All Holographic Four-Point Functions in All Maximally Supersymmetric CFTs

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The anti-de Sitter/conformal field theory (AdS/CFT) correspondence is a remarkable tool for analytically studying strongly coupled physics. Thanks to the AdS/CFT correspondence, strongly correlated quantum systems can be understood as weakly coupled gravity theories, which live in a holographic spacetime with an extra emergent dimension and constant negative curvature. Fundamental observables such as correlation functions are identified with scattering amplitudes in curved spacetime, which, in principle, can be computed by using standard perturbation theory. Unfortunately, such holographic calculations are notoriously difficult, even for tree-level processes involving four external particles. Despite relentless efforts over the past two decades, a full solution to this problem was not found. In this article, we introduce a powerful new method that solves this long-standing problem. We give a closed-form formula for all such four-point functions in a class of theories that constitute the best-known paradigms of AdS/CFT. These models exhaust the theories compatible with maximal supersymmetry, and they live in three, four, and six dimensions. Pivotal to our construction is the use of symmetries. We show that in a judiciously chosen limit, symmetry principles dictate a drastic simplification in holographic correlators, allowing them to be directly computed. Having solved this limit, we further show that the full correlators can be recovered from this special configuration by using only symmetries. In addition to providing valuable explicit expressions that have a wide range of applications in AdS/CFT, our analysis leads to several important conceptual lessons. Our results point out remarkable simplicities underlying the holographic correlators, as well as concrete ways to search for such structures. Moreover, our construction identifies previously unknown elegant underlying organizing principles for holographic correlators. These qualitative features of holographic correlators also echo the exciting progress in the scattering amplitude program in flat space, suggesting tantalizing prospects of future cross-fertilization of ideas.

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I. INTRODUCTION

Understanding nonperturbative phenomena at strong coupling is one of the most challenging open problems of modern physics. Analytic results at strong coupling are scarce and extremely difficult to obtain. However, a remarkable relation conjectured by Maldacena, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–3], provides a rare window through which we can gain analytic insight into strongly coupled physics. The conjecture identifies CFT, with a gravitational theory (M-theory or string theory) that lives in a space with constant negative curvature—AdS space. CFT is located at the boundary of AdS space, and the correspondence gives a concrete realization of the holographic principle [4,5]. The most useful limit to exploit this correspondence is when the bulk gravitational theory becomes weakly coupled, and the dual CFT is strongly coupled. Then, through the weakly coupled bulk description, we can study strong-coupling physics by performing perturbative calculations. This amazing conjecture has withstood numerous tests and has led to a myriad of important theoretical progress and applications (cf. Refs. [6–12] for reviews of the many aspects of the AdS/CFT correspondence).

On the other hand, even though 22 years have passed since its discovery, we are still far from harnessing the full computational power of this correspondence, which becomes particularly clear when considering the most fundamental observables in the CFT—namely, correlation functions of local operators. Various theoretical data can be extracted from these CFT correlators. Via the correspondence, these correlation functions are mapped to scattering amplitudes in AdS space. However, computing these correlators using AdS perturbation theory, in general, is an enormously difficult undertaking, and it is severely underexplored, even at tree level.

We can make the calculations more tractable by delineating a more specialized, yet still sufficiently general and highly nontrivial problem. In order to have better analytic control, we can first restrict ourselves to theories that have the maximal amount of supersymmetry, which leads to the three best-studied paradigms of AdS/CFT [13]:

(i) M-theory on $\mathrm{AdS}_4 \times S^7$ dual to the 3D $\mathcal{N} = 8$ Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [14], with superconformal group $OSp(8|4)$;

(ii) IIB string theory on $\mathrm{AdS}_5 \times S^5$ dual to 4D $\mathcal{N} = 4$ super Yang-Mills theory, with superconformal group $PSU(2,2|4)$;

(iii) M-theory on $\mathrm{AdS}_7 \times S^4$ dual to the 6D $\mathcal{N} = (2,0)$ theory, with superconformal group $OSp(8^*|4)$.

In these examples, the bulk geometry also contains an additional sphere $S^{\mathrm{d}-1}$ as the internal space. From the boundary perspective, the isometry $SO(\mathrm{d})$ of the sphere is interpreted as a global symmetry of the theory, known as R-symmetry. We can further simplify the computation by taking the limit where string theory or M-theory becomes weakly coupled classical supergravity. However, even in this favorable regime and for the above theories, only limited results are available in the literature. The basic CFT operators to consider are the so-called local one-half Bogomol'nyi-Prasad-Sommerfield (BPS) operators. Such operators preserve half of the total supersymmetry and are dual to scalar supergravity fields in AdS. Two- and three-point functions are fully determined by superconformal symmetry [15]. Therefore, starting at four points, we begin to probe the nontrivial dynamics due to strong coupling. We focus on such correlators. The correlators, in principle, can be computed from a diagrammatic expansion in AdS (in terms of the "Witten diagrams") by following a standard procedure similar to the one used for flat-space quantum field theories. But to implement this procedure, one needs to extract all the relevant vertices from a highly complicated Kaluza-Klein (KK) reduction of the theory on the internal manifold $S^{\mathrm{d}-1}$. Moreover, there is an explosion of diagrams when considering operators dual to higher KK modes. These difficulties render the algorithm near impossible after just a few low-lying cases [16–23].

This situation becomes even more troublesome when contrasted with the progress made for flat-space scattering amplitudes (see, e.g., Refs. [24,25] for textbook presentations). Since holographic correlators are on-shell scattering amplitudes in anti-de Sitter space, it would be truly surprising if no interesting structures were found in these objects. Motivated by this analogy with flat-space amplitudes and benefiting from developments in the conformal bootstrap, a new method was proposed in Refs. [26,27]. This method uses the Mellin representation formalism [28,29], where correlators are rephrased as Mellin amplitudes. The Mellin space manifests the scattering amplitude nature of holographic correlators in a way similar to momentum space in flat space. By solely using symmetry principles and consistency conditions, Refs. [26,27] obtained a stunningly simple formula for all tree-level four-point Mellin amplitudes for $\mathrm{AdS}_5 \times S^5$, as the solution to an algebraic bootstrap problem. This method eschewed the explicit details of the effective Lagrangian and avoided diagrams altogether. The general formula was later confirmed in a large number of explicit examples [30–33] and also provided essential data for studying correlators at one loop [35–46]. The remarkable success of the method on $\mathrm{AdS}_5 \times S^5$ was partially replicated on $\mathrm{AdS}_7 \times S^4$, where the bootstrap problem was set up in Refs. [47,48]. Unfortunately, the problem was too difficult to be solved in general, and only partial solutions for small weights were obtained [47,48]. Moreover, the same approach for AdS_5 and AdS_7 was not applicable to $\mathrm{AdS}_4 \times S^7$ because of a difference in the superconformal structure of correlators. A complementary method was subsequently given in Ref. [48], which introduced superconformal Ward identities (WI) in Mellin space, and it can be applied to any spacetime dimensions. However, this method also becomes cumbersome for more general correlators, and only the simplest $\mathrm{AdS}_4 \times S^7$ stress-tensor four-point function was explicitly written down [48]. In summary, the bootstrap methods are only successful in various limited domains, and they fail to generate a complete general picture. Moreover, the bootstrap nature of these methods prevents us from looking more deeply and understanding the microscopic organizing principles of holographic correlators.

In this paper, we achieve these unfinished goals and thereby fully solve this long-standing problem. We develop a unifying method for all three theories by borrowing new ideas from flat-space amplitudes. This method leads to a constructive derivation for all tree-level four-point functions with arbitrary conformal dimensions, in all backgrounds with maximal superconformal symmetry. The result for $\mathrm{AdS}_7 \times S^4$ was already reported in an earlier publication [49], while for $\mathrm{AdS}_5 \times S^5$, our results give a proof of Refs. [26,27]. The derivation presented here shows a surprising universality in holographic correlators across diverse dimensions, and it underscores the pivotal role of symmetries in their common organizing principles. Our method starts with the crucial observation that symmetries dictate a massive simplification in Mellin amplitudes in a judiciously chosen kinematic limit. This limit is achieved by requiring that the R-symmetry polarizations, i.e., the polarizations on the internal sphere, of two scattering particles are aligned. We call such configurations maximally R-symmetry violating (MRV), in analogy with maximally helicity violating (MHV) in flat space. The MRV amplitudes display two striking features: There are no singularities in the u-channel, and the amplitude develops a factor of 2 zeros in the u Mandelstam-Mellin variable. The first property follows from the fact that the supergravity-field exchanges in the u-channel are suppressed by the special choice of R-symmetry configuration. The second property is the manifestation of the decoupling of low-lying unprotected operators in this limit. Both features are universal for correlators in all three theories, and they are dictated by superconformal symmetry. However, at the level of individual Witten diagrams, the consequence of the u-channel zeros is highly nontrivial, and it requires a conspiracy of the exchanged field inside each individual supermultiplet. Imposing the presence of zeros fixes the contribution of all component fields in the multiplet up to an overall constant, which can be easily computed by using the known bulk cubic couplings of scalar fields. This process allows us to write down all MRV amplitudes in these theories. However, the study of the MRV amplitudes serves a greater purpose. From the MRV limit, we can use R-symmetry to restore the generic R-symmetry polarizations in the multiplet exchange amplitudes. This operation determines the full correlators up to the addition of possible contact interactions, which can be mixed into the exchange amplitudes under field redefinitions. However, the contact terms are not arbitrary once a choice for the exchange amplitudes is made, and they are uniquely fixed by requiring that the correlators satisfy superconformal Ward identities. Remarkably, we find a prescription to recover exchange amplitudes from the MRV limit such that no explicit contact terms are present. Using this procedure, we construct all tree-level four-point functions in $\mathrm{AdS}_4\times S^7$, $\mathrm{AdS}_5\times S^5$, and $\mathrm{AdS}_7\times S^4$, and write them in a closed-form formula, which exhibits remarkable simplicity.

The rest of the paper is organized as follows. In Sec. II A, we review the basic kinematics of four-point functions of one-half BPS operators. In Sec. II B, we review the traditional diagrammatic expansion method and various bootstrap methods. We study the properties of the MRV limit in Sec. III and present an efficient algorithm for constructing all MRV amplitudes. In Sec. IV, we show how to recover the full amplitude from the MRV limit and present the general result for all four-point functions in the three maximally superconformal backgrounds. In Sec. V, we address the absence of contact terms by studying superconformal Ward identities in Mellin space. We also study these Ward identities and Mellin amplitudes near the flat-space limit. We conclude in Sec. VI and outline a few future directions.

II. GENERALITIES

A. Kinematics

We focus on the one-half BPS local operators in superconformal field theories which have 16 supercharges. Such operators $\mathcal{O}_k^{I_1\ldots I_k}$ transform in the rank-k symmetric traceless representation of an $SO(\mathrm{d})$ R-symmetry group, with $k=2,3\ldots$. They have protected conformal dimension $\Delta_k=\epsilon k$, where ϵ is related to the spacetime dimension d via $\epsilon=(d-2/2)$. It is convenient to keep track of the R-symmetry indices by contracting them with null vectors,

$$\mathcal{O}_k(x,t)=\mathcal{O}_k^{I_1,\ldots,I_k}(x)t_{I_1}\ldots t_{I_k},\qquad t\cdot t=0. \tag{1}$$

The four-point functions are denoted by

$$G_{k_1k_2k_3k_4}(x_i,t_i)=\langle\mathcal{O}_{k_1}\mathcal{O}_{k_2}\mathcal{O}_{k_3}\mathcal{O}_{k_4}\rangle, \tag{2}$$

and they are functions of both the spacetime coordinates x_i and internal coordinates t_i. We often leave the k_i dependence in $G_{k_1k_2k_3k_4}(x_i,t_i)$ implicit to avoid overloading the notation. We can assume, without loss of generality, that the weights k_i are ordered as $k_1\le k_2\le k_3\le k_4$. Then, we need to further distinguish two possibilities:

$$\begin{aligned}&k_1+k_4\ge k_2+k_3\quad(\text{case I}),\\&k_1+k_4<k_2+k_3\quad(\text{case II}).\end{aligned} \tag{3}$$

We can extract a kinematic factor

$$G(x_i,t_i)=\prod_{i<j}\left(\frac{t_{ij}}{x_{ij}^{2\epsilon}}\right)^{\gamma_{ij}^0}\left(\frac{t_{12}t_{34}}{x_{12}^{2\epsilon}x_{34}^{2\epsilon}}\right)^{\mathcal{E}}\mathcal{G}(U,V;\sigma,\tau), \tag{4}$$

such that the correlators can be written as a function of the cross ratios,

$$\begin{aligned}U&=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad V=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2},\\\sigma&=\frac{t_{13}t_{24}}{t_{12}t_{34}},\qquad \tau=\frac{t_{14}t_{23}}{t_{12}t_{34}}.\end{aligned} \tag{5}$$

Here, $x_{ij}=x_i-x_j$, $t_{ij}=t_i\cdot t_j$, and $\mathcal{E}$ is the extremality,

$$\mathcal{E}=\frac{k_1+k_2+k_3-k_4}{2}\quad(\text{case I}),\qquad \mathcal{E}=k_1\quad(\text{case II}). \tag{6}$$

The exponents are given by

$$\begin{aligned}&\gamma_{12}^0=\gamma_{13}^0=0,\qquad \gamma_{34}^0=\frac{\kappa_s}{2},\qquad \gamma_{24}^0=\frac{\kappa_u}{2},\\&\gamma_{14}^0=\frac{\kappa_t}{2},\qquad \gamma_{23}^0=0\ (\text{I}),\qquad \gamma_{14}^0=0,\qquad \gamma_{23}^0=\frac{\kappa_t}{2}\ (\text{II})\end{aligned} \tag{7}$$

where

$$\begin{aligned}\kappa_s&\equiv|k_3+k_4-k_1-k_2|,\\\kappa_t&\equiv|k_1+k_4-k_2-k_3|,\\\kappa_u&\equiv|k_2+k_4-k_1-k_3|.\end{aligned} \tag{8}$$

Since t_i can only appear in $G(x_i, t_i)$ as polynomials of t_{ij}, and $G(x_i, \lambda_i t_i) = (\prod_i \lambda_i^{k_i}) G(x_i, t_i)$ under rescaling, it is clear from Eq. (4) that $\mathcal{G}(U, V; \sigma, \tau)$ is a polynomial in σ and τ of degree $\mathcal{E}$. Writing $G(x_i, t_i)$ as in Eq. (4) exploits only the bosonic part of the superconformal group. Fermionic generators imply further constraints, known as the superconformal Ward identities. It is useful to introduce the following change of variables:

$$U = z\bar{z}, \qquad V = (1-z)(1-\bar{z}),$$
$$\sigma = \alpha\bar{\alpha}, \qquad \tau = (1-\alpha)(1-\bar{\alpha}). \tag{9}$$

The superconformal Ward identity reads [50]

$$(z\partial_z - \epsilon\alpha\partial_\alpha)\mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha})|_{\alpha=1/z} = 0. \tag{10}$$

Because $\mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha})$ is symmetric under $z \leftrightarrow \bar{z}$ and $\alpha \leftrightarrow \bar{\alpha}$, three more identities follow from the above identity by replacing z with $\bar{z}$ and α with $\bar{\alpha}$.

B. Methods for computing holographic correlators

1. Traditional method: Diagrammatic expansion

The traditional recipe to calculate holographic correlators follows from a standard diagrammatic expansion in AdS. More precisely, one obtains the effective action on AdS_{d+1}, by performing a Kaluza-Klein reduction of the D-dimensional supergravity theory on S^{D-d-1}. For tree-level four-point functions, the relevant information to be extracted from the effective action are the cubic and quartic vertices. One then uses these vertices to write down all the possible exchange and contact Witten diagrams, and the four-point correlator is given by the sum

$$\mathcal{G}_{\text{tree}} = \mathcal{G}^{(s)}_{\text{exch}} + \mathcal{G}^{(t)}_{\text{exch}} + \mathcal{G}^{(u)}_{\text{exch}} + \mathcal{G}_{\text{con}}. \tag{11}$$

Here, the number of exchanged fields in a specific four-point function is always finite. They are dictated by two selection rules on the cubic couplings. The first is an R-symmetry selection rule, which says that the R-symmetry representation carried by the exchanged fields (say, in the s-channel) must appear in the common tensor product of the external representations (i.e., the overlap of the tensor product of rank k_1, k_2 symmetric traceless representations, and that of k_3, k_4). The second is a cutoff on the conformal twist of the exchanged fields,

$$\Delta - \ell < \epsilon \min\{k_1 + k_2, k_3 + k_4\}, \tag{12}$$

which arises from the requirement that the effective action must remain finite. We organize the relevant exchanged fields into superconformal multiplets in the table below [51–54], where the superprimary scalar field s_p is the bulk dual of the one-half BPS operator $\mathcal{O}_p$. The fields $A_{p,\mu}$ and $C_{p,\mu}$ are vector fields in AdS, and $A_{2,\mu}$ is the graviphoton field dual to the R-symmetry currents on the boundary. The fields $\varphi_{p,\mu\nu}$ are the symmetric traceless spin-2 tensor fields, which include the graviton with $p = 2$, dual to the stress-tensor operator. Finally, t_p and r_p are scalar fields.

Field	s_p	$A_{p,\mu}$	$\varphi_{p,\mu\nu}$	$C_{p,\mu}$	t_p	r_p
ℓ	0	1	2	1	0	0
Δ	ϵp	$\epsilon p+1$	$\epsilon p+2$	$\epsilon p+3$	$\epsilon p+4$	$\epsilon p+2$
d_1	p	$p-2$	$p-2$	$p-4$	$p-4$	$p-4$
d_2	0	2	0	2	0	4

In the table, the quantum numbers d_1, d_2 are associated with the R-symmetry representation of the component fields, and they appear in the Dynkin labels as

$$SO(5){:}\,[d_1, d_2], \qquad SU(4){:}\left[\frac{d_2}{2}, d_1, \frac{d_2}{2}\right],$$
$$SO(8){:}\left[d_1, \frac{d_2}{2}, 0, 0\right]. \tag{13}$$

We can write the exchange contributions more explicitly as

$$\mathcal{G}^{(s)}_{\text{exch}} = \sum_p \mathcal{V}^{(s)}_p, \tag{14}$$

$$\begin{aligned}\mathcal{V}^{(s)}_p = {}& \lambda_s \mathcal{Y}_{\{p,0\}} W^{(s)}_{\epsilon p,0} + \lambda_A \mathcal{Y}_{\{p-2,2\}} W^{(s)}_{\epsilon p+1,1} \\ & + \lambda_\varphi \mathcal{Y}_{\{p-2,0\}} W^{(s)}_{\epsilon p+2,2} + \lambda_C \mathcal{Y}_{\{p-4,2\}} W^{(s)}_{\epsilon p+3,1} \\ & + \lambda_t \mathcal{Y}_{\{p-4,0\}} W^{(s)}_{\epsilon p+4,0} + \lambda_r \mathcal{Y}_{\{p-4,4\}} W^{(s)}_{\epsilon p+2,0},\end{aligned} \tag{15}$$

where $\mathcal{V}^{(s)}_p$ is the contribution from the multiplet p. Here, $W^{(s)}_{\Delta,\ell}$ are the standard exchange Witten diagrams in the s-channel with dimension Δ and spin ℓ, and $\mathcal{Y}_{\{d_1,d_2\}}$ are R-symmetry polynomials of σ and τ (see Supplemental Material [55] for details), associated with the exchanged irreducible representation labeled by the R-symmetry quantum numbers $\{d_1, d_2\}$. Historically, such R-symmetry structures were obtained by gluing together three-point spherical harmonics. However, it is more convenient to obtain them by solving the two-particle, quadratic, R-symmetry Casimir equation [56], making $\mathcal{Y}_{\{d_1,d_2\}}$ the compact analogues of conformal blocks. The coefficients λ_{field} in Eq. (15) are pure numbers, which can be fixed by using the explicit cubic vertices and appropriately taking into account the normalization of $\mathcal{Y}_{\{d_1,d_2\}}$. Finally, $\mathcal{G}_{\text{con}}$ contains contact Witten diagrams up to four derivatives and all possible R-symmetry structures. The simplest zero-derivative contact Witten diagram is denoted by the $\bar{D}$ function $\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}$ in the literature, and higher-derivative contact diagrams can be related to the zero-derivative ones by using differential recursion relations. The contact diagram contribution could, in principle, be computed when the quartic vertices are known.

Though clear physically, the traditional method suffers from several severe practical drawbacks. First of all, extracting the vertices, especially the quartic vertices, from the effective action is extremely hard. The general quartic vertices are only known for IIB supergravity on $\mathrm{AdS}_5 \times S^5$ [57], where their complicated expressions covered 15 pages. Second, as one increases the external dimensions (more precisely, the extremality $\mathcal{E}$), one is greeted by a proliferation of exchange Witten diagrams. Finally, the exchange Witten diagrams are only tractable in position space when the quantum numbers are fine-tuned. When the spectrum satisfies the conditions

$$\Delta_1 + \Delta_2 - (\Delta - \ell) \in 2\mathbb{Z}_+ \quad \text{or} \\ \Delta_3 + \Delta_4 - (\Delta - \ell) \in 2\mathbb{Z}_+, \tag{16}$$

the exchange Witten diagrams can be written as a finite sum of contact diagrams [58], which is the case for $\mathrm{AdS}_5 \times S^5$ and $\mathrm{AdS}_7 \times S^4$. However, the conditions are not satisfied by the $\mathrm{AdS}_4 \times S^7$ background. These practical difficulties make it clear that this brute-force approach is extremely cumbersome at best and unlikely to yield any general result unless powerful, underlying, organizing principles can be identified.

2. Bootstrap methods

In recent years, a number of powerful bootstrap methods [26,27,47,48,59,60] have been developed to efficiently compute holographic correlators, which have superseded the traditional method. These bootstrap methods exploit symmetries and self-consistency conditions and fix the correlators by making no reference to the explicit details of the effective Lagrangian. Below, we give an overview of these methods and discuss their respective strengths and limitations.

Position-space method.—A first improvement of the traditional algorithm was made in Refs. [26,27] and was termed the position-space method. The idea is to leave λ_{field} in Eq. (15) as an unfixed parameter and parametrize the most general contact contribution $\mathcal{G}_{\text{con}}$ with unknown coefficients. In models where the truncation conditions (20) are satisfied, one can write the exchange Witten diagrams in terms of a finite number of $\bar{D}$ functions. Furthermore, the $\bar{D}$ functions can be uniquely decomposed as

$$R_\Phi(z,\bar{z})\Phi(U,V) + R_{\log U}(z,\bar{z})\log U \\ + R_{\log V}(z,\bar{z})\log V + R_1(z,\bar{z}), \tag{17}$$

where $\Phi(U,V)$ is the scalar box diagram in four dimensions, and the coefficient functions $R_X(z,\bar{z})$ are rational functions of z and $\bar{z}$. One then imposes the superconformal Ward identities (10), which can be cast into the same form (17) by using differential recursion relations of $\Phi(U,V)$. The superconformal Ward identities uniquely fix all the unknown coefficients in the ansatz up to an overall rescaling factor. This method has the advantage of being very concrete, and it sidesteps the need for obtaining the complicated vertices. On the other hand, the method is applied on a case-by-case basis, and it loses steam for higher-weight external operators. The position-space method can be applied to supergravity theories on $\mathrm{AdS}_5 \times S^5$ [26,27], $\mathrm{AdS}_7 \times S^4$ [47], and $\mathrm{AdS}_3 \times S^3 \times K3$ [60,61] backgrounds [62]. However, it is not applicable to 11D supergravity on $\mathrm{AdS}_4 \times S^7$, where the exchange Witten diagrams do not truncate. Finally, the expressions of holographic correlators in position space are usually highly complicated and beg for a more transparent representation, which we now introduce.

Intermezzo: Mellin space.—A useful tool for holographic correlators is the Mellin representation formalism [28,29]. This formalism is exploited in the methods below and will also be used later as the language of this paper. In the Mellin representation,

$$\begin{aligned} \mathcal{G}_{\text{tree}} &= \int_{-i\infty}^{i\infty} \frac{ds dt}{(4\pi i)^2} U^{\frac{s}{2}-a_s} V^{\frac{t}{2}-a_t} \mathcal{M}(s,t;\sigma,\tau)\Gamma_{\{k_i\}}, \\ \Gamma_{\{k_i\}} &= \Gamma\left[\frac{\epsilon(k_1+k_2)-s}{2}\right]\Gamma\left[\frac{\epsilon(k_3+k_4)-s}{2}\right]\Gamma \\ &\times \left[\frac{\epsilon(k_1+k_4)-t}{2}\right]\Gamma\left[\frac{\epsilon(k_2+k_3)-t}{2}\right]\Gamma \\ &\times \left[\frac{\epsilon(k_1+k_3)-u}{2}\right]\Gamma\left[\frac{\epsilon(k_2+k_4)-u}{2}\right], \end{aligned} \tag{18}$$

where $a_s = (\epsilon/2)(k_1+k_2) - \epsilon\mathcal{E}$, $a_t = (\epsilon/2)\min\{k_1+k_4, k_2+k_3\}$, and $s+t+u = \epsilon\sum_{i=1}^4 k_i \equiv \epsilon\Sigma$, the analytic structure of the holographic correlators becomes particularly clear. The Mellin amplitudes of exchange Witten diagrams are a sum over simple poles,

$$\mathcal{M}^{(s)}_{\Delta,\ell}(s,t) = \sum_{m=0}^{\infty} \frac{\mathcal{Q}_{m,\ell}(t,u)}{s-\Delta+\ell-2m}, \tag{19}$$

where $\mathcal{Q}_{m,\ell}(t,u)$ are degree-ℓ polynomials in t and u. The residues $\mathcal{Q}_{m,\ell}(t,u)$ vanish for $m \geq m_0$ when the conditions (20) are satisfied,

$$\Delta_1 + \Delta_2 = \Delta - \ell + 2m_0 \quad \text{or} \quad \Delta_3 + \Delta_4 = \Delta - \ell + 2m_0, \tag{20}$$

truncating the infinite series into a finite sum, in order to be consistent with the large-N expansion [27]. On the other hand, contact diagrams with $2L$ derivatives have Mellin amplitudes that are polynomials in the Mandelstam variables of degree L. Note that, in the literature, it is also conventional to write Eq. (19) as

$$\mathcal{M}^{(s)}_{\Delta,\ell}(s,t) = \sum_{m=0}^{\infty} \frac{\tilde{\mathcal{Q}}_{m,\ell}(t)}{s-\Delta+\ell-2m} + \mathcal{P}_{\ell-1}(s,t), \tag{21}$$

where one Mandelstam variable is eliminated from $\mathcal{Q}_{m,\ell}(t,u)$ and $\mathcal{P}_{\ell-1}(s,t)$ is a degree-$(\ell-1)$ polynomial. In Eq. (19), we have absorbed the regular terms into the numerator, which is related to the fact that exchange Witten diagrams are not uniquely defined. We can add to them any contact terms with degree $\ell - 1$, which corresponds to choosing different on-shell equivalent cubic couplings.

The Mellin algebraic bootstrap method.—A more elegant method was formulated in Refs. [26,27,47], which rephrased the task of computing holographic four-point functions as solving an algebraic bootstrap problem in Mellin space. This method exploits the special structure of the correlators as dictated by the superconformal Ward identities,

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{D} \circ \mathcal{H}, \qquad (22)$$

where $\mathcal{G}_0$ is a protected part of the correlator that does not contribute to the Mellin amplitude, $\mathcal{D}$ is a differential operator determined by superconformal symmetry, and $\mathcal{H}$ is known as the reduced correlator. We can define a reduced Mellin amplitude $\tilde{\mathcal{M}}$ from $\mathcal{H}$ and translate the differential operator $\mathcal{D}$ into a difference operator $\hat{\mathcal{D}}$ in Mellin space. Then, we have

$$\mathcal{M} = \hat{\mathcal{D}} \circ \tilde{\mathcal{M}}, \qquad (23)$$

which implements the superconformal symmetry at the level of Mellin amplitudes. The bootstrap problem is formulated by further imposing Bose symmetry, analytic properties, and a flat-space limit on the Mellin amplitude $\mathcal{M}$. Such algebraic bootstrap problems are highly constraining, and they fix the correlators uniquely up to an overall constant. The bootstrap problem for $\mathrm{AdS}_5 \times S^5$ was fully solved in Refs. [26,27] for arbitrary four-point functions, and it led to an extremely compact answer. The merit of this approach is that one can treat all external dimensions on the same footing and obtain the correlators without computing any diagrams. However, the analytic structure of the reduced amplitude $\tilde{\mathcal{M}}$ is not as transparent as that of the full amplitude $\mathcal{M}$. This fact sometimes makes it difficult to find a general efficient ansatz for $\tilde{\mathcal{M}}$, such as in $\mathrm{AdS}_7 \times S^4$, and the problem is solved only on a case-by-case basis [47,48]. Moreover, for $d = 3$, the differential operator $\mathcal{D}$ is nonlocal, which makes it difficult to interpret in Mellin space.

Mellin superconformal Ward identities.—Complementary to the above Mellin algebraic bootstrap method is another Mellin-space technique that can be applied to any spacetime dimensions, which was first developed in Ref. [48]. This method can be viewed as the Mellin-space parallel of the position-space method. We can translate Eqs. (11), (14), and (15) into

$$\begin{aligned}
&\mathcal{M}(s,t;\sigma,\tau) = \mathcal{M}^{(s)}_{\text{exch}} + \mathcal{M}^{(t)}_{\text{exch}} + \mathcal{M}^{(u)}_{\text{exch}} + \mathcal{M}_{\text{con}},\\
&\mathcal{M}^{(s)}_{\text{exch}}(s,t;\sigma,\tau) = \sum_p \mathcal{S}^{(s)}_p(s,t;\sigma,\tau),\\
&\mathcal{S}^{(s)}_p = \lambda_s \mathcal{Y}_{\{p,0\}} \mathcal{M}^{(s)}_{\epsilon p,0} + \lambda_A \mathcal{Y}_{\{p-2,2\}} \mathcal{M}^{(s)}_{\epsilon p+1,1}\\
&\qquad + \lambda_\varphi \mathcal{Y}_{\{p-2,0\}} \mathcal{M}^{(s)}_{\epsilon p+2,2} + \lambda_C \mathcal{Y}_{\{p-4,2\}} \mathcal{M}^{(s)}_{\epsilon p+3,1}\\
&\qquad + \lambda_t \mathcal{Y}_{\{p-4,0\}} \mathcal{M}^{(s)}_{\epsilon p+4,0} + \lambda_r \mathcal{Y}_{\{p-4,4\}} \mathcal{M}^{(s)}_{\epsilon p+2,0},
\end{aligned} \qquad (24)$$

with unfixed λ_{field}, and $\mathcal{M}_{\text{con}}$ will be taken as an arbitrary degree-1 polynomial in s, t, and a degree-$\mathcal{E}$ polynomial in σ, τ. Then, we impose the superconformal constraints from the superconformal Ward identities (10). Implementing these constraints in Mellin space may appear difficult as only U and V appear in the definition (18), which is invariant under $z \leftrightarrow \bar{z}$. However, the superconformal Ward identity (10) breaks the symmetry of z and $\bar{z}$ and creates complicated branch cuts when rewritten in terms of U and V. The observation of Ref. [48] is that we can take the sum of a holomorphic and an antiholomorphic copy [64],

$$\begin{aligned}
&(z\partial_z - \epsilon\alpha\partial_\alpha)\mathcal{G}(z,\bar{z};\alpha,\bar{\alpha})|_{\alpha=1/z} = 0,\\
&(\bar{z}\partial_{\bar{z}} - \epsilon\alpha\partial_\alpha)\mathcal{G}(z,\bar{z};\alpha,\bar{\alpha})|_{\alpha=1/\bar{z}} = 0.
\end{aligned} \qquad (25)$$

Then, the coefficients can always be written in terms of polynomials in U and V, which are easy to interpret as difference operators in Mellin space. These difference equations (graded by different powers of the spectator cross ratio $\bar{\alpha}$) constitute the Mellin superconformal Ward identities. By imposing these identities, one fixes all the coefficients in the ansatz up to an overall constant. Note that in Mellin space, exchange Witten diagrams can be easily written down for any spacetime dimension and conformal dimension. This fact greatly extends the range of applicability of this method. Using this Mellin-space technique, Ref. [48] obtained the first four-point correlator in $\mathrm{AdS}_4 \times S^7$ for the stress-tensor multiplet, where all other methods had fallen short. On the other hand, the method suffers from the same shortcomings as the position-space approach, in that it is difficult to go beyond individual correlators.

Other approaches.—There are other methods for computing holographic correlators by incorporating bootstrap ideas. By using factorization and supersymmetric twistings, Ref. [65] computed the five-point function of one-half BPS operators in the stress-tensor multiplet for IIB supergravity on $\mathrm{AdS}_5 \times S^5$. In AdS_3, there is also a method to construct four-point functions from the heavy-heavy-light-light limit by using crossing and consistency with superconformal OPE [66–68]. This approach complements the bootstrap method in AdS_3 [60].

III. MAXIMALLY R-SYMMETRY VIOLATING LIMIT

A. Properties of the MRV amplitudes

While the full Mellin amplitudes appear rather complicated, there are special limits where the amplitudes simplify drastically and give a hint for their underlying organizing principles. One such limit is the MRV limit, introduced in Ref. [49]. In the ordering of $k_1 \leq k_2 \leq k_3 \leq k_4$, the (u-channel) MRV limit is reached by setting $t_1 = t_3$ for the auxiliary R-symmetry null vectors. This choice of null vectors means that in $G(x_i, t_i)$, t_1 cannot be contracted with

t_3, and no t_{13} can appear. In terms of the R-symmetry cross ratios, it corresponds to setting $\sigma = 0$, $\tau = 1$. We denote the MRV amplitude as [69]

$$\mathrm{MRV}(s,t) = \mathcal{M}(s,t;0,1). \tag{26}$$

Note that the MRV limit can also be defined in other channels: In the s-channel, it corresponds to $t_1 = t_2$, and in the t-channel, it amounts to $t_2 = t_3$ (case I) or $t_1 = t_4$ (case II) [70]. The three limits are related by Bose symmetry. Restricting the amplitudes to the MRV limit suppresses certain R-symmetry representations in that channel. For example, all the u-channel supergravity field exchanges are suppressed in the $\sigma = 0$, $\tau = 1$ limit because the R-symmetry polynomials all contain at least one power of t_{13}. This observation gives the first simplifying property of MRV amplitudes: The MRV amplitudes have no poles in the u-channel.

Moreover, in such special R-symmetry configurations, we see the following interesting phenomenon: The superprimary is absent, whereas superdescendants are present [71]. In particular, let us consider the long supermultiplet where the superprimary is a double-trace operator of the schematic form $[:\mathcal{O}_{k_2}\partial^J\mathcal{O}_{k_4}:]_{\{d_1,d_2\}}$. In order for all superdescendants (in particular, the operator acted with $Q^4\bar{Q}^4$, which has maximal deviation in R-symmetry from the superprimary) to have R-symmetry charges admissible in the tensor products of $\mathcal{O}_{k_1}\times\mathcal{O}_{k_3}$ and $\mathcal{O}_{k_2}\times\mathcal{O}_{k_4}$, the representation of the superprimary must satisfy $d_1 + d_2 \leq 2\mathcal{E} + \kappa_t + \kappa_u - 4$. This requirement implies that in the MRV configuration, the R-symmetry polynomial associated with $\{d_1, d_2\}$ vanishes. Moreover, one can show that the only superdescendant that contributes to this limit is $Q^4\bar{Q}^4[:\mathcal{O}_{k_2}\partial^J\mathcal{O}_{k_4}:]_{\{d_1,d_2\}}$. Therefore, we expect to see long operators (albeit not superprimaries) in the u-channel MRV configuration with conformal twist of at least $\epsilon(k_2+k_4)+4$, which is reflected by the double pole at $u = \epsilon(k_2+k_4)+4$ in the $\Gamma_{\{k_i\}}$ factor in Eq. (18). Upon doing the inverse Mellin integral, we see a logarithmic singularity, which is the hallmark of an unprotected long operator. On the other hand, this lower bound for logarithmic singularities cannot be further lowered because $\epsilon(k_2+k_4)$ is the minimal twist of the double-trace operators constructed from $\mathcal{O}_{k_2}$ and $\mathcal{O}_{k_4}$ for the superprimaries of the long multiplets. Thus, we have the second important property of the MRV amplitudes: The MRV amplitudes contain a factor of zeros, $(u - \epsilon k_2 - \epsilon k_4)(u - \epsilon k_2 - \epsilon k_4 - 2)$.

These zeros are precisely needed to cancel one of the double poles in $\Gamma_{\{k_i\}}$, such that no logarithmic singularities at these twists show up.

B. All MRV amplitudes

These two properties of MRV amplitudes have profound consequences in understanding the structure of holographic correlators. In fact, the u-channel zeros are satisfied by each individual supermultiplet exchange in the s-channel (and separately, in the t-channel), which gives rise to an efficient way to fix the relative values of λ_{field} inside each multiplet. More precisely, we choose the contact terms in the exchange Witten diagrams (19) by setting $t = \epsilon\Sigma - u - (\Delta - \ell) - 2m$ in the numerators $\mathcal{Q}_{m,\ell}(t,u)$,

$$P^{(s)}_{\Delta,\ell}(s,u) = \sum_{m=0}^{\infty}\frac{\mathcal{Q}_{m,\ell}(\epsilon\Sigma - u - (\Delta-\ell) - 2m, u)}{s - \Delta + \ell - 2m}. \tag{27}$$

This choice corresponds to the so-called Polyakov-Regge blocks [72,73] (see also Refs. [74–76] for related blocks), which have improved u-channel Regge behavior,

$$P^{(s)}_{\Delta,\ell}(s,u) \to \frac{1}{s}, \qquad s\to\infty, \qquad u \text{ fixed}. \tag{28}$$

For simplicity, we focus on case I of Eq. (3) in what follows, in addition to the ordering $k_1 \leq k_2 \leq k_3 \leq k_4$. However, in the next section, when we assemble the ingredients into the final results and express them in terms of $\kappa_s, \kappa_t, \kappa_u$, the expressions will be valid for any ordering of k_i thanks to Bose symmetry. By using the result (A.10) in the Supplemental Material [55], the $SO(\mathtt{d})$ R-symmetry polynomials take the following values in the MRV limit:

$$\begin{aligned}
\mathcal{Y}^{\mathrm{MRV}}_{\{p,0\}} &\equiv \mathcal{Y}_{\{p,0\}}(0,1) = \frac{(\frac{\kappa_t}{2}!)(\frac{p+k_2-k_1}{2}!)\Gamma[\frac{\mathtt{d}+p+k_2-k_1-2}{2}]\Gamma[\frac{\mathtt{d}+p+k_4-k_3-2}{2}]}{(\frac{\kappa_u}{2}!)(\frac{p+k_1-k_2}{2}!)\Gamma[\frac{\mathtt{d}-k_1-k_3+k_2+k_4-2}{2}]\Gamma[\frac{2p+\mathtt{d}-2}{2}]},\\
\mathcal{Y}^{\mathrm{MRV}}_{\{p,2\}} &\equiv \mathcal{Y}_{\{p,2\}}(0,1) = -\frac{(p+k_2-k_1+\mathtt{d}-2)(p+k_4-k_3+\mathtt{d}-2)}{(\mathtt{d}+p-2)(\mathtt{d}+2p-2)}\mathcal{Y}^{\mathrm{MRV}}_{\{p,0\}},\\
\mathcal{Y}^{\mathrm{MRV}}_{\{p,4\}} &\equiv \mathcal{Y}_{\{p,4\}}(0,1) = \frac{4(\mathtt{d}-3)(\frac{p+k_2-k_1+\mathtt{d}-2}{2})_2(\frac{p+k_4-k_3+\mathtt{d}-2}{2})_2}{(\mathtt{d}-2)(-(\mathtt{d}+p))_2(-\frac{(\mathtt{d}+2p)}{2})_2}\mathcal{Y}^{\mathrm{MRV}}_{\{p,0\}}.
\end{aligned}\tag{29}$$

Requiring the presence of zeros at every pole $s = \epsilon p + 2m$ imposes strong constraints on λ_{field} and solves them in terms of λ_s,

$$\lambda_A^{(p)} = \frac{\mathcal{Y}^{\mathrm{MRV}}_{\{p,0\}}}{\mathcal{Y}^{\mathrm{MRV}}_{\{p-2,2\}}} \frac{\epsilon(k_1-k_2+p)(k_3-k_4+p)}{2p(p\epsilon+2)} \lambda_s^{(p)},$$

$$\lambda_\varphi^{(p)} = \frac{\mathcal{Y}^{\mathrm{MRV}}_{\{p,0\}}}{\mathcal{Y}^{\mathrm{MRV}}_{\{p-2,0\}}} \frac{\epsilon^2(k_1-k_2+p)(k_3-k_4+p)(k_1\epsilon-k_2\epsilon+p\epsilon+2)(k_3\epsilon-k_4\epsilon+p\epsilon+2)}{16(p\epsilon+1)(p\epsilon+2)^2(p\epsilon+3)} \lambda_s^{(p)},$$

$$\lambda_C^{(p)} = \frac{\mathcal{Y}^{\mathrm{MRV}}_{\{p,0\}}}{\mathcal{Y}^{\mathrm{MRV}}_{\{p-4,2\}}} \frac{\epsilon^3(k_1-k_2+p-2)(k_1-k_2+p)(k_3-k_4+p-2)(k_3-k_4+p)\big(\epsilon(k_1-k_2+p)+2\big)\big(\epsilon(k_3-k_4+p)+2\big)}{32(p-2)((p-1)\epsilon+1)((p-1)\epsilon+2)(p\epsilon+2)^2(p\epsilon+3)} \lambda_s^{(p)},$$

$$\lambda_r^{(p)} = \frac{\mathcal{Y}^{\mathrm{MRV}}_{\{p,0\}}}{\mathcal{Y}^{\mathrm{MRV}}_{\{p-4,0\}}} \frac{\epsilon^2(\epsilon+2)(k_1-k_2+p-2)(k_1-k_2+p)(k_3-k_4+p-2)(k_3-k_4+p)}{8(p-2)(p-1)(\epsilon+1)((p-1)\epsilon+2)(p\epsilon+2)} \lambda_s^{(p)},$$

$$\lambda_t^{(p)} = \frac{\mathcal{Y}^{\mathrm{MRV}}_{\{p,0\}}}{\mathcal{Y}^{\mathrm{MRV}}_{\{p-4,4\}}} \frac{\epsilon^4(k_1-k_2+p-2)(k_1-k_2+p)(k_3-k_4+p-2)(k_3-k_4+p)}{256((p-2)\epsilon+1)((p-2)\epsilon+2)((p-1)\epsilon+1)}$$
$$\times \frac{\big(\epsilon(k_1-k_2+p-2)+2\big)\big(\epsilon(k_1-k_2+p)+2\big)\big(\epsilon(k_3-k_4+p-2)+2\big)\big(\epsilon(k_3-k_4+p)+2\big)}{((p-1)\epsilon+2)^2((p-1)\epsilon+3)(p\epsilon+2)(p\epsilon+3)} \lambda_s^{(p)}. \quad (30)$$

Here, we have added a superscript to the coefficients $\lambda^{(p)}_{\mathrm{field}}$ to emphasize that they belong to the pth multiplet. Inserting the solutions into $\mathcal{S}_p^{(s)}$ in Eq. (24) leads to a great simplification. We obtain the following contribution from each supermultiplet to the MRV limit:

$$\mathcal{S}_p^{(s)}(s,t;0,1) = \sum_{m=0}^{\infty} \frac{4\lambda_s^{(p)}(p\epsilon+1)(p\epsilon-\epsilon+1)}{(k_1-k_2-p)(k_4-k_3+p)(k_1\epsilon-k_2\epsilon-p\epsilon-2)(k_3\epsilon-k_4\epsilon-p\epsilon-2)}$$
$$\times \frac{(u-\epsilon k_2-\epsilon k_4)(u-\epsilon k_2-\epsilon k_4-2)}{(p+1)_{-2}(m+p\epsilon-\epsilon)_2} \left(\frac{f_{m,0}|_{\Delta_E=\epsilon p}}{s-\epsilon p-2m}\right), \quad (31)$$

where the u-channel zeros are factored out, leaving just a sum over simple poles with constant residues. The terms in the brackets are just the scalar exchange Mellin amplitude at each simple pole, with f_{m,ℓ_E} defined in Supplemental Material [55]. Notice that the MRV amplitude for each multiplet does *not* depend on the R-symmetry group $SO(\mathtt{d})$.

To write down the full MRV amplitude, we just need to sum over all multiplets, which is restricted to be finite by the selection rules

$$p - \max\{|k_1-k_2|, |k_3-k_4|\} = 2, 4, \ldots 2\mathcal{E}-2. \quad (32)$$

The strength of the contribution from each multiplet, captured by $\lambda_s^{(p)}$, can be determined from the three-point functions of the superprimaries,

$$\langle \mathcal{O}_{k_1}(x_1,t_1)\mathcal{O}_{k_2}(x_2,t_2)\mathcal{O}_{k_3}(x_3,t_3)\rangle = C^{(\epsilon)}_{k_1k_2k_3}(\alpha_1,\alpha_2,\alpha_3)\frac{t_{12}^{\alpha_3}t_{13}^{\alpha_2}t_{23}^{\alpha_1}}{x_{12}^{2\epsilon\alpha_3}x_{13}^{2\epsilon\alpha_2}x_{23}^{2\epsilon\alpha_1}}, \quad (33)$$

where

$$\alpha_1 = \frac{1}{2}(k_2+k_3-k_1), \qquad \alpha_2 = \frac{1}{2}(k_1+k_3-k_2),$$
$$\alpha_3 = \frac{1}{2}(k_1+k_2-k_3). \quad (34)$$

The three-point coefficients read [77–81]

$$C^{(\frac{1}{2})}_{k_1k_2k_3} = \frac{\pi}{n^{\frac{3}{4}}} \frac{2^{-\alpha-\frac{1}{4}}}{\Gamma[\frac{\alpha}{2}+1]} \prod_{i=1}^{3} \frac{\sqrt{\Gamma[k_i+2]}}{\Gamma[\frac{\alpha_i+1}{2}]}, \quad (35)$$

$$C^{(1)}_{k_1k_2k_3} = \frac{\sqrt{k_1k_2k_3}}{n}, \quad (36)$$

$$C^{(2)}_{k_1k_2k_3} = \frac{2^{2\alpha-2}}{(\pi n)^{\frac{3}{2}}}\Gamma[\alpha]\prod_{i=1}^{3}\frac{\Gamma[\alpha_i+\frac{1}{2}]}{\sqrt{\Gamma[2k_i-1]}}, \quad (37)$$

where $\alpha = \alpha_1+\alpha_2+\alpha_3$. $\lambda_s^{(p)}$ is given in terms of $C^{(\epsilon)}_{k_1k_2k_3}$ by

$$\lambda_s^{(p)} = \left(\frac{(\frac{p+k_1-k_2}{2}!)(\frac{p+k_4-k_3}{2}!)}{p!(\frac{k_1+k_4-k_2-k_3}{2}!)}\right) C^{(\epsilon)}_{k_1k_2p} C^{(\epsilon)}_{k_3k_4p}, \quad (38)$$

where the number in the brackets is a gluing factor for the R-symmetry due to the fact that we have normalized the

R-symmetry polynomials to have unit coefficients for $\sigma^{\mathcal{E}}$. The MRV amplitudes are then simply given by

$$\mathrm{MRV}(s,t) = \mathrm{MRV}^{(s)}(s,t) + \mathrm{MRV}^{(t)}(s,t), \quad (39)$$

where

$$\mathrm{MRV}^{(s)}(s,t) = \sum_p \mathcal{S}_p^{(s)}(s,t;0,1), \quad (40)$$

with the summation over p inside the finite range (32) and $\mathrm{MRV}^{(t)}(s,t)$ related to $\mathrm{MRV}^{(s)}(s,t)$ by Bose symmetry. Note that no additional contact terms are allowed in the MRV amplitudes, which follows from the simple fact that contact terms are, at most, linear in the Mandelstam variables, while the requisite zeros are already quadratic. The absence of additional contact terms tells us something quite remarkable about the structure of supergravity theories in AdS: Supersymmetry in the MRV limit not only determines the relative cubic couplings of components within the same multiplet, but its implication reaches quartic couplings as well. It is also worth pointing out that the MRV amplitudes have an improved u-channel Regge behavior compared to a Witten diagram exchanging a spinning field and with generic choices of contact terms. The MRV amplitudes behave in the same way as the Polyakov-Regge blocks.

IV. ALL TREE-LEVEL CORRELATORS FROM THE MRV LIMIT

A. Full amplitudes from MRV amplitudes

Much more information can be extracted from the MRV limit. In fact, in constructing the MRV amplitudes, we have determined the whole polar part of the full Mellin amplitude. This statement follows from the fact that all R-symmetry polynomials (29) are nonvanishing in the MRV limit. We can therefore restore the full σ, τ dependence in Eq. (24) by using R-symmetry [84]. More precisely, we can write

$$\begin{aligned}\tilde{\mathcal{S}}_p^{(s)} &= \lambda_s \mathcal{Y}_{\{p,0\}} P^{(s)}_{\epsilon p,0} + \lambda_A \mathcal{Y}_{\{p-2,2\}} P^{(s)}_{\epsilon p+1,1} \\ &\quad + \lambda_\varphi \mathcal{Y}_{\{p-2,0\}} P^{(s)}_{\epsilon p+2,2} + \lambda_C \mathcal{Y}_{\{p-4,2\}} P^{(s)}_{\epsilon p+3,1} \\ &\quad + \lambda_t \mathcal{Y}_{\{p-4,0\}} P^{(s)}_{\epsilon p+4,0} + \lambda_r \mathcal{Y}_{\{p-4,4\}} P^{(s)}_{\epsilon p+2,0}, \quad (41)\end{aligned}$$

where we have used the Polyakov-Regge blocks, and this corresponds to a specific choice of contact terms. Various $\lambda^{(p)}_{\text{field}}$ have been obtained in Eqs. (30) and (38). It follows that $\tilde{\mathcal{S}}_p^{(s)}$ gives the correct residues for any σ and τ.

However, note that the s-channel Polyakov-Regge blocks are not symmetric in t and u. More precisely, the Bose symmetry in exchanging 1 and 2 is broken by the choice of contact terms, which can be easily seen by the fact that the s-channel Polyakov-Regge blocks have improved Regge behavior in the u-channel but not in the t-channel. To restore the s-channel Bose symmetry in the s-channel multiplet exchange, we give the following simple prescription [49]. The amplitude $\tilde{\mathcal{S}}_p^{(s)}$ takes the form of a sum over simple poles at $s = \epsilon p + 2m$. For each term in the sum, the numerator contains a quadratic factor in u of the form

$$u^2 + \alpha(i,j;m,p)u + \beta(i,j;m,p). \quad (42)$$

We can restore Bose symmetry by eliminating m from this factor from the relation

$$t + u + \epsilon p + 2m = \epsilon\Sigma \quad (43)$$

where we have substituted the pole values of s into the relation among the three Mandelstam variables. This process gives a symmetric s-channel exchange, which we denote as $\mathcal{S}_p^{(s)}$. Using the other generators of Bose symmetry, we can similarly obtain $\mathcal{S}_p^{(t)}$ and $\mathcal{S}_p^{(u)}$. Note that our prescription is not equivalent to simply using the Mellin exchange amplitudes from the Supplemental Material [55], which have already been symmetrized (or antisymmetrized) in Eq. (41). The difference is obvious in the MRV limit, as the symmetrized bosonic Mellin exchange amplitudes do not have improved u-channel Regge behavior. In principle, having specified the polar part of the amplitude, there is still the possibility of adding contact terms. The truly distinguishing feature of our prescription, however, is that the *full* Mellin amplitude can be written as a sum of exchange amplitudes over multiplets, with *no* additional contact terms [85]. The Mellin amplitudes are given by

$$\mathcal{M}(s,t;\sigma,\tau) = \mathcal{M}_s(s,t;\sigma,\tau) + \mathcal{M}_t(s,t;\sigma,\tau) + \mathcal{M}_u(s,t;\sigma,\tau), \quad (44)$$

$$\mathcal{M}_s = \sum_p \mathcal{S}_p^{(s)}(s,t;\sigma,\tau), \qquad \mathcal{M}_t = \sum_p \mathcal{S}_p^{(t)}(s,t;\sigma,\tau),$$
$$\mathcal{M}_u = \sum_p \mathcal{S}_p^{(u)}(s,t;\sigma,\tau), \quad (45)$$

with the multiplet amplitudes $\mathcal{S}_p^{(s)}$, $\mathcal{S}_p^{(t)}$, and $\mathcal{S}_p^{(u)}$ obtained with the above prescription. The absence of contact terms can be proven by the superconformal Ward identities, as we discuss in detail in Sec. V.

Let us now rewrite the Mellin amplitude $\mathcal{M}(s,t;\sigma,\tau)$ into a different form that is more suitable for presentation. As we have seen, the Mellin amplitude has a series of simple poles at $s = \epsilon p_s + 2m$, $t = \epsilon p_t + 2m$, $u = \epsilon p_u + 2m$, with

$$p_s - \max\{|k_1 - k_2|, |k_3 - k_4|\} = 2, 4, \ldots, 2\mathcal{E} - 2, \quad (46)$$

$$p_t - \max\{|k_1 - k_4|, |k_2 - k_3|\} = 2, 4, \ldots, 2\mathcal{E} - 2, \quad (47)$$

$$p_u - \max\{|k_1 - k_3|, |k_2 - k_4|\} = 2, 4, \ldots, 2\mathcal{E} - 2. \quad (48)$$

A series of poles $s = \epsilon p_s + 2m$ truncates if

$$\epsilon(k_1 + k_2) - \epsilon p_s = 2m_0, \qquad m_0 \in \mathbb{Z}_+, \\ \text{or} \quad \epsilon(k_3 + k_4) - \epsilon p_s = 2n_0, \qquad n_0 \in \mathbb{Z}_+. \quad (49)$$

The sum over m is from 0 to $m_0 - 1$ or from 0 to $n_0 - 1$ if only one of them is an integer. In the case when both m_0 and n_0 are integers, m is summed over from 0 to $\min\{m_0, n_0\} - 1$. The truncation of poles in t and u is analogous. In the following, we write $\mathcal{M}_s(s, t; \sigma, \tau)$ as a sum over poles, and we decompose the numerators into different R-symmetry structures spanned by the monomials of σ, τ,

$$\mathcal{M}_s(s, t; \sigma, \tau) = \sum_{i,j} \sigma^i \tau^j \sum_{s_0} \frac{R_s^{i,j}(t, u)}{s - s_0}. \quad (50)$$

The residues $R_{s_0}^{i,j}(t, u)$ are a sum over supergravity multiplets labeled by the Kaluza-Klein level p in the finite set (32),

$$R_{s_0}^{i,j}(t, u) = \sum_p \mathcal{R}_{p,m}^{i,j}(t, u), \quad \epsilon p + 2m = s_0, \qquad m \in \mathbb{N}. \quad (51)$$

The other two channels $\mathcal{M}_t(s, t; \sigma, \tau)$ and $\mathcal{M}_u(s, t; \sigma, \tau)$, are similar and can be obtained from $\mathcal{M}_s(s, t; \sigma, \tau)$ by Bose symmetry. Using our method described above, we calculate $\mathcal{R}_{p,m}^{i,j}(t, u)$ for all correlators in $\mathrm{AdS}_4 \times S^7$, $\mathrm{AdS}_5 \times S^5$, and $\mathrm{AdS}_7 \times S^4$. We present their explicit expressions in the next subsection.

B. All Mellin amplitudes for all maximally supersymmetric CFTs

Let us define a set of convenient combinations $u^\pm$, $t^\pm$,

$$u^\pm = u \pm \frac{\epsilon}{2}\kappa_u - \frac{\epsilon}{2}\Sigma, \qquad t^\pm = t \pm \frac{\epsilon}{2}\kappa_t - \frac{\epsilon}{2}\Sigma, \quad (52)$$

where we recall that $\epsilon = (d - 2/2)$. We find that the residues from each multiplet take the universal form

$$\mathcal{R}_{p,m}^{i,j}(t, u) = K_p^{i,j}(t, u) L_{p,m}^{i,j} N_p^{i,j} \quad (53)$$

in any spacetime dimension, and we give the expressions for $K_p^{i,j}$, $L_{p,m}^{i,j}$, and $N_p^{i,j}$ in each background below.

$\mathrm{AdS}_5 \times S^5$: Let us begin with the case of $d = 4$, where the bulk theory is IIB supergravity on $\mathrm{AdS}_5 \times S^5$. The above procedure gives the following result:

$$\begin{aligned} K_p^{i,j} = {} & 2i(2i + \kappa_u)t^- t^+ + 2j(2j + \kappa_t)u^- u^+ \\ & - 2j\kappa_u t^+ u^- - 2i\kappa_t u^+ t^- \\ & + \frac{1}{4}(2p - \kappa_t - \kappa_u)(2p + \kappa_t + \kappa_u)(u^- t^- + 4ij) \\ & + \frac{1}{2}(\kappa_u + \kappa_t - 2p)(\kappa_u + \kappa_t + 2p)(it^- + ju^-) \\ & + 4ij(t^+ \kappa_u + u^+ \kappa_t) - 8ijt^+ u^+, \end{aligned} \quad (54)$$

$$L_{p,m}^{i,j} = \frac{(-1)^{i+j+\frac{2p-\kappa_t-\kappa_u}{4}} \prod_{i=1}^4 \sqrt{k_i}}{n^2 i! j! m! \Gamma[p + m + 1] \Gamma[\frac{k_1+k_2-2m-p}{2}] \Gamma[\frac{k_3+k_4-2m-p}{2}]}, \quad (55)$$

and

$$N_p^{i,j} = \frac{2^{-3} p \Gamma[\frac{2p+\Sigma-\kappa_s-4l}{4}]}{\Gamma[\frac{\kappa_u+2+2i}{2}] \Gamma[\frac{2(p+2)-\Sigma+\kappa_s+4l}{4}] \Gamma[\frac{\kappa_t+2+2j}{2}]}, \quad (56)$$

where $i + j + l = \mathcal{E}$. Note that $L_{p,m}^{i,j}$ contains two Gamma factors, $\Gamma[k_1 + k_2 - 2m - p/2]\Gamma[k_3 + k_4 - 2m - p/2]$, in the denominator. Since $k_i + k_j - p \in 2\mathbb{Z}_+$ by cubic vertex selection rules, they implement the truncation of poles in the Mellin amplitude.

All tree-level four-point functions for $\mathrm{AdS}_5 \times S^5$ were given in Refs. [26,27] after solving the bootstrap problem and were written in terms of the reduced Mellin amplitude. The full amplitude can be obtained by acting with the superconformal difference operator $\hat{R}$ (see Refs. [26,27] for details). Upon comparing the residues, we find that above expressions reproduce the known result.

$\mathrm{AdS}_7 \times S^4$: Next, we turn to $d = 6$, which corresponds to 11D supergravity on $\mathrm{AdS}_7 \times S^4$. The full solution to all four-point functions was recently obtained in Ref. [49]. The residue factors are given by

$$\begin{aligned} K_p^{i,j} = {} & 2i(2i + \kappa_u)t^- t^+ + 2j(2j + \kappa_t)u^- u^+ + 2j(1 - \kappa_u)t^+ u^- + 2i(1 - \kappa_t)u^+ t^- \\ & + \frac{1}{4}(2p - \kappa_t - \kappa_u)(2p - 2 + \kappa_t + \kappa_u)(u^- t^- + 16ij) \\ & + (\kappa_u + \kappa_t - 2p)(\kappa_u + \kappa_t + 2p - 2)(it^- + ju^-) \\ & + 8ij(t^+(\kappa_u - 1) + u^+(\kappa_t - 1)) - 8ijt^+ u^+, \end{aligned} \quad (57)$$

$$L_{p,m}^{i,j} = \frac{(-1)^{i+j+\frac{2p-\kappa_t-\kappa_u}{4}}\pi^{-\frac{3}{2}}\Gamma[\frac{k_1+k_2-p+1}{2}]\Gamma[\frac{k_3+k_4-p+1}{2}]\Gamma[\frac{k_1+k_2+p}{2}]\Gamma[\frac{k_3+k_4+p}{2}]}{n^3 m! i! j! \prod_{a=1}^4 \sqrt{(2k_a-2)!}\,\Gamma[2p+m]\Gamma[k_1+k_2-m-p]\Gamma[k_3+k_4-m-p]}, \tag{58}$$

$$N_p^{i,j} = \frac{2^{\Sigma-6}(2p-1)\Gamma[\frac{2(p-1)+\Sigma-\kappa_s-4l}{4}]}{\Gamma[\frac{\kappa_u+2+2i}{2}]\Gamma[\frac{2(p+2)-\Sigma+\kappa_s+4l}{4}]\Gamma[\frac{\kappa_t+2+2j}{2}]}. \tag{59}$$

The Gamma functions $\Gamma[k_1+k_2-m-p]\Gamma[k_3+k_4-m-p]$ in $L_{p,m}^{i,j}$ also ensure that the number of poles in the $\mathrm{AdS}_7\times S^4$ Mellin amplitudes is finite.

$\mathrm{AdS}_4\times S^7$: Finally, we consider $d=3$, which corresponds to 11D supergravity on $\mathrm{AdS}_4\times S^7$. The only correlator that has been obtained in the literature is the four-point function of the stress-tensor multiplet [48]. Here, we present new results, which generalize to four-point functions of arbitrary one-half BPS operators:

$$\begin{aligned} K_p^{i,j} = {} & 2i(2i+\kappa_u)t^-t^+ + 2j(2j+\kappa_t)u^-u^+ - 2j(2+\kappa_u)t^+u^- - 2i(2+\kappa_t)u^+t^- \\ & + \frac{1}{4}(2p-\kappa_t-\kappa_u)(4+2p+\kappa_t+\kappa_u)(u^-t^-+ij) \\ & + \frac{1}{4}(\kappa_u+\kappa_t-2p)(\kappa_u+\kappa_t+2p+4)(it^-+ju^-) \\ & + 2ij(t^+(2+\kappa_u)+u^+(2+\kappa_t)) - 8ijt^+u^+, \end{aligned} \tag{60}$$

$$L_{p,m}^{i,j} = \frac{\sqrt{\pi}\prod_{i=1}^4\sqrt{(k_i+1)!}}{n^{\frac{3}{2}} i! j! m! \Gamma[\frac{p+2m+3}{2}]\Gamma[\frac{k_1+k_2-p+2}{4}]\Gamma[\frac{k_3+k_4-p+2}{4}]\Gamma[\frac{k_1+k_2+p+4}{4}]\Gamma[\frac{k_3+k_4+p+4}{4}]}\frac{(-1)^{i+j+\frac{2p-\kappa_t-\kappa_u}{4}}}{\Gamma[\frac{k_1+k_2-4m-p}{4}]\Gamma[\frac{k_3+k_4-4m-p}{4}]}, \tag{61}$$

$$N_p^{i,j} = \frac{2^{-\frac{11+\Sigma}{2}}(1+p)\Gamma[\frac{2(p+2)+\Sigma-\kappa_s-4l}{4}]}{\Gamma[\frac{\kappa_u+2+2i}{2}]\Gamma[\frac{2(p+2)-\Sigma+\kappa_s+4l}{4}]\Gamma[\frac{\kappa_t+2+2j}{2}]}. \tag{62}$$

Unlike the previous two cases, the Gamma function factors $\Gamma[k_1+k_2-4m-p/4]\Gamma[k_3+k_4-4m-p/4]$ in $L_{p,m}^{i,j}$ do not guarantee that the Mellin amplitudes will have a finite number of poles. Upon setting $k_i=2$, we reproduce the result of Ref. [48].

Clearly, the Mellin amplitude residues in the three maximally supersymmetric backgrounds are highly similar. In fact, we can accentuate their similarity by writing a formula that interpolates M-theory and string-theory amplitudes. More precisely, we can modify $K_p^{i,j}$, $L_{p,m}^{i,j}$, and $N_p^{i,j}$ by introducing ϵ dependence as follows:

$$\begin{aligned} K_p^{i,j} = {} & 2i(2i+\kappa_u)t^-t^+ + 2j(2j+\kappa_t)u^-u^+ - 2j\left(\frac{2}{\epsilon}-2+\kappa_u\right)t^+u^- - 2i\left(\frac{2}{\epsilon}-2+\kappa_t\right)u^+t^- \\ & + \frac{1}{4}(2p-\kappa_t-\kappa_u)\left(2p+\frac{4}{\epsilon}-4+\kappa_t+\kappa_u\right)(u^-t^-+4\epsilon^2 ij) \\ & + \frac{\epsilon}{2}(\kappa_u+\kappa_t-2p)\left(\kappa_u+\kappa_t+2p+\frac{4}{\epsilon}-4\right)(it^-+ju^-) \\ & + 4\epsilon ij\left(t^+\left(\kappa_u+\frac{2}{\epsilon}-2\right)+u^+\left(\kappa_t+\frac{2}{\epsilon}-2\right)\right) - 8ijt^+u^+, \end{aligned} \tag{63}$$

$$\begin{aligned} L_{p,m}^{i,j} = {} & \frac{\pi^{-\frac{(\epsilon-1)(2\epsilon+5)}{6}}2^{\frac{2(\epsilon-1)(2\epsilon-1)}{3}}\prod_{i=1}^4\left(\sqrt{k_i+\frac{1}{\epsilon}-1}\,\Gamma[\frac{2}{3}((1+\epsilon)k_i+2-\epsilon)]^{\frac{1}{3}(\frac{1}{\epsilon}-\epsilon)}\right)}{n^{1+\epsilon}\Gamma[2-\epsilon+m+\epsilon p]} \\ & \times\frac{(\Gamma[\frac{(1+\epsilon)(k_1+k_2+p)}{6}+\frac{2(2-\epsilon)}{3}]\Gamma[\frac{(1+\epsilon)(k_3+k_4+p)}{6}+\frac{2(2-\epsilon)}{3}])^{-\frac{2}{3}(\frac{1}{\epsilon}-\epsilon)}}{i!j!m!} \\ & \times\frac{(-1)^{i+j+\frac{2p-\kappa_t-\kappa_u}{4}}(\Gamma[\frac{(1+\epsilon)(k_1+k_2-p)}{6}+\frac{1}{2}]\Gamma[\frac{(1+\epsilon)(k_3+k_4-p)}{6}+\frac{1}{2}])^{-\frac{2}{3}(\frac{1}{\epsilon}-\epsilon)}}{\Gamma[\frac{\epsilon}{2}(k_1+k_2-p)-m]\Gamma[\frac{\epsilon}{2}(k_3+k_4-p)-m]}, \end{aligned} \tag{64}$$

and

$$N_p^{i,j} = \frac{2^{\Sigma(\epsilon-1)-4-\epsilon}\Gamma[\frac{1}{4}(\frac{4}{\epsilon}-4+2p+\Sigma-\kappa_s-4l)](-(\frac{5\epsilon^2-15\epsilon+6}{\epsilon})p+1-\epsilon)}{\Gamma[\frac{\kappa_u+2+2i}{2}]\Gamma[\frac{2(p+2)-\Sigma+\kappa_s+4l}{4}]\Gamma[\frac{\kappa_t+2+2j}{2}]}. \quad (65)$$

When substituting $\epsilon=\frac{1}{2}$, 1, and 2, the above formulas reduce to the results in respective dimensions. Of course, such interpolation formulas that go through the three physical ϵ values are far from being unique, and we do not expect, on any grounds, that M-theory and string-theory correlators should be physically connected. Nevertheless, what we wish to highlight are the similarities of analytic structures in the residues, which allow them to be compactly encapsulated in a single set of formulas. We also want to mention that the above sum over the multiplets p can be performed in a closed form, and this leads to a hypergeometric series. However, we think that it is better to leave the sum unperformed, which makes the analytic structure more clear.

C. Examples

Let us demonstrate our general formulas with a few illuminating examples. The simplest example has $k_i=2$, which corresponds to the stress-tensor four-point functions. The extremality $\mathcal{E}$ is 2. Therefore, i, j run from 0 to 2, and the Mellin amplitudes are degree-2 polynomials in σ, τ. There is only one value $p=2$ in the range of summation (32), which means only the stress-tensor multiplet contributes. Using our formulas, we find that, for $\epsilon=1$,

$$\begin{aligned}
\mathcal{M}^{\mathrm{AdS}_5}_{2222}(s,t;\sigma,\tau) &= \mathcal{M}^{\mathrm{AdS}_5}_{2222,s}(s,t;\sigma,\tau)+\mathcal{M}^{\mathrm{AdS}_5}_{2222,t}(s,t;\sigma,\tau)+\mathcal{M}^{\mathrm{AdS}_5}_{2222,u}(s,t;\sigma,\tau),\\
\mathcal{M}^{\mathrm{AdS}_5}_{2222,s}(s,t;\sigma,\tau) &= -\frac{2}{n^2}\left(\frac{(t-4)(u-4)+(s+2)((t-4)\sigma+(u-4)\tau)}{s-2}\right),\\
\mathcal{M}^{\mathrm{AdS}_5}_{2222,t}(s,t;\sigma,\tau) &= \tau^2\mathcal{M}^{\mathrm{AdS}_5}_{2222,s}\left(t,s;\frac{\sigma}{\tau},\frac{1}{\tau}\right),\qquad \mathcal{M}^{\mathrm{AdS}_5}_{2222,u}(s,t;\sigma,\tau)=\sigma^2\mathcal{M}^{\mathrm{AdS}_5}_{2222,s}\left(u,t;\frac{1}{\sigma},\frac{\tau}{\sigma}\right),
\end{aligned} \quad (66)$$

where $s+t+u=8$. For $\epsilon=2$, we get

$$\begin{aligned}
\mathcal{M}^{\mathrm{AdS}_7}_{2222}(s,t;\sigma,\tau) &= \mathcal{M}^{\mathrm{AdS}_7}_{2222,s}(s,t;\sigma,\tau)+\mathcal{M}^{\mathrm{AdS}_7}_{2222,t}(s,t;\sigma,\tau)+\mathcal{M}^{\mathrm{AdS}_7}_{2222,u}(s,t;\sigma,\tau),\\
\mathcal{M}^{\mathrm{AdS}_7}_{2222,s}(s,t;\sigma,\tau) &= -\frac{1}{n^3}\left(\frac{(t-8)(u-8)+(s+2)((t-8)\sigma+(u-8)\tau)}{s-4}\right.\\
&\qquad\left.+\frac{(t-8)(u-8)+(s+2)((t-8)\sigma+(u-8)\tau)}{4(s-6)}\right),\\
\mathcal{M}^{\mathrm{AdS}_7}_{2222,t}(s,t;\sigma,\tau) &= \tau^2\mathcal{M}^{\mathrm{AdS}_7}_{2222,s}\left(t,s;\frac{\sigma}{\tau},\frac{1}{\tau}\right),\qquad \mathcal{M}^{\mathrm{AdS}_7}_{2222,u}(s,t;\sigma,\tau)=\sigma^2\mathcal{M}^{\mathrm{AdS}_7}_{2222,s}\left(u,t;\frac{1}{\sigma},\frac{\tau}{\sigma}\right),
\end{aligned} \quad (67)$$

where $s+t+u=16$. These two correlators, respectively, reproduce the results of Refs. [17,18]. When $\epsilon=\frac{1}{2}$, we have

$$\begin{aligned}
\mathcal{M}^{\mathrm{AdS}_4}_{2222}(s,t;\sigma,\tau) &= \mathcal{M}^{\mathrm{AdS}_4}_{2222,s}(s,t;\sigma,\tau)+\mathcal{M}^{\mathrm{AdS}_4}_{2222,t}(s,t;\sigma,\tau)+\mathcal{M}^{\mathrm{AdS}_4}_{2222,u}(s,t;\sigma,\tau),\\
\mathcal{M}^{\mathrm{AdS}_4}_{2222,s}(s,t;\sigma,\tau) &= \sum_{m=0}^{\infty}-\frac{3((t-2)(u-2)+(s+2)((t-2)\sigma+(u-2)\tau))}{\sqrt{2\pi}n^{\frac{3}{2}}\Gamma(\frac{1}{2}-m)^2 m!\Gamma(m+\frac{5}{2})(s-1-2m)},\\
\mathcal{M}^{\mathrm{AdS}_4}_{2222,t}(s,t;\sigma,\tau) &= \tau^2\mathcal{M}^{\mathrm{AdS}_4}_{2222,s}\left(t,s;\frac{\sigma}{\tau},\frac{1}{\tau}\right),\mathcal{M}^{\mathrm{AdS}_4}_{2222,u}(s,t;\sigma,\tau)=\sigma^2\mathcal{M}^{\mathrm{AdS}_4}_{2222,s}\left(u,t;\frac{1}{\sigma},\frac{\tau}{\sigma}\right),
\end{aligned} \quad (68)$$

where $s+t+u=4$. These equations reproduce the result of Ref. [48], where the contact terms have now been automatically absorbed in the exchange contribution according to our prescription.

Another interesting case is the next-to-next-to-extremal correlators with $k_1=k_2=2$ and $k_3=k_4=k$. Let us give only the explicit result for $\mathrm{AdS}_4\times S^7$, which has not appeared in the literature. This family of correlators will be the starting point for constructing the four-point function $\langle 2222\rangle$ at one loop. These correlators also have $\mathcal{E}=2$. Therefore, $p=2$ for the s-channel exchanges, while $p=k$ for the t- and u-channel exchanges. We have

$$\mathcal{M}^{\mathrm{AdS}_4}_{22kk,s}(s,t;\sigma,\tau)=\sum_{m=0}^{\infty}-\frac{3k}{8\sqrt{2\pi}n^{\frac{3}{2}}m!\Gamma[\frac{k-2m-1}{2}]\Gamma[\frac{1-2m}{2}]\Gamma[\frac{5+2m}{2}]}\times\frac{(2t-k-2)(2u-k-2)+4(s+2)\big(\sigma(t-\frac{k}{2}-1)+\tau(u-\frac{k}{2}-1)\big)}{s-1-2m},\tag{69}$$

where $s+t+u=2+k$, and

$$\mathcal{M}^{\mathrm{AdS}_4}_{22kk,t}(s,t;\sigma,\tau)=\sum_{m=0}^{\infty}-\frac{3k\tau\Gamma[\frac{k}{2}+1]}{8\sqrt{2}n^{\frac{3}{2}}m!\Gamma[\frac{k-1}{2}]\Gamma[\frac{1-2m}{2}]^2\Gamma[\frac{k+3+2m}{2}]}\times\frac{(2t+k+2)(2u-k-2)+2(s-k)\big(\sigma(k+2t+2)+\tau(2u-k-2)\big)}{t-\frac{k}{2}-2m},\tag{70}$$

$$\mathcal{M}^{\mathrm{AdS}_4}_{22kk,u}(s,t;\sigma,\tau)=\mathcal{M}^{\mathrm{AdS}_4}_{22kk,t}(s,u;\tau,\sigma).\tag{71}$$

Note that when k is odd, the pole series in $\mathcal{M}^{\mathrm{AdS}_4}_{22kk,s}$ truncates, while if k is even, this does not happen.

Finally, let us give an example with higher extremality $\mathcal{E}=3$. We consider the case with $k_i=3$. In the sum over multiplets, p now takes values 2 and 4 according to Eq. (32). Using our formulas, we get

$$\begin{aligned}\mathcal{M}^{\mathrm{AdS}_4}_{3333,s}(s,t;\sigma,\tau)=&\sum_{m=0}^{\infty}-\frac{27\big((t-3)(u-3)+(s+2)\big((t-3)\sigma+(u-3)\big)\big)}{4n^{\frac{3}{2}}\sqrt{2\pi}m!\Gamma[1-m]^2\Gamma[\frac{2m+5}{2}](s-1-2m)}\\&+\sum_{m=0}^{\infty}\frac{48}{5n^{\frac{3}{2}}\sqrt{2\pi}m!\Gamma[\frac{1-2m}{2}]^2\Gamma[\frac{2m+7}{2}](s-2-2m)}\\&\times\bigg[(t-3)(u-3)+4(s+3)\big((s+2)\sigma\tau-(t-3)\sigma^2-(u-3)\tau^2\big)\\&+(s+3)\big((t-3)\sigma+(u-3)\tau\big)-4\bigg((t-3)\left(u-\frac{15}{4}\right)\sigma+(u-3)\left(t-\frac{15}{4}\right)\tau\bigg)\bigg],\end{aligned}\tag{72}$$

where $s+t+u=6$. The other two channels are related by crossing symmetry,

$$\begin{aligned}\mathcal{M}^{\mathrm{AdS}_4}_{3333,t}(s,t;\sigma,\tau)&=\tau^3\mathcal{M}^{\mathrm{AdS}_4}_{3333,s}\left(t,s;\frac{\sigma}{\tau},\frac{1}{\tau}\right),\\\mathcal{M}^{\mathrm{AdS}_4}_{3333,u}(s,t;\sigma,\tau)&=\sigma^3\mathcal{M}^{\mathrm{AdS}_4}_{3333,s}\left(u,t;\frac{1}{\sigma},\frac{\tau}{\sigma}\right).\end{aligned}\tag{73}$$

V. SUPERCONFORMAL WARD IDENTITIES

A. Ward identities in Mellin space

In the previous section, we constructed the polar part of the general Mellin amplitudes for the backgrounds $\mathrm{AdS}_4\times S^7$, $\mathrm{AdS}_5\times S^5$, and $\mathrm{AdS}_7\times S^4$, and claimed that no further contact terms are needed. In order to show that these contact terms are absent, we need to show that these amplitudes satisfy the superconformal WI. Note that since the WI are not heavily used in our construction, this also serves as a nontrivial check of our results. In the cases of $\mathrm{AdS}_5\times S^5$ and $\mathrm{AdS}_7\times S^4$, one can efficiently impose the WI by requiring the existence of a reduced amplitude $\tilde{\mathcal{M}}$, as discussed in Sec. II B. However, for $\mathrm{AdS}_4\times S^7$, this is not possible. Below, we develop an efficient method to impose the WI in Mellin space at the level of the full amplitude, expanding on Ref. [48]. We start by recalling the WI (10) in space-time,

$$(z\partial_z-\epsilon\alpha\partial_\alpha)\mathcal{G}(z,\bar{z};\alpha,\bar{\alpha})|_{\alpha=1/z}=0.\tag{74}$$

In order to write this relation in Mellin space, we first note

$$z\partial_z=U\partial_U-\frac{z}{1-z}V\partial_V.\tag{75}$$

In Mellin space, $U\partial_U$ and $V\partial_V$ have a very simple, multiplicative action, which follows from the definition (18),

$$U\partial_U\to\left(\frac{s}{2}-a_s\right)\times,\qquad V\partial_V\to\left(\frac{t}{2}-a_t\right)\times.\tag{76}$$

On the other hand, z does not. In order to proceed, we write the Mellin amplitude in terms of the R-symmetry cross ratios α, $\bar{\alpha}$ and expand it in powers of α:

$$\mathcal{M}(s,t,\alpha,\bar{\alpha})=\sum_{q=0}^{\mathcal{E}}\alpha^q\mathcal{M}^{(q)}(s,t,\bar{\alpha}).\tag{77}$$

In terms of the components $\mathcal{M}^{(q)}(s,t,\bar{\alpha})$, the WI take the form

$$\sum_{q=0}^{\mathcal{E}}\left((1-z)z^{\mathcal{E}-q}\left(\frac{s}{2}-a_s-q\right)-z^{\mathcal{E}-q+1}\left(\frac{t}{2}-a_t\right)\right)\mathcal{M}^{(q)}(s,t,\bar{\alpha})=0. \tag{78}$$

We can obtain an inequivalent relation by replacing $z\to\bar{z}$,

$$\sum_{q=0}^{\mathcal{E}}\left((1-\bar{z})\bar{z}^{\mathcal{E}-q}\left(\frac{s}{2}-a_s-q\right)-\bar{z}^{\mathcal{E}-q+1}\left(\frac{t}{2}-a_t\right)\right)\mathcal{M}^{(q)}(s,t,\bar{\alpha})=0. \tag{79}$$

Considering two independent linear combinations of the relations above, we arrive at

$$\sum_{q=0}^{\mathcal{E}}\left((\zeta_{\pm}^{(\mathcal{E}-q)}-\zeta_{\pm}^{(\mathcal{E}-q+1)})\left(\frac{s}{2}-a_s-q\right)-\zeta_{\pm}^{(\mathcal{E}-q+1)}\left(\frac{t}{2}-a_t\right)\right)\mathcal{M}^{(q)}(s,t,\bar{\alpha})=0, \tag{80}$$

where we have defined

$$\zeta_{+}^{(n)}=z^n+\bar{z}^n,\qquad \zeta_{-}^{(n)}=\frac{z^n-\bar{z}^n}{z-\bar{z}}. \tag{81}$$

The crucial observation is that, while z and $\bar{z}$ by themselves do not have a simple action in Mellin space, $\zeta_{\pm}^{(n)}$, which should be interpreted as operators, do. Indeed, for each n, $\zeta_{\pm}^{(n)}$ are simply polynomials of U and V, while powers of U and V act in Mellin space as shift operators. This observation leads to the following representation in Mellin space:

$$\zeta_{+}^{(0)}=2,\qquad \zeta_{-}^{(0)}=0, \tag{82}$$

$$\zeta_{+}^{(1)}=1+\hat{U}-\hat{V},\qquad \zeta_{-}^{(1)}=1, \tag{83}$$

$$\zeta_{+}^{(2)}=1-2\hat{V}+\hat{U}^2+\hat{V}^2-2\widehat{\mathrm{UV}},\qquad \zeta_{-}^{(2)}=1+\hat{U}-\hat{V}, \tag{84}$$

and so on, where $\widehat{U^mV^n}$ is the shift operator corresponding to U^mV^n, given by

$$\widehat{U^mV^n}\circ\mathcal{M}(s,t)=\frac{\Gamma_{\{k_i\}}(s-2m,t-2n)}{\Gamma_{\{k_i\}}(s,t)}\mathcal{M}(s-2m,t-2n). \tag{85}$$

Note that for a given extremality $\mathcal{E}$, only operators up to $\zeta_{\pm}^{(\mathcal{E}+1)}$ appear.

1. An example

The simplest example is that of $k_i=2$, namely, the correlator of the stress-tensor multiplet. So let us work out this case in detail. We focus on Eq. (80), involving ζ_-, which has not been explicitly considered before. In this case, the extremality $\mathcal{E}=2$, and we can decompose the Mellin amplitude as

$$\mathcal{M}(s,t,\alpha,\bar{\alpha})=\mathcal{M}^{(0)}(s,t)+\alpha\mathcal{M}^{(1)}(s,t)+\alpha^2\mathcal{M}^{(2)}(s,t), \tag{86}$$

where the dependence on $\bar{\alpha}$ has not been explicitly shown since it acts as a spectator. The WI take the form

$$\zeta_{-}^{(1)}((s+t-8\epsilon)\mathcal{M}^{(2)}(s,t)+(2\epsilon-s)\mathcal{M}^{(1)}(s,t))+\zeta_{-}^{(2)}((s+t-6\epsilon)\mathcal{M}^{(1)}(s,t)-s\mathcal{M}^{(0)}(s,t))+\zeta_{-}^{(3)}(s+t-4\epsilon)\mathcal{M}^{(0)}(s,t)=0, \tag{87}$$

or, explicitly, after acting with the shift operators,

$$\begin{aligned}
&(t-4\epsilon)\mathcal{M}^{(0)}(s,t)-\frac{2(s-4\epsilon)^2(t-4\epsilon)^2}{(s+t-4\epsilon-2)^2(s+t-4(\epsilon+1))}\mathcal{M}^{(0)}(s-2,t-2)\\
&+\frac{(s^2-2s(4\epsilon+1)+8\epsilon(2\epsilon+1))^2}{(s+t-4\epsilon-2)^2(s+t-4(\epsilon+1))}\mathcal{M}^{(0)}(s-4,t)-\frac{(t-4\epsilon)^2(s+2t-8\epsilon-4)}{(s+t-4\epsilon-2)^2}\mathcal{M}^{(0)}(s,t-2)\\
&+\frac{(t^2-2t(4\epsilon+1)+8\epsilon(2\epsilon+1))^2}{(s+t-4\epsilon-2)^2(s+t-4(\epsilon+1))}\mathcal{M}^{(0)}(s,t-4)+\frac{(s-4\epsilon)^2(t-4\epsilon)}{(s+t-4\epsilon-2)^2}\mathcal{M}^{(0)}(s-2,t)\\
&+\frac{(s-4\epsilon)^2(s+t-6\epsilon-2)}{(s+t-4\epsilon-2)^2}\mathcal{M}^{(1)}(s-2,t)-\frac{(t-4\epsilon)^2(s+t-6\epsilon-2)}{(s+t-4\epsilon-2)^2}\mathcal{M}^{(1)}(s,t-2)\\
&+(t-4\epsilon)\mathcal{M}^{(1)}(s,t)+(s+t-8\epsilon)\mathcal{M}^{(2)}(s,t)=0.
\end{aligned} \tag{88}$$

Note this equation gives $\mathcal{M}^{(2)}(s,t)$ in terms of $\mathcal{M}^{(0)}(s,t)$ and $\mathcal{M}^{(1)}(s,t)$, which is a general phenomenon: For a general extremality $\mathcal{E}$, we can use the WI involving ζ_- to solve for $\mathcal{M}^{(\mathcal{E})}(s,t)$ in terms of the other ones. Returning to Eq. (88), for

$d = 4, 6$, we can simply plug in the results given in Sec. IV C and check that they indeed satisfy this relation for $\epsilon = 1$ and $\epsilon = 2$, respectively. For $d = 3$, we can resume the expression given in Eq, (68) to obtain

$$\mathcal{M}^{\mathrm{AdS}_4}_{2222,s}(s,t;\sigma,\tau) = \left(-\frac{3(t-2)(u-2)}{2\sqrt{2}\pi^{3/2}n^{3/2}(s-1)s(s+2)\Gamma[1-\frac{s}{2}]} + \frac{3\sqrt{2}(t-2)(t+u-6)}{\pi^{3/2}n^{3/2}(s-1)s^2(s+2)^2\Gamma[-\frac{s}{2}-1]}\sigma + \frac{3\sqrt{2}(u-2)(t+u-6)}{\pi^{3/2}n^{3/2}(s-1)s^2(s+2)^2\Gamma[-\frac{s}{2}-1]}\tau\right)h(s), \tag{89}$$

where we have introduced

$$h(s) = \sqrt{\pi}(s^2+3s-4)\Gamma\left[1-\frac{s}{2}\right] + 8\Gamma\left[\frac{3-s}{2}\right]. \tag{90}$$

Adding the contributions in the t- and u-channels, we can obtain the corresponding expressions for $\mathcal{M}^{(q)}(s,t)$, for $q = 0, 1, 2$. Plugging them into Eq. (88), we can check that, indeed, the identity is satisfied for $\epsilon = 1/2$.

We have checked the above WI for a vast variety of examples. We have found that our answer satisfies the WI in each case, without the addition of a contact term. This result actually proves that by using the representation we have chosen, our result provides the full answer and not just the polar part of the amplitude.

B. WI and the flat-space limit

It is illuminating to study the superconformal Ward identities and the Mellin amplitudes around the flat-space limit, where s, t are large. In the flat-space limit, shift operators act multiplicatively. Indeed, in this limit, $\mathcal{M}(s-2m, t-2n) \sim \mathcal{M}(s,t)$ plus higher-order derivative corrections, and one can explicitly check that

$$\widehat{U^m V^n} \circ \mathcal{M}(s,t) = \frac{s^{2m}t^{2n}}{(s+t)^{2(m+n)}} + \cdots, \tag{91}$$

which leads to the following rule for the operators $\zeta^{(n)}_{\pm}$ to leading order:

$$\zeta^{(n)}_{+} = \frac{2s^n}{(s+t)^n} + \cdots, \qquad \zeta^{(n)}_{-} = \frac{ns^{n-1}}{(s+t)^{n-1}} + \cdots. \tag{92}$$

Plugging these expressions into Eq. (80) and taking the flat-space limit, we observe that the equation for ζ_+ is trivially satisfied to leading order, while the remaining equation gives

$$\sum_{q=0}^{\mathcal{E}} \frac{s^{\mathcal{E}-q}}{(s+t)^{\mathcal{E}-q}} \mathcal{M}^{(q)}_{\mathrm{flat}}(s,t,\bar{\alpha}) = 0. \tag{93}$$

However, this equation simply implies that, in the flat-space limit,

$$\mathcal{M}_{\mathrm{flat}}\left(s,t;\frac{s+t}{s},\bar{\alpha}\right) = 0,$$
$$\mathcal{M}_{\mathrm{flat}}\left(s,t;\alpha,\frac{s+t}{s}\right) = 0, \tag{94}$$

as a consequence of the superconformal Ward identities, in any number of dimensions. The second relation follows from replacing $\alpha \to \bar{\alpha}$.

From our results, we can study the explicit form of the amplitudes in the flat-space limit. In all cases, we find

$$\lim_{s,t\to\infty} \mathcal{M}(s,t;\sigma,\tau) = \mathcal{N}_{\{k_i\}} \frac{\Theta^{\mathrm{flat}}_4(s,t;\sigma,\tau)}{stu} P_{\{k_i\}}(\sigma,\tau),$$

with $s+t+u=0$ in the flat-space limit, and

$$\Theta^{\mathrm{flat}}_4(s,t;\sigma,\tau) = (tu + ts\sigma + su\tau)^2. \tag{95}$$

Note that $P_{\{k_i\}}(\sigma,\tau)$ is an R-symmetry polynomial explicitly given by

$$P_{\{k_i\}} = \sum_{\substack{i+j+k=\mathcal{E}-2\\ 0\le i,j,k\le\mathcal{E}-2}} \frac{(\mathcal{E}-2)!\sigma^i\tau^j}{i!j!k!(i+\frac{\kappa_u}{2})!(j+\frac{\kappa_t}{2})!(k+\frac{\kappa_s}{2})!}.$$

The form of the flat-space limit is completely universal, and the prefactor Θ^{flat}_4 and the polynomials $P_{\{k_i\}}(\sigma,\tau)$ do not depend on the number of dimensions. Furthermore, rewriting Θ^{flat}_4 in terms of $\alpha, \bar{\alpha}$ and using $s+t+u=0$, we obtain

$$\Theta^{\mathrm{flat}}_4(s,t;\alpha,\bar{\alpha}) = (s+t-s\alpha)^2(s+t-s\bar{\alpha})^2, \tag{96}$$

which neatly factorizes into a holomorphic and an antiholomorphic part. Note that the presence of this factor implies the relations (94) indeed hold. For $d = 4, 6$, the presence of the prefactor $\Theta^{\mathrm{flat}}_4(s,t;\alpha,\bar{\alpha})$ in the flat-space limit has also been discussed in Refs. [86,87]. In those cases, the solutions to the WI can be written as a shift operator acting on a reduced amplitude, and we can show that the flat-space limit of such a shift operator always contains the prefactor $\Theta^{\mathrm{flat}}_4(s,t;\alpha,\bar{\alpha})$.

VI. CONCLUSION

In this paper, we solved a long-standing problem in AdS/CFT. We developed a constructive method that gives all tree-level four-point holographic correlators in all theories with maximal superconformal symmetry. Our method exploits the remarkable simplicity of the Mellin amplitude in a special R-symmetry limit. The simplicity arises as a result of superconformal symmetry and makes it possible to directly compute all the amplitudes in this limit. This limit also hides new powerful organizing principles for holographic correlators, which allow us to reconstruct the full amplitudes from this limit. The reconstruction procedure is based purely on symmetries, and it is universal for all spacetime dimensions. This method allows us to derive results for different backgrounds on the same footing, which overcomes the limitations of all previous methods. For $d = 4$, our result constitutes a proof for the conjecture [26,27]. For $d = 6$, we reproduce the results recently reported in Ref. [49], and for $d = 3$, we provide new results. Our results lead to an array of interesting questions, applications, and avenues for future research. We list a few below.

(i) The multitude of four-point functions constructed in this paper contain a wealth of theoretical data. These data include anomalous dimensions and three-point function coefficients, and are of great importance for studying strongly coupled, superconformal field theories. For example, in $d = 3$, part of the data can be compared with other exact results, obtained from topological twisting and supersymmetric localization [88–91]. Moreover, these data can also be used to calibrate the numerical bootstrap bounds at large central charge [88,89,92].

(ii) The data from bulk tree-level supergravity are also the necessary input for studying AdS quantum gravity using conformal bootstrap techniques. A systematic procedure to compute loop corrections is given in Ref. [35], which works in a similar fashion as the amplitude unitarity method in flat space. The algorithm takes tree-level data as input and outputs loop-level amplitudes, which capture the quantum corrections. While the computation of loop-level correlators is quite advanced in $AdS_5 \times S^5$ [36–46,93], it is still in its infancy for $AdS_7 \times S^4$ [94]. Similar progress for $AdS_4 \times S^7$ at one loop is yet to be made. Systematically understanding the structure of general loop-level correlators will constitute an important next step in advancing the general program of holographic correlators.

(iii) Moreover, our results contain a fascinating feature that could lead to great progress at higher points, therefore extending this program in another important direction. In our construction, we give a prescription for restoring Bose symmetry in the exchange amplitudes, which remarkably, at the same time, expresses the full amplitude as a sum over exchange amplitudes with no extra contact terms. The absence of contact terms is a clear indication of on-shell reconstructibility in AdS, and a similar phenomenon was also observed at the level of the five-point function [65]. In flat space, such reconstructibility leads to efficient recursive algorithms that generate higher-point amplitudes from lower-point ones. It would be interesting to have a better understanding of the observed reconstructibility in AdS and to explore similar recursive computational methods.

(iv) In this paper, we showed that the MRV limit of amplitudes encodes important physics. We can also study various other limits of the general four-point correlators. One particularly interesting limit is to take k_i to be large, where we would expect to see the semiclassical behavior of membranes or strings scattering in AdS.

(v) It would also be interesting to generalize our techniques to study theories with less supersymmetry. Some initial progress using bootstrap methods has been reported in Ref. [59] for the simplest four-point functions. We expect that using the MRV limit will greatly facilitate the analysis and allow for a similar solution for general correlators in these theories. It would be very interesting to see whether the same organizing principles will continue to hold for theories with less supersymmetry and, in particular, whether intrinsic contact terms are absent.

(vi) On a technical note, we have also initiated a study of the Mellin superconformal Ward identities (and their solutions) around the flat-space limit. It may be interesting to pursue this further to construct the solution to the superconformal Ward identities for the $d = 3$ case, where the solution in position space is not tractable because of the appearance of nonlocal differential operators.

(vii) Finally, our results and techniques for holographic correlators have demonstrated many similarities to those for flat-space scattering amplitudes. The MRV notion that played a key role in this paper was, in fact, motivated by the similar MHV concept in flat space. We believe that there is a promising future-research avenue that further explores and exploits such connections and will greatly benefit both fields of research through the exchange of ideas. For example, there has been some progress in understanding gravitational MHV amplitudes through twistor actions in the presence of a cosmological constant (see Ref. [95] and references therein). Relatedly, there have also been some attempts [96,97] to develop a Cachazo-He-Yuan formalism [98] in AdS by using ambitwistor string techniques. It would be very interesting to make a connection between these formalisms and the results of this paper.

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