Study of the Density Parameters of the Dark Energy and the Cosmological Constant in the context of Brans-Dicke Theory

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Abstract. The discovery of supernova projects at the end of the 20^{th} century altered the theory of current cosmology and sparked the hunt for dark energy and dark matter. The time evolution of the density parameter is examined here by considering a test function. Using the Brans-Dicke frame work, the calculations are advanced. The density parameter of dark energy (Ω_D) and matter (Ω_m) are connected by mathematical descriptions and found to produce a unit value when Ω_D and Ω_m are in fraction. As the calculations are built on the structure of the scale factor, this present article is attempt to describe the framework in terms of scale factor (a) as well. The results coming from the test function model show nice resemblance with known observations. The use of Brans-Dicke theory makes provision for in detail study of Ω_D and Ω_m.
Keywords: Cosmological constant, density parameter, time evolution, dark energy.

1 Introduction

Discovery of the accelerated expansion of the universe and dark energy unveils the possible options to express the cosmological models of the universe. Those models or the endeavours involve the density parameters of the dark energy for the description of the necessary constructions [1, 2, 3].
To start working with the Friedmann–Robertson–Walker (FRW) cosmology, Einsteins' field equations are the primary steps [1, 2, 3]. Cosmological constant (ΛCDM) model is basically one of such model [1, 2, 3]. But due to several constraints (e.g. fine-tuning, etc.), the candidate of Λ becomes weaker [2, 3]. In such cases scalar field models are the next immediate choices. Quintessence models are also scalar field models and differnt potentials based v(ϕ) such models are there among the speculated dark energy models [3, 4, 5]. In this article we have chosen a scalar field (ϕ) to form dark energy model using an alternative approach of the Brans-Dicke theory.
According to Mach's principle the inertial forces observed locally in an accelerated laboratory may be interpreted as gravitational effects having their origin in distant matter accelerated relative to the laboratory [6, 7, 8]. The role and relevance of Mach's principle is associated with the equivalence principle of General theory of relativity. In other words, the concept of inertia must be rooted in acceleration with respect to the general mass distribution of the universe. One must not consider inertial masses of various elementary particles as fundamental constants. However, it can be considered to represent the particle's interaction with the cosmic field.
Thus to offer an alternate approach, Brans-Dicke theory depicts a scalar field (say ϕ) when describing the gravitation G. According to Brans-Dicke (BD) theory one may obtain the perfect field equations by replacing G by $\frac{1}{\phi}$ as the field $\langle\phi\rangle \simeq \frac{1}{G}$ [9].

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Einstein field equation is expressed as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} \tag{1}$$

With the Brans-Dicke consideration the field equation takes the form as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi}{\phi}({T_M}^{\mu\nu} + {T_\phi}^{\mu\nu}) \tag{2}$$

In the present article this scalar field is used with special form to test a defined model [10], [11] with new function.

2 Metric and field equations

In generalized Brans-Dicke's theory the action parameter is given by -

$$\mathcal{A} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R + \frac{\omega(\phi)}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mathcal{L}_m\right) \tag{3}$$

In the given expression, g represents the determinant of the metric tensor $g^{\mu\nu}$, $\mathcal{L}_m$ is the Lagrangian for matter, R is the Ricci scalar, ϕ is the Brans-Dicke's scalar field, and ω is a dimensionless parameter that depends on the scalar field (ϕ) in the generalized Brans-Dicke's theory. In a uniform and isotropic universe, the line element appears as follows:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2sin^2\theta d\xi^2\right] \tag{4}$$

In this context, $a(t)$ represents the scale factor, k denotes the spatial curvature parameter, and t is cosmic time. The value of k determines whether the universe is closed ($k = 1$), flat ($k = 0$), or open ($k = -1$). For the purposes of this work, all calculations has been done considering the spatial curvature parameter ($k = 0$).
In a uniform and isotropic space-time, as described by equation (4), the BD theory's field equations for a cosmos containing a perfect fluid are represented by the following three equations.

$$3\frac{\dot{a}^2 + k}{a^2} + 3\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \frac{\omega(\phi)}{2}\frac{\dot{\phi}^2}{\phi^2} = \frac{\rho}{\phi} \tag{5}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} + \frac{\omega(\phi)}{2}\frac{\dot{\phi}^2}{\phi^2} + 2\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} + \frac{\ddot{\phi}}{\phi} = -\frac{\gamma\rho}{\phi} \tag{6}$$

$$\frac{\ddot{\phi}}{\phi} + 3\frac{\dot{a}\dot{\phi}}{a\phi} = \frac{\rho}{\phi}\frac{(1-3\gamma)}{2\omega+3} - \frac{\dot{\phi}}{\phi}\frac{\dot{\omega}}{2\omega+3} \tag{7}$$

The value of the parameter for EOS of the cosmic fluid can be expressed by γ ($\equiv \frac{p}{\rho}$) in these field equations.The dimension of $\frac{p}{\rho}$ is known as Equation Of State. Hence γ behaves like and EOS. So progressive with the article we can mention γ as EOS. In this work, we investigate this parameter as a function of time.

3 Theoretical framework

3.1 Theoretical model

On combining the equations (5), (6) and (7), we obtained an expression for K=0 as,

$$\dot{\omega} + \left(2\frac{\ddot{\phi}}{\dot{\phi}} + 6\frac{\dot{a}}{a} - \frac{\dot{\phi}}{\phi}\right)\omega - 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)\frac{\phi}{\dot{\phi}} = 0 \tag{8}$$

The following assumptions about ϕ and ω are made in this investigation. We have suggested that these parameters exhibit power-law behavior with respect to ϕ and a.

$$\phi = \phi_0\left(\frac{a}{a_0}\right)^n \qquad\qquad \omega = \omega_0\left(\frac{\phi}{\phi_0}\right)^m \tag{9}$$

Now, on using the equations (9) in equation (8), We obtain the equation

$$m\omega n + (n + 4 - 2q)\omega - \frac{6}{n}(1 - q) = 0 \tag{10}$$

Here, is the deceleration parameter (q$\equiv -\frac{\ddot{a}a}{\dot{a}^2}$). At $t = t_0$ writing $\omega = \omega_0$ and q=q_0, we acquire,

$$m = \frac{6(1 - q_0)}{n^2\omega_0} - \frac{n + 4 - 2q_0}{n} \tag{11}$$

Now, on combining equations (5) and equation (9) for zero spatial curvature, one obtain,

$$\rho = \phi H^2 \left(3 + 3n - \frac{\omega}{2}n^2\right) \tag{12}$$

In this context, the Hubble parameter is $H \equiv \frac{\dot{a}}{a}$. Once we replace all of the parameters in equation (12), utilizing their values at $t = t_0$, we derive:

$$\omega_0 = \frac{2}{n^2H_0^2}\left(3H_0^2 + 3nH_0^2 - \frac{\rho_0}{\phi_0}\right) \tag{13}$$

On using equation (13) in equation (11) one gets,

$$m = \frac{3(1 - q_0)H_0^2}{3H_0^2 + 3nH_0^2 - \frac{\rho_0}{\phi_0}} - \frac{n + 4 - 2q_0}{n} \tag{14}$$

For zero spatial curvature, we derive the following by stating $\frac{\ddot{a}}{a} = -qH^2$ and replacing the equation (9) and (12) into equation (6).

$$\gamma = \frac{2q - 1 - 0.5\omega n^2 - n - n^2 + nq}{3 + 3n - 0.5\omega n^2} \tag{15}$$

As a result, the current EOS value has been given by:

$$\gamma_0 = \frac{2q_0 - 1 - 0.5\omega_0 n^2 - n - n^2 + nq_0}{3 + 3n - 0.5\omega_0 n^2} \tag{16}$$

The values of m and ω_0 from equations (13) and (14) respectively must be extracted before using the notation ω given in equation (9).
In this model, the time evolution of various cosmological parameters (ϕ, ω, ρ, γ) has been investigated by employing an empirical expression for the scale factor.

$$a = a_0 Exp\left[\alpha\left\{\left(\frac{t}{t_0}\right)^{\beta} - 1\right\}\right] \tag{17}$$

In order to produce a deceleration parameter that gradually shifts from positive to negative, the scale factor was deliberately set. This transition marks a shift in the cosmos from a phase of decelerating expansion to one of accelerating expansion, consistent with findings from numerous recent astrophysical observations [12, 13]. For our present study, we have selected $a_0 = 1$ for all calculations. Here, it is crucial for both parameters α and β to have the same sign, ensuring that the scale factor increases with time. Equation (17) could potentially be leveraged for determining the deceleration parameter (q) and the Hubble parameter (H):

$$H = \frac{\dot{a}}{a} = \frac{\alpha\beta}{t_0}\left(\frac{t}{t_0}\right)^{\beta-1} \qquad q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1-\beta}{\alpha\beta}\left(\frac{t}{t_0}\right)^{-\beta} - 1 \tag{18}$$

For $1 > \beta > 0$ and $\alpha > 0$, we observe that $q \to +\infty$ as $t \to 0$ and $q \to -1$ as $t \to \infty$, indicating a clear change in the sign of the parameter (q) over time. At $t = t_0$. The values of α and β may be found as follows when $H = H_0$ and $q = q_0$ are known:

$$\alpha = \frac{H_0 t_0}{1 - H_0 t_0(1 + q_0)} \qquad \beta = 1 - H_0 t_0(1 + q_0) \tag{19}$$

Finding the progression of a cosmological characteristic in relation to the redshift parameter (z) is frequently required where $z = \frac{a_0}{a}$-1. Now, by utilizing equation (17), one can derive the relationship between the redshift parameter and time as:

$$Z = \frac{a_0}{a} - 1 = Exp\left[-\alpha\{\left(\frac{t}{t_0}\right)^{\beta} - 1\}\right] - 1 \tag{20}$$

We adopted the aforementioned value for different cosmological parameters throughout our present study:

$$H_0 = \frac{72\frac{Km}{s}}{Mpc} = 2.33 \times 10^{-18} sec^{-1}. \quad q_0 = -0.55, \quad \Omega_{D0} = 0.7, \quad \rho_0 = 9.9 \times 10^{-27} Kgm^{-3}.$$

$$\phi_0 = \frac{1}{G_0} = 1.498 \times 10^{10} Kgs^2 m{-}3, \quad t_0 = 1.4 \times 10^{10} Years = 4.42 \times 10^{17} s$$

3.2 The density parameter evolution methodology
Since the matter component, which includes both dark matter and baryonic matter, is recognized as pressureless dust according to [14, 15], dark energy is primarily responsible for the universe's cumulative pressure$P$. Thus, a hypothetical illustration of the relationship may attainable:

$$p = \gamma\rho = \gamma_D \rho_D \tag{21}$$

Here,ρ_D is the density of dark energy and γ_D is the equation of state of the dark energy. Using equation (21), Ω_D can be expressed as

$$\Omega_D = \frac{\rho_D}{\rho} = \frac{\gamma}{\gamma_D} \tag{22}$$

Then the subsequent empirical formulation for the EOS parameter of dark energy, can take the form,

$$\gamma_D = \gamma f(t) \tag{23}$$

Where, $f(t)$ is a time-varying function, and we have chosen two empirical expressions for this. This choice was influenced by the observed parallelism within the cosmos total matter-energy content [16] and the chronological progression of the EOS parameters for dark energy [17, 18, 19]. Thus, $f(t)$ is presented as an analogy between γ_D and γ.

3.3 Test function
Regarding the test function, we suggest employing this form of f(t) to determine the value of the density parameter.

$$f(t) = \frac{A}{2}\left(\frac{t}{t_0}\right)^{2\mu} \tag{24}$$

Here, A and μ are dimensionless constants. Now, by using equation (22), (23) and (24) we obtained,

$$\Omega_D = \frac{\rho_D}{\rho} = \frac{\gamma}{\gamma_D} = 2A^{-1}\left(\frac{t}{t_0}\right)^{-2\mu} \tag{25}$$

Substituting $\Omega_D = \Omega_{D0}$ at the current time, i.e., $t = 0$, into equation (25), we obtained:

$$A = \frac{2}{\Omega_{D0}} \tag{26}$$

Now, by using equation (25) and (26) we obtained,

$$\Omega_D = \Omega_{D0}\left(\frac{t}{t_0}\right)^{-2\mu} \tag{27}$$

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By using equation (27), Ω_m is given by,

$$\Omega_m = \frac{\rho_m}{\rho} = \frac{\rho - \rho_D}{\rho} = 1 - \Omega_{D0} \left(\frac{t}{t_0}\right)^{-2\mu} \tag{28}$$

Utilizing equations (27) and (28), we can express ρ_D and ρ_m respectively:

$$\rho_D = \rho\Omega_D = \rho\Omega_{D0} \left(\frac{t}{t_0}\right)^{-2\mu} \tag{29}$$

$$\rho_m = \rho\Omega_m = \rho \left[1 - \Omega_{D0} \left(\frac{t}{t_0}\right)^{-2\mu}\right] \tag{30}$$

On using equations (24) and (26) in (23), γ_D becomes,

$$\gamma_D = \frac{\gamma}{\Omega_D} = \frac{\gamma}{\Omega_{D0}} \left(\frac{t}{t_0}\right)^{2\mu} \tag{31}$$

Using equation (20) in (27) and (28), the formulas for Ω_D and Ω_m can be expressed using the z as follows,

$$\Omega_D = \Omega_{D0} \left(1 + \frac{1}{\alpha} ln \frac{1}{z+1}\right)^{-\frac{2\mu}{\beta}} \tag{32}$$

$$\Omega_m = 1 - \Omega_{D0} \left(1 + \frac{1}{\alpha} ln \frac{1}{z+1}\right)^{-\frac{2\mu}{\beta}} \tag{33}$$

According to various astrophysical observations [20, 21], the current value of Ω_{D0} is approximately 0.7. During the evolution of the universe's density parameters, there was a recent period when Ω_D and Ω_m approached a specific value of 0.5, with the corresponding z falling within a certain range of $0 < z < 1$, based on results by current studies [22, 23].
Recent observations reveal that the universe transformed trajectory at $z = 0.9818$, or around 7.2371×10^9 years ago, evolving from an era of slower expansion towards its present faster growth [21]. We acquire the subsequent formula for μ based on the formulations (32) and (33), in which z_c indicates the redshift parameter that existed when the entire cosmos reached $\Omega_m = \Omega_D$.

$$\mu = \frac{\beta ln(2\Omega_{D0})}{2ln\left[1 + \frac{1}{\alpha} ln\left(\frac{1}{z_c+1}\right)\right]} \tag{34}$$

Now, utilizing mathematical equations (9), (17), (27), and (28), Ω_D and Ω_m could potentially be interpreted as functions corresponding to ϕ.

$$\Omega_D = \Omega_{D0} \left(1 + \frac{1}{n\alpha} ln \frac{\phi}{\phi_0}\right)^{-\frac{2\mu}{\beta}} \tag{35}$$

$$\Omega_m = 1 - \Omega_{D0} \left(1 + \frac{1}{n\alpha} ln \frac{\phi}{\phi_0}\right)^{-\frac{2\mu}{\beta}} \tag{36}$$

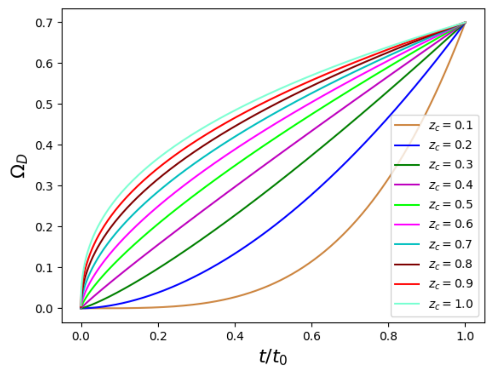

Figure 1: Illustration showing Ω_D vs time for ten distinguished redshift parameter (z_c) values

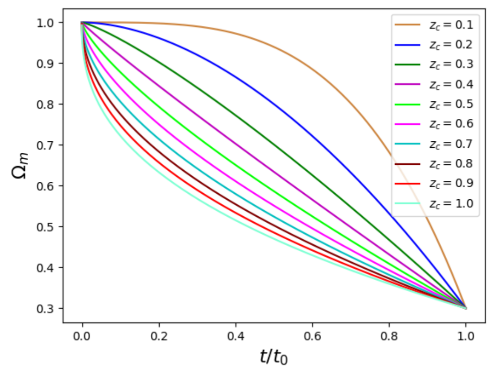

Figure 2: Illustration showing Ω_m vs time for ten distinguished redshift parameter (z_c) values

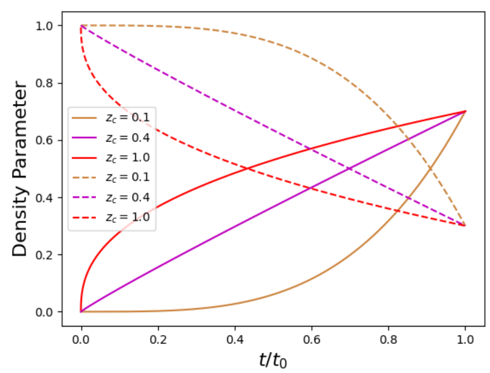

Figure 3: Illustration Ω_D and Ω_m denoted by solid line and dashed line respectively for three different values of the parameters (z_c) as a function over time.

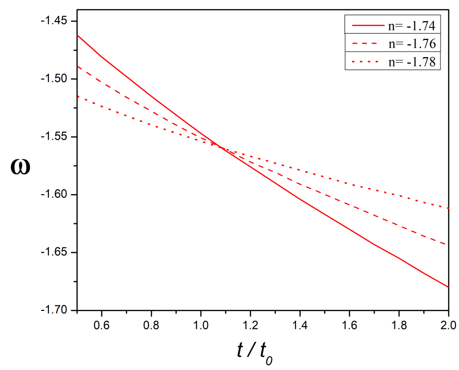

Figure 4: Brans-Dicke parameter against time plots for three different values of parameters n.

4 Selection of values for the parameter n

The temporal variations of ϕ are controlled by the quantity n. Given that both m and ω_0 rely on n, variations in ω, ρ, and γ are affected by n. The G can be expressed using ϕ [14, 19]. Therefore, employing equations (9) and (17), we acquire:

$$G = \frac{1}{\phi} = \frac{1}{\phi_0}\left(\frac{a}{a_0}\right)^{-n} = \frac{1}{\phi_0}Exp\left[-n\alpha\left\{\left(\frac{t}{t_0}\right)^{\beta} - 1\right\}\right] \tag{37}$$

According to certain research findings, the gravitational constant experiences a rise over time [24, 25, 26]. Therefore, G would result in a function that rises with time for negative values of the parameter n. Now, using equation (2) in equation (37), we obtained,

$$\frac{\dot{G}}{G} = -n\frac{\dot{a}}{a} = -nH = -n\frac{\alpha\beta}{t_0}\left(\frac{t}{t_0}\right)^{\beta-1} \tag{38}$$

For present time i.e., t=t_0 the expression for the equation (38) can be expressed as,

$$n = -\frac{t_0}{\alpha\beta}\left(\frac{\dot{G}}{G}\right)_{t=t_0} \tag{39}$$

Using equation (39), it becomes feasible to deduce the parameter n from experimental data regarding $\left(\frac{\dot{G}}{G}\right)_{t=t_0}$. As indicated in a study, the maximum achievable value of $\left|\frac{\dot{G}}{G}\right|_{t=t_0}$ is 4×10^{-10} per year[9]. Now, using equation (39) one gets the value of n $\sim$ -5.43 for this upper limit.

- **Table -1**

n	ω_0	m	γ_0	$\gamma_{D0}(z_c = 0.1)$	$(\frac{\dot{G}}{G})_{t=t_0}(Yr^{-1})$
-1.66	-1.525	-0.140	-1.486	-2.122	1.221×10^{-10}
-1.68	-1.532	-0.115	-1.287	-1.839	1.235×10^{-10}
-1.70	-1.538	-0.093	-1.095	-1.564	1.250×10^{-10}
-1.72	-1.543	-0.073	-0.909	-1.299	1.265×10^{-10}
-1.74	-1.547	-0.055	-0.730	-1.043	1.279×10^{-10}
-1.76	-1.551	-0.038	-0.557	-0.796	1.294×10^{-10}
-1.78	-1.554	-0.024	-0.392	-0.559	1.309×10^{-10}
-1.80	-1.557	-0.011	-0.232	-0.332	1.324×10^{-10}

5 Conclusion

Using the concept of Brans-Dicke theory with a chosen function, corresponding coupling parameter Ω_D and Ω_m are derived.

Ω_D increases with t for increasing restshift and Ω_m decreases with t for increasing redshift that are presented in figure(1) and figure(2). And figure(3) provides a combine result from figure(1) and figure(2).

Treating $\frac{\dot{G}}{G}$ using the Weinberg's prescription, the upper limit of parameter n is moderated by this work from **-5.5** [10] to **-5.43**. This n properly provides the result of $\frac{\dot{G}}{G}$.

The fluctuation of the BD parameter ω over time, taking into account various values of the parameter n is shown in the graphical representation (Figure:4). Only negative $\boldsymbol{n}$ values are under consideration here during the plot. And for the larger negative $\boldsymbol{n}$ values there is a slower rate of decrease. The trend of negative ω with respect to time is consistent with findings from the BD theory [27, 28].

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