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Curved space-times and string theory

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Abstract

We discuss the feasibility of different kinds of curved space-times as backgrounds for string theory. This thesis is divided into two parts. First, we focus on the double copy procedure and apply it to the Born-Infeld solution in open string theory in the approximation when field-strength derivative terms can be ignored. This process leads us to a generalised version of the Schwarzschild metric which looks non-singular at the origin. Whether this metric is a solution to some closed string equations of motion is also discussed. In the second half of the thesis, Kundt space-times are studied. We show how it is possible to obtain solutions to the Weyl anomaly equations for a background formed by a Kundt metric and a dilaton. The inclusion of a B-field is mentioned but not implemented.

Contents

1 Introduction

The search for new solutions to string theory equations of motion is a crucial process in the understanding of the capabilities and connections to previously found results of the theory at hand. In this context, it is not only important to classify these solutions but also to find relations between them that might allow us to comprehend how such solutions may be generalised or reproduced by a systematic procedure. Therefore, we split the following explanation into two very well-differentiated parts.

First, we focus on a method derived from relations between scattering amplitudes that allows to link together solutions to gravity and gauge theory equations of motion. This method is commonly known as the *double copy* procedure and it lead us to find a generalisation of the Schwarzschild metric that looks formally non-singular at the origin starting from the Born-Infeld electric field created by a point charge. The latter is the leading-order solution to the open string effective action when the field-strength derivative terms are ignored. It would be great news if the metric that we found were a solution to some closed string equations of motion. Alas, this is not to be expected as generalisations of the double copy ansatz may be needed to account for the presence of non-trivial α'-corrections as well as a non-vanishing dilaton field on the closed-string side of the problem. It is also important to note that neglecting further corrections to the Born-Infeld solution is a sensitive choice, as these may induce major changes on the resulting double copy metric.

In the second part of our thesis, we focus on a family of metrics known as Kundt spacetimes, which have been given a lot of attention over the years due to their very interesting and important properties. The Kundt family of metrics is very wide and encapsulates some of the most relevant exact solutions to Einstein's equations of General Relativity as well as β-function equations of motion in higher dimensions. Our aim in these notes is to give a general idea of the feasibility of a Kundt metric to be a solution to the Weyl anomaly equations in the presence of additional fields, but also to make some general points on the symmetries of these metrics under certain coordinate transformations. Our work shows that it is possible to find solutions to the leading-order terms of the Weyl anomaly equations in the presence of a metric and a dilaton field under some conditions. This result contributes as a bit of evidence that Kundt metrics may be consistent backgrounds for string theory, but a general proof is still missing.

2 Generalised Schwarzschild metric from double copy of point-like charge solution in Born-Infeld theory

2.1 Preliminaries

The (classical) double copy is a procedure to construct gravity solutions from gauge theory ones. It originated from the KLT relations [1] in string theory and BCJ duality [2, 3, 4, 5] associated to scattering amplitudes in field theory, which allow to understand closed-string scattering amplitudes as products of its open-string counterparts. The underlying reason of why this is allowed remained unclear as an explicit relationship between the lagrangians of both theories was missing. In general terms, the double copy idea is sometimes referred to as "duality between color and kinematics" and it presents applications to a wide range of theories, including supersymmetric ones (see [4] for a review). To show schematically how the double copy for amplitudes works, let us consider two m-point L-loop gauge theory amplitudes in a space-time of arbitrary dimension d [3, 4, 6]:

$$\mathcal{A}_m^{(L)} \propto g^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \, d^d p_l, \tag{2.1}$$

$$\tilde{\mathcal{A}}_m^{(L)} \propto \tilde{g}^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{\tilde{n}_i \tilde{c}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \, d^d p_l, \tag{2.2}$$

where the sums are only over triple-vertex graphs; $g, \tilde{g}$ are the coupling constants; $n_i, \tilde{n}_i$ are kinematic numerators; and $c_i, \tilde{c}_i$ are the colour factors associated to each graph. The $c_i, \tilde{c}_i$ depend on each other through linear relations derived from their Lie algebra, which can be schematically written as $c_k = c_i - cj$, for some triplet of colour factors $\{c_i, c_j, c_k\}$ (same for $\tilde{c}_l$). The BCJ conjecture then claims that it is possible to find some kinematic numerators that also obey linear relations in a one-to-one correspondence with those of the colour factors. Since the relations satisfied by both sets of entities are the same, replacing colour factors by kinematic numerators does not violate gauge-invariance and allows for the construction of new objects. If this duality between colour and kinematics is present in at least one of the two amplitudes shown above, we can replace the colour factors in one of them with the kinematic numerators of the other and construct the following m-point L-loop amplitude [3, 4, 6]:

$$\mathcal{M}_m^{(L)} \propto i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \, d^d p_l, \tag{2.3}$$

which is a typical gravity amplitude after replacing the gauge theory coupling constants by their gravity analogue κ. Notice that if we wanted to go the other way around and obtain a gauge theory amplitude from (2.3), schematically the operation reduces to substituting one of the kinematic numerators by an appropriate set of colour factors. On this spirit, we can ask what we would obtain by performing such a substitution on an amplitude such as (2.1) or (2.2). In this case, for a coupling constant y the result would look like:

$$\mathcal{T}_m^{(L)} \propto i^L y^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{\tilde{c}_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \, d^d p_l, \tag{2.4}$$

which corresponds to a biadjoint scalar theory [6].

A remarkable encounter was that this double copy idea also works when applied to the exact solutions of the underlying gravity and gauge theory equations of motion. A sufficient condition is that the space-time metric on the gravity side admits a Kerr-Schild decomposition as:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi k_\mu k_\nu, \tag{2.5}$$

where $\bar{g}_{\mu\nu}$ is a background metric, ϕ is a scalar field depending on the space-time coordinates and k_μ is a null vector field with respect to both g and $\bar{g}$. The Kerr-Schild family of metrics, first introduced in [7], are specially nice to deal with since quantities such as the Ricci tensor and scalar linearise on the field ϕ [6]:

$$R^\mu{}_\nu = \frac{1}{2} \left[\partial^\mu \partial_\alpha (\phi k^\alpha k_\nu) + \partial_\nu \partial^\alpha (\phi k_\alpha k^\mu) - \partial^2 (\phi k^\mu k_\nu)\right], \tag{2.6}$$

$$R = \partial_\mu \partial_\nu (\phi k^\mu k^\nu). \tag{2.7}$$

One can naively define a gauge field from the scalar ϕ and null vector k_μ as $A_\mu \equiv \phi k_\mu$ and see that if we demand the stationary vacuum Einstein's equations to be satisfied for the metric $g_{\mu\nu}$, then the abelian field strength $F_{\mu\nu} = 2\partial_{(\mu} A_{\nu)}$ solves the vacuum Maxwell equations $\partial_\mu F^{\mu\nu} = 0$. There is also evidence to support that the same works for non-abelian theories [6]. Furthermore, this implies that the scalar field ϕ solves an abelian version of the biadjoint scalar equations of motion, $\partial^2 \phi = 0$. This fact may be used to give a physical interpretation to the double copy procedure from that of the scalar field ϕ, as this can then be understood as a scalar propagator integrated over a given source [6]. The scalar propagator ϕ stays the same after constructing the single copy gauge field and the double copy metric. This is an analogous behaviour to the denominators of the amplitudes of equations (2.1) to (2.4) remaining intact [3, 5, 6].

It is common in the literature to refer to the gauge fields coming from a metric via the double copy as "single copy gauge fields" and to the process of transitioning from gravity to gauge theory as the "single copy". The expression "double copy" would therefore be assigned to the inverse process, while the resulting metric will be called "double copy metric". On the other hand, the transition from gauge theory to the biadjoint scalar theory is usually known as the "zeroth copy", and hence the field ϕ is the "zeroth copy scalar field".

There are multiple examples of the application of the double copy method to exact solutions of the equations of motion of General Relativity and Electromagnetism. In Section 2, we give a review of a few of them. Maybe, the simplest use case is on the relation between the Schwarzschild metric and the Coulomb potential $A_\mu = (\phi(r), 0, 0, 0)$ created by a point charge in four space-time dimensions, where $\phi = Q/r$ [6]. By means of a gauge transformation we can write the gauge field as $A_\mu = \phi(r)\, k_\mu$ for $k_\mu = (1,\ x_i/r)$ a null vector field. Then we may construct the metric $g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu$, which is the Schwarzschild metric with mass $M = 2Q$ in Kerr-Schild form.

So far almost all examples of the double copy started with linear Maxwell fields. The validity and physical origins of the classical double copy construction at the full non-linear, quantum and string theory levels are not clearly understood at present but one might speculate that it may extend beyond the leading order in α' and relate exact open-string and closed-string backgrounds. In [8], a first naive attempt to study such an extension was made and will be explained in the following sections.

Gauge theory equations of motion appear as the leading order approximation to the effective field equations for the massless vector field in the open string theory [9]. The tree-level open string effective action is given, in the abelian case, by the Born-Infeld [10, 11] term $\sqrt{\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}$ [12] plus terms depending on derivatives of the field strength $F_{\mu\nu}$ (for a review see [13]). We may attempt to first ignore all derivative corrections and generalize the Maxwell's theory Coulomb solution to its Born-Infeld counterpart [10, 11]. The corresponding electric field is $E = Q/\sqrt{r_0^4 + r^4}$, where $r_0^2 = T^{-1}Q$ and $T = \frac{1}{2\pi\alpha'}$ is the string tension. In contrast to the Coulomb case here the field is non-singular at the origin. This may be interpreted as a consequence of the inclusion of the α' corrections that are expected to "regularize" point-like singularities in string theory [14]. Since the field of the Born-Infeld solution is approximately constant near the origin, this suggests that it may be possible to consider it as an approximation to a solution of the full (tree-level) open string effective field equations in the region close to $r = 0$.

One may wonder whether this regular Born-Infeld solution may double-copy to a generalization of the Schwarzschild metric that will also be non-singular at the origin.[1] Making the simplest assumption that the form (2.5) with a Minkowski background of the standard "leading-

[1] One may argue that to discuss a possibility of a double copy for Born-Infeld fields one should be assuming that there exists its non-abelian version that satisfies some form of color/kinematics duality.

order" double copy ansatz is not modified by the α'-corrections, the resulting metric with the potential ϕ corresponding to the regular Born-Infeld solution will look formally non-singular at $r = 0$. However, as we will find in Section 4, the corresponding curvature invariants happen to diverge at the origin. This has to do with too slow $\phi \sim r$ decay of the scalar potential at $r \to 0$.

We do not expect this singular α'-dependent double-copy metric to solve a closed-string generalization of the Einstein equations. First, the string-theory generalization of the double copy ansatz may require its non-trivial α'-modification. One may also need to generalize the double copy ansatz to allow for a non-zero dilaton field [15, 16] which is expected to be non-trivial for the closed-string generalization of the Schwarzschild solution beyond the leading order in α'. Finally, our use of the Born-Infeld solution as an approximation to the exact open-string solution may be too naive: it is possible that (a resummation of) the derivative corrections in the open-string equations may lead to a subtle modification of the Born-Infeld solution resulting in a non-singular double-copy metric.

We organise this part of the thesis as follows. In Section 2.2 we give a brief review of some well-known space-time metrics and gauge fields that can be related via the double copy method. Section 2.3 deals with basic aspects of open string effective field theory focusing on the solution obtained when all field-strength derivative terms are ignored, i. e. the Born-Infeld solution. In Section 2.4 we will present the double copy metric corresponding to the Born-Infeld solution and discuss the singularity of the corresponding curvature invariants. Section 2.5 contains a prospect of two possible lines of work that we would like to pursue in the future in relation to the double copy. We give some concluding remarks in Section 2.6 and present three technical appendices (A, B and C) extracted from our paper [8], in which Sections 2.3 and 2.4 are also based on.

2.2 The Classical Double Copy. Known Relations

2.2.1 Coulomb and Schwarzschild

The Schwarzschild metric is a particular case of the more general family of metrics of the form:

$$ds^2 = -\big[1 - \phi(r)\big]\,dt^2 + \frac{dr^2}{1 - \phi(r)} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\ , \tag{2.8}$$

for $\phi = 2M/r$. It can be written in Kerr-Schild form via a change of coordinates to $(\bar{t}, x_i)$, where $\bar{t} \equiv t + 2M\ln(r - 2M)$ is an Eddington-Finkelstein time variable and x_i are the usual cartesian coordinates [6]:

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu \ , \qquad k_\mu \equiv \big(1, \frac{x^i}{r}\big) \ , \qquad k_\mu k^\mu = 0 \ . \tag{2.9}$$

The double copy prescription then allows us to construct an abelian vector field as

$$A_\mu = \phi(r)\, k_\mu \ , \qquad \phi = \frac{Q}{r} \ . \tag{2.10}$$

where we assume that the charge Q in the gauge theory side is defined in terms of the mass M on the gravity side as $Q \equiv 2M$, ignoring the presence of normalization constants on both parameters. The electric potential in (2.10) is gauge-equivalent to the Coulomb potential, $\tilde{A}_\mu = \phi(1,0,0,0)$. The gauge transformation between them is $\tilde{A}_\mu = A_\mu - \partial_\mu \chi$, for $\chi(x^i) = Q\ln(r^2)/2$.

For general $\phi(r)$, the change of coordinates bringing the metric (2.8) to the Kerr-Schild form (2.9) can be found by looking for radial null geodesics of (2.8). Setting $-(1-\phi)dt^2 + \frac{dr^2}{1-\phi} = 0$ gives the following integral representation for t (denoted by $t^*(r)$):

$$t^*(r) = \pm \int \frac{dr}{1-\phi(r)} \ . \tag{2.11}$$

In the Schwarzschild case of $\phi = 2M/r$ this gave $t^*(r) = r + 2M\ln(r - 2M)$. The Kerr-Schild form of (2.8) is then obtained by changing from (t, r, θ, ϕ) to $(\bar{t}, x_i)$ coordinates where x_i are the standard cartesian ones and $\bar{t} \equiv t - r + t^*(r)$. To perform the change of coordinates it is sufficient to use the differential of $\bar{t} = t - r + t^*$, i.e.

$$d\bar{t} = dt + \frac{\phi(r)}{1-\phi(r)}\, dr. \tag{2.12}$$

We can also consider the analogue of the Schwarzschild metric in higher dimensions [6, 18], known as the Tangherlini metric, which may be written in Kerr-Schild form as:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\mu}{r^{d-3}}\ k_\mu k_\nu, \tag{2.13}$$

where d is the number of space-time dimensions and μ is related to the mass. The single copy of the Tanguerlini solution is naturally the higher-dimensional generalisation of the Coulomb field:

$$A^\mu = \left(\frac{gT_a}{\Omega_{d-2} r^{d-3}},\ 0,\ 0,\ 0\right), \tag{2.14}$$

for Ω_{d-2} the area of the $(d-2)$-dimensional sphere. It was noted in [18] that the classical double copy is expected to remain valid in higher-dimensions, as the results for scattering amplitudes, do not depend on the number of dimensions.

2.2.2 Coulomb and JNW Solution

A general statement of the double copy idea can be formulated by saying that gravity states are to be identified as tensor products of gauge theory states. But the tensor product of two spin-1 states will generally involve more than just a symmetric second-rank tensor, and the classical double copy as stated above does not capture this fact [16]. In general, we will expect a second-rank tensor with no specific symmetry properties. This can be split into its irreducible parts as sum of an antisymmetric, symmetric traceless tensors and a scalar. Using the string theory terminology, these would correspond to the NS-NS two-form B-field, the graviton field and the dilaton, [15, 16]. A discussion of this point can be found in [16]. Here, it is suggested that the double copy of the Coulomb field is the JNW solution [19] in General Relativity, which describes a metric minimally coupled to a scalar field and includes the Schwarzschild solution as a particular case. The novelty here is that the JNW solution does not admit a Kerr-Schild form. The possibility of a classical double copy not constrained by a Kerr-Schild ansatz was discussed in some detail in [15]. The JNW metric and dilaton can be written in the following form [15, 16]:

$$ds^2 = -\left(1-\frac{\rho_0}{\rho}\right)^{\gamma} dt^2 + \left(1-\frac{\rho_0}{\rho}\right)^{-\gamma} d\rho^2 + \left(1-\frac{\rho_0}{\rho}\right)^{1-\gamma} \rho^2 d\Omega^2, \tag{2.15}$$

and

$$\phi = \frac{\kappa}{2}\frac{Y}{4\pi\rho_0} \ln\left(1-\frac{\rho_0}{\rho}\right), \tag{2.16}$$

with the definitions:

$$\rho_0 \equiv 2G\sqrt{M^2+Y^2} \quad \text{and} \quad \gamma \equiv \frac{M}{\sqrt{M^2+Y^2}}, \tag{2.17}$$

where M is related to the graviton and Y to the scalar field or dilaton. The event horizon of this solution can be found on this set of coordinates at $\rho = \rho_0$. We can recover the Schwarzschild case for $Y = 0$ and positive M.

The JNW metric cannot be cast into the usual Kerr-Schild form, but a double-copy-inspired analysis is still possible. Following [16], we can rewrite closed string effective theory in a way that T-duality is manifest, sometimes known as "double field theory" or "DFT". The metric and B-field of closed string theory would then be encapsulated under a "generalised metric" that can be casted in a form reminiscent of Kerr-Schild. Studying the equations of motion in this setting it is possible to extract the form of the single copy gauge field and compute the field strength with the JNW data. In the limit where the radial distance to the origin goes to infinity we recover the Coulomb electric field. Whether it is possible to employ this procedure the other way around, i. e. starting with some gauge theory solution and working out what would the closed string theory fields be is not clear. As far as we know, the consistency of the method shown in [16] is only valid in the transition from gravity to gauge theory, but not the other way around.

2.2.3 Kerr Black Hole and Maxwell with a magnetic field

The Kerr space-time metric, describing a rotating uncharged black hole, admits a Kerr-Schild decomposition with background metric being Minkowski and the ϕ and k_μ fields given by [6]:

$$\phi(r) = \frac{2MGr^3}{r^4 + a^2 z^2} \qquad \text{and} \qquad k_\mu = \left(1,\ \frac{rx + ay}{r^2 + a^2},\ \frac{ry - ax}{r^2 + a^2},\ \frac{z}{r}\right), \tag{2.18}$$

where r is defined via the relation:

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{a^2} = 1, \qquad \text{outside of the region} \quad x^2 + y^2 \leq a,\ z = 0, \tag{2.19}$$

for a a constant. r is equal to zero inside such region. The single copy gauge field can be read directly from here:

$$A^a_\mu = \frac{g}{4\pi}\ \phi(r)\ c_a k_\mu, \tag{2.20}$$

which is also a solution to Maxwell's equations, as in the cases of Schwarzschild and Coulomb studied previously, but including as well a magnetic component. One can think of this [6] as a consequence of the rotation characteristic of a Kerr black hole. The extension to higher dimensions of the Kerr black hole is the Myers-Perry black hole [17], which also admits a Kerr-Schild decomposition. Since the number of rotation planes depends of whether the dimensionality of the spacetime is even or odd, we have two different expressions for the double copy fields. If we call d the number of spacetime dimensions, the zeroth copy field ϕ for the Myers-Perry black hole is [6]:

$$\phi(r) = \frac{\mu r^2}{\Pi F}, \qquad \text{for odd } d, \tag{2.21}$$

$$\phi(r) = \frac{\mu r}{\Pi F}, \qquad \text{for even } d, \tag{2.22}$$

and the single copy gauge field k_μ may be written as:

$$k_\mu dx^\mu = dt + \sum_{i=1}^{(d-1)/2} \frac{r(x_i dx_i + y_i dy_i) + a_i(x_i dy_i - y_i dx_i)}{r^2 + a_i^2}, \qquad \text{for odd } d, \tag{2.23}$$

$$k_\mu dx^\mu = dt + \frac{zdz}{r} + \sum_{i=1}^{(d-2)/2} \frac{r(x_i dx_i + y_i dy_i) + a_i(x_i dy_i - y_i dx_i)}{r^2 + a_i^2}, \qquad \text{for even } d, \tag{2.24}$$

where we have the following definitions for Π and F:

$$\Pi \equiv \prod_{i=1}^{(d-2)/2} r^2 + a_i^2, \qquad F \equiv 1 - \sum_{i=1}^{(d-1)/2} \frac{a_i^2(x_i^2 + y_i^2)}{(r^2 + a_i^2)^2}, \tag{2.25}$$

for a_i a rotation parameter, (x_i, y_i) a pair of spatial coordinates, z a spatial coordinate unpaired from (x_i, y_i) and r a radial coordinate. The latter is given by the condition that k_μ is null [17]:

$$\sum_{i=1}^{(d-1)/2} \frac{x_i^2 + y_i^2}{(r^2 + a_i^2)^2} = 0, \qquad \text{for odd } d, \tag{2.26}$$

$$\frac{z^2}{r^2} + \sum_{i=1}^{(d-2)/2} \frac{x_i^2 + y_i^2}{(r^2 + a_i^2)^2} = 0, \qquad \text{for even } d. \tag{2.27}$$

2.2.4 Magnetic Monopole and Taub-NUT spacetime

We have seen how the single copy field of the Schwarzschild metric may be identified with the electric field created by a point-like charge. In the same way, we can prove that the field created by a magnetic monopole can be understood as the single copy of a general Taub-NUT spacetime [20, 21, 22, 23]. This family of metrics are stationary solutions to Einstein's equations and show some very interesting properties (see [22, 23] for more). A general Taub-NUT metric may be written as [23, 24, 27]:

$$ds^2 = -f(r)(dt - 2N\,\cos\theta\, d\phi)^2 + f(r)^{-1}dr^2 + (r^2 + N^2)d\Omega_{(2)}^2, \tag{2.28}$$

where N is the Taub-NUT charge, with no known Newtonian analogue and the function $f(r)$ is given by [23]:

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2 + N^2}, \tag{2.29}$$

with $r_\pm \equiv M \pm r_0$, $r_0^2 \equiv M^2 + N^2$, M being the mass of the source in the Newtonian limit. In Kerr-Schild form, the Taub-NUT metric can be expressed as [23, 26]:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} = \eta_{\mu\nu} + \kappa(\phi k_\mu k_\nu + \psi l_\mu l_\nu), \tag{2.30}$$

where the vector fields k_μ and l_μ are null with respect to both the background Minkowski metric and the full metric. They also satisfy the identities $(k \cdot D)k_\mu = 0$ and $(l \cdot D)l_\mu = 0$, where D is the covariant derivative taken with respect to those two metrics, [22, 23]. A couple of comments are in place. First, we see that this expression is not of the simple Kerr-Schild form that we have seen previously. Instead, we have a slightly generalized version of the ansatz, including two sets of null vectors and scalar fields. Secondly, the reader may also find the Taub-NUT metric in Kerr-Schild form with the Minkowski background metric replaced by a general de Sitter metric $\bar{g}_{\mu\nu}$ as in [22]. We will however continue our work with the form given in (2.30).

There is a particular coordinate system where the Einstein equations linearise for the Taub-NUT metric. We will denote this set of coordinates by $(\tilde{\tau},\ \tilde{\sigma},\ p,\ q)$, following the notation of [22]. In these coordinates, the metric acquires the form [22]:

$$d\tilde{s}^2 = -\frac{\bar{\Delta}_p(d\tilde{\tau}+q^2d\tilde{\sigma})^2+\bar{\Delta}_q(d\tilde{\tau}+p^2d\tilde{\sigma})^2}{q^2-p^2} - 2(d\tilde{\tau}+q^2d\tilde{\sigma})dp - 2(d\tilde{\tau}+p^2d\tilde{\sigma})dq, \tag{2.31}$$

with the definitions $\bar{\Delta}_p \equiv \gamma - \epsilon p^2 + \lambda p^4$ and $\bar{\Delta}_q \equiv -(\gamma - \epsilon q^2 + \lambda q^4)$, where ϵ is a constant, while γ relates to the angular momentum and λ is a cosmological constant. As explained in [27], not all of the constants involved carry physical meaning. In fact, they are to be considered as arbitrary integration constants in the general case and there even exists a family of scale transformations that allows to assign a discrete value to the constant ϵ [27].

Under the $(\tilde{\tau},\ \tilde{\sigma},\ p,\ q)$ coordinate system, the expressions for the null vectors and scalar fields are as follows [22, 23]:

$$k_\mu = (1,\ q^2,\ 0,\ 0), \quad l_\mu = (1,\ p^2,\ 0,\ 0), \quad \phi = \frac{2Np}{q^2-p^2}, \quad \psi = \frac{2Mq}{q^2-p^2}. \tag{2.32}$$

In order to obtain our single copy gauge field, we can perform the substitutions $M\kappa/2 \to (c_aT^a)g_s$ and $N\kappa/2 \to (c_aT^a)\tilde{g}_s$, which are a naive generalization of the one used in the Schwarzschild case above. Also, naively, our gauge field may be defined as [23]:

$$A^a_\mu = c^a(\phi k_\mu + \phi l_\mu) = \frac{2c^a}{q^2-p^2}(Np+Mq,\ Npq^2+Mqp^2,\ 0,\ 0). \tag{2.33}$$

We can see more clearly what this gauge field corresponds to if we perform a change to spherical coordinates, $(\tilde{\tau},\ \tilde{\sigma},\ p,\ q) \to (t,\ r,\ \theta,\ \phi)$. This job can be completed in three steps. First, the variables $\tilde{\tau}$ and $\tilde{\sigma}$ are related to p, q and two new variables τ and σ by [22]:

$$d\tilde{\tau} = d\tau + \frac{p^2}{\bar{\Delta}_p}dp - \frac{q^2}{\bar{\Delta}_q}dq \quad \text{and} \quad d\tilde{\sigma} = d\sigma - \frac{dp}{\bar{\Delta}_p} + \frac{dq}{\bar{\Delta}_q}. \tag{2.34}$$

Second, the variables (τ, σ, p, q) relate to the usual spherical variables through the identities [22]:

$$\tau = t + a\phi, \quad \sigma = \frac{\phi}{a}, \quad q = r \quad \text{and} \quad p = a\cos\theta, \tag{2.35}$$

where we have defined a such that $a^2 \equiv \gamma$. Finally, take the limit where the constant γ, or equivalently a, goes to zero. This constant is related to the angular momentum, so it vanishing will imply spherical instead of spheroidal symmetry. With these changes, the gauge field of equation (2.33) can be written as:

$$A = 2c^a\, \frac{Na\cos\theta + Mr}{r^2 - a^2\cos^2\theta}\, dt + 2c^a\, \frac{M}{r(\lambda r^2 - \epsilon)}\, dr - 2c^a\, \frac{N\sin\theta}{(\epsilon - \lambda a^2\cos^2\theta)\cos\theta}\, d\theta +$$

$$+ 2c^a\, \frac{Nr^2\cos\theta + Mar(1+\cos^2\theta) + Na^2\cos\theta}{r^2 - a^2\cos^2\theta}\, d\phi, \tag{2.36}$$

which, in the limit $a \to 0$, becomes:

$$A = \frac{2c^a M}{r}\, dt + \frac{2c^a M}{r(\lambda r^2 - \epsilon)}\, dr - \frac{2c^a N\sin\theta}{\epsilon\cos\theta}\, d\theta + 2c^a N\cos\theta\, d\phi \tag{2.37}$$

The field-strength built from this gauge field has only four non-vanishing components, of which only two are independent. We can easily compute them to be:

$$F_{rt} = \partial_r A_t = -\frac{2c^a M}{r^2} \quad \text{and} \quad F_{\theta\phi} = \partial_\theta A_\phi = -2c^a N\sin\theta. \tag{2.38}$$

The first of these components may be identified with the Coulomb solution [22, 23]. In the Newtonian limit, the charge M is the mass of the source. The component $F_{\theta\phi}$, on the other hand, exclusively depends on the NUT charge N and corresponds to a magnetic monopole, [23].

2.2.5 PP-waves

PP-waves are exact solutions of both Einstein equations in General Relativity and gauge theory equations of motion. They have very interesting properties and, therefore, have been given a lot of attention in the literature, see [28, 29, 30, 31, 32, 33]. From the gravity side, a plane-wave metric in Brinkmann coordinates can be written as:

$$ds^2 = dudv + K(u,\ x^i)du^2 + dx^i dx_i, \tag{2.39}$$

where u and v are light-cone coordinates, x^i for $i = 1,\ 2, \ldots,\ D-2$ are "transverse" coordinates, and $K(u,x)$ is a function independent of the coordinate v. There is an alternative coordinate system to represent plane-waves in General Relativity, the *Einstein-Rosen* coordinates, but their non-global character makes Brinkmann coordinates easier to work with [28]. Metrics of this form possess a null Killing propagation vector, represented by the components of the derivative along the light-cone direction v [34]. In the context of the classical theory of gravity, space-times of this kind have been named of *plane-fronted waves with parallel rays*, or *pp-waves* for short, a term introduced by Ehlers and Kundt in [35]. They are a time-dependent Ricci-flat solution to any action built from Einstein-Hilbert plus any corrections with arbitrary powers of Riemann tensor contractions and derivatives which are gravitationally covariant. Moreover, they solve the corresponding equations of motion at all orders [34, 36, 37]. Therefore, in particular, the pp-wave metric given above is an exact solution of closed string theory provided that $\partial_i \partial^i K = 0$.

In the context of the double-copy procedure, it is evident that we can construct a plane-wave gauge field from the metric of equation (2.39) even without the Kerr-Schild ansatz [18, 38]. We may just define [18]:

$$A^a_\mu \equiv \phi^a(u, x^i) k_\mu, \tag{2.40}$$

where $\phi(u, x^i) \propto K(u, x^i)$ and k_μ is defined such that $k_\mu dx^\mu \equiv du$. This gauge field is of the plane-wave type, standing as an exact solution to Maxwell's equations and, more generally, to open string theory to all orders in the α'-expansion [39], provided that ϕ is harmonic in transverse space, a condition that is already fulfilled from the gravity side.

The case of the double copy procedure for pp-waves can also be explored from the viewpoint of amplitudes, as was done for example in [28]. There is also an alternative method, the so-called *Weyl double copy*, which instead of relating metrics and gauge fields it works on the field-strengths and Weyl tensors. It has proven to solve some inconveniences of the Kerr-Schild double copy [23] and has been applied to pp-waves as well [38].

2.2.6 Gauge and Gravity Shock-waves

We can transform the Schwarzschild solution to obtain a shockwave-like solution to Einstein's equations, a procedure first carried out by Aichelburg and Sexl in 1971 [40, 41]. This new solution in General Relativity is therefore known as the Aichelburg-Sexl shockwave and admits a representation in Kerr-Schild form, so it is in principle possible to discuss it in the context of the double copy procedure. This has in fact been verified and represents one of the simplest time-dependent cases where the double copy recipe applies [6, 42], together with the even simpler case of pp-waves discussed previously.

Following from the general expression of a metric in Kerr-Schild form (2.5), the Aichelburg-Sexl shock-wave metric corresponds to the following choice of the background metric $\tilde{g}_{\mu\nu}$, scalar field ϕ and vector field k_μ [41]:

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu}, \quad k_\mu = (1, \ -1, \ 0, \ 0) \quad \text{and} \quad \phi = \frac{\kappa^2 M}{8\pi} \ln\left(\frac{\rho}{\rho_0}\right) \delta(t-x), \tag{2.41}$$

where $\rho_0 \equiv Constant$ and $\rho \equiv \sqrt{y^2+z^2}$ if the shock-wave is propagating along the direction give by the x coordinate. We define κ here such that $\kappa^2 \equiv 16\pi G$ [41]. We can read off from here the expression of the corresponding single copy gauge field to be [41]:

$$A^a_\mu \propto \frac{gc^a}{4\pi} \ln\left(\frac{\rho}{\rho_0}\right) \delta(t-x)\, k_\mu, \tag{2.42}$$

which relates to a gauge theory shock-wave, as one could intuitively foresee [41].

2.2.7 Further Developments

After exploring all the cases shown above, an evident question is whether or not the double copy still works when the space-time is not four-dimensional. We already commented without much detail on the cases of the Schwarzschild and Kerr black holes in higher dimensions, i. e. the Tangherlini and Myers-Perry space-times, respectively. There are a number of works dealing with other higher-dimensional metrics, such as D-Type space-times or black branes (see, for example, [6, 23]).

On the other side of the spectrum, little work has been done in lower dimensions. We may cite [43], where the workings of the double copy in three dimensions were studied. The kind of black hole solutions living on a three-dimensional space-time are collectively known as *BTZ black holes* [44]. These do not have a Newtonian limit due to the lack of propagating degrees of freedom for the graviton, but admit a Kerr-Schild form and so are suitable for a double copy analysis. The single copy of the BTZ black hole corresponds to a constant charge density extending to all space [43]. A remarkable detail is that the mass on the gravity side and the charge on the gauge theory side are not related to each other. What is more, the mass does not play any role in the dynamics of the single-copy theory. Starting from gauge theory, one can also ask what is the double copy of a point-charge living in three space-time dimensions. The answer is gravity coupled to a dilaton field [43].

Attempts to accommodate the double copy idea to metrics that do no admit a Kerr-Schild decomposition have been carried out in some works, see [16] for instance. Another approach is to elevate the double copy procedure from the metric and gauge field to the level of field strengths and Weyl tensors by means of the Weyl double copy [23, 38]. It was shown to be consistent with the classical double copy, proving itself useful to find new relations and double copy interpretations where the classical procedure could not reach. In terms of spinors, the Weyl double copy relation reads:

$$C_{ABCD} = \frac{1}{S}\, f_{(AB}f_{CD)}, \tag{2.43}$$

where C is the spinor for the Weyl tensor, f the analogue for the electromagnetic field strength and S is a function associated to the zeroth copy.

Another interesting line of research asks whether the double copy idea may be used to find the gravity (gauge theory) analogues of certain symmetries known to be present in the gauge theory (gravity) side. As an example, [23] studies what are the gravity analogues of S-duality and charge conjugation in electromagnetism. S-duality refers to the invariance of Maxwell's equations under a suitable exchange of the electric and magnetic fields, while charge conjugation may be schematically understood as a sign inversion on the electric charge. Both were shown to be present as well in the more general Yang-Mills theory. We can interpret these symmetries as solution generating techniques in the sense that given a solution to Maxwell's theory, we can construct a different solution by simple application of the corresponding symmetry transformation. In this spirit, we know of at least two solution generating techniques in General Relativity that allow to build solutions to Einstein's equations from previously known ones: the Ehlers transformation [35] and the Buchdahl's reciprocal transformation [45]. The former was shown to be the double copy analogue of S-duality, while the latter was identified with charge conjugation [23].

Finally, we will briefly mention a correspondence found recently in [46], dubbed the *Newman-Penrose map*. This maps enhances the double copy idea in that it applies to Kerr-Schild metrics that need not be time-independent or vacuum solutions. However, it also has some drawbacks. For instance, since the Newman-Penrose construction is based on the tetrad formalism, there is no obvious higher-dimensional extension, which in the Kerr-Schild double copy is evident. Furthermore, the set of Kerr-Schild metrics that the Newman-Penrose map was applied to in [46] must be such that the Kerr-Schild vector is tangent to a congruence with a non-vanishing expansion that is null, geodesic and shear-free. It was argued [46] that such a constraint is not very restrictive, since most of the cases of interest are of this form. In essence, the Newman-Penrose idea may be summarised as follows. For any metric that admits a Kerr-Schild decomposition and satisfies the properties listed above for corresponding Kerr-Schild vector, one can find a complex scalar field Φ that is harmonic for which $A = \tilde{k}\Phi$ is self-dual[2] and solves the vacuum Maxwell equations. Here, $\tilde{k}$ is a one-form understood as a differential operator, which plays the role of the Kerr-Schild vector in the classical double copy. The authors of [46] outlined the possible lines of research that could be followed to uncover the full potential of this new relationship in the spirit of the double copy. To cite a couple of them, it would be interesting to see if the Newman-Penrose map also applies to complex Kerr-Schild metrics or if it may provide a work with non-abelian gauge fields.

[2]Another work that comments on how to apply the Kerr-Schild double-copy idea to self-dual solutions is [6].

2.3 Open string effective action and the Born-Infeld solution

In this section we give an explanation of how the Born-Infeld solution of modified electrodynamics [10, 11] may be obtained from bosonic open string effective theory. The explanation follows closely that of [8], where we investigated what the double copy of such a solution might be.

The effective action for the abelian gauge field in the bosonic open string theory has the following structure [12, 47, 48] after considering reduction to four dimensions and setting $T^{-1} \equiv 2\pi\alpha'$:

$$S = c\int d^4x \,\sqrt{-\det(\eta_{\mu\nu} + T^{-1}F_{\mu\nu})}\,\Big[1 + T^{-3}\,B^{\mu\nu\rho\sigma\lambda\gamma}(T^{-1}F)\,\partial_\mu F_{\nu\rho}\partial_\sigma\,F_{\lambda\gamma} + \mathcal{O}(\partial^4 F)\Big], \tag{2.44}$$

where the ∂F-independent part is the Born-Infeld action and B is a particular function of the field-strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Explicitly, the leading order α'^5 derivative terms are [48]

$$\begin{aligned} S =& c\int d^4x\Big(\sqrt{-\det(\eta + T^{-1}F)} - \frac{1}{48\pi}\,T^{-5}\Big[(\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\mu\nu})F_{\rho\sigma}F^{\rho\sigma} \\ &+ 8(\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\nu\lambda})F_{\lambda\rho}F^{\rho\mu} + 4(\partial_\alpha F_{\mu\nu})(\partial^\beta F^{\mu\nu})F_{\beta\lambda}F^{\alpha\lambda}\Big] + \mathcal{O}(T^{-7})\Big)\ . \end{aligned} \tag{2.45}$$

The resulting equation for $F_{\mu\nu}$ may be written as:

$$\begin{aligned} &2\partial_\mu\Big[\frac{\partial\sqrt{-\det(\eta + T^{-1}F)}}{\partial F_{\mu\nu}}\Big] - \frac{1}{12\pi}T^{-5}\Big[(\partial_\alpha F_{\lambda\gamma})(\partial^\alpha F^{\lambda\gamma})(\partial_\mu F^{\mu\nu}) \\ &\quad + 2(\partial_\mu\partial_\alpha F_{\lambda\gamma})(\partial^\alpha F^{\lambda\gamma})F^{\mu\nu} + 4\partial_\mu\big[(\partial_\alpha F_{\sigma\gamma})(\partial^\mu F^{\sigma\gamma})F^{\alpha\nu}\big] \\ &\quad + 4\partial_\mu\big[(\partial_\alpha F_{\beta\gamma})(\partial^\alpha F^{\gamma\mu})F^{\nu\beta} + (\partial_\alpha F_{\gamma\lambda})(\partial^\alpha F^{\nu\gamma})F^{\lambda\mu}\big]\Big] + \mathcal{O}(T^{-7}) = 0\ . \end{aligned} \tag{2.46}$$

The Born-Infeld equation corresponding to the vanishing of the first term here is equivalent to $(\eta - T^{-2}F^2)^{-1}_{\ \lambda\mu}\,\partial^\lambda F^{\mu\nu} = 0$.

Ignoring the contributions of the derivative correction terms in (2.45) let us look for a point-like charge solution of the Born-Infeld term in (2.46). In the purely electric case the Born-Infeld part of (2.46) reduces to $\partial_i\left(E_i/\sqrt{1 - T^{-2}E^2}\,\right) = 0$. If the electric field is spherically symmetric (corresponding to a point-like charge), i.e. has only a radial component depending on r, one finds [10][3]

$$E_r = F_{0r} = -\partial_r A_0(r) = \frac{Q}{\sqrt{r_0^4 + r^4}}\ , \qquad\qquad r_0^2 \equiv T^{-1}Q. \tag{2.47}$$

In contrast to the standard Coulomb solution the Born-Infeld solution is regular at $r = 0$. Since the electric field (2.47) is approximately constant near $r = 0$, one may hope that at least near

[3] The Born-Infeld solution has been extensively studied in the literature. A few works that we can cite include [13, 49, 50, 51, 52, 53, 54].

the origin this background may be trusted as a solution to the full open string effective action, including the derivative corrections. A further discussion of this point is presented in Appendix A.

Our aim below will be to construct the double copy metric corresponding to the scalar potential in (2.47) that generalizes the Schwarschild metric which is the double copy of the Coulomb potential.

2.4 Double copy of the Born-Infeld solution

To determine the classical double copy metric for the Born-Infeld solution in (2.47) we will follow the general steps given in the introduction. To do this, we will need to compute the gauge field A_μ. Its zeroth component, the electric potential, is given by (2.47) and can be calculated by integrating that equation over r with the boundary condition $A_0\big|_{r\to\infty} \to 0$. The result reads:

$$A_0(r) \equiv \phi(r) = \int_r^\infty dr'\, E_r(r') = \frac{Q}{r}\ {}_2F_1\Big(\frac{1}{4},\ \frac{1}{2},\ \frac{5}{4}, -\frac{r_0^4}{r^4}\Big) = \frac{Q}{r}\Big[1 - \frac{r_0^4}{10r^4} + \mathcal{O}(\frac{r_0^8}{r^8})\Big]\ , \qquad (2.48)$$

where ${}_2F_1$ is the standard hypergeometric function, also known as Gauss's hypergeometric function. Under our assumptions in the previous section, this is the only non-zero component of A_μ, i. e. the gauge field takes the form $A_\mu = (\phi, 0, 0, 0)$. In analogy with the Schwarzschild case of Section 2.1, we require our gauge field to be of the form (2.10). We can achieve this via a gauge transformation and the result is:

$$A_\mu = \phi(r)\, k_\mu = \phi(r)\,\Big(1,\ \frac{x_i}{r}\Big)\ . \qquad (2.49)$$

We now use (2.9) with $\phi(r)$ given by (2.48) to build the double copy metric as $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ with $x^\mu = (\bar{t}, x_i)$ and $\bar{t}$ related to t as in (2.12). This metric can be written in the Schwarzschild-like form (2.8) with $\phi(r)$ as in (2.48) by using (2.12) and the transformation relations between cartesian and spherical coordinates. The fact that our metric admits such a representation may indicate that it is indeed a generalization of the Schwarzschild solution for non-zero $r_0^2 = 2\pi\alpha' Q$. Furthermore, the components of our double copy metric look non-singular due to the fact that the series expansion of the function $\phi(r)$ near $r = 0$ is regular:

$$\phi(r) = c_0 + c_1 r + c_5 r^5 + \mathcal{O}(r^9) = \frac{Q}{r_0}\Gamma(\tfrac{5}{4})\Big[\Gamma(\tfrac{1}{4})\sqrt{\pi} - \frac{r}{\Gamma(\frac{1}{4})\, r_0} + \frac{r^5}{5r_0^5}\Big] + \mathcal{O}(r^9)\ . \qquad (2.50)$$

This became a striking discovery after we found out that the corresponding curvature invariants still presented a singular behaviour for small r. To give an example, the scalar curvature is given by

$$R = \frac{2\phi(r)}{r^2} - \frac{2Q(r^4+2r_0^4)}{r(r^4+r_0^4)^{3/2}} = \frac{2Q\,\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}{r_0\sqrt{\pi}}\frac{1}{r^2} - \frac{4Q[\Gamma(\frac{1}{4})+2\Gamma(\frac{5}{4})]}{r_0^2\,\Gamma(\frac{1}{4})}\frac{1}{r} + \mathcal{O}(r^3)\ . \tag{2.51}$$

This singularity is due to the presence of the first two (c_0 and $c_1 r$) terms in the $r \to 0$ expansion of ϕ in (2.50). The same happens for the contraction of two Riemann tensors. To be more explicit, let us start with a general ϕ that admits the following regular Taylor expansion near the origin:

$$\phi(r) = c_0 + c_1 r + c_2 r^2 + c_3 r^3 + c_4 r^4 + c_5 r^5 + \mathcal{O}(r^6), \tag{2.52}$$

Hence, for a metric as in (2.8) we obtain a curvature squared that reads

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big|_{r\to 0} = \frac{4c_0^2}{r^4} + \frac{8c_0c_1}{r^3} + \frac{4c_1^2+8c_0c_2}{r^2} + \frac{8(c_1c_2+c_0c_3)}{r} \tag{2.53}$$

$$+\,4(c_2^2+2c_1c_3+2c_0c_4) + 8(c_2c_4+c_1c_4+c_0c_5)r + \mathcal{O}(r^2)\ . \tag{2.54}$$

As we had previously announced, it is indeed the c_0 and c_1 coefficients on the expansion of $\phi(r)$ that are responsible for the non-regular behaviour. For completeness, substituting the expression for $\phi(r)$ that we obtained in (2.50) we find (see also (B.5))

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{4[\phi(r)]^2}{r^4} + \frac{8Q^2(r^8+r_0^4r^4+\frac{1}{2}r_0^8)}{r^2(r^4+r_0^4)^3} \tag{2.55}$$

$$= \frac{4Q^2\,\Gamma(\frac{1}{4})^2\Gamma(\frac{5}{4})^2}{\pi r_0^2}\frac{1}{r^4} - \frac{32Q^2\Gamma(\frac{5}{4})^2}{\sqrt{\pi}\,r_0^3}\frac{1}{r^3} + \frac{4Q^2}{r_0^4}\frac{\Gamma(\frac{1}{4})^2+16\Gamma(\frac{5}{4})^2}{\Gamma(\frac{1}{4})^2}\frac{1}{r^2} + \frac{16Q^2\,\Gamma(\frac{5}{4})^2}{5r_0^7\sqrt{\pi}}r + \mathcal{O}(r^3)\ .$$

The Ricci tensor is also non-trivial for r_0 different from zero (see Appendix B). It is worth pointing out that, in the case of $\phi(r)$ as in the Schwarzschild metric, both (2.51) and (2.55) reduce to $R = 0$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12Q^2}{r^6}$, respectively, as expected.

If $\phi\big|_{r\to 0} = c_0 \neq 0$ then the metric (2.8) has a conical singularity at $r = 0$. This is not a real issue because we can always set $c_0 = 0$ by changing the integration constant in (2.48) (or by a gauge transformation of the potential (2.49)) and then redefine the double copy metric (2.9) using ϕ in this gauge. However, this would also entail that instead of setting $\phi\big|_{r\to\infty} = 0$ we will have to consider the limit $\phi\big|_{r\to\infty} = -c_0$. The real problem is that $\phi\big|_{r\to 0} = c_1 r = c_1\sqrt{x_i^2}$ is non-analytic in cartesian coordinates and this effectively produces a singularity in the curvature invariants. This $c_1 r$ term cannot be eliminated by a gauge transformation as it is responsible for the non-zero constant value of the Born-Infeld electric field $E_r = -\partial_r\phi$ in (2.47) at $r = 0$ (see also Appendix C). Finally, it is not clear if there is some generalization of the Einstein equations for which the metric (2.8) with ϕ in (2.48) is a solution.

2.5 Further Work

We will briefly discuss two lines of work that might be interesting to pursue in the future. Mainly, the application of the classical double copy to the Kerr-Schild-Kundt and Robinson-Trautman families of metrics. We will first give a short explanation of their properties and an outline of past works on the subject to motivate our interest on these space-times . A third line of work that could potentially be of great value in the future, but which we will not consider in this section, is a rigorous analysis of whether the double copy ansatz should be modified for gravity and gauge theory solutions that include α'-corrections. In case that a modification is needed, we would need to reconsider our work on the Born-Infeld solution of Sections 3 and 4, but most importantly, it will widen our understanding of the nature of the double copy procedure itself and possibly lead to new applications.

2.5.1 Kerr-Schild-Kundt Metrics

There is a certain family of metrics called "universal" that solve nearly any covariant gravity equations, including or not any set order of quantum corrections [55]. As this is a very restrictive property, there are not many examples of such metrics and finding them is not an easy task. However, the examples that can be found in the literature do admit a Kerr-Schild form which can be written as follows [56]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu, \tag{2.56}$$

where the background metric is of the de Sitter or anti-de Sitter type and the vector field λ_μ and function V satisfy

$$\lambda \cdot \lambda = 0, \qquad \lambda \cdot \partial V = 0, \tag{2.57}$$

as well as

$$\nabla_\mu \lambda_\nu = \xi_{(\mu}\lambda_{\nu)}, \qquad \xi \cdot \lambda = 0, \tag{2.58}$$

where ξ is an auxiliary vector field defined by the first of these relations. Due to (2.57) and (2.58), λ is non-expanding, shear-free and non-twisting. This means that the metric (2.56) defines a Kundt space-time. Since it is also of the Kerr-Schild form, these metrics have been called "Kerr-Schild-Kundt", or "KSK" for short, metrics [56]. Two examples of this class are the AdS-plane waves and the AdS-spherical waves.

Kundt space-times appear in the study of universal solutions to the Einstein-Yang-Mills theory [57, 58, 59]. It has been shown [59, 57] that a necessary and sufficient condition for a metric and gauge field with a non-zero field-strength to be a solution to the full Einstein-Yang-Mills theory with an arbitrary number of higher-order corrections is that both fields are of the so-called "vanishing scalar invariant" type (or *VSI* for short) satisfying a couple of simple tensor relations. A gauge field with a non-vanishing field-strength is VSI if and only if said field-strength is aligned in a degenerate Kundt space-time and null, where the gauge group is chosen to be compact and semi-simple [60]. On the other hand, the metric is Kundt and satisfies the properties outlined below (2.58).

The fact that the Kundt family of metrics may be written in Kerr-Schild form is motivation enough for us to consider it in the context of the double copy. Such a program was followed in some detail in [58] in the context of the Einstein-Yang-Mills theory. Even though the authors did not pursue a strict application of the double copy idea, they did show that there is a relation between the metric and gauge field in some Einstein-Maxwell-Yang-Mills theories. Therefore, it would be interesting to know what the single copy of a KSK metric is on its own, i. e. not necessarily inside any theory of gravity coupled to electromagnetism. Once this is achieved, it would indeed be interesting to see what equations such a gauge field satisfies and under what conditions it relates to solutions of an Einstein-Yang-Mills theory.

2.5.2 Robinson-Trautman metric

The Robinson-Trautman family of metrics (or *RT metrics* for short) are solutions to the four-dimensional vacuum equations of motion in General Relativity where one of the principal null vector fields of the Weyl tensor is shear and twist-free, has non-zero divergence and is parallel to a congruence of null geodesics [61, 62]. The line element for such a metric may be written as [30]:

$$ds^2 = 2r^2 e^{\Phi} dz\, d\bar{z} - 2dtdr - H(z, \bar{z}; t)dt^2, \tag{2.59}$$

where r is an affine parameter, t is a retarded time coordinate, $(z, \bar{z})$ is a system of conformally-flat coordinates (which can be taken to be the stereographic coordinates on the unit sphere) and H is a function given by:

$$H(z, \bar{z}; t) = r\partial_t \Phi - \nabla^2 \Phi - \frac{2m(t)}{r} - \frac{\Lambda r^2}{3}, \tag{2.60}$$

with m understood sometimes as the physical mass of the system and Λ being a cosmological constant. The function Φ is in principle unknown and depends on the problem at hand. The remarkable fact about these metrics, first studied in [63, 64], is that they bear a close relationship to the so-called Liénard-Wiechert fields (or *L-W* for short), describing the motion of a point charge on a given worldline [65, 66]. The analogy comes into place after noticing that a sufficient and necessary condition for a regular solution to Maxwell's equations to be a Liénard-Wiechert field is that the Maxwell tensor admits a principal null vector with the same properties as those listed above characteristic of the principal null vector of a Robinson-Trautman metric [62]. In fact, an analysis reminiscent of the single copy procedure was carried out in [62] where it was shown how it is possible to extract L-W fields from RT metrics. Therefore, we ask whether there is a double copy relationship between the Liénard-Wiechert and Robinson-Trautman solutions. Here, we will only perform a very basic attempt on writing a general R-T metric in Kerr-Schild form and leave the rest of the discussion for future work. It should be noted that a Kerr-Schild-like form for this family of metrics has already been studied in the past (see [67]).

We aim to write the line element (2.59) as:

$$ds^2 = (\bar{g}_{\mu\nu} + \phi\ k_\mu k_\nu) dx^\mu dx^\nu. \tag{2.61}$$

Let us first try to attempt this on the coordinates $x^\mu = (t, r, z, \bar{z})$. We can then expand (2.61) and identify the coefficients of the coordinate differentials with (2.59). We obtain the following equations:

$$\begin{aligned}
&\bar{g}_{00} + \phi(k_0)^2 = -H, \qquad \bar{g}_{11} + \phi(k_1)^2 = 0, \qquad \bar{g}_{22} + \phi(k_2)^2 = 0,\\
&\bar{g}_{33} + \phi(k_3)^2 = 0, \qquad \bar{g}_{01} + \phi k_0 k_1 = -1, \qquad \tilde{g}_{02} + \phi k_0 k_2 = 0,\\
&\bar{g}_{03} + \phi k_0 k_3 = 0, \qquad \bar{g}_{12} + \phi k_1 k_2 = 0, \qquad \bar{g}_{13} + \phi k_1 k_3 = 0,\\
&\bar{g}_{23} + \phi k_2 k_3 = r^2 e^\Phi.
\end{aligned} \tag{2.62}$$

A solution may be found with the following assumptions:

$$k_2 = k_3 = 0, \qquad \bar{g}_{0i} = \bar{g}_{12} = \bar{g}_{13} = \bar{g}_{22} = \bar{g}_{33} = 0. \tag{2.63}$$

We obtain:

$$k_0 = k_1, \qquad \bar{g}_{23} = r^2 e^\Phi, \qquad \bar{g}_{00} = 1 - H. \tag{2.64}$$

Finally, setting $k_1 \equiv 1$, the Kerr-Schild form of the RT metric in (2.59) is $ds^2 = (\bar{g}_{\mu\nu} + \phi k_\mu k_\nu) dx^\mu dx^\nu$, for

$$\phi = -1, \qquad k_\mu = (1, 1, 0, 0), \qquad \bar{g}_{\mu\nu} = \begin{pmatrix} 1-H & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r^2 e^\Phi \\ 0 & 0 & r^2 e^\Phi & 0 \end{pmatrix}. \tag{2.65}$$

The background metric $\bar{g}_{\mu\nu}$ is not Minkowski. Recall that in spherical coordinates the Minkowski metric is given by:

$$ds^2_{Minkowski} = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.66}$$

If we take $(z, \bar{z})$ to be the stereographic coordinates on the sphere, i. e. $z = e^{i\varphi}\cot(\theta/2)$, then:

$$dz d\bar{z} = \frac{d\theta^2 + \sin^2\theta d\varphi^2}{4\sin^4(\theta/2)}, \tag{2.67}$$

so we end up with

$$ds^2_{Minkowki} = -dt^2 + dr^2 + 4r^2\sin^4(\theta/2) dz d\bar{z}. \tag{2.68}$$

Direct substitution on the equations (2.62) has the same effect as equating this line element with that of $\bar{g}_{\mu\nu}$ in (2.65), so what we are left with the following constraints on the functions H and Φ:

$$H(z, \bar{z}; t) = 2 \qquad \text{and} \qquad e^{\Phi} = 4\sin^4(\theta/2), \tag{2.69}$$

thus reducing the generality of our solution.

Whether the Kerr-Schild metric with parameters as in (2.65) will have some interesting double copy application needs further work. It is of course not ideal for the simplest analysis that the background metric is not Minkowski and whether it does relate to the Liénard-Wiechert family of fields is not clear.

2.6 Concluding remarks

We studied the known applications of the classical double copy procedure on its Kerr-Schild formulation. This idea has gained increasing interest in the last few years mainly due to its promise of bringing gravity and gauge theory closer together. It is too early to ascertain whether the double copy does have some underlying physical interpretation or if it is merely a useful mathematical game. Regardless of the answer to this question, it is undeniable that the several applications found so far in the literature and the increasing number of possibilities to be explored are enough to keep our hopes up on this active area of research.

In Section 1 we introduced the concept of double copy from the scattering amplitudes point of view and quickly went on to explain the classical Kerr-Schild double copy on which we focused on the rest of the document. We presented several examples of application of this idea on the first six titles within Section 2 and summarised other different and highly interesting lines of work on the last subsection.

In the next two sections, we narrowed the analysis to our work in [8]. The bosonic sector of the open-string effective action may be approximated to the Born-Infeld action when field-strength derivative terms are ignored. The solution to its equations of motion is a generalization of the Coulomb solution that includes α'-corrections. Near the origin, i. e. $r = 0$, the Born-Infeld solution is approximately constant, and so it may be trusted to some extent as a solution to the full open string effective theory. We asked whether the classical double copy of the Born-Infeld solution could yield a metric that may be interpreted as a generalisation to the Schwarzschild solution in General Relativity. Our result does indeed have some relation to Schwarzschild and it reduces to it in the $\alpha' \to 0$ limit. One remarkable property of our double copy metric is that although all of its components are regular at the origin, the corresponding curvature invariants are not. We traced back the reason for this divergence to be associated to the slow decay of the scalar potential of the Born-Infeld electric field as the radial distance approaches the origin. We do not speculate on the possibility for this double copy metric to be a solution to some closed-string effective equations of motion.

Finally, we presented a summary of some interesting future lines of work. This includes the concept of "universality" in gravity theories and the double copy of the Robinson-Trautmann family of metrics. Another open question that has not been mentioned here is whether we will be able to map exact open-string and closed-string solutions via the double copy. This program is highly non-trivial and might require an extension of the double copy idea that applies to all orders in the α' expansion.

3 Kundt metrics as consistent backgrounds for string theory

3.1 Preliminaries

Kundt spacetimes have a number of applications in general relativity in four dimensions and have been shown to be relevant as well in higher dimensions in the study of string theory. In general, an n-dimensional spacetime is said to belong to the Kundt class if it admits a null vector ℓ satisfying the following conditions [68]:

$$\nabla_b \ell_a \, \ell^b \propto \ell_a, \quad \nabla_a \ell^a = 0, \quad \nabla_{(b}\ell_{a)} \nabla^{(b}\ell^{a)} = 0, \quad \nabla_{[b}\ell_{a]} \nabla^{[b}\ell^{a]} = 0. \tag{3.1}$$

This guarantees that ℓ is geodesic, expansion-free, shear-free and twist-free within some kinematic frame [68]. A general n-dimensional Kundt metric may be written in the following way [69, 70, 71]:

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = -2dudv + K(u,v,x^k)du^2 + 2A_i(u,v,x^k)dudx^i + G_{ij}(u,x^k)dx^i dx^j, \tag{3.2}$$

where $i,j = 1,\ldots,n-2$ and G_{ij} is a metric on an $(n-2)$-dimensional transverse space. Under certain conditions, coordinate transformations may be used to simplify the form of this metric. For instance, in the absence of the $dudx^i$ components and for a function K of the form $K(u,v,x^k) = H(u)v + \tilde{K}(u,x^k)$ the metric in (3.2) is equivalent to one of the same form but whose components do not show any explicit dependence on the coordinate v (see Section 3.3). Identities of this sort are useful in the search for new solutions of equations of motion where backgrounds of this kind are present.

Kundt spacetimes have several important subclasses that have been studied extensively in the literature. Some of the most well-known examples include exact solutions to Einstein's equations of General Relativity such as pp-waves [69, 72, 73, 74], for which $A_i = 0$, K is independent of v and G_{ij} is flat euclidean space in $n-2$ dimensions. Moreover, some CSI (Constant Scalar Invariants) and all VSI (Vanishing Scalar Invariants) spacetimes also belong to the Kundt class [72, 75]. For (3.2) to be CSI, we require that K is a polynomial of at most second-order in v and the coefficient of the v^2-term be constant. On the other hand, (3.2) being VSI demands that K is linear in v and the transverse metric is flat and euclidean [70].

In four dimensions, all CSI metrics that are not locally homogeneous define Kundt spacetimes. In fact, any four-dimensional lorentzian spacetime metric that is not determined by its curvature invariants must belong to the Kundt class [68, 76]. In the case where K and A_i are independent of v, i. e. when there is no explicit dependence on the coordinate v on the components of the metric, (3.2) is the most general form of a spacetime with a covariantly constant null Killing vector field [70]. If only A_i is independent of v, then (3.2) is a suitable expression for any metric with $Sim(n-2)$ holonomy, and therefore has a recurrent vector field. These metrics, also known as Kundt-RNV metrics, are said to be of *Walker form* [70].

In higher dimensions, there has been progress on the study of Kundt metrics as consistent backgrounds for string theory. In fact, several VSI spacetimes have been found to be such consistent backgrounds at all orders in σ-model perturbation theory [77], where pp-waves are also included as a particular case. Kundt spacetimes in higher dimensions also play a role in the search for supersymmetric solutions to supergravity theories where they may lead to the discovery of new physical phenomena [70].

There is currently no systematic approach to the search for new solutions of General Relativity and string theory. The same is true even if we limit ourselves to the study of one particular set of backgrounds, such as the Kundt class of metrics. We aim to provide evidence on the feasibility of Kundt spacetimes as consistent backgrounds for string theory starting from the case of a simplified Kundt metric coupled to a dilaton field. If the $dudx^i$ components of the metric are linear in the coordinate v and the transverse space is flat, solutions to the β-function equations at leading order can be found for a dilaton linear on the coordinate v. Two different sets of solutions arise depending on whether the coefficient function of the v term in the dilaton is identically equal to zero or not.

The structure of this part of the thesis is as follows. Section 3.2 is devoted to a brief review of a few important examples of Kundt metrics and their classification. In Section 3.3, we analyse some of the properties of a metric such as (3.2). We focus on whether or not it is possible to simplify this expression under certain constraints. In particular, we look at when and how the $dudx^i$ terms may be eliminated as well as to the possibility of removing the v-dependence from the metric components. Section 3.4 explores the solutions of the Weyl anomaly equations in string theory for a background formed by a metric such as (3.2) and a general dilaton $\phi = \phi(u, v, x^k)$. We proceed with our calculations under the assumption that the functions A_i are linear in the coordinate v. Finally, in Section 3.5 we wrap up our discussion with some concluding remarks and point towards possible future work. We also include a couple of appendices, D and E. The first one serves as a reference for some mathematical properties and expressions regarding the metric (3.2) that were used in the derivation of our results. The second one extends Section 3.5 by showing some calculations in relation to future investigations in the subject matter of the content of that section.

3.2 Kundt Metrics. Particular Cases

3.2.1 VSI and CSI metrics

Vanishing Scalar Invariants spacetimes (VSI, for short) are Lorentzian space-time metrics with the property that all scalar curvature invariants constructed from the Riemann tensor and its derivatives are equal to zero at all orders [72, 68]. This condition is satisfied if and only if there exists a null direction ℓ^α with the same properties as the null vector involved in the definition of a general Kundt metric given above (3.2) (see Theorem 1 in [69]). Therefore, all VSI spacetimes belong to the Kundt class, regardless of the space-time dimension, and can be written as in (3.2) but with the following values for the functions K, A_i and the transverse metric G_{ij} [72, 68]:

$$K(u, v, x^k) = a(u, x^k)v^2 + b(u, x^k)v + c(u, x^k), \tag{3.3}$$

$$A_i(u, v, x^k) = X_i(u, x^k) + v\ Y_i(u, x^k), \tag{3.4}$$

$$G_{ij}(u, x^k) = \delta_{ij}. \tag{3.5}$$

VSI spacetimes are part of a more general family known as Constant Scalar Invariants (CSI, for short) spacetimes, made up of metrics whose scalar curvature invariants are not necessarily vanishing but constants. In contrast to the VSI case, not all CSI spacetimes belong to the Kundt class in an arbitrary number of dimensions. A useful result known as the CSI_k conjecture (see [69, 68, 75]) helps with the classification of CSI metrics by asserting that all CSI space-times are either locally homogeneous or of the higher-dimensional Kundt-CSI class. This conjecture is true in four dimensions [75] and Kundt space-times that are CSI can be written as in (3.2) with K and A_i as in (3.3) and (3.4), respectively, but without any further assumptions on the transverse metric components G_{ij} [75, 68, 72].

CSI and VSI space-times have very important physical applications as some of them are exact solutions to the equations of motion of General Relativity and string theory. In fact, all four-dimensional VSI space-times are exact solutions to string theory equations of motion at all orders in σ-model perturbation theory [69]. From a more strictly mathematical point of view, these families of metrics also provide useful insight in the classification of space-times and of whether these can be uniquely characterised by their curvature invariants [69].

3.2.2 PP-waves

The family of metrics known as *pp-waves* has been briefly introduced in 2.2.5 in the context of the classical double copy. This section will be focused on their role as a member of the wider family of Kundt metrics. In Brinkmann coordinates, a general pp-wave metric takes the form given in (2.39), but we will rewrite it here in line with the general expression on (3.2):

$$ds^2 = -2dudv + K(u, x^k)du^2 + \delta_{ij}dx^i dx^j, \tag{3.6}$$

where the sum over the i, j indices is implicit and x^i are a set of transverse coordinates with $i = 1, 2, \ldots, D-2$ in D dimensions. The function K is independent of v and $\{u, v\}$ are a couple of light-cone coordinates. The transverse metric is that of flat space as all VSI metrics belonging to the Kundt class allow for a coordinate transformation $(u, v, x^i) \to (u, v, f^i(u, x^k))$ that transforms the general transverse metric $G_{ij}(u, x^k)$ into δ_{ij} (see [78, 79]). We mentioned in 2.2.5 that the Brinkmann form of the metric is easier to work with when it comes to curvature calculations and regularity. However, Einstein-Rosen coordinates are still used in the study of symmetries as these are more manifest in this coordinate frame [28].

PP-waves stand as one of the simplest examples of a Kundt metric that is an exact solution to string theory equations of motion at all orders in perturbation theory [29, 74, 77, 80]. This is still true even in the presence of certain radiation fields and of a dilaton [29]. PP-waves are a particular case of a VSI metric for which we impose that the null vector ℓ in the definition of a general Kundt metric above (3.2) is also a Killing vector. This requires K and A_i in (3.3) and (3.4), respectively, to be independent of the coordinate v [68, 69]. According to our explanation in section 3.3, it should be possible then to eliminate the $dudx^i$ components of the metric via a change of coordinates and we would end up with the expression given in (3.6).

3.2.3 Gyratons

Gyratons stand as a particular case of a VSI metric and are in close relationship with the D-dimensional pp-wave family of metrics. Their physical interpretation is that of the field created by a beam of radiation or null matter that propagates with a non-zero angular momentum. The angular momentum and energy of the beam are kept finite by imposing that the radius of the cross-section of the source is negligible and that it lasts for a controlled amount of time [72, 81, 82]. The metric of such a space-time is given by that of a pp-wave metric but where the $dudx^i$ terms cannot be eliminated, that is:

$$ds^2 = -2dudv + K(u, x^k)du^2 + 2A_i(u, x^k)dudx^i + \delta_{ij}dx^i dx^j. \tag{3.7}$$

In the pp-wave case, we could eliminate the A_i functions, at least locally (see [72] and section 3.3), but in the gyraton case this is not possible. The reason having to do with the properties of the exterior vacuum pp-wave manifold (for a more complete explanation, see [72]).

In order for (3.7) to satisfy the Einstein equations of General Relativity in vacuum, the following two conditions must be satisfied [81]:

$$\partial_j F_i^{\ j} = 0, \tag{3.8}$$

$$\partial_i \partial^i K - 2\partial_u \partial_i A^i = \frac{1}{2}\, F^2, \tag{3.9}$$

where $F_{ij} \equiv \partial_i A_j - \partial_j A_i$ and $i, j = 1, 2, \ldots, D-2$. These equations allow for an interpretation of the A_i and K functions as the electromagnetic vector and scalar potentials, respectively, in the flat transverse space [72]. We should also mention that there exist generalisations of the metric (3.7) with an explicit dependence on the variable v and which fall as well within the Kundt family [72].

3.3 Kundt Metrics. Coordinate transformations

In N-dimensions, a general Kundt metric takes the form [69, 70, 71]:

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = -2dudv + K(u, v, x^k)du^2 + 2A_i(u, v, x^k)dudx^i + G_{ij}(u, x^k)dx^i dx^j, \tag{3.10}$$

where we will call "transverse space" the spacetime spanned by the coordinates x^k with metric $G_{ij}(u, x^k)$ and latin indices running from 1 to $N-2$. This Kundt metric has a null, geodesic, non-expanding, shear-free vector field. Let us consider the most general case of (3.10) and show when the $dudx^i$ terms can be removed by a change of coordinates. Define v' and x'^k as [86]:

$$v = v' + h(u, x'^k), \qquad x^i = f^i(u, x'^k), \tag{3.11}$$

where h and f^i are arbitrary functions. The K and A_i functions in (3.10) will now depend on u, v' and x'^k after the change of coordinates. For the sake of clarity, we will write this dependence explicitly in the expressions to come. The line element becomes:

$$ds^2 = -2dudv' + K' du^2 + 2A'_m dudx'^m + G'_{mn} dx'^m dx'^n, \tag{3.12}$$

with the following definitions:

$$K' \equiv K(u, v' + h, x'^k) + 2A_i(u, v' + h, x'^k)\partial_u f^i + G_{ij}\partial_u f^i \partial_u f^j - 2\partial_u h, \tag{3.13}$$

$$A'_m \equiv A_i(u, v' + h, x'^k)\partial'_m f^i + G_{ij}\partial'_m f^i \partial_u f^j - \partial'_m h, \tag{3.14}$$

$$G'_{mn} \equiv \partial'_m f^i \partial'_n f^j G_{ij}, \tag{3.15}$$

where $\partial'_m f^i = \partial f^i / \partial x'^m$. For the sake of clarity, since everything depends on the x' coordinates we can redefine $x' \to x$ and get rid of the primes on top of the coordinates and derivatives. For the $dudx'^m \to dudx^m$ term to vanish we just need to impose $A'_m = 0$, i. e.:

$$A_i(u, v' + h, x^k)\partial_m f^i + G_{ij}\partial_m f^i \partial_u f^j - \partial_m h = 0. \tag{3.16}$$

This differential equation does not have a solution when A_i is a general function of v and its other arguments. However, let us explore the case where A_i is linear in v, i. e.

$$A_i(u, v, x^k) = X_i(u, x^k)v + Y_i(u, x^k). \tag{3.17}$$

Then, applying the change of coordinates on v, we have:

$$A_i(u, v', x^k) = X_i(u, x^k)v' + hX_i(u, x^k) + Y_i(u, x^k), \tag{3.18}$$

and (3.16) becomes:

$$-\partial_m h + X_i h \partial_m f^i + G_{ij}\partial_u f^i \partial_m f^j + Y_i \partial_m f^i + (X_i \partial_m f^i)v' = 0, \tag{3.19}$$

which is satisfied if:

$$X_i \partial_m f^i = 0, \tag{3.20}$$

$$-\partial_m h + X_i h \partial_m f^i + G_{ij}\partial_u f^i \partial_m f^j + Y_i \partial_m f^i = 0. \tag{3.21}$$

We can simplify (3.21) by using (3.20) and obtain:

$$\partial_m h = Y_i \ \partial_m f^i + G_{ij}\partial_m f^i \partial_u f^j. \tag{3.22}$$

Notice that this equation is formally the same as (3.16) for the case when A_i is independent of v:

$$\partial_m h = A_i(u, x^k)\partial_m f^i + G_{ij}\partial_m f^i \partial_u f^j. \tag{3.23}$$

If we want to eliminate A_i, which is equivalent to eliminating the $2(N-2)$ functions X_i and Y_i, it does not seem possible to do so by using the $2(N-2)$ equations (3.20) and (3.22) if we only know the $N-1$ functions h, f_i. However, in the case where A_i does not depend on v, it seems reasonable that under certain conditions we would be able to solve (3.23) and simplify our line element to:

$$ds^2 = -2dudv + K(u, v, x^k)du^2 + G_{ij}(u, x^k)dx^i dx^j. \tag{3.24}$$

We reiterate that for the wider case where A_i is a general function of u, x^i and v, we cannot remove the $dudx^i$ terms of (3.10) without any loss of generality [4].

A case of interest is that of spacetimes where the function K is a polynomial on the coordinate v. These metrics, when considered together with the right bosonic fields, can solve certain supergravity and superstring equations. For K quadratic on v:

$$K(u, v, x^k) = \lambda v^2 + \mathrm{H}(u, x^k)v + \tilde{K}(u, x^k), \tag{3.25}$$

all curvature invariants of the metric (3.24) will be constants. Metrics with this property are often called *CSI* (or *Constant Curvature Invariants*) spacetimes. Let us focus for now on a function K linear in v with a coefficient H only dependent on u:

$$K(u, v, x^k) = \mathrm{H}(u)v + \tilde{K}(u, x^k). \tag{3.26}$$

Here, it is possible to further simplify the form of our metric by a simple change of coordinates. We formalise this statement in the following theorem:

Theorem 1. *"A metric of the Kundt family as shown in* (3.10) *with no $dudx^i$ components is equivalent, up to a coordinate transformation, to a metric of the same form but whose components are independent of the coordinate v if the function K is linear on v with a coefficient only dependent on the coordinate u. In other words, a metric of the form* (3.24) *with K given by* (3.26) *is equivalent, up to a coordinate transformation, to a metric of the same form and same K but with* $\mathrm{H} = 0$*."*

Proof. Introduce two new coordinates (U, V) defined such that $v \equiv f(u)V$ and $U \equiv \int du\ f(u)$, where f is an arbitrary function. Then, the line element of our metric becomes:

$$ds^2 = -2dUdV + [2V(\mathrm{H}\frac{f}{2} - f') + \tilde{K}]\frac{1}{f^2}dU^2 + G_{ij}dx^i dx^j. \tag{3.27}$$

Therefore, if $f(u) \equiv \exp\big(\int du\ \frac{1}{2}\mathrm{H}\big)$, then:

$$ds^2 = -2dUdV + \bar{K}dU^2 + G_{ij}dx^i dx^j, \tag{3.28}$$

where we have defined $\bar{K}(U, x^k) \equiv \tilde{K}(u, x^k)/f^2$. Notice that this metric is formally the same as (3.24) but with K independent of v or, equivalently, with K as in (3.26) but for $\mathrm{H} = 0$. In fact, all components of the new metric are independent of the coordinate v. □

[4] In fact, the $dudx^i$ do carry physical meaning. They are important in the case of *gyratonic* solutions to General Relativity [82, 87, 88, 89]. These metrics have been studied since as early as 1970 [90] by solving Einstein's equations with a source moving in a straight line at the speed of light. They are related to the Aichelburg-Sexl family representing a shockwave [82]. See section 3.2.3 for more.

3.4 Weyl Anomaly Equations

We will attempt to solve the Weyl anomaly equations for a dilaton $\phi = \phi(u, v, x^k)$ and the metric of equation (3.10) when A_i depends on v. The Christoffel symbols and Ricci tensor components for this case are shown in Appendix D. We will employ an ansatz where A_i is linear in the coordinate v:

$$A_i(u, v, x^k) = X_i(u, x^k)v + Y_i(u, x^k). \tag{3.29}$$

The β-function equations to consider are[5]:

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi = 0, \tag{3.30}$$

$$-\frac{1}{2}\nabla^2 \phi + G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi = c, \tag{3.31}$$

for the metric and dilaton, respectively, where $c = (D - 26)/6$ for the critical bosonic string, D being the spacetime dimension [85]. We will work in the case where the transverse space part of the metric is flat, i. e. $G_{ij} = \delta_{ij}$. The (vv) component of the equations for the metric implies that the dilaton has to be linear on v:

$$\phi(u, v, x^k) = q(u, x^k)v + p(u, x^k). \tag{3.32}$$

Then, the (vi) equation gives the following relation between X_i and q:

$$\partial_i q = -\frac{1}{2}\, q X_i, \tag{3.33}$$

while the (ij) equation gives the following two equalities:

$$2\partial_i \partial_j q + q(\partial_i X_j + \partial_j X_i) = 0, \tag{3.34}$$

$$2\partial_i \partial_j p + q(\partial_j Y_i + \partial_i Y_j) - \frac{1}{2}(X_i X_j + \partial_i X_j + \partial_j X_i) = 0. \tag{3.35}$$

The (uv) component of (3.30) becomes the following differential equation for the function K:

$$-\frac{1}{2}\partial_v^2 K + q\partial_v K + k_1 v + k_2 = 0, \tag{3.36}$$

where we have defined k_1 and k_2 as:

$$k_1 \equiv -qX^2 - X^i \partial_i q, \tag{3.37}$$

$$k_2 \equiv \frac{1}{2}\partial_i X^i + \frac{1}{2}X^2 + 2\partial_u q - X^i(\partial_i p + qY_i). \tag{3.38}$$

[5]The consistency of these equations with each other can be tested by taking the derivative of (3.30) and using the Bianchi identity. For a formal explanation on how they are obtained in the first place, see for instance [83, 84, 85].

The solution for K can thus be written as:

$$K(u,v,x^k) = c_2(u,x^k) + \frac{c_1(u,x^k)}{2q}\ e^{2qv} - \frac{k_1 + 2k_2 q}{2q^2}\ v - \frac{k_1}{2q}\ v^2. \tag{3.39}$$

Note that this expression for K is not compatible with the case $q = 0$, where (3.36) becomes

$$-\frac{1}{2}\partial_v^2 K + \bar{k}_2 = 0, \tag{3.40}$$

for

$$\bar{k}_2 = \frac{1}{2}\partial_i X^i + \frac{1}{2}X^2 - X^i \partial_i p, \tag{3.41}$$

and the solution for K is:

$$K(u,v,x^k) = \bar{k}_2 v^2 + c_1(u,x^k)v + c_2(u,x^k). \tag{3.42}$$

This expression for K is not a particular case of (3.39). Therefore, in the following we will examine the cases $q = 0$ and $q \neq 0$ separately.

3.4.1 Case $q \neq 0$

For $q \neq 0$, we can use (3.33) in (3.34) to obtain the following relation between $\partial_i X_j$ and X_i:

$$\partial_i X_j = -\frac{1}{2}X_i X_j, \tag{3.43}$$

which allows us to simplify (3.35) to:

$$2\partial_i\partial_j p = -q(\partial_i Y_j + \partial_j Y_i). \tag{3.44}$$

The dilaton equation, (3.31), provides two different relations:

$$X^2 = 0, \tag{3.45}$$

$$-\frac{1}{2}\partial_i\partial^i p + \partial_i p\ \partial^i p + 2qY^i\partial_i p + q^2 Y^2 - 2q\partial_u p - \frac{q}{2}\ \partial_i Y^i + \frac{1}{8}\ X^2 - q^2 c_2 - c = 0. \tag{3.46}$$

The first of these equations implies $X_i = 0$, so $A_i = Y_i$, $k_1 = 0$, $k_2 = 2\partial_u q$, and from (3.33) we get $\partial_i q = 0$, so $q = q(u)$. Also, (3.34) (or (3.43)) is automatically satisfied and we can use (3.35) to simplify (3.46) to

$$\partial_i p\ \partial^i p + 2qY^i\partial_i p + q^2 Y^2 - 2q\partial_u p - q^2 c_2 - c = 0, \tag{3.47}$$

which gives the following expression for c_2:

$$c_2 = -\frac{c}{q^2} + \frac{1}{q^2}\ \partial_i p\ \partial^i p + \frac{2}{q}\ Y^i\partial_i p + Y^2 - \frac{2\partial_u p}{q}. \tag{3.48}$$

The (ui) component of the equations for the graviton is

$$q\partial_i c_2 + 2\partial_u\partial_i p - (qY^j + \partial^j p)(\partial_i Y_j - \partial_j Y_i) + \frac{1}{2}\partial^j(\partial_i Y_j - \partial_j Y_i) = 0, \tag{3.49}$$

which can be simplified by using (3.48) to give

$$\partial^j(\partial_i Y_j - \partial_j Y_i) = 0. \tag{3.50}$$

An expression for Y_i that solves both (3.44) and (3.50) is

$$Y_i(u, x^k) = -\frac{\partial_i p}{q} + Z_i(u), \tag{3.51}$$

where Z_i are arbitrary functions of the variable u. With all the information we have gathered so far, the (uu) component of (3.30) gives the following equation:

$$-\frac{1}{4q}\,\partial_i\partial^i c_1 + \frac{\partial^i p}{q}\,\partial_i c_1 - \frac{1}{2}Z^i\partial_i c_1 + \frac{1}{2}\partial_u c_1 + c_1\left(Z^i\partial_i p + \frac{\partial_i\partial^i p}{2q} - \frac{\partial_i p\,\partial^i p}{q} - \partial_u p + \frac{\partial_u q}{2q}\right) = 0. \tag{3.52}$$

To summarise our results, our initial metric and dilaton can now be written as

$$ds^2 = -2dudv + K(u, v, x^k)du^2 + 2Y_i(u, x^k)dudx^i + \delta_{ij}dx^i dx^j \tag{3.53}$$

$$\phi(u, v, x^k) = q(u)v + p(u, x^k), \tag{3.54}$$

where

$$K(u, v, x^k) = c_2(u, x^k) + \frac{c_1(u, x^k)}{2q}e^{2qv} - 2v\,\frac{\partial_u q}{q}, \tag{3.55}$$

$$c_2(u, x^k) = -\frac{c}{q^2} + Z^2 - \frac{2\partial_u p}{q}, \tag{3.56}$$

subject to (3.52), which is the only remaining differential condition to our problem. The simplest solution to this equation is $c_1 = 0$. A sightly more general case is given by $c_1 \equiv Constant \neq 0$, which implies

$$Z^i\partial_i p + \frac{\partial_i\partial^i p}{2q} - \frac{\partial_i p\partial^i p}{q} - \partial_u p + \frac{\partial_u q}{2q} = 0, \tag{3.57}$$

admitting $q, p \equiv Constant$ as a solution, for instance. This restricts Y_i and c_2 to be functions of u only; the dilaton only depends on v and the function K only has v dependence on the exponential factor on its second term in (3.55).

3.4.2 Case $q = 0$

In this case, equation (3.33) is automatically satisfied, the dilaton is $\phi = p(u, x^k)$, independent of v, and K is given by (3.42). Then, the equation for the dilaton provides the following relation

$$\partial_i p\;\partial^i p - \frac{1}{2}X^i\partial_i p - \frac{1}{2}\partial_i\partial^i p - c = 0, \tag{3.58}$$

the (ij) component of (3.30) becomes

$$\partial_i \partial_j p = \frac{1}{4} \left(X_i X_j + \partial_i X_j + \partial_j X_i\right), \tag{3.59}$$

The (ui) and (uu) components of the equations for the graviton are quite complicated in the general case. Instead of solving them head-on, let us use them to check if a particular Kundt metric that we know is a solution to the vacuum Einstein equations also solves our Weyl anomaly equations for some non-trivial dilaton field. We will focus on the four-dimensional metric of equation (1) of [91], which can be written in the following alternative form by a rescaling of the variable v:

$$ds^2 = -2dudv + \left(\frac{v^2}{x^2} - 8xh\right) du^2 + 2 \cdot \frac{2v}{x}\, dudx + dx^2 + dy^2, \tag{3.60}$$

where $h = h(u, x, y)$ has to be harmonic on the transverse space coordinates for the metric to be a solution to the vacuum Einstein equations. According to our notation, the functions K and A_i that we have been using so far become in this case:

$$K(u, v, x^k) = \frac{v^2}{x^2} - 8xh(u, x, y), \qquad A_x = X_x(x)v = \frac{2v}{x}, \qquad A_y = 0. \tag{3.61}$$

The (uv) component of the equations for the graviton (or just matching this expression for K with (3.42)), we get $\partial_x p = 0$, so $p = p(u, y)$. The equation for the dilaton is

$$\partial_i p \, \partial^i p - \frac{\partial_x p}{x} - \frac{1}{2}\partial_i \partial^i p = c. \tag{3.62}$$

The (ij) component of the equations for the graviton gives $\partial_y^2 p = 0$, so p is linear in y:

$$p(u, y) = P(u)y + Q(u), \tag{3.63}$$

where P, Q are arbitrary functions of u. The (ui) component of the graviton equations then implies that p has to be independent of u, so P, Q become constants and our dilaton is:

$$\phi(y) = Py + Q. \tag{3.64}$$

Furthermore, (3.62) then implies that P is related to the constant c by

$$P^2 = c. \tag{3.65}$$

Finally, the (uu) graviton equation is

$$(\partial_x^2 + \partial_y^2)h - 2P\partial_y h = 0. \tag{3.66}$$

If we avoid the case of a trivial dilaton, given by $P = 0$, this equation can be satisfied by assuming that h is harmonic in the transverse space coordinates and that it is independent of the variable y. It follows then that h has to be linear on x:

$$h(u,x) = h_1(u)x + h_2(u). \tag{3.67}$$

A more general solution of (3.66) may be obtained by assuming that h can be decomposed in the following way:

$$h(u,x,y) = X(u,x)Y(u,y), \tag{3.68}$$

where X, Y are arbitrary functions. Then, it follows that X and Y must satisfy

$$\partial_x^2 X - \lambda(u)X = 0, \tag{3.69}$$

$$\partial_y^2 Y - 2P\partial_y Y + \lambda(u)Y = 0, \tag{3.70}$$

where $\lambda(u)$ is a general function of the variable u. These are second-order differential equations that can be solved to give

$$X(u,x) = X_{01}(u)e^{-x\sqrt{\lambda}} + X_{02}(u)e^{x\sqrt{\lambda}}, \tag{3.71}$$

$$Y(u,y) = Y_{01}(u)e^{y(P-\sqrt{P^2-\lambda})} + e^{y(P+\sqrt{P^2-\lambda})}, \tag{3.72}$$

for $X_{01}, X_{02}, Y_{01}, Y_{02}$ arbitrary functions of u. We see that for h to be real, we need a constant P such that P^2 is always greater than or equal to the maximum value of λ as a function of u. We have thus shown that the metric (3.60) can indeed solve our Weyl anomaly equations for a non-trivial dilaton field (3.64) if the function $h(u,x,y)$ is a solution of (3.66), such as the ones shown in (3.67) and (3.68).

3.5 Concluding remarks

At the beginning, we gave an introduction to the family of Kundt metrics and briefly delved into some of its most important subclasses. Later, we explored the mathematical structure of a general Kundt space-time showing that there are situations where the $dudx^i$ terms of the metric may be eliminated by a change of coordinates without any loss of generality. Later on, we solved the string β-function equations for a background given by a general Kundt metric and a dilaton field. We focused on the case where the coefficients of the $dudx^i$ components of the metric are linear in the coordinate v and the transverse space is flat. The equations soon lead us to a dilaton also linear on the coordinate v. This split our attention into two main cases depending on whether the coefficient function of v in the dilaton vanishes or not. We found particular solutions in both cases under some further assumptions.

The fact that solutions to these equations exist implies that Kundt metrics may be consistent backgrounds for string theory. However, further information is needed to know where such backgrounds are solvable and/or may be associated with relevant and new interesting phenomena. For this purpose, we present Appendix E. Here, we take a look at three different sets of equations and properties, all of them related to the metric of equation (3.60), i.e. the case of a dilaton independent of the variable v. The first aspect we look at is null geodesics, where a solution is obtained. The second is a massless scalar field equation of motion on this metric, which is solvable under our separable variables ansatz of equation (E. 11) for a slight variation of the du^2 component of our metric. Finally, we also look at the string equations of motion on this background and show how a solution may be obtained. The implications of our results are left open for the moment to further research on the subject.

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Appendix A Born-Infeld solution as an approximation to open-string solution

Assuming the same ansatz for $F_{\mu\nu}$ (no magnetic field, time-independent electric field) that led to the Born-Infeld solution in (2.47), only the $\nu = 0$ component of the equations (2.46) is non-trivial and may be written as (ignoring higher order terms in (2.46))

$$\partial_i\Big(\frac{E_i}{\sqrt{1-T^{-2}E^2}}\Big) + \frac{1}{6\pi}T^{-3}\Big[\frac{1}{2}\,(\partial_k E_i)(\partial_k E_i)\partial_j E_j + (\partial_i\partial_j E_k)(\partial_j E_k)E_i$$

$$+\, 2\partial_j[(\partial_i E_k)(\partial_j E_k)E_i] + \partial_i[(\partial_j E_k)(\partial_j E_i)E_k] + \partial_i[(\partial_j E_k)(\partial_j E_k)E_i]\Big] = 0. \qquad \text{(A.1)}$$

Assuming further that E_i is spherically-symmetric we get ($E \equiv E_r(r)$)

$$\partial_r\Big[\frac{r^2 E}{\sqrt{1-T^{-2}E^2}}\Big] + \frac{3}{4\pi}T^{-3}\partial_r E\Big[2rE^2 + r^3(\partial_r E)^2 + 2r^2E(\partial_r E + r\partial_r^2 E)\Big] = 0\ . \qquad \text{(A.2)}$$

From here we may find the leading correction to the Born-Infeld solution coming from the presence of the field strength derivative terms in the open string effective action. Setting $E(r) = E^{(0)}(r) + E^{(1)}(r)$, where $E^{(0)}(r)$ is the Born-Infeld solution (2.47) we obtain from (A.2) the following first-order differential equation for $E^{(1)}$ $(r_0^2 = T^{-1}Q)$:

$$\frac{dE^{(1)}}{dr} + \frac{2(r^4 - 2r_0^4)}{r(r^4 + r_0^4)}\,E^{(1)} = \frac{3r_0^6 r^7(7r^8 - 6r_0^4 r^4 + r_0^8)}{\pi(r^4 + r_0^4)^6}\ . \qquad \text{(A.3)}$$

Its solution may be written as:

$$E^{(1)} = -\frac{r_0^6 r^4(7r^8 + 2r_0^4 r^4 + r_0^8)}{2\pi(r^4 + r_0^4)^5} = -\frac{r^4}{2\pi r_0^6} + \frac{3r^8}{2\pi r_0^{10}} + \mathcal{O}(r^{12})\ . \qquad \text{(A.4)}$$

Its expansion near the origin starts at order r^4 so it does not change the $E\big|_{r\to 0} = \frac{Q}{r_0^2} = \text{const}$ behaviour of the Born-Infeld field (2.47) near the origin, suggesting it can be trusted near $r = 0$. Equivalently, the derivative terms do not alter the leading $c_1 r$ term in the scalar potential (2.50) that was found to be responsible for the singularity of the double-copy metric.

Appendix B Curvature tensor for the double copy metric

The curvature tensor for the metric of the form (2.8) can be computed for general function $\phi(r)$ with the non-trivial components being

$$R^t{}_{rtr} = \frac{\phi''}{2(1-\phi)}, \qquad R^t{}_{\theta\theta t} = R^r{}_{\theta\theta r} = -\frac{r}{2}\,\phi', \qquad R^t{}_{\varphi\varphi t} = R^r{}_{\varphi\varphi r} = R^t{}_{\theta\theta t}\sin^2\theta,$$

$$R^r{}_{ttr} = \frac{1}{2}\,(1-\phi)\,\phi'', \qquad R^\theta{}_{tt\theta} = R^\phi{}_{tt\phi} = \frac{1-\phi}{2r}\,\phi', \qquad R^\theta{}_{r\theta r} = R^\varphi{}_{r\varphi r} = \frac{\phi'}{2r(1-\phi)},$$

$$R^\theta{}_{\varphi\theta\varphi} = R^\varphi{}_{\theta\varphi\theta}\sin^2\theta = \phi(r)\sin^2\theta. \qquad \text{(B.1)}$$

For the Ricci tensor and scalar we get:

$$R_{tt} = -\frac{1-\phi}{2r}(2\phi' + r\phi''), \qquad R_{rr} = \frac{2\phi' + r\phi''}{2r - 2r\phi}, \qquad R_{\theta\theta} = \frac{R_{\varphi\varphi}}{\sin^2\theta} = \phi + r\phi' \ , \tag{B.2}$$

$$R = \frac{2\phi}{r^2} + \frac{4\phi'}{r} + \phi'' \ . \tag{B.3}$$

The explicit form of the Ricci tensor corresponding to the metric (2.8) with ϕ in (2.48) is

$$R_{tt} = \frac{Q\,r_0^4\,(1-\phi)}{r^7(1+\frac{r_0^4}{r^4})^{3/2}} = \frac{Q\,r_0\sqrt{\pi} - Q^2\,\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}{r_0^3\sqrt{\pi}}\frac{1}{r} + \frac{4Q^2\,\Gamma(\frac{5}{4})}{r_0^4\Gamma(\frac{1}{4})} + \mathcal{O}(r^3),$$

$$R_{rr} = \frac{Q\,r_0^4}{(1-\phi)(1+\frac{r_0^4}{r^4})^{3/2}} = -\frac{Q\sqrt{\pi}}{r_0[r_0\sqrt{\pi} - Q\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})]}\frac{1}{r} + \frac{4Q^2\pi\,\Gamma(\frac{5}{4})\Gamma(\frac{1}{4})}{r_0^2[r_0\sqrt{\pi} - Q\,\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})]^2} + \mathcal{O}(r),$$

$$R_{\theta\theta} = \frac{R_{\varphi\varphi}}{\sin^2\theta} = \ \phi(r) - \frac{Q}{r}\Big(1+\frac{r_0^4}{r^4}\Big)^{-1/2} = \frac{Q\,\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}{r_0\sqrt{\pi}} - \frac{r_0^2\,\Gamma(\frac{1}{4}) + 8Q\Gamma(\frac{5}{4})}{2r_0^2\,\Gamma(\frac{1}{4})}r + \mathcal{O}(r^3). \tag{B.4}$$

The curvature squared invariant is

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{4\phi^2}{r^4} + \frac{4\phi'^2}{r^2} + \phi''^2 = \frac{4\phi^2}{r^4} + \frac{8Q^2(r^8 + r_0^4 r^4 + \frac{1}{2}r_0^8)}{r^2(r^4 + r_0^4)^3} \ , \tag{B.5}$$

with its expansion at $r \to 0$ given in (2.55). The Weyl tensor squared is also singular at $r \to 0$

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = \frac{(2\phi - 2r\phi' + r^2\phi'')^2}{3r^4} = \frac{4Q^2\Gamma(\frac{1}{4})^2\Gamma(\frac{5}{4})^2}{3\pi r_0^2}\frac{1}{r^4} + \mathcal{O}(r^{-3}). \tag{B.6}$$

Appendix C Gauge transformation of the vector potential near $r = 0$

Given the vector potential $A_\mu = \phi(r)(1,\ x_i/r)$ with $\phi\big|_{r\to 0} = c_0 + c_1 r + c_5 r^5 + ...$ as in (2.49),(2.50), let us see if there is a gauge transformation that eliminates c_0 and c_1 terms, i.e. if A_μ can be transformed into

$$\tilde{A}_\mu = \tilde{\phi}(r)\big(1, \frac{x_i}{r}\big), \qquad \tilde{\phi}(r)\big|_{r\to 0} = \tilde{c}_5 r^5 + ... \ . \tag{C.1}$$

The relation $\tilde{A}_\mu = A_\mu - \partial_\mu\chi$ implies

$$\partial_0\chi = c_0 + c_1 r + (c_5 - \tilde{c}_5)r^5 + ..., \qquad \partial_i\chi = \frac{c_0 x_i}{r} + c_1 x_i + (c_5 - \tilde{c}_5)x_i r^4 + ... \ . \tag{C.2}$$

These equations lead to

$$\chi(t,x) = \big[c_0 + c_1 r + (c_5 - \tilde{c}_5)r^5\big]\,t + f(x) \ , \tag{C.3}$$

$$\partial_i f(x) = \frac{c_0 x_i}{r} + c_1 x_i + (c_5 - \tilde{c}_5)x_i r^4 - \Big[\frac{c_1 x_i}{r} + 5(c_5 - \tilde{c}_5)x_i r^3\Big]\,t. \tag{C.4}$$

The left-hand side of (C.4) is time-independent, so it is consistent only if $c_5 = \tilde{c}_5$ and $c_1 = 0$. Thus c_1 cannot be eliminated by a gauge transformation.

Appendix D General Christoffel Symbols and Curvature Components

The metric of equation (3.10) with contravariant components [72]

$$G^{vi} = G^{ij}A_j, \qquad G^{vv} = -K + G^{ij}A_iA_j, \qquad G^{uv} = -1, \qquad G^{uu} = 0, \qquad G^{ui} = 0, \quad \text{(D. 1)}$$

has the following non-zero Christoffel connection components [72]:

$$\Gamma_{u\ u}^{\ u} = \frac{1}{2}\partial_v K, \quad \text{(D. 2)}$$

$$\Gamma_{u\ i}^{\ u} = -\Gamma_{v\ i}^{\ v} = \frac{1}{2}\partial_v A_i, \quad \text{(D. 3)}$$

$$\Gamma_{u\ v}^{\ i} = \frac{1}{2}G^{ij}\partial_v A_j, \quad \text{(D. 4)}$$

$$\Gamma_{u\ u}^{\ i} = G^{ij}\partial_u A_j - \frac{1}{2}G^{ij}\partial_j K - \frac{1}{2}A^i\partial_v K, \quad \text{(D. 5)}$$

$$\Gamma_{u\ j}^{\ i} = \frac{1}{2}G^{ik}(\partial_u G_{jk} + \partial_j A_k - \partial_k A_j) - \frac{1}{2}A^i\partial_v A_j, \quad \text{(D. 6)}$$

$$\Gamma_{i\ k}^{\ j} = \bar{\Gamma}_{i\ k}^{\ j}, \quad \text{(D. 7)}$$

$$\Gamma_{u\ v}^{\ v} = \frac{1}{2}A^i\partial_v A_i - \frac{1}{2}\partial_v K, \quad \text{(D. 8)}$$

$$\Gamma_{u\ u}^{\ v} = -\frac{1}{2}(A^2 - K)\partial_v K - \frac{1}{2}\partial_u K + A^i\partial_u A_i - \frac{1}{2}A^i\partial_i K, \quad \text{(D. 9)}$$

$$\Gamma_{u\ i}^{\ v} = -\frac{1}{2}(A^2 - K)\partial_v A_i - \frac{1}{2}\partial_i K + \frac{1}{2}A^j(\partial_u G_{ij} + \partial_i A_j - \partial_j A_i), \quad \text{(D. 10)}$$

$$\Gamma_{i\ j}^{\ v} = \frac{1}{2}(\partial_u G_{ij} - \partial_j A_i - \partial_i A_j) + \frac{1}{2}A^k(\partial_j G_{ik} + \partial_i G_{jk} - \partial_k G_{ij}). \quad \text{(D. 11)}$$

Likewise, the non-zero Ricci tensor components are [72]:

$$R_{vv} = 0, \quad \text{(D. 12)}$$

$$R_{vi} = -\frac{1}{2}\partial_v^2 A_i, \quad \text{(D. 13)}$$

$$R_{vu} = -\frac{1}{2}\partial_v^2 K + \frac{1}{2}\partial_i(G^{ij}\partial_v A_j) + \frac{1}{4}G^{ij}\partial_v^2(A_iA_j) + \frac{1}{2}G^{ij}\partial_i(\ln\sqrt{G})\partial_v A_j, \quad \text{(D. 14)}$$

$$R_{ij} = \bar{R}_{ij} + \bar{\Gamma}_{i\ j}^{\ k}\partial_v A_k - \frac{1}{2}(\partial_v A_i\partial_v A_j + \partial_v\partial_j A_i + \partial_v\partial_i A_j), \quad \text{(D. 15)}$$

$$\begin{aligned} R_{uu} = {} & \frac{1}{2}(\partial_v^2 K - G^{kl}\partial_v A_k\partial_v A_l)(K - A^2) \\ & - \frac{1}{2}(\partial_v K)(\partial_i A^i + A^i\partial_i \ln\sqrt{G} - \partial_u \ln\sqrt{G}) - \frac{1}{2}(G^{ij}A_j\partial_v A_i)^2 \\ & + \frac{1}{2}G^{ij}\partial_v A_j[\partial_i K + 2A^l(\partial_l A_i - \partial_i A_l)] + A^i(\partial_v\partial_u A_i - \partial_v\partial_i K) \\ & - \frac{1}{2}\partial_i(G^{ij}\partial_j K) - \frac{1}{2}G^{ij}\partial_j K\partial_i \ln\sqrt{G} - \frac{1}{2}G^{ij}G^{kl}\partial_l A_j(\partial_i A_k - \partial_k A_i) \\ & + \partial_i(G^{ij}\partial_u A_j) + G^{ij}\partial_u A_j\partial_i \ln\sqrt{G} - \frac{1}{4}G^{ij}G^{kl}\partial_u G_{ik}\partial_u G_{jl} - \partial_u^2 \ln\sqrt{G}, \end{aligned} \quad \text{(D. 16)}$$

$$\begin{aligned}
R_{ui} = & \frac{1}{2}A^j(\partial_v\partial_i A_j - \partial_v\partial_j A_i - \partial_v A_i \partial_v A_j) - \frac{1}{2}\partial_v\partial_i K + \frac{1}{2}G^{jk}\partial_v A_k \partial_i A_j \\
& + \frac{1}{2}\partial_k[G^{jk}(\partial_u G_{ij} + \partial_i A_j - \partial_j A_i - A_j\partial_v A_i)] \\
& + \frac{1}{2}G^{jk}(\partial_u G_{ij} + \partial_i A_j - \partial_j A_i - A_j \partial_v A_i)\partial_k \ln\sqrt{G} \\
& + \frac{1}{2}G^{jk}G^{lm}\partial_k G_{im}(\partial_j A_l - \partial_l A_j + A_j\partial_v A_l - A_l \partial_v A_j) \\
& + \frac{1}{2}\partial_v\partial_u A_i + \frac{1}{2}\partial_v A_i \partial_u \ln\sqrt{G} - \partial_u\partial_i \ln\sqrt{G} - \frac{1}{4}G^{jk}G^{lm}\partial_i G_{km}\partial_u G_{jl}, \qquad \text{(D. 17)}
\end{aligned}$$

where $A^2 \equiv G^{ij}A_iA_j$.

Appendix E Further properties

E.1 Null geodesics

We may also investigate some other properties of the metric (3.60). Let us start by a study of its massless null geodesics. From the geodesics equation, we obtain the following four equations:

$$\ddot{u} = -\frac{1}{x^2}\ \dot{u}\ (v\dot{u} + 2x\dot{x}), \qquad \text{(E. 1)}$$

$$\ddot{v} = -\frac{2v}{x^2}\ \dot{u}\dot{v} + \frac{2}{x}\ \dot{v}\dot{x} + \frac{2v}{x^2}\dot{x}^2 + -8x\partial_y h\ \dot{u}\dot{y} + 4\left(\frac{v^2}{x^3} + 2h - 2x\partial_x h\right)\dot{u}\dot{x} + \left(\frac{v^3}{x^4} - 8v\partial_x h - 4x\partial_u h\right)\dot{u}^2, \qquad \text{(E. 2)}$$

$$\ddot{x} = \frac{1}{x^3}\ \dot{u}(v^2\dot{u} - 4x^3h\dot{u} - 2x^2\dot{v} + 4vx\dot{x} - 4x^4\partial_x h\ \dot{u}), \qquad \text{(E. 3)}$$

$$\ddot{y} = -4x\ \partial_y h\ \dot{u}^2, \qquad \text{(E. 4)}$$

and from the condition of null geodesics, i.e. $ds^2 = 0$, we get

$$-2\dot{u}\dot{v} + \left(\frac{v^2}{x^2} - 8xh\right)\dot{u}^2 + \frac{4v}{x}\ \dot{u}\dot{x} + \dot{x}^2 + \dot{y}^2 = 0. \qquad \text{(E. 5)}$$

Everywhere in these equations, the dot represents derivative with respect to some affine parameter τ. A simple solution can be obtained by assuming $u = Constant$. Then, (E. 1) is automatically satisfied. Equations (E. 3) and (E. 4) imply that x and y are, respectively, linear in the affine parameter:

$$x(\tau) = x_0 + x_1\tau, \qquad y(\tau) = y_0 + y_1\tau, \qquad \text{(E. 6)}$$

where x_0, x_1, y_0, y_1 are constants. Equation (E. 5) gives a relation between the linear coefficients of x and y:

$$x_1^2 + y_1^2 = 0. \qquad \text{(E. 7)}$$

Finally, (E. 2) provides a second order ordinary differential equation:

$$\ddot{v} - \frac{2x_1}{x_0 + x_1\tau}\ \dot{v} - \frac{2x_1^2}{(x_0 + x_1\tau)^2}\ v = 0, \qquad \text{(E. 8)}$$

which solves to

$$v(\tau) = v_0 \; x^{3/2+\sqrt{17}/2} + v_1 \; x^{3/2-\sqrt{17}/2}, \tag{E. 9}$$

where v_0, v_1 are constants and $x = x(\tau)$ is the linear function of τ shown above. The domain of $v(\tau)$ expands over all values of τ such that $x = x_0 + x_1\tau \geq 0$, i.e. for $\tau \geq -x_0/x_1$.

E.2 Massless scalar field

We can also examine the solutions of a massless scalar equation, $-\nabla^2 T = 0$, for some scalar field $T = T(u, v, x, y)$. For the metric of equation (3.60), this takes the form

$$-2\partial_u\partial_v T + \left(\frac{3v^2}{x^2} + 8xh\right)\partial_v^2 T + \frac{4v}{x}\;\partial_v\partial_x T + \partial_x^2 T + \partial_y^2 T + \frac{4v}{x^2}\partial_v T + \frac{2}{x}\partial_x T = 0. \tag{E. 10}$$

A solution to this equation may be obtained by decomposing the scalar field T as

$$T(u, v, x, y) = \varphi(u, v)\Phi(x, y), \tag{E. 11}$$

where φ, Φ are functions of their respective arguments and Φ is harmonic, i.e. $(\partial_x^2 + \partial_y^2)\Phi = 0$. Then, if neither Φ nor $\partial_x\Phi$ vanish, (E. 10) gives

$$-2\partial_u\partial_v\varphi + \left(\frac{3v^2}{x^2} + 8xh\right)\partial_v^2\varphi + \frac{4v}{x^2}\;\partial_v\varphi = 0, \tag{E. 12}$$

$$\partial_v\varphi + \frac{1}{2v}\;\varphi = 0. \tag{E. 13}$$

We can use the second of these equations to simplify the first to

$$\partial_u\varphi + \left(\frac{v}{4x^2} + \frac{6xh}{v}\right)\varphi = 0, \tag{E. 14}$$

Equations (E. 13) and (E. 14) each gives an expression for $\partial_u\partial_v\varphi$. Equating both, we obtain the following for h:

$$h = -\frac{v^2}{24x^3}. \tag{E. 15}$$

This expression for h is incompatible with its definition in the metric of equation (3.60), since there is an explicit v-dependence. Setting h to zero in (E. 14) would then imply $\varphi = 0$ after equating both expressions for $\partial_u\partial_v\varphi$, which in turn leads to the trivial solution $T = 0$. If we relax the condition that h is independent of v and consider a general $h = h(u, v, x, y)$ with v-dependence, then we can still live with (E. 15) and move forward. By direct substitution of (E. 15) into (E. 14), we get that φ has to be independent of u and therefore we can solve (E. 13) to obtain

$$\varphi(v) = \frac{\varphi_0}{\sqrt{v}}, \tag{E. 16}$$

for some constant φ_0. Our final solution for T then takes the form

$$T(v, x, y) = \frac{\varphi_0}{\sqrt{v}}\;\Phi(x, y), \qquad \text{for} \qquad (\partial_x^2 + \partial_y^2)\Phi = 0. \tag{E. 17}$$

E.3 String equations of motion

The metric (3.60) as a target space metric for string theory leads to the following string equations of motion (conformal gauge and light-cone worldsheet coordinates, $\sigma^{\pm} = \tau \pm \sigma$, are used here):

$$\begin{aligned}
u: \quad & vx(\partial_+ u \partial_- v + \partial_+ v \partial_- u) - 2vx\partial_+ x \partial_- x + x^2(\partial_+ v \partial_- x + \partial_+ x \partial_- v) \\
& + x[(v^2 - 8x^3 h)\partial_+ \partial_- u - x^2 \partial_+ \partial_- v + 2vx\partial_+ \partial_- x] - 4x^4(\partial_+ u \partial_- y + \partial_+ y \partial_- u)\partial_y h \\
& - [v^2 + 4x^3(h + x\partial_x h)](\partial_+ u \partial_- x + \partial_+ x \partial_- u) - 4x^4 \partial_u h \; \partial_+ u \; \partial_- u = 0, \qquad \text{(E. 18)} \\
v: \quad & v\partial_+ u \partial_- u + x(\partial_+ u \partial_- x + \partial_+ x \partial_- u + x\partial_+ \partial_- u) = 0, \qquad \text{(E. 19)} \\
x: \quad & v^2 \partial_+ u \; \partial_- u + 2vx^2 \partial_+ \partial_- u \\
& + x^2(4hx\partial_- u \partial_+ u + \partial_+ u \partial_- v + \partial_+ v \partial_- u + x\partial_+ \partial_- x + 4x^2 \partial_x h \; \partial_+ u \; \partial_- u) = 0, \qquad \text{(E. 20)} \\
y: \quad & \partial_+ \partial_- y + 4x\partial_y h \; \partial_+ u \; \partial_- u = 0. \qquad \text{(E. 21)}
\end{aligned}$$

These equations can be solved, for example, for $u = u(\sigma^+)$, so $\partial_- u = 0$. Then, the equation for x, tells us that $\partial_- x = 0$, so $x = x(\sigma^+)$, and the same for the equation for x, from where we get $\partial_- v = 0$, so $v = v(\sigma^+)$. The equations for u and y simplify to:

$$8x \; \partial_+ u \; \partial_- y \; \partial_y h = 0, \qquad \text{(E. 22)}$$

$$\partial_+ \partial_- y = 0. \qquad \text{(E. 23)}$$

Both of these equations may be solved for $\partial_- y = 0$ (other possibilities also hold). In this case, we attain a solution for our system of equations if all variables u, v, x, y depend only on σ^+. A different solution may be obtained if one starts with the opposite assumption that u is only a function of σ^-. Then, the analogue solution to the one showed here is that where all coordinates only depend on σ^-.