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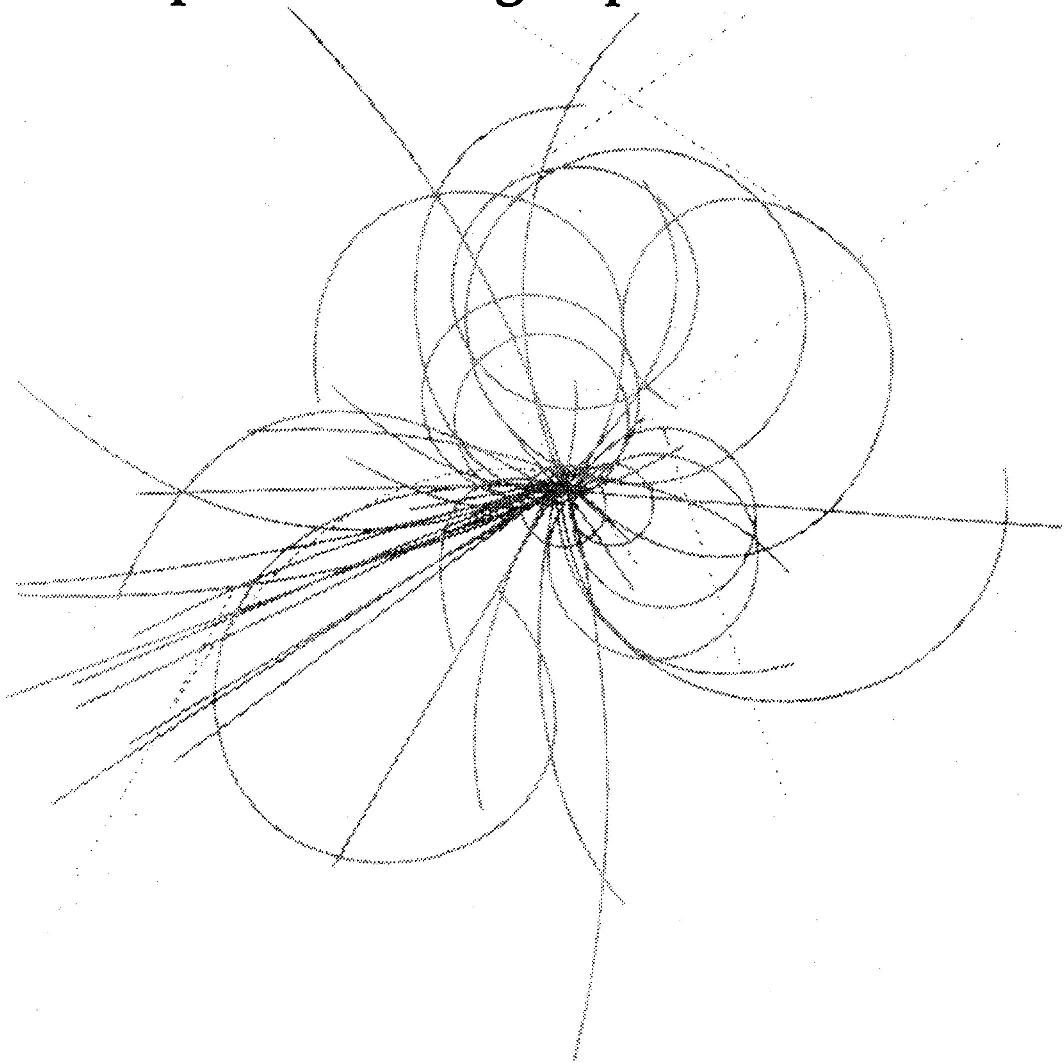

Recent Efforts on Nonlinear Dynamics

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*Presented at XVth International Conference on High Energy Accelerators, July 20–24, 1992, Hamburg Germany.

†Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

RECENT EFFORTS ON NONLINEAR DYNAMICS

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ABSTRACT

This is a brief survey of the recent efforts made during the past few years on the subject of nonlinear dynamics. A list of contributions on this subject, including some on nonlinear spin dynamics, to this conference is given in Refs. [1-13, 47-54].

1 INTRODUCTION

The motion of a charged particle in a circular accelerator is typically predominantly linear in the sense that it can be rather accurately described by the Courant-Snyder analysis [14]. With increasingly demanding requirements on accelerator performance, however, the weak and subtle nonlinear effects become important, and the subject of nonlinear dynamics becomes a focus of study efforts.

In the 1960s and 1970s, nonlinear dynamics are treated using a by-now-traditional approach based on the canonical perturbation theory and its equivalents. [15] Concepts such as tune shifts and resonance widths were introduced and calculated, and important advances were made. However, these approaches do not extend easily to higher orders in the nonlinear perturbation. This was addressed by the Lie algebra technique, introduced for accelerator applications in the early 1980s by Dragt. [16] The Lie algebra technique provides a framework in nonlinear dynamics which is a natural generalization of the Courant-Snyder analysis for the linear dynamics. The concept of Lie maps, for example, is a generalization of the Courant-Snyder matrix, and the concept of normal forms parallels the Courant-Snyder transformation and the Courant-Snyder invariant. Armed with this analysis tool, the question is then how to generate the Lie maps efficiently to sufficiently high orders to represent the accelerator.

This question was resolved in the second half of the 1980s when the differential algebra (DA) was introduced to the field by Berz. [17] DA is a technique to obtain Taylor expansion expressions of any output quantity in terms of the input parameters once an algorithm relating the outputs and the inputs is given. The DA and the Lie algebra framework constitute a powerful combination of tools, based on which a proliferation of ideas and techniques are being developed as we speak.

Equally important to the advance in nonlinear dynamics is the fact that the computing power has been increasing exponentially. This has much extended the numerical capacity which allowed new simulation and analysis techniques to be explored.

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Nonlinear dynamics is currently an active topic in accelerator physics. This survey hopes to catch a glimpse of these (mostly on-going) developments. The following categories of the recent efforts come to mind:

(a) Long term stability One of the most challenging problems in nonlinear dynamics is to predict the long term stability of particle motion. Since it is unpractical to perform long-term tracking simulations for all cases of interest, methods to extract some weak signals from shorter-term tracking results to serve as early warning signs of possible long-term particle loss becomes a topic of importance. (The fact that this is not an easy task can be appreciated by noting that, when a long-term loss occurs in a tracking simulation, the amplitude of the particle often behaves normally for a long time until the very last 20 turns of its lifetime.) Recent efforts in this direction include searching for :

- the Lyapunov exponent, [18, 19]
- time evolution of a Lie invariant, [20]
- a mathematical bound in canonical transformation. [21]

(b) Maps Another way to deal with the long-term problem is to search for a one-turn map that faithfully represents the accelerator. Once found, it can be used for long-term tracking because it takes much less computer time to track a map then to track all the accelerator elements. One challenge, then, is how to write a computer code that best meets specific accelerator physics goals, is efficient in manipulating a large number of indices for the maps and is most user friendly. This effort has caught the interest of computer experts and accelerator physicists alike. [22]

As to the application of maps, their application to the traditional analysis (tune shifts, resonance widths, etc.) is not in question. Given a high-order map, these traditional quantities can be calculated to high orders. Another notable application is the search of high order achromat designs. [23] More of debate is whether these maps also can be trusted for long term tracking.

A straightforward application of DA yields an expression of $\mathbf{X_f}$ as an N-th order Taylor series of $\mathbf{X_i}$ for a pre-chosen N, where $\mathbf{X_i}$ and $\mathbf{X_f}$ are the initial and final values of the phase space variables $(x, x', y, y', z, \delta)$ for one turn around the accelerator. However, the Taylor map is symplectic only up to order N. The search for alternative symplectic representation of maps has been a focus of attention. Various proposals were made:

- generating function [24],
- kick maps [25],
- interpolation maps [26],
- monomial maps [2],
- dynamic rescaling [4].

Once a symplectic map is found, the hope is that it can be tracked for long-term purposes. It remains to be seen whether this hope can be fulfilled.

(c) Nonlinear dynamics experiments Testing the theories in existing accelerators is critical in confirming the theories. Systematic nonlinear dynamics experiments have been performed since the mid-1980's and this trend continued in recent years with increased sophistication. [27-31]

(d) Nonlinear spin dynamics. Polarized beams have been a focus of attention recently, with applications to LEP, HERA and SSC. Nonlinear dynamics also plays an important role here. For each nonlinear dynamics technique developed for orbital motions, there is an extention of the technique that deals with the spin dynamics. Recent efforts on spin nonlinear dynamics include Refs.[9-13] and the Siberian snake experiments at Indiana University Cyclotron [32] and the AGS. [33]

An outline of some of the above-mentioned developments are given next. Interested readers should consult the original papers for details.

2 LYAPUNOV EXPONENT

Stability of the motion of a particle is related to chaos of its motion in phase space. The short-term dynamic aperture, for example, is related to large-scale chaos (in the sense of Chirikov criterion of overlapping resonances). It is conceivable that the long-term dynamic aperture is related to weak local chaos (in the sense of Arnold diffusion). Detection of a weak local chaos therefore provides a possible early signal of long-term instability.

A large-scale chaos can be revealed by tracking a single particle. To detect a local chaos, two particles initially very close to each other in phase space are tracked and their distance registered as a function of time. A local chaos is identified when this distance grows exponentially. It occurs often that when a single particle is tracked, nothing obviously alarming is observed, indicating no large-scale chaos, but local chaos is detected by running two neighboring particles. According to the Lyapunov method, a particle with this initial amplitude is likely to be lost eventually.

Figure 1 shows the survival plots [34] (the number of turns a particle survives as a function of its initial launching amplitude) in an LHC simulation.[18] The survival plots can be used to give an upper bound of dynamic aperture. With the detection threshold of Lyapunov exponentiation carefully chosen, the Lyapunov method presumably gives a lower bound because local chaos is a necessary-but-not-sufficient condition for long-term particle loss.

Some insight can be gained from a slight variation of this method, as shown in Fig. 2. [18]. The unperturbed tunes are represented as a star. The instantaneous tunes are obtained by sampling 1000 turns

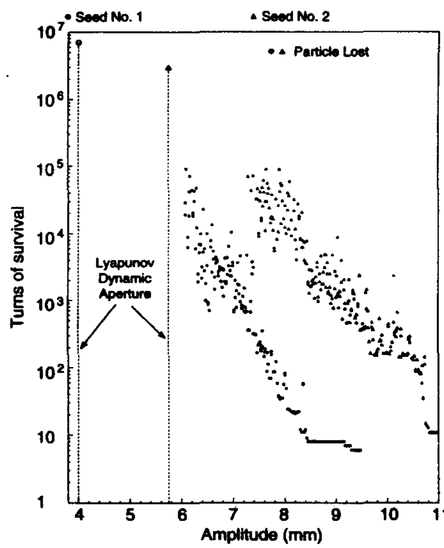

Figure 1: Survival plot for the LHC illustrating the dynamic apertures detemined by detecting the Lyapunov exponentiation compared with tracking results for two random number sets of multipole errors.

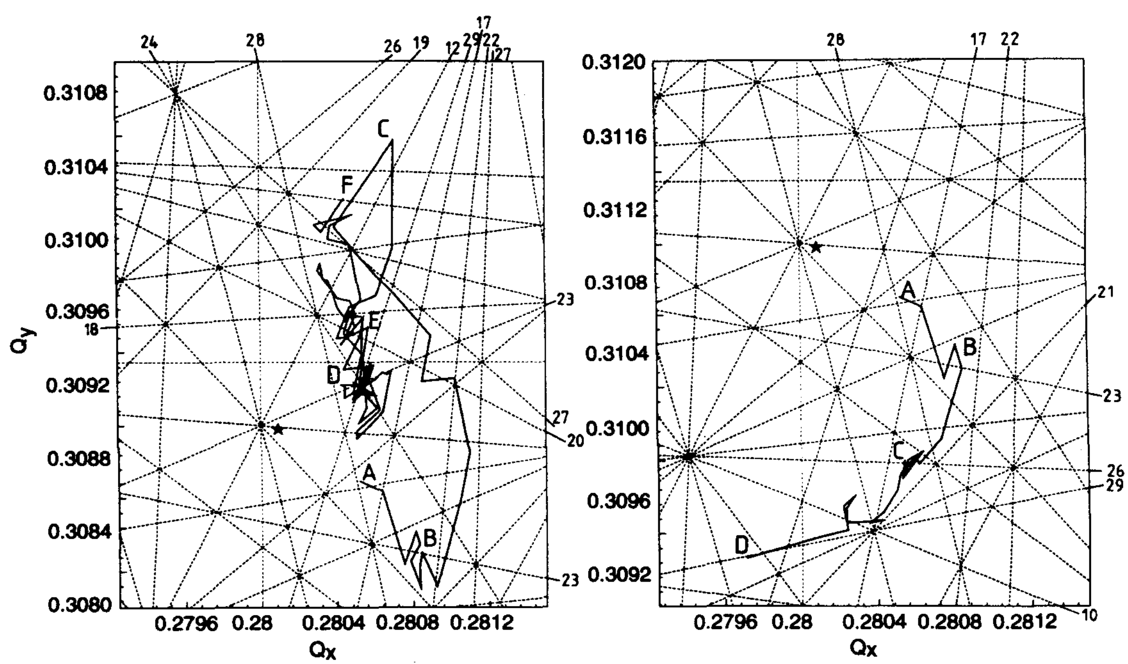

Figure 2: The wondering paths of the instantaneous tunes of two neighboring particles in the (Q_x, Q_y) plane for the LHC.

for each reading on the tune path. The two neighboring particles start with the same tunes, follow each other closely for 4000 turns and then depart to take on their own destiny, encountering very different sets of resonances. Resonances up to the 29-th order are indicated by dotted lines in Fig. 2. Eventually both particles are lost, one in 65000 turns, the other in 15000 turns.

3 TIME EVOLUTION OF LIE INVARIANTS

Another proposal to detect an early warning of long-term particle loss is to look for the time evolution of an alleged invariant, such as the Taylor expression obtained using Lie algebra and DA up to some convenient order. The idea is that any deviation from the alleged invariance could be attributed, at least partially, to chaos. Near the dynamic aperture, the second order (Courant-Snyder) invariant typically varies markedly with time, indicating it is not a good invariant. A higher order invariant is then used instead. Using a 6-th order invariant, Fig. 3 indicates that the long-term dynamic aperture is between 6.0 and 6.6 mm for the simulated SSC. The same accelerator model was tested using the Lyapunov method, and it is found that local chaos also seems to occur between 6.0 and 6.6 mm.

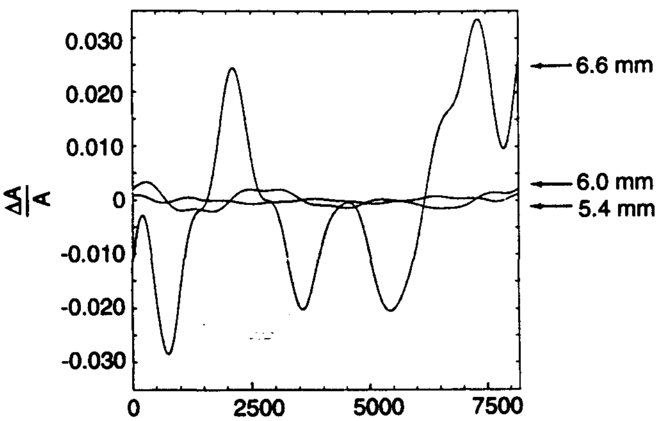

Figure 3: The 6-th order invariant as a function of turn number for the SSC. Three different launching amplitudes x=y=5.4, 6.0 and 6.6 mm are used.

4 BOUND ON CANONICAL TRANSFORMATION

Consider a 2-D system which is nearly integrable and we have an approximate expression $J=(J_1,J_2)$ of the invariants. If $J_{1,2}$ are truely invariants, we have a stable system (away from resonances), but being approximate invariants, their values vary with time. Consider a region R in the (J_1,J_2) space; let $\delta J_{1,2}$ be the maximum amount of variation in R during N turns. Then it can be shown [35] that particles in R will stay in R for $N(\Delta J_1/\delta J_1)$ turns [or $N(\Delta J_2/\delta J_2)$ turns, whichever is smaller] if they stay away from the boundary of R by a distance $\Delta J_{1,2}$. Once an approximate expression is found for the invariants, therefore, this technique allows prediction of an lower bound of the particle lifetime.

To apply the above observation, one needs to find a good expression of the invariants to make δJ as small as possible. One example of invariant expressions is the Taylor expression used in Fig. 3. Another example is to apply an interpolation technique which, by tracking a mesh of initial conditions for, say, 1000 turns and interpolating between the results, allows a numerical relation between the invariants J as functions of I and ϕ.[21] This technique was applied to the ALS (ignoring synchrotron radiation) and the region $R=(2.51<J_1<2.82$ µm, $1.34<J_2<1.64$ µm). It was found by a numerical search (not a simple procedure due to the necessarily granular nature of this search) that $\delta J=(2.8, 4.0)\times10^{-6}$ µm during $N=10^4$ turns. This means particles in the sub-region ($2.54<J_1<2.79$ µm, $1.38<J_2<1.60$ µm) will not get out of region R in 10^8 turns.

5 TAYLOR MAP TRACKING

As mentioned, a straightforward application of the DA yields an N-th order Taylor map, which is symplectic up to order N. Once obtained, this Taylor map gives a straightforward means to track particles. However, it was observed [36] in the SSC application that the Taylor map must be of sufficiently high order (N>11) in order to describe the long-term behavior accurately. Furthermore, the reason of needing N>11 is not due to dynamics, but simply due to kinematics, i.e., the map must be sufficiently symplectic. A relatively low-ordered Taylor map (N=7) suffices as long as it is somehow symplectified, even though artificially. For long-term purposes, therefore, it is important to keep the map symplectic. A similar observation was obtained when a detailed comparison was made between one-trun Taylor map and element-by-element tracking for the cases of LHC and SSC [42].

6 KICK MAP

One way to symplectify an N-th order Taylor map is to represent it by a sequence of nonlinear "kicks" connected by linear maps. Both the linear and the kick maps are symplectic; thus the total map is symplectic. The representation is such that the total map is the same as the original Taylor map up to the N-th order. Beyond the N-th order, the Taylor map is terminated, but the kick map contains additional artificial terms when expanded as Taylor series. The simplest linear map is the Courant-Snyder type. The simplest kick is of the type $\Delta x'=f(x)$ in the 1-D case. More sophisticated alternative forms of both linear and kick maps can be employed to reduce the number of kicks required to represent the Taylor map.

In general, the Taylor expansion of a Lie map does not terminate; it terminates if and only if it can be written as a kick map. A study of kick map (called a "jolt" map in Ref. [2]) is carried out for a simple example of an anharmonic oscillatior with Hamiltonian $H=(p^2+q^2)/2 - q^4/4$. It is found that a 7-th order Taylor map, a Lie map of the form $\exp(:f_4:)\exp(:f_6:)\exp(:f_8:)$, and a kick map consisting of 9 kicks, all represent the motion reasonably well for one turn. Although the Taylor map sems to be most accurate as one-turn map, it is not suitable for long-term tracking because it is non-symplectic.

Kick maps can be rather efficient; for the case of 6-D phase space, 9 kicks can represent a 3-rd order Taylor map, and 48 and 68 kicks can represent an 7-th and 11-th order map respectively [37]. On the other hand, for a given Taylor map, its kick map representation is not unique. How to choose a kick map optimally is a subtle issue.

7 INTERPOLATION MAP

Another way to assure symplecticity is to deal with a generating function $G(I,\phi')$ that brings (I,ϕ) for one turn to (I',ϕ')—the map is exactly symplectic even if the generating function is approximate. One approach, similar to that used to find the long-term bound for the ALS, is based on fitting a one-turn tracking of a mesh of initial conditions and interpolation in between to obtain G [26]. Unlike the Taylor maps, a generating function, tracking can only be implicit in the sense that a generating function necessarily mixes the old and new dynamic variables. This technique is not based on an expansion in terms of some small parameters, and is sometimes referred to as nonperturbative.

8 MONOMIAL MAP

Although a Lie map $\exp(:F(X):)$ is formally symplectic, when it is applied for tracking purposes, it is truncated to some order and symplecticity is restored only when, after truncation, a generating function is introduced [24]. It was then discovered [38] that if the function F(X) contains only a single monomial term in X, there is an exact formula which does not require expansion. In the 1-D case with phase space coordinates (q,p), this remarkable formula reads

$$
\begin{aligned}
\exp(\alpha :q^n p^m:)\, q &= q\,[\,1 + (n-m)\,\alpha\, q^{n-1} p^{m-1}\,]^{m/(m-n)} && \text{if } m \neq n \\
&\ q \exp(\,-\alpha\, m\, q^{m-1} p^{m-1}\,) && \text{if } m = n \\
\exp(\alpha :q^n p^m:)\, p &= p\,[\,1 + (n-m)\,\alpha\, q^{n-1} p^{m-1}\,]^{n/(m-n)} && \text{if } m \neq n \\
&\ p \exp(\,\alpha\, m\, q^{m-1} p^{m-1}\,) && \text{if } m = n
\end{aligned}
\tag{1}
$$

with the convergence condition $|(n-m)\alpha q^{n-1}p^{m-1}| < 1$. Equation (1) is exact and is symplectic. Given a Lie map for which F(X) is Taylor series in X, we can factorize the Lie map into a product of monomial maps, each being a Lie map with a monomial exponent. Symplectic tracking can then be performed using Eq.(1).

It is worth mentioning here that the various forms of maps (the Taylor map, the Lie map, the Dragt-Finn factorized Lie map, the normal form Lie map, the monomial map, the kick map, and perhaps to a lesser degree, the interpolation map) can be transformed into one another if desired. One needs only to determine which map to use for the current applications.

9 DYNAMIC RESCALING

Still another proposal to symplectify a Taylor map is by an artificial rescaling of the phase space coordinates instead of adding artificial higher order terms [4]. Strictly speaking, this proposal enforces not the complete symplecticity condition but a weaker condition that phase space volume is preserved (Liouville theorem). To apply this method, a particle is tracked with a Taylor map and its corresponding element-by-element map for, say, 1000 turns. The phase space amplitudes as a function of time for the two tracking results are slightly different partly because of the non-symplecticity of the Taylor map. The difference is then minimized by applying scale factors f_x, f_y and f_s to the Taylor tracking results. These scale factors are different for different initial conditions. This allows a partial symplectification up to the dynamic aperture. Having obtained the scale factors, the particle can conceivably be tracked for long term purposes.

10 IUCF EXPERIMENT

In the Indiana University Cyclotron, the electron-cooled beam has a small transverse emittance and a small energy spread, which makes it ideal for probing the transverse and longitudinal phase spaces. To explore the transverse phase space, the electron-cooled beam is kicked and its subsequent motion detected by 2 position monitors. Cooling is turned off just before the kick is applied. The 3-rd and 4-th order resonances were studied with tunes ν_x close to 11/3 and 15/4 respectively. The measured phase spaces are shown in Fig. 4 [27]. These explorations of the phase space using a "pencil" beam give detailed information about the effective Hamiltonian near those resonances. In particular, the island tune for the 4-th order islands is found to be 0.0013. Continuation of this experiment includes the study of 2-D coupled nonlinear resonances.

To explore the longitudinal phase space, the rf phase is first shifted, and the beam phase relative to the rf and the beam energy error (using a beam position monitor located with large dispersion) are measured. The signals can be frequency-analyzed to find the synchrotron tune ν_s. Fig. 5 shows ν_s as a function of the synchrotron oscillation amplitude which is determined by the rf phase shift. The solid curve is the theoretical prediction, which is found to agree well with the data. The response of the beam to a sinusoidal modulation of the synchronous phase has also been studied, yielding good agreement between experiment and theory. Once the longitudinal exploration is established, one could next study the dynamic aperture as a function of synchrotron oscillation amplitude.

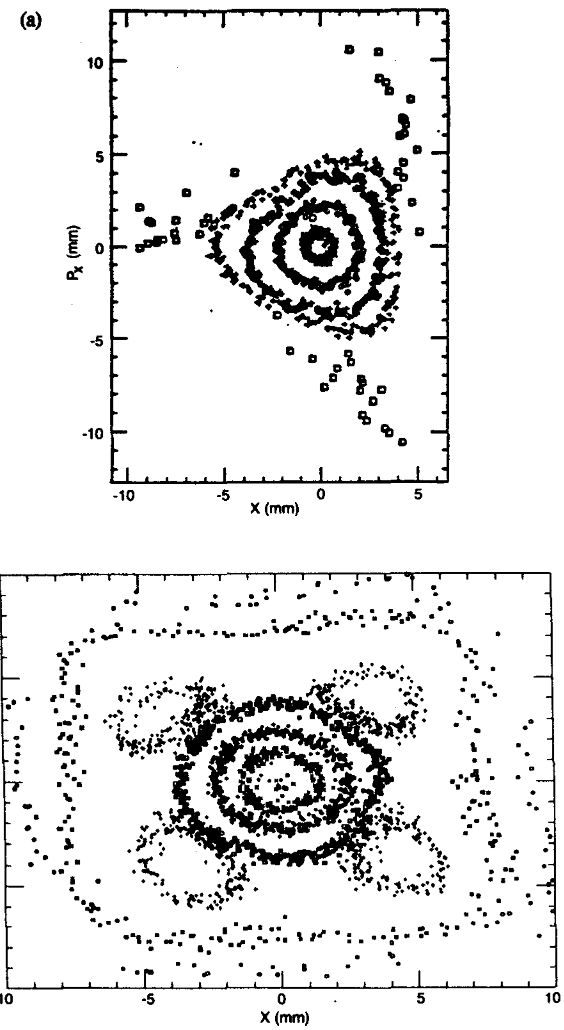

Figure 4: Transverse phase space behavior in the IUCF (a) near ν_x=11/3, and (b) near ν_x=15/4.

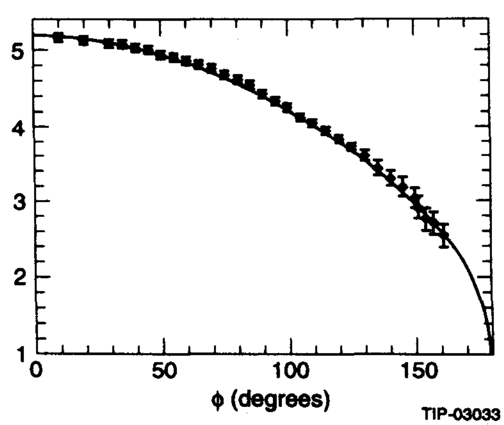

Figure 5: Measured synchrotron tune versus synchrotron amplitude for the IUCF. The solid curve is theoretical prediction.

11 SPS EXPERIMENT

Long-term diffusion effects in the presence of intentional nonlinearities (8 sextupoles) and tune ripple were studied experimentally at SPS. The beam is kicked horizontally and beam loss occurred mostly in the vertical dimension. Figure 6 shows some of the results. In the absence of tune ripple, the beam has a long lifetime. When a tune ripple with an amplitude of 1.65×10^{-3} and frequency of 9 Hz is introduced, the lifetime drops to 7 hr. When the same rippling amplitude is divided into two rippling frequencies at 9 Hz and 180 Hz, the lifetime drops further to 2 hr. The observations that a tune ripple of 10^{-3} causes significant diffusion and that richness of rippling frequency enhances the diffusion agree qualitatively with the simulation results. A quantitative comparison of the experimental and the simulation results has been inconclusive so far.

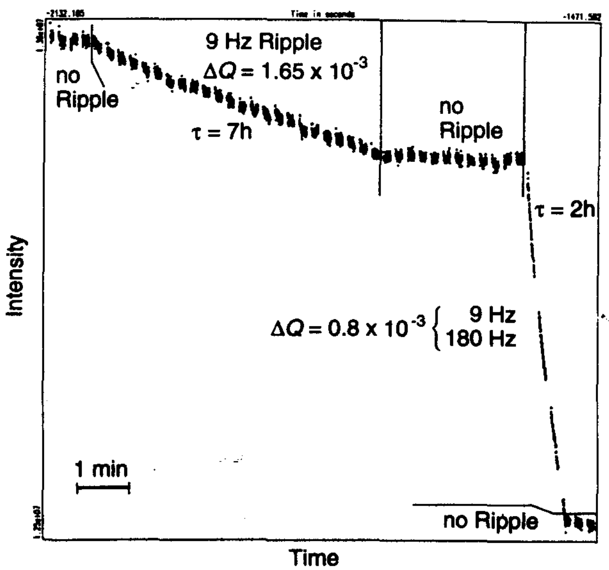

Figure 6: Diffusion measurements at SPS in the presence of nonlinearities and tune ripple.

12 TEVATRON EXPERIMENT

This experiment explores the transverse dynamics using a set of 16 sextupoles. The recent efforts include the study of the slow diffusion effects and the island modulation effects. To study the slow diffusion effects, the beam is kicked horizontally and flying wire scan was made once per minute for 30 minutes. A diffusion constant D(J) is then obtained as a function of amplitude J by fitting the observations. The experiment was repeated for two values of tunes, one slightly above 2/5 and one slightly below 2/5, and two different sextupole configurations. It is found that D(J) depends strongly on J, and is very different for different sextupole configurations and the choice of tune.

The strong dependence of D(J) on J is consistent with the strong dependence of the survival time τ on the launching amplitude J as seen in the survival plots. On the other hand, a diffusion model may have its limitations. An attempt was made to explain the survival plots phenomenologically using a diffusion model [39]. It is found that if the diffusion constant is defined from $\langle\tau\rangle(J)$, a simple-minded model leads to an internal inconsistency in that it predicts a $\langle\tau^2\rangle(J)$ which conflicts with the tracking results.

In another study, the portion of the beam trapped by the islands are observed while the islands are modulated by a set of quadrupoles. The dynamics is equivalent to a driven pendulum. These experiments become quite sophisticated in that both the amplitude and frequency of the tune modulation were then ramped with time (chirping). Detailed beam dynamics can be pursued along this line. In particular, the island tune is related to the modulation frequency when particles are resonantly driven out from the islands, and is found to be about 0.063.

13 THIRD ORDER ACHROMATS

An n-th order achromat means a beamline which is an identity transformation to n-th order in **X**. Second order achromats have been well explored in the past [40]. Using the program MARYLIE, [24] various designs of third order achromats are proposed. [23] One example consists of a string of 5 cells, with each cell schematically shown in Fig. 7. A total of 14 variables is needed to make the achromat: 1 type of bend, 2 types of quadrupoles, 3 types of sextupoles and 8 types of octupoles. The tunes per cell are $\nu_x=\nu_y=1/5$. In this design, 2 of the 8 octupole types are used for compensating the $2\nu_x-2\nu_y=0$ resonance; otherwise 6 octupole types would be sufficient. If the path length isochrony (z dependence on δ) is not required, one may use one less sextupole and one less octupole types. The following table gives a few examples of 3-rd order achromats (the first 5 columns are per-cell values):

ν_x	ν_y	# quad	# sext	# oct	isochrony	# cell/achr
1/5	1/5	2	3	8	yes	5
1/5	1/6	2	3	6	yes	30
1/5	1/6	2	2	5	no	30
1/7	2/7	2	3	6	yes	7

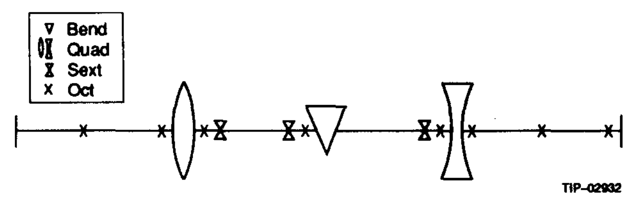

Figure 7: A cell of a 3-rd order achromat.

14 HIGHER ORDER ERRORS OF A BEAMLINE

Having obtained a design of a beamline, such as an achromat of some kind, the higher-order residual errors can be calculated using a Lie algebraic technique.[3, 41]. Having obtained the beamline Lie map exp(:F:), the error terms are identified as the undesired higher-order terms in F that are not taken care of by the design. These error terms give an explicit handle on the cause of the errors as well as how to correct them if desired. This useful technique has been applied to the TBF final focus. The error terms identified agree with tracking results.

15 SECOND ORDER PATH LENGTH

The path length variation can be written as, in the absence of sextupoles,

$$\Delta L/L = \alpha_c\,\delta + \langle\gamma_x\rangle\,\varepsilon_x/4 + \langle\gamma_y\rangle\,\varepsilon_y/4 \quad (2)$$

where $\gamma_{x,y}$ are the Twiss parameters, $\varepsilon_{x,y}$ are the betatron emittances and $\langle\rangle$ means averaging over the accelerator circumference. The leading term is given by the momentum compaction α_c. For a low-α_c lattice, however, the path length dependence on betatron amplitude is not negligible and its effect on beam stability in the synchrotron and betatron motions can be important. This effect is studied in Ref. [1] for the APS. It was found that the inclusion of sextupoles reduces this effect substantially, as shown in Fig. 8, and the net effect due to the 2-nd order path length is found to be tolerable even though APS has a small α_c.

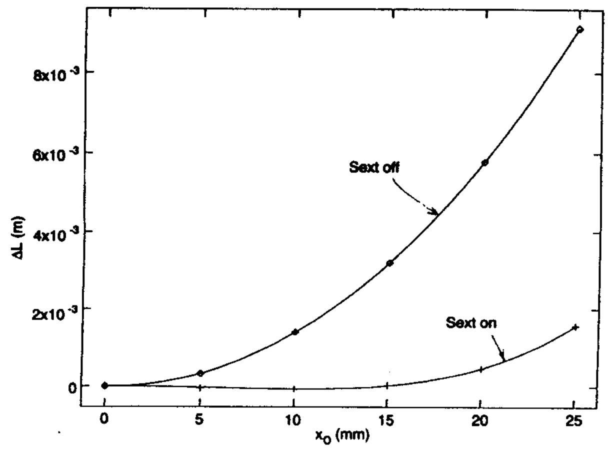

Figure 8: Path length dependence on betatron amplitude for the APS with and without sextupoles.

16 PARTICLE DISTRIBUTION EVOLUTION

It is often important in practice to know the growth of tails in the particle distribution and the slow emittance growth in the presence of nonlinearities. This is most conveniently done by studying the distribution of a multi-particle system instead of individual particles because the statistical fluctuation is avoided. In Ref. [43], a tracking algorithm is developed for the beam distribution in phase space. One example application is seen in Fig. 9.

If we are interested in the dependence of the particle distribution ρ on amplitude J, and not in angle ϕ, an iteration algorithm is developed [8] to give the time evolution of $\rho(J)$ based on a perturbation expansion on the Vlasov equation. The ϕ-dependences are averaged out analytically. Without the ϕ-dependences, this approach is potentially faster than tracking the entire distribution function as done in Ref. [43]. The time evolution of $\rho(J)$ contains information on diffusion.

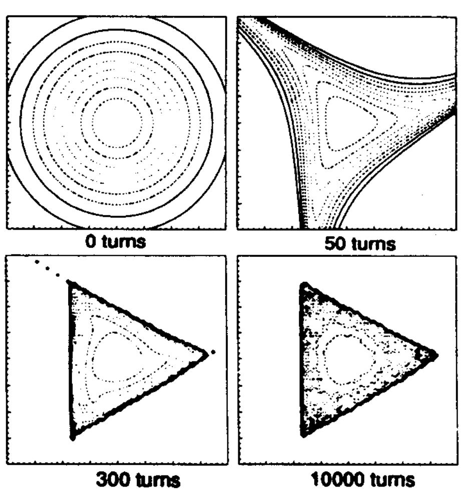

Figure 9: Time evolution of the particle distribution in the presence of sextupoles and a tune close to 1/3. The result is obtained by tracking the distribution instead of a collection of single particles.

17 NONLINEAR SPIN DYNAMICS

One way to study the nonlinear spin dynamics is by tracking simulations. Refs..[10] and [11] describe such efforts for the HERA electron storage ring using the program SITROS [44]. Ref.[12] describes the spin tracking for a proton synchrotron using spinor formalism. The particle energy is ramped through an isolated depolarization resonance. Synchrotron oscillation can be included. Simulations with complete and partial Siberian snakes are also demonstrated. The results agree well with expectation.

Another way to study the nonlinear spin dynamics is to exploit the nonlinear dynamics tools developed for orbital motion. For example, a Lie algebra technique is used in Ref. [45]; a perturbation technique is used in Ref. [46]; the DA technique is applied in Ref. [9]; and the interpolation map technique is adopted in Ref. [13].

In Ref. [9], the one-turn maps for orbital and spin motions are first obtained as a Taylor expansion in terms of the betatron coordinates. Using DA, this map then gives a Taylor expansion of the spin tune and the equilibrium polarization direction as functions of the betatron invariants and the particle energy error. This approach can be useful for proton synchrotrons and has been applied to the Moscow Kaon Factory Booster design.

Nonlinear dynamics applied to electron storage rings is more involved, as studied in Refs. [13, 45, 46]. The equilibrium polarization is given by the Derbenev-Kondratenko formula in terms of the equilibrium spin direction **n** and the spin chromaticity $\mathbf{d}=\gamma\partial\mathbf{n}/\partial\gamma$. The leading order in **d**, which is independent of $J_{x,y,s}$, gives the linear depolarization resonances. Higher order terms in **d** give higher order depolarization resonances.

The method of Ref. [13] is as follows. Assume the orbit motion has been integrated and $(\mathbf{J},\psi)$ are the action-angle variables with $J_{x,y,s}$ the invariants. Using an interpolation technique, **J** is known in terms of **I** and ϕ. Tracking a mesh of initial conditions for one turn and using interpolations, one obtains the one-turn spin map $T(\mathbf{J},\psi)$. Away from depolarization resonances, the vector **n** is given by $T(\mathbf{J},\psi)\mathbf{n}(\mathbf{J},\psi)=\mathbf{n}(\mathbf{J},\psi+2\pi\nu)$ and **d** is then computed by numerical differentiating near-by trajectories, or by the DA technique, or using a spline interpolation as done in the program SODOM. Averaging over the circumference and the beam distribution in $(\mathbf{J},\psi)$ then gives the equilibrium polarization.

This method is applied to a toy FODO ring with vertical bends. Only linear orbital motions are considered, although the analysis is not restricted to it. It is found that the $J_{x,y}$ dependences of **d** are negligible, but the J_s dependence is important because the synchrotron energy deviation is non-negligible compared with 440 MeV, the spacing between integer depolarization resonances. This means the synchrotron sideband resonances $\gamma a=47+m\nu_s$ are excited. In Fig. 10, the dotted curve is the result using a linear analysis using the program SLIM; the solid curve is a theoretical

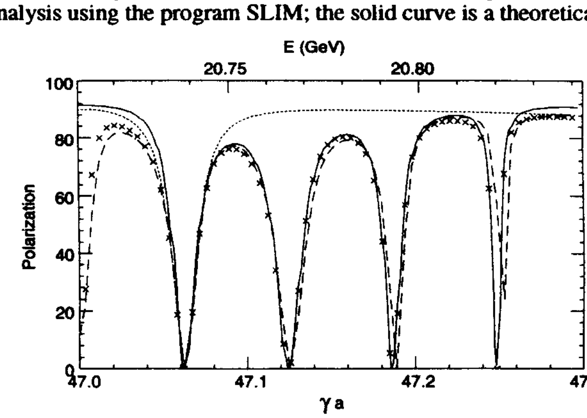

Figure 10: Comparison of calculated equilibrium polarization using SODOM with other methods.

prediction normalized at the first sideband; the dashed curve is a 7-th order perturbation calculation using SMILE; [46] and the crosses are the SODOM results. As can be seen, SODOM agrees well with existing approaches. Nonlinear orbital motions can be included next.

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