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On quantum groups and Lie bialgebras related to $sl(n)$

A Stolin and I Pop

Department of Mathematical Sciences, University of Gothenburg, SE-405 30 Gothenburg, Sweden

Abstract. Given an arbitrary field $\mathbb{F}$ of characteristic 0, we study Lie bialgebra structures on $sl(n, \mathbb{F})$, based on the description of the corresponding classical double. For any Lie bialgebra structure δ, the classical double $D(sl(n, \mathbb{F}), \delta)$ is isomorphic to $sl(n, \mathbb{F}) \otimes_{\mathbb{F}} A$, where A is either $\mathbb{F}[\varepsilon]$, with $\varepsilon^2 = 0$, or $\mathbb{F} \oplus \mathbb{F}$ or a quadratic field extension of $\mathbb{F}$. In the first case, the classification leads to quasi-Frobenius Lie subalgebras of $sl(n, \mathbb{F})$. In the second and third cases, a Belavin-Drinfeld cohomology can be introduced which enables one to classify Lie bialgebras on $sl(n, \mathbb{F})$, up to gauge equivalence. The Belavin-Drinfeld untwisted and twisted cohomology sets associated to an r-matrix are computed.

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1. Introduction

Following [3], we recall that a quantized universal enveloping algebra (or a quantum group) over a field k of characteristic zero is a topologically free topological Hopf algebra H over the formal power series ring $k[[\hbar]]$ such that $H/\hbar H$ is isomorphic to the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over k.

The quasi-classical limit of a quantum group is a Lie bialgebra. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ together with a cobracket δ which is compatible with the Lie bracket. Given a quantum group H, with comultiplication Δ, the quasi-classical limit of H is the Lie bialgebra $\mathfrak{g}$ of primitive elements of $H/\hbar H$ and the cobracket is the restriction of the map $(\Delta - \Delta^{21})/\hbar (\mathrm{mod} \hbar)$ to $\mathfrak{g}$.

The operation of taking the semiclassical limit is a functor $SC : QUE \to LBA$ between categories of quantum groups and Lie bialgebras over k. The existence of universal quantization functors was proved by Etingof and Kazhdan [4, 5]. They used Drinfeld's theory of associators to construct quantization functors for any field k of characteristic zero. More precisely, let $(\mathfrak{g}, \delta)$ be a Lie bialgebra over k. Then one can associate a Lie bialgebra $\mathfrak{g}_\hbar$ over $k[[\hbar]]$ defined as $(\mathfrak{g} \otimes_k k[[\hbar]], \hbar\delta)$. According to Theorem 2.1 of [5] there exists an equivalence $\widehat{Q}$ between the category $LBA_0(k[[\hbar]])$ of topologically free over $k[[\hbar]]$ Lie bialgebras with $\delta \equiv 0 \, (\mathrm{mod}\, \hbar)$ and the category $HA_0(k[[\hbar]])$ of topologically free Hopf algebras cocommutative modulo $\hbar$. Moreover, for any $(\mathfrak{g}, \delta)$ over k, one has the following: $\widehat{Q}(\mathfrak{g}_\hbar) = U_\hbar(\mathfrak{g})$.

Due to this equivalence, the classification of quantum groups whose quasi-classical limit is $\mathfrak{g}$ is equivalent to the classification of Lie bialgebra structures on $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$. Since any cobracket over $\mathbb{C}[[\hbar]]$ can be extended to one over $\mathbb{C}((\hbar))$ and conversely, any cobracket over $\mathbb{C}((\hbar))$, multiplied by an appropriate power of $\hbar$, can be restricted to a cobracket over $\mathbb{C}[[\hbar]]$, this in turn reduces to the problem of finding Lie bialgebras on $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\hbar))$. Denote, for the sake of simplicity, $\mathbb{K} := \mathbb{C}((\hbar))$ and $\mathfrak{g}(\mathbb{K}) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{K}$.

As a first step towards classification, following ideas of [6], we proved in [8] that for any Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$, the associated classical double is of the form $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$, where A

is one of the following associative algebras: $\mathbb{K}[\varepsilon]$, where $\varepsilon^2 = 0$, $\mathbb{K} \oplus \mathbb{K}$ or $\mathbb{K}[j]$, where $j^2 = \hbar$.

As it was shown in [8], the classification of Lie bialgebras with classical double $\mathfrak{g}(\mathbb{K}[\varepsilon])$ leads to the classification of quasi-Frobenius Lie algebras over $\mathbb{K}$, which is a complicated and still open problem.

Unlike this case, the classification of Lie bialgebras with classical double $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ can be achieved by cohomological and combinatorial methods. In [8], we introduced a Belavin-Drinfeld cohomology theory which proved to be useful for the study of Lie bialgebra structures. To any non-skewsymmetric r-matrix r_{BD} from the Belavin-Drinfeld list [1], we associated a cohomology set $H^1_{BD}(\mathfrak{g}, r_{BD})$. We proved the existence of a one-to-one correspondence between any Belavin-Drinfeld cohomology and gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$. In case $\mathfrak{g} = sl(n)$, we showed that for any non-skewsymmetric r-matrix r_{BD}, the cohomology set $H^1_{BD}(sl(n), r_{BD})$ has only one class, which is represented by the identity.

Regarding the classification of Lie bialgebras whose classical double is isomorphic to $\mathfrak{g}(\mathbb{K}[j])$, with $j^2 = \hbar$, a cohomology theory can be introduced too. Our result states that there exists a one-to-one correspondence between Belavin-Drinfeld twisted cohomology and gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ whose classical double is isomorphic to $\mathfrak{g}(\mathbb{K}[j])$. In [8], we proved that the twisted cohomology corresponding to the Drinfeld-Jimbo r-matrix has only one class, represented by a certain matrix J (not the identity). A deeper investigation was done in the subsequent article [9] where twisted cohomologies for $sl(n)$ associated to generalized Cremmer-Gervais r-matrices were studied.

The aim of the present article is the study of Lie bialgebra structures on $sl(n, \mathbb{F})$, for an arbitrary field $\mathbb{F}$ of characteristic zero. Again the idea is to use the description of the classical double. We will show that for any Lie bialgebra structure δ, the classical double $D(sl(n, \mathbb{F}), \delta)$ is isomorphic to $sl(n, \mathbb{F}) \otimes_{\mathbb{F}} A$, where A is either $\mathbb{F}[\varepsilon]$, with $\varepsilon^2 = 0$, or $\mathbb{F} \oplus \mathbb{F}$ or a quadratic extension of $\mathbb{F}$. In the first case, the classification leads to quasi-Frobenius Lie subalgebras of $sl(n, \mathbb{F})$. In the second and third cases, we will introduce a Belavin-Drinfeld cohomology which enables one to classify Lie bialgebras on $sl(n, \mathbb{F})$, up to gauge equivalence. In the particular case $\mathbb{F} = \mathbb{C}((\hbar))$ we recover the classification of quantum groups whose classical limit is $sl(n, \mathbb{C})$ obtained in [8,9].

2. Description of the classical double

From the general theory of Lie bialgebras it is known that for each Lie bialgebra structure δ on a fixed Lie algebra L one can construct the corresponding classical double $D(L, \delta)$. As a vector space, $D(L, \delta) = L \oplus L^*$. Moreover, since the cobracket of L induces a Lie bracket on L^*, there exists a Lie algebra structure on $L \oplus L^*$, induced by the bracket and cobracket of L, and such that the canonical symmetric nondegenerate bilinear form Q on this space is invariant.

Let $\mathbb{F}$ be an arbitrary field of zero characteristic. Let us assume that δ is a Lie bialgebra structure on $sl(n, \mathbb{F})$. Then one can construct the corresponding classical double $D(sl(n, \mathbb{F}), \delta)$.

Similarly to Lemma 2.1 from [6], one can prove that $D(sl(n, \mathbb{F}), \delta)$ is a direct sum of regular adjoint $sl(n)$-modules. Combining this result with Prop. 2.2 from [2], one obtains the following

Theorem 2.1. *There exists one associative, unital, commutative algebra A of dimension 2 over $\mathbb{F}$, such that $D(sl(n, \mathbb{F}), \delta) \cong sl(n, \mathbb{F}) \otimes_{\mathbb{F}} A$.*

The symmetric invariant nondegenerate bilinear form Q on $sl(n, \mathbb{F}) \otimes_{\mathbb{F}} A$ is given in the following way. For arbitrary elements $f_1, f_2 \in sl(n, \mathbb{F})$ and $a, b \in A$ we have

$$Q(f_1 \otimes a, f_2 \otimes b) = K(f_1, f_2) \cdot t(ab)$$

where K denotes the Killing form on $sl(n, \mathbb{F})$ and $t : A \longrightarrow \mathbb{F}$ is a trace function.

Let us now investigate the algebra A. Since A is unital and of dimension 2 over $\mathbb{F}$, one can choose a basis $\{e, 1\}$, where 1 denotes the unit. Moreover, there exist p and q in $\mathbb{F}$ such that $e^2 + pe + q = 0$. Let $\Delta = p^2 - 4q \in \mathbb{F}$. We distinguish the following cases:

(i) Assume $\Delta = 0$. Let $\varepsilon := (e+p)/2$. Then $\varepsilon^2 = 0$ and $A = \mathbb{F}\varepsilon \oplus \mathbb{F} = \mathbb{F}[\varepsilon]$.

(ii) Assume Δ is the square of a nonzero element of $\mathbb{F}$. In this case, one can choose $e' \in \mathbb{F}^*$ such that $e'^2 = \Delta$. Then $A = \mathbb{F} \oplus e'\mathbb{F} = \mathbb{F} \oplus \mathbb{F}$.

(iii) Assume Δ is not a square of an element of $\mathbb{F}$. Then $A = \mathbb{F} + e'\mathbb{F}$, where $e' = (e+p)/2$ and $e'^2 = \Delta/4 \in \mathbb{F}$. Thus A is a quadratic field extension of $\mathbb{F}$.

Summing up the above observations, we get

Theorem 2.2. *Let δ be an arbitrary Lie bialgebra structure on $sl(n,\mathbb{F})$. Then $D(sl(n,\mathbb{F}),\delta) \cong sl(n,\mathbb{F}) \otimes_{\mathbb{F}} A$, where $A = \mathbb{F}[\varepsilon]$ and $\varepsilon^2 = 0$, $A = \mathbb{F} \oplus \mathbb{F}$ or A is a quadratic field extension of $\mathbb{F}$.*

The classification of Lie bialgebras with classical double $sl(n,\mathbb{F}[\varepsilon])$ leads to the classification of quasi-Frobenius Lie algebras over $\mathbb{F}$. More precisely, due to the correspondence between Lie bialgebras and Manin triples (see [3]), the following result holds:

Proposition 2.3. *There exists a one-to-one correspondence between Lie bialgebra structures on $sl(n,\mathbb{F})$ whose corresponding double is isomorphic to $sl(n,\mathbb{F}[\varepsilon])$ and Lagrangian subalgebras W of $sl(n,\mathbb{F}[\varepsilon])$ complementary to $sl(n,\mathbb{F})$.*

Similarly to Theorem 3.2 from [7], one can prove

Proposition 2.4. *Any Lagrangian subalgebra W of $sl(n,\mathbb{F}[\varepsilon])$ complementary to $sl(n,\mathbb{F})$ is uniquely defined by a subalgebra L of $sl(n,\mathbb{F})$ together with a nondegenerate 2-cocycle B on L.*

We recall that a Lie algebra is called quasi-Frobenius if there exists a nondegenerate 2-cocycle on it. The complete classification of quasi-Frobenius Lie subalgebras of $sl(n,\mathbb{F})$ is not generally known for large n.

3. Belavin-Drinfeld untwisted cohomologies

Unlike the previous case, the classification of Lie bialgebras with classical double $sl(n,\mathbb{F})\oplus sl(n,\mathbb{F})$ can be achieved by cohomological and combinatorial methods.

Lemma 3.1. *Any Lie bialgebra structure δ on $sl(n,\mathbb{F})$ for which the associated classical double is isomorphic to $sl(n,\mathbb{F}) \oplus sl(n,\mathbb{F})$ is a coboundary $\delta = dr$ given by an r-matrix satisfying $r + r^{21} = f\Omega$, where $f \in \mathbb{F}^*$ and $\mathrm{CYB}(r) = 0$.*

We may suppose that $f = 1$. Naturally we want to classify all such r up to $GL(n,\mathbb{F})$-equivalence. Let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$. Any Lie bialgebra structure δ over $\mathbb{F}$ can be extended to a Lie bialgebra structure $\overline{\delta}$ over $\overline{\mathbb{F}}$.

According to [1], the Lie bialgebra structures on a simple Lie algebra $\mathfrak{g}$ over an algebraically closed field are coboundaries given by non-skewsymmetric r-matrices. Suppose we have fixed a Cartan subalgebra $\mathfrak{h}$ and the corresponding root system. Any r-matrix depends on a discrete and a continuous parameter. The discrete parameter is an admissible triple $(\Gamma_1, \Gamma_2, \tau)$, i.e. an isometry $\tau : \Gamma_1 \longrightarrow \Gamma_2$ where $\Gamma_1, \Gamma_2 \subset \Gamma$ such that for any $\alpha \in \Gamma_1$ there exists $k \in \mathbb{N}$ satisfying $\tau^k(\alpha) \notin \Gamma_1$. The continuous parameter is a tensor $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfying

$$r_0 + r_0^{21} = \Omega_0, \quad (\tau(\alpha) \otimes 1 + 1 \otimes \alpha)(r_0) = 0, \quad \forall\, \alpha \in \Gamma_1$$

Here Ω_0 denotes the Cartan part of the quadratic Casimir element Ω. Then the associated r-matrix is given by the following formula

$$r = r_0 + \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + \sum_{\alpha \in (Span\Gamma_1)^+} \sum_{k \in \mathbb{N}} e_\alpha \wedge e_{-\tau^k(\alpha)}$$

Now, let us assume that δ is a Lie bialgebra structure on $sl(n,\mathbb{F})$. Then its extension $\overline{\delta}$ has a corresponding r-matrix. Up to $GL(n,\overline{\mathbb{F}})$-equivalence, we have the Belavin-Drinfeld classification. We may therefore assume that our r-matrix is of the form $r_X = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r)$, where $X \in GL(n,\overline{\mathbb{F}})$ and r satisfies the system $r + r^{21} = \Omega$ and $\mathrm{CYB}(r) = 0$.

Let $\sigma \in Gal(\overline{\mathbb{F}}/\mathbb{F})$. Since

$$\delta(a) = [r_X, a \otimes 1 + 1 \otimes a]$$

for any $a \in sl(n,\mathbb{F})$ we have

$$(\sigma \otimes \sigma)(\delta(a)) = [\sigma(r_X), a \otimes 1 + 1 \otimes a]$$

and $(\sigma \otimes \sigma)(\delta(a)) = \delta(a)$. Consequently, $\sigma(r_X) = r_X + \lambda\Omega$, for some $\lambda \in \overline{\mathbb{F}}$. Let us show that $\lambda = 0$. Really,

$$\Omega = \sigma(\Omega) = \sigma(r_X) + \sigma(r_X^{21}) = r_X + r_X^{21} + 2\lambda\Omega$$

Thus $\lambda = 0$ and $\sigma(r_X) = r_X$. Consequently,

$$(\mathrm{Ad}_{X^{-1}\sigma(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma(X)})(\sigma(r)) = r$$

We recall the following

Definition 3.2. Let r be an r-matrix. The *centralizer* $C(r)$ of r is the set of all $X \in GL(n,\overline{\mathbb{F}})$ satisfying $(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r) = r$.

Using the same argumets as in the proof of Theorem 4.3 [8], it follows that $\sigma(r) = r$ and $X^{-1}\sigma(X) \in C(r)$, for any $\sigma \in Gal(\overline{\mathbb{F}}/\mathbb{F})$.

Definition 3.3. Let r be a non-skewsymmetric r-matrix from the Belavin-Drinfeld list and $C(r)$ its centralizer. We say that $X \in GL(n,\overline{\mathbb{F}})$ is a *Belavin-Drinfeld cocycle* associated to r if $X^{-1}\sigma(X) \in C(r)$, for any $\sigma \in Gal(\overline{\mathbb{F}}/\mathbb{F})$.

The set of Belavin-Drinfeld cocycles associated to r will be denoted by $Z_{BD}(sl(n,\mathbb{F}), r)$. Note that this set contains the identity.

Definition 3.4. Two cocycles X_1 and X_2 in $Z_{BD}(sl(n,\mathbb{F}), r)$ are called *equivalent* if there exists $Q \in GL(n,\mathbb{F})$ and $C \in C(r)$ such that $X_1 = QX_2C$.

Definition 3.5. Let $H^1_{BD}(sl(n,\mathbb{F}), r)$ denote the set of equivalence classes of cocycles from $Z_{BD}(sl(n,\mathbb{F}), r)$. We call this set the *Belavin-Drinfeld cohomology* associated to the r-matrix r. The Belavin-Drinfeld cohomology is said to be *trivial* if all cocycles are equivalent to the identity, and *non-trivial* otherwise.

Combining the above definitions with the preceding discussion, we obtain

Proposition 3.6. *For any non-skewsymmetric r-matrix r, there exists a one-to-one correspondence between $H^1_{BD}(sl(n,\mathbb{F}), r)$ and gauge equivalence classes of Lie bialgebra structures on $sl(n,\mathbb{F})$ with classical double isomorphic to $sl(n,\mathbb{F}) \oplus sl(n,\mathbb{F})$ and $\overline{\mathbb{F}}$-isomorphic to $\delta = dr$.*

The Belavin-Drinfeld cohomology set can be computed as in [8] and the following result holds.

Theorem 3.7. *For any non-skewsymmetric r-matrix r, $H^1_{BD}(sl(n,\mathbb{F}), r)$ is trivial. Any Lie bialgebra structure on $sl(n,\mathbb{F})$ with classical double $sl(n,\mathbb{F}) \oplus sl(n,\mathbb{F})$ is of the form $\delta = dr$, where r is an r-matrix which is, up to a multiple from $\mathbb{F}^*$, $GL(n,\mathbb{F})$-equivalent to a non-skewsymmertric r-matrix from the Belavin-Drinfeld list.*

4. Belavin-Drinfeld twisted cohomologies

We focus on the study of Lie bialgebra structures on $sl(n,\mathbb{F})$ whose classical double is isomorphic to $sl(n,\mathbb{F})\otimes_{\mathbb{F}} A$, where A is a quadratic extension of $\mathbb{F}$. We may suppose that $A=\mathbb{F}(\sqrt{d})$, where d is not a square in $\mathbb{F}$. We will show that Lie bialgebras of this type can also be classified by means of certain cohomology sets.

Twisted cohomologies associated to r-matrices for $sl(n,\mathbb{F})$ can be defined as in [8], where we studied the particular case $\mathbb{F}=\mathbb{C}((\hbar))$. First, similarly to Prop. 5.3 of [8], one can prove the following

Proposition 4.1. *Any Lie bialgebra structure on $sl(n,\mathbb{F})$ with classical double isomorphic to $sl(n,\mathbb{F}[\sqrt{d}])$ is given by an r-matrix r' which satisfies $CYB(r')=0$ and $r'+r'_{21}=\sqrt{d}\Omega$.*

Over $\overline{\mathbb{F}}$, all r-matrices are gauge equivalent to the ones from Belavin-Drinfeld list. It follows that there exists a non-skewsymmetric r-matrix r and $X\in GL(n,\overline{\mathbb{F}})$ such that $r'=\sqrt{d}(\mathrm{Ad}_X\otimes\mathrm{Ad}_X)(r)$.

The field $\mathbb{F}[\sqrt{d}]$ is endowed with a conjugation $\overline{a+b\sqrt{d}}=a-b\sqrt{d}$. Denote by σ_2 its lift to $Gal(\overline{\mathbb{F}}/\mathbb{F})$. If $X\in GL(n,\mathbb{F}[\sqrt{d}])$, then $\sigma_2(X)=\overline{X}$. Now let us consider the action of σ_2 on r'. We have $\sigma_2(r')=r'+\lambda\Omega$, for some $\lambda\in\overline{\mathbb{F}}$. Let us show that $\lambda=-\sqrt{d}$. Indeed, since $r'+r'_{21}=\sqrt{d}\Omega$, we also have $\sigma_2(r')+\sigma_2(r'_{21})=-\sqrt{d}\Omega$. Combining these relations with $\sigma_2(r')=r'+\lambda\Omega$, we get $\lambda=-\sqrt{d}$ and therefore $\sigma_2(r')=r'-\sqrt{d}\Omega=-r'_{21}$.

Recall now that $r'=\sqrt{d}(\mathrm{Ad}_X\otimes\mathrm{Ad}_X)(r)$. Then condition $\sigma_2(r')=-r'_{21}$ implies

$$(\mathrm{Ad}_{X^{-1}\sigma_2(X)}\otimes\mathrm{Ad}_{X^{-1}\sigma_2(X)})(\sigma_2(r))=r^{21}$$

For any $\sigma\in Gal(\overline{\mathbb{F}}/\mathbb{F}[\sqrt{d}])$, $\sigma(r')=r'$, which in turn implies

$$(\mathrm{Ad}_{X^{-1}\sigma(X)}\otimes\mathrm{Ad}_{X^{-1}\sigma(X)})(\sigma(r))=r$$

Now, using the same type of arguments as in the proof of Theorem 4.3 [8], one can deduce that $\sigma(r)=r$ and therefore the following result holds.

Proposition 4.2. *Any Lie bialgebra structure on $sl(n,\mathbb{F})$ with classical double isomorphic to $sl(n,\mathbb{F}[\sqrt{d}])$ is given by $r'=\sqrt{d}(\mathrm{Ad}_X\otimes\mathrm{Ad}_X)(r)$, where r is, up to a multiple from $\mathbb{F}^*$, a non-skewsymmetric r-matrix from the Belavin-Drinfeld list and $X\in GL(n,\overline{\mathbb{F}})$ satisfies $(\mathrm{Ad}_{X^{-1}\sigma_2(X)}\otimes\mathrm{Ad}_{X^{-1}\sigma_2(X)})(r)=r^{21}$ and, for any $\sigma\in Gal(\overline{\mathbb{F}}/\mathbb{F}[\sqrt{d}])$, $(\mathrm{Ad}_{X^{-1}\sigma(X)}\otimes\mathrm{Ad}_{X^{-1}\sigma(X)})(r)=r$.*

Definition 4.3. Let r be a non-skewsymmetric r-matrix from the Belavin-Drinfeld list. We say that $X\in GL(n,\overline{\mathbb{F}})$ is a *Belavin-Drinfeld twisted cocycle* associated to r if

$$(\mathrm{Ad}_{X^{-1}\sigma_2(X)}\otimes\mathrm{Ad}_{X^{-1}\sigma_2(X)})(r)=r^{21}$$

and for any $\sigma\in Gal(\overline{\mathbb{F}}/\mathbb{F}[\sqrt{d}])$,

$$(\mathrm{Ad}_{X^{-1}\sigma(X)}\otimes\mathrm{Ad}_{X^{-1}\sigma(X)})(r)=r$$

The set of Belavin-Drinfeld twisted cocycle associated to r will be denoted by $\overline{Z}_{BD}(sl(n,\mathbb{F}),r)$. Let us analyse for which admissible triples this set is non-empty.

Let $S\in GL(n,\mathbb{F})$ be the matrix with 1 on the second diagonal and 0 elsewhere. Let us denote by s the automorphism of the Dynkin diagram given by $s(\alpha_i)=\alpha_{n-i}$ for all $i\leq n-1$.

Proposition 4.4. *Let r be a non-skewsymmetric r-matrix associated to an admissible triple (Γ_1,Γ_2,τ). If $\overline{Z}_{BD}(sl(n,\mathbb{F}),r)\neq\emptyset$, then $s(\Gamma_1)=\Gamma_2$ and $s\tau=\tau^{-1}s$.*

Definition 4.5. Let X_1 and X_2 be two Belavin-Drinfeld twisted cocycles associated to r. We say that they are *equivalent* if there exist $Q \in GL(n, \mathbb{F})$ and $C \in C(r)$ such that $X_2 = QX_1C$.

The set of equivalence classes of twisted cocycles corresponding to a non-skewsymmetric r-matrix r will be denoted by $\overline{H}^1_{BD}(sl(n, \mathbb{F}), r)$.

Remark 4.6. If two twisted cocycles X_1 and X_2 are equivalent, then the corresponding r-matrices $\sqrt{d}(\mathrm{Ad}_{X_1} \otimes \mathrm{Ad}_{X_1})(r)$ and $\sqrt{d}(\mathrm{Ad}_{X_2} \otimes \mathrm{Ad}_{X_2})(r)$ are gauge equivalent via Q.

Remark 4.7. In fact, by obvious reasons it is better to denote $\overline{H}^1_{BD}(sl(n, \mathbb{F}), r)$ by $\overline{H}^1_{BD}(sl(n, \mathbb{F}), r, d)$. However, we fix d and the notation $\overline{H}^1_{BD}(sl(n, \mathbb{F}), r)$ is not misleading.

Proposition 4.8. *There exists a one-to-one correspondence between the twisted cohomology set $\overline{H}^1_{BD}(sl(n, \mathbb{F}), r)$ and gauge equivalence classes of Lie bialgebra structures on $sl(n, \mathbb{F})$ with classical double isomorphic to $sl(n, \mathbb{F}[\sqrt{d}])$ and $\overline{\mathbb{F}}$-isomorphic to $\delta = dr$.*

Let r_{DJ} be the Drinfeld-Jimbo r-matrix. Having fixed a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the associated root system, we choose a system of generators e_α, $e_{-\alpha}$, h_α where $\alpha > 0$ such that $K(e_\alpha, e_{-\alpha}) = 1$. Denote by Ω_0 the Cartan part of Ω. Then

$$r_{DJ} = \sum_{\alpha > 0} e_\alpha \otimes e_{-\alpha} + \frac{1}{2}\Omega_0$$

The twisted cohomology corresponding to r_{DJ} can be studied in the same manner as was done in [8] (see Prop. 7.15). Let $J \in GL(n, \mathbb{F}[\sqrt{d}])$ denote the matrix with entries $a_{kk} = 1$ for $k \leq m$, $a_{kk} = -\sqrt{d}$ for $k \geq m + 1$, $a_{k,n+1-k} = 1$ for $k \leq m$, $a_{k,n+1-k} = \sqrt{d}$ for $k \geq m + 1$, where $m = [(n + 1)/2]$.

Theorem 4.9. *The Belavin-Drinfeld twisted cohomology $\overline{H}^1_{BD}(sl(n), r_{DJ})$ is non-empty and consists of one element, the class of J.*

Proof. Let X be a twisted cocycle associated to r_{DJ}. Then X is equivalent to a twisted cocycle $P \in GL(n, \mathbb{F}[\sqrt{d}])$, associated to r_{DJ}. We may therefore assume from the beginning that $X \in GL(n, \mathbb{F}[\sqrt{d}])$ and it remains to prove that all such cocycles are equivalent. The proof will be done by induction.

For $n = 2$, consider

$$J = \begin{pmatrix} 1 & 1 \\ \sqrt{d} & -\sqrt{d} \end{pmatrix}$$

and let $X = (a_{ij}) \in GL(2, \mathbb{F}[\sqrt{d}])$ satisfy $\overline{X} = XSD$ with

$$D = \mathrm{diag}(d_1, d_2) \quad \in GL(2, \mathbb{F}[\sqrt{d}])$$

The identity is equivalent to the following system:

$$\overline{a_{11}} = a_{12}d_1, \quad \overline{a_{12}} = a_{11}d_2, \quad \overline{a_{21}} = a_{22}d_1, \quad \overline{a_{22}} = a_{21}d_2$$

Assume that $a_{21}a_{22} \neq 0$. Let $a_{11}/a_{21} = a' + b'\sqrt{d}$. Then $a_{12}/a_{22} = a' - b'\sqrt{d}$. One can immediately check that $X = QJD'$, where

$$Q = \begin{pmatrix} a' & b' \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{F}), \quad D' = \mathrm{diag}(a_{21}, a_{22}) \in \mathrm{diag}(2, \mathbb{F}[\sqrt{d}])$$

For $n = 3$, set

$$J = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \sqrt{d} & 0 & -\sqrt{d} \end{pmatrix}$$

and let $X = (a_{ij}) \in GL(3, \mathbb{F}[\sqrt{d}])$ satisfy $\overline{X} = XSD$ with $D = \mathrm{diag}(d_1, d_2, d_3) \in GL(3, \mathbb{K}[\sqrt{d}])$. The identity is equivalent to the following system:

$$\begin{aligned} &\overline{a_{11}} = d_1 a_{13}, \quad \overline{a_{21}} = d_1 a_{23}, \quad \overline{a_{31}} = d_1 a_{33}, \\ &\overline{a_{12}} = d_2 a_{12}, \quad \overline{a_{22}} = d_2 a_{22}, \quad \overline{a_{32}} = d_2 a_{32}, \\ &\overline{a_{13}} = d_3 a_{11}, \quad \overline{a_{23}} = d_3 a_{21}, \quad \overline{a_{33}} = d_3 a_{31} \end{aligned}$$

Assume that $a_{21}a_{22}a_{23} \neq 0$. Let

$$a_{11}/a_{21} = b_{11} + b_{13}\sqrt{d}, \quad a_{31}/a_{21} = b_{31} + b_{33}\sqrt{d}$$

Then

$$a_{13}/a_{23} = b_{11} - b_{13}\sqrt{d}, \quad a_{33}/a_{23} = b_{31} - b_{33}\sqrt{d}$$

On the other hand, let $b_{12} := a_{12}/a_{22}$ and $b_{32} := a_{32}/a_{22}$. Note that $b_{12} \in \mathbb{F}$, $b_{32} \in \mathbb{F}$. One can immediately check that $X = QJD'$, where

$$Q = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 1 & 1 & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in GL(3, \mathbb{F}), \quad D' = \mathrm{diag}(a_{21}, a_{22}, a_{23}) \in \mathrm{diag}(3, \mathbb{F}[\sqrt{d}])$$

Assume $n > 3$. Denote the constructed above $J \in GL(n, \mathbb{F}[\sqrt{d}])$ by J_n. We are going to prove that if $X \in GL(n, \mathbb{F}[\sqrt{d}])$ satisfies $\overline{X} = XSD$, then using elementary row operations with entries from $\mathbb{F}$ and multiplying columns by proper elements from $\mathbb{F}[\sqrt{d}]$ we can bring X to J_n.

We will need the following operations on a matrix $M = \{m_{pq}\} \in \mathrm{Mat}(n)$

1. $u_n(M) = \{m_{pq},\ p, q = 2, 3, \ldots, n-1\} \in \mathrm{Mat}(n-2)$;
2. $g_n(M) = \{m_{pq}\} \in \mathrm{Mat}(n+2)$, where m_{pq} are already defined for $p, q = 1, 2, \ldots n$, $m_{00} = m_{n+1,n+1} = 1$ and the rest $m_{0,a} = m_{a,0} = m_{n+1,a} = m_{a,n+1} = 0$.

It is clear that $u_n(X)$ satisfies the twisted cocycle condition. However, its determinant might vanish. To avoid this complication, we note that columns $2, 3, \ldots, n-1$ of X are linearly independent. Applying elementary row operations (in fact, they are permutations) we obtain a new cocycle X_1, which is equivalent to X and such that $u_n(X_1)$ is a cocycle in $GL(n-2, \mathbb{F}[\sqrt{d}])$. Then, by induction, there exist $Q_{n-2} \in GL(n-2, \mathbb{F})$ and a diagonal matrix D_{n-2} such that

$$Q_{n-2} \cdot u_n(X_1) \cdot D_{n-2} = J_{n-2}$$

Consider

$$X_n = g_{n-2}(Q_{n-2}) \cdot X_1 \cdot g_{n-2}(D_{n-2})$$

Clearly, X_n is a twisted cocycle equivalent to X and $u_n(X_n) = J_{n-2}$.

Applying elementary row operations with entries from $\mathbb{F}$ and multiplying by a proper diagonal matrix we can obtain a new cocycle $Y_n = (y_{pg})$ equivalent to X with the following properties:

1. $u_n(Y_n) = J_{n-2}$;
2. $y_{12} = y_{13} = \ldots = y_{1,n-1} = 0$ and $y_{n2} = y_{n3} = \ldots = y_{n,n-1} = 0$;
3. $y_{11} = y_{1n} = 1$, here we use the fact that if $y_{pq} = 0$, then $y_{p,n+1-q} = 0$.

It follows from the cocycle condition $\overline{Y_n} = Y_n \cdot S \cdot \mathrm{diag}(h_1, \ldots, h_n)$ that $h_1 = h_n = 1$ and hence, $y_{n1} = \overline{y_{nn}}$.

Now, we can use the first row to achieve $y_{n1} = -y_{nn} = \sqrt{d}$ and after that, we use the first and the last rows to "kill" $\{y_{k1},\ k = 2, \ldots, n-1\}$. Then the set $\{y_{kn},\ k = 2, \ldots, n-1\}$ will be "killed" automatically. We have obtained J_n from X and thus, have proved that X is equivalent to J_n. □

Now investigate twisted cohomologies associated to arbitrary non-skewsymmetric r-matrices. The following two results will prove to be useful for our study.

Lemma 4.10. *Assume $X \in \overline{Z}_{BD}(sl(n), r)$. Then there exists a twisted cocycle $Y \in GL(n, \mathbb{F}[\sqrt{d}])$, associated to r, and equivalent to X.*

Proof. We have $X \in GL(n, \overline{\mathbb{F}})$ and for any

$$\sigma \in Gal(\overline{\mathbb{F}}/\mathbb{F}[\sqrt{d}]), \quad X^{-1}\sigma(X) \in C(r)$$

On the other hand, the Belavin-Drinfeld cohomology for $sl(n)$ associated to r is trivial. This implies that X is equivalent to the identity, where in the equivalence relation we consider $\mathbb{F}[\sqrt{d}]$ instead of $\mathbb{F}$. So there exists $Y \in GL(n, \mathbb{F}[\sqrt{d}])$ and $C \in C(r)$ such that $X = YC$. On the other hand, $Y \in \overline{Z}_{BD}(sl(n), r)$ since

$$(\mathrm{Ad}_{X^{-1}\sigma_2(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma_2(X)})(r) = r^{21} \quad \Longrightarrow \quad (\mathrm{Ad}_{Y^{-1}\sigma_2(Y)} \otimes \mathrm{Ad}_{Y^{-1}\sigma_2(Y)})(r) = r^{21}$$ □

Proposition 4.11. *Let r be a non-skewsymmetric r-matrix associated to an admissible triple $(\Gamma_1, \Gamma_2, \tau)$ satisfying $s(\Gamma_1) = \Gamma_2$ and $s\tau = \tau^{-1}s$. If $X \in \overline{Z}_{BD}(sl(n, \mathbb{F}), r)$, then there exist $R \in GL(n, \mathbb{F})$ and $D \in \mathrm{diag}(n, \overline{\mathbb{F}})$ such that $X = RJD$.*

Proof. According to Lemma 4.10, $X = YC$, where $Y \in GL(n, \mathbb{F}[\sqrt{d}])$ and $C \in C(r)$. Since

$$(\mathrm{Ad}_{Y^{-1}\sigma_2(Y)} \otimes \mathrm{Ad}_{Y^{-1}\sigma_2(Y)})(r) = r^{21}, \quad (\mathrm{Ad}_S \otimes \mathrm{Ad}_S)(r) = r^{21}$$

it follows that $S^{-1}Y^{-1}\sigma_2(Y) \in C(r)$. On the other hand, by Lemma 4.11 from [8], $C(r) \subset \mathrm{diag}(n, \overline{\mathbb{F}})$. We get $S^{-1}Y^{-1}\sigma_2(Y) \in \mathrm{diag}(n, \overline{\mathbb{F}})$. Now Theorem 4.9 implies that $Y = RJD_0$, where $R \in GL(n, \mathbb{F})$ and $D_0 \in \mathrm{diag}(n, \overline{\mathbb{F}})$. Consequently, $X = RJD_0C = RJD$ with $D = D_0C \in \mathrm{diag}(n, \overline{\mathbb{F}})$. □

Let T denote the automorphism of $\mathrm{diag}(n, \overline{\mathbb{F}})$ defined by $T(D) = SD^{-1}S\overline{D}$.

Lemma 4.12. *Let r be a non-skewsymmetric r-matrix with centralizer $C(r)$. Let $X = RJD$, with $R \in GL(n, \mathbb{F})$ and $D \in \mathrm{diag}(n, \mathbb{F}[\sqrt{d}])$. Then $X \in \overline{Z}_{BD}(sl(n, \mathbb{F}), r)$ if and ony if $D \in T^{-1}(C(r))$.*

Proof. Let us first note that $X \in \overline{Z}_{BD}(sl(n, \mathbb{F}), r)$ if and only if for any $\sigma \in Gal(\overline{\mathbb{F}}/\mathbb{F}[\sqrt{d}])$, $X^{-1}\sigma(X) \in C(r)$ and $SX^{-1}\overline{X} \in C(r)$. We have $X = RJD$ which implies

$$\overline{X} = \overline{RJD} = RJS\overline{D} = RJDD^{-1}S\overline{D} = XD^{-1}S\overline{D} = XST(D)$$

We immediately get that $SX^{-1}\overline{X} \in C(r)$ if and only if $T(D) \in C(r)$. □

Lemma 4.13. *Let $X_1 = R_1JD_1$ and $X_2 = R_2JD_2$ be two Belavin-Drinfeld twisted cocycles associated to r. Then X_1 and X_2 are equivalent if and only if $D_2D_1^{-1} \in C(r) \cdot Ker(T)$.*

Proof. Assume the two cocycles are equivalent. There exist $Q \in GL(n, \mathbb{F})$ and $C \in C(r)$ such that $X_2 = QX_1C$. Then

$$Q = R_2JD_2C^{-1}D_1^{-1}J^{-1}R_1^{-1}$$

Since $Q = \overline{Q}$ and $\overline{J} = JS$, we get

$$D_2C^{-1}D_1^{-1} = S\overline{D_2C^{-1}D_1^{-1}}S$$

Thus $D_2C^{-1}D_1^{-1} \in Ker(T)$. On the other hand, $C \in C(r) \subset \mathrm{diag}(n, \overline{\mathbb{F}})$, so $D_2C^{-1}D_1^{-1} = D_2D_1^{-1}C^{-1}$. We have obtained that $D_2D_1^{-1} \in C(r) \cdot Ker(T)$. Conversely, if this condition is satisfied, then write $D_2D_1^{-1} = D_0C$, where $C \in C(r)$ and $D_0 \in Ker(T)$. Denote $Q := R_2JD_0J^{-1}R_1^{-1}$. Then, by construction, $Q = \overline{Q}$ and $X_2 = QX_1C$. □

By lemmas 4.12 and 4.13, we get

Proposition 4.14. *Let r be a non-skewsymmetric r-matrix associated to an admissible triple $(\Gamma_1, \Gamma_2, \tau)$ satisfying $s(\Gamma_1) = \Gamma_2$ and $s\tau = \tau^{-1}s$. Then*

$$\overline{H}^1_{BD}(sl(n, \mathbb{F}), r) = \frac{T^{-1}(C(r))}{C(r) \cdot Ker(T)}$$

At this point, one needs the explicit description of the centralizer and its preimage under T.

Lemma 4.15. *Let r be a non-skewsymmetric r-matrix associated to an admissible triple $(\Gamma_1, \Gamma_2, \tau)$. Then the following hold:*

(a) $C(r)$ consists of all diagonal matrices $D = \mathrm{diag}(d_1, \ldots, d_n)$ such that $d_i = s_is_{i+1}\ldots s_n$, where $s_i \in \overline{\mathbb{F}}$ satisfy the condition: $s_i = s_j$ if $\alpha_i \in \Gamma_1$ and $\tau(\alpha_i) = \alpha_j$.

(b) $T^{-1}(C(r))$ consists of all diagonal matrices $D = \mathrm{diag}(d_1, \ldots, d_n)$ such that $d_i = s_is_{i+1}\ldots s_n$, where $s_i \in \overline{\mathbb{F}}$ satisfy the condition: $\overline{s}_is_{n-i} = \overline{s}_js_{n-j}$ if $\alpha_i \in \Gamma_1$ and $\tau(\alpha_i) = \alpha_j$.

Proof. Part (a) can be proved in the same way as Lemma 5.5 from [8] and (b) follows immediately from (a). □

Let us make the following remark. Any admissible triple $(\Gamma_1, \Gamma_2, \tau)$ can be viewed as a union of strings

$$\alpha_{i_1} \xrightarrow{\tau} \alpha_{i_2} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \alpha_{i_k}, \quad \tau(\alpha_{i_k}) \notin \Gamma_1$$

The above lemma implies that elements of $C(r)$ have the property that $s_{i_1} = s_{i_2} = \ldots = s_{i_k}$, i.e. s_i is constant on each string. In turn, elements of $T^{-1}(C(r))$ satisfy

$$\overline{s}_{i_1}s_{n-i_1} = \overline{s}_{i_2}s_{n-i_2} = \ldots = \overline{s}_{i_k}s_{n-i_k}$$

i.e. $\overline{s}_is_{n-i}$ is constant on each string.

Theorem 4.16. *Suppose r is a non-skewsymmetric r-matrix with admissible triple $(\Gamma_1, \Gamma_2, \tau)$ satisfying $s\tau = \tau^{-1}s$. Let $str(\Gamma_1, \Gamma_2, \tau)$ denote the number of symmetric strings not containing the middlepoint. Then*

$$\overline{H}^1_{BD}(sl(n, \mathbb{F}), r) = \left(\frac{\mathbb{F}^*}{N_{\mathbb{F}(\sqrt{d})/\mathbb{F}}(\mathbb{F}(\sqrt{d}))^*}\right)^{str(\Gamma_1,\Gamma_2,\tau)}$$

Proof. Let $\varphi : (\overline{\mathbb{F}}^*)^n \to \mathrm{diag}(n, \overline{\mathbb{F}})$ be the map

$$\varphi(s_1, \ldots, s_{n-1}, s_n) = \mathrm{diag}(s_1 \ldots s_n, s_2 \ldots s_n, \ldots, s_{n-1}s_n, s_n)$$

Consider $\widetilde{T} = \varphi^{-1}T\varphi$. Since $Ker(T) = \varphi Ker(\widetilde{T})$, we have

$$\frac{T^{-1}(C(r))}{Ker(T) \cdot C(r)} \cong \frac{\widetilde{T}^{-1}\varphi^{-1}(C(r))}{Ker(\widetilde{T}) \cdot \varphi^{-1}(C(r))}$$

We make the following remarks:

(i) $(s_1, \ldots, s_n) \in Ker(\widetilde{T})$ if and only if $\overline{s}_i s_{n-i} = 1$ for all $i \leq n-1$ and $\overline{s}_n = s_1 \ldots s_n$.
(ii) $(s_1, \ldots, s_n) \in \varphi^{-1}(C(r))$ is equivalent to s_i is constant on each string of the given triple.
(iii) $(s_1, \ldots, s_n) \in \widetilde{T}^{-1}\varphi^{-1}(C(r))$ implies that $\overline{s}_i s_{n-i}$ is constant on each string.

Step 1.
Suppose that the admissible triple is the disjoint union of two symmetric strings

$$\alpha_{i_1} \xrightarrow{\tau} \alpha_{i_2} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \alpha_{i_k}, \quad \alpha_{n-i_k} \xrightarrow{\tau} \alpha_{n-i_{k-1}} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \alpha_{n-i_1}$$

Here we recall that τ has the property that $\tau(\alpha_{n-j}) = \alpha_{n-i}$ if $\tau(\alpha_i) = \alpha_j$.

Let $(s_1, \ldots, s_n) \in \widetilde{T}^{-1}\varphi^{-1}(C(r))$. Then

$$\overline{s}_{i_1} s_{n-i_1} = \ldots = \overline{s}_{i_k} s_{n-i_k} =: t, \quad \overline{s}_{n-i_1} s_{i_1} = \ldots = \overline{s}_{n-i_k} s_{i_k} = \overline{t}$$

One can check that $(s_1, \ldots, s_n) \in Ker(\widetilde{T}) \cdot \varphi^{-1}(C(r))$. Indeed, let us assume first that $n = 2m+1$. Then $(s_1, \ldots, s_n)$ is the product of the following elements:

$$(s_1, \ldots, s_m, (\overline{s}_m)^{-1}, \ldots, (\overline{s}_1)^{-1}, \overline{s_1 \ldots s_m}), \quad (1, \ldots, 1, s_{m+1}\overline{s}_m, \ldots, s_{n-1}\overline{s}_1, s_n(\overline{s_1 \ldots s_m})^{-1}$$

The first factor belongs to $Ker(\widetilde{T})$ and the second is in $\varphi^{-1}(C(r))$ since the $n-i_1$, $\ldots$, $n-i_k$ coordinates have the constant value t.

Suppose that $n = 2m$. Consider

$$\left(s_1, \ldots, s_{m-1}, r_m, (\overline{s}_{m+1})^{-1}, \ldots, (\overline{s}_1)^{-1}, \overline{s}_n\right), \quad (1, \ldots, 1, s_m/r_m, s_{m+1}\overline{s}_{m-1}, \ldots, s_{n-1}\overline{s}_1)$$

where

$$r_m = \frac{\overline{s_1 \ldots s_{m-1}} s_n}{s_1 \ldots s_{m-1}\overline{s}_n}$$

The first factor is in $Ker(\widetilde{T})$ since $r_m \overline{r}_m = 1$ and the second is in $\varphi^{-1}(C(r))$ since neither $n-i_1$, $\ldots$, $n-i_k$ can be m, and the corresponding coordinates all equal t.

Step 2.
Let us assume that the admissible triple includes a symmetric string

$$\alpha_{i_1} \xrightarrow{\tau} \alpha_{i_2} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \alpha_{i_k} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \alpha_{n-i_k} \xrightarrow{\tau} \alpha_{n-i_{k-1}} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \alpha_{n-i_1}$$

not containing the middlepoint. Let $(s_1, \ldots, s_n) \in \widetilde{T}^{-1}\varphi^{-1}(C(r))$. Then

$$\overline{s}_{i_1} s_{n-i_1} = \ldots = \overline{s}_{i_k} s_{n-i_k} = \overline{s}_{n-i_1} s_{i_1} = \ldots = \overline{s}_{n-i_k} s_{i_k} = t$$

We note that $t \in \mathbb{F}$ since $t = \overline{t}$.

Case 1. Assume there exists $q \in \mathbb{F}(\sqrt{d})$ such that $t = q\overline{q}$. Then $(s_1, \ldots, s_n) \in Ker(\widetilde{T}) \cdot \varphi^{-1}(C(r))$. Indeed, one can make the same construction as in Step 1, except for the positions $i_1, \ldots, i_k$, $n - i_1, \ldots, n - i_k$ where we consider instead the decomposition

$$(\ldots, s_{i_l}, \ldots, s_{n-i_l}, \ldots) = (\ldots, s_{i_l}/q, \ldots, s_{n-i_l}/q, \ldots) \cdot (\ldots, q, \ldots, q, \ldots)$$

Case 2. Assume for any $q \in \mathbb{F}(\sqrt{d})$, $t \neq q\overline{q}$. Then it follows that $(s_1, \ldots, s_n) \notin Ker(\widetilde{T}) \cdot \varphi^{-1}(C(r))$. Indeed, let us assume the contrary, i.e. we may write $s_i = p_i r_i$, where $\overline{p}_i p_{n-i} = 1$ for all $i \leq n - 1$, $\overline{p}_n = p_1 \ldots p_n$ and

$$r_{i_1} = \ldots = r_{i_k} = r_{n-i_1} = \ldots = r_{n-i_k}$$

It follows that

$$t = \overline{s}_{i_1} s_{n-i_1} = \overline{r}_{i_1} r_{n-i_1} = \overline{r}_{i_1} r_{i_1}$$

which is a contradiction.

Step 3.
Let us suppose that the admissible triple includes a symmetric string

$$\alpha_{i_1} \xrightarrow{\tau} \alpha_{i_2} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \alpha_{i_k} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \alpha_{n-i_k} \xrightarrow{\tau} \alpha_{n-i_{k-1}} \xrightarrow{\tau} \ldots \xrightarrow{\tau} \alpha_{n-i_1}$$

containing the middlepoint. In this case

$$\overline{s}_{i_1} s_{n-i_1} = \ldots = \overline{s}_{i_k} s_{n-i_k} = \overline{s}_{n-i_1} s_{i_1} = \ldots = \overline{s}_{n-i_k} s_{i_k} = t$$

Moreover, $t = s_m \overline{s}_m$, where s_m is the coordinate corresponding to the middlepoint α_m. Then again $(s_1, \ldots, s_n) \in Ker(\widetilde{T}) \cdot \varphi^{-1}(C(r))$ since we may proceed as in Step 2, case 1 by taking $q = s_m$. □

Example 4.17. For $\mathbb{F} = \mathbb{R}$ and $d = -1$, it follows that given an r-matrix r with admissible triple $(\Gamma_1, \Gamma_2, \tau)$ we have

$$\overline{H}^1_{BD}(sl(n, \mathbb{R}), r) = (\mathbb{Z}_2)^{str(\Gamma_1, \Gamma_2, \tau)}$$

Example 4.18. Let us consider $\mathbb{F} = \mathbb{C}((\hbar))$ and $d = \hbar$. Then $N(\mathbb{F}(\sqrt{d})) = \mathbb{F}$ and Theorem 4.16 implies that $\overline{H}^1_{BD}(sl(n, \mathbb{C}((\hbar))), r)$ is trivial (consists of one element) for any r-matrix r satisfying the condition of Proposition 4.4 and empty otherwise. We have thus generalized our previous results [9], where we proved that twisted cohomologies for $sl(n)$ associated to generalized Cremmer-Gervais r-matrices are trivial.

This result completes classification of quantum groups which have $sl(n, \mathbb{C})$ as the classical limit. Summarizing, we have the following picture:

1. According to [4, 5], there exists an equivalence between the category $HA_0(\mathbb{C}[[\hbar]])$ of topologically free Hopf algebras cocommutative modulo $\hbar$ and the category $LBA_0(\mathbb{C}[[\hbar]])$ of topologically free over $\mathbb{C}[[\hbar]]$ Lie bialgebras with $\delta \equiv 0 \,(\mathrm{mod}\, \hbar)$.
2. To describe the category $LBA_0(\mathbb{C}[[\hbar]])$, it is sufficient (multiplying by a proper power of $\hbar^N$) to classify Lie bialgebra structures on the Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\hbar))$.

3. Following [6], only three classical Drinfeld doubles are possible, namely

$$D(\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\hbar))) = \mathfrak{g} \otimes_{\mathbb{C}} A_k, \quad k = 1, 2, 3$$

Here

$$A_1 = \mathbb{K}[\varepsilon], \quad \varepsilon^2 = 0, \quad A_2 = \mathbb{K} \oplus \mathbb{K}, \quad A_3 = \mathbb{K}(\sqrt{\hbar}) \quad \text{with} \quad \mathbb{K} = \mathbb{C}((\hbar))$$

4. Lie bialgebra structures related to the case A_1 are in a one-to-one correspondence with quasi-Frobenius subalgebras of $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\hbar))$.
5. Now we turn to the case $D(\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\hbar))) = \mathfrak{g} \otimes_{\mathbb{C}} A_2$ with $\mathfrak{g} = sl(n)$. Up to multiplication by $\hbar^N$ and conjugation by an element of $GL(n, \mathbb{K})$, the related Lie bialgebra structures are defined by the Belavin-Drinfeld data (see [1] and Section 2, the main ingredient is the triple $\tau : \Gamma_1 \to \Gamma_2$) and an additional data called a Belavin-Drinfeld cohomology. In the case $\mathfrak{g} = sl(n)$, the cohomology consists of *one element* independently of the Belavin-Drinfeld data. As a representative of this cohomology class one can choose the *identity matrix*.
6. Finally, in the case A_3 and $\mathfrak{g} = sl(n)$ the description is as follows. Up to multiplication by $\hbar^N$ and conjugation by an element of $GL(n, \mathbb{K})$, the related Lie bialgebra structures are defined by the Belavin-Drinfeld data and an additional data called a *twisted* Belavin-Drinfeld cohomology. In this case the twisted cohomology consists of *one element* if $\tau : \Gamma_1 \to \Gamma_2$ satisfies the condition of Proposition 4.4 and is *empty* otherwise (no Lie bialgebra structures of the type A_3 if τ does not satisfy the condition of Proposition 4.4). If the cohomology class is non-empty, it can be represented by the matrix J introduced before Theorem 4.9.

Appendix A.

Throughout the paper we use the following convenient notations for the arXiv references:

[8] Stolin A and Pop I 2013 arXiv:1303.4046

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[8] Bibliographic description is given in Appendix A
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