arbitrariness in the computations because one can decide both in what way to split the full metric into the background and the fluctuating fields, but also around which specific background one wants to expand in fluctuations. Some of the often used backgrounds are flat background and Einstein spaces (e.g. the sphere and the hyperboloid)[8]. Common choices for splitting the metric are the so-called linear split, $g = \overline{g} + h$, and the exponential split, $g = \overline{g}\exp(h)$, although some other choices have been studied too.

Arbitrariness in parametrizing the fields does not stop at choosing how to split the metric. Just as one can decompose gauge fields into transverse and longitudinal components so can one decompose the metric field into the trace and the trace-free part. The trace-free part may then be further decomposed in a way that generalizes flat-spacetime decomposition into irreducible representations of the Lorentz group. This decomposition is called York- or transverse-traceless decomposition and it takes the following form for the fluctuation field,

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \overline{\nabla}_\mu \xi^T_\nu + \overline{\nabla}_\nu \xi^T_\mu + \overline{\nabla}_\mu \overline{\nabla}_\nu \sigma - \frac{1}{d}\overline{g}_{\mu\nu}\overline{\nabla}^2\sigma + \frac{1}{d}\overline{g}_{\mu\nu} h^{Tr} , \qquad (3.36)$$

where $\overline{\nabla}$ is the Levi-Civita connection of the metric $\overline{g}$, and the fields on the RHS satisfy the following properties,

$$h^{TT\ \mu}_{\ \mu} = 0 \ \ \textit{(traceless)} \qquad (3.37)$$

$$\overline{\nabla}_\nu h^{TT\ \nu}_{\ \mu} = 0 \ \ \textit{(transverse)} \qquad (3.38)$$

$$\overline{\nabla}_\nu \xi^{T\ \nu} = 0 \ \ \textit{(transverse)}\,. \qquad (3.39)$$

When quantizing the theory, one has to decide whether to path-integrate over the field h, or over the “component fields” in the York decomposition.[9]

Lightning Review of the State of the Art

Before moving to the two central sections of this chapter we briefly review some of the main directions in the Asymptotic Safety research currently undergoing intense develoment. We will focus on the three directions to which the author has had the opportunity to contribute during his doctoral training and which are directly related to the following sections[10].

In the past twenty-two years there has been a significant accumulation of evidence for the existence of the gravitational fixed point [103]–[115]. The main question now is what to do with this information, how can we make it useful? Just as asymptotic freedom would not be more than a nice idea in a Universe with too many fermions, so too one has to check whether the gravitational fixed point survives contact with reality, meaning inclusion of realistic matter models. As already

[8]Note that the flat-space expansion by no means assumes the fluctuations are perturbative.

[9]In the latter case one needs to pay attention to determinants that arise in going from h to the new fields h^{TT}, h^{Tr},...

[10]It should go without saying that there are equally important questions being studied by other groups worldwide which we do not mention in the following discussion. Among others, the list of these questions contains for example cosmological models and phenomenology based on the Asymptotic Safety scenario, fixed point studies in the ADM framework, high order vertex expansion computations for pure gravity, and much more.

argued before, this is also extremely important if one ever hopes to do serious phenomenology of quantum gravity, given that Planck scale will remain unreachable for the forseeable future.

Recent years have brought papers studying truncations ranging from gravity with non-interacting matter (minimally coupled to gravity) to interacting matter and non-minimal interactions with gravity [116]–[122]. At the time of writing of this thesis all papers on the topic seem to suggest that the Standard Model is compatible with the existence of the gravity-matter fixed point. It is worth pointing out that there is some controversy regarding the question of whether the gravity-matter fixed point can exist for an arbitrary number of fields, or whether too much matter can destroy the fixed point. Early works have suggested that there may be such a thing as too much matter [116], and this criterion was claimed to be rather strong, forbidding most GUT or SUSY models. This is another example of how Asymptotic Safety conjecture can be rather powerful. Some later works have instead suggested that the fixed point will always exist, because gravity is claimed to always rule supreme and tame the UV behavior of the rest of the matter sector, no matter how large [123]. The main reason for the discrepancy of the results is the fact that computations were done in different setups (different truncations, gauges, etc). This underlines the point that the results obtained via FRG are usually not as definitive as perturbation theory results. The way this problem will be solved in the future is probably through increasingly refined truncations until finally some convergence is demonstrated.

Besides having power to exclude models, recently there have also been some very interesting hints about the potential predictivity of Asymptotically Safe gravity coupled to matter systems (see the review [124]). It seems to be the case that the Asymptotic Safety of gravity-matter systems is capable of putting bounds on the Higgs mass [80], curing the U(1) Landau pole [79], [80], "predicting" the mass of the top quark [86] and more [81], [125]. The research is still in the early stages and we do not have the robustness of perturbation theory on our side, but these results look incredibly promising.

The following section is based on the paper [2] that formed a small contribution to this large research program. In particular, in section 3.2 we provide the answer to the question whether there can exist a fixed point at which gravity is asymptotically safe, but all other matter is asymptotically free. Our computations give strong indications that this cannot happen, as expected based on general arguments discussed in the beginning of section 3.2. This result is important because it demonstrates that one should not put too much trust in truncations that include only non-interacting matter. In fact, the RG flow of the gravity-matter interaction present in the kinetic term will generate operators compatible with the symmetries of the kinetic operator whose fixed point values will be non-vanishing. This guarantees that dynamics at the fixed point will always be nontrivial.

Second major research program is related to the concept of background independence in quantum gravity. The main reason for caring about this question is not of philosophical, but of technical nature. Let us look at the question in more detail.

Suppose we start from some effective average action, $\Gamma_k[h, \bar{g}]$, and we want to write the Wetterich equation. The simplest approximation that one can do is called background field approximation (BFA). Having computed the Hessian, $\Gamma_k[h, \bar{g}]^{(2)}$, the BFA consists of putting $h = 0$ on the RHS of the Wetterich equation. This approximation turns out to be more robust than one might initially expect, and in fact it was the basis for most research on Asymptotic Safety in the first fifteen years or so since the seminal work by Reuter[11]. This technique generally goes in pair with truncations containing operators which are curvature invariants constructed with the background metric. Trace on the RHS is usually dealt with via heat kernel asymptotic expansion methods. Most important for our discussion is the fact that in this approach one is studying the flow of the "background couplings". This name refers to the Newton coupling and other gravitational couplings constructed with the background metric alone.

Now suppose we only look at one operator, such as the Ricci scalar, R. In the bare action the coupling in front of R is $1/16\pi G_N$, where G_N stands for Newton's constant. Let us expand the full metric g into $\bar{g}$ and h, and then further expand $R = \bar{R} + O_1^{\mu\nu} h_{\mu\nu} + \mathcal{O}(h^2)$. If diffeomorphisms were perfectly preserved in quantum gravity the coupling in front of $\bar{R}$ would be the same as the coupling in front of all the other higher order operators starting with O_1. In this case BFA would contain all the information contained in any higher order vertex, so higher order vertex expansions would not be necessary. This, however, is clearly not the case because the fields $\bar{g}$ and h are independent arguments of the EAA $\Gamma_k[h, \bar{g}]$. In fact, because $\bar{g}$ and h are independent we should think of all the copies of the Newton's coupling that appear in front of different operators in the above expansion of the Ricci scalar as independent couplings. Each of these Newton couplings will have a different behavior under the RG flow, but all of them will be related. The relation comes in the form of Ward identities. In fact, considering linear split for example, the full metric equals $g = \bar{g} + h$, so it is invariant under the shift $\bar{g} \to \bar{g} + \epsilon$ compensated by an equal and opposite shift in the fluctuation field, $h \to h - \epsilon$. If the full quantum effective action is to maintain this invariance, "split Ward identities" [12] have to be imposed s.t. there is still an implicit relation between the copies of the Newton's coupling (and similarly for all the other couplings). In the context of FRG this discussion is complicated by the fact that the diff-invariance is not only broken by gauge fixing, but also by the cutoff action. This implies that looking at on-shell quantities is not enough to recover diff-invariance, one in principle also has to take the RG scale k to zero. In practice one cannot obtain exact diff-invariance because one is always

[11] Extensive introduction to the BFA and related techniques can be found in [102].

[12] "Split" in the name of Ward identities refers to the name "split symmetry" of the related symmetry of the metric. Note that we do not differentiate between Nielsen, Slavnov-Taylor and Ward identities but universally refer to all of them as Ward identities.

constrained to work in a truncation. Split Ward identities that are modified so as to keep track of the breaking of diff-invariance due to the regulator term are dubbed modified split Ward identities or msWI.

To recapitulate, one can go beyond BFA by considering vertex expansion. It is expected to give different and probably more precise results than BFA. Nonetheless, different quantum "copies" of the same underlying classical coupling will have different RG behavior and the information about their relation is contained in the msWI.

There are a couple of different ways of dealing with msWI's. First one is the so-called bi-metric approach [126], [127]. Second one is to neglect msWI's altogether and to focus on high-order vertex expansions, the idea being that in a sufficiently high order truncation one should see signs of convergence of the behavior of various copies of the same coupling[113]. Finally, one may try to solve the msWI's and use that to rewrite the EAA as a functional of only one variable, g [3], [113], [128]–[131]. We postpone the rest of this discussion for the section 3.3 which is a contribution in the last described direction of solving msWI's in some specific contexts.

Last active area of the Asymptotic Safety research that we will comment on here is related to the so-called functional truncations. This subset of the Asymptotic Safety research program deals with truncations such as the $f(R)$ truncation, $\Gamma_k = \int_x \sqrt{|g|} f(R)$, and the analysis is usually done within the BFA [108], [112], [132]–[136]. As compared to simpler truncations, e.g. the Einstein-Hilbert truncation, functional truncations try to improve robustness in a very different way from vertex expansions. While vertex expansions gain robustness by considering higher order vertices and flow of the "quantum" couplings, functional truncations (as considered thus far, in BFA) attempt to gain robustness by considering infinite dimensional subspaces of the theory space. The truncations such as the $f(R)$ truncation cannot be and are not motivated from the EFT point of view. If such a study is performed on a space which distinguishes curvature invariants then one is considering infinitely many irrelevant operators while at the same time neglecting infinitely many more relevant operators. If, on the other hand, the study is performed on an Einstein space, as is almost always the case, then curvature invariants cannot be disentangled and one is studying the flow of an unknown mixture of operators [13]. One motivation for the $f(R)$ truncations comes from the need to check the stability of the fixed points under the inclusion of irrelevant operators. In fact, functional truncations seem to be telling us that irrelevant operators are generically pushed further towards irrelevancy at the FP, which is good news for the Asymptotic Safety program, as it assures us that dimensionality of the critical surface remains (very) low [14]. In addition, these works often found convergence of the results for large enough

[13] For example, if $Ricci = \frac{1}{d} g R$ then $tr(Ricci^2) = R^2$ and thus when studying the flow of the Starobinsky operator R^2 one automatically studies an unknown admixture of $Ricci^2$ and other quadratic operators.

[14] This gives a tentative answer to a possible question, "how do I know that including R^{256} will not ruin the existence of a fixed point"? We are confident that it will not, because fixed point was shown

expansion order in R.

The author has done research in this field studying Wetterich equation and fixed point solutions in a number of different setups. We do not report any of the findings in this thesis because they are all incomplete at the time of writing of this thesis. It is fair to say that the $f(R)$ studies are much more subtle than the standard BFA truncations because the quality of the results seems to significantly depend on the background (hyperboloid vs sphere), regulator function (optimized cutoff vs exponential cutoff), and other choices s.a. whether to work with a pure cutoff or a spectrally adjusted one.

If there is a lesson of the $f(R)$ truncations for the rest of the research directions it is to take seriously the fact that one is not in perturbation theory anymore, and to think deeply and carefully when constructing a truncation. Simply including very many operators need not be a good choice, in fact it may render the analysis more difficult without giving much in return. One may be better off studying less operators, but thinking more deeply about the underlying physics, msWI's, and so on. In practice this would mean for example doing vertex expansions, keeping derivative structure of the vertices, and so on.

This concludes the general introduction to the FRG and its applications to quantum gravity. The following two sections are rather technical, but all of the physical ideas and mathematical concepts have been covered in this introduction so the reader should have no problem in understanding the following discussions.

3.2 Lessons From an Interacting Matter-Gravity Fixed Point

We could summarize the main message of the section 3.1 in the statement that an interacting fixed point of the renormalization group flow, triggered by quantum gravity fluctuations in the vicinity of the Planck scale, could underlie a predictive quantum field theory of gravity and matter [103]. In such a setting, it is critical to characterize the interaction structure of that fixed point. This is typically done with the functional renormalization group that provides a framework to extract the scale-dependence of the running couplings from the scale dependence of the effective dynamics. As quantum fluctuations can generically induce all couplings compatible with the global symmetries of the model, the underlying space of couplings – the theory space – is infinite dimensional. For practical reasons, the functional renormalization group requires a truncation of that space to a (typically) finite-dimensional subspace.

In section 3.1 we commented on the fact that the extended truncations in the case of pure gravity provide compelling evidence for the existence of an interacting fixed point [103]–[115]. Consistently with the hypothesis of "weakly-nonperturbative"

to exist and be robust in Taylor expansions to very high order in R, but also because fixed points were found by numerically solving for the fixed point functionals $f(R)$.

fixed point, the canonical dimensionality of couplings has proven to be a strong indicator of relevance and therefore can be used as a powerful guide to set up reliable truncations [108], [137]. Including matter fields into the setting enlarges the theory space considerably while still yielding promising hints of a fixed point [116]–[119]. Thus, the task of finding good truncations guided by physical insight becomes even more critical. Similar to pure-gravity studies, results in gravity-matter models suggest that canonical dimensionality of couplings remains a reasonably good guiding principle to determine which couplings are likely to become UV attractive at the asymptotically safe fixed point, see [120]–[122] for example.

A particular class of canonically irrelevant matter-gravity interactions are of interest. Those are couplings where symmetry-based arguments imply that no free fixed point should exist under the impact of gravity. Those directions provide critical tests for the viability of the Asymptotic Safety paradigm. While they are expected to not feature a fixed point at vanishing coupling, they are not guaranteed to feature a fixed point at a real value of the coupling at all.

Specifically, it was conjectured [138] that the interactions compatible with the global symmetries of the kinetic terms of matter fields cannot become asymptotically free when quantum gravity is present. Explicitly, such a pattern was already confirmed for a subset of fermion self-interactions [139], [140], scalar self-interactions [141], scalar-ghost interactions [142], scalar-fermion interactions [143] and vector self-interactions [79]. Here, we will find further evidence for this conjecture, as we will demonstrate that nonminimal couplings follow the same pattern. The coupling that we focus on differs from those included in previous studies; namely, nonminimal interactions of the form $\phi^n R^m$ for a scalar field ϕ have been explored in [120], [121], [144]–[146]. Such nonminimal interactions violate the global shift symmetry $\phi \to \phi + a$ of the kinetic term for a scalar. Based on this symmetry argument alone one can infer that they will feature a fixed point at a vanishing value. Explicit calculations support this result, e.g. [120], [121], [144]–[146].

On the other hand, a class of nonminimal interactions starting with

$$S_{\phi,Ric}[\phi; g] = \bar{\sigma} \int d^4x \sqrt{g}\, R^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi, \tag{3.40}$$

is compatible with the shift symmetry. Therefore, we expect that the corresponding coupling $\bar{\sigma}$ cannot feature a fixed point at vanishing value. As it cannot be zero at a fixed point its study constitutes a nontrivial test of Asymptotic Safety. Moreover, a large back-reaction onto the fixed-point value in the gravity sector would constitute a sign of possible instabilities of typically used truncations. We thus perform a nonminimal test of the viability of the Asymptotic Safety scenario by working in a truncation of the renormalization group flow where we discover the existence of an interacting fixed point for a corresponding nonminimal coupling. As a key finding, we observe nontrivial indications of stability of the properties of the fixed point under the impact of nonminimal derivative interactions, further strengthening the case for asymptotic safety in gravity-matter systems.

3.2.1 Technical Setup

The functional RG provides a way to derive the explicit beta functions in a truncation of the full theory space. We choose to work with the Wetterich equation described in section 3.1. For additional reviews and introductions see [147]–[151]. Wetterich equation describes the RG flow of the EAA, Γ_k, which contains effects of high-momentum quantum fluctuations. Upon a change of the momentum scale k, further quantum fluctuations are integrated out in the underlying path integral resulting in a change of the effective dynamics encoded in Γ_k. The scale dependence is encoded in scale dependence of the couplings so beta functions can be read-off from $k\partial_k\Gamma_k$ by projecting onto the appropriate field monomial in the effective dynamics.

The Wetterich equation,

$$\partial_t\Gamma_k \equiv k\,\partial_k\Gamma_k = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\partial_t R_k\right], \tag{3.41}$$

is formally exact, as explained in the previous section, and it generates only one-loop diagrams which makes it very practical. The supertrace STr implements a summation over the eigenvalues of the full, regularized propagator $\left(\Gamma_k^{(2)} + R_k\right)^{-1}$, where R_k is the regularization kernel and $\Gamma_k^{(2)}$ is shorthand for the second functional derivative of the flowing action with respect to the fields, and is matrix-valued in field space.

For our functional RG study of the nonminimal coupling, we employ the background field method in a linear split of the metric,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{3.42}$$

into a background metric $\bar{g}_{\mu\nu}$ and a fluctuation field $h_{\mu\nu}$. The gauge-fixing of the fluctuations is then performed with respect to the background field. We choose a standard gauge-fixing condition,

$$S_{\mathrm{gf}}[h;\bar{g}] = \frac{1}{32\pi\bar{G}_0\alpha}\int \mathrm{d}^4x\,\sqrt{\bar{g}}F^{\mu}[h;\bar{g}]\bar{g}_{\mu\nu}F^{\nu}[h;\bar{g}], \tag{3.43}$$

$$F^{\mu}[h;\bar{g}] = \left(\bar{g}^{\mu\kappa}\bar{D}^{\lambda} - \tfrac{1+\beta}{4}\bar{g}^{\kappa\lambda}\bar{D}^{\mu}\right)h_{\kappa\lambda}\,, \tag{3.44}$$

where $\bar{G}_0$ is the background Newton coupling. The computation of the beta functions that we perform below can be done either with or without the York decomposition. We have done the calculation in both ways to serve as an independent consistency check. In the latter case the calculation significantly simplifies if one chooses to work in the gauge $\beta \to \alpha \to 0$, so this is what we do. The reason for the simplification comes from the fact that the effective decomposition of the metric becomes,

$$h_{\mu\nu} = h^{\text{TT}}{}_{\mu\nu} + \tfrac{1}{4}\bar{g}_{\mu\nu}h^{\text{Tr}}\,, \tag{3.45}$$

with the trace h^{Tr} and the transverse-traceless mode h^{TT} of the fluctuation field being the only relevant degrees of freedom of gravity in this computation (in the sense that the vector degrees of freedom have a vanishing propagator for this gauge choice). As explained in the introduction, the "TT" gravitational degree of freedom satisfies $\bar{D}_\mu h^{\text{TT}\mu\nu} = 0$ and $h^{\text{TT}\mu}{}_\mu = 0$. The gauge-fixing is supplemented by the ghost action

$$S_{\text{gh}}[h, c, \bar{c}; \bar{g}] = \int \mathrm{d}^4x\, \sqrt{\bar{g}}\, \bar{c}_\mu \tfrac{\delta F^\mu}{\delta h_{\alpha\beta}} \delta_c^{\text{Q}} h_{\alpha\beta}, \tag{3.46}$$

where we use $\delta_c^{\text{Q}} h$ to denote the quantum gauge transformation of h with transformation parameter c. For the linear split employed here we have

$$\delta_c^{\text{Q}} h_{\mu\nu} = 2\bar{g}_{\rho(\mu}\bar{D}_{\nu)}c^\rho + c^\rho \bar{D}_\rho h_{\mu\nu} + 2h_{\rho(\mu}\bar{D}_{\nu)}c^\rho\,. \tag{3.47}$$

Note that this form of the ghost action immediately implies that there are no higher graviton-ghost interactions present. This would not be the case for more general splits of the metric.

Similar to the gauge-fixing and ghost actions, the cutoff term is a function of the background Laplacian. These choices break the split symmetry, which encodes that $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$ can be combined into a full metric. Accordingly, the flow of $\bar{\sigma}$ read-off from the background term $\sqrt{\bar{g}}\bar{R}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$ will differ from the flows of the fluctuation terms, e.g. $\frac{1}{2}\bar{\Delta}h^{\text{TT}\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi$, where $\bar{\Delta} = -\bar{D}^2$. This is precisely related to the discussion of the modified split Ward identities that we have started in the section 3.1 and that we will discuss in detail in section 3.3.

Regarding the choice of the background, as we restrict ourselves exclusively to fluctuation couplings of the graviton to itself, to the ghosts or to the scalar field, we choose a flat background without loss of generality in the following discussion.

The setup of our truncation is as follows. We use the classical action $S_{\text{class}}[h, c, \bar{c}, \phi; \bar{g}]$ as the generator for vertices,

$$\begin{aligned} S_{\text{class}}[h, c, \bar{c}, \phi; \bar{g}] = S_{\text{EH}}[\bar{g} + h] + S_{\text{gf}}[h; \bar{g}] + S_{\text{gh}}[h, c, \bar{c}; \bar{g}] \\ + S_{\phi,\text{kin}}[\phi; \bar{g} + h] + S_{\phi,Ric}[\phi; \bar{g} + h], \end{aligned} \tag{3.48}$$

with the Einstein-Hilbert action,

$$S_{\text{EH}}[g] = -\tfrac{1}{16\pi \bar{G}_0} \int \mathrm{d}^4x\, \sqrt{g}\, R, \tag{3.49}$$

as well as the gauge-fixing, S_{gf}, and ghost action, S_{gh}, from equations (3.43) and (3.46), the kinetic part of the action for the scalar field,

$$S_{\phi,\text{kin}}[\phi; g] = \frac{1}{2} \int \mathrm{d}^4x \sqrt{g}\, g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi, \tag{3.50}$$

and the nonminimal part of the action for the scalar field, $S_{\phi,Ric}$, from equation (3.40). Included in the same shift-symmetric theory space with the same canonical dimension -2 is a coupling of the curvature scalar to $\partial_\mu\phi\partial^\mu\phi$. In our study, it is set to zero, as it does not yield a contribution to the TT-graviton-two-scalar vertex on a flat background, but only contributes to the coupling between two scalars and h^{Tr}. Based on the general expectation that the TT-graviton mode should dominate we focus on σ as the coupling more likely to feature a significant backreaction on the flow of couplings included in previous truncations, and therefore providing a more meaningful test of the robustness of results in previous truncations.

To generate the vertices we employ the decomposition of h into h^{TT} and h^{Tr} according to equation (3.45), and then expand the classical action polynomially in the fields,

$$S_{\text{class}}[\Phi; \bar{g}] = \sum_{n=0}^{\infty} \frac{1}{n!} S^{(n;0)}_{\text{class}}[\Phi; \bar{g}]\Phi^n, \quad \Phi = (h^{\text{TT}}, h^{\text{Tr}}, c, \bar{c}, \phi). \tag{3.51}$$

For each new order in this polynomial expansion we introduce a new coupling according to the following prescription [15]

$$S^{(n)}_{\text{EH}} \to (32\pi \bar{G}_n)^{\frac{n}{2}-1} \cdot 32\pi\bar{G}_0 \cdot S^{(n)}_{\text{EH}}, \quad n \geq 2, \tag{3.52}$$

$$S^{(2;0)}_{\text{gf}} \to 32\pi\bar{G}_0 \cdot S^{(2;0)}_{\text{gf}}, \tag{3.53}$$

$$S^{(1,1,1;0)}_{\text{gh}} \to (32\pi\bar{g}^c_3)^{\frac{1}{2}} \cdot S^{(1,1,1;0)}_{\text{gh}}, \tag{3.54}$$

$$S^{(2;n)}_{\phi,\text{kin}} \to (32\pi\bar{g}_{n+2})^{\frac{n}{2}} \cdot S^{(2;n)}_{\phi,\text{kin}}, \quad n \geq 1, \tag{3.55}$$

$$S^{(2;n)}_{\phi,Ric} \to (32\pi\bar{g}_{n+2})^{\frac{n}{2}} \frac{\bar{\sigma}_{n+2}}{\bar{\sigma}} \cdot S^{(2;n)}_{\phi,Ric}, \quad n \geq 0, \tag{3.56}$$

where $S^{(n_1,\ldots,n_m)}_i$ refers to functional derivatives with respect to the arguments, i.e.,

$$S^{(n_1,\ldots,n_m)}_i = \frac{\delta^{n_1}}{\delta\phi_1^{n_1}} \cdots \frac{\delta^{n_m}}{\delta\phi_m^{n_m}} S[\phi_1, \ldots, \phi_m]. \tag{3.57}$$

Finally, we rescale the scalar field and the gravity degrees of freedom with a wave function renormalization,

$$h^{\text{TT}} \to \sqrt{Z_{\text{TT}}}\, h^{\text{TT}}, \quad h^{\text{Tr}} \to \sqrt{Z_{\text{Tr}}}\, h^{\text{Tr}}, \quad \phi \to \sqrt{Z_\phi}\, \phi, \tag{3.58}$$

[15] Choice is made s.t. the background couplings do not appear in front of the fluctuation operators, and such that all fluctuating couplings have the same dimensionality as their background counterparts.

and switch to dimensionless couplings,

$$\bar{G}_n = \frac{G_n}{k^2}, \quad \bar{g}_3^c = \frac{g_3^c}{k^2}, \quad \bar{g}_n = \frac{g_n}{k^2}, \quad \bar{\sigma}_n = \frac{\sigma_n}{k^2}. \tag{3.59}$$

In order to extract the beta-functions, we need to specify how to project the flow onto the corresponding field monomials in our truncated theory space. The general idea is to employ a simultaneous vertex and derivative expansion [16], distinguishing different couplings via the order in the fields and the derivatives. However, for a given order, there typically remains a large degeneracy. For couplings involving a graviton we expect the TT-mode of the graviton, h^{TT}, to be less affected by technical choices (such as the choice of gauge or regulator) than the trace h^{Tr}. Therefore, we construct the projections such that they project onto the TT-mode if applicable, and thereby reduce this degeneracy significantly. We derive the anomalous dimensions, η_{TT}, η_{Tr} and η_ϕ, as well as beta functions for g_3 and σ_3. For the anomalous dimensions we project on

$$\Gamma_{Z_{\mathrm{TT}}} = \tfrac{1}{2} Z_{\mathrm{TT}} \int \mathrm{d}^4x\, h^{\mathrm{TT}}{}_{\mu\nu} \Box h^{\mathrm{TT}\mu\nu}, \tag{3.60}$$

$$\Gamma_{Z_{\mathrm{Tr}}} = -\tfrac{3}{16} Z_{\mathrm{Tr}} \int \mathrm{d}^4x\, h^{\mathrm{Tr}} \Box h^{\mathrm{Tr}}, \tag{3.61}$$

$$\Gamma_{Z_\phi} = \tfrac{1}{2} Z_\phi \int \mathrm{d}^4x\, \phi \Box \phi, \tag{3.62}$$

where $\Box = -\partial^2$. These are the only linearly independent invariants at this order. The interaction monomial for g_3 is given by

$$\Gamma_{g_3} = \tfrac{1}{2}\sqrt{32\pi \tfrac{g_3}{k^2} Z_{\mathrm{TT}}} Z_\phi \int \mathrm{d}^4x\, h^{\mathrm{TT}\mu\nu} \phi\, \partial_\mu \partial_\nu \phi, \tag{3.63}$$

which is the only linearly independent invariant involving one TT-graviton, two scalars and two derivatives. To calculate the flow of σ_3, i.e. the nonminimal coupling of one graviton to two scalars induced by the interaction (3.40), we project on

$$\Gamma_{\sigma_3} = -\tfrac{1}{2}\tfrac{\sigma_3}{k^2}\sqrt{32\pi \tfrac{g_3}{k^2} Z_{\mathrm{TT}}} Z_\phi \int \mathrm{d}^4x\, \Box h^{\mathrm{TT}\mu\nu}\, \phi\, \partial_\mu \partial_\nu \phi. \tag{3.64}$$

This invariant is one of two linearly independent ones at this order. A possible choice for the other is given by lowering the number of derivatives acting on the graviton,

$$\Gamma_{h^{\mathrm{TT}}\phi\partial^4\phi} \sim \int \mathrm{d}^4x\, h^{\mathrm{TT}\mu\nu}\, \phi\, \partial_\mu \partial_\nu \Box \phi. \tag{3.65}$$

Using our basis, we project onto the interaction monomial (3.64), projecting out the other (3.65). Note that the interaction (3.64) directly arises from (3.40), whereas the other interaction (3.65) would arise from a higher derivative term $\int \mathrm{d}^4x \sqrt{g}\, \phi\, \Delta^2 \phi$. For the evaluation of the flow equation (3.41) we need to choose a regulator. Our

[16] See overview of the FRG in section 3.1 for nomenclature.

results are obtained with a Litim-type [152] regulator,

$$R_k^h = 32\pi\bar{G}_0 \cdot (Z_{\mathrm{TT}}\Pi_{\mathrm{TT}} + Z_{\mathrm{Tr}}\Pi_{\mathrm{Tr}})(S^{(2)}_{\mathrm{EH}} + S^{(2)}_{\mathrm{gf}})\, r_k(\tfrac{\Box}{k^2}), \tag{3.66}$$

$$R_k^c = S^{(0,1,1;0)}_{\mathrm{gh}}\, r_k(\tfrac{\Box}{k^2}), \tag{3.67}$$

$$R_k^\phi = Z_\phi S^{(2;0)}_{\phi,\mathrm{kin}}\, r_k(\tfrac{\Box}{k^2}), \tag{3.68}$$

where Π_{TT} is the projector onto the TT-mode,

$$\Pi_{\mathrm{TT}\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = h^{\mathrm{TT}}{}_{\mu\nu}, \tag{3.69}$$

Π_{Tr} is the projector onto the Tr-mode,

$$\Pi_{\mathrm{Tr}\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = \tfrac{1}{4}\bar{g}_{\mu\nu} h^{\mathrm{Tr}}, \tag{3.70}$$

and r_k is the Litim regulator shape function,

$$r_k(z) = \tfrac{1}{z}(1-z)\theta(1-z). \tag{3.71}$$

3.2.2 Fixed-Point Analysis 1: Identifying Newton's Avatars

Our calculation distinguishes between all the couplings appearing in (3.52)-(3.56), but in practice the analysis becomes easier and more insightful if we identify some of them to reduce the dimensionality of the parameter space. The same goal may be achieved as well by analyzing a subset of the beta functions considering some of the couplings as parameters. We refer the reader to [2] for full expressions of the beta functions and of the anomalous dimensions, including individual diagram contributions and explicit dependence on all distinct avatars of the Newton coupling.

We begin by treating g_3 and the other avatars of the Newton coupling as parameters and show that σ_3 can only feature an interacting fixed point. The beta function for σ_3, under the identification $\sigma_5 = \sigma_4 = \sigma_3$, $G_3 = g_3^c = g_5 = g_4 = g_3$ and with all anomalous dimensions set to zero reads

$$\beta_{\sigma_3} = 2\sigma_3 - \tfrac{43}{216\pi} g_3 + \tfrac{1225}{648\pi} g_3\sigma_3 - \tfrac{341}{432\pi} g_3\sigma_3^2 + \tfrac{83}{60\pi} g_3\sigma_3^3. \tag{3.72}$$

The second term is crucial because it remains non-vanishing even if σ_3 is set to zero. Accordingly, σ_3 cannot feature a free fixed point, as soon as g_3 features an interacting fixed point. This property is in line with arguments elaborated in [138] and observations in gravity-matter systems where interactions that respect the symmetry of the kinetic terms, to wit shift symmetry and $\mathbb{Z}_2$ reflection symmetry, are induced at the UV fixed point by gravity.

As expected, the fixed-point value grows as a function of increasing Newton coupling, see Figure 3.1. The figure includes the leading order (LO) result, defined by setting $\eta = 0$ wherever it appears, next-to-leading order (NLO) result, in which all η's that arise from scale derivatives of the regulator are set to zero, but no other,

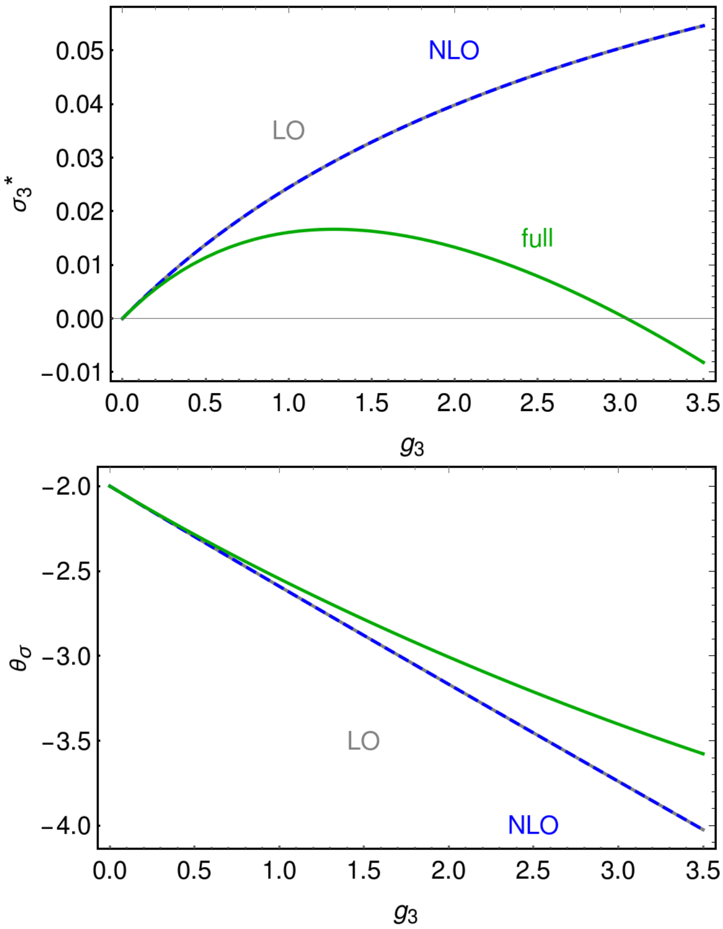

FIGURE 3.1: The top panel shows the fixed-point value for σ_3 as a function of the Newton coupling g_3, which is treated as a parameter here. The bottom panel shows the critical exponent as a function of g_3. The anomalous dimensions are set to zero (gray, dashed line; LO), included perturbatively (blue, dashed line; NLO) and included fully (green, solid line; full).

and finally the full case where full dependency on η is kept (which introduces a non-polynomial dependence of beta functions on the couplings). This nomenclature should not be confused with the similar names given to the perturbative schemes, such as 2-1-0, in Chapter 2. The NLO case corresponds to the prescription to recover universal one-loop beta functions for canonically marginal couplings from the FRG, and is therefore also referred to as the perturbative approximation. Incidentally, the LO and NLO case agree due to our definition of the corresponding interaction term in equation (3.64). As the term comes with a prefactor that is a product of σ_3 and g_3, the factor $\sqrt{Z_{\rm TT}}Z_\phi$ is absorbed in the definition of g_3. Hence, β_{σ_3} does not contain any explicit η-terms, except those that arise from the scale-derivative of the regulator.

The critical exponent is defined as $\theta_\sigma = -\frac{\partial \beta_{\sigma_3}}{\partial \sigma_3}\Big|_{\sigma_3^*}$, i.e. $\theta_\sigma < 0$ signals irrelevance. As g_3 increases, the interaction $\sim \sigma_3$ is pushed further into irrelevance, cf. lower panel in Figure 3.1. This is in line with the anomalous dimensions becoming more positive, see Figure 3.2, which adds a contribution to critical exponents that shifts

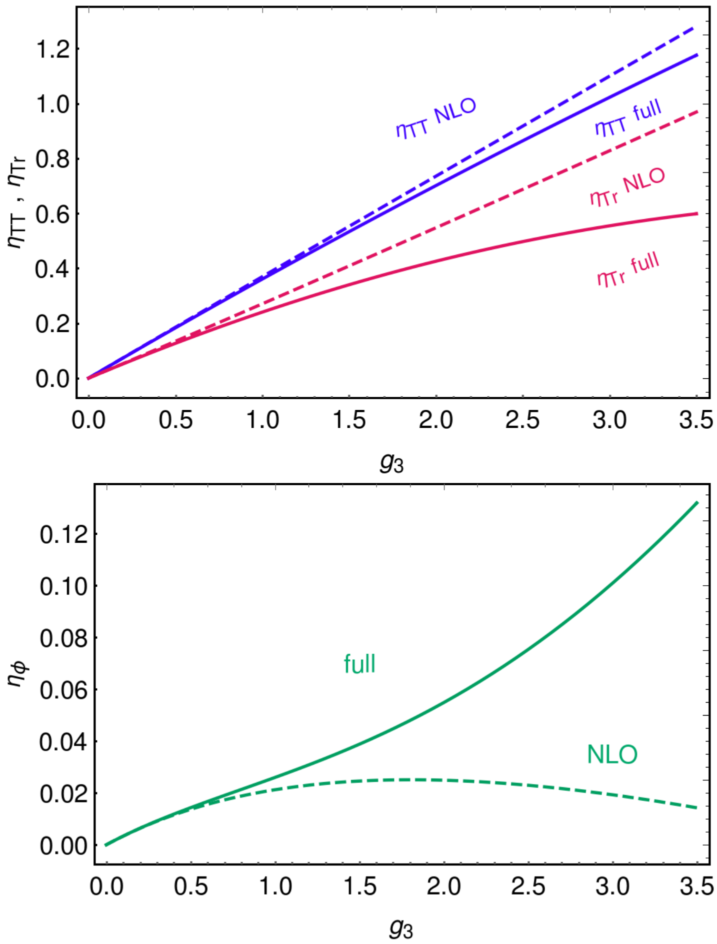

FIGURE 3.2: We show the anomalous dimensions of the TT-mode, the Tr-mode and the scalar respectively. In each figure the anomalous dimension is evaluated perturbatively (dashed line; NLO) and evaluated fully (solid line; full).

these towards irrelevance. Even though the non-universal fixed-point value shows a significant dependence on the approximation (LO/NLO vs. full; cf. Figure 3.1), the critical exponent is reasonably robust. This signals stability of the fixed point of the gravity-matter system in two ways. First of all, it supports the main guiding principle that is used to set up truncations, namely the assumption that canonically irrelevant couplings are not likely to be shifted into relevance. Secondly, an increasingly negative critical exponent implies that the fixed point for σ_3 remains real if g_3 is increased further. The reason is that fixed points can only become complex in pairs, i.e. when two distinct fixed points collide. At such fixed-point collisions, the critical exponent has to become zero. An increasingly negative critical exponent implies that the system is protected from such collisions along the eigendirection corresponding to that exponent. Accordingly, the so-called "weak-gravity bound", which has been observed in other induced interactions [79], [138], [143] is avoided here; no instability is expected even in the strong-quantum gravity regime, at increasingly large g_3.

Similarly to the case of induced four-fermion interactions [138], [139], there is instead a bound in the unphysical regime at $g_3 < 0$. While we do not explicitly include results including a cosmological constant or graviton mass parameter here, we have checked that there is no value for those couplings that shifts the bound at $g_3 < 0$ to positive values of g_3.

Considering Figure 3.1 we note that for rather large values of $g_3 \gtrsim 3$ the sign of σ_3^* changes. A priori there seems to be no preferred sign for $\bar{\sigma}$ in (3.40). However, considering the stability of the conformal mode might provide us with a preferred choice. For $\bar{\sigma}$ being zero the kinetic term of the conformal mode has the wrong sign, leading to the standard conformal mode instability. By turning on $\bar{\sigma}$ the conformal mode and the scalar are coupled. Therefore the stability analysis might change depending on the sign of $\bar{\sigma}$. This is similar to the pure gravity case when adding an R^2 term with the right sign [153]. We caution, however, that the question of stability cannot be answered in a truncation to finite order in the fields, as higher order terms could potentially induce global stability.

3.2.3 Fixed-Point Analysis 2: Distinguishing Newton's Avatars

The beta function for σ_3 depends on the gravity-scalar couplings g_3, g_4, and g_5 as well as on the three-graviton and four-graviton couplings G_3, G_4, and on the ghost-graviton coupling g_3^c. In the following analysis we explore the dependence of σ_3^* on both g_3 and G_3 separately by identifying higher order avatars of each of these couplings with its lower order partners.

It is worth repeating that in a classical setting that respects diffeomorphism invariance, these should all be equal and agree with the corresponding background couplings but due to the presence of the gauge-fixing and regulator terms diffeomorphism invariance is broken and encoded in modified split Ward Identities. Ultimately, the symmetry-breaking by the regulator implies that the UV initial condition of the flow should contain just the right amount of symmetry-breaking such that an invariant effective action can be recovered in the IR (see the introduction in [149]).

Despite the fact that some steps towards imposing the msWI's have been performed (e.g. in [3], [109], [113], [126]–[131], [154]–[157]) these are not yet at a level where they could be applied in our context, so here we simply explore the dependence of σ_3^* on g_3 and G_3 without attempting to solve the problem of diff-invariance.

Our first observation is that the non-universal fixed-point value switches sign across the plane spanned by g_3 and G_3 if the graviton self-interaction dominates the graviton scalar interaction, $G_3 > g_3$, cf. Figure 3.3. We also notice that, interestingly, the critical exponent shows the largest slope in the direction $g_3 \approx G_3$. This identification therefore leads to a strong deviation from canonical scaling. This behavior is linked to the behavior of the anomalous dimensions, which grow increasingly positive as a function of increasing G_3, see Figure 3.4. Accordingly, a larger G_3 moves the system further towards the border where the truncation becomes unreliable. An estimate of the border is given by $\eta = 2$, where the regulator does not suppress UV modes reliably [117]. In practice in our beta functions this effect becomes noticeable

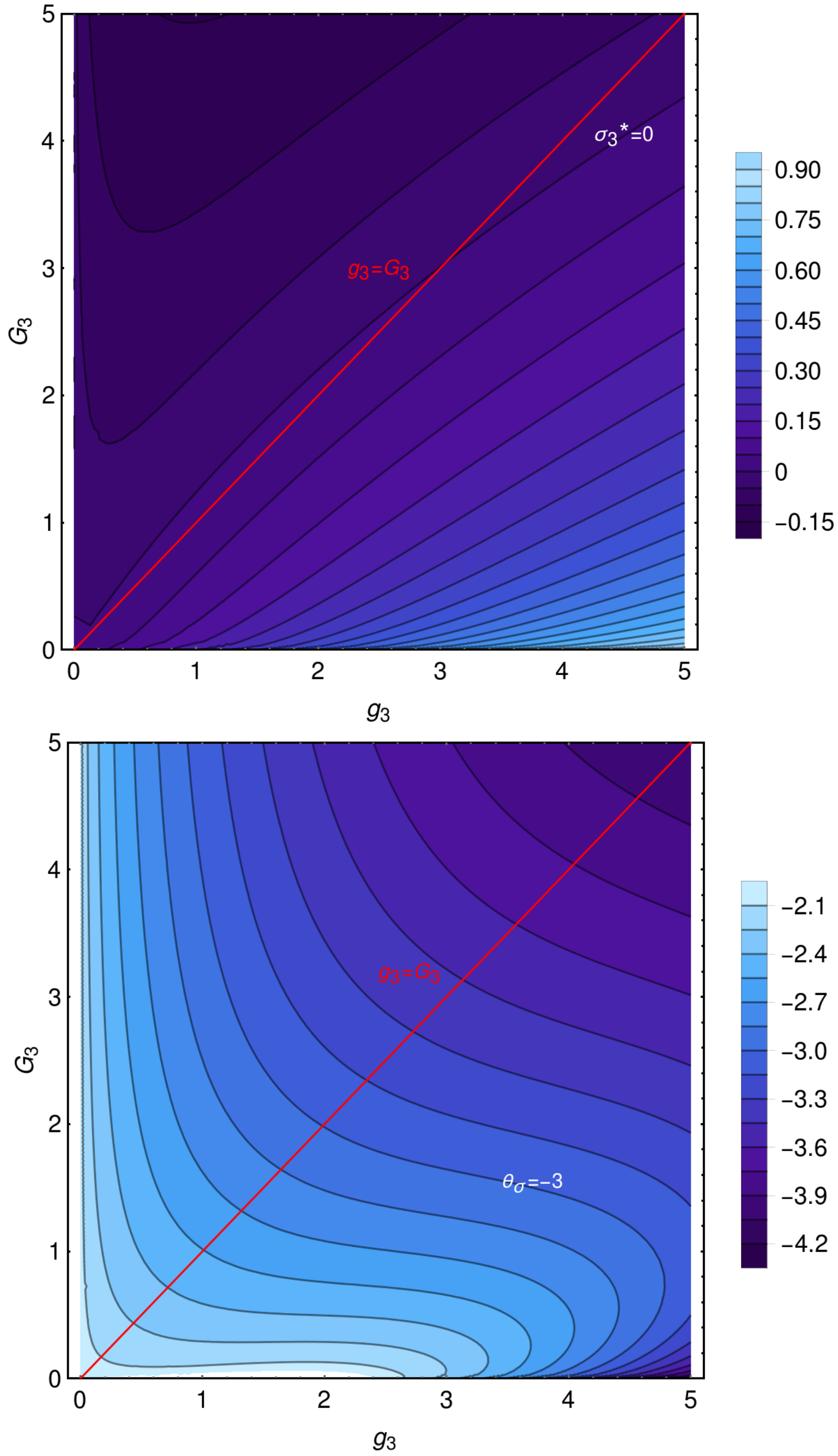

FIGURE 3.3: Fixed-point value for σ_3 (top panel) and critical exponent (bottom panel) as a function of the Newton couplings g_3 and G_3 which are treated as independent parameters here. The anomalous dimensions are included fully.

at $\eta \approx 3$ which are the smallest values of η for which some terms in the beta functions start changing signs[17]. In Figure 3.4 the role of the different Newton couplings

[17]For particular diagrams that get affected see [2].

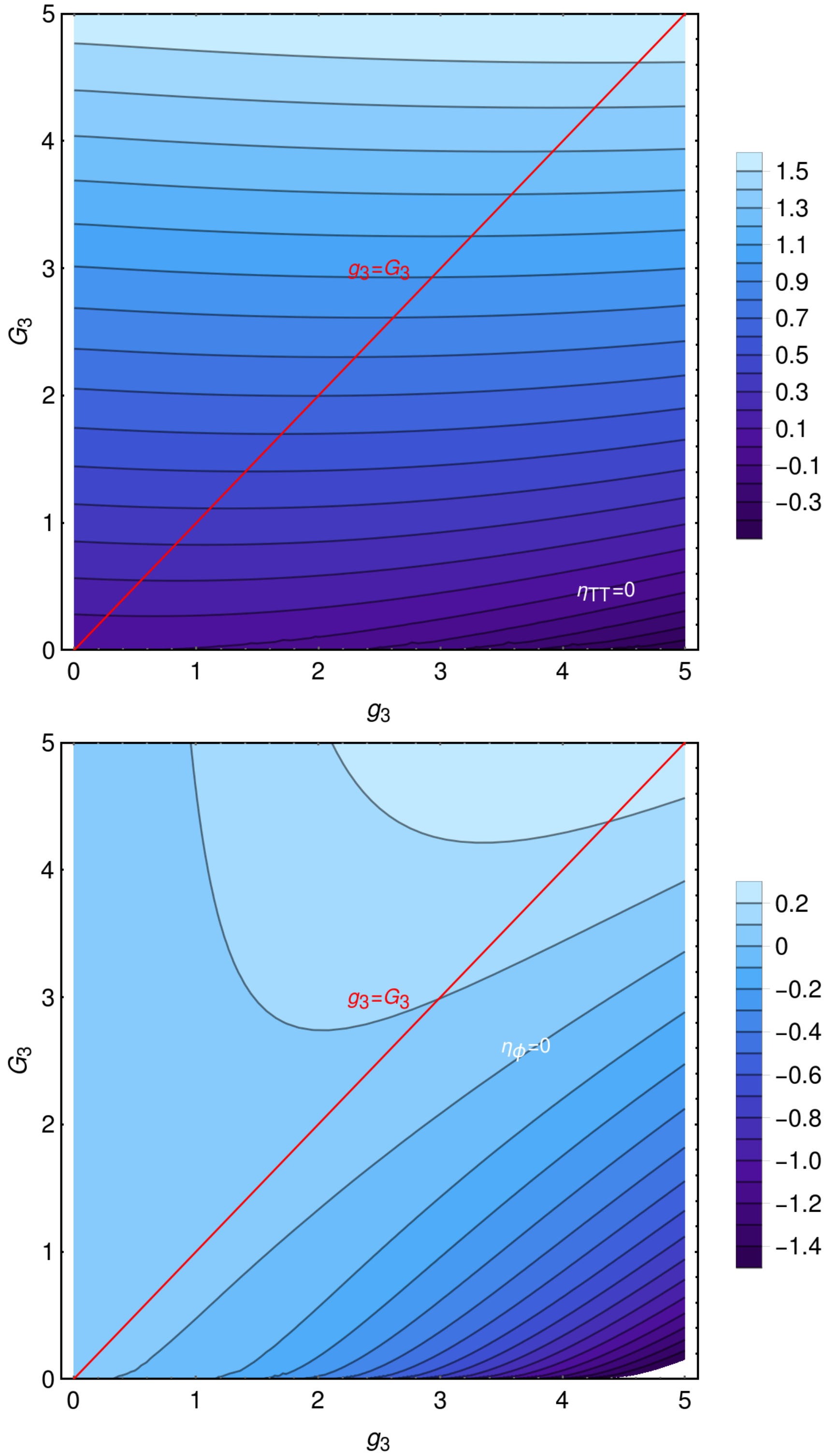

FIGURE 3.4: Anomalous dimension of the TT-mode (upper panel) and the scalar (bottom panel) as a function of the Newton couplings g_3 and G_3 which are treated as independent parameters here. The anomalous dimensions are included fully.

becomes clear as one of them effectively determines the strength of the graviton propagator while the other one determines the strength of the scalar propagator.

One might tentatively associate the instability of the system for large values of

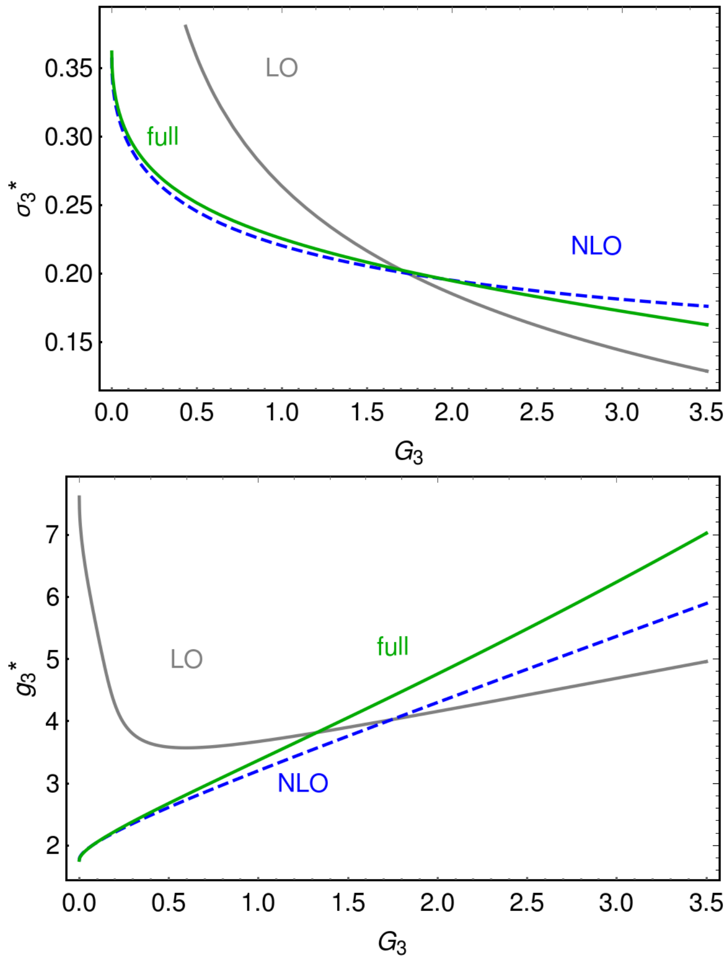

FIGURE 3.5: Fixed-point values for σ_3 (upper panel) and g_3 (lower panel) as a function of the Newton coupling G_3. The anomalous dimensions are set to zero (gray, dashed line; LO), included perturbatively (blue, dashed line; NLO) and included fully (green, solid line; full).

G_3 to a type of weak-gravity bound, however that idea should be examined more carefully in a truncation including a beta function for G_3.

We now investigate how large the backcoupling of the induced σ_3^* into the flow of the Newton coupling g_3 is. As we do not calculate the flow of G_3 here, we will adopt the fixed-point value in the state-of-the-art pure-gravity truncation employed in [113], which is $G_3^* = 0.83$, for the remainder of our study.[18] The inclusion of one scalar is not expected to change the fixed-point value of G_3 by much [116]–[119], [145], [158]. Keeping this in mind we observe that g_3^* appears to deviate considerably, as $g_3^* = 3.17$ at $G_3 = 0.83$. Incidentally, we observe that our results appear to favor a regime of values for $G_3 \approx 1$ over $G_3 \approx 3$, as the fixed-point results for g_3 and σ_3 are in approximate agreement for $G_3 \approx 1$ comparing the LO, NLO and full case, cf. figures 3.5, 3.6, 3.7. In this regime of values for G_3, our truncation appears reasonably robust, as LO, NLO and full results are in semi-quantitative agreement

[18] Note that these results were obtained in the gauge $\alpha = 0$, $\beta = 1$.

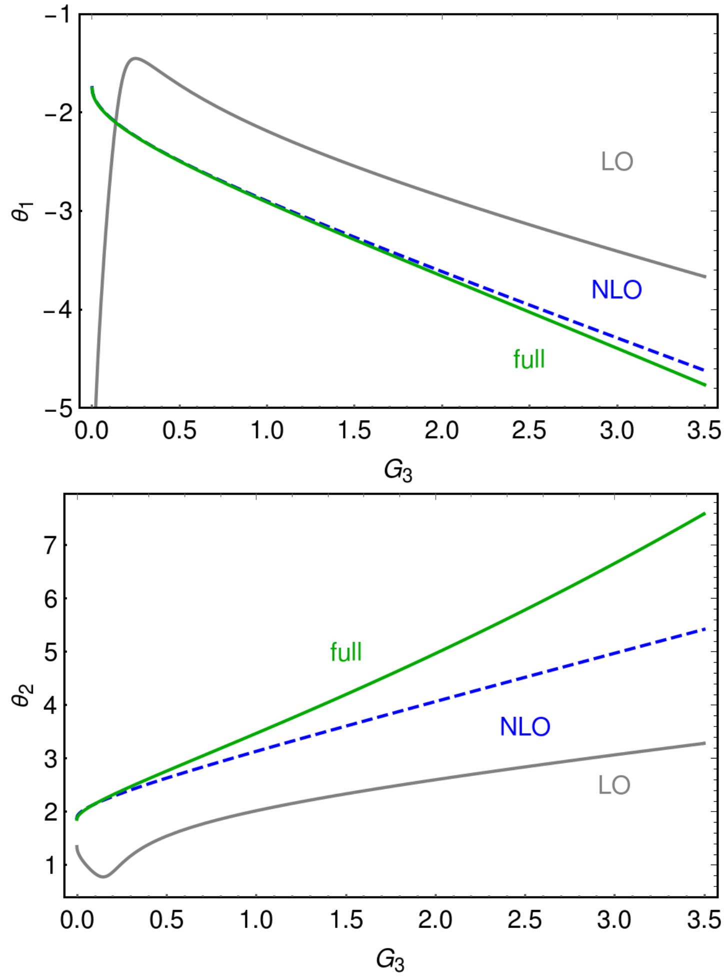

FIGURE 3.6: Critical exponents as a function of the Newton coupling G_3. The anomalous dimensions are set to zero (gray, dashed line; LO), included perturbatively (blue, dashed line; NLO) and included fully (green, solid line; full).

with the same qualitative dependence on G_3.

This analysis reinforces our main point, that interactions in the gravity sector necessarily percolate into the matter sector. Even unconventional gravity-matter-interactions are generated. As highlighted in Table 3.1, the induced interactions couple back into the gravity-system and in turn impact the gravitational fixed-point values. Interestingly, the system is rather robust under the inclusion of σ_3. The gravitational fixed-point values are essentially unaffected, with the exception of the leading-order result. The same is true for the critical exponents, which do not change by more than 10%, and in fact show increasing stability at increasing order of the approximation, with only a 5% change of the critical exponent in the full beta function under the inclusion of σ_3. On the other hand, the matter system itself appears to be less robust, with a significant change in the anomalous dimension, and even a change of sign. Interestingly, the anomalous dimension becomes negative under the inclusion of σ_3, which contributes to a shift of matter couplings into relevance at the Gaussian fixed point. For couplings that are marginally irrelevant in the Standard

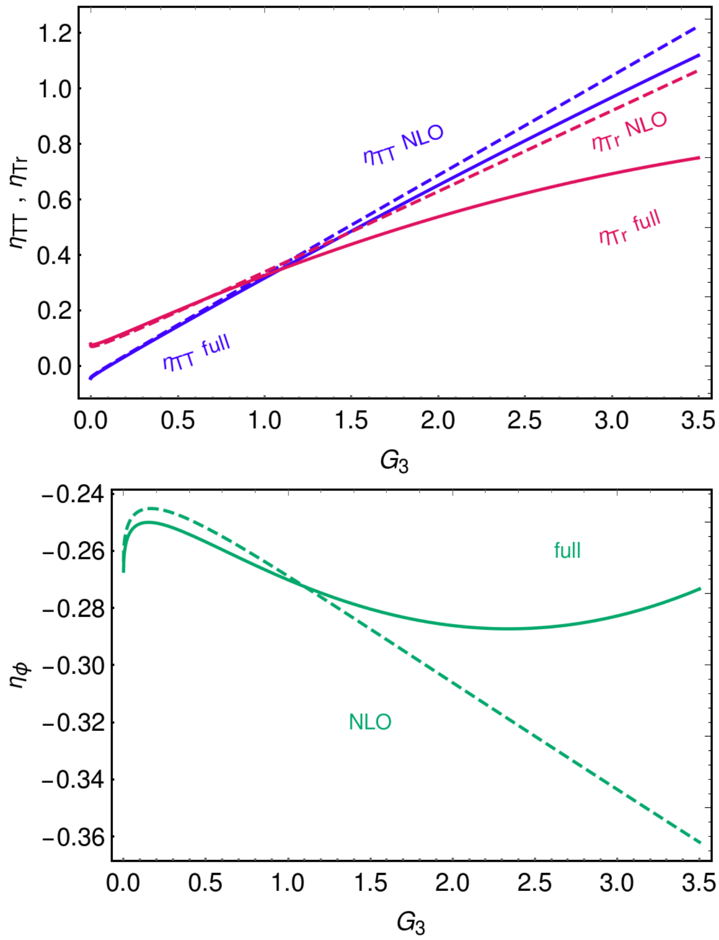

FIGURE 3.7: Anomalous dimensions of the TT-mode (blue line) and Tr-mode (red line) are presented in the upper panel, and the scalar (green line) is shown in the lower panel. The anomalous dimensions are evaluated perturbatively (dashed line; NLO) and evaluated fully (solid line; full).

Model case, a shift into relevance at the Gaussian fixed point implies the potential existence of a predictive, quantum-gravity induced ultraviolet completion [80], [86], [124]. (The above discussion is saying that if a SM coupling is marginally relevant at the GFP, but then gets shifted into irrelevancy at the GFP, this is equivalent to shifting it to relevancy at the UV NGFP and thus it becomes predictive. leave like this or change?)

We tentatively conclude that the inclusion of σ_3 appears to support the scenario that a predictive, quantum-gravity induced UV completion of the Standard Model might be viable.

3.2.4 Discussion

In line with expectations based on symmetry arguments [138], we have confirmed that an asymptotically safe regime in gravity is incompatible with a free matter model. This result was first explicit confirmation that the unavoidable presence of matter interactions in an asymptotically safe matter-gravity model also extends to

system & order	g_3^*	σ_3^*	θ_1	θ_2	η_{TT}	η_{tr}	η_ϕ
$g_3@LO$	2.51	-	2	-	0	0	0
$g_3, \sigma_3@LO$	3.61	.29	1.88	-2.05	0	0	0
$g_3@NLO$	3.01	-	3.11	-	.34	.14	.11
$g_3, \sigma_3@NLO$	3.01	.23	2.96	-2.77	.27	.29	-.26
$g_3@full$	3.17	-	3.07	-	.33	.12	.11
$g_3, \sigma_3@full$	3.14	.23	3.22	-2.78	.26	.28	-.27

TABLE 3.1: We set $G_3 = 0.83$ and compare results for g_3^* with and without σ_3 in the different approximations.

mixed matter-gravity interactions. A direct consequence is that matter systems can only appear free under the impact of asymptotically safe gravity in appropriately chosen truncations. The general structure of the tentative gravity-matter fixed point for Standard Model matter coupled to gravity is that of a hybrid fixed point, namely it is free in interactions which break some of the global symmetries of the kinetic terms. All other interactions are generically finite in the UV. In this section we have confirmed this expectation in a so far unexplored direction in theory space, namely in nonminimal derivative interactions. We have found that the gravity-matter system features a fixed point with finite nonminimal interactions.

This constitutes a nontrivial test of the asymptotic safety scenario in gravity-scalar systems; while the inclusion of another set of nonminimal couplings was part of earlier studies, these particular couplings always feature the free fixed point as they are protected from the impact of quantum gravity by global symmetries of the scalar. On the other hand, the coupling that we explore is part of the shift symmetric theory space and as such not protected by symmetry. Accordingly it is necessarily nonzero at the fixed point. As such, the nonminimal interaction that we've studied corresponds to an interaction that might have destroyed the asymptotically safe scale invariant regime, as there is no a priori reason for the fixed-point equation to have real fixed-point values. The most important results of our study are thus that there is a real fixed point, that the additional coupling is even more irrelevant than canonical power-counting would suggest, and that the backcoupling into the gravitational fixed-point properties is subdominant.

One important message of the above analysis can be read off directly from figures 3.6 and 3.7. In perturbation theory the anomalous dimensions should be relatively close to 0 for the analysis to remain trustworthy, and similarly critical exponents should be very close to their classical values. Our analysis goes beyond perturbation theory, but even though it is difficult to quantify its reliability we should hope to stay as close to the perturbative regime as possible. Figures 3.6 and 3.7 are telling us that either we shouldn't expect g_3 and G_3 to take similar values, or we should accept large quantum effects (as compared to perturbation theory).

Since we know from section 3.1 that different copies of the couplings will run in different ways, the best way forward is to try to understand the information stored in modified split Ward identities. In fact, having witnessed the importance of the diffeomorphism invariance for practical computations in this section, we will look more deeply into this subject in the following section.

3.3 Background Independence in Quantum Gravity

We have witnessed the importance of dealing with the background independence in practical computations in quantum gravity in the last section. Information about the symmetry breaking is stored in (modified) split-symmetry Ward identities, and making use of them is crucial even for interpreting results about the flow of different avatars of Newton's constant. In this section we construct a procedure for solving msWI's. We successfuly use it in the context of conformally reduced gravity and we discuss its possible extensions to full gravity and its potential limitations.

3.3.1 Background Field Method and Background Independence

Almost all work on covariant quantum gravity is based on the background field method. One begins by splitting the metric into background and quantum parts

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{3.73}$$

and then performs a functional integral over $h_{\mu\nu}$. In doing so one has to gauge-fix the invariance under diffeomorphisms. It is very convenient to choose linear background gauges of the form

$$F_\mu = \bar{\nabla}_\rho h^\rho{}_\mu - \frac{\beta+1}{d}\bar{\nabla}_\mu h^\rho{}_\rho \,. \tag{3.74}$$

The advantage of such gauges is that they break "quantum" diffeomorphisms

$$\delta_v^{(Q)}\bar{g}_{\mu\nu} = \mathcal{L}_v g_{\mu\nu}\,; \qquad \delta_v^{(Q)} h_{\mu\nu} = 0 \tag{3.75}$$

as required, while preserving "background" diffeomorphisms

$$\delta_v^{(B)}\bar{g}_{\mu\nu} = \mathcal{L}_v \bar{g}_{\mu\nu}\,; \qquad \delta_v^{(B)} h_{\mu\nu} = \mathcal{L}_v h_{\mu\nu} \,. \tag{3.76}$$

The classical action, regarded as a functional of $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$, is invariant under the "split symmetry"

$$\delta\bar{g}_{\mu\nu} = \epsilon_{\mu\nu}(x)\,, \qquad \delta h_{\mu\nu} = -\epsilon_{\mu\nu}(x)\,, \tag{3.77}$$

simply because $g_{\mu\nu}$ is. The gauge condition (3.74) breaks this symmetry and consequently the effective action is a functional of two separate arguments $\Gamma(h;\bar{g})$. This is not a very serious drawback, because one expects that the n-point functions of h or $\bar{g}$ lead to the same physical results, once one goes on shell, as is the case in YM theory [159], [160].

Background Field Method in the FRG Context

The problem is more serious when one tries to calculate the Effective Average Action (EAA) Γ_k, which is defined by introducing in the functional integral a cutoff term

$$\Delta S_k(h_{\mu\nu};\bar{g}_{\mu\nu}) = \frac{1}{2}\int d^dx\sqrt{\bar{g}}\,h_{\mu\nu}\mathcal{R}_k^{\mu\nu\rho\sigma}h_{\rho\sigma}\,, \tag{3.78}$$

where $\mathcal{R}_k^{\mu\nu\rho\sigma}$ is an infrared regulator constructed with the background metric. This introduces further breaking of the split symmetry, which is not merely a gauge artifact and spoils "background-independence", the notion that physical results should not depend on the choice of the background metric.

Is it possible to repackage the information contained in the EAA into a functional of a single metric? Pragmatically, much work on the renormalization group for gravity has concentrated on the functional $\bar{\Gamma}_k(\bar{g}) = \Gamma_k(0;\bar{g})$ where one simply sets the classical fluctuation field to zero [98]. By using the covariant Schwinger-DeWitt formalism, one can sometimes compute beta functions without specifying the background [106], so that the result can be said to be background-independent [19].

More recently there have been several calculations of beta functions based either on "bi-metric" truncations [126], [127], [154] or truncation depending on a flat background and up to four powers of $h_{\mu\nu}$ [109], [116], [117], [161], [162]. These calculations highlight the problem of the split symmetry breaking and raise the question of how to physically interpret the results. In the bi-metric case split-invariance has been imposed in the IR limit [156]. Alternatively, one can try to solve simultaneously the modified split Ward identity and the flow equation. This was achieved in the conformally reduced case [128], [129], [163]. Other related ideas have been discussed in [164]–[166].

More progress has been made recently for the special case when

$$\epsilon_{\mu\nu} = 2\epsilon\bar{g}_{\mu\nu} \tag{3.79}$$

i.e. when the background is simply rescaled by a constant factor [130], [131], [157]. In this case it was possible to write the anomalous Ward identity explicitly. By making judicious choices in the gauge-fixing and cutoff terms, it has been reduced to the simple form

$$\delta_\epsilon\Gamma_k = \epsilon\partial_t\Gamma_k\,, \tag{3.80}$$

where $t = \log k$ and the RHS is just the "beta functional" of the theory. The definition of the EAA contains a large degree of arbitrariness, and in order to arrive at (3.80) one has to make several specific choices. First and foremost, the split between the background and the quantum field will not be of the standard linear form (3.73) but rather of the exponential form, see (3.129) below. Further specific choices have to be

[19] Notice difference in meaning; background independence in this context refers to the fact that the computation is done, in a sense, for all backgrounds simultaneously.

made in the gauge-fixing and in the cutoff term. In particular one has to use a "pure" cutoff, namely one that does not contain any running parameters [145], [167]. From the point of view of reducing the number of variables that the EAA depends on, this relation can be used to eliminate only the dependence on a single real degree of freedom, namely the total volume of the background.

Goals

The main purpose of this section is to study the generalization of the above result to the case when the infinitesimal transformation parameter ϵ in (3.79) is not a constant, in other words when the background is subjected to a Weyl transformation.

A secondary aim is to highlight the relation between certain results concerning the fate of global and local scale transformations in quantum gravity. Several results can be more easily discussed in the context of Conformally Reduced (CORE) gravity, where only the spin-zero, conformal degree of freedom of the metric is dynamical. In [168] the Wetterich equation was applied to CORE gravity and it was noted that (a certain realization of) Weyl transformations could be either preserved or not, depending on the choice of cutoff. Subsequently, several studies have focused on split symmetry transformations in CORE gravity [128]. Our treatment will follow closely [169], where it was shown (albeit in a single-metric context) how to maintain Weyl invariance in the functional RG. With some changes, the results of that paper can be adapted to the present case.

Weyl group

Because of the many different ways in which the Weyl group can be realized in a physical theory, it will be useful to clarify the meaning of the phrase "Weyl transformation". The abstract Weyl group $\mathcal{W}$ is just the multiplicative group of positive real functions on a manifold. It can be realized on fields in several ways and to avoid the danger of misunderstandings one should specify what realization one is talking about. If the bare (classical) action is Weyl invariant, then in addition to fixing the gauge for diffeomorphisms one should also fix the Weyl gauge. In this context, as with diffeomorphisms, one will have to distinguish between "quantum" Weyl transformations

$$\delta^{(Q)}_\epsilon g_{\mu\nu} = 2\epsilon g_{\mu\nu}\,; \qquad \delta^{(Q)}_\epsilon \bar{g}_{\mu\nu} = 0\,; \qquad \delta^{(Q)}_\epsilon h_{\mu\nu} = 2\epsilon g_{\mu\nu} \tag{3.81}$$

and "background" Weyl (BW) transformations

$$\delta^{(B)}_\epsilon g_{\mu\nu} = 2\epsilon g_{\mu\nu}\,; \qquad \delta^{(B)}_\epsilon \bar{g}_{\mu\nu} = 2\epsilon \bar{g}_{\mu\nu}\,; \qquad \delta^{(B)}_\epsilon h_{\mu\nu} = 2\epsilon h_{\mu\nu}\,. \tag{3.82}$$

What we will mostly be interested in here is a different realization, which we shall call "split Weyl transformations" (SW)

$$\delta_\epsilon^{(S)} g_{\mu\nu} = 0 \, ; \qquad \delta_\epsilon^{(S)} \bar{g}_{\mu\nu} = 2\epsilon \bar{g}_{\mu\nu} \, ; \qquad \delta_\epsilon^{(S)} h_{\mu\nu} = -2\epsilon \bar{g}_{\mu\nu} \, . \tag{3.83}$$

Note that any bare action is invariant under (3.83) simply because $g_{\mu\nu}$ is invariant under those transformations. In fact, SW transformations are a subgroup of the split transformation (3.77).

3.3.2 CORE gravity

Several authors have considered CORE gravity as an interesting theoretical toy model in which to test various ideas related to the use of the FRG equation. This has been done both in the "single field" [168], [170], [171] and in the bi-field approximation [126]–[129], [154], [163].

Definitions

In CORE gravity one considers only metrics belonging to a single conformal class. Fixing a "fiducial" or reference metric $\hat{g}_{\mu\nu}$ in this class, every other metric can be obtained by a Weyl transformation

$$g_{\mu\nu} = e^{2\sigma} \hat{g}_{\mu\nu} . \tag{3.84}$$

Given any action $S(g)$, one obtains an action $S'(\sigma; \hat{g}) = S(g(\sigma, \hat{g}))$. Insofar as $\hat{g}$ is kept fixed, the dependence on it is often not indicated. In this way gravity is reduced to a scalar field theory. For the field σ one has an additive background-quantum split

$$\sigma = \bar{\sigma} + \omega \, . \tag{3.85}$$

Thus, we can define a background metric

$$\bar{g}_{\mu\nu} = e^{2\bar{\sigma}} \hat{g}_{\mu\nu} , \tag{3.86}$$

and the full metric is obtained from the background metric by means of the Weyl transformation

$$g_{\mu\nu} = e^{2\omega} \bar{g}_{\mu\nu} \, . \tag{3.87}$$

Since we have three different metrics in total, there are several Weyl transformations we can perform in this setting. If the classical action is Weyl-invariant to begin with, its CORE reduction is constant and the CORE theory is topological. This is a somewhat trivial case, but one could still discuss the fate of the transformations in the quantum theory. Background Weyl transformations are defined by

$$\delta^{(B)} g_{\mu\nu} = 2\epsilon g_{\mu\nu} \, ; \quad \delta^{(B)} \bar{g}_{\mu\nu} = 2\epsilon \bar{g}_{\mu\nu} \, ; \quad \delta^{(B)} \hat{g}_{\mu\nu} = 0 \tag{3.88}$$

and therefore

$$\delta^{(B)}\sigma = \epsilon\,; \quad \delta^{(B)}\omega = 0\,; \quad \delta^{(B)}\overline{\sigma} = \epsilon\,. \tag{3.89}$$

For a generic gravitational action, its CORE reduction is not constant, but is by construction invariant under the SW transformations, which are defined by

$$\delta^{(S)}g_{\mu\nu} = 0\,; \quad \delta^{(S)}\overline{g}_{\mu\nu} = 2\epsilon\overline{g}_{\mu\nu}\,; \quad \delta^{(S)}\hat{g}_{\mu\nu} = 0 \tag{3.90}$$

and therefore

$$\delta^{(S)}\sigma = 0\,; \quad \delta^{(S)}\omega = -\epsilon\,; \quad \delta^{(S)}\overline{\sigma} = \epsilon\,. \tag{3.91}$$

One can define a third realization of the Weyl group, acting on the fiducial metric in such a way as to maintain the background (as well as the full) metric invariant. For want of a better name, these transformations shall be called "fiducial Weyl (FW) transformations":

$$\delta^{(F)}g_{\mu\nu} = 0\,; \quad \delta^{(F)}\overline{g}_{\mu\nu} = 0\,; \quad \delta^{(F)}\hat{g}_{\mu\nu} = 2\epsilon\hat{g}_{\mu\nu} \tag{3.92}$$

and

$$\delta^{(F)}\sigma = -\epsilon\,; \quad \delta^{(F)}\omega = 0\,; \quad \delta^{(F)}\overline{\sigma} = -\epsilon\,. \tag{3.93}$$

In CORE gravity one does not generally consider such transformations, because the fiducial metric is kept fixed, but we mention them here for later reference.

Weyl calculus

Let us briefly digress to review a computational technique, dubbed Weyl calculus, that we rely on extensively for the rest of this discussion [172]–[178]. We define a new variable $\overline{\chi}$ as the inverse square root of the conformal factor of the background metric $\overline{\chi} = e^{-\overline{\sigma}}$. It transforms under Weyl transformations as

$$\delta\overline{\chi} = -\epsilon\overline{\chi}\,, \tag{3.94}$$

hence it can be identified with the background value of a dilaton field. We can use $\overline{\chi}$ to construct a pure-gauge abelian gauge field $\kappa_\mu = -\overline{\chi}^{-1}\partial_\mu\overline{\chi} = \partial_\mu\overline{\sigma}$, transforming under Weyl transformations as

$$\delta\kappa_\mu = \partial_\mu\epsilon\,. \tag{3.95}$$

Let $\overline{\nabla}_\mu$ be the covariant derivative with respect to the Levi-Civita connection of the metric $\overline{g}$ and $\hat{\nabla}_\mu$ be the covariant derivative with respect to the Levi-Civita connection of the metric $\hat{g}$. They are related by

$$\hat{\Gamma}_\mu{}^\lambda{}_\nu = \bar{\Gamma}_\mu{}^\lambda{}_\nu - \delta^\lambda_\mu\kappa_\nu - \delta^\lambda_\nu\kappa_\mu + \overline{g}_{\mu\nu}\overline{g}^{\lambda\tau}\kappa_\tau\,. \tag{3.96}$$

The connection coefficients $\hat{\Gamma}$ are invariant under background Weyl transformations, as is obvious since the metric $\hat{g}$ is. We say that a tensor t has weight α if it transforms under background Weyl transformation as

$$\delta t = \alpha\, \epsilon\, t\, . \tag{3.97}$$

Here we do not write tensor indices, as they are the same on both sides of the equation. For example, the background metric has weight 2, as does the the fluctuation $h^T_{\mu\nu}$. For any tensor t of weight α, we define the Weyl-covariant derivative as

$$\mathcal{D}_\mu t = \hat{\nabla}_\mu t - \alpha \kappa_\mu t\, . \tag{3.98}$$

It is a tensor with the same weight as t. We note in particular the special cases

$$\mathcal{D}_\rho \overline{g}_{\mu\nu} = 0\, ; \qquad \mathcal{D}_\rho \overline{\chi} = 0. \tag{3.99}$$

The fields $\overline{\sigma}$ and ω transform inhomogeneously and therefore have to be treated separately. Their Weyl-covariant derivatives are defined as

$$\mathcal{D}_\rho \overline{\sigma} = \partial_\rho \overline{\sigma} - \kappa_\rho = 0\, ; \qquad \mathcal{D}_\rho \omega = \partial_\rho \omega + \kappa_\rho \tag{3.100}$$

and are *in*variant (reflecting the absence of a homogeneous term in their transformation).

Cutoffs

In CORE gravity, we introduce the cutoff in the functional integration over ω. It has the general form

$$\Delta S_k = \frac{1}{2}\int d^d x \sqrt{\hat{g}}\; \omega \mathcal{R}_k(\overline{\sigma}, \hat{g})\omega. \tag{3.101}$$

The cutoff kernel $\mathcal{R}_k$ is a function of a Laplace-type operator $\mathcal{O}$ constructed with the fiducial metric and the background conformal factor. There are several choices for this operator, including the use of Weyl-covariant derivatives. Here we consider one natural choice of cutoff; for some other possibilities see [3].

We start with the cutoff defined by using $\mathcal{O} = \overline{\Delta}$ in (3.101), where $\overline{\Delta} = -\overline{g}^{\mu\nu}\bar{\nabla}_\mu \bar{\nabla}_\nu$ is the Laplacian of the background metric. For dimensional reasons, it can be written as

$$\mathcal{R}_k(\overline{\Delta}) = k^d r(y), \tag{3.102}$$

where r is a dimensionless function of the dimensionless variable $y = \overline{\Delta}/k^2$, that goes rapidly to zero for $y > 1$ and tends to 1 for $y \to 0$. The result of the transformations discussed in the previous section is

$$\begin{aligned}
\delta^{(S)}\Delta S_k &= -\frac{1}{2}\int d^dx\sqrt{\bar{g}}(\epsilon\mathcal{R}_k\omega + \omega\mathcal{R}_k\epsilon) \\
&\quad +\frac{1}{2}\int d^dx\sqrt{\bar{g}}\omega\left[\epsilon d\mathcal{R}_k + \epsilon\frac{\partial\mathcal{R}_k}{\partial\bar{\sigma}} + \partial_\mu\epsilon\frac{\partial\mathcal{R}_k}{\partial(\partial_\mu\bar{\sigma})} + \dots\right]\omega\,, \qquad (3.103)\\
\delta^{(F)}\Delta S_k &= 0\,, \qquad (3.104)\\
\delta^{(B)}\Delta S_k &= \frac{1}{2}\int d^dx\sqrt{\bar{g}}\omega\left[\epsilon d\mathcal{R}_k + \epsilon\frac{\partial\mathcal{R}_k}{\partial\bar{\sigma}} + \partial_\mu\epsilon\frac{\partial\mathcal{R}_k}{\partial(\partial_\mu\bar{\sigma})} + \dots\right]\omega. \qquad (3.105)
\end{aligned}$$

We clearly see that only $\delta^{(F)}$ provides a simple transformation rule for the cutoff, in fact, it is trivial [20]. This type of cutoff has also been used in [128] where the split Ward identity has been studied. The infinite series of terms appearing in the other two expressions, however, precludes deriving a useful expression for the modified split-Weyl Ward identities (msWWI). In [128] this problem was circumvented by considering only constant $\bar{\sigma}$, in which case the terms involving $\partial\epsilon$ drop out and a manageable expression was obtained.

Here we shall try to avoid such restrictions on the fields by introducing a SW-covariant derivative. The general definition of Weyl-covariant derivatives was given in (3.98). For the CORE case it is sufficient to note that $\partial_\mu\bar{\sigma}$ transforms as a gauge field under $\delta^{(S)}$. Since the field ω transforms by a shift, one can define its covariant derivative $\mathcal{D}_\mu\omega = \partial_\mu\omega + \partial_\mu\bar{\sigma}$. It is *invariant* under $\delta^{(S)}$, so that the second covariant derivative $\mathcal{D}_\nu\mathcal{D}_\mu\omega = \hat{\nabla}_\nu\mathcal{D}_\mu\omega$ is also invariant and the Laplacian $\overline{\Delta^W} = -\bar{g}^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu$ transforms simply as $\delta^{(S)}\overline{\Delta^W} = -2\epsilon\overline{\Delta^W}$.

As we shall see in more detail below, it will be useful to consider an "extended" transformation $\delta^{(E)}$ which agrees with $\delta^{(S)}$ on all fields but acts also on the cutoff by

$$\delta^{(E)}k = -\epsilon k\,, \qquad (3.106)$$

as dictated by dimensional analysis. Thus, acting on any functional of the fields and k,

$$\delta^{(E)} = \delta^{(S)} - \int d^dx\,\epsilon\,k\frac{\delta}{\delta k}\,. \qquad (3.107)$$

Note that since ϵ is generally not constant, we cannot assume that k is constant either. For the time being we take this just as a mathematical fact and defer a discussion of its physical meaning until later.

The cutoff is now a function

$$\mathcal{R}_k(\overline{\Delta^W}) = k^d r(y), \quad \text{with} \quad y = \frac{1}{k^2}\overline{\Delta^W}\,. \qquad (3.108)$$

[20] For this reason, in [168], where only transformations of the type $\delta^{(F)}$ were considered, this was called a "Weyl-preserving" cutoff.

We can write the $\delta^{(F)}$, $\delta^{(B)}$ and $\delta^{(S)}$ transformations of the cutoff terms for x-dependent k as follows,

$$\begin{aligned}
\delta^{(S)}\Delta S_k &= \int d^dx\epsilon k\frac{\delta}{\delta k}\Delta S_k - \frac{1}{2}\int d^dx\sqrt{\bar{g}}(\epsilon\mathcal{R}_k\omega + \omega r_0\epsilon), && (3.109)\\
\delta^{(F)}\Delta S_k &= -\frac{1}{2}\int d^dx\sqrt{\bar{g}}\omega\left[\partial_\mu\epsilon\frac{\partial\mathcal{R}_k\omega}{\partial(\partial_\mu\bar{\sigma})} + \partial_\mu\partial_\nu\epsilon\frac{\partial\mathcal{R}_k\omega}{\partial(\partial_\mu\partial_\nu\bar{\sigma})} + \dots\right], && (3.110)\\
\delta^{(B)}\Delta S_k &= \int d^dx\epsilon k\frac{\delta}{\delta k}\Delta S_k + \frac{1}{2}\int d^dx\sqrt{\bar{g}}\omega k^d\sum_{n=1}^{\infty} r_n y^n\epsilon, && (3.111)
\end{aligned}$$

These expressions can be arrived at via following reasoning. The crucial point is the transformation of the function $r(y)$, acting on a tensor t of weight α, under $\delta^{(E)}$. As usual, we start by expanding $r(y)$ in Taylor series

$$r(y) = \sum_{n=0}^{\infty} r_n y^n \,. \tag{3.112}$$

Then, we act on t and find how each term of $r(y)t$ in the series transforms. The thing to bear in mind is that y contains the covariant derivative $\mathcal{D}_\mu$ which acts on all the fields including $k(x)$. The variation of the first term gives $\delta^{(E)}(r_0t) = \alpha\epsilon r_0 t$. Since $\delta^{(E)}$ goes through the covariant derivative, the variation of the second term gives

$$\delta^{(E)}(yt) = \alpha\epsilon(yt). \tag{3.113}$$

From this expression, we obtain how y transforms. Since $\delta^{(E)}(yt) = (\delta^{(E)}y)t + y(\delta^{(E)}t)$, we get that

$$\delta^{(E)}y = \alpha[\epsilon, y]. \tag{3.114}$$

When this last result is applied to the third term in the series, we get

$$\delta^{(E)}(y^2t) = \delta^{(E)}y(yt) + y(\delta^{(E)}yt) = \alpha[\varepsilon, y]yt + y(\alpha\epsilon yt) = \alpha\epsilon y^2 t. \tag{3.115}$$

If we proceed by induction, we arrive at

$$\delta^{(E)}(y^n t) = \alpha\epsilon(y^n t). \tag{3.116}$$

Thus, we realize that $\delta^{(E)}(r(y)t) = \alpha\epsilon r(y)t$. That is, $r(y)$ maps a tensor of weight α to another tensor of weight α under $\delta^{(E)}$. Consequently, when analysing the variation of ΔS_k under $\delta^{(E)}$, we take the transformation of $r(y)t$ as just the transformation property of t. If we want to extend this result to the field ω, we have to take into account that it transforms by a shift and that $\mathcal{D}_\mu\omega$ is invariant under $\delta^{(E)}$. Therefore, in this case we have

$$\delta^{(E)}r(y)\omega = -r_0\epsilon. \tag{3.117}$$

These explicit outcomes lead us to the following conclusions

$$\delta^{(E)}\Delta S_k^T = 0, \;\; \delta^{(E)}\Delta S_k^{gh} = 0, \;\; \delta^{(E)}\Delta S_k^{\omega} = -\frac{1}{2}\int d^dx\sqrt{\hat{g}}\,(\epsilon\mathcal{R}_k\omega + \omega r_0\epsilon)\,. \tag{3.118}$$

It will become clear later that transformations involving linear terms in ω in (3.109)-(3.111) do not contribute to the variation of the EAA, so they are harmless for the derivation of a Ward identity. On the other hand, the transformations involving the functional derivative with respect to k lead to Ward identities with a known and compact form, as we will now show. We now derive the Ward identity associated to $\delta^{(S)}$ for this type of cutoff.

The modified split-Weyl Ward Identity

We start from the generating functional W_k, defined by

$$e^{W_k(j;\bar{\sigma},\hat{g})} = \int \mathcal{D}\omega\, e^{-S-\Delta S_k+\int j\omega}\,. \tag{3.119}$$

It is convenient to assume that j is a scalar density, to avoid the appearance of $\sqrt{\hat{g}}$. Taking into account that S is invariant under $\delta^{(S)}$, the variation of W_k is

$$\delta^{(S)}W_k(j;\hat{g},\bar{\sigma}) = -\langle\delta^{(S)}\Delta S_k\rangle - \int d^dx j\epsilon\,. \tag{3.120}$$

From the definition of the EAA

$$\Gamma_k(\langle\omega\rangle;\bar{\sigma},\hat{g}) = -W_k + \int d^dx j\langle\omega\rangle - \Delta S_k(\langle\omega\rangle)\,, \tag{3.121}$$

its transformation is

$$\delta^{(S)}\Gamma_k = -\delta^{(S)}W_k - \int d^dx j\epsilon - \delta^{(S)}\Delta S_k(\langle\omega\rangle). \tag{3.122}$$

The terms coming from the source cancel in the variation of Γ_k, and we end up just with

$$\delta^{(S)}\Gamma_k = \langle\delta^{(S)}\Delta S_k\rangle - \delta^{(S)}\Delta S_k(\langle\omega\rangle). \tag{3.123}$$

Similarly, the linear terms in ω coming from $\delta^{(S)}\Delta S_k$ cancel out, and we find

$$\delta^{(S)}\Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\omega\delta\omega} + \mathcal{R}_k\right)^{-1}\int d^dx\epsilon k\frac{\delta\mathcal{R}_k}{\delta k}\,, \tag{3.124}$$

where we have used the relation $\left(\frac{\delta^2\Gamma_k}{\delta\omega\delta\omega} + \mathcal{R}_k\right)^{-1} = \langle\omega(x)\omega(y)\rangle - \langle\omega(x)\rangle\langle\omega(y)\rangle$. Equation (3.124) tells us that the split symmetry in S is broken at the quantum level due to the introduction of the cutoff action.

On the other hand, in the appendix of [3] we have shown that the effective action, for an x-dependent scale, satisfies the flow equation

$$\int d^d x \delta k \frac{\delta \Gamma_k}{\delta k} = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 \Gamma_k}{\delta\omega\delta\omega} + \mathcal{R}_k \right)^{-1} \int d^d x \delta k \frac{\delta \mathcal{R}_k}{\delta k} . \qquad (3.125)$$

Therefore, the variation of the effective action with respect to the transformation $\delta^{(S)}$ is proportional to the functional derivative with respect to the scale k

$$\delta^{(S)} \Gamma_k = \int d^d x \epsilon k \frac{\delta \Gamma_k}{\delta k} . \qquad (3.126)$$

This last expression states that Γ_k is invariant under the extended transformation $\delta^{(E)}$ defined in the previous section:

$$\delta^{(E)} \Gamma_k = 0 . \qquad (3.127)$$

The EAA can thus be written in terms of the invariant quantities $\hat{k} = e^{\bar{\sigma}} k$ and $\sigma = \bar{\sigma} + \langle \omega \rangle$ as

$$\Gamma_k(\omega; \bar{\sigma}, \hat{g}) = \hat{\Gamma}_{\hat{k}}(\sigma; \hat{g}). \qquad (3.128)$$

In this way we have been able to reduce by one the number of functions that the EAA depends upon. We will now try to extend this result to the case of full gravity.

3.3.3 Full gravity

The discussion of CORE gravity makes it clear that the exponential parametrization of the conformal factor is the most natural one because Weyl transformations then act additively on the field.

Exponential parametrization

The last observation suggests that it may be convenient to parametrize the full metric fluctuation exponentially rather than additively as in (3.73). There is a number of advertised advantages of the exponential split so its use has become rather widespread, e.g. [110], [112], [118], [136], [144], [145], [179]–[190].

Henceforth we split the metric as follows

$$g_{\mu\nu} = \bar{g}_{\mu\rho} (e^{\mathbf{X}})^\rho{}_\nu \quad \text{where} \quad X^\rho{}_\nu = \bar{g}^{\rho\sigma} h_{\sigma\nu} . \qquad (3.129)$$

In dealing with exponentials, it is convenient to suppress indices and treat two-index tensors as matrices, independent of the position of the indices. Thus (3.129) will be written $\mathbf{g} = \bar{\mathbf{g}} e^{\mathbf{X}}$ and $\mathbf{X} = \bar{\mathbf{g}}^{-1} \mathbf{h}$. We decompose the fluctuation field into its tracefree and trace parts:

$$\mathbf{X} = \mathbf{X}^T + 2\omega \mathbf{1} \qquad (3.130)$$

where $\mathbf{X}^T$ is traceless and (following [144], [183]) we have defined $\omega = h/2d$, with $h = \mathrm{tr}\mathbf{X} = \bar{g}^{\mu\nu}h_{\mu\nu}$. Then we can write

$$\mathbf{g} = \bar{\mathbf{g}}\, e^{2\omega} e^{\mathbf{X}^T} , \tag{3.131}$$

which is the obvious generalization of (3.87). If the background metric undergoes the finite transformation $\bar{\mathbf{g}} \to \bar{\mathbf{g}}e^{2\epsilon}$, invariance of the full metric can be maintained by the compensating transformation $\omega \to \omega - \epsilon$, while $\mathbf{X}^T$ is left invariant. Then $\delta^{(S)}\mathbf{h}^T = \delta^{(S)}(\bar{\mathbf{g}}\mathbf{X}^T) = 2\epsilon\bar{\mathbf{g}}X^T = 2\epsilon\mathbf{h}^T$, which implies that

$$\delta^{(S)}h^{T\mu}{}_{\nu} = 0\,, \qquad \delta^{(S)}h^T_{\mu\nu} = 2\epsilon h^T_{\mu\nu}\,, \qquad \delta^{(S)}\omega = -\epsilon\,. \tag{3.132}$$

Here we will consider background metrics belonging to a single conformal equivalence class. As in the CORE case, we choose a fiducial metric $\hat{g}_{\mu\nu}$ in this class and parametrize all the others by their conformal factor $e^{\bar{\sigma}}$:

$$\bar{\mathbf{g}} = \hat{\mathbf{g}}e^{2\bar{\sigma}}\,. \tag{3.133}$$

Note that we can then write [21]

$$\mathbf{g} = \hat{\mathbf{g}}\, e^{2\sigma} e^{\mathbf{X}^T} = \bar{\mathbf{g}}\, e^{2\omega} e^{\mathbf{X}^T}\,; \qquad \bar{\mathbf{g}} = \hat{\mathbf{g}}\, e^{2\bar{\sigma}} \tag{3.134}$$

where e^{σ} is the conformal factor of the full metric, which can be split into a background part $e^{\bar{\sigma}}$ and a quantum part e^{ω}, related again as in (3.85). The basic transformation rule $\delta^{(S)}\bar{\mathbf{g}} = 2\epsilon\bar{\mathbf{g}}$ implies

$$\delta^{(S)}\bar{\sigma} = \epsilon \tag{3.135}$$

and this together with (3.132) implies $\delta^{(S)}\sigma = 0$. Thus with these definitions the invariance of the full metric can be expressed again as a simple shift invariance, albeit of the arguments of exponentials.

Gauge fixing

We can use the Weyl calculus, explained in subsection 3.3.2, to write SW-invariant functionals. Let us consider a gauge fixing term

$$S_{GF} = \frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\, F_\mu Y^{\mu\nu} F_\nu\,, \tag{3.136}$$

where F_μ is of the form (3.74). As shown in [130], F_μ is invariant under *global* SW transformations, *i.e.* transformations (3.79) with constant ϵ. It is easy to extend this result to *local* transformations simply replacing the derivative $\overline{\nabla}$ by the Weyl-covariant derivative $\mathcal{D}$ defined in (3.98). Thus the gauge condition is now

$$F_\mu = \mathcal{D}_\rho h^\rho{}_\mu - 2(\beta+1)\mathcal{D}_\mu\omega\,. \tag{3.137}$$

[21] A somewhat similar splitting in the gravitational path integral has been advocated in [191]–[193].

(we recall that $\omega = h/2d$) and we have

$$\delta^{(S)} F_\mu = 0\,. \tag{3.138}$$

In [130] invariance of the gauge-fixing action was obtained by choosing $Y^{\mu\nu}$ to contain a power of the background Laplacian. Here we note that the covariant derivative $\mathcal{D}$ has separate dependencies on $\bar{g}_{\mu\nu}$ and $\bar{\sigma}$, *i.e.* it cannot be written in terms of $\bar{g}_{\mu\nu}$ alone. Thus the gauge-fixing action is a functional $S_{GF}(h^{T\mu}{}_\nu, \omega; \bar{g}_{\mu\nu}, \bar{\sigma})$. Given that there is already a separate dependence on $\bar{\sigma}$, aside from the dependence through $\bar{g}_{\mu\nu}$, we may as well use it to define

$$Y^{\mu\nu} = e^{-(d-2)\bar{\sigma}} \bar{g}^{\mu\nu}\,. \tag{3.139}$$

This makes the gauge-fixing action invariant, without introducing additional derivatives. In particular, there is no need to introduce an auxiliary Nielsen-Kallosh ghost.

In order to derive the Faddeev-Popov operator, we start from the transformation of the full metric under an infinitesimal diffeomorphism η, $\delta_\eta \mathbf{g} = \mathcal{L}_\eta \mathbf{g}$. The "quantum" gauge transformation of the background $\bar{\mathbf{g}}$ and fluctuation field $\mathbf{X}$ satisfy

$$\delta_\eta^{(Q)} \bar{\mathbf{g}} = 0\,; \qquad e^{-\mathbf{X}} \delta_\eta^{(Q)} e^{\mathbf{X}} = e^{-\mathbf{X}} \bar{\mathbf{g}}^{-1} \mathcal{L}_\eta \mathbf{g} = e^{-\mathbf{X}} \bar{\mathbf{g}}^{-1} \mathcal{L}_\eta \bar{\mathbf{g}} e^{\mathbf{X}} + e^{-\mathbf{X}} \mathcal{L}_\eta e^{\mathbf{X}}\,. \tag{3.140}$$

Under any variation δ, $e^{-\mathbf{X}} \delta e^{\mathbf{X}} = \frac{1 - e^{-ad_{\mathbf{X}}}}{ad_{\mathbf{X}}} \delta \mathbf{X}$, so using this on both sides we obtain

$$\delta_\eta^{(Q)} \mathbf{X} = \frac{ad_{\mathbf{X}}}{e^{ad_{\mathbf{X}}} - \mathbf{1}} \bar{\mathbf{g}}^{-1} \mathcal{L}_\eta \bar{\mathbf{g}} + \mathcal{L}_\eta \mathbf{X}\,. \tag{3.141}$$

The Faddeev-Popov operator, acting on a ghost field C^μ, is defined by

$$\Delta_{FP\mu\nu} C^\nu = \mathcal{D}_\rho \left((\delta_C^{(Q)} \mathbf{X})^\rho{}_\mu - \frac{1+\beta}{d} \delta^\rho{}_\mu \mathrm{tr}(\delta_C^{(Q)} \mathbf{X}) \right) \tag{3.142}$$

where the infinitesimal transformation parameter η has been replaced by the ghost C^μ. The full ghost action then has the form [189]

$$S_{gh}(C^*_\mu, C_\mu; \bar{g}_{\mu\nu}, \bar{\sigma}) = -\int d^d x \sqrt{\bar{g}}\, C^*_\mu Y^{\mu\nu} \Delta_{FP\nu\rho} C^\rho\,. \tag{3.143}$$

The infinitesimal diffeomorphism parameter η^μ, and hence the ghost field C^μ, can be assumed to be invariant under $\delta^{(S)}$. Then, a straightforward calculation shows that $\delta_C^{(Q)} \mathbf{X}$ is invariant. Consequently, also $\Delta_{FP\mu}{}^\nu C_\nu$ is invariant. Assuming that the antighost C^*_μ is also invariant, the transformation of $Y^{\mu\nu}$ then exactly cancels the transformation of the integration measure, and we conclude that S_{gh} is SW-invariant[22]. Note that this statement refers to the full ghost action, containing infinitely many interaction vertices that are bilinear in the ghosts and contain arbitrary powers of $h_{\mu\nu}$.

[22]These transformation of the ghost C_μ and antighost C^*_μ agree with those of [130] when ϵ is constant.

Cutoffs

We now have to generalize the cutoff choice discussed in [130], from constant to non-constant rescalings of the metric. As before, it will be useful to consider also transformations where the cutoff itself changes (see Eq. (3.106)). Our ultimate goal is to arrive at an EAA that is invariant under the extended transformations, and the way to achieve it is to concoct the cutoff term in such a way that it is itself invariant. This issue has been addressed previously in a slightly different context in [169], [194]. Here we shall use the same techniques to write cutoff actions that are invariant, except for a single term that has to do with the inhomogeneous transformation properties of ω. We shall see that this is not an obstacle for the construction of an invariant EAA.

In order to construct diffeomorphism- and Weyl-invariant cutoffs we use a Weyl-covariant second order differential operator. For definiteness we adopt a "type I" cutoff (in the terminology of [106]) depending on the Laplacian

$$\overline{\Delta^W} = -\bar{g}^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu\,. \tag{3.144}$$

The cutoff terms for all the fields have the structure

$$\begin{aligned}\Delta S_k^T(h^T;\bar{g},\bar{\sigma}) &= \frac{1}{2}\int d^dx\sqrt{\bar{g}}\,h^{T\mu}{}_\nu\mathcal{R}_k(\overline{\Delta^W})h^{T\nu}{}_\mu\,,\\ \Delta S_k^\omega(\omega;\bar{g},\bar{\sigma}) &= \frac{1}{2}\int d^dx\sqrt{\bar{g}}\,\omega\,\mathcal{R}_k(\overline{\Delta^W})\omega\,,\\ \Delta S_k^{gh}(C^*,C;\bar{g},\bar{\sigma}) &= \int d^dx\sqrt{\bar{g}}\,C^*_\mu\mathcal{R}_k(\overline{\Delta^W})C^\mu\,,\end{aligned} \tag{3.145}$$

where

$$\mathcal{R}_k(\overline{\Delta^W}) = k^d r(y)\,,\qquad y = \frac{1}{k^2}\overline{\Delta^W}. \tag{3.146}$$

We have chosen the cutoff terms to be diagonal in field space, without loss of generality. Except for the introduction of the Weyl-covariant derivatives, these cutoffs are the same as in [130].

Note that we write the cutoff in terms of the mixed fluctuation so that all the fields have weight zero, *i.e.* they are invariant, except for ω that transforms by a shift. For a general tensor of weight α, the operator $\overline{\Delta}^W$ generates a tensor of weight $\alpha-2$. Thus we can write

$$\delta^{(E)}\overline{\Delta}^W = -2\epsilon\overline{\Delta}^W + \alpha[\epsilon,\overline{\Delta}^W]\,. \tag{3.147}$$

This implies that $r(y)$ maps a tensor of weight α to another tensor of weight α under $\delta^{(E)}$. Therefore, by simple counting, the cutoff terms for h^T and C are invariant under the extended transformations $\delta^{(E)}$. Using (3.107), there follows that

$$\delta^{(S)}\Delta S_k^{(i)} = \int d^dx\,\epsilon\,k\frac{\delta}{\delta k}\Delta S_k^{(i)}\qquad\text{for } i\in T, gh \tag{3.148}$$

where the functional variation with respect to k acts only on the cutoffs $\mathcal{R}_k$.

The case $i = \omega$ works a little differently, because ω does not transform homogeneously:

$$\delta^{(S)} \Delta S_k^{\omega} = \int d^d x\, \epsilon\, k \frac{\delta}{\delta k} \Delta S_k^{\omega} - \frac{1}{2} \int d^d x \sqrt{\bar{g}}\, (\epsilon \mathcal{R}_k \omega + \omega r_0 \epsilon) \ . \tag{3.149}$$

Thus this term is *not* invariant under $\delta^{(E)}$.

The modified split-Weyl Ward identity

We now have all the ingredients that are needed to derive the Ward identity for the SW tranformations $\delta^{(S)}$. One could follow step by step the derivation given in [130], which was based on the integro-differential equation satisfied by the EAA. Alternatively, we follow here the logic of [157]. We subject W_k to a SW transformation, with fixed sources and fixed k. Since the actions S, S_{GF} and S_{gh} are invariant by construction, and assuming the measure to be invariant, the only variations come from the cutoff and source terms:

$$\delta^{(S)} W_k = -\langle \delta^{(S)} \Delta S_k^T \rangle - \langle \delta^{(S)} \Delta S_k^{\omega} \rangle - \langle \delta^{(S)} \Delta S_k^{gh} \rangle - \int d^d x j \epsilon. \tag{3.150}$$

The variations of the cutoff terms have been given in (3.148,3.149). Their expectation values involve two- and one-point functions, that we can reexpress in terms of connected two-point functions and one-point functions as follows

$$\begin{aligned}
&-\frac{1}{2}\mathrm{Tr} \int \epsilon k \frac{\delta \mathcal{R}_k}{\delta k} \frac{\delta^2 W_k}{\delta j_T \delta j_T} - \frac{1}{2} \int d^d x \sqrt{\bar{g}} \frac{\delta W_k}{\delta j_T} \int \epsilon k \frac{\delta \mathcal{R}_k}{\delta k} \frac{\delta W_k}{\delta j_T} \\
&-\frac{1}{2}\mathrm{Tr} \int \epsilon k \frac{\delta \mathcal{R}_k}{\delta k} \frac{\delta^2 W_k}{\delta j \delta j} + \frac{\delta W_k}{\delta j} \int \epsilon k \frac{\delta \mathcal{R}_k}{\delta k} \frac{\delta W_k}{\delta j} - \int d^d x \sqrt{\bar{g}} \epsilon \mathcal{R}_k \frac{\delta W_k}{\delta j} \\
&-\mathrm{Tr} \int \epsilon k \frac{\delta \mathcal{R}_k}{\delta k} \frac{\delta^2 W_k}{\delta J \delta J_*} + 2 \frac{\delta W_k}{\delta J_*} \int \epsilon k \frac{\delta \mathcal{R}_k}{\delta k} \frac{\delta W_k}{\delta J}
\end{aligned}$$

where we use the shorthand

$$\int f \delta k \frac{\delta}{\delta k} = \int d^d x\, f(x) \delta k(x) \frac{\delta}{\delta k(x)} \,, \tag{3.151}$$

and we suppress indices for notational clarity. The variation of the EAA can be computed inserting these variations in (3.121). The source terms cancel out, as does the term linear in ω from (3.149) and the variations of the cutoff terms evaluated on the classical fields. There remain only the terms with the connected two-point functions, that can be re-expressed in terms of the EAA:

$$\begin{aligned}\delta^{(S)}\Gamma_k &= \frac{1}{2}\mathrm{STr}\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi}+\mathcal{R}_k\right)^{-1}\int \epsilon k\frac{\delta\mathcal{R}_k}{\delta k} \\ &= \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta h^T\delta h^T}+\mathcal{R}_k\right)^{-1}\int \epsilon k\frac{\delta\mathcal{R}^T_k}{\delta k}+\frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\omega\delta\omega}+\mathcal{R}_k\right)^{-1}\int \epsilon k\frac{\delta\mathcal{R}_k}{\delta k} \\ &\quad -\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta C^*\delta C}+\mathcal{R}_k\right)^{-1}\int \epsilon k\frac{\delta\mathcal{R}_k}{\delta k}+\dots . \end{aligned} \tag{3.152}$$

Here we use the "superfield" notation $\phi = (h^T_{\mu\nu}, \omega, C^*_\mu, C_\mu)$, and the ellipses indicate further mixing terms that arise in the inversion of the Hessian. Comparing (3.152) and local ERGE derived in [3] we see that

$$\delta^{(S)}\Gamma_k = \int \epsilon k\frac{\delta\Gamma_k}{\delta k} , \tag{3.153}$$

where we recall that the variation on the LHS involves only the field arguments of Γ_k and leaves k fixed. We have thus arrived at a remarkably simple result: with our choices for the gauge and cutoff terms, the anomalous variation in the msWWI is given by the "beta functional" of the theory, as expressed by the RHS of the local ERGE.

The reduced flow equation

Recalling the definition (3.107), we can rewrite (3.153) simply as

$$\delta^{(E)}\Gamma_k = 0 . \tag{3.154}$$

This is a statement of invariance of the EAA under a particular realization of the Weyl group. At the level of finite transformations

$$\Gamma_k(h^{T\mu}{}_\nu, C^*_\mu, C^\mu, \omega; \bar\sigma, \hat g_{\mu\nu}) = \Gamma_{\Omega^{-1}k}(h^{T\mu}{}_\nu, C^*_\mu, C^\mu, \omega - \log\Omega; \bar\sigma + \log\Omega, \hat g_{\mu\nu}) . \tag{3.155}$$

We can therefore rewrite the action entirely in terms of SW-invariant variables. Having chosen some of the fields to be invariant obviously simplifies the task. The choice of variables that we find both conceptually most satisfying and technically most useful is the following:

$$\hat k = e^{\bar\sigma}k\,;\quad h^{T\mu}{}_\nu\,;\quad C^*_\mu\,;\quad C^\mu\,;\quad \sigma=\bar\sigma+\omega\,;\quad \hat g_{\mu\nu} . \tag{3.156}$$

In the spirit of Weyl's theory, we are using the background dilaton field $\bar\chi = e^{-\bar\sigma}$ as unit of length and measure everything in its units [23].

[23] We avoid the alternative definition $\hat k = e^{\omega}k$ used in [130] because we find it awkward to have a dynamical variable in the cutoff scale. Another possible invariant metric would be $\tilde g_{\mu\nu} = e^{2\omega}\bar g_{\mu\nu}$. Note the relation between invariants: $\tilde g_{\mu\nu} = e^{2\sigma}\hat g_{\mu\nu}$. The alternative definition $\hat h^T_{\mu\nu} = e^{2\omega}h^T_{\mu\nu}$ would lead to a more complicated (off-diagonal) Jacobian.

The solution of the msWWI is therefore a functional

$$\hat{\Gamma}_{\hat{k}}(h^{T\mu}{}_{\nu}, C^*_\mu, C^\mu, \sigma; \hat{g}_{\mu\nu}) = \Gamma_k(h^{T\mu}{}_{\nu}, C^*_\mu, C^\mu, \omega; \overline{\sigma}, \hat{g}_{\mu\nu}) \,. \tag{3.157}$$

As expected the msWWI eliminates the dependence of the EAA on the dynamical variable ω and on the background variable $\overline{\sigma}$, replacing them by the single invariant σ. In the process one also has to redefine the background as well as the cutoff.

We must now rewrite the flow equation in terms of the new variables. Taking the total variation of both sides of (3.155), regarded as functionals of all their arguments, and comparing the coefficients of each differential, one obtains the following transformation rules:

$$\begin{aligned} k\frac{\delta\Gamma}{\delta k} &= \hat{k}\frac{\delta\hat{\Gamma}}{\delta\hat{k}}\,; \qquad \frac{\delta\Gamma}{\delta\hat{g}_{\mu\nu}} = \frac{\delta\hat{\Gamma}}{\delta\hat{g}_{\mu\nu}}\,; \qquad \frac{\delta\Gamma}{\delta\overline{\sigma}} = \frac{\delta\hat{\Gamma}}{\delta\sigma} + \hat{k}\frac{\delta\hat{\Gamma}}{\delta\hat{k}} \\ \frac{\delta\Gamma}{\delta h^{T\mu}{}_{\nu}} &= \frac{\delta\hat{\Gamma}}{\delta h^{T\mu}{}_{\nu}}\,; \qquad \frac{\delta\Gamma}{\delta\omega} = \frac{\delta\hat{\Gamma}}{\delta\sigma}\,; \qquad \frac{\delta\Gamma}{\delta C^*_\mu} = \frac{\delta\hat{\Gamma}}{\delta C^*_\mu}\,; \qquad \frac{\delta\Gamma}{\delta C^\mu} = \frac{\delta\hat{\Gamma}}{\delta C^\mu}\,. \end{aligned} \tag{3.158}$$

The reduced flow equation for the functional $\hat{\Gamma}$ has the form $\hat{k}\frac{\delta\hat{\Gamma}}{\delta\hat{k}} = \ldots$. Its RHS is the RHS of the local ERGE, that we must rewrite in terms of the new variables. In terms of the invariant "superfield" $\hat{\phi} = (h^{T\mu}{}_{\nu}, \sigma, C^*_\mu, C_\mu)$, one obtains

$$\begin{aligned} \hat{k}\frac{\delta\hat{\Gamma}_{\hat{k}}}{\delta\hat{k}} &= \frac{1}{2}\mathrm{STr}\left(\frac{\delta^2\hat{\Gamma}_{\hat{k}}}{\delta\hat{\phi}\delta\hat{\phi}} + \hat{\mathcal{R}}_{\hat{k}}\right)^{-1}\hat{k}\frac{\delta\hat{\mathcal{R}}_{\hat{k}}}{\delta\hat{k}} \\ &= \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\hat{\Gamma}_{\hat{k}}}{\delta\hat{h}^T\delta\hat{h}^T} + \hat{\mathcal{R}}_{\hat{k}}\right)^{-1}\hat{k}\frac{\delta\hat{\mathcal{R}}^T_{\hat{k}}}{\delta\hat{k}} + \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\hat{\Gamma}_{\hat{k}}}{\delta\sigma\delta\sigma} + \hat{\mathcal{R}}_{\hat{k}}\right)^{-1}\hat{k}\frac{\delta\hat{\mathcal{R}}_{\hat{k}}}{\delta\hat{k}} \\ &\quad -\mathrm{Tr}\left(\frac{\delta^2\hat{\Gamma}_{\hat{k}}}{\delta C^*\delta C} + \hat{\mathcal{R}}_{\hat{k}}\right)^{-1}\hat{k}\frac{\delta\hat{\mathcal{R}}_{\hat{k}}}{\delta\hat{k}} + \ldots\,. \end{aligned} \tag{3.159}$$

We see that the reduced flow equation has exactly the same form of the original one, except for being formulated in terms of invariant variables.

We note that this equation could be derived by rewriting the cutoff action as a functional the new variables:

$$\begin{aligned} \Delta\hat{S}^T_{\hat{k}}(h^{T\mu}{}_{\nu}; \hat{g}_{\mu\nu}) &= \Delta S_k(h^{T\mu}{}_{\nu}; \hat{g}_{\mu\nu}, \overline{\sigma}) = \frac{1}{2}\int d^dx\sqrt{\hat{g}}\, h^{T\mu}{}_{\nu}\, \hat{\mathcal{R}}_{\hat{k}}(\hat{\Delta}) h^{T\nu}{}_{\mu} \\ \Delta\hat{S}^\sigma_{\hat{k}}(\sigma; \hat{g}_{\mu\nu}) &= \Delta S^\omega_k(\omega; \hat{g}_{\mu\nu}, \overline{\sigma}) = \frac{1}{2}\int d^dx\sqrt{\hat{g}}\,\omega\, \hat{\mathcal{R}}_{\hat{k}}(\hat{\Delta})\omega\,, \\ \Delta\hat{S}^{gh}_{\hat{k}}(C^*_\mu, C^\mu; \hat{g}_{\mu\nu}) &= \Delta S_k(C^*_\mu, C^\mu; \hat{g}_{\mu\nu}, \overline{\sigma}) = \int d^dx\sqrt{\hat{g}}\, C^*_\mu \hat{g}^{\mu\nu}\hat{\mathcal{R}}_{\hat{k}}(\hat{\Delta})\hat{C}_\nu\,, \end{aligned} \tag{3.160}$$

where now

$$\hat{\mathcal{R}}^{(i)}_{\hat{k}}(\hat{\Delta}) = u^d r(y)\,, \qquad y = \frac{1}{\hat{k}^2}\hat{\Delta}\,, \qquad \hat{\Delta} = \frac{1}{\overline{\chi}^2}\overline{\Delta} = \hat{g}^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu\,, \qquad i \in \{T, \omega, gh\} \tag{3.161}$$

The global form of the reduced ERGE

In the discussion, we have introduced the extended transformations where besides the fields, also the cutoff is subjected to Weyl transformation. This is natural from the point of view of dimensional analysis, but it leads to the consequence that the cutoff cannot be regarded as constant anymore. The ERGE can be easily generalized to the case of non-constant cutoff, but its physical interpretation becomes then less clear. The flow of the FRGE in theory space would depend on a function, instead of a single parameter, which would be somewhat reminiscent of the “many-fingered time” of General Relativity. It would be interesting to explore a possible connection of the local ERGE with the notion of local RG [14], [195], which would then give it a direct physical meaning. We refrain from doing so here. Instead, we have noted that the solution of the msWWI implies that also the cutoff has to be replaced, as an argument of the EAA, by the quantity $\hat{k}$. Unlike k, it is invariant under (extended) SW transformations. It is therefore consistent to assume that

$$\hat{k} = k e^{\bar{\sigma}} = \text{constant} \tag{3.162}$$

If $\hat{k}$ is constant, we can replace

$$\hat{k}(x)\frac{\delta\hat{\Gamma}_{\hat{k}}}{\delta\hat{k}(x)} \qquad \text{by} \qquad \hat{k}\frac{d\hat{\Gamma}_{\hat{k}}}{d\hat{k}}$$

and the reduced ERGE becomes again an ordinary differential equation, whose solution are curves in theory space depending on the single parameter $\hat{k}$. In this way the local ERGE can be seen just as an intermediate mathematical construction.

3.3.4 Discussion

Let us summarize the main steps of our procedure. We started from the exponential parametrization (3.129) and demanded that physical results should not change under the SW transformations (3.132), (3.135). This is part of the requirement of background-independence. The classical action is trivially invariant under these transformations, because it is formulated in terms of a single metric, but the quantum effective action cannot be. In particular, the EAA, which is an effective action depending on an external cutoff scale k, cannot be invariant, because the gauge-fixing term and even more importantly the cutoff term are not. There is therefore a kind of anomaly. By making certain choices, we have however been able to define the EAA in such a way that the only source of non-invariance is the presence of the cutoff k. One can then extend the definition of the SW transformations by also transforming k, which henceforth had been kept fixed. The natural transformation is (3.106), on account of the dimensionality of k. This implies that k cannot be assumed to be constant. We have therefore generalized the ERGE by allowing the cutoff to be a positive function on spacetime. The resulting local ERGE has the same form as the usual one, except for the appearance of functional derivatives with respect to k in place of ordinary derivatives. One finds that the RHS of the msWWI is identical to the RHS of the local ERGE. Then, the msWWI just expresses the fact that the EAA is invariant under the extended transformations. This is the central result of this section. It establishes that the msWWI and the flow equation are manifestly compatible and can be solved simultaneously. The solution of the msWWI consists in writing the EAA as a functional $\hat{\Gamma}$ of a new set of variables that are invariant under the extended SW transformations, as in (3.157). We have then shown that the functional $\hat{\Gamma}$ satisfies a local ERGE (3.159) that is formally identical to the one satisfied by the original EAA.

The main aim of this analysis was to reduce the number of independent arguments that the EAA depends on, when the background field method is used. Let us discuss how this goal has been achieved. In the case of CORE gravity the fiducial metric is always held fixed, so the EAA can be seen as a functional $\Gamma_k(\omega;\bar{\sigma})$, where $\bar{\sigma}$ is the background field (the conformal factor of the background metric) and ω is the quantum field (such that $\bar{\sigma}+\omega$ is the conformal factor of the full dynamical metric). The msWWI (3.128) shows that the EAA can be rewritten in terms of a functional of σ alone, thereby reducing the number of scalar fields that it depends on from two to one, as desired.

Let us now see how the counting works in full gravity. In principle, we begin from an EAA $\Gamma_k(h^{T\mu}{}_\nu,\omega;\bar{g}_{\mu\nu})$, depending on 9+1+10=20 functions, instead of the desired 10 (we do not count here the ghosts, which are irrelevant for this discussion). We wanted to reduce this number by one by solving the msWWI. However, in order to apply our techniques, we had to reparametrize the background metric by splitting off its conformal part as in (3.133). This can only be done by first choosing a reference metric $\hat{g}_{\mu\nu}$ in the same conformal class as the background. In this way the EAA has become a functional $\Gamma_k(h^{T\mu}{}_\nu,\omega;\bar{\sigma},\hat{g}_{\mu\nu})$ depending on 9+1+1+10=21 functions. The solution of the msWWI (3.153) allows us to rewrite the EAA as a functional $\hat{\Gamma}_{\hat{k}}(h^{T\mu}{}_\nu,\sigma;\hat{g}_{\mu\nu})$ where, just as in the CORE case, the two functions ω and $\bar{\sigma}$ have been replaced by the single function σ. But now this depends again on 9+1+10 functions: it appears that we have merely traded the original dependence on the background metric with the dependence on the fiducial metric.

Why is full gravity different from CORE gravity? The only difference is that in the discussion of full gravity we keep track of the dependence on the fiducial metric, whereas in CORE gravity this is ignored. In fact the same issue would be present also in CORE gravity if we took into account the dependence on the fiducial metric.

In both cases, the problem is that the fiducial metric $\hat{g}_{\mu\nu}$ is another artificial choice that enters in the definition of the EAA, just like the background metric, and no physical result should depend on it. The two fields $\bar{\sigma},\hat{g}_{\mu\nu}$ would count as 10 independent variables if Γ_k was invariant under FW transformations. This invariance, however, is broken by the cutoffs. Alternatively, we could at least try to solve the corresponding modified FW WI. Unfortunately, as seen in Eq. (3.110), the transformation of the cutoff for ω under such FW transformations is very complicated and there is no hope of achieving a practical solution. In addition, in full gravity the transformation of the other cutoffs is similarly intractable [24]

$$\delta^{(F)}\Delta S_k^T = -\frac{1}{2}\int d^dx\sqrt{\bar{g}}h^{T\mu}{}_\nu\left[\partial_\lambda\epsilon\frac{\partial\mathcal{R}_k^T}{\partial(\partial_\lambda\bar{\sigma})}+\partial_\lambda\partial_\alpha\epsilon\frac{\partial\mathcal{R}_k^T}{\partial(\partial_\lambda\partial_\alpha\bar{\sigma})}+\cdots\right]h^{T\nu}{}_\mu, \tag{3.163}$$

$$\delta^{(F)}\Delta S_k^{gh} = -\int d^dx\sqrt{\bar{g}}C^*_\mu\left[\partial_\lambda\epsilon\frac{\partial\mathcal{R}_k^{gh}}{\partial(\partial_\lambda\bar{\sigma})}+\partial_\lambda\partial_\alpha\epsilon\frac{\partial\mathcal{R}_k^{gh}}{\partial(\partial_\lambda\partial_\alpha\bar{\sigma})}+\cdots\right]C^\mu. \tag{3.164}$$

Thus, with the chosen cutoff, there is no way to solve exactly the mFWWI. Additional approximations may allow one to do so. It was shown in [3] how to obtain a simple and exactly solvable mFWWI by using a different type of cutoff. In that case it is the msWWI that is too complicated to solve.

Thus we conclude that the task of reducing the number of functions in the EAA by one cannot be solved by the methods used here in generality. If there was any separate physical argument selecting a preferred metric $\hat{g}_{\mu\nu}$ within its conformal class, then the methods discussed here would provide the desired reduction of the independent variables. This solution could probably be extended to the full $\bar{g}_{\mu\nu}$ dependence of the EAA (as opposed to just its conformal factor) by using the $GL(4)$-invariant formulation discussed in [196]–[198]. The methods proposed are not restricted to the Wetterich equation but can be extended also to proper time-type flow equations, both approximate [199] and exact [101].

[24]We observe that if we used the cutoff (3.102), as in [128], invariance under FW transformations would be trivial. In that case we would not have been able to solve the msWWI, though.

Chapter 4

Conclusions

Reception of the LHC results by the high energy physics community has been, by and large, pessimistic (and not only because certain bottles of cognac had to change hands [200]). Altarelli had put it like this [201]:

"A Higgs particle has been discovered which is compatible with the elementary, weakly coupled Higgs boson of the minimal SM version of the EW symmetry breaking sector. No clear signal of new physics has been found by ATLAS, CMS and LHCb. On the basis of naturalness one was expecting a more complicated reality. Nature appears to disregard our notion of naturalness and rather indicates an alternative picture where the SM, with a few additional ingredients, is valid up to large energies."

Our perspective on these results is less negative than that of GUT, SUSY or composite Higgs communities. We too dislike that the LHC did not provide us with any glimpse of new physics, but our guiding principle is the asymptotic safety, and not naturalness, so we were not shocked to foundation by the LHC results.

Future work is going to be in muddier waters than it would have been had the LHC revealed some elements of new physics. This is as true for the asymptotic safety as it is for other paradigms. However, this would appear to be a relatively stronger position for the asymptotically safe models than it is for the theories based on naturalness. For one thing this is the case because fundamental scalars pose no problem to asymptotically safe theories, and as we have seen in section 2.1 these may even be required ingredients. Second thing is that a generic asymptotically safe model has high probability of being very predictive with respect to the usual effective field theory models.

We have elaborated repeatedly throughout this thesis on how this predictivity arises. A generic UV fixed point will have a certain number of relevant directions and an infinite number of irrelevant ones. If we conjecture that the nature is asymptotically safe, we are demanding that its RG flow lies on an n-dimensional hypersurface in the coupling space, where n is the number of relevant directions. One ought to measure only n couplings in order to be able to fully predict all others. An optimist will notice that this opens up a lot of potential, since in principle one could explain relations between various SM parameters that we currently think of as independent; i.e. it may not be necessary to go to high energy scales in order to test some asymptotically safe models.

Consider the "*desert hypothesis*"—the idea that besides a few right-handed neutrinos and quantum gravity there's nothing else to be added to the SM—which is an asymptotic safety equivalent of MSSM[1]. In the context of the asymptotic safety this is a minimalistic and predictive model, which makes it anything but boring. If we could reliably establish the existence and the location of the (gravity induced) UV fixed point and then integrate its RG flow to the IR, we would obtain certain relations between the SM parameters at experimentally reachable energy scales. If these relations would be shown not to hold it would be a clear indication that either nature is not asymptotically safe or that there is more new physics "at the bottom".

The aim of this thesis was to provide a view on some recent developments in the field of asymptotically safe QFT from multiple angles; it is author's sincere hope that the resulting picture is easier to interpret than the works of cubists. The route taken in the thesis was such that both simplicity and the robustness of the results were decreasing with section numbers.

We began, in section 2.1, with a recently-introduced asymptotically-safe gauge-Yukawa system called LISA. In section 2.2 we have checked the robustness of the parametrically-weak perturbative UV fixed point in LISA against the instanton effects. In the process we were able to model the vacuum of LISA as an instanton fluid. We did this by mimicking the standard pure-YM computation of Diakonov and Petrov, and in fact we were able to characterize the instanton fluid by computing its instanton density per unit volume and its average instanton size. These variables are of particular interest due to their relation to nuclear observables.

Having confirmed the existence of perturbative UV fixed points as a matter of principle, we proceeded to relax the assumptions and move from the Veneziano limit to phenomenologically more interesting theories. In section 2.3 we have studied close to 400 000 models whose Lagrangian is given by the Standard Model coupled to new vector-like fermions charged under the SM gauge group, and uncharged scalars. Despite reporting negative results—our search has not produced a single reliable UV-safe and phenomenologically interesting candidate—we have stressed the importance of continuing this line of model building in the future. Limitations of our models were explained in detail, and here we only reiterate their intentional (over)simplicity. By studying models which are so simple we have managed to obtain robust results and pave the way for future computations which we believe will be more fruitful. Nonetheless, one has to face the possibility that particle physics models have ignored gravity for too long, and that gravitational contributions to the gauge, Yukawa, and scalar beta functions may be crucial for developing phenomenologically relevant UV fixed points. In particular, we've shown that studied models that do develop perturbative FPs appear to suffer from triviality problem in the $U(1)$-sector—a problem that may be simply curable by coupling the gauge theory to gravity.

[1]Minimal Supersymmetric Standard Model—an unusually simple SUSY model used, among other things, to look for SUSY signals in colliders [202]–[204]. Notice that by our standards it is not a minimalistic model at all, requiring measurement of 120 new parameters w.r.t. SM.

The most studied (tentatively) asymptotically safe gauge theory is gravity. The central result of the studies to date is a sweet-sour one: the gravitational fixed point probably exists, but in the semi-perturbative regime. We have defined the concept of semi-perturbativity as being "weakly non-perturbative", in the sense that studying such dynamics cannot be done in perturbation theory, but one may still use (semi-) analytic techniques such as functional renormalization group, and rely on the usual perturbative thinking, e.g. regarding the operator ordering. Thus, even though the UV fixed point is non-Gaussian, a priori one probably wouldn't have expected such a fortunate scenario, in fact most of the popular approaches to quantum gravity assume that metric is a poor description of UV degrees of freedom and thus introduce new fundamental objects, e.g. strings and branes, spin-foam, or causal sets. This is analogous to going from quarks to mesons in the IR QCD. By changing degrees of freedom one may again write Lagrangians with perturbative couplings, the difference being, of course, that in the case of gravity "mesons" would be the more fundamental objects. We find it very encouraging that all the Asymptotic Safety research to date seems to imply that there is a non-Gaussian UV FP in the gravitational Einstein-Hilbert equivalence class, and that the FP is weakly-interacting enough so that the perturbative reasoning is useful for its exploration.

Following all of the above observations, in section 3.2 we have studied a specific nonminimal derivative interaction between gravity and a scalar field to confirm a hypothesis that a generic combined matter-gravity fixed point cannot admit a fully asymptotically free matter sector. In fact, what we have shown is that at a generic FP there will be gravity-induced matter-gravity interactions. This is in line with general arguments and with expectations from similar previous computations and it reinforces the point that models aiming to explore the existence of realistic matter-gravity fixed points should not assume negligible matter interactions in the deep UV.

One of the observations that we've made in the above computation is the dependence of the results on various "copies" of the quantum Newton coupling. These couplings are related through so-called modified split Ward identities or msWI's. A good way of dealing with these identites is not yet known, but it is of central importance for obtaining quantitativelly robust results with the functional RG. In the final section, 3.3, we have built on a few previous works and we've constructed a procedure to deal with the msWI's. This procedure works well in the case of conformally reduced gravity, but at this stage it is not clear if it may also be successfully extended to the case of full gravity or if a different strategy will have to be adopted. It is interesting that in solving msWI's with our technique we were naturally led to the concept of a local functional RG equation.

Considering the above discussion it is clear that there exist innumerable [2] ways of extending our works and the other asymptotic safety works, so instead of attempting to count them we leave them to reader's imagination.

[2]More precisely: a very large numer.

Bibliography

[1] F. Sannino and V. Skrinjar, "Safe and free instantons", 2018. arXiv: 1802.10372 [hep-th].

[2] A. Eichhorn, S. Lippoldt, and V. Skrinjar, "Nonminimal hints for asymptotic safety", *Phys. Rev.*, vol. D97, no. 2, p. 026 002, 2018. DOI: 10.1103/PhysRevD.97.026002. arXiv: 1710.03005 [hep-th].

[3] C. M. Nieto, R. Percacci, and V. Skrinjar, "Split weyl transformations in quantum gravity", *Phys. Rev.*, vol. D96, no. 10, p. 106 019, 2017. DOI: 10.1103/PhysRevD.96.106019. arXiv: 1708.09760 [gr-qc].

[4] D. Barducci, M. Fabbrichesi, C. M. Nieto, R. Percacci, and V. Skrinjar, "In search of a uv completion of the standard model - 378.000 models that don't work", 2018. arXiv: 1807.05584 [hep-ph].

[5] D. J. Gross and F. Wilczek, "Asymptotically free gauge theories. 1", *Phys. Rev.*, vol. D8, pp. 3633–3652, 1973. DOI: 10.1103/PhysRevD.8.3633.

[6] H. D. Politzer, "Reliable perturbative results for strong interactions?", *Phys. Rev. Lett.*, vol. 30, pp. 1346–1349, 1973. DOI: 10.1103/PhysRevLett.30.1346.

[7] K. G. Wilson, "Renormalization group and critical phenomena. 1. renormalization group and the kadanoff scaling picture", *Phys. Rev.*, vol. B4, pp. 3174–3183, 1971. DOI: 10.1103/PhysRevB.4.3174.

[8] ——, "Renormalization group and critical phenomena. 2. phase space cell analysis of critical behavior", *Phys. Rev.*, vol. B4, pp. 3184–3205, 1971. DOI: 10.1103/PhysRevB.4.3184.

[9] S. Weinberg, "Ultraviolet divergences in quantum theories of gravitation", *General Relativity: An Einstein centenary survey, Eds. Hawking, S.W., Israel, W; Cambridge University Press*, pp. 790–831, 1979.

[10] D. F. Litim and F. Sannino, "Asymptotic safety guaranteed", *JHEP*, vol. 12, p. 178, 2014. DOI: 10.1007/JHEP12(2014)178. arXiv: 1406.2337 [hep-th].

[11] O. Antipin, M. Gillioz, J. Krog, E. Mølgaard, and F. Sannino, "Standard model vacuum stability and weyl consistency conditions", *JHEP*, vol. 08, p. 034, 2013. DOI: 10.1007/JHEP08(2013)034. arXiv: 1306.3234 [hep-ph].

[12] H. Osborn, "Derivation of a four dimensional c theorem for renormalisable quantum field theories", *Phys. Lett. B*, vol. 222, pp. 97–102, 1989. DOI: 10.1016/0370-2693(89)90729-6.

[13] I. Jack and H. Osborn, "Analogs for the c theorem for four-dimensional renormalizable field theories", *Nucl. Phys.*, vol. B343, pp. 647–688, 1990. DOI: 10.1016/0550-3213(90)90584-Z.

[14] H. Osborn, "Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories", *Nucl. Phys.*, vol. B363, pp. 486–526, 1991. DOI: 10.1016/0550-3213(91)80030-P.

[15] I. Jack and H. Osborn, "Constraints on rg flow for four dimensional quantum field theories", *Nucl. Phys.*, vol. B883, pp. 425–500, 2014. DOI: 10.1016/j.nuclphysb.2014.03.018. arXiv: 1312.0428 [hep-th].

[16] O. Antipin, M. Gillioz, E. Mølgaard, and F. Sannino, "The a theorem for gauge-yukawa theories beyond banks-zaks fixed point", *Phys. Rev.*, vol. D87, no. 12, p. 125 017, 2013. DOI: 10.1103/PhysRevD.87.125017. arXiv: 1303.1525 [hep-th].

[17] N. A. Dondi, F. Sannino, and V. Prochazka, "Conformal data of fundamental gauge-yukawa theories", 2017. arXiv: 1712.05388 [hep-th].

[18] M. Shifman, *Advanced topics in quantum field theory.* Cambridge, UK: Cambridge Univ. Press, 2012, ISBN: 9781139210362, 9780521190848.

[19] E. V. Shuryak, "The qcd vacuum, hadrons and the superdense matter", *World Sci. Lect. Notes Phys.*, vol. 71, pp. 1–618, 2004, [World Sci. Lect. Notes Phys.8,1(1988)].

[20] S. Coleman, *Aspects of symmetry: Selected Erice lectures*. Cambridge University Press, 1988.

[21] T. Schäfer and E. V. Shuryak, "Instantons in qcd", *Rev. Mod. Phys.*, vol. 70, pp. 323–426, 1998. DOI: 10.1103/RevModPhys.70.323. arXiv: hep-ph/9610451 [hep-ph].

[22] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, "Abc's of instantons", *Sov. Phys. Usp.*, vol. 25, p. 195, 1982, [Usp. Fiz. Nauk136,553(1982)]. DOI: 10.1070/PU1982v025n04ABEH004533.

[23] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, "Pseudoparticle solutions of the yang-mills equations", *Phys. Lett.*, vol. 59B, pp. 85–87, 1975. DOI: 10.1016/0370-2693(75)90163-X.

[24] G. 't Hooft, "Computation of the quantum effects due to a four-dimensional pseudoparticle", *Phys. Rev.*, vol. D14, pp. 3432–3450, 1976, [Erratum: Phys. Rev.D18,2199(1978)]. DOI: 10.1103/PhysRevD.18.2199.3,10.1103/PhysRevD.14.3432.

[25] C. W. Bernard, N. H. Christ, A. H. Guth, and E. J. Weinberg, "Instanton parameters for arbitrary gauge groups", *Phys. Rev.*, vol. D16, p. 2967, 1977. DOI: 10.1103/PhysRevD.16.2967.

[26] C. W. Bernard, "Gauge zero modes, instanton determinants, and qcd calculations", *Phys. Rev.*, vol. D19, p. 3013, 1979. DOI: 10.1103/PhysRevD.19.3013.

[27] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, "Instanton density in a theory with massless quarks", *Nucl. Phys.*, vol. B163, pp. 46–56, 1980. DOI: 10.1016/0550-3213(80)90389-2.

[28] D. Diakonov and V. Yu. Petrov, "A theory of light quarks in the instanton vacuum", *Nucl. Phys.*, vol. B272, pp. 457–489, 1986. DOI: 10.1016/0550-3213(86)90011-8.

[29] C. G. Callan Jr., R. F. Dashen, and D. J. Gross, "Toward a theory of the strong interactions", *Phys. Rev.*, vol. D17, p. 2717, 1978. DOI: 10.1103/PhysRevD.17.2717.

[30] E. V. Shuryak, "The role of instantons in quantum chromodynamics. 2. hadronic structure", *Nucl. Phys.*, vol. B203, pp. 116–139, 1982. DOI: 10.1016/0550-3213(82)90479-5.

[31] D. Diakonov and V. Yu. Petrov, "Instanton based vacuum from feynman variational principle", *Nucl. Phys.*, vol. B245, pp. 259–292, 1984. DOI: 10.1016/0550-3213(84)90432-2.

[32] W. E. Caswell and F. Wilczek, "On the gauge dependence of renormalization group parameters", *Phys. Lett.*, vol. 49B, pp. 291–292, 1974. DOI: 10.1016/0370-2693(74)90437-7.

[33] T. Schäfer, "Instantons in qcd with many colors", *Phys. Rev.*, vol. D66, p. 076 009, 2002. DOI: 10.1103/PhysRevD.66.076009. arXiv: hep-ph/0206062 [hep-ph].

[34] D. F. Litim, M. Mojaza, and F. Sannino, "Vacuum stability of asymptotically safe gauge-yukawa theories", *JHEP*, vol. 01, p. 081, 2016. DOI: 10.1007/JHEP01(2016)081. arXiv: 1501.03061 [hep-th].

[35] H. Gies and M. M. Scherer, "Asymptotic safety of simple yukawa systems", *Eur. Phys. J.*, vol. C66, pp. 387–402, 2010. DOI: 10.1140/epjc/s10052-010-1256-z. arXiv: 0901.2459 [hep-th].

[36] H. Gies, S. Rechenberger, and M. M. Scherer, "Towards an asymptotic-safety scenario for chiral yukawa systems", *Eur. Phys. J.*, vol. C66, pp. 403–418, 2010. DOI: 10.1140/epjc/s10052-010-1257-y. arXiv: 0907.0327 [hep-th].

[37] M. Fabbrichesi, R. Percacci, A. Tonero, and O. Zanusso, "Asymptotic safety and the gauged su(n) nonlinear sigma model", *Phys. Rev.*, vol. D83, p. 025 016, 2011. DOI: 10.1103/PhysRevD.83.025016. arXiv: 1010.0912 [hep-ph].

[38] F. Bazzocchi, M. Fabbrichesi, R. Percacci, A. Tonero, and L. Vecchi, "Fermions and goldstone bosons in an asymptotically safe model", *Phys. Lett.*, vol. B705, pp. 388–392, 2011. DOI: 10.1016/j.physletb.2011.10.029. arXiv: 1105.1968 [hep-ph].

[39] M. Fabbrichesi, R. Percacci, A. Tonero, and L. Vecchi, "The electroweak s and t parameters from a fixed point condition", *Phys. Rev. Lett.*, vol. 107, p. 021 803, 2011. DOI: 10.1103/PhysRevLett.107.021803. arXiv: 1102.2113 [hep-ph].

[40] H. Gies, S. Rechenberger, M. M. Scherer, and L. Zambelli, "An asymptotic safety scenario for gauged chiral higgs-yukawa models", *Eur. Phys. J.*, vol. C73, p. 2652, 2013. DOI: 10.1140/epjc/s10052-013-2652-y. arXiv: 1306.6508 [hep-th].

[41] B. Bajc and F. Sannino, "Asymptotically safe grand unification", *JHEP*, vol. 12, p. 141, 2016. DOI: 10.1007/JHEP12(2016)141. arXiv: 1610.09681 [hep-th].

[42] A. D. Bond, G. Hiller, K. Kowalska, and D. F. Litim, "Directions for model building from asymptotic safety", *JHEP*, vol. 08, p. 004, 2017. DOI: 10.1007/JHEP08(2017)004. arXiv: 1702.01727 [hep-ph].

[43] S. Abel and F. Sannino, "Framework for an asymptotically safe standard model via dynamical breaking", *Phys. Rev.*, vol. D96, no. 5, p. 055 021, 2017. DOI: 10.1103/PhysRevD.96.055021. arXiv: 1707.06638 [hep-ph].

[44] ——, "Radiative symmetry breaking from interacting uv fixed points", *Phys. Rev.*, vol. D96, no. 5, p. 056 028, 2017. DOI: 10.1103/PhysRevD.96.056028. arXiv: 1704.00700 [hep-ph].

[45] R. Mann, J. Meffe, F. Sannino, T. Steele, Z.-W. Wang, and C. Zhang, "Asymptotically safe standard model via vectorlike fermions", *Phys. Rev. Lett.*, vol. 119, no. 26, p. 261 802, 2017. DOI: 10.1103/PhysRevLett.119.261802. arXiv: 1707.02942 [hep-ph].

[46] G. M. Pelaggi, A. D. Plascencia, A. Salvio, F. Sannino, J. Smirnov, and A. Strumia, "Asymptotically safe standard model extensions?", *Phys. Rev.*, vol. D97, no. 9, p. 095 013, 2018. DOI: 10.1103/PhysRevD.97.095013. arXiv: 1708.00437 [hep-ph].

[47] G. M. Pelaggi, F. Sannino, A. Strumia, and E. Vigiani, "Naturalness of asymptotically safe higgs", *Front.in Phys.*, vol. 5, p. 49, 2017. DOI: 10.3389/fphy.2017.00049. arXiv: 1701.01453 [hep-ph].

[48] S. Ipek, A. D. Plascencia, and J. Turner, "Assessing perturbativity and vacuum stability in high-scale leptogenesis", 2018. arXiv: 1806.00460 [hep-ph].

[49] A. D. Bond and D. F. Litim, "Theorems for asymptotic safety of gauge theories", *Eur. Phys. J.*, vol. C77, no. 6, p. 429, 2017, [Erratum: Eur. Phys. J.C77,no.8,525(2017)]. DOI:

10.1140/epjc/s10052-017-4976-5, 10.1140/epjc/s10052-017-5034-z. arXiv: 1608.00519 [hep-th].

[50] M. Gell-Mann and F. E. Low, "Quantum electrodynamics at small distances", *Phys. Rev.*, vol. 95, pp. 1300–1312, 1954. DOI: 10.1103/PhysRev.95.1300.

[51] M. Gockeler, R. Horsley, V. Linke, P. E. L. Rakow, G. Schierholz, and H. Stuben, "Is there a landau pole problem in qed?", *Phys. Rev. Lett.*, vol. 80, pp. 4119–4122, 1998. DOI: 10.1103/PhysRevLett.80.4119. arXiv: hep-th/9712244 [hep-th].

[52] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori, and A. Strumia, "Higgs mass and vacuum stability in the standard model at nnlo", *JHEP*, vol. 08, p. 098, 2012. DOI: 10.1007/JHEP08(2012)098. arXiv: 1205.6497 [hep-ph].

[53] O. Antipin, N. A. Dondi, F. Sannino, A. E. Thomsen, and Z.-W. Wang, "Gauge-yukawa theories: beta functions at large n_f", *Phys. Rev.*, vol. D98, no. 1, p. 016 003, 2018. DOI: 10.1103/PhysRevD.98.016003. arXiv: 1803.09770 [hep-ph].

[54] J. L. Cardy, "Is there a c theorem in four-dimensions?", *Phys. Lett.*, vol. B215, pp. 749–752, 1988. DOI: 10.1016/0370-2693(88)90054-8.

[55] A. D. Bond, D. F. Litim, G. Medina Vazquez, and T. Steudtner, "Uv conformal window for asymptotic safety", *Phys. Rev.*, vol. D97, no. 3, p. 036 019, 2018. DOI: 10.1103/PhysRevD.97.036019. arXiv: 1710.07615 [hep-th].

[56] A. Codello, M. Safari, G. P. Vacca, and O. Zanusso, "Functional perturbative rg and cft data in the epsilon-expansion", *Eur. Phys. J.*, vol. C78, no. 1, p. 30, 2018. DOI: 10.1140/epjc/s10052-017-5505-2. arXiv: 1705.05558 [hep-th].

[57] D. M. Hofman and J. Maldacena, "Conformal collider physics: energy and charge correlations", *JHEP*, vol. 05, p. 012, 2008. DOI: 10.1088/1126-6708/2008/05/012. arXiv: 0803.1467 [hep-th].

[58] M. J. Duff, "Observations on conformal anomalies", *Nucl. Phys.*, vol. B125, pp. 334–348, 1977. DOI: 10.1016/0550-3213(77)90410-2.

[59] D. G. C. McKeon and C. Zhao, "Multiple couplings and renormalization scheme ambiguities", *Nucl. Phys.*, vol. B932, pp. 425–438, 2018. DOI: 10.1016/j.nuclphysb.2018.05.017. arXiv: 1711.04758 [hep-ph].

[60] M. E. Machacek and M. T. Vaughn, "Two loop renormalization group equations in a general quantum field theory. 1. wave function renormalization", *Nucl. Phys.*, vol. B222, pp. 83–103, 1983. DOI: 10.1016/0550-3213(83)90610-7.

[61] ——, "Two loop renormalization group equations in a general quantum field theory. 2. yukawa couplings", *Nucl. Phys.*, vol. B236, pp. 221–232, 1984. DOI: 10.1016/0550-3213(84)90533-9.

[62] ——, "Two loop renormalization group equations in a general quantum field theory. 3. scalar quartic couplings", *Nucl. Phys.*, vol. B249, pp. 70–92, 1985. DOI: 10.1016/0550-3213(85)90040-9.

[63] M.-x. Luo, H.-w. Wang, and Y. Xiao, "Two loop renormalization group equations in general gauge field theories", *Phys. Rev.*, vol. D67, p. 065 019, 2003. DOI: 10.1103/PhysRevD.67.065019. arXiv: hep-ph/0211440 [hep-ph].

[64] M. Cirelli, N. Fornengo, and A. Strumia, "Minimal dark matter", *Nucl. Phys.*, vol. B753, pp. 178–194, 2006. DOI: 10.1016/j.nuclphysb.2006.07.012. arXiv: hep-ph/0512090 [hep-ph].

[65] E. W. Kolb, D. J. H. Chung, and A. Riotto, "Wimpzillas!", *AIP Conf. Proc.*, vol. 484, no. 1, pp. 91–105, 1999, [,592(1999)]. DOI: 10.1063/1.59655. arXiv: hep-ph/9810361 [hep-ph].

[66] E. Molinaro, F. Sannino, and Z.-W. Wang, "Safe pati-salam", 2018. arXiv: 1807.03669 [hep-ph].

[67] F. Sannino and I. M. Shoemaker, "Asymptotically safe dark matter", *Phys. Rev.*, vol. D92, no. 4, p. 043 518, 2015. DOI: 10.1103/PhysRevD.92.043518. arXiv: 1412.8034 [hep-ph].

[68] S. P. Robinson and F. Wilczek, "Gravitational correction to running of gauge couplings", *Phys. Rev. Lett.*, vol. 96, p. 231 601, 2006. DOI: 10.1103/PhysRevLett.96.231601. arXiv: hep-th/0509050 [hep-th].

[69] A. R. Pietrykowski, "Gauge dependence of gravitational correction to running of gauge couplings", *Phys. Rev. Lett.*, vol. 98, p. 061 801, 2007. DOI: 10.1103/PhysRevLett.98.061801. arXiv: hep-th/0606208 [hep-th].

[70] D. J. Toms, "Quantum gravity and charge renormalization", *Phys. Rev.*, vol. D76, p. 045 015, 2007. DOI: 10.1103/PhysRevD.76.045015. arXiv: 0708.2990 [hep-th].

[71] D. Ebert, J. Plefka, and A. Rodigast, "Absence of gravitational contributions to the running yang-mills coupling", *Phys. Lett.*, vol. B660, pp. 579–582, 2008. DOI: 10.1016/j.physletb.2008.01.037. arXiv: 0710.1002 [hep-th].

[72] Y. Tang and Y.-L. Wu, "Gravitational contributions to the running of gauge couplings", *Commun. Theor. Phys.*, vol. 54, pp. 1040–1044, 2010. DOI: 10.1088/0253-6102/54/6/15. arXiv: 0807.0331 [hep-ph].

[73] D. J. Toms, "Quantum gravity, gauge coupling constants, and the cosmological constant", *Phys. Rev.*, vol. D80, p. 064 040, 2009. DOI: 10.1103/PhysRevD.80.064040. arXiv: 0908.3100 [hep-th].

[74] ——, "Quantum gravitational contributions to quantum electrodynamics", *Nature*, vol. 468, pp. 56–59, 2010. DOI: 10.1038/nature09506. arXiv: 1010.0793 [hep-th].

[75] ——, "Quadratic divergences and quantum gravitational contributions to gauge coupling constants", *Phys. Rev.*, vol. D84, p. 084 016, 2011. DOI: 10.1103/PhysRevD.84.084016.

[76] J.-E. Daum, U. Harst, and M. Reuter, "Running gauge coupling in asymptotically safe quantum gravity", *JHEP*, vol. 01, p. 084, 2010. DOI: 10.1007/JHEP01(2010)084. arXiv: 0910.4938 [hep-th].

[77] U. Harst and M. Reuter, "Qed coupled to qeg", *JHEP*, vol. 1105, p. 119, 2011. DOI: 10.1007/JHEP05(2011)119. arXiv: 1101.6007 [hep-th].

[78] S. Folkerts, D. F. Litim, and J. M. Pawlowski, "Asymptotic freedom of yang-mills theory with gravity", *Phys.Lett.*, vol. B709, pp. 234–241, 2012. DOI: 10.1016/j.physletb.2012.02.002. arXiv: 1101.5552 [hep-th].

[79] N. Christiansen and A. Eichhorn, "An asymptotically safe solution to the u(1) triviality problem", *Phys. Lett.*, vol. B770, pp. 154–160, 2017. DOI: 10.1016/j.physletb.2017.04.047. arXiv: 1702.07724 [hep-th].

[80] A. Eichhorn and F. Versteegen, "Upper bound on the abelian gauge coupling from asymptotic safety", 2017. arXiv: 1709.07252 [hep-th].

[81] A. Eichhorn, A. Held, and C. Wetterich, "Quantum-gravity predictions for the finestructure constant", *Phys. Lett.*, vol. B782, pp. 198–201, 2018. DOI: 10.1016/j.physletb.2018.05.016. arXiv: 1711.02949 [hep-th].

[82] J. F. Donoghue, "Leading quantum correction to the newtonian potential", *Phys. Rev. Lett.*, vol. 72, pp. 2996–2999, 1994. DOI: 10.1103/PhysRevLett.72.2996. arXiv: gr-qc/9310024 [gr-qc].

[83] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, "Quantum corrections to the schwarzschild and kerr metrics", *Phys. Rev.*, vol. D68, p. 084 005, 2003, [Erratum: Phys. Rev.D71,069904(2005)]. DOI: 10.1103/PhysRevD.68.084005,10.1103/PhysRevD.71.069904. arXiv: hep-th/0211071 [hep-th].

[84] N. E. J Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, "Quantum gravitational corrections to the nonrelativistic scattering potential of two masses", *Phys. Rev.*, vol. D67, p. 084 033, 2003, [Erratum: Phys. Rev.D71,069903(2005)]. DOI: 10.1103/PhysRevD.71.069903,10.1103/PhysRevD.67.084033. arXiv: hep-th/0211072 [hep-th].

[85] N. E. J. Bjerrum-Bohr, J. F. Donoghue, B. R. Holstein, L. Planté, and P. Vanhove, "Bending of light in quantum gravity", *Phys. Rev. Lett.*, vol. 114, no. 6, p. 061 301, 2015. DOI: 10.1103/PhysRevLett.114.061301. arXiv: 1410.7590 [hep-th].

[86] A. Eichhorn and A. Held, "Top mass from asymptotic safety", 2017. arXiv: 1707.01107 [hep-th].

[87] M. Shaposhnikov and C. Wetterich, "Asymptotic safety of gravity and the higgs boson mass", *Phys. Lett.*, vol. B683, pp. 196–200, 2010. DOI: 10.1016/j.physletb.2009.12.022. arXiv: 0912.0208 [hep-th].

[88] J. Laiho and D. Coumbe, "Evidence for asymptotic safety from lattice quantum gravity", *Phys. Rev. Lett.*, vol. 107, p. 161 301, 2011. DOI: 10.1103/PhysRevLett.107.161301. arXiv: 1104.5505 [hep-lat].

[89] J. Laiho and D. Coumbe, "Asymptotic safety and lattice quantum gravity", *PoS*, vol. LATTICE2011, p. 005, 2011.

[90] J. Laiho, S. Bassler, D. Coumbe, D. Du, and J. T. Neelakanta, "Lattice quantum gravity and asymptotic safety", *Phys. Rev.*, vol. D96, no. 6, p. 064 015, 2017. DOI: 10.1103/PhysRevD.96.064015. arXiv: 1604.02745 [hep-th].

[91] J. Laiho, S. Bassler, D. Du, J. T. Neelakanta, and D. Coumbe, "Recent results in euclidean dynamical triangulations", *Acta Phys. Polon. Supp.*, vol. 10, pp. 317–320, 2017. DOI: 10.5506/APhysPolBSupp.10.317. arXiv: 1701.06829 [hep-th].

[92] J. Ambjorn, J. Jurkiewicz, and R. Loll, "Emergence of a 4-d world from causal quantum gravity", *Phys. Rev. Lett.*, vol. 93, p. 131 301, 2004. DOI: 10.1103/PhysRevLett.93.131301. arXiv: hep-th/0404156 [hep-th].

[93] ——, "Reconstructing the universe", *Phys. Rev.*, vol. D72, p. 064 014, 2005. DOI: 10.1103/PhysRevD.72.064014. arXiv: hep-th/0505154 [hep-th].

[94] J. Ambjorn, A. Gorlich, J. Jurkiewicz, and R. Loll, "Planckian birth of the quantum de sitter universe", *Phys. Rev. Lett.*, vol. 100, p. 091 304, 2008. DOI: 10.1103/PhysRevLett.100.091304. arXiv: 0712.2485 [hep-th].

[95] ——, "The nonperturbative quantum de sitter universe", *Phys. Rev.*, vol. D78, p. 063 544, 2008. DOI: 10.1103/PhysRevD.78.063544. arXiv: 0807.4481 [hep-th].

[96] J. Ambjorn, A. Goerlich, J. Jurkiewicz, and R. Loll, "Nonperturbative quantum gravity", *Phys. Rept.*, vol. 519, pp. 127–210, 2012. DOI: 10.1016/j.physrep.2012.03.007. arXiv: 1203.3591 [hep-th].

[97] C. Wetterich, "Exact evolution equation for the effective potential", *Phys.Lett.*, vol. B301, pp. 90–94, 1993. DOI: 10.1016/0370-2693(93)90726-X.

[98] M. Reuter, "Nonperturbative evolution equation for quantum gravity", *Phys. Rev.*, vol. D57, pp. 971–985, 1998. DOI: 10.1103/PhysRevD.57.971. arXiv: hep-th/9605030 [hep-th].

[99] A. Bonanno and M. Reuter, "Proper time flow equation for gravity", *JHEP*, vol. 02, p. 035, 2005. DOI: 10.1088/1126-6708/2005/02/035. arXiv: hep-th/0410191 [hep-th].

[100] D. F. Litim and D. Zappala, "Ising exponents from the functional renormalisation group", *Phys. Rev.*, vol. D83, p. 085 009, 2011. DOI: 10.1103/PhysRevD.83.085009. arXiv: 1009.1948 [hep-th].

[101] S. P. de Alwis, "Exact rg flow equations and quantum gravity", *JHEP*, vol. 03, p. 118, 2018. DOI: 10.1007/JHEP03(2018)118. arXiv: 1707.09298 [hep-th].

[102] R. Percacci, *An Introduction to Covariant Quantum Gravity and Asymptotic Safety*, ser. 100 Years of General Relativity. WSP, 2017, vol. 3, ISBN: 9789813207172, 9789813207196. DOI: 10.1142/10369.

[103] M. Reuter, "Effective average actions and nonperturbative evolution equations", 1996. arXiv: hep-th/9602012 [hep-th].

[104] M. Reuter and F. Saueressig, "Renormalization group flow of quantum gravity in the einstein-hilbert truncation", *Phys. Rev.*, vol. D65, p. 065 016, 2002. DOI: 10.1103/PhysRevD.65.065016. arXiv: hep-th/0110054 [hep-th].

[105] D. F. Litim, "Fixed points of quantum gravity", *Phys. Rev. Lett.*, vol. 92, p. 201 301, 2004. DOI: 10.1103/PhysRevLett.92.201301. arXiv: hep-th/0312114 [hep-th].

[106] A. Codello, R. Percacci, and C. Rahmede, "Investigating the ultraviolet properties of gravity with a wilsonian renormalization group equation", *Annals Phys.*, vol. 324, pp. 414–469, 2009. DOI: 10.1016/j.aop.2008.08.008. arXiv: 0805.2909 [hep-th].

[107] D. Benedetti, P. F. Machado, and F. Saueressig, "Asymptotic safety in higher-derivative gravity", *Mod. Phys. Lett.*, vol. A24, pp. 2233–2241, 2009. DOI: 10.1142/S0217732309031521. arXiv: 0901.2984 [hep-th].

[108] K. Falls, D. Litim, K. Nikolakopoulos, and C. Rahmede, "A bootstrap towards asymptotic safety", 2013. arXiv: 1301.4191 [hep-th].

[109] N. Christiansen, B. Knorr, J. Meibohm, J. M. Pawlowski, and M. Reichert, "Local quantum gravity", *Phys. Rev.*, vol. D92, p. 121 501, 2015. DOI: 10.1103/PhysRevD.92.121501. arXiv: 1506.07016 [hep-th].

[110] H. Gies, B. Knorr, and S. Lippoldt, "Generalized parametrization dependence in quantum gravity", *Phys. Rev.*, vol. D92, no. 8, p. 084 020, 2015. DOI: 10.1103/PhysRevD.92.084020. arXiv: 1507.08859 [hep-th].

[111] H. Gies, B. Knorr, S. Lippoldt, and F. Saueressig, "Gravitational two-loop counterterm is asymptotically safe", *Phys. Rev. Lett.*, vol. 116, no. 21, p. 211 302, 2016. DOI: 10.1103/PhysRevLett.116.211302. arXiv: 1601.01800 [hep-th].

[112] N. Ohta, R. Percacci, and G. P. Vacca, "Flow equation for f(r) gravity and some of its exact solutions", *Phys. Rev.*, vol. D92, no. 6, p. 061 501, 2015. DOI: 10.1103/PhysRevD.92.061501. arXiv: 1507.00968 [hep-th].

[113] T. Denz, J. M. Pawlowski, and M. Reichert, "Towards apparent convergence in asymptotically safe quantum gravity", 2016. arXiv: 1612.07315 [hep-th].

[114] W. B. Houthoff, A. Kurov, and F. Saueressig, "Impact of topology in foliated quantum einstein gravity", *Eur. Phys. J.*, vol. C77, p. 491, 2017. DOI: 10.1140/epjc/s10052-017-5046-8. arXiv: 1705.01848 [hep-th].

[115] B. Knorr and S. Lippoldt, "Correlation functions on a curved background", *Phys. Rev.*, vol. D96, no. 6, p. 065 020, 2017. DOI: 10.1103/PhysRevD.96.065020. arXiv: 1707.01397 [hep-th].

[116] P. Dona, A. Eichhorn, and R. Percacci, “Matter matters in asymptotically safe quantum gravity”, *Phys. Rev.*, vol. D89, no. 8, p. 084 035, 2014. DOI: 10.1103/PhysRevD.89.084035. arXiv: 1311.2898 [hep-th].

[117] J. Meibohm, J. M. Pawlowski, and M. Reichert, “Asymptotic safety of gravity-matter systems”, *Phys. Rev.*, vol. D93, no. 8, p. 084 035, 2016. DOI: 10.1103/PhysRevD.93.084035. arXiv: 1510.07018 [hep-th].

[118] P. Donà, A. Eichhorn, P. Labus, and R. Percacci, “Asymptotic safety in an interacting system of gravity and scalar matter”, *Phys. Rev.*, vol. D93, no. 4, p. 044 049, 2016, [Erratum: Phys. Rev.D93,no.12,129904(2016)]. DOI: 10.1103/PhysRevD.93.129904, 10.1103/PhysRevD.93.044049. arXiv: 1512.01589 [gr-qc].

[119] J. Biemans, A. Platania, and F. Saueressig, “Renormalization group fixed points of foliated gravity-matter systems”, *JHEP*, vol. 05, p. 093, 2017. DOI: 10.1007/JHEP05(2017)093. arXiv: 1702.06539 [hep-th].

[120] G. Narain and R. Percacci, “Renormalization group flow in scalar-tensor theories. i”, *Class.Quant.Grav.*, vol. 27, p. 075 001, 2010. DOI: 10.1088/0264-9381/27/7/075001. arXiv: 0911.0386 [hep-th].

[121] G. Narain and C. Rahmede, “Renormalization group flow in scalar-tensor theories. ii”, *Class. Quant. Grav.*, vol. 27, p. 075 002, 2010. DOI: 10.1088/0264-9381/27/7/075002. arXiv: 0911.0394 [hep-th].

[122] A. Eichhorn and S. Lippoldt, “Quantum gravity and standard-model-like fermions”, *Phys. Lett.*, vol. B767, pp. 142–146, 2017. DOI: 10.1016/j.physletb.2017.01.064. arXiv: 1611.05878 [gr-qc].

[123] N. Christiansen, D. F. Litim, J. M. Pawlowski, and M. Reichert, “Asymptotic safety of gravity with matter”, *Phys. Rev.*, vol. D97, no. 10, p. 106 012, 2018. DOI: 10.1103/PhysRevD.97.106012. arXiv: 1710.04669 [hep-th].

[124] A. Eichhorn, “Status of the asymptotic safety paradigm for quantum gravity and matter”, in *Black Holes, Gravitational Waves and Spacetime Singularities Rome, Italy, May 9-12, 2017*, 2017. arXiv: 1709.03696 [gr-qc]. [Online]. Available: http://inspirehep.net/record/1623009/files/arXiv:1709.03696.pdf.

[125] A. Eichhorn and A. Held, “Mass difference for charged quarks from quantum gravity”, 2018. arXiv: 1803.04027 [hep-th].

[126] E. Manrique and M. Reuter, “Bimetric truncations for quantum einstein gravity and asymptotic safety”, *Annals Phys.*, vol. 325, pp. 785–815, 2010. DOI: 10.1016/j.aop.2009.11.009. arXiv: 0907.2617 [gr-qc].

[127] E. Manrique, M. Reuter, and F. Saueressig, “Bimetric renormalization group flows in quantum einstein gravity”, *Annals Phys.*, vol. 326, pp. 463–485, 2011. DOI: 10.1016/j.aop.2010.11.006. arXiv: 1006.0099 [hep-th].

[128] J. A. Dietz and T. R. Morris, “Background independent exact renormalization group for conformally reduced gravity”, *JHEP*, vol. 04, p. 118, 2015. DOI: 10.1007/JHEP04(2015)118. arXiv: 1502.07396 [hep-th].

[129] P. Labus, T. R. Morris, and Z. H. Slade, “Background independence in a background dependent renormalization group”, *Phys. Rev.*, vol. D94, no. 2, p. 024 007, 2016. DOI: 10.1103/PhysRevD.94.024007. arXiv: 1603.04772 [hep-th].

[130] R. Percacci and G. P. Vacca, “The background scale ward identity in quantum gravity”, *Eur. Phys. J.*, vol. C77, no. 1, p. 52, 2017. DOI: 10.1140/epjc/s10052-017-4619-x. arXiv: 1611.07005 [hep-th].

[131] N. Ohta, “Background scale independence in quantum gravity”, *PTEP*, vol. 2017, no. 3, 033E02, 2017. DOI: 10.1093/ptep/ptx020. arXiv: 1701.01506 [hep-th].

[132] A. Codello and R. Percacci, “Fixed points of higher derivative gravity”, *Phys. Rev. Lett.*, vol. 97, p. 221 301, 2006. DOI: 10.1103/PhysRevLett.97.221301. arXiv: hep-th/0607128 [hep-th].

[133] A. Codello, R. Percacci, and C. Rahmede, “Ultraviolet properties of f(r)-gravity”, *Int.J.Mod.Phys.*, vol. A23, pp. 143–150, 2008. DOI: 10.1142/S0217751X08038135. arXiv: 0705.1769 [hep-th].

[134] D. Benedetti and F. Caravelli, “The local potential approximation in quantum gravity”, *JHEP*, vol. 1206, p. 017, 2012. DOI: 10.1007/JHEP06(2012)017, 10.1007/JHEP10(2012)157. arXiv: 1204.3541 [hep-th].

[135] J. A. Dietz and T. R. Morris, “Asymptotic safety in the f(r) approximation”, *JHEP*, vol. 1301, p. 108, 2013. DOI: 10.1007/JHEP01(2013)108. arXiv: 1211.0955 [hep-th].

[136] N. Ohta, R. Percacci, and G. P. Vacca, “Renormalization group equation and scaling solutions for f(r) gravity in exponential parametrization”, 2015. arXiv: 1511.09393 [hep-th].

[137] K. Falls, D. F. Litim, K. Nikolakopoulos, and C. Rahmede, “Further evidence for asymptotic safety of quantum gravity”, *Phys. Rev.*, vol. D93, no. 10, p. 104 022, 2016. DOI: 10.1103/PhysRevD.93.104022. arXiv: 1410.4815 [hep-th].

[138] A. Eichhorn and A. Held, “Viability of quantum-gravity induced ultraviolet completions for matter”, 2017. arXiv: 1705.02342 [gr-qc].

[139] A. Eichhorn and H. Gies, “Light fermions in quantum gravity”, *New J.Phys.*, vol. 13, p. 125 012, 2011. DOI: 10.1088/1367-2630/13/12/125012. arXiv: 1104.5366 [hep-th].

[140] J. Meibohm and J. M. Pawlowski, “Chiral fermions in asymptotically safe quantum gravity”, *Eur. Phys. J.*, vol. C76, no. 5, p. 285, 2016. DOI: 10.1140/epjc/s10052-016-4132-7. arXiv: 1601.04597 [hep-th].

[141] A. Eichhorn, “Quantum-gravity-induced matter self-interactions in the asymptotic-safety scenario”, *Phys. Rev.*, vol. D86, p. 105 021, 2012. DOI: 10.1103/PhysRevD.86.105021. arXiv: 1204.0965 [gr-qc].

[142] ——, “Faddeev-popov ghosts in quantum gravity beyond perturbation theory”, *Phys. Rev.*, vol. D87, no. 12, p. 124 016, 2013. DOI: 10.1103/PhysRevD.87.124016. arXiv: 1301.0632 [hep-th].

[143] A. Eichhorn, A. Held, and J. M. Pawlowski, “Quantum-gravity effects on a higgs-yukawa model”, *Phys. Rev.*, vol. D94, no. 10, p. 104 027, 2016. DOI: 10.1103/PhysRevD.94.104027. arXiv: 1604.02041 [hep-th].

[144] R. Percacci and G. P. Vacca, “Search of scaling solutions in scalar-tensor gravity”, *Eur. Phys. J.*, vol. C75, no. 5, p. 188, 2015. DOI: 10.1140/epjc/s10052-015-3410-0. arXiv: 1501.00888 [hep-th].

[145] P. Labus, R. Percacci, and G. P. Vacca, “Asymptotic safety in o(n) scalar models coupled to gravity”, *Phys. Lett.*, vol. B753, pp. 274–281, 2016. DOI: 10.1016/j.physletb.2015.12.022. arXiv: 1505.05393 [hep-th].

[146] Y. Hamada and M. Yamada, “Asymptotic safety of higher derivative quantum gravity non-minimally coupled with a matter system”, *JHEP*, vol. 08, p. 070, 2017. DOI: 10.1007/JHEP08(2017)070. arXiv: 1703.09033 [hep-th].

[147] J. Berges, N. Tetradis, and C. Wetterich, "Nonperturbative renormalization flow in quantum field theory and statistical physics", *Phys. Rept.*, vol. 363, pp. 223–386, 2002. DOI: 10.1016/S0370-1573(01)00098-9. arXiv: hep-ph/0005122 [hep-ph].

[148] J. M. Pawlowski, "Aspects of the functional renormalisation group", *Annals Phys.*, vol. 322, pp. 2831–2915, 2007. DOI: 10.1016/j.aop.2007.01.007. arXiv: hep-th/0512261 [hep-th].

[149] H. Gies, "Introduction to the functional rg and applications to gauge theories", *Lect. Notes Phys.*, vol. 852, pp. 287–348, 2012. DOI: 10.1007/978-3-642-27320-9_6. arXiv: hep-ph/0611146 [hep-ph].

[150] B. Delamotte, "An introduction to the nonperturbative renormalization group", *Lect. Notes Phys.*, vol. 852, pp. 49–132, 2012. DOI: 10.1007/978-3-642-27320-9_2. arXiv: cond-mat/0702365 [COND-MAT].

[151] J. Braun, "Fermion interactions and universal behavior in strongly interacting theories", *J.Phys.*, vol. G39, p. 033 001, 2012. DOI: 10.1088/0954-3899/39/3/033001. arXiv: 1108.4449 [hep-ph].

[152] D. F. Litim, "Optimized renormalization group flows", *Phys. Rev.*, vol. D64, p. 105 007, 2001. DOI: 10.1103/PhysRevD.64.105007. arXiv: hep-th/0103195 [hep-th].

[153] A. Bonanno and M. Reuter, "Modulated ground state of gravity theories with stabilized conformal factor", *Phys. Rev.*, vol. D87, no. 8, p. 084 019, 2013. DOI: 10.1103/PhysRevD.87.084019. arXiv: 1302.2928 [hep-th].

[154] E. Manrique, M. Reuter, and F. Saueressig, "Matter induced bimetric actions for gravity", *Annals Phys.*, vol. 326, pp. 440–462, 2011. DOI: 10.1016/j.aop.2010.11.003. arXiv: 1003.5129 [hep-th].

[155] N. Christiansen, B. Knorr, J. M. Pawlowski, and A. Rodigast, "Global flows in quantum gravity", *Phys. Rev.*, vol. D93, no. 4, p. 044 036, 2016. DOI: 10.1103/PhysRevD.93.044036. arXiv: 1403.1232 [hep-th].

[156] D. Becker and M. Reuter, "En route to background independence: broken split-symmetry, and how to restore it with bi-metric average actions", *Annals Phys.*, vol. 350, pp. 225–301, 2014. DOI: 10.1016/j.aop.2014.07.023. arXiv: 1404.4537 [hep-th].

[157] T. R. Morris, "Large curvature and background scale independence in single-metric approximations to asymptotic safety", *JHEP*, vol. 11, p. 160, 2016. DOI: 10.1007/JHEP11(2016)160. arXiv: 1610.03081 [hep-th].

[158] D. Becker, C. Ripken, and F. Saueressig, "On avoiding ostrogradski instabilities within asymptotic safety", 2017. arXiv: 1709.09098 [hep-th].

[159] L. F. Abbott, M. T. Grisaru, and R. K. Schaefer, "The background field method and the s matrix", *Nucl. Phys.*, vol. B229, pp. 372–380, 1983. DOI: 10.1016/0550-3213(83)90337-1.

[160] C. Becchi and R. Collina, "Further comments on the background field method and gauge invariant effective actions", *Nucl. Phys.*, vol. B562, pp. 412–430, 1999. DOI: 10.1016/S0550-3213(99)00555-6. arXiv: hep-th/9907092 [hep-th].

[161] I. Donkin and J. M. Pawlowski, "The phase diagram of quantum gravity from diffeomorphism-invariant rg-flows", 2012. arXiv: 1203.4207 [hep-th].

[162] A. Codello, G. D'Odorico, and C. Pagani, "Consistent closure of renormalization group flow equations in quantum gravity", *Phys. Rev.*, vol. D89, no. 8, p. 081 701, 2014. DOI: 10.1103/PhysRevD.89.081701. arXiv: 1304.4777 [gr-qc].

[163] J. A. Dietz, T. R. Morris, and Z. H. Slade, "Fixed point structure of the conformal factor field in quantum gravity", *Phys. Rev.*, vol. D94, no. 12, p. 124 014, 2016. DOI: 10.1103/PhysRevD.94.124014. arXiv: 1605.07636 [hep-th].

[164] M. Safari and G. P. Vacca, "Covariant and background independent functional rg flow for the effective average action", *JHEP*, vol. 11, p. 139, 2016. DOI: 10.1007/JHEP11(2016)139. arXiv: 1607.07074 [hep-th].

[165] ——, "Covariant and single-field effective action with the background-field formalism", *Phys. Rev.*, vol. D96, no. 8, p. 085 001, 2017. DOI: 10.1103/PhysRevD.96.085001. arXiv: 1607.03053 [hep-th].

[166] C. Wetterich, "Gauge invariant flow equation", 2016. arXiv: 1607.02989 [hep-th].

[167] G. Narain and R. Percacci, "On the scheme dependence of gravitational beta functions", *Acta Phys. Polon.*, vol. B40, pp. 3439–3457, 2009. arXiv: 0910.5390 [hep-th].

[168] P. F. Machado and R. Percacci, "Conformally reduced quantum gravity revisited", *Phys. Rev.*, vol. D80, p. 024 020, 2009. DOI: 10.1103/PhysRevD.80.024020. arXiv: 0904.2510 [hep-th].

[169] R. Percacci, "Renormalization group flow of weyl invariant dilaton gravity", *New J. Phys.*, vol. 13, p. 125 013, 2011. DOI: 10.1088/1367-2630/13/12/125013. arXiv: 1110.6758 [hep-th].

[170] M. Reuter and H. Weyer, "Background independence and asymptotic safety in conformally reduced gravity", *Phys. Rev.*, vol. D79, p. 105 005, 2009. DOI: 10.1103/PhysRevD.79.105005. arXiv: 0801.3287 [hep-th].

[171] ——, "Conformal sector of quantum einstein gravity in the local potential approximation: non-gaussian fixed point and a phase of unbroken diffeomorphism invariance", *Phys. Rev.*, vol. D80, p. 025 001, 2009. DOI: 10.1103/PhysRevD.80.025001. arXiv: 0804.1475 [hep-th].

[172] F. Englert, C. Truffin, and R. Gastmans, "Conformal invariance in quantum gravity", *Nucl. Phys.*, vol. B117, pp. 407–432, 1976. DOI: 10.1016/0550-3213(76)90406-5.

[173] E. S. Fradkin and G. A. Vilkovisky, "Conformal off mass shell extension and elimination of conformal anomalies in quantum gravity", *Phys. Lett.*, vol. 73B, pp. 209–213, 1978. DOI: 10.1016/0370-2693(78)90838-9.

[174] R. Floreanini and R. Percacci, "Average effective potential for the conformal factor", *Nucl. Phys.*, vol. B436, pp. 141–162, 1995. DOI: 10.1016/0550-3213(95)00479-C. arXiv: hep-th/9305172 [hep-th].

[175] ——, "The renormalization group flow of the dilaton potential", *Phys. Rev.*, vol. D52, pp. 896–911, 1995. DOI: 10.1103/PhysRevD.52.896. arXiv: hep-th/9412181 [hep-th].

[176] M. E. Shaposhnikov and I. I. Tkachev, "Quantum scale invariance on the lattice", *Phys. Lett.*, vol. B675, pp. 403–406, 2009. DOI: 10.1016/j.physletb.2009.04.040. arXiv: 0811.1967 [hep-th].

[177] M. Shaposhnikov and D. Zenhausern, "Quantum scale invariance, cosmological constant and hierarchy problem", *Phys. Lett.*, vol. B671, pp. 162–166, 2009. DOI: 10.1016/j.physletb.2008.11.041. arXiv: 0809.3406 [hep-th].

[178] C. Pagani and R. Percacci, "Quantization and fixed points of non-integrable weyl theory", *Class. Quant. Grav.*, vol. 31, p. 115 005, 2014. DOI: 10.1088/0264-9381/31/11/115005. arXiv: 1312.7767 [hep-th].

[179] H. Kawai and M. Ninomiya, "Renormalization group and quantum gravity", *Nucl. Phys.*, vol. B336, pp. 115–145, 1990. DOI: 10.1016/0550-3213(90)90345-E.

[180] H. Kawai, Y. Kitazawa, and M. Ninomiya, "Ultraviolet stable fixed point and scaling relations in (2+epsilon)-dimensional quantum gravity", *Nucl. Phys.*, vol. B404, pp. 684–716, 1993. DOI: 10.1016/0550-3213(93)90594-F. arXiv: hep-th/9303123 [hep-th].

[181] T. Aida, Y. Kitazawa, J. Nishimura, and A. Tsuchiya, "Two loop renormalization in quantum gravity near two-dimensions", *Nucl. Phys.*, vol. B444, pp. 353–380, 1995. DOI: 10.1016/0550-3213(95)00071-Y. arXiv: hep-th/9501056 [hep-th].

[182] A. Eichhorn, "On unimodular quantum gravity", *Class.Quant.Grav.*, vol. 30, p. 115 016, 2013. DOI: 10.1088/0264-9381/30/11/115016. arXiv: 1301.0879 [gr-qc].

[183] R. Percacci and G. P. Vacca, "Asymptotic safety, emergence and minimal length", *Class. Quant. Grav.*, vol. 27, p. 245 026, 2010. DOI: 10.1088/0264-9381/27/24/245026. arXiv: 1008.3621 [hep-th].

[184] A. Nink, "Field parametrization dependence in asymptotically safe quantum gravity", *Phys. Rev.*, vol. D91, no. 4, p. 044 030, 2015. DOI: 10.1103/PhysRevD.91.044030. arXiv: 1410.7816 [hep-th].

[185] M. Demmel and A. Nink, "Connections and geodesics in the space of metrics", *Phys. Rev.*, vol. D92, no. 10, p. 104 013, 2015. DOI: 10.1103/PhysRevD.92.104013. arXiv: 1506.03809 [gr-qc].

[186] A. Nink and M. Reuter, "The unitary conformal field theory behind 2d asymptotic safety", 2015. arXiv: 1512.06805 [hep-th].

[187] A. Codello and G. D'Odorico, "Scaling and renormalization in two dimensional quantum gravity", *Phys. Rev.*, vol. D92, no. 2, p. 024 026, 2015. DOI: 10.1103/PhysRevD.92.024026. arXiv: 1412.6837 [gr-qc].

[188] N. Ohta, R. Percacci, and A. D. Pereira, "Gauges and functional measures in quantum gravity i: einstein theory", *JHEP*, vol. 06, p. 115, 2016. DOI: 10.1007/JHEP06(2016)115. arXiv: 1605.00454 [hep-th].

[189] ——, "Gauges and functional measures in quantum gravity ii: higher derivative gravity", *Eur. Phys. J.*, vol. C77, no. 9, p. 611, 2017. DOI: 10.1140/epjc/s10052-017-5176-z. arXiv: 1610.07991 [hep-th].

[190] A. Eichhorn, "The renormalization group flow of unimodular f(r) gravity", *JHEP*, vol. 1504, p. 096, 2015. DOI: 10.1007/JHEP04(2015)096. arXiv: 1501.05848 [gr-qc].

[191] G. 't Hooft, "Probing the small distance structure of canonical quantum gravity using the conformal group", 2010. arXiv: 1009.0669 [gr-qc].

[192] ——, "The conformal constraint in canonical quantum gravity", 2010. arXiv: 1011.0061 [gr-qc].

[193] ——, "A class of elementary particle models without any adjustable real parameters", *Found. Phys.*, vol. 41, pp. 1829–1856, 2011. DOI: 10.1007/s10701-011-9586-8. arXiv: 1104.4543 [gr-qc].

[194] A. Codello, G. D'Odorico, C. Pagani, and R. Percacci, "The renormalization group and weyl-invariance", *Class. Quant. Grav.*, vol. 30, p. 115 015, 2013. DOI: 10.1088/0264-9381/30/11/115015. arXiv: 1210.3284 [hep-th].

[195] H. Osborn, "Local renormalization group equations in quantum field theory", in *2nd JINR Conference on Renormalization Group Dubna, USSR, September 3-6, 1991*, 1991, pp. 128–138.

[196] R. Floreanini and R. Percacci, "Canonical algebra of gl(4) invariant gravity", *Class. Quant. Grav.*, vol. 7, p. 975, 1990. DOI: 10.1088/0264-9381/7/6/007.

[197] ——, "Mean field quantum gravity", *Phys. Rev.*, vol. D46, pp. 1566–1579, 1992. DOI: 10.1103/PhysRevD.46.1566.

[198] R. Percacci, "Gravity from a particle physicists' perspective", *PoS*, vol. ISFTG, p. 011, 2009. arXiv: 0910.5167 [hep-th].

[199] R. Floreanini and R. Percacci, "The heat kernel and the average effective potential", *Phys. Lett.*, vol. B356, pp. 205–210, 1995. DOI: 10.1016/0370-2693(95)00799-Q. arXiv: hep-th/9505172 [hep-th].

[200] K. Jepsen, *Winners declared in susy bet*, https://www.symmetrymagazine.org/article/winners-declared-in-susy-bet, Blog, 2016.

[201] G. Altarelli, "Theoretical implications of lhc results", *Nucl. Instrum. Meth.*, vol. A742, pp. 56–62, 2014. DOI: 10.1016/j.nima.2013.10.084.

[202] S. Dimopoulos and H. Georgi, "Softly broken supersymmetry and su(5)", *Nucl. Phys.*, vol. B193, pp. 150–162, 1981. DOI: 10.1016/0550-3213(81)90522-8.

[203] collaboration ATLAS, *Susy searches - public results*, https://twiki.cern.ch/twiki/bin/view/AtlasPublic/SupersymmetryPublicResults.

[204] collaboration CMS, *Cms susy physics results*, https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResultsSUS.