



Rastall gravity is equivalent to Einstein gravity

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ABSTRACT

Rastall gravity, originally developed in 1972, is currently undergoing a significant surge in popularity. Rastall gravity purports to be a modified theory of gravity, with a non-conserved stress-energy tensor, and an unusual non-minimal coupling between matter and geometry, the Rastall stress-energy satisfying $[T_R]^{ab}_{;b} = \frac{\lambda}{4} g^{ab} R_{;b}$. Unfortunately, a deeper look shows that Rastall gravity is completely equivalent to Einstein gravity – usual general relativity. The gravity sector is completely standard, based as usual on the Einstein tensor, while in the matter sector Rastall's stress-energy tensor corresponds to an artificially isolated part of the physical conserved stress-energy.

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1. Introduction

Rastall gravity [1], despite its somewhat mixed 45-year history, is currently undergoing a significant surge in popularity. Some 19 closely related articles have appeared so far in 2017 [2–20]. (See also [21–23].)

Unfortunately, as I shall argue below, Rastall gravity is completely equivalent to standard Einstein gravity – general relativity – all that is going on is that one is artificially splitting the physical conserved stress-energy tensor into two non-conserved pieces.

Historically, in 1972 Rastall tentatively suggested [1] that it might prove profitable to consider a covariantly non-conserved stress-energy tensor, one with $\nabla_b [T_R]^{ab} \neq 0$. In particular, he then suggested the phenomenological model $\nabla_b [T_R]^{ab} = F^a$, where F^a is some vector field vanishing in flat spacetime.

A (somewhat weak) plausibility argument then led him to consider $\nabla_b [T_R]^{ab} \propto g^{ab} \nabla_a R$. Ultimately Rastall posited the existence of a constant λ such that for Rastall's non-conserved stress energy tensor

$$\nabla_b [T_R]^{ab} = \frac{\lambda}{4} g^{ab} \nabla_b R. \quad (1)$$

(For future convenience, I have chosen a slightly different normalization than Rastall.) The full Rastall equations of motion (EOM) are then [1]:

$$G_{ab} + \frac{1}{4} \lambda R g_{ab} = \kappa [T_R]_{ab}, \quad (2)$$

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whence

$$(\lambda - 1) R = \kappa T_R. \quad (3)$$

So already at this stage it is clear that the case $\lambda = 1$ is special.

There are numerous and extensive claims in the literature that Rastall's approach amounts to introducing a deep non-minimal coupling between gravity and matter. Unfortunately, as we shall see below, in terms of the underlying physics, this approach proves simply to be a content-free rearrangement of the matter sector. As gravity, there is absolutely nothing new in this proposal.

Similar comments can be found in a little-known 1982 paper by Lindblom and Hiscock [24]. As per the discussion below, in this particular 35-year-old article the authors emphasize the construction of a conserved stress-energy tensor, algebraically built from the Rastall stress-energy [24].¹

2. Rastall gravity in vacuum

First, we observe that in vacuum Rastall's equation reduces to

$$G_{ab} + \frac{1}{4} \lambda R g_{ab} = 0; \quad (\lambda - 1) R = 0. \quad (4)$$

If $\lambda \neq 1$ this implies

¹ Where Lindblom and Hiscock differ from the current analysis is by introducing the explicit (and quite radical) assumption that laboratory equipment couples only to the non-conserved Rastall stress-energy, not to the conserved stress-energy tensor [24]. This allows them to place stringent phenomenological constraints on Rastall's λ parameter: $|\lambda| < 10^{-15}$. I will not be exploring this particular route in the current article.

1 $G_{ab} = 0;$ (5)

2 whereas if $\lambda = 1$ one obtains

3 $G_{ab} = \Lambda g_{ab}.$ (6)

4 This is either the standard vacuum Einstein EOM, or at worst the
5 Einstein EOM + (arbitrary cosmological constant). The vacuum
6 solution is simply an Einstein spacetime. (For $\lambda \neq 1$ this vacuum
7 degeneracy between the Rastall and Einstein theories was already
8 noted by Rastall some 45 years ago [1].)9 **3. Adding matter: generic case ($\lambda \neq 1$)**10 Since $R = \frac{\kappa T_R}{\lambda-1}$, we construct the geometrical Einstein tensor in
11 terms of Rastall's stress-energy as

12 $G_{ab} = \kappa \left([T_R]_{ab} + \frac{1}{4} \frac{\lambda}{1-\lambda} T_R g_{ab} \right).$ (7)

13 Therefore, if we choose to define

14 $T_{ab} = [T_R]_{ab} + \frac{1}{4} \frac{\lambda}{1-\lambda} T_R g_{ab},$ (8)

15 then this quantity is covariantly conserved. Thus it is this stress
16 energy that should be considered physical, and in terms of this
17 physical stress-energy tensor

18 $G_{ab} = \kappa T_{ab}$ (9)

19 is the usual Einstein equation.

20 We can of course invert this construction using

21 $T = T_R + \frac{\lambda}{1-\lambda} T_R = \frac{1}{1-\lambda} T_R;$ (10)

22 so that

23 $T_R = (1-\lambda)T.$ (11)

24 We see

25 $[T_R]_{ab} = T_{ab} - \frac{1}{4} \lambda T g_{ab}.$ (12)

26 That is, from the Rastall stress-energy $[T_R]_{ab}$, (and knowledge of
27 the Rastall coupling λ), one can always reconstruct the physical
28 stress-energy T_{ab} , and vice versa. So, (at least for $\lambda \neq 1$), all that
29 is going on is that Rastall has simply mis-identified the physical
30 stress-energy. In terms of the true physical conserved stress-
31 energy T_{ab} one just has standard Einstein gravity.² Indeed, one can
32 easily jump back and forth using equations (8) and (12). Sometimes this very simple observation is hidden very deeply in very
33 technical, very specific, and very turgid calculations.³34 **4. Adding matter: special case ($\lambda = 1$)**35 This is the only case that is even mildly interesting. Ironically, it
36 was already considered (and rejected) by Rastall 45 years ago [1].
37 For $\lambda = 1$ the Rastall EOM reduce to

38 $G_{ab} + \frac{1}{4} R g_{ab} = \kappa [T_R]_{ab}; \quad T_R = 0;$ (13)

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Note that the existence of an automatic implied consistency condition for Rastall
stress-energy: $\nabla_{[c} \nabla^{b]} [T_R]_{ab} = 0$. This might at first glance look “deep”; unfortunately it is not “deep”. Observe that one trivially has $\nabla_{[c} \nabla^{b]} [T_R]_{ab} = \frac{\lambda}{4} \nabla_{[c} \nabla_{a]} R = 0$.³ For traceless matter, such as electromagnetic stress-energy, the whole process trivializes, $[T_R]_{ab} \rightarrow T_{ab}$.

$$[T_R]^{ab} = (1 - \lambda)T^{ab} + \lambda[T_{tf}]^{ab}. \quad (21)$$

Non-conservation of Rastall's $[T_R]^{ab}$ is then seen to be an automatic consequence of non-conservation of $[T_{tf}]^{ab}$; this does not render the Rastall stress-energy any more physical, if anything it further emphasizes the purely formal and artificial book-keeping status of the Rastall stress-energy.

6. Relation of Rastall matter to perfect fluids

For perfect fluids the inter-relation between Rastall density and pressure, and conserved density and pressure, is simply:

$$\rho_R = \left(1 - \frac{\lambda}{4}\right)\rho + \frac{3}{4}\lambda p; \quad (22)$$

and

$$p_R = \left(1 - \frac{\lambda}{4}\right)p - \frac{1}{4}\lambda\rho. \quad (23)$$

Conversely

$$\rho = \left(\frac{1 - \frac{\lambda}{4}}{1 - \lambda}\right)\rho_R - \frac{3}{4(1 - \lambda)}\lambda p_R; \quad (24)$$

and

$$p = \left(\frac{1 - \frac{\lambda}{4}}{1 - \lambda}\right)p_R - \frac{1}{4(1 - \lambda)}\lambda\rho_R. \quad (25)$$

Again we see a simple rearrangement of the matter sector, and no change to the gravity sector.

7. Relation of Rastall matter to w -matter

Another quite popular matter model is w -matter, where one considers a perfect fluid and defines

$$w = \frac{p}{\rho}; \quad w_R = \frac{p_R}{\rho_R}. \quad (26)$$

However when doing so it is easy to see that

$$w_R = \frac{w + \frac{\lambda}{4}(1 - 3w)}{1 - \frac{\lambda}{4}(1 - 3w)}, \quad (27)$$

and conversely

$$w = \frac{w_R - \frac{\lambda}{4}(1 + w_R)}{1 - \frac{3\lambda}{4}(1 + w_R)}. \quad (28)$$

That is, moving back and forward from ordinary w -matter to the Rastall version of w -matter is simply equivalent to redefining the value of the w parameter and switching $w \leftrightarrow w_R$. The gravity sector is completely unaffected by this procedure.

Consider for example reference [26]. Those authors consider a Kerr back hole immersed in quintessence (w -matter). The only effect of the presence of the w -matter is that in the vacuum Kerr spacetime metric one should replace the mass parameter m by

$$m \rightarrow m(r) = m + kr^{-3w}. \quad (29)$$

(Here k is some constant characterizing the density of the infalling w -matter.) If one now considers Rastall-type w -matter, then the only *mathematical* change is that one should use equation (28) to simply replace $w \rightarrow w(w_R, \lambda)$. No actual physics is changing – the form of the spacetime metric is invariant. One is merely re-parameterizing the physics by using (w_R, λ) instead of w .

8. Lagrangian/action formulation: lack thereof

Several authors have noted the absence of any widely accepted and complete Lagrangian formulation (or action formulation) for Rastall gravity [2,12,13,16,20], making it at best a phenomenological model, (as we have seen, a trivial phenomenological model). Recently in reference [19] the authors attempted to develop a Lagrangian, but their approach does not fully reproduce Rastall's equations.

Now one can certainly write a variational principle for the physical conserved stress-energy. For the usual matter Lagrangian we have

$$S_m = \int \sqrt{-g} L_m d^4x; \quad T_{ab} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{ab}}. \quad (30)$$

But a similar construction for Rastall stress-energy is lacking. Can one find a "Rastall action" S_R such that

$$[T_R]_{ab} = \frac{-2}{\sqrt{-g}} \frac{\delta S_R}{\delta g^{ab}}? \quad (31)$$

The key difficulty is this: Since one is attempting to artificially split the total conserved stress-energy into two non-conserved pieces, the price of doing so will be to somehow artificially split the matter action into two pieces that individually lead to non-conserved stress-energies. This would require the introduction of some non-dynamical background field [33].

Even if this could be done, it would not be particularly useful. The Einstein-Hilbert action would be unaffected by any such construction – the gravity sector remains that of standard Einstein gravity. In contrast, the matter sector would be artificially subdivided into two contributions, neither of which is individually covariantly conserved, but whose sum is covariantly conserved. We again see that Rastall gravity is simply a repackaging of standard Einstein gravity. Seeking a Lagrangian/action formulation is not a useful exercise.

An alternative approach might be to relax what one means by a variational principle, and adopt the modified construction

$$[T_R]_{ab} = \frac{-2}{\sqrt{-g}} \left\{ \frac{\delta S_m}{\delta g^{ab}} - \frac{1}{4} \lambda g_{ab} g^{cd} \frac{\delta S_m}{\delta g^{cd}} \right\}. \quad (32)$$

This construction certainly has variational ingredients, but it does not amount to what most people would consider to be a variational principle.

For another example, consider reference [25]. In that article the authors first split the total action S into an Einstein-Hilbert term S_{EH} plus the rest – S_{not-EH} . But then those authors artificially split S_{not-EH} into pieces they choose to call S_{matter} and $S_{interaction}$. Now the standard definition of the stress-energy tensor is based on the metric variation of S_{not-EH} , and immediately leads to a conserved stress-energy tensor. If one artificially splits $S_{not-EH} = S_{matter} + S_{interaction}$ then neither T_{matter}^{ab} nor $T_{interaction}^{ab}$ need individually be covariantly conserved – only their sum need be conserved. But the split into S_{matter} and $S_{interaction}$ is only a semantic choice – no actual physics is involved. Worse, it is not at all clear how their suggested procedure would actually implement Rastall gravity, since they are *ab initio* assuming $T_R^{ab} = T_{matter}^{ab}$ then in view of equation (8) those authors would need to impose

$$T_{interaction}^{ab} = \frac{1}{4} \frac{\lambda}{1 - \lambda} T_{matter} g^{ab}. \quad (33)$$

This is a very strong constraint on their assumed split between the matter and interaction terms; one that they do not even attempt

1 to justify or discuss in the rest of their article. The rest of reference [25] does not actually address any of the points I have raised
 2 in this article; those authors appear to have significantly misinterpreted my actual comments.
 3

4 9. Discussion

5 In summary, we have seen that generically Rastall gravity is
 6 simply an essentially trivial re-arrangement of the matter sector in
 7 Einstein gravity; as gravity there is absolutely nothing new. Even in the non-generic case, one at best obtains “unimodular (fixed
 8 modulus) gravity”, ordinary Einstein gravity plus an arbitrary cosmological constant. It is perhaps sobering to realize that, just because an idea has been in circulation for 45 years, does not mean it
 9 has been fully debugged. In closing, Rastall gravity is not so much
 10 wrong, as it is of rather limited utility.

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 13 by the Royal Society of New Zealand.

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