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TIME AND CLOCKS IN QUANTUM COSMOLOGY

Ph.D. Dissertation

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Abstract

In this thesis we shall apply the effective approach first to set up a generalised method for the effective semiclassical truncation of a general class of two–component dynamical systems (applicable to homogeneous cosmology) and provide a general construction for switching clocks in such systems. Next we shall put this method in use and evaluate relational dynamics of the closed Friedman–Robertson–Walker cosmological model filled with a minimally coupled massive scalar field in a semiclassical approximation. We have chosen this model as an interesting representative on account of three of its characteristics. Firstly, the model features a non–trivial coupling of the relational clock to the evolving degrees of freedom, secondly, a temporally global clock variable does not exist for such a system, and finally, this model universe is non–integrable, which is a typical feature for generic dynamical systems. The effective approach is especially well–designed for addressing the concept of relational evolution in this context as it makes a resolution of the evolution possible through transformations between different clocks and still manages to provide a (locally) sensible time evolution in transient observables as long as we do not leave the domain of semiclassical approximation. Evidence will be provided that relational evolution generically breaks down in the region of maximal expansion and a good relational evolution appears to be only a transient and semiclassical phenomenon. This is a direct consequence of a defocusing of classical trajectories which leads to rapid spreading of states that are initially sharply peaked and to a mixing of internal time directions in this region. These results are qualitatively compared to the previous work on this model. Attention is paid to conceptual issues raised earlier in the literature. A short qualitative overview of other FRW models is offered.

Abstrakt

Táto dizertačná práca sa zaoberá popisom neintegrabilných dynamických kvantových systémov v efektívnej formulácii relačnej dynamiky. Pre triedu dvojkomponentových dynamických systémov sme v semiklasickej aproximácii vytvorili zovšeobecnený postup redukcie a proces zámeny hodín. Postup sme konkrétne použili na vyhodnotenie relačnej dynamiky uzavretého Friedmanovho–Robertsonovho–Walkerovho (FRW) kozmologického modelu vyplneného hmotným skalárnym poľom s minimálnou väzbou. Tento model je z hľadiska napredovania relačnej dynamiky veľmi zaujímavý a dôležitý, pretože v ňom relačný čas má netriviálnu väzbu s vyvíjajúcimi sa stupňami voľnosti, neexistuje tu globálna časová veličina a v neposlednom rade je tento model ako typický zástupca všeobecných dynamických systémov neintegrabilný. Práve efektívna formulácia relačnej dynamiky sa ukazuje ako veľmi vhodný nástroj pre popis takýchto systémov, nielenže je vývoj systému zabezpečený mechanizmom zámeny hodín, navyše pokiaľ neopustíme doménu semiklasickej aproximácie je možné lokálne rozuzliť problematický časový vývoj v „prechodných“ veličinách. Ukážeme, že chaotickosť zvoleného modelu spôsobuje rozplynutie počiatočne si blízkych klasických trajektórií a teda relačný vývoj sa pre prípad všeobecne zvoleného systému „pokazí“ v intervale maximálnej expanzie. „Slušný“ časový vývoj, možný len pre dostatočne vhodne zvolené trajektórie, je len dočasným a semiklasickým fenoménom. Zhrnieme konceptuálne problémy vybraného modelu rozoberané v dostupnej literatúre a vlastné výsledky kvalitatívne porovnáme doterajšími. Text doplníme o krátku kvalitatívnu diskusiu k ďalším FRW modelom.

I would like to express gratitude to my advisor Franz Hinterleitner for granting me freedom in my research, his support and great patience and for revising the manuscript; to Martin Bojowald for introducing and guiding me to and within the effective framework, and also for his support both at the Institute for Gravitation and the Cosmos of the Pennsylvania State University during fall semester of 2010 and ever since; to Philipp Höhn and Artur Tsobanjan for the fruitful collaboration and their friendship as collateral damage.

Thanks to Marlene for carefully emending the text, herding all the stray articles and punctuation marks and catching up with all the run–ons:)

Ďakujem rodine a kamarátom za (morálnu) podporu, mamine, ocinovi, Katke a Ivke za pevné nervy, chápavo načúvajúce uši a povzbudivé slová, keď ich najviac bolo treba.

A Myšiakovi za všetko a viac.

rodičom

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'Were I to await perfection, my book would never be finished.'

- Tai T'ung, 'Principles of Chinese Writing'

Preface

Quantum Mechanics and General Relativity stand on strong foundation stones, both very well respected, proved correct by numerous experiments and continuously give testable predictions to high order of accuracy. Neither of the theories ever failed in its region of validity and thus we have complete trust in both. The quest of bringing the theories together is made even more complicated by this fact for neither of the theories presents as the one true theory, correct in itself, and thus suggesting it is the other one, that needs to be bent and shaped to fit their joint picture perfectly.

Both theories emerged as completely strange for their day, with brand new ideas and concepts to lead the way; General Relativity, set in the realm of differential geometry, equipped with new mathematical tools conceived just for the occasion of Einstein's new physics. Quantum Mechanics stands on the shoulders of Schrödinger's equation that came as almost a providential insight not having a tiny lead to where the idea came from but thoroughly supported by all the experimental data ever since. It would be very strange and foolish to expect two such profoundly different theories to agree with one another in any easy way.

Understandably, there are various efforts to provide the bridge between the quantum world of tiny and cosmologically huge via gravity quantisation, as we all believe there must be a way.

With reference to a starting point of our chosen quantum theory of gravity there are two main options to choose from. There is the primary approach to follow — one is to start with classical theory and apply heuristic quantisation rules, i.e. General Relativity $\rightarrow$ 'quantum General Relativity' (with subchoices of covariant quantisation — preservation of four–dimensional covariance at every step, or canonical quantisation — split of space–time into space and time at the classical level). There the advantage is our familiarity with the starting point (one does not arrive at the unified theory of all interactions though and this may be seen as a disadvantage by many of the–holy–grail–of–physics seekers). Opting for the secondary approach one is to start with fundamental quantum framework of all interactions and try to derive General Relativity in limiting situations, i.e. String Theory. The upside of such approach is that it automatically yields the 'Theory of Everything' (TOE), but on the downside, not to step on any toes, the starting point is entirely speculative.

Just as there are many different approaches that are tackling the problem, so the expectations of the theory of quantum gravity vary from one such approach to the next. However, any one of the theories has to satisfy at least these two requirements to become a valid contestant: i) the theory needs to be internally consistent and ii) the theory must reproduce the known physical theories when taken at the appropriate limit (i.e. the theory must yield General Relativity for the Planck constant approaching zero, alternatively making the gravitational constant vanish should give us Quantum Mechanics).

And what is the motivation for this great undertaking? The first reason would be the quest to find The Holy Grail of Physics, the Theory of Everything, the Unification Theory of all interactions. Apart from the wish for the physics of the universe to remain beautiful in its simplicity, the reductionist point of view has been amply supported historically as well; moreover, the Unification could hopefully cure divergence problems in Quantum Field Theory (Canonical Quantum Gravity and String Theory are two candidates for such a divergence–free theory). The second reason is to get to the understanding of cosmology and Black Holes, where singularities live and the breakdown of General Relativity happens, as at the moment there is still an acute lack of understanding of the early universe (i.e. initial conditions near the Big Bang/Bounce) and the final stages of Black Holes. Yet another reason would be to give the answer to the multifaceted problem of time.

This thesis deals with a much simpler problem than reconciling the jumpy Quantum Mechanics with smoothness of General Relativity. The techniques that we shall put in use in this work (stemming from the primary approach) are not a rigorous attempt at quantising gravity, they merely shed a first light beyond the classical theory, but the results are convincing enough for these efforts to be worth pursuing.

Quantum Gravity:
The troublesome double–act
Of Little and Large

—

from 'Cosmological Haiku' by Peter Coles

Chapter 1

Introduction

“How can a unitary evolution in a ‘classical’ time emerge from the full quantum theory?”: that is The question. The question impersonating one of the greatest puzzles in quantum gravity and cosmology, ‘the many–faceted problem of time’, has been extensively studied in works of [1–11].

The root of all evil in this case is the absence of the time coordinate in the quantum theory and the need to employ dynamical degrees of freedom as the internal clock variables [12–17]. Clock variables of these types (i.e. relational clock variables) are not at all of the likes of our well–known classical and perfect monotonic clocks measuring the proper time of some observer. They are rather true quantum degrees of freedom, occasionally running backwards even classically, suffering the fate of being a subject to quantum fluctuations; this phenomenon is referred to as the *global problem of time* [1–7]. Such imperfect clocks are in general coupled to other quantum degrees of freedom in the system; this coupling causes back–reaction and complicates the interpretation of the evolution of the rest of degrees of freedom in a chosen clock [18–20]. To be specific, resolution of unitary relational quantum evolution requires for the degrees of freedom and the clock to be separated, however this may prove to be impossible for the highly quantum states [6, 7, 11, 18–20]. Thus recovering a unitary evolution remains a difficult task even in the semiclassical regime.

One generally achieves the extraction of valid dynamical information from finite dimensional systems as in (loop) quantum cosmology by deparametrisations in specific matter degrees of freedom, such as dust or free scalar fields (or model specific geometrical degrees of freedom [21]), which assume the role of internal clocks. Great progress has been made in this arena [22–26]. However, the standard free scalar field [24–26], as well as the recently discussed dust fields

[27, 28] are rather special clocks, as they decouple from the other degrees of freedom, yield a 'time–independent' Hamiltonian and correspond to the 'ideal clock limit' of [18].

Extracting relational quantum dynamics from generic models in quantum cosmology is not an easy undertaking. Despite the fact that dynamical systems in general are non–integrable and this characteristics has severe implications for the relational evolution, this point has not been addressed in the literature on relational dynamics properly. Examination of an evolution of a system is impeded in particular by the fact, that the Hamiltonian constraint is the only global constant of motion [29–31]. Thus the question arises, whether it is even relevant to discuss relational evolution in such a case. Fortunately, as the relational observables can and do still exist implicitly and locally, the relational evolution is locally (in 'time') meaningful.

Analysis of generic situations in quantum cosmology features many technical difficulties, such as construction of a positive–definite inner product on the space of solutions to the quantum constraints known as the *Hilbert space problem* [1–5]. The effective techniques have been developed [22, 23, 32–34] not as a way of tackling the problem head–on, but to sidestep the issue altogether. These techniques allow us to extract qualitative and generic features from the systems that would otherwise be too intricate to be solved exactly. Based on the techniques, an *effective approach to the problem of time* has been introduced [6, 7] (see also [35] for a brief and [11] for comprehensive summary) which permits to evaluate the relational quantum dynamics of systems featuring the *global time problem* in the semiclassical regime. This effective approach allows us to depart from the idealised relational clock references and to study employment of more general degrees of freedom as imperfect clocks, including non–monotonic clocks as well as clocks coupled to evolving degrees of freedom. In this sense, (temporally) local time evolution with (temporally) local relational observables is made feasible and provides a way to singlehandedly avoid clock pathologies stemming from evolving the system in imperfect clocks by explicitly switching back and forth between different internal times.[1] In this manner the tools for covering the semiclassical evolution trajectories by patches of local relational times are provided.

In this work we will take a step out of deparametrisations with 'ideal clocks'. Instead, we shall explore a generic situation by considering (more realistically) coupled clock degrees of freedom in a non–integrable cosmological model. To be specific, even though observationally a flat universe seems to be favoured [36], we

[1] This effective approach was used to extract the relational evolution in two simple toy models with decoupled clocks in [7].

will investigate the closed FRW model filled with a minimally coupled massive scalar field in order to specifically address the issue of relational evolution. This model universe has been studied extensively in the literature [37–48], in particular, because it constitutes a simple cosmology which 'generically' produces inflation.[2] We will include a short overview of general FRW models as well.

In the main chapters of this work we shall provide explanation of some of the quantum troubles and — at least in the semiclassical regime — make some progress with regards to the relational evolution of this model universe by means of the effective approach. We will focus on the region of maximal expansion which features a chaotic scattering and thus proves to be especially challenging for relational dynamics.[3] In addition, conceptual issues raised earlier in the literature regarding the initial value problem and the semiclassical limit [8–10, 46, 48] will be addressed.

The primary result of this work is strong evidence that the quantum relational evolution in the investigated models, while possible for sufficiently semiclassical states, generically breaks down in the region of maximal expansion and the breakdown of semiclassicality is ultimately rooted in non–integrability. These results shed a first light on the breakdown of relational quantum evolution in generic models.

The rest of this thesis is organised as follows: Chapter 2 introduces the problem of time and relational dynamics and Chapter 3 provides effective formulation of constrained quantum systems. Chapter 4 details the effective semiclassical truncation of a general class of two–component dynamical systems (applicable to homogeneous cosmology) and provides the general construction for switching clocks in such systems. Chapter 5 is dedicated to a thorough examination the closed FRW model universe minimally coupled to a massive scalar field both classically and within the effective framework and ends with a short discussion on general FRW models with the additional feature of a cosmological constant. We will conclude the work in Chapter 6 with a discussion and an outlook. Also, this thesis suffers from Appendicitis as some details of the analysis in chapters 4 and 5 have been removed from the main body. Finally, a recipe for an effective semiclassical relational evolution of a quantum system is included in the Cookbook.

[2] The classical dynamics of this model has been studied in a great depth [39, 41, 42, 44], however its complete and consistent quantisation is still pending in any approach to quantum cosmology, obstacles to the complete quantisation stemming from the non–integrability of the system and an absence of a global internal clock.

[3] Report on singularity avoidance in this model within the framework of semiclassical gravity has been previously published by [49], resolution of the classically singular region through a quantum bounce in effective loop quantum cosmology was studied in [47].

Z: "Tak nech tam aj dačo vyskúmaš… Snáď vyriešiš aspoň svetový mier!"

M: "Idem na to! Jediný problém je, že používam komplexný čas, takže riešenie môže vyjsť čisto imaginárne…"

Chapter 2

The problem of time and relational dynamics

Time itself can not be observed. This is an inescapable fact. One can observe and measure only change and motion of objects. Heraclitus' Panta Rhei, absolute Newtonian time, Mach's and Einstein's (also) temporal relativity, quantum–mechanically preferred clocks, gravity's proper time in a chosen spacetime, many–fingered time... — the understanding of the concept of time has been evolving with humanity. In the following chapter we will discuss the way time is featured in Quantum Mechanics and General Relativity and show how the problem of time arises when one tries to unite the two in the elusive full theory of quantum gravity. A conceptual solution in the form of relational dynamics will be offered.

2.1 What time is it?

2.1.1 Time in Quantum Mechanics

In the realm of Quantum Mechanics, time is an external parameter and all events unfold on the ready–made canvas that is our universe, with this preferred time (direction)[1] as an absolute element. This means that the scalar product is conserved in time (unitarity), which in turn expresses conservation of total probability that is precisely the reason why the absolute nature of time is crucial for interpretation of Quantum Mechanics. Introduction of time as a quantum operator is problematic, mainly because of the explicit presence of t in the

[1] [50] discusses the difference between the problem and the arrow of time

Schrödinger equation

$$i\hbar\partial_t\psi(q,t) = \hat{H}(\hat{q},\hat{p})\psi(q,t), \tag{2.1.1}$$

where it is a background parameter. In fact it is Newton's absolute time, and we have it at our disposal before we solve any equations of motion.

The difficulty with promoting time to the space of operators was explained by Pauli in [51]. The idea is simple: if time $\hat{t}$ is a self–adjoint operator with a continuous spectrum and we consider the Hamiltonian operator $\hat{H}$ as a time translator, these two should satisfy a general commutator relation from which it then follows that $\hat{H}$ is also an unbounded and continuous operator. However, as this has not been observed, we shall abandon our attempts at establishing the concept of the time operator altogether.

The distinguished time t is of key importance in Quantum Mechanics, see e.g. Chapter 3.2 of [52]: (i) the observables are measured for a fixed t, (ii) the quantum state of a system, characterised by a set of observables is defined in a fixed t for which the observables are evaluated, and (iii) the scalar product is evaluated for a fixed t.

This concept of time is smoothly reconciled in Special Relativity with appointing a t parameter for every inertial observer; General Relativity, however, has a very different view on the matter.

2.1.2 Time in General Relativity

Meanwhile, time in General Relativity is coupled to space in a concept of space–time, a dynamical object that influences material clocks (i.e. their way of showing proper time) and these clocks react on metric and change geometry (metric itself is a clock [46]). A quantisation of the metric can thus be seen as a quantisation of the concept of time [53]. GR is a fully constrained covariant theory governed by a Hamiltonian constraint (classical constraints do not contain any time parameter) instead of a true Hamiltonian (with respect to a distinguished time variable) and the time evolution of a given system only makes sense once the Einstein equations for the system have been solved.

Physical time is given to us via proper time of the observer. As all observers have been created equal and a preferred proper time does not exist, all observers are equipped with their own proper time defined in an unambiguous manner as

$$\tau = \int \sqrt{-g_{\alpha\beta}u^{\alpha}(s)u^{\beta}(s)}\mathrm{d}s. \tag{2.1.2}$$

Another notion of time arises in canonical formulation by introducing the decomposition of space–time into foliations of spatial hypersurfaces, there however the 'time t' is not a distinguished time with a unique physical meaning — the t is a reparametrisation invariant parameter that can be chosen at liberty. General relativity thus in fact sees physics as purely relational.

2.1.3 Time in Quantum Gravity

The fact that in General Relativity there does not exist a background metric and we do not define any hidden space–time structure, while quantum mechanically all of physics is tied to the background Minkowski space–time metric is the reason for the concept of time in quantum gravity being very unclear.

Naturally, quantising efforts need to manage this issue in some way, and apart from searching for a novel concept of time and thus solving the puzzle once and for all, one of these three approaches is usually pursued [1, 5, 53, 54]: 'time *before* quantisation' or 'time *after* quantisation'; else one is left with '*timeless* options'.

Following the 'time *before* quantisation' approach one takes ADM variables, decomposes into embedding variables and true degrees of freedom, solves constraints on the classical level, inserts the solution into action, and derives the equations of motion from the Hamiltonian.

Some of the positive aspects of this approach include the fact that the time variable is isolated on the classical level, the resulting formalism is similar to ordinary Quantum Field Theory, and along with the distinguished time variable come both the natural Hilbert space and observables.

There are however many more obstacles: for one, the multiple choice problem manifests — swhich variables shall we choose for the embedding? Secondly, it is not possible to find a global time variable. The true Hamiltonian depends (non–trivially) on the time variable and this fact prohibits a general rigorous definition. Moreover the presence of quantum anomalies may spoil the consistency of the approach and finally 'the problem of construction' arises as the decoupling of the time variable from the other 'true' evolving degrees of freedom has been managed only for special cases. All in all it may seem that this approach does not solve the problem of time in the general case.

To employ the 'time *after* quantisation' procedure, one needs to use commutation rules to arrive at the wave functionals; dynamics is then implemented via quantisation of constraints, and as a result one obtains the Wheeler–DeWitt equation as the equation for the wave functional. However the factor ordering problem, the need for regularisation (and possibly renormalisation as well), the potential presence of anomalies, and the choice of a Hilbert space and inner product all present problems associated with the equation in question. Additionally, as Kiefer writes in [53], ‘the standard concept of time in quantum theory emerges only in semiclassical approximation — the Wheeler–deWitt equation itself is ‘timeless’.’

Timeless, ‘frozen’ dynamics is a result of an attempt to directly follow Dirac’s quantisation procedure and apply the Hamiltonian constraint on the physical states (that are to vanish) — this leads to $i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle = 0$. As physical quantities (=observables) are to be gauge invariant, they are required to commute with all the constraints in quantum theory (we shall not make any distinction between the Hamiltonian and other constraints). The apparent notion of ‘frozen dynamics’ where ‘nothing is moving’ constitutes the problem of time [11].

2.2 Relational dynamics

The dynamics can still be extracted from quantum systems with all due respect to the principles of General Relativity, and that is by employing the concept of the relational dynamics. Evolution of the system is hence not being measured against some distinguished time parameter, as one rather uses other internal degrees of freedom as a ’temporal’ reference. Such a clock does not need to coincide or be linked with any observer’s proper time and this evolution is then interpreted locally, with respect to the chosen local internal clock. Ultimately, in quantum gravity we end up studying coupled systems of dynamical internal clocks with gravitational degrees of freedom, see a reference to the concept of ’evolving constants of motion’ by Rovelli in [12–14]. These would be relational Dirac observables which measure the interaction between the chosen internal clock and other degrees of freedom of the system.

Once we make a choice of the clock variable, our notion of time and thus evolution of the system has been settled as these are inherently connected.

However, even looking at and working in this setting of the ‘relational time fix’, there are still many technical problems that persist: the Hilbert space problem (which Hilbert space and which inner product on such space to choose?), the multiple choice problem (which internal variable is to be our chosen clock?), the global time problem, problem of observables (how to construct a sufficient set of observables?), the operator–ordering problem, and the problem of non–integrability (a problem that is usually overlooked in the literature and shall be addressed in the thesis).

The relational evolution is made difficult to interpret as the physical clocks are neither universal nor perfect and moreover are rarely globally valid (this is the impersonification of the global problem of time). A perfect clock is a function whose increment is monotonic (such clock always runs forward) and coincides with an increment of some observer's proper time. The imperfect clocks are not a problem at the classical level, yet promoting the system to a quantum one is a bit more challenging. Classically, when our chosen clock starts to run slowly, or even backwards, we just make a different choice of a local time coordinate and patch the evolutions (in a time–wise orderly fashion) together. Quantum uncertainties however, make the patching in the full quantum theory highly problematic. Nevertheless, in the semiclassical regime a similar strategy to classical treatment is possible, and as this approach provides enough information about the system and thus offers a helpful insight, it may indeed be of use when the time comes to tackle the problem in all generality [7]. To be specific, what we need and search for in a good clock function is that it (i) is sufficiently fast and (ii) interacts with measured degrees of freedom on a sufficiently small scale to obtain a good resolution of the relational quantum evolution of our chosen system.

In the following text we shall employ a notion of internal clocks as a measure of relational time, moreover, we will work in the effective approach to the problem of time developed in [6, 7] that takes advantage of using an internal (local) time (or clock), when global time is not defined and switching to a new suitable local internal time when neccessary, just as one would switch between local coordinates on a manifold.

It is obvious that if one was able to work out the procedure of covering the manifold with maps in local internal times in the full theory, this could solve the problem of time in Quantum Gravity altogether.

Chapter 3

Effective framework

The biggest advantage that the use of the methods and techniques of the effective framework offers, lies in its representation independence. Despite the fact that the method employs an idea of circumventing the Hilbert space problem altogether, it still allows us to study a quantum constrained system and its dynamics in great detail. We shall sum up the neccessary techniques in the following chapter, for the proper introduction to the matter refer to founding papers [6, 7, 32–34], else see references [11, 22, 23] for reviews.

3.1 Effective equations

Effective equations of a quantum system describe the system by a study of the expectation values of the operators in a quantum state. When comparing these equations to classical equations of motion, one can immediately see what effect the quantum corrections have. In addition, if such equations can be set up and solved, the quantum nature of the system will manifest itself.

We will work in the semiclassical approximation to the problem and use the fact that there is a clear hierarchy to the magnitude of contributions of various quantum corrections. Also longer (i.e. cosmological) evolution times may lead to greater changes in quantum states, highlighting the quantum corrections and thus the quantum nature of the system even further.

The key idea of this algebraic approach is to extract representation independent information, with operator algebras of standard quantum mechanics serving as a motivation.

Let us work with an algebra $\mathcal{A}$ with elements $\hat{a}_i$ that satisfy the commutation relations

$$[\hat{a}_i, \hat{a}_j] = i\hbar A_{ijk}\hat{a}_k. \tag{3.1.1}$$

The expectation value that a state ϕ assigns to the operator $\hat{a}$ will be denoted $\langle\hat{a}\rangle_\phi$. The full information about the system is then hidden in the expectation values of the (physical) variables and in the countably infinite set of *quantum moments* [6, 7, 22, 23, 32–34]

$$\Delta(a_1^{m_1}...a_n^{m_n}) := \langle(\hat{a}_1 - \langle\hat{a}_1\rangle)^{m_1}...(\hat{a}_n - \langle\hat{a}_n\rangle)^{m_n}\rangle_{Weyl}. \tag{3.1.2}$$

The subscript 'Weyl' denotes totally symmetrised ordering of the product of operators inside the brackets. There are two big advantages to working directly with expectation values and moments: immediately treating physical variables of interest and having a clear hierarchy of order of magnitude of the quantum corrections (when evaluated in the semiclassical approximation) that are supplied via quantum moments.

The natural phase space structure is carried by the Poisson bracket defined by

$$\{\langle\hat{A}\rangle, \langle\hat{B}\rangle\} = \frac{\langle[\hat{A}, \hat{B}]\rangle}{i\hbar} \tag{3.1.3}$$

for any pair of operators $\hat{A}$ and $\hat{B}$ of the algebra $\mathcal{A}$, extended to the moments using the Leibniz rule and linearity.

In this thesis we will be inspecting systems generated by pairs of canonical variables $(\hat{q}_1, \hat{p}_1; \hat{q}_2, \hat{p}_2; \cdots \hat{q}_n, \hat{p}_n)$ where the commutation relations (3.1.1) take the form of

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}. \tag{3.1.4}$$

The quantum state is then described through the values it assigns to the *expectation values* $\langle\hat{q}_i\rangle$, $\langle\hat{p}_i\rangle$, $\langle\hat{q}_j\rangle$, $\langle\hat{p}_j\rangle, ...$ and the (countably) infinite set of *moments* [6, 7, 22, 23, 32–34]

$$\Delta(q_1^{a_1}p_1^{b_1}q_2^{a_2}p_2^{b_2}\cdots) := \langle(\hat{q}_1 - \langle\hat{q}_1\rangle)^{a_1}(\hat{p}_1 - \langle\hat{p}_1\rangle)^{b_1}(\hat{q}_2 - \langle\hat{q}_2\rangle)^{a_2}(\hat{p}_2 - \langle\hat{p}_2\rangle)^{b_2}\cdots\rangle_{\mathrm{Weyl}}, \tag{3.1.5}$$

defined for $\sum_i(a_i + b_i) \geq 2$, where the latter quantity will be referred to as the *order* of a given moment. Totally symmetrised ordering of the product of the operators inside the bracket needs to be consistently employed.

Relation (3.1.3) implies the classical Poisson bracket for the expectation values to be

$$\{\langle\hat{q}_i\rangle, \langle\hat{p}_i\rangle\} = \delta_{ij}. \tag{3.1.6}$$

Assuming a true Hamiltonian generating the evolution of the system in some definitive time t, we will derive the Hamiltonian flow generating the evolution of expectation values and quantum moments from the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle = \frac{\langle[\hat{A},\hat{H}]\rangle}{i\hbar} + \frac{\partial\langle\hat{A}\rangle}{\partial t} \tag{3.1.7}$$

and the Poisson structure (3.1.3) as

$$\frac{\mathrm{d}\langle\hat{A}\rangle}{\mathrm{d}t} = \left\{\langle\hat{A}\rangle, H_Q\right\} + \frac{\partial\langle\hat{A}\rangle}{\partial t}, \tag{3.1.8}$$

where $H_Q = \langle H(\hat{q}_i,\hat{p}_i)\rangle_{Weyl}$ [32, 33].

Taylor expansion gives us the Hamiltonian flow of the quantum Hamiltonian function as a function of expectation values and quantum moments:

$$\begin{aligned} H_Q(\langle\hat{q}_i\rangle,\langle\hat{p}_i\rangle,\Delta(\cdots)) &= \langle H(\langle\hat{q}_i\rangle + (\hat{q}_i - \langle\hat{q}_i\rangle)),(\langle\hat{p}_i\rangle + (\hat{p}_i - \langle\hat{p}_i\rangle))\rangle_{Weyl} \\ &= H(\langle\hat{q}_i\rangle,\langle\hat{p}_i\rangle) \\ &+ \sum_{a_1=0}^{\infty}\sum_{b_1=0}^{\infty}\sum_{a_2=0}^{\infty}\sum_{b_2=0}^{\infty}\cdots\frac{1}{a_1!\,b_1!\,a_2!\,b_2!\cdots}\frac{\partial^{a_1+b_1+a_2+b_2+\cdots}H(\langle\hat{q}_i\rangle,\langle\hat{p}_i\rangle)}{\partial^{a_1}\langle\hat{q}_1\rangle\partial^{a_1}\partial^{a_1}\langle\hat{p}_1\rangle\langle\hat{q}_2\rangle\partial^{a_1}\langle\hat{p}_2\rangle\cdots} \\ &\qquad .\Delta(q_1^{a_1}p_1^{b_1}q_2^{a_2}p_2^{b_2}\cdots). \end{aligned}$$

Equations of motion on the quantum phase space are then

$$\begin{aligned} \frac{\mathrm{d}\langle\hat{q}_i\rangle}{\mathrm{d}t} &= \{\langle\hat{q}_i\rangle, H_Q\}, \\ \frac{\mathrm{d}\langle\hat{p}_i\rangle}{\mathrm{d}t} &= \{\langle\hat{p}_i\rangle, H_Q\}, \\ \frac{\mathrm{d}\Delta(\cdots)}{\mathrm{d}t} &= \{\Delta(\cdots), H_Q\}. \end{aligned} \tag{3.1.9}$$

Thus we have ‘simplified’ our problem from solving a single partial differential equation into solving (in general and full theory) infinitely many (in general coupled) ordinary differential equations (3.1.9).

For the moments to represent a true quantum state an equivalent of the Schwarz inequalities must be satisfied, thus we impose

$$\langle(\hat{A}-\langle\hat{A}\rangle)^2\rangle\langle(\hat{B}-\langle\hat{B}\rangle)^2\rangle \geq \frac{1}{4}|\langle -i[\hat{A},\hat{B}]\rangle|^2 + \frac{1}{4}|\langle[(\hat{A}-\langle\hat{A}\rangle),(\langle(\hat{B}-\langle\hat{B}\rangle)]_+\rangle|^2, \tag{3.1.10}$$

where $\hat{A},\hat{B}$ are Hermitean operators representing any two observables and $[.,.]_+$ denotes an anticommutator. This, in fact, is a generalised uncertainty relation.

3.2 Effective constraints

In all of the further work we will consider quantum systems that are governed by one single constraint $\hat{C}$ that plays the role of the Hamiltonian constraint. When following Dirac's quantisation programme, one demands that physical states satisfy $\hat{C}|\psi\rangle = 0$, thus

$$C(\langle\hat{q}_i\rangle, \langle\hat{p}_j\rangle, \Delta(\cdots)) := \langle\hat{C}\rangle = \langle C(\hat{q}_i, \hat{p}_j)\rangle = 0. \qquad (3.2.1)$$

Imposing such a first class constraint will take care of one canonical pair of variables, however the (infinite tower of the) moments of the eliminated pair will remain unconstrained. In order to restrict all of the quantum modes of the constraint operator, one needs to impose a further set of constraints, where in fact for physical states the expectation values must vanish as

$$\langle f(\hat{q}_i, \hat{p}_i)\hat{C}^n\rangle = 0, \qquad n > 0$$

for any phase space function f.

It turns out [32, 33] that it is sufficient to impose these conditions in form of

$$C_{pol}(\langle\hat{q}_i\rangle, \langle\hat{p}_j\rangle, \Delta(\cdots)) := \langle(\widehat{pol} - \langle\widehat{pol}\rangle)\hat{C}\rangle = \langle\widehat{pol}\hat{C}\rangle = 0 \qquad (3.2.2)$$

for all polynomials $\widehat{pol}$ in the basic operators.

The set (C, C_{pol}) is thus a complete set of first class constraints. These are independent and there are infinitely many constraints for infinitely many quantum variables on the quantum phase space [32].

Notice that quantum constraints, unlike the quantum moments, are not defined via totaly symmetric ordering of the quantum operators, as under Weyl–ordering the constraints would neither form a closed set nor be first class [11, 32].

The non–symmetric ordering in (3.2.2) has two important consequences: quantum constraint functions induce quantum gauge transformations and some quantum constraints take complex values, for a comprehensive discussion see [11].

To summarise, at the moment we have to treat a quantum system of an infinite number of degrees of freedom that is governed by an infinite number of equations of motion and is subject to an infinite number of constraint equations. The first step, however, should not be to reduce the system via action of the 'C_H' constraint; we should rather take advantage of the fact that our system can be approximated by a finite number of degrees of freedom in the semiclassical approximation [32, 55–57].

3.3 Semiclassical approximation

Constructing the effective constraints on a quantum system yields an infinite system (infinitely many constraints for infinitely many variables). A (some) truncation of a system is needed in order to be able to extract any significant information about the system. It is very useful to employ a semiclassical approximation, as one takes advantage of the clear hierarchy that the quantum moments follow when evaluated in the semiclassical expansion for a class of Gaussian states in an ordinary Schrödinger representation of a quantum particle [58]. This hierarchy holds for more general states as well.

To any given order in $\hbar$, only finitely many constraints as well as quantum moments are contributory. Referring to the Poisson structure defined in (3.1.1), one can see that in a truncation to a certain order, the Poisson bracket of a moment of that order with all other moments can be consistently neglected, as the evolution of moments of higher order is slower, than that of lower orders. As regards the order of a moment, we assume that a moment of the order N is of (the same semiclassical) order (as) $\hbar^{N/2}$.

Note that the semiclassical expansion used as described will be an important consequence for counting the degrees of freedom: only expectation values of operators form a symplectic submanifold on the quantum phase space as quantum moments to any order feature a degenerate Poisson tensor[1] [33]. Thus in the effective approach one works with constrained systems on Poisson manifolds and not every independent first class constraint generates an independent flow.

We choose to truncate the system by both truncating the degrees of freedom and the system of constraints at some finite order in the semiclassical expansion. Once the system of constraints has been truncated, we shall refer to these constraints as the 'effective' ones. In the following work we shall focus on the leading quantum correction that corresponds to truncating the system at the order $\hbar$, which effectively means cutting out all the contributions of the moments from the order of $\hbar^{3/2}$ and higher.

[1]Consider the canonical pair $\hat{q}, \hat{p}$. The Poisson structure to the order of $\hbar$ is clearly degenerate as there is an odd number of second order moments associated to it [11].

Chapter 4

Effective relational dynamics

4.1 Leading order quantum corrections

The effective relational dynamics has been set up in quite some detail in [6, 7, 22, 23, 32], both general aspects and several toy models have been explored therein.

Although at first we treated the system and computed the model from first principles, see Appendix for details, in this chapter we shall provide the setup in the generalised form that we derived for the classical Hamiltonian constraints that are of the form

$$C_{class} = p_1^2 - p_2^2 - V(q_1, q_2)\,, \tag{4.1.1}$$

where $V(q_1, q_2)$ is polynomial, or at least has a convergent power series expansion in q_1 and q_2. This class of Hamiltonian constraints covers several homogeneous cosmological models and the treatment of such models is a natural step forward in the investigation of the validity of the effective approach as regards to more comprehensive systems.

The fact that none of the terms involve noncommuting variables allows us to take the corresponding constraint operator in the form of

$$\hat{C} = \hat{p}_1^2 - \hat{p}_2^2 - V(\hat{q}_1, \hat{q}_2). \tag{4.1.2}$$

We will systematically impose the constraint conditions (3.2.2) by demanding

$$\langle \hat{q}_1^a \hat{p}_1^b \hat{q}_2^c \hat{p}_2^d \hat{C} \rangle = 0\,, \tag{4.1.3}$$

for all non–negative integer values of a, b, c, d.

As has already been established in the theoretical introduction, we will be interested not in the full system, but only its truncation at the order of $\hbar$ — the semiclassical approximation. As it transpires, our quantum system is then described by four expectation values of the form $a = \langle \hat{a} \rangle \propto \hbar^0$, four spreads of the form $(\Delta a)^2 = \langle (\hat{a} - a)^2 \rangle \propto \hbar$ and six covariances of the form $\Delta(ab) = \langle (\hat{a} - a)(\hat{b} - b) \rangle_{\mathrm{Weyl}} \propto \hbar$. One needs to bear in mind that due to the symmetrisation process $\Delta(ab) = \Delta(ba)$. Also, a is used both to refer to the expectation value of the quantum operator $\hat{a}$ and the classical variable as well, it should be clear from the context which one is represented.

After the truncation, only five nontrivial constraint functions survive to govern the system. We will have a closer look at how to arrive at this set of constraints. First, we will derive the $\langle \hat{C} \rangle$ constraint. The trick is to perform the substitution $\hat{a} = a + (\hat{a} - a)$ and then Taylor–expand Eq.(4.1.3) around the expectation values,

$$\begin{aligned} C := \langle \hat{C} \rangle \;=\;& \langle \hat{p}_1 \rangle^2 - (\hat{p}_1 - \langle \hat{p}_1 \rangle)^2 - \langle \hat{p}_2 \rangle^2 + (\hat{p}_2 - \langle \hat{p}_2 \rangle)^2 + \langle V(\hat{q}_1, \hat{q}_2) \rangle \\ & - \frac{1}{2} \langle \ddot{V}(\hat{q}_1, \hat{q}_2) \rangle (\hat{p}_1 - \langle \hat{p}_1 \rangle)^2 - \frac{1}{2} \langle V''(\hat{q}_1, \hat{q}_2) \rangle (\hat{p}_2 - \langle \hat{p}_2 \rangle)^2 \\ & - \langle \dot{V}'(\hat{q}_1, \hat{q}_2) \rangle (\hat{p}_1 - \langle \hat{p}_1 \rangle)(\hat{p}_2 - \langle \hat{p}_2 \rangle), \end{aligned} \tag{4.1.4}$$

where the shorthand notation concerning the potential has been introduced as follows: the dots over the potential V indicate partial derivative with respect to the variable q_1 and primes then label differentiation with respect to the second variable, q_2. We can also omit the explicit reference to the arguments without loss of clarity and the notation becomes e.g.: $\dot{V} = \frac{\partial V}{\partial q_1}(q_1, q_2)$.

When calculating the second order constraints that are in fact the quantum flows generated by the quantum moments, the question of ordering arises. Let us examine the procedure of obtaining these constraints by means of calculating one of the constraints imposed, C_{q_1},

$$\begin{aligned} C_{q_1} &= \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle) \hat{C} \rangle \\ &= \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1^2 - \hat{p}_2^2 - V(\hat{q}_1, \hat{q}_2) \rangle. \end{aligned} \tag{4.1.5}$$

We will work explicitly with the first term as there the issue of ordering arises:

$$\begin{aligned} \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle) {\hat{p}_1}^2 \rangle &= \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\langle \hat{p}_1 \rangle + (\hat{p}_1 - \langle \hat{p}_1 \rangle))^2 \rangle \\ &= \langle \hat{q}_1 - \langle \hat{q}_1 \rangle \rangle \langle \hat{p}_1 \rangle^2 \\ &\quad + 2 \langle \hat{p}_1 \rangle \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle) \rangle + \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle)^2 \rangle, \end{aligned} \tag{4.1.6}$$

where the first term vanishes on account of $\langle \hat{q}_1 - \langle \hat{q}_1 \rangle \rangle = \langle \hat{q}_1 \rangle - \langle \hat{q}_1 \rangle = 0$.

In order to be able to write the resulting expression in terms of moments, the remaining terms need to be ordered symmetrically, which will be done using the canonical commutation relations. The term then becomes

$$\begin{aligned} \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle) \hat{p_1}^2 \rangle &= \langle \hat{p}_1 \rangle (2 \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle) \rangle_{Weyl} + i\hbar) \\ &\quad + \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle)^2 \rangle_{Weyl}, \end{aligned}$$

where

$$\begin{aligned} \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle)^2 \rangle_{Weyl} &= \frac{1}{3} \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle)^2 \\ &+ (\hat{p}_1 - \langle \hat{p}_1 \rangle)(\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle) \\ &+ (\hat{p}_1 - \langle \hat{p}_1 \rangle)^2 (\hat{q}_1 - \langle \hat{q}_1 \rangle) \rangle \end{aligned}$$

is a third order moment to be discarded in the truncation to the order $\hbar$, and

$$\begin{aligned} \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle) \rangle_{Weyl} &= \frac{1}{2} \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle) + (\hat{p}_1 - \langle \hat{p}_1 \rangle)(\hat{q}_1 - \langle \hat{q}_1 \rangle) \rangle \\ &= \langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle) \rangle + \frac{1}{2} i\hbar. \end{aligned}$$

Proceeding in this fashion with the rest of the terms of the constraint, one arrives at the full expression for the constraint C_{q_1}

$$\begin{aligned} C_{q_1} := \langle (\hat{q}_1 - q_1)\hat{C} \rangle = \; & 2p_1 \Delta(q_1 p_1) + i\hbar p_1 - 2p_2 \Delta(q_1 p_2) \\ & - \dot{V}(\Delta q_1)^2 - V' \Delta(q_1 q_2), \end{aligned} \tag{4.1.7}$$

where we have adopted the $\langle \hat{x} \rangle = x$, $\langle (\hat{q}_1 - \langle \hat{q}_1 \rangle)(\hat{p}_1 - \langle \hat{p}_1 \rangle) \rangle = \Delta(q_1 p_1)$ and the $(\hat{q}_1 - \langle \hat{q}_1 \rangle)^2 = (\Delta q_1)^2$ labeling.

The full set of non–zero constraint flows truncated to the order $\hbar$ expressed in symmetrically ordered moments reads as follows:

$$\begin{aligned} C := \quad \langle \hat{C} \rangle \quad &= p_1^2 - p_2^2 + (\Delta p_1)^2 - (\Delta p_2)^2 - V - \frac{1}{2} \ddot{V} (\Delta q_1)^2 \\ & - \frac{1}{2} V'' (\Delta q_2)^2 - \dot{V}' \Delta(q_1 q_2), \\ C_{q_1} := \langle (\hat{q}_1 - q_1)\hat{C} \rangle &= 2p_1 \Delta(q_1 p_1) + i\hbar\, p_1 - 2p_2 \Delta(q_1 p_2) - \dot{V}(\Delta q_1)^2 - V' \Delta(q_1 q_2), \\ C_{p_1} := \langle (\hat{p}_1 - p_1)\hat{C} \rangle &= 2p_1 (\Delta p_1)^2 - 2p_2 \Delta(p_1 p_2) - \dot{V}(\Delta(q_1 p_1) - \frac{i\hbar}{2}) - V' \Delta(p_1 q_2), \\ C_{q_2} := \langle (\hat{q}_2 - q_2)\hat{C} \rangle &= 2p_1 \Delta(p_1 q_2) - 2p_2 \Delta(q_2 p_2) - i\hbar\, p_2 - \dot{V} \Delta(q_1 q_2) - V' (\Delta q_2)^2, \\ C_{p_2} := \langle (\hat{p}_2 - p_2)\hat{C} \rangle &= 2p_1 \Delta(p_1 p_2) - 2p_2 (\Delta p_2)^2 - \dot{V} \Delta(q_1 p_2) - V' (\Delta(q_2 p_2) - \frac{i\hbar}{2}). \end{aligned} \tag{4.1.8}$$

4.2 The choice of a local clock: the *Zeitgeist*

We will be studying systems that possess only a non–global internal time variable, as these systems feature the global time problem. By the non–global internal time we mean a clock variable, whose hypersurfaces of equal time can be intersected more than once (or not at all) by a classical trajectory and the clock will thus pass through more extrema in its classical evolution. An example of such a system is a closed FRW model filled with a massive scalar field that will be studied in detail in Chapter 5. In this model, evolution with respect to local time variables is required in order to make any headway.

The system (4.1.8) of constraint functions is quite easy to solve, however the integration and interpretation of the Poisson flows these constraints generate is much more complicated. We shall take advantage of the general feature of these flows: as a consequence of the degeneracy in the Poisson tensor, a non–trivial combination of the constraints has a vanishing flow. Following the reasoning and methods first proposed in [32, 33] it is helpful to fix three out of the four independent gauge flows at order $\hbar$ and interpret the single remaining quantum flow as the dynamics of the system.

As has been argued before, there are not to be any distinguished clocks and thus one can treat all the clock choices equally. To advance, let us make an arbitrary choice of q_1 as the clock variable. We shall impose three 'q_1–gauge' conditions, so as to 'project the relational clock q_1 to a classical parameter' [6, 7]

$$\phi_1 := (\Delta q_1)^2 = 0, \quad \phi_2 := \Delta(q_1 q_2) = 0, \quad \phi_3 := \Delta(q_1 p_2) = 0. \tag{4.2.1}$$

After employing the gauge, the constraint C_{q_1} evaluates the covariance of the clock and its conjugate momentum as $\Delta(q_1 p_1) = -\frac{i\hbar}{2}$. One gauge flow remains, which preserves both the constraints and the above gauge conditions and is generated by the 'Hamiltonian' constraint[1]

$$C_H := C + \beta C_{p_1} + \gamma C_{p_2} + \epsilon C_{q_2}. \tag{4.2.2}$$

To find this gauge flow, one needs to evaluate the prefactors β, γ and ϵ. Let us make use of the simple requirement that each of the gauge conditions commutes with the Hamiltonian constraint, i.e. $\{C_H, \phi_i\} = 0$.

[1] see Appendix for details on how to obtain C_H

It is quite helpful to work out the general Poisson algebra betwen constraints and gauge conditions featuring derivatives of the potential:

	$\boldsymbol{\phi_1}$	$\boldsymbol{\phi_2}$	$\boldsymbol{\phi_3}$
$\mathbf{C}$	$2i\hbar$	$-2\Delta(q_2p_1)$	$-2\Delta(p_1p_2)$
$\mathbf{C_{p_1}}$	$4i\hbar p_1$	$-4p_1\Delta(q_2p_1)+2\Delta(q_2p_2)$ $+V'(\Delta q_2)^2-i\hbar p_2$	$-4p_1\Delta(p_1p_2)+2p_2(\Delta p_2)^2$ $-V'(\Delta(q_2p_2)+\frac{i\hbar}{2})$
$\mathbf{C_{q_2}}$	0	$-2p_1(\Delta q_2)^2$	$-2p_1\Delta(q_2p_2)-i\hbar p_1$
$\mathbf{C_{p_2}}$	0	$-2p_1\Delta(q_2p_2)+i\hbar p_1$	$-2p_1(\Delta p_2)^2$

Table 4.1: The Poisson algebra between constraints and gauge conditions

Using the results of the Poisson algebra of Table 4.1 one will arrive at the general expression for the (multiplication) prefactors in the Hamiltonian constraint

$$C_H := C - \frac{1}{2p_1}C_{p_1} - \frac{p_2}{2p_1^2}C_{p_2} - \frac{V'}{4p_1^2}C_{q_2}\,. \tag{4.2.3}$$

For consistency of this interpretation we require that the values of these variables satisfy *positivity* conditions

$$\begin{aligned}&q_2, p_2, (\Delta q_2)^2, (\Delta p_2)^2, \Delta(q_2p_2) \in \mathbb{R},\\ &(\Delta p_2)^2, (\Delta q_2)^2 \geq 0,\\ &(\Delta q_2)^2(\Delta p_2)^2 - (\Delta(q_2p_2))^2 \geq \frac{1}{4}\hbar^2,\end{aligned} \tag{4.2.4}$$

for the proof see Appendix B in [7].

In [6, 7] it has been shown that the expectation value of the clock picks up a specific imaginary contribution

$$\Im[q_1] = -\frac{\hbar}{2p_1} \tag{4.2.5}$$

in order for the constraint C_H of (4.2.3) to be consistently satisfied and for the evolving variables to remain real along the flow generated by it. This "*complex imaginary time*" feature is precisely the reason, why the relational evolution breaks down and one needs to resort to patching the regions evolved in local times (this issue will be discussed further in Section 4.3 and Chapter 5).

The gauge–fixing conditions, positivity conditions (4.2.4) and the interpretation of the remaining flow as the evolution of the system in the internal clock q_1 will be referred to as the "*Zeitgeist*" associated with q_1. The observables of the system measured in a selected clock are now temporally local and we shall refer to them as "*fashionables*". These aptly named variables are given as a correlation of moments and expectation values with the expectation value of the clock and can be interpreted as describing approximate unitary evolution in q_1. They are valid only as long as their corresponding *Zeitgeist* is and "fall out of fashion" with its breakdown.

Once we have obtained the Hamiltonian constraint, we are finally able to construct the equations of motion (3.1.9) and evolve the system. As the previously chosen temporality–resolving variable experiences the fate of the imperfect clock and its corresponding *Zeitgeist* is doomed to break down, one needs to alternatively choose q_2 as a clock function, working in an analogy with the sequence of steps starting with the gauge conditions (4.2.1) and arriving at a slightly altered expression for the Hamiltonian constraint C_H, as the subtraction sign in front of the momentum of the new clock changes the expression to be

$$C_H := C - \frac{1}{2p_2}C_{p_2} - \frac{p_1}{2p_2^2}C_{p_1} - \frac{\dot{V}}{4p_2^2}C_{q_1}, \tag{4.2.6}$$

with the derivative label switching from prime to dot.

The gauge–fixing conditions and the imaginary contribution to the clock and positivity conditions are all obtained by simple switching between the "1" and "2" labels.

Notice that relational evolution *in* a chosen clock is not only most conveniently interpreted in the corresponding *Zeitgeist*, but, furthermore, in every *Zeitgeist* we evolve a *different* set of relational observables, *fashionables*, as these go in and out of fashion every time we pick a fresh clock variable (see [6] and especially Sec. IV C in [7]).

4.3 Breakdown of a *Zeitgeist* and clock transformations

We have already indicated that the failure of the *Zeitgeist* is inevitable. For the class of Hamiltonian constraints that we are considering in this work, a globally valid clock along the gauge orbits does not exist.

The breakdown of the evolution generally happens, when in the q_1–gauge $\{q_1, C_{class}\} = 2p_1 = 0$, i.e. the evolution of the clock vanishes and its direction reverses. This is a direct consequence of the fact, that the momentum is not a constant of motion in this system. Looking from the effective point of view, the failure of the *Zeitgeist* happens when the *Zeitgeist* with its interpretation leave the domain of validity of the semiclassical approximation. As the momentum of the clock approaches zero, the imaginary contribution to the clock (4.2.5) becomes divergently large. In addition, the equations of motion become singular in p_1 and the moments diverge too, as the coefficients in (4.2.3) also diverge when one is approaching the turning point. We could interpret this situation intuitively by stating that the breakdown of the *Zeitgeist* happens as the clock's momentum becomes small (when compared with the scales of the system), i.e. the clock variable becomes too slow to resolve and will lead to large fluctuations in the evolution of the other (faster) evolving degrees of freedom. Such fluctuations are divergent in the interval where the clock approaches the concept of a maximally "imperfect" clock and ultimately "stops". This feature of the effective treatment is in fact an analogy to the evolution non–unitarity in q_1. Requiring the clock's spread to vanish in q_1–*Zeitgeist* leads to an inconsistency in the turning region — in general there is loss of normalisation, the clock can no longer be projected to a classical parameter and mixing of internal time directions occurs [6, 7]. This statement concurs with the analysis in [18–20] where it was shown that a good resolution of relational observables and evolution requires the clock to be essentially decoupled from the other evolving degrees of freedom and its momentum to be large. [2]

However, this does not indicate, that the situation is desperate and that we have lost the semiclassicality altogether. The semiclassicality only breaks with respect to the chosen set of gauge conditions, and should we evolve the system in some other, say q_2, *Zeitgeist*, the approximation would hold again, as long as we would not overstep the region of validity of $\{q_2, C_{class}\} = 2p_2 \neq 0$. We can

[2] However, large energies (or momenta) are a complex issue in gravitational physics due to black hole formation. Consequently, there is a limit on the clock's energy and thus on the accuracy of physical clocks [59, 60].

select q_2 to serve as a clock, fix the corresponding gauge, set up the equations of motion and evolve the system analogously with the procedure explained in the previous section for the q_1 clock. This new clock may (and probably will) turn out to be only of local nature as well, resulting in our gauge eventually failing again, however at this point we might find it safe to use the first (in this case q_1) gauge again. The crucial point in this idea is to find such a sequence of clock variables, where the semiclassical approximation holds for respective clocks before/after the clock change.[3]

It is precisely because of the transient nature of the non–global internal clocks that we refer to the gauge and its dynamical interpretation as the q_1–*Zeitgeist* (ghost of the time) and the transient relational observables as *fashionables*, as they are the true physical observables only while their corresponding *Zeitgeist* is valid and fall out of fashion with its breakdown.

In both [6, 7] it has been shown, that the clock–variable choice/change amounts to a mere gauge transformation and by implementing it as such, we have actually solved the multiple–choice aspect of the problem of time. In order to be able to consistently change between two clock variables one thus needs a method for transferring the relational information between the two gauge frameworks. The transformations that follow have been developed in a generalised notation to use for the systems governed by a single Hamiltonian constraint, featuring the potential of the form of (4.1.1), for the ease of computation when dealing with various models of this class. The details of the procedure of arriving at these transformations will be reviewed in the following section.

4.3.1 Gauge transformations between two different *Zeitgeister*

The gauge transformations between *Zeitgeister* are based on idea explained in [7] and have been computed with a substantial help from MATHEMATICA 8.

First, let us provide a quick outlook on the situation: the two component systems we are studying truncated to the order $\hbar$ are described by fourteen kinematical degrees of freedom. The truncated system of constraints comprises of five functionally independent conditions $C_i = 0$, which leaves us with a nine–dimensional constraint surface. The five constraint functions in general generate four independent vector fields X_{C_i}, which integrate to a four dimensional

[3] We shall see that for the class of cosmological models considered in this work one will in general fail to secure this condition in the region of maximal expansion and as a consequence the breakdown of relational evolution will occur.

gauge orbit. Once we have chosen our clock variable and imposed the three corresponding *Zeitgeist* conditions $\phi_i = 0$, these will take care of three of the four gauge conditions and we are left with one combination of the vector fields that preserves the gauge; this will be interpreted as the dynamics of the corresponding clock variable. As regards the geometrical interpretation of the situation, these one–dimensional orbits are formed by the intersection of the surface defined by the set of gauge conditions $\phi_i = 0$ with the integral orbits of the set of vector fields X_{C_i} on the constraint surface. It is obvious that a different set of gauge conditions, e.g. $\phi'_i = 0$ associated with a different clock will lead to a different evolution flow, as the intersections with gauge orbits will be different. What we need to do to be able to switch between different clocks in the system is to go from one surface $\phi_i = 0$ to another $\phi'_i = 0$ without leaving a given gauge orbit. We will try to do this in a most natural way that comes to mind, which is we will find such a combination of vector fields X_{C_i}, whose integral curves intersect both $\phi_i = 0$ and $\phi'_i = 0$.

We shall work with q_1–gauge conditions, however it will prove more convenient to use the set in the form of $\phi_1 := (\Delta q_1)^2 = 0, \phi_2 := \Delta(q_1 q_2) = 0$ and $\phi_3 := \Delta(q_1 p_1) = -i\hbar/2$ rather than the original set (4.2.1). Note, however, that both ϕ_3 conditions are equivalent (this follows from C_{p_1} once the other two gauge conditions are imposed on the set of constraints). The same exchange shall be performed for the q_2–gauge conditions. Looking for such gauge transformations we ask that these fulfill the following requirement: to transform from q_1–gauge to q_2–gauge we need to find a combination of the vector fields $G = \sum_i \xi_i X_{C_i}$, such that a (possibly finite) integral of its flow transforms the variables as

$$\begin{cases} (\Delta q_2)^2 = (\Delta q_2)^2_0 \\ \Delta(q_1 q_2) = 0 \\ \Delta(q_2 p_2) = \Delta(q_2 p_2)_0 \end{cases} \rightarrow \begin{cases} (\Delta q_2)^2 = 0 \\ \Delta(q_1 q_2) = 0 \\ \Delta(q_2 p_2) = -i\hbar/2, \end{cases} \tag{4.3.1}$$

where the subscript '0' labels the value of the corresponding variable prior to the gauge transformation.

This transformation is unique up to the two Hamiltonian constraints that it preserves, to fix this freedom, we opt for the numeric prefactor of the vector field X_C to be zero. After we are rid of C, three independent flows remain. The first flow that we search for must satisfy the condition $G_1(\Delta(q_2 p_2)) = G_1(\Delta(q_1 q_2)) = 0$ on the constraint surface, and rescale the flow so that $G_1((\Delta q_2)^2) = 1$. The condition for the second one will be of the form $G_2((\Delta q_2)^2) = G_2(\Delta(q_1 q_2)) = 0$ on the constraint surface, and rescale the flow so that $G_2(\Delta(q_2 p_2)) = 1$. The required condition then amounts to integrating the flow along

$$G = -(\Delta q_2)_0^2 G_1 - ((\Delta(q_2p_2))_0 + i\hbar/2)G_2.$$

The condition $\Delta(q_1q_2) = 0$ is shared by both gauge choices and is preserved by the G by construction. We will use the condition for simplification of the form of the gauge transformation fields G_1 and G_2. After engaging MATHEMATICA 8 one can see the explicit effect of G_1 and G_2 on the free variables of the q_2–gauge

$$\begin{aligned}
G_1(q_1) &= -\frac{p_1\dot{V} + 2p_2V'}{4p_1p_2^2}, & G_2(q_1) &= -\frac{1}{p_1}\,,\\
G_1(p_1) &= -\frac{p_1\ddot{V} + p_2\dot{V}'}{4p_2^2}, & G_2(p_1) &= 0\,,\\
G_1(q_2) &= \frac{V'}{4p_2^2}, & G_2(q_2) &= \frac{1}{p_2}\,,\\
G_1(p_2) &= -\frac{p_1\dot{V}' + p_2V''}{4p_2^2}, & G_2(p_2) &= 0\,,\\
G_1\left((\Delta q_1)^2\right) &= -\frac{p_1^2}{p_2^2}, & G_2\left((\Delta q_1)^2\right) &= 0\,,\\
G_1\left((\Delta p_1)^2\right) &= -\dot{V}\frac{p_1\dot{V} + 2p_2V'}{4p_1p_2^2}, & G_2\left((\Delta p_1)^2\right) &= -\frac{\dot{V}}{p_1}\,,\\
G_1\left(\Delta(q_1p_1)\right) &= -\frac{p_1\dot{V} + p_2V'}{2p^2}, & G_2\left(\Delta(q_1p_1)\right) &= -1\,.
\end{aligned}$$

The transformations between the two gauges that would respect the conditions summed up in (4.3.1) would in fact translate into following the integral curve of the vector field G, for the length of the interval of the flow parameter equal to unity. Let us denote the flow of G by α_G^s, where s is the flow parameter. If $\mathcal{C}$ is the constraint surface, then the scalar functions transform through $\alpha_G^s.f(x) = f(\alpha_G^s(x)), x \in \mathcal{C}$. The translated functions vary differentiably along the flow as

$$\frac{\mathrm{d}}{\mathrm{d}s}(\alpha_G^s.f)(x) = G(\alpha_G^s.f)(x). \tag{4.3.2}$$

Provided $f(x)$ is smooth along the field G, it is possible to construct the solution to the equation above as a derivative power series

$$\alpha_G^s.f(x) = \sum_{n=0}^{\infty}\frac{s^n}{n!}G^n(f)(x), \tag{4.3.3}$$

where $G^n(f)$ is the n^{th} derivative of f along the G.

In our case, the interest lies in transformations to the order of $\hbar$, moreover $s = 1$ and $G = aG_1 + bG_2$, where a, b are constants of the order of $\hbar$. For truncation at the order of $\hbar$ it thus follows that $G_1(f)$ and $G_2(f)$ are classical for all of the moments and expectation values. We can thus make an approximation for the transformation in the leading–order term in the form of

$$\alpha_G^1 . f(x) = f(x) + G(f)(x) + \mathcal{O}(\hbar^2). \tag{4.3.4}$$

The evolving variables in the q_2–*Zeitgeist* (appearing on the left hand side with the subscript "new") in terms of those in q_1 gauge (appearing on the right–hand side) are given by

$$\begin{aligned}
q_{1\text{ new}} &= q_1 + \frac{i\hbar}{2p_1} + \frac{p_1\dot{V} + 2p_2V'}{4p_1p_2^2}(\Delta q_2)^2 + \frac{1}{p_1}\Delta(q_2p_2), \\
q_{2\text{ new}} &= q_2 - \frac{i\hbar}{2p_2} - \frac{V'}{4p_2^2}(\Delta q_2)^2 - \frac{1}{p_2}\Delta(q_2p_2), \\
p_{1\text{ new}} &= p_1 + \frac{p_1\ddot{V} + p_2\dot{V}'}{4p_2^2}(\Delta q_2)^2, \\
p_{2\text{ new}} &= p_2 + \frac{p_1\dot{V}' + p_2V''}{4p_2^2}(\Delta q_2)^2, \\
(\Delta q_1)^2_{\text{new}} &= \frac{p_1^2}{p_2^2}(\Delta q_2)^2, \\
\Delta(q_1p_1)_{\text{new}} &= \Delta(q_2p_2) + \frac{p_1\dot{V} + p_2V'}{2p_2^2}(\Delta q_2)^2, \\
(\Delta p_1)^2_{\text{new}} &= \frac{1}{p_1^2p_2^2}\Bigg[p_2^4(\Delta p_2)^2 + p_2^2(p_1\dot{V} + p_2V')\Delta(q_2p_2) \\
&\qquad + \frac{1}{4}(p_1\dot{V} + p_2V')^2(\Delta q_2)^2\Bigg],
\end{aligned} \tag{4.3.5}$$

while the reverse transformation is given by:

$$
\begin{aligned}
q_{1\text{ new}} &= q_1 - \frac{i\hbar}{2p_1} + \frac{\dot{V}}{4p_1^2}(\Delta q_1)^2 - \frac{1}{p_1}\Delta(q_1p_1), \\
q_{2\text{ new}} &= q_2 + \frac{i\hbar}{2p_2} - \frac{2p_1\dot{V} + p_2V'}{4p_1^2p_2}(\Delta q_1)^2 + \frac{1}{p_2}\Delta(q_1p_1), \\
p_{1\text{ new}} &= p_1 - \frac{p_1\ddot{V} + p_2\dot{V}'}{4p_1^2}(\Delta q_1)^2, \\
p_{2\text{ new}} &= p_2 - \frac{p_1\dot{V}' + p_2V''}{4p_1^2}(\Delta q_1)^2, \\
(\Delta q_2)^2_{\text{new}} &= \frac{p_2^2}{p_1^2}(\Delta q_1)^2, \\
\Delta(q_2p_2)_{\text{new}} &= \Delta(q_1p_1) - \frac{p_1\dot{V} + p_2V'}{2p_1^2}(\Delta q_1)^2, \\
(\Delta p_2)^2_{\text{new}} &= \frac{1}{p_1^2p_2^2}\Bigg[p_1^4(\Delta p_1)^2 - p_1^2(p_1\dot{V} + p_2V')\Delta(q_1p_1) \\
&\qquad + \frac{1}{4}(p_1\dot{V} + p_2V')^2(\Delta q_1)^2\Bigg]. \qquad (4.3.6)
\end{aligned}
$$

Few points to bear in mind: as has been expected, the two transformations can be inverted into each other up to the order of $\hbar^{3/2}$, preserve the positivity conditions and are consistent with imaginary contribution to the expectation value of the clock.

Chapter 5

Closed Friedman–Robertson–Walker model minimally coupled to a massive scalar field

Let us now extend the application of the effective framework to quantum cosmology described in chapters 3 and 4. We shall review characteristics of the closed Friedman–Robertson–Walker model (see e.g. [61–63]) first classically, then discuss the problems of quantisation of this model and finally provide a detailed treatise of its effective dynamics.

5.1 Classical dynamics

The action of a homogeneous massive scalar field $\phi(t)$ minimally coupled to a (homogeneous and isotropic) closed Friedman–Robertson–Walker spacetime, of topology $\mathbb{R} \times \mathbb{S}^3$ and described by the metric

$$\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\Omega^2, \tag{5.1.1}$$

(where $\mathrm{d}\Omega^2$ is the line element on a unit $\mathbb{S}^3$), is given by

$$S[a,\phi] = \frac{1}{2}\int \mathrm{d}t\, N a^3 \left(-\left(\frac{1}{aN}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 + \frac{1}{a^2} + \left(\frac{1}{N}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 - m^2\phi^2 \right). \tag{5.1.2}$$

Variation of the action with respect to lapse N, field ϕ and scale factor a yields the Friedman equation, 'Klein–Gordon' equation and Raychaudhuri equation, respectively, (where $\dot{} = N^{-1}\frac{\mathrm{d}}{\mathrm{d}t}$)

$$\dot{a}^2 = -1 + a^2\left(\dot{\phi}^2 + m^2\phi^2\right), \tag{5.1.3}$$

$$\ddot{\phi} + \frac{3\dot{a}}{a}\dot{\phi} + m^2\phi = 0, \tag{5.1.4}$$

$$\ddot{a} = a\left(m^2\phi^2 - 2\dot{\phi}^2\right). \tag{5.1.5}$$

These equations of motion are clearly not all independent (e.g., differentiating (5.1.3) and combining it with (5.1.4) gives the Raychaudhuri equation (5.1.5)). Despite the apparent simplicity, this model possesses a rich solution space. We refer to [37, 39–44] as we do not intend to review all the details here, but only wish to summarise and pinpoint those classical aspects that are of consequence for our further discussion in the quantum theory.

This model has enjoyed a lot of press, mainly on account of the mass term that generically produces inflation.[1] Indeed, one can study various phases of cosmological evolution in this model, as the equation of state of the scalar field varies itself throughout evolution [41, 66]. Let us start with a small value of the scale factor a. The scalar field $\phi(t)$ gradually decreases with increasing a, generating an inflationary phase[2] and evolves to the equilibrium value $\phi \approx 0$ around which the field begins to oscillate with frequency m.[3] The model universe experiences a matter–dominated era in which $a \propto t^{2/3}$ [37, 39–41, 48]. Now the scale factor begins to oscillate between points of regular (non–global) maxima $a_{max,k}$ and (non–global) minima $a_{min,k}$ [39–42]. A generic solution will evolve to a point of maximal expansion—the turning point—a_{max} (with a possibility of few oscillations around this point) and eventually recollapse into a Big Crunch singularity [37, 39, 41]. Thus, clearly, both ϕ and a will generically fail to be globally valid internal clock functions in this model.[4]

Two typical classical solutions are displayed in Figure 5.1.

Relational evolution offers an even worse scenario: in [37–39, 41, 44] it has been shown that there exists a countably infinite discrete set of periodic solutions

[1] This is quite a fortunate setting, considering [64, 65].

[2] The larger the initial value ϕ_0 of the scalar field the longer the inflationary period [37].

[3] A solution that expands out to a length scale of the order of 10^{60} Planck lengths requires at least 10^{60} such oscillations of ϕ, see [38, 48].

[4] For small masses m, the scalar field $\phi(t)$ is still a monotonically increasing function of t as in the massless case and thus a good global clock (see the discussion in [42], esp. the region in configuration space called 'region 0').

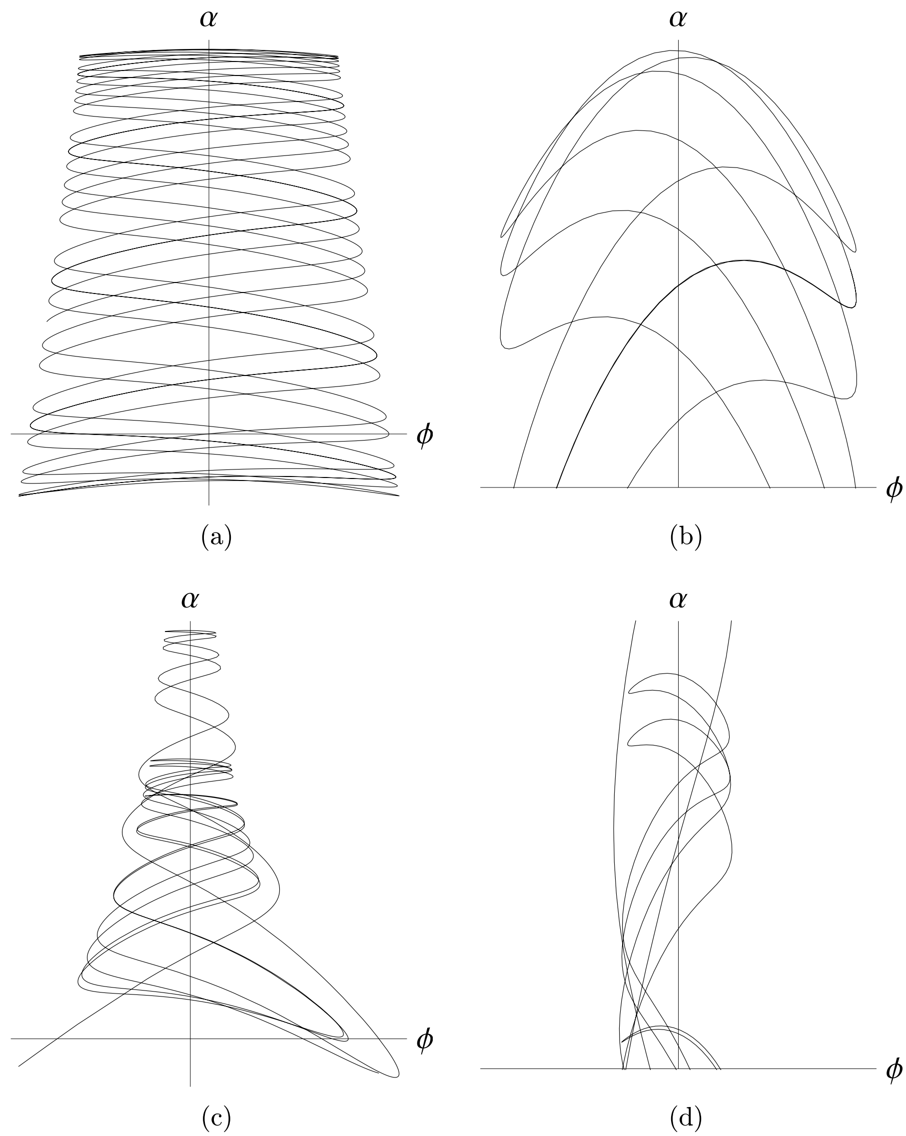


Figure 5.1: Two typical classical solutions to the closed FRW spacetime — both ϕ and a generically fail to be globally valid internal clock functions in this model. Here we used $\alpha = \ln(a)$ as appropriate for the canonical discussion following (5.1.6), (5.1.7). (a) and (c) show extended segments of (both the expanding and re–contracting branch of) relational evolution up to the point of maximal expansion $\alpha_{max} = \ln(a_{max})$. The (new) scale factor α oscillates between points of regular (non–global) maxima $\alpha_{max,k} = \ln(a_{max,k})$ and (non–global) minima $\alpha_{min,k} = \ln(a_{min,k})$; (b) shows the relational configuration space trajectory of (a), but zoomed into the region of α_{max}, displaying the non–global extrema in greater detail, while (d) depicts a close–up of an intermediate section of trajectory (c).

which bounce without ever encountering a spacetime singularity. Moreover, there exists an uncountably infinite discrete set of perpetually bouncing aperiodic solutions[5] which exhibits an interesting fractal–like behaviour, see [39, 42, 44]. It is thus the non–integrability and chaotic nature [39, 42, 44] of the system (5.1.3), (5.1.4) that is responsible for our problems in the quantum regime.

The Hamiltonian formulation for this model universe makes the comparison of the classical with effective features (in Section 5.3 below) possible. Moreover after its careful inspection one can also see wherefrom the absence of the global clock arises.

For the ease of the further calculations we shall perform a variable transformation $\alpha = \ln(a)$ and henceforth work with α. The reasoning behind this transformation would comprise of three arguments. Firstly, in the quantum theory, one completely avoids the factor ordering problem in the Hamiltonian constraint [5, 48]; secondly, the resulting quantum Hamiltonian constraint (5.2.6) is immediately of the form (5.1.6) and thus the effective constructions of the Section 4.1 are directly applicable; and thirdly, the domains for our variables are now $\alpha \in (-\infty, \infty)$ and $\phi \in (-\infty, \infty)$ and thus our working configuration space is $\mathcal{Q} = \mathbb{R}^2$, which is arguably simpler to quantise than $\mathcal{Q} = \mathbb{R} \times \mathbb{R}_+$ [5, 67], as for instance, $\hat{p}_a$ is not self–adjoint on $L^2(\mathbb{R}_+, da)$, or, when choosing $L^2(\mathbb{R}, da)$ instead, one would have to give meaning to $a < 0$. On the other hand, $\hat{p}_\alpha$ is self–adjoint on $L^2(\mathbb{R}, d\alpha)$ and $\alpha \in (-\infty, \infty)$. Once we have chosen to work with this transformation, we also have the advantage of the Big Bang and Big Crunch singularities appearing now at $\alpha \to -\infty$ which is sufficient for us since in the effective approach, we shall be focussing on the regime of maximal expansion of the scale factor a.[6]

Choosing a gauge $N = e^{3\alpha}$, it is straightforward to arrive at the Hamiltonian constraint corresponding to the system (5.1.2) as[7]

$$C_H = p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} = 0, \tag{5.1.6}$$

which is precisely of the form of (4.1.1). The term $m^2\phi^2 e^{6\alpha}$ acts as the coupling term between the relational clock, i.e. either α or ϕ, and the evolving configuration variable, i.e. either ϕ or α, respectively. In fact, the squared mass m^2 can be

---

[5] (of measure zero in the space of solutions [43, 44])

[6] Apparently, only a full quantisation can cope with the classically singular regime, however see [47]. Furthermore, when discussing the quantum dynamics in sections 5.2 and 5.3 below, note that small (big) fluctuations in α do not necessarily translate into small (big) fluctuations in a.

[7] see also [48, 68]

interpreted as the coupling constant, while the factor $e^{6\alpha}$ can in certain regimes be treated as an adiabatic factor [18, 48]. This coupling term will have a great effect on quantum relational evolution. Using the symplectic structure on $T^*\mathcal{Q}$, the corresponding canonical equations of motion read

$$\begin{aligned} \dot{\alpha} = &\ \{\alpha, C_H\} &= -2p_\alpha, \\ \dot{p}_\alpha = &\ \{p_\alpha, C_H\} &= \ 4e^{4\alpha} - 6m^2\phi^2 e^{6\alpha}, \\ \dot{\phi} = &\ \{\phi, C_H\} &= \ 2p_\phi, \\ \dot{p}_\phi = &\ \{p_\phi, C_H\} &= -2m^2\phi e^{6\alpha}, \end{aligned} \tag{5.1.7}$$

where now the overdot refers to differentiation with respect to the time coordinate t. As a consequence of our gauge choice of $N = e^{3\alpha}$, note that henceforth t does not coincide with the proper times τ of comoving observers in (5.1.1). Figure 5.2 depicts the behaviour of the canonical variables for a rather benign solution.

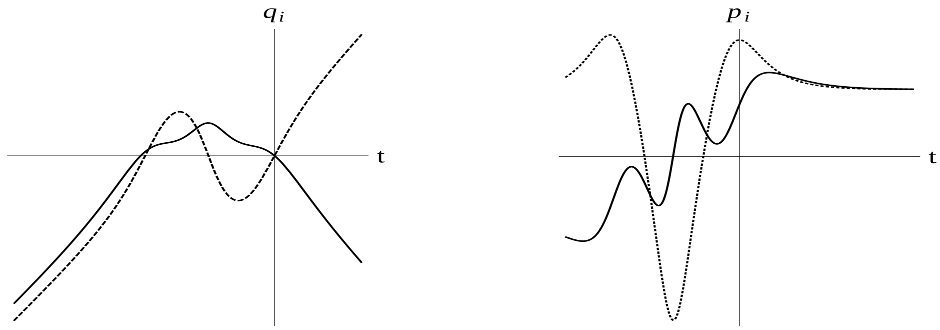
In the work of Hájíček, see [68], where the investigation of the finite dimensional parametrised systems in the quantum theory via relational evolution is performed, a direct link has been established between the unitarity of the system and the requirement of the existence of a (temporally) global internal clock already at the classical level. This, in turn, was shown as equivalent to the classical system to be reducible.

As an example, the system governed by (5.1.2, 5.1.6) was considered and it was shown that [68]: firstly, the constraint surface $\mathcal{C}$ defined by (5.1.6) in $T^*\mathcal{Q}$ is of topology $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$ and thus connected but not simply connected, and secondly, the flow of C_H on $\mathcal{C}$ does not have any critical points, but incontractible cycles (around $\mathbb{S}^1$).

These incontractible cycles correspond to the periodically bouncing solutions [37, 39, 42, 44] and as such prevent the system from being reducible and possessing a global clock.

5.1.1 Non–integrable systems

We shall now comment on a few aspects of our chosen non–integrable model universe with regard to the relational evolution. These comments are of the utmost importance for the further investigation of the model universe in question, as the standard literature on relational evolution does not provide any space for the discussion of non–integrability, in spite of the fact, that it is a typical feature in generic dynamical systems (that are in fact chaotic) [29–31] and that



(a) $\alpha(t)$ thick plot, $\phi(t)$ dashed plot

(b) $p_\alpha(t)$ thick plot, $p_\phi(t)$ dashed plot

Figure 5.2: Graphs of the canonical variables governed by (5.1.7) for a rather benign classical solution. Notice how α features quasi–turning points closely to the turning points of ϕ (also manifested in p_α having a local minimum close to the zeros of p_ϕ).

the aftermath of the non–integrability in relational dynamics can not be simply ignored. Thus we believe that the results of the present work are a first step towards a more general discussion of the fate of relational dynamics, specifically in the quantum theory.

A general system that does not possess the full set of constants of motion is known as a non–integrable system. To be specific, the only global constant of motion (i.e. Dirac observable) that our model possesses is the Hamiltonian itself [29–31]; this Hamiltonian constraint (5.1.6) in fact coincides with the first integral of motion defined by the Friedman equation (5.1.3). Even though globally valid Dirac observables are not available, one can still derive locally valid Dirac observables and make the relational evolution work at least locally (in time).[8] This is where we expect the effective approach to relational evolution [6, 7] to be of benefit since it enables one to make sense of local time evolution and (temporally) local relational observables (*fashionables* in the terminology of [6, 7]) in the semiclassical regime. Note that in contrast to integrable systems, even if locally a complete set of relational observables is derived, this set in general does not characterise the orbit anymore on account of the fact that chaotic systems typically possess ergodic orbits. These orbits come arbitrarily

[8]For instance, in Eq. (5.6) of [48] the relational observable $\phi(a)$ is given for the matter dominated phase of expansion where $a \propto \tau^{2/3}$ and τ is proper time.

close to any point on the energy surface (which for constrained systems is the constraint surface [29–31]).

The evolution is furthermore plagued by the instability of initial data, another generic feature of chaotic systems. The instability in chaotic systems manifests as some initial data based trajectory in the system, that approaches arbitrarily close to a closed (periodic or nonperiodic) orbit, will typically diverge from such orbit to the point of becoming entirely uncorrelated.[9] Detailed discussion on the closed orbits of the present model was presented in [39, 42, 44] with a very nice demonstration of the fractal structure in the space of initial data and the consequent divergence of solutions first arbitrarily close to each other in [44]. In fact, defocusing of nearby trajectories occurs in the present model also for trajectories not arbitrarily close to a closed orbit.

Let us examine e.g. Figure 5.3 that shows how neighbouring trajectories defocus in the region of maximal expansion already for a rather well–behaved classical solution. For generic solutions exhibiting more oscillations in both ϕ and α [42], this feature will get more pronounced. Such fanning out of the trajectories will be particularly relevant in the quantum theory, since it constitutes the ultimate cause of a generic breakdown of semiclassicality and relational evolution.

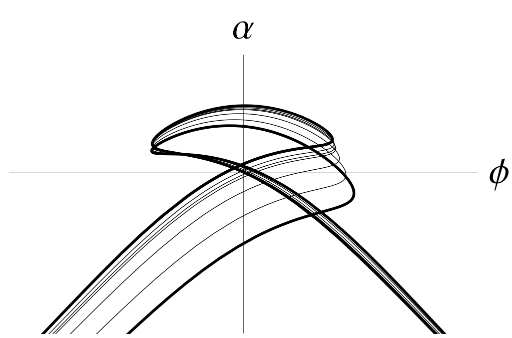


Figure 5.3: Defocusing of nearby trajectories, caustics develop along the extrema of ϕ (see also [40]).

In the classical theory we have an equal choice of using either α or ϕ as a global clock function despite the fact, that the turning points of the clock variables lead

[9]Clearly, such a statement depends strongly on the time coordinate which is potentially dangerous in general relativity, however, there exists a very general definition of chaotic behaviour which takes this into account and essentially requires a defocusing of trajectories (without any statement to the rate of the defocusing), as well as ergodicity [69].

to the relational observables to be multi–valued, as we can always remedy the situation by selecting the gauge parameter so as to provide the ordering to the correlations [7]. Still, one will find calculations less demanding should one choose to employ α as an internal clock for large parts of the evolution, as the scalar field exhibits as highly oscillatory at large volumes and thus proves more difficult to work with. However in quantum theory one will not be able to work with either of the variables globally on account of non–unitarity of the chosen model.

In the classical case, one using C_H imposes suitable initial data at some fixed time $t = t_0$, with the relational initial value problem (IVP) solved in the following manner: if α was used as a clock, then one could choose $\phi(\alpha_0)$ and $p_\phi(\alpha_0)$ at some value $\alpha_0 = \alpha(t_0)$, which corresponds to a point in the phase space. Should one choose, for instance, the negative sign solution for the initial clock momentum, $p_\alpha(t_0)$ (via the constraint (5.1.6)), this could be some configuration on the expanding branch of cosmic evolution. Indeed, in relativistic systems subject to constraints quadratic in the momenta, a relational IVP additionally requires an initial internal time direction in order to relationally evolve [7, 70]. We continue with evolution of the system by evolving the data through maximal extension into the big crunch singularity, with the expanding branch being the logical predecessor of the contracting branch. In contrast to earlier work [8–10, 46, 48] on the quantum theory of (5.1.2), we shall perform the same IVP construction for sufficiently semiclassical states in the effective framework in Section 5.3 below.

5.2 Troubles for Hilbert space quantisations

As has been hinted previously, quantum dynamics of the present model will not be easy to extract. In general, moving from the classical to the quantum version of the problem is a highly non–trivial undertaking whatever our setting is, however, when we consider chaotic models, the challenge moves to a completely different level [29–31]. There has been great attention paid to better our understanding of at least the semiclassical solutions to the present model in various approaches [37, 38, 40, 46–49], however real progress (even research, for that matter) within relational dynamics has not been made yet, mainly because (non–trivial) exact quantum solutions are not known. In order to be able to compare with the effective relational dynamics of Section 5.3, we ideally would like to extract (at least approximate) dynamical information from the Hilbert space or path–integral quantisations carried out thus far.

In the present section, we will briefly summarise the hindrances to obtaining the relational dynamics from any of the previously employed approaches.

Let us first recall the result of [7], where effective relational dynamics was studied in a toy model lacking global clocks — semiclassically (at the order of $\hbar$), the system was shown to be equivalent with the dynamics in local internal time Schrödinger regime by carrying out a local deparametrisation of the classical model and its subsequent quantisation. Local deparametrisation of a model governed by a quadratic constraint of the form $C = p_t^2 - H^2(t, q, p)$ was performed via choosing a local clock, in this case t, and factorising the constraint as $C = C_+ C_- = (p_t + H)(p_t - H)$.

The standard quantisation of $C_\pm$ constraint yields a Schrödinger equation with 'time–dependent' square–root Hamiltonian $\hat{H}$ (defined by a spectral decomposition)

$$i\hbar\partial_t\psi(t, q) = \pm\hat{H}(t, \hat{q}, p)\psi(t, q), \tag{5.2.1}$$

which, if t is a non–global internal clock, is only locally valid (in 'time' t) on account of non–unitarity.

Naturally a question arises as to whether a similar construction would not work for the present model so that one would compare the local dynamics of the Schrödinger regime with the effective results.

In fact, this question has already been considered in an early work on quantum cosmology by Blyth and Isham [71], in which they investigated a reduced quantisation of FRW models filled with a homogeneous scalar field. Various choices have been employed, with relational time variables (chosen before quantisation) which all yield distinct time–dependent Schrödinger equations with square–root Hamiltonians that describe precisely the desired Schrödinger regimes. The motivation behind the chosen quantisation procedure (quantisation by reduction, rather than Dirac quantisation) was to avoid the non–positive definiteness of Klein–Gordon type inner products.

Regarding the relation between the Schrödinger regime and a Dirac quantisation yielding a Wheeler–DeWitt (WDW) equation (with quantised $\hat{t}$),

$$\hat{H}^2(\hat{t}, \hat{q}, \hat{p})\tilde{\psi}(q, t) = \hat{p}_t^2\tilde{\psi}(q, t) = -\hbar^2\partial_t^2\tilde{\psi}(q, t), \tag{5.2.2}$$

it was noted in [71] that (5.2.2) does not follow from (5.2.1) when $\hat{H}$ is explicitly time dependent, because acting with $\pm\hat{H}$ on both sides of (5.2.1)—rather than (5.2.2)—yields

$$\hat{H}^2(t, \hat{q}, \hat{p})\psi(t, q) = -\left(\hbar^2\partial_t^2 \pm i\hbar\partial_t\hat{H}(t, \hat{q}, \hat{p})\right)\psi(t, q). \tag{5.2.3}$$

Note, that in [6] it was shown that to the order of $\hbar$ the expectation value versions[10] of (5.2.1) and (5.2.2) are solved by the same state $\psi = \tilde{\psi}$ if the expectation value of the 'internal time operator' $\hat{t}$ in (5.2.2) is complex with an imaginary part coinciding with the effective result (4.2.5)

$$\Im[\langle\hat{t}\rangle] = -\frac{\hbar}{2\langle\hat{p}_t\rangle}. \tag{5.2.4}$$

Thus, away from classical turning points, the solution to the relativistic WDW equation can be locally approximated by the Schrödinger regime up to the semiclassical order.

Note, however, that only $k = 1$, $m = 0$ and $k \leq 0$, $m \neq 0$ FRW models have been treated explicitly in [71], with the reason for these choices as follows: when treating the closed model with coupled massive scalar field as we do (5.1.6), the classical Hamiltonian for evolution in $t = \alpha$ time is given by $H(\alpha; \phi, p_\phi) = \sqrt{p_\phi^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha}}$, while the one for evolution in $t = \phi$ time reads $H(\phi; \alpha, p_\alpha) = \sqrt{p_\alpha^2 + e^{4\alpha} - m^2\phi^2 e^{6\alpha}}$. As a result our quantum Hamiltonian $\hat{H}(t,\ldots)$ is not only 'time–dependent', it also fails to commute with itself at different 'times', $[\hat{H}(t,\ldots), \hat{H}(t',\ldots)] \neq 0$ for both $t = \alpha, \phi$. The unfortunate consequence then is the fact, that the 'energy' eigenstates at a given 'time' are no longer eigenstates at later 'times' and subsequently the formal solution to (5.2.1) involves a Dyson time–ordering

$$\psi(t,q) = \hat{U}(t,t_0)\psi(t_0,q) = T\left[\exp\left(\mp\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(s,\hat{q},\hat{p})ds\right)\right]\psi(t_0,q). \tag{5.2.5}$$

Unfortunately, we have to conclude that the construction of the explicit time–evolution operator $\hat{U}(t,t_0)$ with either $\hat{H}(\alpha;\hat{\phi},\hat{p}_\phi)$ or $\hat{H}(\phi;\hat{\alpha},\hat{p}_\alpha)$ does not seem feasible for this non–integrable system even to the order of $\hbar$ and thus we will not make any further attempts at constructing a local Schrödinger regime.

In another approach to the problem of obtaining the relational dynamics from the quantum theory, one could try to solve the WDW equation and consider a suitable inner product so as to compute expectation values which may then be compared to their effective counterparts. Following this line of argument, the canonical Dirac quantisation was considered, e.g., see [40, 46, 48]. The standard quantisation of (5.1.6) leads to a Klein–Gordon type hyperbolic partial

[10] Assuming a standard $t = const$ Schrödinger theory inner product away from any turning points.

differential equation (setting for now $\hbar = 1$),

$$\left(\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} - e^{4\alpha} + m^2\phi^2 e^{6\alpha}\right)\psi(\alpha,\phi) = 0, \tag{5.2.6}$$

with variable mass $M^2 = e^{4\alpha}(e^{2\alpha}m^2\phi^2 - 1)$ in the 2D Lorentzian superspace metric

$$ds^2 = -d\alpha^2 + d\phi^2. \tag{5.2.7}$$

Thus, $\alpha = const$ is a 'spacelike' slice in minisuperspace.

In [37, 40, 44, 45, 48] WKB approximation has been extensively studied and despite attacking the idea from various perspectives, all papers have reported the breakdown of the semiclassical regime in the region of maximal expansion. A WKB approximation $\psi = \sum_n C_n(\alpha,\phi)\exp(\pm iS_n(\alpha,\phi))$ is valid only if the amplitude C_n varies much slower than the phase S_n [37, 40, 45, 48, 72]. As has been pointed out in [40, 44], the caustics resulting from focussing of nearby classical trajectories (see also Figure 5.3 above) cause $|C_n|^2 \to \infty$, while $|C_n|^2$ goes rapidly to zero where classical trajectories defocus. See for instance the region of maximal expansion in Figure 5.3, which leads to a generic breakdown of the WKB approximation. As this is relevant for a qualitative comparison to the effective results displayed in Section 5.3 below, we shall provide a summary of the most important features of these semiclassical constructions.

Kiefer in [48] imposed initial data for ψ on a 'spacelike' slice $\alpha = const$ in order to construct wave packets in minisuperspace, approximately solving (5.2.6) via a Born–Oppenheimer (with expansion parameter m_p^{-1}) and a subsequent WKB approximation. It is possible to construct classically expanding and contracting universes represented by tubelike standing waves as solutions if some additional conditions were imposed on account of 'normalisability', namely $\psi \to 0$ as $\alpha \to \infty$. This condition is not a normalisation condition, however, as normalisation would require an inner product.

The turning point $\alpha_{max}(n)$ of the individual oscillator modes in the wave packet depends strongly on the mode n. The reflection of the wave packet at the average $\alpha_{max} = \alpha_{max}(\bar{n})$ is thus described by a scattering phase shift that is dependent on the mass and is an integer multiple of π only for discrete values of m [48]. This means, that one can construct the narrow wave tubes on the expanding and re–contracting branch only for these special values of m, provided one keeps away from the classical turning region, i.e. only for $\alpha << \alpha_{max}$.

Furthermore, Hawking applied the 'no–boundary–proposal' [73] to the present model [37], providing boundary conditions for the path–integral for quantum gravity which should specify the quantum state of the universe uniquely by requiring that the wave function of any compact spatial three–geometry $G^{(3)}$ be given by a path–integral over all compact Euclidean four–geometries $G^{(4)}$ having $G^{(3)}$ as the only boundary. An IVP is thereby rendered superfluous and the resulting wave functions turn out to be non–normalisable.

The following semiclassical wave function can be interpreted as a superposition of quantum states peaked around an ensemble of non–singular bouncing solutions with a long inflationary period which corresponds to the aforementioned set of measure zero periodic and aperiodic solutions [39, 42, 44].[11] Numerical evidence for these results was given in [38], while report on similar outcomes with special attention to singular classical trajectories was given in [45]. Page approximated the Hawking wave function in [40] by starting from the canonical constraint (5.2.6) and translating the 'no–boundary–condition' into sufficient Cauchy data. Unfortunatelly also this WKB approximation breaks down due to caustics at the extrema of ϕ [40].

Regarding the classical determinism, mentioned in Section 5.1.1, of having the re–contracting branch as the logical successor of the expanding one, it was maintained in [8–10, 46, 48] that:

(1) The quantum and classical IVP are very different from one another. Initial data have to be imposed on all of the minisuperspace–slices $\alpha = const$, implying that both the expanding and the re–contracting branch have to be present 'initially' (in α). It is not possible to distinguish between the 'initial' and 'final state' anymore.

(2) Extension of classical paths through the turning region of α into the re–collapsing phase in quantum cosmology is completely without meaning. The WKB approximation does not provide the complete classical trajectory. Obtaining the latter could only be accomplished through continuous measurement by higher degrees of freedom (which would suppress the scattering at α_{max}).

These statements are, however, partially dependent on the construction used in [48], namely, (a), on the way of obtaining the semiclassical limit by means of a WKB approximation, (b), on singular use of α, rather than ϕ, as the internal clock and, (c), on the 'final condition', $\psi \to 0$ as $\alpha \to \infty$. Let us discuss each of these points separately.

[11] This is in agreement with standard results on the semiclassical limit of quantum models which are classically chaotic. Semiclassical states are typically concentrated on the closed orbits of measure zero [29–31].

(a) Even though a WKB approximation represents one of the possible ways of extracting semiclassical information from a quantum model, it is not the most general of the semiclassical approximations and it inevitably breaks down for chaotic systems [44]. Meanwhile, the semiclassical approximation employed in the effective approach is very general in nature. We shall see in the following section that semiclassicality can be achieved in the classical turning region, however, only for sufficiently peaked initial effective states. A fairly classical trajectory with the expanding branch serving as a logical predecessor of the re–collapsing branch can thus be obtained without the spoiling effect of the decoherence in additional degrees of freedom.

(b) The chaotic scattering of the wave packet around α_{max} [46, 48] manifests non–unitarity in the α evolution. As different modes have different turning points there is an interference of segments of the wave packet before and after the turning interval of the non–global time function resulting in a superposition of internal time directions.[12] This ultimately leads to a breakdown of the evolution in the non–global clock and of inner products based on its level surfaces before the classical turning point. In agreement with the analysis in [18] that concerns the reconstruction of the unitary Schrödinger and Heisenberg picture from relational quantum dynamics, this turns out to be only locally feasible for the states that are sufficiently semiclassical and for the clock degrees of freedom far enough from their turning points. Unitarity in a chosen time variable is equivalent to preservation of such inner products in the evolution with respect to the time variable and apparently is not possible here. What one could do instead would be to switch to a new degree of freedom to serve in a relational evolution as an alternate clock, should this one behave sufficiently semiclassically in the problematic turning region of the first clock [6, 7]. The breakdown of the relational evolution occurs in a case when such a degree of freedom is unavailable. In the present model universe, ϕ may be used for sufficiently well–behaved and semiclassical states as an intermediate clock in the turning regions of α. Although unclear as to how to manage the idea at the level of the WDW equation, this is precisely what can and will be carried out in the effective framework in Section 5.3. At the effective level, we will lose the non–local IVP and single evolution generator of [8–10, 46, 48] to a local IVP imposed on one of the (expanding or re–contracting) branches alone and the necessity of two evolution generators, one in α, the other in ϕ time.

(c) In fact, it is the 'final condition' which prevents narrow wave packets

[12] i.e. of positive and negative frequencies associated to the spectrum of the momentum conjugate to the internal clock function

around α_{max}; only exponentially (in α) decreasing modes are allowed and the data for both the expanding and re–contracting branch must be present initially at α_0, although it is subsequently scattered at α_{max} [8–10, 46, 48]. On the other hand, we can not impose any final condition in the effective approach. This yields semiclassical trajectories in the region of maximal expansion for sufficiently benign states.

As it is not clear what the physical inner product should be for the present model, let us consider some (naïve) possibilities.

(i) Since the operator $\hat{H}^2 = -\partial_\phi^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha}$ is not generally non–negative, evolution with respect to α is non–unitary and a standard Schrödinger type inner product is clearly not preserved.

(ii) Group averaging [74, 75] is commonly employed in constructing physical inner products in quantum cosmology. As it requires integrating over the flow of the quantum constraint, this does not seem practical on account of the classical non–integrability.

(iii) There exists a method going back to DeWitt [70, 76] which leads to a conserved quadratic form on $\mathcal{H}_{phys}$ from $\mathcal{H}_{aux}$ which in the present case amounts $L^2(\mathbb{R}^2, d\alpha d\phi)$:

Theorem 5.2.1. *Let $(\mathcal{Q}, \eta)$ be an n–dimensional configuration manifold with volume form η, and $\hat{C}$ be a second–order differential operator on $\mathcal{C}_0^2(\mathcal{Q}, \mathbb{C})$ (space of twice differentiable complex functions with compact support on $\mathcal{Q}$) that is symmetric with respect to the scalar product on $L^2(\mathcal{Q}, \eta)$. Then, for any $\Psi, \Phi \in \mathcal{C}_0^2(\mathcal{Q}, \mathbb{C})$, there is a vector field $\vec{J}[\Psi, \Phi]$ on $\mathcal{Q}$ such that*

$$(\hat{C}\Psi)^*\Phi - \Psi^*(\hat{C}\Phi) = Div_\eta \vec{J}. \tag{5.2.8}$$

Clearly, if both Ψ, Φ are annihilated by a hyperbolic $\hat{C}$, $\vec{J}$ defines a conserved current on the space of solutions to $\hat{C}\psi = 0$. It is not difficult to show, that for the constraint (5.2.6) $\vec{J}$ is just given by the standard Klein–Gordon current vector,

$$J^a = g^{ab}[(\partial_a \Psi^*)\Phi - \Psi^*(\partial_a \Phi)], \tag{5.2.9}$$

where g^{ab} is the inverse 2D minisuperspace metric (5.2.7), such that the conserved quadratic form provided by the theorem coincides with the Klein–Gordon inner product. Here one can not restrict the attention globally to positive or negative frequency modes (on subspaces thereof the Klein–Gordon product would be positive definite) as it was possible in the case of a Klein–Gordon particle.

It is a consequence of the non–existence of a global clock, with subsequent mixing of positive and negative frequencies in α time in the turning region of α. What is more, the Klein–Gordon charge is identically zero for real Ψ, Φ and thus conserved trivially. The semiclassical solutions of [37, 38, 40, 48] are real. Consequently, it is not possible to use the Klein–Gordon inner product even as an approximation for known semiclassical states on only the 'negative' (i.e. expanding) or 'positive frequency' (i.e. re–collapsing) that branch away from the turning region in which frequencies mix. It thus remains unclear what the correct physical inner product should be and how the Hilbert space problem could be solved.

Thus it indeed seems that (at the moment) relational dynamics of a given non–integrable model is only practically solvable in the effective approach (and also there only in a limited regime) since it sidesteps many technical difficulties associated with the Hilbert space quantisation [7]. This shall be the topic of the next section.

5.3 Effective dynamics

Following the general procedure laid down in Chapter 3 and Section 4.1, we shall now turn to the effective treatment of the closed FRW model.

The non–integrability will not become an issue for us; since we are only interested in semiclassical solutions, we shall refrain from attempting to solve the full quantum dynamics of the model but will only 'expand around' classical trajectories.

Even though we shall focus our attention on rather well–behaved trajectories it will already be evident what would (and ultimately will) happen for more generic and complicated solutions.

Using the potential $V(\alpha, \phi) = e^{4\alpha} - m^2\phi^2 e^{6\alpha}$ in (4.1.8), the constraint (5.1.6) translates to order $\hbar$ into the following five quantum constraint functions

$$\begin{aligned}
C =\ & p_\phi^2 + (\Delta p_\phi)^2 - p_\alpha^2 - (\Delta p_\alpha)^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} \\
& + m^2 e^{6\alpha}(\Delta\phi)^2 + 12m^2\phi e^{6\alpha}\Delta(\alpha\phi) + (18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha})(\Delta\alpha)^2, \\
C_\alpha =\ & 2p_\phi\Delta(\alpha p_\phi) - 2p_\alpha\Delta(\alpha p_\alpha) - i\hbar\, p_\alpha + 2m^2\phi e^{6\alpha}\Delta(\alpha\phi) \\
& + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})(\Delta\alpha)^2, \\
C_\phi =\ & 2p_\phi\Delta(\phi p_\phi) + i\hbar\, p_\phi - 2p_\alpha\Delta(\phi p_\alpha) + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha\phi) \\
& + 2m^2\phi e^{6\alpha}(\Delta\phi)^2, \\
C_{p_\alpha} =\ & 2p_\phi\Delta(p_\alpha p_\phi) - 2p_\alpha(\Delta p_\alpha)^2 + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})(\Delta(\alpha p_\alpha) - \frac{1}{2}i\hbar) \\
& + 2m^2\phi e^{6\alpha}\Delta(\phi p_\alpha), \\
C_{p_\phi} =\ & 2p_\phi(\Delta p_\phi)^2 - 2p_\alpha\Delta(p_\alpha p_\phi) + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\phi) + 2m^2\phi e^{6\alpha}\Delta(\phi p_\phi) \\
& - i\hbar\, m^2\phi e^{6\alpha}.
\end{aligned} \tag{5.3.1}$$

Due to the degeneracy in the quantum Poisson structure the five constraints (5.3) generate only four independent gauge flows. To remove redundant degrees of freedom, we choose a relational clock and a corresponding *Zeitgeist*, thereby fixing three of the four independent gauge flows. We are then left with just one (Hamiltonian) constraint governing the evolution of the system.

5.3.1 Evolution in α

Let us make use of the advantage that the α clock provides, i.e. the evolution of the system being technically less demanding if we consider the system first in the α–*Zeitgeist*

$$(\Delta\alpha)^2 = \Delta(\phi\alpha) = \Delta(\alpha p_\phi) = 0, \tag{5.3.2}$$

which, as can be easily checked by solving C_α, leads to a saturation of the generalised uncertainty relation for the clock degrees of freedom.

After one employs the gauge conditions, the rest of the constraints simplify to form the following system:

$$\begin{aligned}
C =\ & p_\phi^2 + (\Delta p_\phi)^2 - p_\alpha^2 - (\Delta p_\alpha)^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} + m^2 e^{6\alpha}(\Delta\phi)^2, \\
C_\phi =\ & 2p_\phi\Delta(\phi p_\phi) + i\hbar\, p_\phi - 2p_\alpha\Delta(\phi p_\alpha) + 2m^2\phi e^{6\alpha}(\Delta\phi)^2, \\
C_{p_\alpha} =\ & 2p_\phi\Delta(p_\alpha p_\phi) - 2p_\alpha(\Delta p_\alpha)^2 + 2m^2\phi e^{6\alpha}\Delta(\phi p_\alpha) - i\hbar(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha}), \\
C_{p_\phi} =\ & 2p_\phi(\Delta p_\phi)^2 - 2p_\alpha\Delta(p_\alpha p_\phi) + 2m^2\phi e^{6\alpha}\Delta(\phi p_\phi) - i\hbar\, m^2\phi e^{6\alpha}.
\end{aligned} \tag{5.3.3}$$

The spread $(\Delta p_\alpha)^2$ of the clock variable's conjugate momentum, as well as both its covariances $\Delta(\phi p_\alpha)$ and $\Delta(p_\alpha p_\phi)$, do not represent any physical variables, thus they shall be referred to as the "unphysical" moments. The gauge conditions can be used to solve for these.

Relational evolution of the remaining degrees of freedom in α is generated by the remaining first–class (Hamiltonian) constraint which, by (4.2.3), in the α–*Zeitgeist* reads

$$\begin{aligned} C_H = \;& p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} + \left[1 - \frac{p_\phi^2}{p_\alpha^2}\right](\Delta p_\phi)^2 - \frac{2m^2\phi e^{6\alpha} p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) \\ & + \left[m^2 e^{6\alpha} - \frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2}\right](\Delta\phi)^2 + i\hbar\,\frac{3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha}}{p_\alpha}. \end{aligned} \tag{5.3.4}$$

Through the Poisson structure (3.1.3) this constraint generates the following equations of motion:

$$\begin{aligned} \dot{\alpha} =& - 2p_\alpha + \frac{2p_\phi^2}{p_\alpha^3}(\Delta p_\phi)^2 + \frac{4m^2\phi e^{6\alpha} p_\phi}{p_\alpha^3}\Delta(\phi p_\phi) + \frac{2m^4\phi^2 e^{12\alpha}}{p_\alpha^3}(\Delta\phi)^2 \\ & - i\hbar\,\frac{3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha}}{p_\alpha^2}, \\ \dot{p_\alpha} =\;& 4e^{4\alpha} - 6m^2\phi^2 e^{6\alpha} + \frac{12m^2\phi e^{6\alpha} p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) - i\hbar\,\frac{18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha}}{p_\alpha} \\ & - \left[6m^2 e^{6\alpha} - \frac{12m^4\phi^2 e^{12\alpha}}{p_\alpha^2}\right](\Delta\phi)^2, \\ \dot{\phi} =\;& 2p_\phi - \frac{2p_\phi}{p_\alpha^2}(\Delta p_\phi)^2 - \frac{2m^2\phi e^{6\alpha}}{p_\alpha^2}\Delta(\phi p_\phi), \\ \dot{p_\phi} =& - 2m^2\phi e^{6\alpha} + \frac{2m^2 e^{6\alpha} p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) + \frac{2m^4\phi e^{12\alpha}}{p_\alpha^2}(\Delta\phi)^2 - i\hbar\,\frac{6m^2\phi e^{6\alpha}}{p_\alpha}, \end{aligned}$$

$$
\begin{aligned}
(\Delta\dot{\phi})^2 =&\; 4\left[1-\frac{p_\phi^2}{p_\alpha^2}\right]\Delta(\phi p_\phi) - \frac{4m^2\phi e^{6\alpha}p_\phi}{p_\alpha^2}(\Delta\phi)^2,\\
\Delta(\dot{\phi p_\phi}) =&\; 2\left[1-\frac{p_\phi^2}{p_\alpha^2}\right](\Delta p_\phi)^2 + 2\left[\frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2} - m^2 e^{6\alpha}\right](\Delta\phi)^2,\\
(\Delta\dot{p_\phi})^2 =&\; \frac{4m^2\phi e^{6\alpha}p_\phi}{p_\alpha^2}(\Delta p_\phi)^2 + 4\left[-m^2e^{6\alpha} + \frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2}\right]\Delta(\phi p_\phi). \qquad (5.3.5)
\end{aligned}
$$

It is straightforward to show that the evolving degrees of freedom in the α–*Zeitgeist*, i.e. $\phi, p_\phi, (\Delta\phi)^2, (\Delta\phi p_\phi)$ and $(\Delta p_\phi)^2$, can be consistently chosen to be real if α picks up the imaginary part (4.2.5) (with q_1, p_1 replaced by α, p_α), see [6, 7]. The set (5.3.1) can be solved numerically, yielding the evolution of the transient observables of the α–*Zeitgeist* (i.e. the correlations of the evolving variables with $\Re[\alpha]$).

As the general discussion in Section 4.2 reveals, the α–*Zeitgeist* possesses only a transient validity because α is a non–global clock. To remedy this ailment in the turning region(s) of α, we will choose ϕ as the new clock and evolve the system in the corresponding ϕ–*Zeitgeist*.

5.3.2 Evolution in ϕ

The ϕ–*Zeitgeist*,

$$(\Delta\phi)^2 = \Delta(\alpha\phi) = \Delta(\phi p_\alpha) = 0, \qquad (5.3.6)$$

by solving C_ϕ, leads to a saturation of the generalised uncertainty relation for the canonical pair (ϕ, p_ϕ). The secondary constraints are now given by

$$
\begin{aligned}
C =&\; p_\phi^2 + (\Delta p_\phi)^2 - p_\alpha^2 - (\Delta p_\alpha)^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha}\\
&+ (18m^2\phi^2e^{6\alpha} - 8e^{4\alpha})(\Delta\alpha)^2,\\
C_\alpha =&\; 2p_\phi\Delta(\alpha p_\phi) - 2p_\alpha\Delta(\alpha p_\alpha) - i\hbar\, p_\alpha + (6m^2\phi^2e^{6\alpha} - 4e^{4\alpha})(\Delta\alpha)^2,\\
C_{p_\alpha} =&\; 2p_\phi\Delta(p_\alpha p_\phi) - 2p_\alpha(\Delta p_\alpha)^2 + (6m^2\phi^2e^{6\alpha} - 4e^{4\alpha})\left(\Delta(\alpha p_\alpha) - \frac{i\hbar}{2}\right)\\
C_{p_\phi} =&\; 2p_\phi(\Delta p_\phi)^2 - 2p_\alpha\Delta(p_\alpha p_\phi) + (6m^2\phi^2e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\phi)\\
&- 2i\hbar\, m^2\phi e^{6\alpha}, \qquad (5.3.7)
\end{aligned}
$$

and, again, can be used to solve for the unphysical moments $\Delta(\alpha p_\phi)$, $\Delta(p_\alpha p_\phi)$ and $(\Delta p_\phi)^2$.

The Hamiltonian constraint in the ϕ–*Zeitgeist* reads

$$\begin{aligned} C_H = \; & p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} - \left[1 - \frac{p_\alpha^2}{p_\phi^2}\right](\Delta p_\alpha)^2 \\ & - \frac{p_\alpha}{p_\phi^2}(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\alpha) + i\hbar\,\frac{m^2\phi e^{6\alpha}}{p_\phi} \\ & + \left[18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha} + \frac{(3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha})^2}{p_\phi^2}\right](\Delta\alpha)^2 \end{aligned} \tag{5.3.8}$$

and generates the following set of equations of motion for $\alpha, p_\alpha, (\Delta\alpha)^2, (\Delta p_\alpha)^2$ and $\Delta(\alpha p_\alpha)$ which constitute the evolving degrees of freedom in the ϕ–*Zeitgeist*:

$$\begin{aligned} \dot{\phi} = \; & 2p_\phi - \frac{2p_\alpha^2}{p_\phi^3}(\Delta p_\alpha)^2 + \frac{p_\alpha}{p_\phi^3}\left(12m^2\phi^2 e^{6\alpha} - 8e^{4\alpha}\right)\Delta(\alpha p_\alpha) \\ & - \frac{(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})^2}{2p_\phi^3}(\Delta\alpha)^2 - i\hbar\,\frac{m^2\phi e^{6\alpha}}{p_\phi^2}, \end{aligned}$$

$$\begin{aligned} \dot{p_\phi} = & - 2m^2\phi e^{6\alpha} + \frac{12p_\alpha}{p_\phi^2}m^2\phi e^{6\alpha}\Delta(\alpha p_\alpha) \\ & - \left[36m^2\phi e^{6\alpha} + \frac{12m^2\phi e^{6\alpha}(3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha})}{p_\phi^2}\right](\Delta\alpha)^2 - i\hbar\,\frac{m^2 e^{6\alpha}}{p_\phi}, \end{aligned}$$

$$\dot{\alpha} = - 2p_\alpha + \frac{2p_\alpha}{p_\phi^2}(\Delta p_\alpha)^2 - \frac{6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha}}{p_\phi^2}\Delta(\alpha p_\alpha),$$

$$\begin{aligned} \dot{p_\alpha} = \; & 4e^{4\alpha} - 6m^2\phi^2 e^{6\alpha} + \frac{p_\alpha}{p_\phi^2}(36m^2\phi^2 e^{6\alpha} - 16e^{4\alpha})\Delta(\alpha p_\alpha) - i\hbar\,\frac{6m^2\phi e^{6\alpha}}{p_\phi} \\ & - \Bigg[108m^2\phi^2 e^{6\alpha} - 32e^{4\alpha} \\ & \qquad + \frac{(18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha})(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})}{p_\phi^2}\Bigg](\Delta\alpha)^2, \end{aligned}$$

$$
\begin{aligned}
(\Delta\dot{\alpha})^2 =& -4\left[1-\frac{p_\alpha^2}{p_\phi^2}\right]\Delta(\alpha p_\alpha) - \frac{p_\alpha}{p_\phi^2}(12m^2\phi^2 e^{6\alpha} - 8e^{4\alpha})(\Delta\alpha)^2, \\
\Delta(\dot{\alpha p_\alpha}) =& -2\left[1-\frac{p_\alpha^2}{p_\phi^2}\right](\Delta p_\alpha)^2 \\
& -2\left[18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha} + \frac{(3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha})^2}{p_\phi^2}\right](\Delta\alpha)^2, \\
(\Delta\dot{p_\alpha})^2 =& \ \frac{p_\alpha}{p_\phi^2}(12m^2\phi^2 e^{6\alpha} - 8e^{4\alpha})(\Delta p_\alpha)^2 \\
& -4\left[18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha} + \frac{(3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha})^2}{p_\phi^2}\right]\Delta(\alpha p_\alpha).
\end{aligned} \tag{5.3.9}
$$

Once more, the clock variable ϕ develops a complex nature, in agreement with (4.2.5), $\operatorname{Im}\phi = -\frac{\hbar}{2p_\phi}$, while the evolving degrees of freedom are chosen to be real. After bridging the region of maximal expansion it is once more safe to go back to evolving the system in agreement with Section 5.3.1 in the α–*Zeitgeist* and recover the re–contracting branch.

5.3.3 Numerical results

In this part we shall analyse the numerical behaviour of the effective system that is the truncated closed FRW cosmological model with a minimally coupled massive scalar field. The system starts off peaked about classical trajectories and as this model universe does not possess a globally valid clock function, both the scalar field and the scale factor are possible candidates for only a locally valid clock.

For simplicity, let us concentrate on the well–behaved systems, i.e. those, where trajectories only have few extrema in the scale factor. More general trajectories can in theory be treated as well, only in that case the switching between internal clocks would need to be done many times to evolve the system through the whole trajectory. The 'simple' cases that will be considered are 'easy' enough to make the evolution work throughout its course and rich enough still to exhibit several generally valid characteristics (i.e. changing the internal clocks in the region of maximal expansion will not work in a generic solution) to let us draw relevant conclusions.

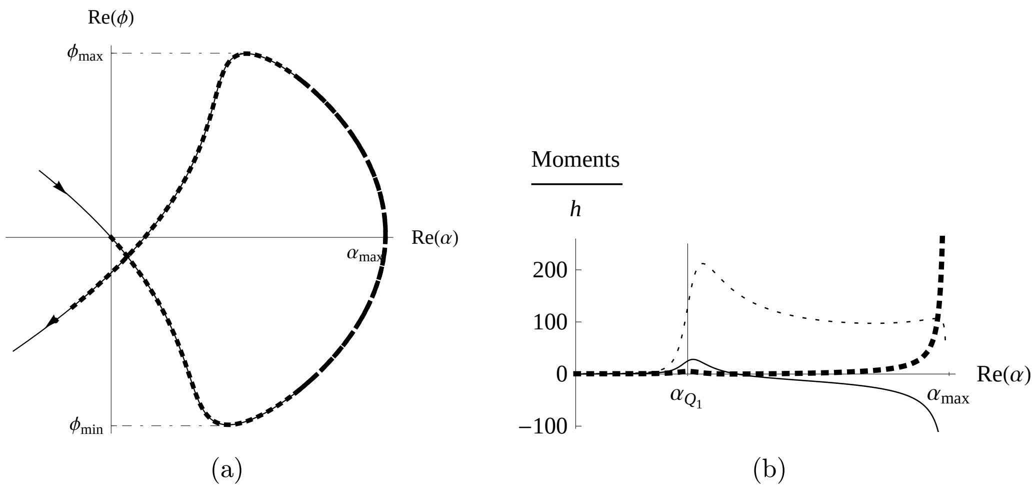


Figure 5.4: (a) Classical trajectory (dotted) and patched effective trajectory: α–gauge (solid), ϕ–gauge (dashed). (b) Moments in α–gauge on the incoming branch: $(\Delta\phi)^2$ (thick, dashed), $(\Delta p_\phi)^2$ (thin, dashed), $\Delta(\phi p_\phi)$ (solid). α_{Q_1} is the quasi–turning point of α on the incoming branch where the clock becomes 'slow' (see text and figure 5.6(a)).

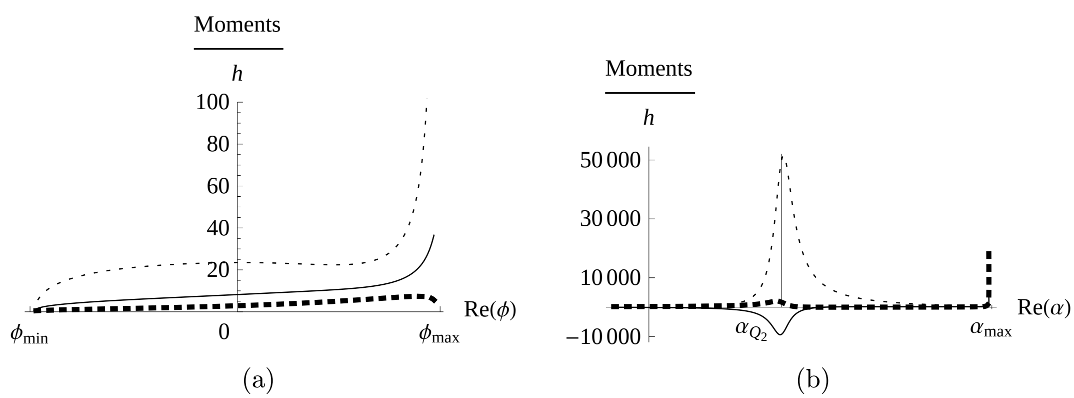


Figure 5.5: (a) Moments in ϕ–gauge: $(\Delta\alpha)^2$ (thick, dashed), $(\Delta p_\alpha)^2$ (thin, dashed), $\Delta(\alpha p_\alpha)$ (solid). (b) Moments in α–gauge on the outgoing branch: $(\Delta\phi)^2$ (thick, dashed), $(\Delta p_\phi)^2$ (thin, dashed), $\Delta(\phi p_\phi)$ (solid). α_{Q_2} is the quasi–turning point of α on the outgoing branch where the clock becomes 'slow' (see text and Figure 5.6(b)).

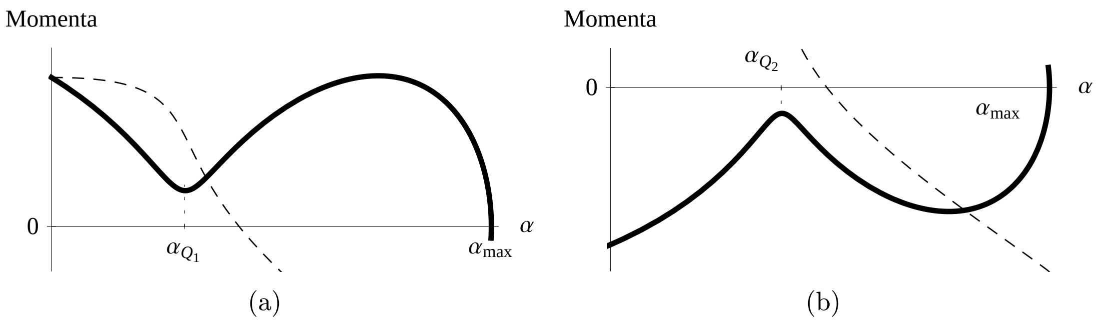


Figure 5.6: (a) Classical momenta on the incoming branch with quasi–turning point α_{Q_1} of the clock α: p_ϕ (dashed), p_α (solid). (b) Classical momenta on the outgoing branch with quasi–turning point α_{Q_2}: p_ϕ (dashed), p_α (solid).

Figure 5.4(a) shows two trajectores in the configuration space obtained by consistently switching gauges and clocks according to the general method developed in Section 4.3.1. One is the classical trajectory of the system, the other, in fact identical to the classical one, is the patched effective trajectory. The effective trajectory has been obtained by first evolving the system in the α–*Zeitgeist*, then switching to ϕ clock between the extremal points $\phi = \phi_{min}$ and $\alpha = \alpha_{max}$, only to finish the evolution in the α–*Zeitgeist* again, once $\alpha = \alpha_{max}$ has been passed and just before $\phi = \phi_{max}$ has been reached.

The quantum scale for this particular numerical evolution has been chosen so that $\sqrt{\hbar} \sim 10^{-4}$ while the expectation values of operators are of the order of unity. The quantum corrections are of the order of $\hbar$ in the leading order and are thus of the magnitude of $\sim 10^{-8}$. This large separation of the quantum corrections and the classical variables has been a deliberate choice: quantum back–reaction is virtually non–existent and the classical variables can evolve independently of the quantum modes.

Moments in figures 5.4(b)–5.5(b) illustrate, how the classical chaoticity of the system penetrates to the quantum system: the initial values of the moments in the α–*Zeitgeist* are close to $\hbar$, however they will start growing at some point to reach values 10^4 greater than the initial one thus leaving the region of semiclassicality.

The “spikes” in the moments, particularly $(\Delta p_\phi)^2$, in figures 5.4(b) and 5.5(b) are a direct consequence of the classical quasi–turning points of the internal clock $\alpha = \alpha_{Q_1}$ and $\alpha = \alpha_{Q_2}$, where $\dot{\alpha} = -2p_\alpha$ is small and the clock α thus becomes ‘too slow’ for resolving the evolution of other degrees of freedom with respect to itself (also see discussion in Section 4.3.1).

An obvious attempt at treating the problem would involve evolving the system through the problematic region using ϕ as the internal clock, this, however, may not always be possible, as the quasi–turning points in α may lie so close to turning points of ϕ, that the ϕ is still a bad clock at the time, and α is already a bad clock itself. Such a situation is illustrated in figures 5.6(a) and 5.6(b) for the incoming and outgoing branches respectively, where one can see the proximity of the local minima in p_α and the points where $p_\phi = 0$.

Drawing from the general discussion of classical solutions in [42], this property features generically in the space of solutions and is indeed the origin of all problems concerning the good resolution of transient quantum observables. Both α *and* ϕ (and their momenta that also feature turning points) are thus ‘bad clocks’ in the immediate vicinity of each other, leading to a poor resolution of relational evolution with a large growth of the moments as a consequence. Unfortunately we are not able to perform any clock change that could cure this ailment, since if α and ϕ fail to resolve the evolution, then neither can any function $f(\alpha)$ or $g(\phi)$ serve as a good clock for the problematic piece of trajectory as $(\Delta f)^2 \propto (\Delta\alpha)^2$ and $(\Delta g)^2 \propto (\Delta\phi)^2$ and since it is not possible for both $(\Delta\alpha)^2$ and $(\Delta\phi)^2$ to vanish in this region, the possibility of a valid f–, or g–*Zeitgeist* does not exist either.

Nevertheless, even if for the particular trajectory presented here the relational evolution is not impeded by an overwhelming growth of moments (as these manage to stay within the bounds of the semiclassical approximation), we shall discuss further that for more generic trajectories this becomes a fundamental problem that makes relational evolution impossible altogether.

There is, in fact, a more general problem underlying the abovementioned issue: an arbitrary classical trajectory in this model will exhibit a structure where local maxima and minima occur at all scales.

Let us stress again, that the first trajectory we investigated was a well–behaved one and on account of the close investigation of this nicely–evolving system that has been performed, one is now quite easily able to identify potential problems and draw relevant conclusions as regards the relational evolution of a more general model. Namely, one is able to predict the difficulties arising from picking such a generalised classical trajectory, that is only slightly more complicated, e.g. one plotted in Figure 5.7.

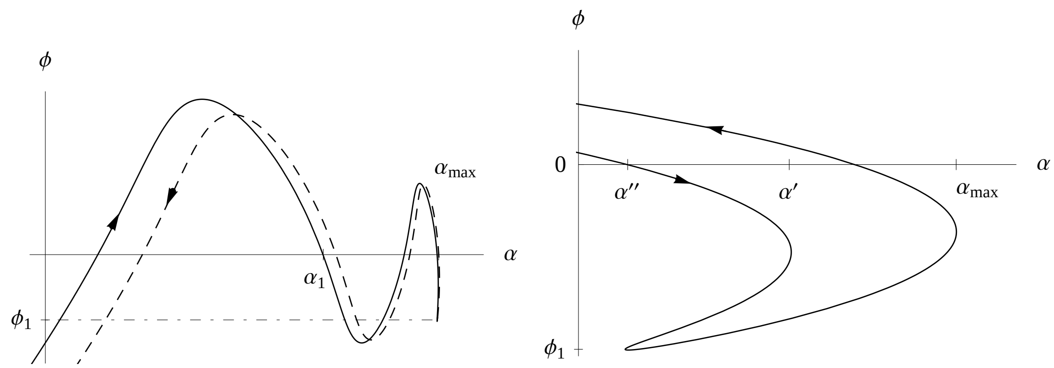


Figure 5.7: Left: a classical configuration–space trajectory of the same model parameters as in Figure 5.4(a), but different initial conditions: incoming branch (solid), outgoing branch (dashed). Right: a closeup of the same trajectory near $\alpha = \alpha_{max}$; there are two other local extrema in α labeled by α' (a maximum) and α'' (a minimum), and ϕ reaches a locally minimal value ϕ_1 very near $\alpha = \alpha''$.

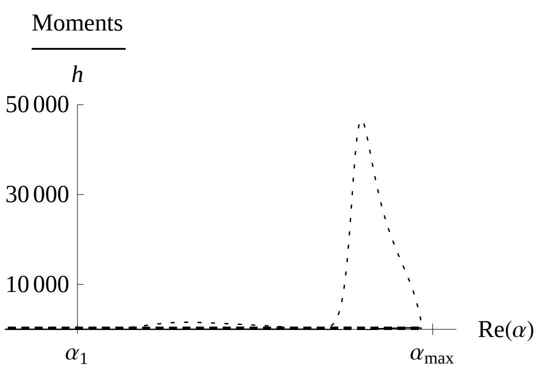


Figure 5.8: Moments in α–gauge on the incoming branch evolved effectively in a state initially peaked around the trajectory in Figure 5.7 : $(\Delta\phi)^2$ (thick, dashed), $(\Delta p_\phi)^2$ (thin, dashed), $\Delta(\phi p_\phi)$ (solid).

This trajectory shares the same m and $\hbar$ parameters, but differs from the trajectory in Figure 5.4(a) in initial conditions. Moreover, the trajectory features three local extrema in the scale factor at $\alpha = \alpha_{max}, \alpha', \alpha''$. The resulting effective system is so unstable already along the incoming branch (when evolved in α–clock), that by the time it reaches the vicinity of the classical turning points in α, the quantum moments are of the order comparable to the separation between the three extrema of α (Figure 5.8). And the situation gets dire when we try to evolve the system further. With the separation between the turning point of ϕ, where $\phi = \phi_1$, and the local minimum in α, where $\alpha = \alpha''$, is of order $\sqrt{\hbar}$, it ceases to be possible to resolve the two points even with well–behaved quantum moments, thus one is not able to resolve the effective evolution of the given system in the chosen quantum scale.

This result is quite general: there is an infinite set of classical trajectories with extrema in α and ϕ for any given choice of the quantum scale, that are separated on or below that scale [42] and for which an effective–procedure–motivated construction of semiclassical states evolving through the region of maximal expansion in a well–behaved manner is fundamentally 'flawed' and thus not feasible. Let us have a closer look at this breakdown of relational evolution in the following section.

5.4 Breakdown of relational evolution

As can be deduced from classical dynamics, the wealth of structure of generic semiclassical trajectories, that we have discussed above is present in all scales; impedes resolution via clock transformations and ultimately leads to a generic breakdown of quantum relational evolution in the region of maximal expansion. Let us discuss this issue in greater depth. Consider an arbitrary classical phase space trajectory and choose an arbitrary open neighbourhood, through which this trajectory passes.[13] We demand that a phase space function must grow monotonically along the trajectory in our chosen open neighbourhood to be deemed a good relational clock. For the situation to be ideal, one would expect the level surfaces of the clock function to be orthogonal to the tangent vector of the trajectory at each point, meaning the trajectory and the clock function should optimally vary at the same order of magnitude. However, this ideal situation does not occur on account of the instability of the initial data in this

[13] As a matter of fact, it is possible to choose any open neighbourhood, as in chaotic systems a generic trajectory passes through any open neighbourhood on a constraint surface.

chaotic model.

A trajectory, initially close to some other particular trajectory, is in the end bound to suffer uncorrelated fate. Thus in any open neighbourhood a phase space function that initially was deeemed a good relational clock function for one trajectory shall ultimately fail to be a good relational clock for any other trajectory. This orbit dependence manifests further, as we investigate different trajectories passing through any arbitrarily small neighbourhood on the constraint surface. Since the system is chaotic, there will be an infinite number of uncorrelated trajectories which pass through the neighbourhood in all directions. Since an internal time direction of a clock function is orthogonal to its level surfaces at every phase space point, we can already conclude that there does not exist any phase space function that could be a good clock for all the uncorrelated trajectories in the entire neighbourhood.

Effective semiclassical trajectories suffer greatly on account of this fact. Quantum moments cause that evolving an effective solution through the quantum phase space is in fact equivalent to evolving a neighbourhood of volume of the order of $\hbar^2$ through the phase space (the order is given by the fact that our system contains four canonical variables). We are of course still working at semiclassical order $\hbar$. As the example above manifests, this may work for the states that are initially highly semiclassical, i.e. that correspond to the unique superposition of there being special classical solutions featuring low numbers of turning points in the clock functions that vary roughly on the same scales. Should we happen to come across initially semiclassical effective states corresponding to a superposition of (in the end) unrelated generic classical orbits, the relational evolution must break down in the region where these classical trajectories fan out. There might, in fact, exist neighbourhoods where the phase space function could be deemed a good clock for the given trajectory, however as such a clock is highly orbit dependent, in the generic case there does not exist any neighbourhood of volume of order $\hbar^2$ (or larger) exists where any phase space function could be a good clock for all classical orbits in the superposition. As this would be a requirement for the relational evolution of a semiclassical state, we have to conclude that at this stage, that it is not possible to perform any clock change and relational evolution must break down altogether. For such trajectories, it is then fundamentally impossible, using the effective method, to construct entire semiclassical states that would evolve seamlessly throughout the region of maximal expansion. In addition, effective relational evolution (in a 'classical' clock) is sure to break down for general (non–semiclassical) effective states: if one attempted to evolve a neighbourhood larger than the considered

$\hbar^2$–order one through the quantum phase space, the above mentioned problems must only intensify.

5.5 Review of FRW models

In the previous text we have detailed both classical and effective treatment of the closed FRW model with a minimally coupled massive scalar field. In the following short section we shall review both open and flat model with a cosmological constant Λ, as well as the closed FRW model featuring Λ.

5.5.1 General FRW setting

General expression for the Hamiltonian constraint for an FRW model filled with a minimally coupled massive scalar field admitting all three possible values for the curvature as well as the possibility of the cosmological constant reads:

$$C_H = p_\phi^2 - p_\alpha^2 - ke^{4\alpha} + (\frac{\Lambda}{3} + m^2\phi^2)e^{6\alpha}, \tag{5.5.1}$$

where $ke^{4\alpha} - (\frac{\Lambda}{3} + m^2\phi^2)e^{6\alpha}$ is the potential $V(\alpha, \phi)$.

Let us discuss different combinations of parameters involved. It is obvious that depending on the choice of the curvature parameter k and cosmological constant Λ, we will get a cosmology either qualitatively very similar to the one discussed in sections 5.1–5.3, or a model universe that is somewhat different.[14] The resulting potential functions for different parameter choices are summed up in Table 5.1, with $|\Lambda| = \lambda$.

5.5.2 Discussion

It is obvious that only overall sign of the $(-\frac{\lambda}{3} + m^2\phi^2)$ bracket is important for the following discussion. FRW models with cosmological constant $\Lambda > 0$ and $\lambda < 3m^2\phi^2$ are qualitatively indistinguishable from their counterparts without Λ. This is in agreement with statements that the mass of the scalar field acts as an effective cosmological constant [37–48].

[14] Indeed, should we opt for a massless case the model would become separable and the non–global clock problem would dissolve immediately. Some examples have been studied in [79], albeit for a model with an ordering issue present that arises from a different Lapse function gauge choice.

	$\Lambda < 0$	$\Lambda = 0$	$\Lambda > 0$
$k = 1$	$e^{4\alpha} - (-\frac{\lambda}{3} + m^2\phi^2)e^{6\alpha}$	$e^{4\alpha} - m^2\phi^2 e^{6\alpha}$	$e^{4\alpha} - (\frac{\lambda}{3} + m^2\phi^2)e^{6\alpha}$
$k = 0$	$-(-\frac{\lambda}{3} + m^2\phi^2)e^{6\alpha}$	$-m^2\phi^2 e^{6\alpha}$	$-(\frac{\lambda}{3} + m^2\phi^2)e^{6\alpha}$
$k = -1$	$-e^{4\alpha} - (-\frac{\lambda}{3} + m^2\phi^2)e^{6\alpha}$	$-e^{4\alpha} - m^2\phi^2 e^{6\alpha}$	$-e^{4\alpha} - (\frac{\lambda}{3} + m^2\phi^2)e^{6\alpha}$

Table 5.1: The potential function $V(\alpha, \phi)$ of various FRW cosmological universes featuring different choices of the curvature parameter and cosmological constant.

FRW models that possess a cosmological constant $\Lambda < 0$ and $\lambda > 3m^2\phi^2$ yield an opposite sign before the 'effective cosmological constant' term, thus promising a slightly different evolution. The work on these models is underway.

After a preliminary treatment of a flat FRW model without Λ, the effective results appear to be qualitatively reproducing those of the closed FRW model; this too however is a work in progress and so is a treatise on the last, open, model universe.

In [80, 81], it has been shown that (most) FRW models with a conformally coupled massive scalar field are non–integrable (save some exceptions in parameter fine tuned flat variants). We have numerically sampled several classical FRW models with minimal coupling to find that these are all non–integrable in, at least, a certain defined interval and thus pose the same problem for the resolution of their relational evolution. There will ultimately occur a breakdown of the relational evolution in the region of maximal expansion precipitating from the wealth of structure of these non–integrable model universes. Even though it would be interesting to check rigorously for integrable exceptions in these models, we can conclude, that since the generic type of the system is non–integrable, any exception due to fine tuning can not be regarded as a valid representative of functionality of the relational evolution, merely a favourable occurrence and could serve as a checking mechanism for the relational calculations at best.

Chapter 6

Conclusion and Outlook

The work presented in this thesis is the first step towards the study of relational quantum dynamics in the generic case of a non–integrable system with the added feature of a non–trivial coupling of the clock function to the evolving variables. To advance our understanding of such systems we have made use of the effective approach of [6, 7] as the framework enables one to resolve temporally local time evolution via *fashionables* – transient relational observables and clock transformations between different *Zeitgeister*.

We have applied the effective approach to the FRW model universes filled with a minimally coupled massive scalar field, with emphasis on the closed FRW model, as quantum dynamics thereof have thus far not been properly studied.

The numerical results that have been obtained for the closed FRW model exhibit sensitivity to initial conditions in the region of maximal expansion already for the well–behaved trajectories and predict a failure of the semiclassical approximation in more generic cases since a semiclassical state peaked on initially nearby classical trajectories will ultimately spread apart. This is in sharp contrast with simpler toy models discussed in [7] where, for one, it is the coherent states that are sharply peaked even in the turning regions of the clock functions, and moreover, the non–global clocks are decoupled there, thus a correlation between the "badness" of these clocks does not exist.

As has been confirmed in this work, the chaotic scattering forming in the region of maximal expansion poses a great challenge for the relational dynamics, since the dynamics is only feasible for sufficiently sharply peaked states and generally breaks down in the region of maximal expansion. It is evident, that it is not advisable to trust the effective semiclassical truncation in this regime anymore, as the quantum moments first leave the truncation order and ultimately diverge. The quantum back–reaction is not to be a culprit as this effect does not play any prominent role in our situation (by construction). After examining the classical trajectory we identify the problem to be rooted in the quasi–turning points of the clock α that immediately follow/precede a turning point of the field ϕ [42], thus making the two clock conjugate momenta small (or vanish) in immediate neighbourhood of each other.[1] This in turn leaves both clocks 'too slow' for a proper resolution of relational evolution [6, 7, 18–20, 77], meaning a change of clock can not remedy the situation. A system more generic, featuring more oscillations in both α and ϕ than the one studied in this thesis, would obviously fare even worse.

In agreement with the discussion in [6] we interpret such a breakdown of the effective semiclassical truncation as an evidence, that the relational evolution generally breaks down on account of mixing of internal time directions and consider this an analogy to how a non–unitarity in a deparametrisation at a Hilbert space level causes a failure of any inner product based on level surfaces of the local clock function.

Both the ultimate breakdown of the WKB approximation to (5.2.6) that has been reported in the earlier literature [40, 44–46, 48] and the generic breakdown of semiclassicality in the region of maximal expansion are in agreement, note however, that contrary to a very specific semiclassical limit obtained by the WKB techniques, the semiclassical approximation we have employed was a very general one. Precisely on account of this fact the arguments of [8–10, 46, 48] regarding the semiclassical limit obtained via WKB approximation are in contrast with our effective approach, where it has been shown possible to obtain the semiclassical states and evolve them through the region of maximal expansion if they were initially sufficiently sharply peaked with sufficiently well–behaved corresponding classical trajectory. All one needs to do is to follow the general construction presented in Chapter 4 and switch the relational clocks as required. As a result one obtains a system with an expanding branch as the logical predecessor of the re–contracting branch, just as one would in the classical system and in disagreement with the discussion in [8–10, 46, 48]. Note, however, that the recovery of a good temporally local relational evolution is highly state–dependent.

Since non–integrability is the generic feature of chaotic systems [29–31], we propose, that the qualitative results we have obtained in Chapter 5 dealing with the breakdown of the semiclassical approximation and of the relational evolution will be present in quantum cosmology and gravity systems generically,

[1] Moreover, one is not able to employ the conjugate momenta as the clock variables as these are of highly oscillatory nature in general.

and thus, generically, a 'good relational evolution' appears to be only a transient and semiclassical phenomenon.

The natural step forward when pursuing the path of relational evolution in semiclassical approximation could be made in two directions, that is, one could either incorporate higher order quantum corrections (that would mean taking into account moments of order $\hbar^{3/2}$ or even higher) into the solutions of one of the previously treated models, or choose rather more complex model universes (e.g. Bianchi models) in order to introduce more degrees of freedom/anisotropy, first investigated in order $\hbar$ with higher order quantum corrections to follow in a sequel.

The main obstacle to the sensible evolution of non–integrable systems is their sensitivity to initial data. However, as Loop Quantum Cosmology (LQC) is equipped with a minimal length scale that arises as a consequence of the minimal area gap [22–26], one might wonder whether the chaotic behaviour of the FRW model universes could be resolved within LQC on account of this feature and therefore provide a nicer, better–behaved theory.

Appendix

Effective relational dynamics of the closed FRW model in detail

In sections 4.1 and 4.2 we have devised a generalised form of the effective calculation for models governed by the classical Hamiltonian constraint of the form (4.1.1). When one is treating a different model however, the full calculation 'from first principles' may be useful.

Let us start the formulation of the problem with noting the standard metric of an FRW spacetime (5.1.1) and the action of a homogeneous massive scalar field minimally coupled to such a spacetime (5.1.2). After fixing $k = 1$ for the closed model and adopting a dot as a shorthand notation for the 't' derivative, one is left with the action in the form that agrees with [48]

$$S = \frac{1}{2} \int \mathrm{d}t N a^3 \left[-\frac{\dot{a}^2}{N^2 a^2} + \frac{1}{a^2} + \frac{\dot{\phi}}{N^2} - m^2 \phi^2 \right]. \tag{A.1}$$

It is straightforward to arrive at a Hamiltonian function

$$C_H = N \left[\frac{p_\phi^2}{2a^3} - \frac{p_a^2}{2a} - \frac{a}{2} + \frac{1}{2} m^2 \phi^2 a^3 \right], \tag{A.2}$$

where the presence of a lapse function N actually reveals that the C_H is not a true Hamiltonian, but a Hamiltonian constraint. The lapse function can easily be eliminated by a gauge choice at a later time. (Note that we shall not employ the usual choice of equating the lapse function to unity.)

The Hamiltonian constraint features two canonical pairs of variables, (a, p_a) and (ϕ, p_ϕ). For the ease of calculations, we shall perform the transformation $\alpha = \ln a$ and work with a new canonical pair, (α, p_α) from this point forward.

The Hamiltonian constraint acquires the form of

$$C_H = N\left[\frac{1}{e^{3\alpha}}[p_\phi^2 - p_\alpha^2] - e^{\alpha} + m^2\phi^2 e^{3\alpha}\right]. \tag{A.3}$$

To keep the calculations as simple as possible, we shall take the advantage of the prefactor $\frac{1}{e^{3\alpha}}$ that has emerged after the scale factor transformation and absorb it in the gauge choice for the lapse function. That leaves us with the classical constraint

$$C_H = p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha}. \tag{A.4}$$

Next we need to get a quantum version of C. A naive approach would be to promote each term to an operator $\hat{C} = \hat{p}_\phi^2 - \hat{p}_\alpha^2 - \widehat{e^{4\alpha}} + m^2\widehat{\phi^2 e^{6\alpha}}$. What actually needs to be done, is the following: one needs to express the operators through their expectation values and spreads, achieving this in two terms through the Taylor expansion around the expectation values of corresponding operators. We want to pursue the semiclassical approach to the order of $\hbar$, thus we drop terms with spreads of order 2 (these would correspond to terms of $\hbar^{3/2}$) and higher. Also, there will not be an ordering issue so far as α and ϕ commute with each other. The desired constraint is then a sum of the terms of the four Taylor expansions around the expectation values to the order of $\hbar$, namely

$$\begin{aligned} C &= \langle\hat{p}_\phi\rangle^2 + (\Delta p_\phi)^2 - \langle\hat{p}_\alpha\rangle^2 - (\Delta p_\alpha)^2 - e^{4\langle\hat{\alpha}\rangle} - 8e^{4\langle\hat{\alpha}\rangle}(\Delta\alpha)^2 + m^2\langle\phi\rangle^2 e^{6\langle\hat{\alpha}\rangle} \\ &\quad + m^2 e^{6\langle\hat{\alpha}\rangle}(\Delta\phi)^2 + 12m^2\langle\phi\rangle e^{6\langle\hat{\alpha}\rangle}\Delta(\alpha\phi) + 18m^2\langle\phi\rangle^2 e^{6\langle\hat{\alpha}\rangle}(\Delta\alpha)^2. \end{aligned} \tag{A.5}$$

To be able to write down the full set of constraints for the model we need to formulate second–order constraints defined by Eq.(3.2.2):

$$\begin{aligned} C_\alpha &= \langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)\hat{C}\rangle, \\ C_\phi &= \langle(\hat{\phi} - \langle\hat{\phi}\rangle)\hat{C}\rangle, \\ C_{p_\alpha} &= \langle(\hat{p_\alpha} - \langle\hat{p_\alpha}\rangle)\hat{C}\rangle, \\ C_{p_\phi} &= \langle(\hat{p_\phi} - \langle\hat{p_\phi}\rangle)\hat{C}\rangle. \end{aligned} \tag{A.6}$$

Let us demonstrate the procedure of obtaining the second order constraints by means of calculating the C_α constraint explicitly,

$$C_\alpha = \langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)\hat{C}\rangle = \langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)(\hat{p}_\phi^2 - \hat{p}_\alpha^2 - \widehat{e^{4\alpha}} + m^2\widehat{\phi^2 e^{6\alpha}})\rangle. \tag{A.7}$$

We will work with the second term as there the issue of ordering arises:

$$\begin{aligned}
\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)\hat{p}_\alpha^2\rangle &= \langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\langle\hat{p}_\alpha\rangle+(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle))^2\rangle \\
&= \langle\hat{\alpha}-\langle\hat{\alpha}\rangle\rangle\langle\hat{p}_\alpha\rangle^2 \\
&\quad +2\langle\hat{p}_\alpha\rangle\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)\rangle+\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)^2\rangle,
\end{aligned} \tag{A.8}$$

where the first term vanishes on account of $\langle\hat{\alpha}-\langle\hat{\alpha}\rangle\rangle=\langle\hat{\alpha}\rangle-\langle\hat{\alpha}\rangle=0$.

In order to be able to write the resulting expression in terms of moments, the remaining terms need to be ordered symmetrically, which will be made possible by use of the canonical commutation relations. The term becomes

$$\begin{aligned}
\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)\hat{p}_\alpha^2\rangle &= \langle\hat{p}_\alpha\rangle(2\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)\rangle_{Weyl}+i\hbar) \\
&\quad +\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)^2\rangle_{Weyl},
\end{aligned}$$

where

$$\begin{aligned}
\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)^2\rangle_{Weyl} &= \frac{1}{3}\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)^2 \\
&+ (\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle) \\
&+ (\hat{\alpha}-(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)^2\langle\hat{\alpha}\rangle)\rangle
\end{aligned}$$

is a third order moment and will be discarded in the truncation to the order $\hbar$, and

$$\begin{aligned}
\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)\rangle_{Weyl} &= \frac{1}{2}\langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)+(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)(\hat{\alpha}-\langle\hat{\alpha}\rangle)\rangle \\
&= \langle(\hat{\alpha}-\langle\hat{\alpha}\rangle)(\hat{p}_\alpha-\langle\hat{p}_\alpha\rangle)\rangle_{Weyl}+\frac{1}{2}i\hbar.
\end{aligned}$$

Proceeding in this fashion one arrives at the full expression for the constraint C_α

$$\begin{aligned}
C_\alpha &= 2p_\phi\Delta(\alpha p_\phi)-2p_\alpha\Delta(\alpha p_\alpha)-i\hbar\, p_\alpha+2m^2\phi e^{6\alpha}\Delta(\alpha\phi) \\
&\quad +(6m^2\phi^2e^{6\alpha}-4e^{4\alpha})(\Delta\alpha)^2,
\end{aligned} \tag{A.9}$$

where the $\langle\hat{x}\rangle=x$ labeling has been adopted.

Consistent truncation at the order $\hbar$ yields these five non-zero constraints on our system:

$$\begin{aligned}
C &= p_\phi^2 + (\Delta p_\phi)^2 - p_\alpha^2 - (\Delta p_\alpha)^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} \\
&\quad + m^2 e^{6\alpha}(\Delta\phi)^2 + 12m^2\phi e^{6\alpha}\Delta(\alpha\phi) + (18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha})(\Delta\alpha)^2, \\
C_\alpha &= 2p_\phi\Delta(\alpha p_\phi) - 2p_\alpha\Delta(\alpha p_\alpha) - i\hbar\, p_\alpha + 2m^2\phi e^{6\alpha}\Delta(\alpha\phi) \\
&\quad + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})(\Delta\alpha)^2, \\
C_\phi &= 2p_\phi\Delta(\phi p_\phi) + i\hbar\, p_\phi - 2p_\alpha\Delta(\phi p_\alpha) + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha\phi) \\
&\quad + 2m^2\phi e^{6\alpha}(\Delta\phi)^2, \\
C_{p_\alpha} &= 2p_\phi\Delta(p_\alpha p_\phi) - 2p_\alpha(\Delta p_\alpha)^2 + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})(\Delta(\alpha p_\alpha) - \frac{1}{2}i\hbar) \\
&\quad + 2m^2\phi e^{6\alpha}\Delta(\phi p_\alpha), \\
C_{p_\phi} &= 2p_\phi(\Delta p_\phi)^2 - 2p_\alpha\Delta(p_\alpha p_\phi) + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\phi) + 2m^2\phi e^{6\alpha}\Delta(\phi p_\phi) \\
&\quad - i\hbar\, m^2\phi e^{6\alpha}. \qquad \text{(A.10)}
\end{aligned}$$

Evolution in α–*Zeitgeist*

To extract the dynamics of the system in the α–*Zeitgeist* we shall impose the corresponding gauge:

$$\phi_1 = (\Delta\alpha)^2 = 0, \phi_2 = \Delta(\phi\alpha) = 0, \phi_3 = \Delta(\alpha p_\phi) = 0. \qquad \text{(A.11)}$$

Imposing the gauge directly on the set of constraints leads to saturation of the uncertainty condition in the C_α constraint:

$$-2p_\alpha\Delta(\alpha p_\alpha) - i\hbar\, p_\alpha = 0 \qquad \to \qquad \Delta(\alpha p_\alpha) = -\frac{i\hbar}{2},$$

with the rest of the constraints appearing in somewhat simplified form:

$$\begin{aligned}
C &= p_\phi^2 + (\Delta p_\phi)^2 - p_\alpha^2 - (\Delta p_\alpha)^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} + m^2 e^{6\alpha}(\Delta\phi)^2, \\
C_\phi &= 2p_\phi\Delta(\phi p_\phi) + i\hbar\, p_\phi - 2p_\alpha\Delta(\phi p_\alpha) + 2m^2\phi e^{6\alpha}(\Delta\phi)^2, \\
C_{p_\alpha} &= 2p_\phi\Delta(p_\alpha p_\phi) - 2p_\alpha(\Delta p_\alpha)^2 + 2m^2\phi e^{6\alpha}\Delta(\phi p_\alpha) - i\hbar(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha}), \\
C_{p_\phi} &= 2p_\phi(\Delta p_\phi)^2 - 2p_\alpha\Delta(p_\alpha p_\phi) + 2m^2\phi e^{6\alpha}\Delta(\phi p_\phi) - i\hbar\, m^2\phi e^{6\alpha}. \qquad \text{(A.12)}
\end{aligned}$$

The quantum version of the Hamiltonian constraint is a linear combination of the four constraints, namely: $C_H = C + \kappa C_\phi + \beta C_{p_\alpha} + \gamma C_{p_\phi}$. Imposing a condition that the Hamiltonian constraint must commute with the gauge we shall make use of the Poisson algebra table:

	$\boldsymbol{\phi_1}$	$\boldsymbol{\phi_2}$	$\boldsymbol{\phi_3}$
$\mathbf{C}$	$-2i\hbar$	$2\Delta(\phi p_\alpha)$	$-2\Delta(p_\alpha p_\phi)$
$\mathbf{C_{p_\alpha}}$	$-4i\hbar p_\alpha$	$4p_\alpha\Delta(\phi p_\alpha) - 2\Delta(\phi p_\phi)$ $- 2m^2\phi e^{6\alpha}(\Delta\phi)^2 + i\hbar p_\phi$	$4p_\alpha\Delta(p_\alpha p_\phi) - 2p_\phi(\Delta p_\phi)^2$ $- 2m^2\phi e^{6\alpha}(2\Delta(\phi p_\phi) + i\hbar)$
$\mathbf{C_\phi}$	0	$2p_\alpha(\Delta\phi)^2$	$2p_\alpha\Delta(\phi p_\phi)i\hbar p_\alpha$
$\mathbf{C_{p_\phi}}$	0	$2p_\alpha\Delta(\phi p_\phi) - i\hbar p_\alpha$	$2p_\alpha(\Delta p_\phi)^2$

Table 6.1: The Poisson algebra of constraints and gauge conditions in α–*Zeitgeist*

and obtain the coefficients

$$\kappa = -\frac{m^2\phi e^{6\alpha}}{2p_\alpha^2}, \qquad \beta = -\frac{1}{2p_\alpha}, \qquad \gamma = -\frac{p_\phi}{2p_\alpha^2},$$

from which the quantum version of the Hamiltonian constraint that accounts for the relevant moments follows as

$$\begin{aligned} C_H &= p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} + \left[1 - \frac{p_\phi^2}{p_\alpha^2}\right](\Delta p_\phi)^2 - \frac{2m^2\phi e^{6\alpha} p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) \\ &\quad + \left[m^2 e^{6\alpha} - \frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2}\right](\Delta\phi)^2 + i\hbar\,\frac{3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha}}{p_\alpha}. \end{aligned} \tag{A.13}$$

The solutions to unphysical moments, that follow from the constraint equations, read:

$$\begin{aligned} \Delta(\phi p_\alpha) &= \frac{p_\phi}{p_\alpha}\Delta(\phi p_\phi) + \frac{m^2\phi e^{6\alpha}}{p_\alpha}(\Delta\phi)^2 + i\hbar\,\frac{p_\phi}{2p_\alpha}, \\ \Delta(p_\alpha p_\phi) &= \frac{p_\phi}{p_\alpha}(\Delta p_\phi)^2 + \frac{m^2\phi e^{6\alpha}}{p_\alpha}\Delta(\phi p_\phi) - i\hbar\,\frac{m^2\phi e^{6\alpha}}{2p_\alpha}, \\ (\Delta p_\alpha)^2 &= \frac{p_\phi^2}{p_\alpha^2}(\Delta p_\phi)^2 + \frac{2m^2\phi e^{6\alpha} p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) + \frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2}(\Delta\phi)^2 \\ &\quad - i\hbar\,\frac{3m^2\phi e^{6\alpha} - 2e^{4\alpha}}{2p_\alpha}. \end{aligned} \tag{A.14}$$

Corresponding equations of motion are generated through $\dot{x} = \{x, C_H\}$ as:

$$\begin{aligned}
\dot{\alpha} &= -2p_\alpha + \frac{2p_\phi^2}{p_\alpha^3}(\Delta p_\phi)^2 + \frac{4m^2\phi e^{6\alpha}p_\phi}{p_\alpha^3}\Delta(\phi p_\phi) + \frac{2m^4\phi^2 e^{12\alpha}}{p_\alpha^3}(\Delta\phi)^2 \\
&\quad -i\hbar\,\frac{3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha}}{p_\alpha^2}, \\
\dot{p_\alpha} &= 4e^{4\alpha} - 6m^2\phi^2 e^{6\alpha} + \frac{12m^2\phi e^{6\alpha}p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) \\
&\quad - \left[6m^2 e^{6\alpha} - \frac{12m^4\phi^2 e^{12\alpha}}{p_\alpha^2}\right](\Delta\phi)^2 - i\hbar\,\frac{18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha}}{p_\alpha}, \\
\dot{\phi} &= 2p_\phi - \frac{2p_\phi}{p_\alpha^2}(\Delta p_\phi)^2 - \frac{2m^2\phi e^{6\alpha}}{p_\alpha^2}\Delta(\phi p_\phi), \\
\dot{p_\phi} &= -2m^2\phi e^{6\alpha} + \frac{2m^2 e^{6\alpha}p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) + \frac{2m^4\phi e^{12\alpha}}{p_\alpha^2}(\Delta\phi)^2 - i\hbar\,\frac{6m^2\phi e^{6\alpha}}{p_\alpha}, \\
(\Delta\dot{\phi})^2 &= 4\left[1 - \frac{p_\phi^2}{p_\alpha^2}\right]\Delta(\phi p_\phi) - \frac{4m^2\phi e^{6\alpha}p_\phi}{p_\alpha^2}(\Delta\phi)^2, \\
\Delta(\dot{\phi p_\phi}) &= 2\left[1 - \frac{p_\phi^2}{p_\alpha^2}\right](\Delta p_\phi)^2 + 2\left[\frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2} - m^2 e^{6\alpha}\right](\Delta\phi)^2, \\
(\Delta\dot{p_\phi})^2 &= \frac{4m^2\phi e^{6\alpha}p_\phi}{p_\alpha^2}(\Delta p_\phi)^2 + 4\left[-m^2 e^{6\alpha} + \frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2}\right]\Delta(\phi p_\phi). \qquad \text{(A.15)}
\end{aligned}$$

The transient validity of the α–*Zeitgeist* on account of α being a non–global clock will result in the breakdown of the relational evolution in the turning region(s) of α, thus we shall choose a new clock variable and evolve the system in the corresponding, here ϕ–, *Zeitgeist*, that will allow us to evolve through the aforementioned turning region(s).

Evolution in ϕ–*Zeitgeist*

Again, imposing the gauge

$$\phi_1 = (\Delta\phi)^2 = 0, \phi_2 = \Delta(\alpha\phi) = 0, \phi_3 = \Delta(\phi p_\alpha) = 0 \qquad \text{(A.16)}$$

on the set of constraints leaves us with the uncertainty relation

$$-2p_\phi\Delta(\phi p_\phi) - i\hbar\, p_\phi = 0 \qquad \rightarrow \qquad \Delta(\phi p_\phi) = -\frac{i\hbar}{2},$$

and the set of four simplified constraints

$$
\begin{aligned}
C &= p_\phi^2 + (\Delta p_\phi)^2 - p_\alpha^2 - (\Delta p_\alpha)^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} \\
&\quad +(18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha})(\Delta\alpha)^2, \\
C_\alpha &= 2p_\phi\Delta(\alpha p_\phi) - 2p_\alpha\Delta(\alpha p_\alpha) - i\hbar\, p_\alpha + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})(\Delta\alpha)^2, \\
C_{p_\alpha} &= 2p_\phi\Delta(p_\alpha p_\phi) - 2p_\alpha(\Delta p_\alpha)^2 + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\left(\Delta(\alpha p_\alpha) - \frac{i\hbar}{2}\right) \\
C_{p_\phi} &= 2p_\phi(\Delta p_\phi)^2 - 2p_\alpha\Delta(p_\alpha p_\phi) + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\phi) \\
&\quad -2i\hbar\, m^2\phi e^{6\alpha}, \qquad \text{(A.17)}
\end{aligned}
$$

that combined such that $C + \kappa C_\alpha + \beta C_{p_\alpha} + \gamma C_{p_\phi}$ yield a quantum version of the Hamiltonian constraint C_H.

Employing the commutation relations of the Hamiltonian constraint with the gauge

	$\boldsymbol{\phi_1}$	$\boldsymbol{\phi_2}$	$\boldsymbol{\phi_3}$
$\mathbf{C}$	$2i\hbar$	$-2\Delta(\alpha p_\phi)$	$-2\Delta(p_\phi p_\alpha)$
$\mathbf{C}_{\mathbf{p}_\phi}$	$4i\hbar p_\phi$	$-4p_\phi\Delta(\alpha p_\phi)+2\Delta(\alpha p_\alpha)-i\hbar p_\alpha$ $+ (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})(\Delta\alpha)^2$	$-4p_\phi\Delta(p_\phi p_\alpha) + 2p_\alpha(\Delta p_\alpha)^2$ $-(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})(\Delta(\alpha p_\alpha)+\frac{i\hbar}{2})$
$\mathbf{C}_\alpha$	0	$-2p_\phi(\Delta\alpha)^2$	$-2p_\phi\Delta(\alpha p_\alpha) - i\hbar p_\phi$
$\mathbf{C}_{\mathbf{p}_\alpha}$	0	$-2p_\phi\Delta(\alpha p_\alpha) + i\hbar p_\phi$	$-2p_\phi(\Delta p_\alpha)^2$

Table 6.2: The Poisson algebra of constraints and gauge conditions in ϕ–*Zeitgeist*

one arrives at the coefficients:

$$
\kappa = \frac{3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha}}{2p_\phi^2}, \qquad \beta = -\frac{p_\alpha}{2p_\phi^2}, \qquad \gamma = -\frac{1}{2p_\phi},
$$

giving the quantum Hamiltonian constraint the form of

$$
\begin{aligned}
C_H &= p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} - \left[1 - \frac{p_\alpha^2}{p_\phi^2}\right](\Delta p_\alpha)^2 \\
&\quad -\frac{p_\alpha}{p_\phi^2}(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\alpha) + i\hbar\,\frac{m^2\phi e^{6\alpha}}{p_\phi} \\
&\quad + \left[18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha} + \frac{(3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha})^2}{p_\phi^2}\right](\Delta\alpha)^2. \qquad \text{(A.18)}
\end{aligned}
$$

Equations of motion follow from the Poisson structure

$$\begin{aligned}
\dot{\phi} &= 2p_\phi - \frac{2p_\alpha^2}{p_\phi^3}(\Delta p_\alpha)^2 + \frac{p_\alpha}{p_\phi^3}\left(12m^2\phi^2 e^{6\alpha} - 8e^{4\alpha}\right)\Delta(\alpha p_\alpha) \\
&\quad - \frac{(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})^2}{p_\phi^3}(\Delta\alpha)^2 - i\hbar\,\frac{m^2\phi e^{6\alpha}}{p_\phi^2}, \\
\dot{p_\phi} &= -2m^2\phi e^{6\alpha} + \frac{12p_\alpha}{p_\phi^2}m^2\phi e^{6\alpha}\Delta(\alpha p_\alpha) \\
&\quad - \left[36m^2\phi e^{6\alpha} + \frac{12m^2\phi e^{6\alpha}(3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha})}{p_\phi^2}\right](\Delta\alpha)^2 - i\hbar\,\frac{m^2 e^{6\alpha}}{p_\phi}, \\
\dot{\alpha} &= -2p_\alpha + \frac{2p_\alpha}{p_\phi^2}(\Delta p_\alpha)^2 - \frac{6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha}}{p_\phi^2}\Delta(\alpha p_\alpha), \\
\dot{p_\alpha} &= 4e^{4\alpha} - 6m^2\phi^2 e^{6\alpha} + \frac{p_\alpha}{p_\phi^2}(36m^2\phi^2 e^{6\alpha} - 16e^{4\alpha})\Delta(\alpha p_\alpha) - i\hbar\,\frac{6m^2\phi e^{6\alpha}}{p_\phi} \\
&\quad - \Bigg[108m^2\phi^2 e^{6\alpha} - 32e^{4\alpha} \\
&\qquad + \frac{(18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha})(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})}{p_\phi^2}\Bigg](\Delta\alpha)^2, \\
\dot{(\Delta\alpha)^2} &= -4\left[1 - \frac{p_\alpha^2}{p_\phi^2}\right]\Delta(\alpha p_\alpha) - \frac{p_\alpha}{p_\phi^2}(12m^2\phi^2 e^{6\alpha} - 8e^{4\alpha})(\Delta\alpha)^2, \\
\dot{\Delta(\alpha p_\alpha)} &= -2\left[1 - \frac{p_\alpha^2}{p_\phi^2}\right](\Delta p_\alpha)^2 \\
&\quad -2\left[18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha} + \frac{(3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha})^2}{p_\phi^2}\right](\Delta\alpha)^2, \\
\dot{(\Delta p_\alpha)^2} &= \frac{p_\alpha}{p_\phi^2}(12m^2\phi^2 e^{6\alpha} - 8e^{4\alpha})(\Delta p_\alpha)^2 \\
&\quad -4\left[18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha} + \frac{(3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha})^2}{p_\phi^2}\right]\Delta(\alpha p_\alpha). \qquad \text{(A.19)}
\end{aligned}$$

To complete the treatment of the system, one needs to make use of the constraint equations to retrieve the solutions to the unphysical moments,

$$\begin{aligned}
\Delta(\alpha p_\phi) &= \frac{p_\alpha}{p_\phi}\Delta(\alpha p_\alpha) - \frac{3m^2\phi^2e^{6\alpha} - 2e^{4\alpha}}{p_\phi}(\Delta\alpha)^2 + i\hbar\,\frac{p_\alpha}{2p_\phi}, \\
\Delta(p_\alpha p_\phi) &= \frac{p_\alpha}{p_\phi}(\Delta p_\alpha)^2 - \frac{3m^2\phi^2e^{3\alpha} - 2e^{4\alpha}}{p_\phi}\Delta(\alpha p_\alpha) + i\hbar\,\frac{3m^2\phi^2e^{6\alpha} - 2e^{4\alpha}}{2p_\phi}, \\
(\Delta p_\phi)^2 &= \frac{p_\alpha^2}{p_\phi^2}(\Delta p_\alpha)^2 - \frac{p_\alpha}{p_\phi^2}(6m^2\phi^2e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\alpha) \\
&\quad + \frac{(3m^2\phi^2e^{6\alpha} - 2e^{4\alpha})^2}{p_\phi^2}(\Delta\alpha)^2 + i\hbar\,\frac{m^2\phi e^{6\alpha}}{p_\phi}. \qquad \text{(A.20)}
\end{aligned}$$

Cookbook

The recipe for an effective semiclassical relational evolution of a quantum system

A systematic method for switching between different clocks in semiclassical regime is now available for investigating the evolution of more interesting, non-integrable systems. Using this method, it is possible to obtain semiclassical solutions which follow classical trajectory if the state is initially sufficiently sharply peaked and corresponding classical trajectory is sufficiently well–behaved. In generic case, effective relational dynamics breaks down in the region of maximal expansion on account of a wealth of structure on all scales of the chaotic model and no change of clock can remedy this. Generically, a good relational evolution appears to be only a transient and semiclassical phenomenon. Nevertheless, the results for the well–behaved trajectory are still valuable as they are the first step beyond a classical treatise and we therefore wish to summarise the main points of the effective treatment in the following 'recipe':

1. Perform effective analogue of Dirac quantisation:
   impose $C_{pol} = \langle \widehat{pol\hat{C}} \rangle = 0$ for all polynomials in basic variables.
2. Truncate the system at your favourite order N of $\hbar^N$.
3. Perform classical type constraint analysis.
4. Partially fix gauge freedom: choose your *Zeitgeist*.
5. Interpret remaining quantum flow as the dynamics of the system with respect to the chosen clock. (The clock picks up imaginary contribution as is required for the consistency and reality of evolving variables.)
6. Pathologies of non-global clocks near their turning points can be avoided by systematic translation between different *Zeitgeister*.
7. Semiclassical evolution is patched by different 'physical coordinate systems'.

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Publications related to this thesis

Part of the material presented in this thesis has appeared in the following publication:

- P. A. Höhn, E. Kubalová and A. Tsobanjan, "Effective relational dynamics of a nonintegrable cosmological model," Phys. Rev. **D 86** 065014 (2012), arXiv:1111.5193[gr-qc] (Chapters 4-6)

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