

SOME CURRENT APPROACHES TO THE CONFINEMENT PROBLEM*

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I summarize a number of ideas about the confinement mechanism which are currently under active investigation. These include confinement via center vortices, monopoles, and calorons, Coulomb confinement and the Gribov horizon, Dyson–Schwinger equations, and vacuum wavefunctionals.

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1. Introduction

Lattice Monte Carlo methods, combined with chiral perturbation theory, have made great strides in recent years towards solving quantum chromodynamics, in the sense of calculating such things as the low-lying hadron spectrum and weak decay constants from first principles (see, *e.g.*, [1]). In view of this progress, it is natural to ask what else is really needed, apart from more computational efforts along the same lines. I would say that what is still missing, despite the advances made so far, is a real understanding of how QCD actually works, especially in regards to the quark confinement mechanism. In this area, however, progress has been slow, and in fact there is even some disagreement in the literature about what it is that we are trying to explain. So I will begin these lectures with a discussion of the possible meanings of the word “confinement”, and I will discuss in particular the question of whether confinement can be distinguished from non-confinement by the broken or unbroken realization of some symmetry, such as gauge symmetry, dual gauge symmetry, or center symmetry. I will then go on to discuss a number of promising ideas about the confinement mechanism, some quite old and some relatively new, that are currently under active investigation. These include the center vortex mechanism, magnetic monopoles and dual

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superconductivity, calorons, and Dyson–Schwinger equations. I also touch on progress regarding vacuum wavefunctionals in $2+1$ dimensions, the color Coulomb potential, and the gluon-chain model. The approach known as AdS/CFT, however, calls for its own lecture series, and will not be covered here.

2. What *is* confinement?

The first question to ask is this: what are people trying to prove, in order to “prove” confinement? And what do they *mean* by that word?

Most efforts to explain confinement have concentrated on trying to derive a static quark potential which rises linearly at large quark separations, going to infinity asymptotically. The majority of order parameters that have been proposed, which would distinguish the confined phase of a gauge theory from other phases, imply the existence of a static quark potential which rises indefinitely to infinity. On the other hand, the term “confinement” is also used to mean the absence of color-charged particle states in the hadron spectrum. But the existence of a linear potential, and the absence of color-charged particle states, are not quite the same thing, and this raises some tricky semantic issues. In the first place, if we identify confinement with an asymptotically linear static quark potential, then we are faced with the fact that real QCD, with light dynamical quarks, does not really have such a potential, due to a process known as “string-breaking”. This is the production of light quark–antiquark pairs which bind to the heavy color sources to form color singlets. As a result, the static quark potential goes flat asymptotically. So shall we say, against all historical usage, that QCD is non-confining? The alternative is to use the word “confinement” to mean the absence of color charged asymptotic particle states. But this alternative also has its problems, because asymptotic particle states are also colorless in a Higgs theory, where there is no linear potential whatever. Should we then describe these so-called “broken” gauge theories, of which the electroweak theory is one example, as confining?

This last point, concerning color confinement in Higgs theories, deserves some elaboration. The fact that color is confined in a non-Abelian lattice gauge-Higgs theory, with the Higgs particle in the fundamental representation of the gauge group, was first pointed out in 1975 by Fradkin and Shenker [2], who based their work on a theorem proven by Osterwalder and Seiler [3]. Consider, for example, an $SU(2)$ lattice gauge-Higgs theory with the lattice action

$$S = \beta \sum_{\text{plaq}} \frac{1}{2} \text{Tr} \left[U U U^\dagger U^\dagger \right] + \gamma \sum_{x, \mu} \frac{1}{2} \text{Tr} \left[\phi^\dagger(x) U_\mu(x) \phi(x + \hat{\mu}) \right] ,$$

where the scalar field ϕ is an $SU(2)$ matrix-valued scalar field¹. The phase diagram of this theory, in the $\beta - \gamma$ coupling plane, is sketched in Fig. 1. At small γ the theory is “confinement-like”, in the sense that a flux tube forms, and the static quark potential is linear up to some string-breaking scale, much as in real QCD. At large γ , the theory is in what is usually called the “Higgs” or “broken” phase. (What is broken? We will discuss that shortly.) The solid line was once thought to be a line of first-order transitions, and this may indeed be the case at large enough β , but extensive computer simulations in the region of $\beta \approx 2$ have shown that the line is actually a “crossover region”, where bulk quantities such as the action density vary rapidly, but without any apparent discontinuity [4]. The line terminates around $\beta = 2$.

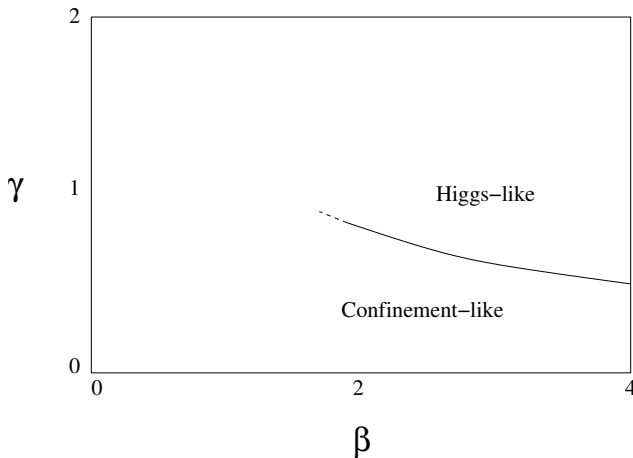


Fig. 1. Phase diagram of the gauge-Higgs model.

Osterwalder and Seiler proved rigorously that there exists a path in the coupling plane, between any point in the confinement-like region at $\beta, \gamma \ll 1$, and any point in the Higgs region at $\beta, \gamma \gg 1$, such that all Green’s functions of all local, gauge invariant observables

$$\langle A(x_1)B(x_2)C(x_3)\dots \rangle \quad (2.1)$$

vary analytically along the path. As emphasized by Fradkin and Shenker, this fact rules out an abrupt transition from a colorless to a color-charged spectrum, and implies that the hadron spectrum in the Higgs-like region at $\beta, \gamma \gg 1$, like the spectrum in the confinement-like region, is colorless. Although the Osterwalder–Seiler theorem does not rule out the existence of

¹ The action can be re-expressed as the theory of a gauge field coupled to a scalar field with a fixed modulus.

a transition to a Coulomb phase somewhere in the phase diagram, computer simulations of the theory have not found any such transitions, and it is now believed that all points in the phase diagram are in the same phase.

This conclusion is at odds with the treatment of the Higgs mechanism as found in many textbooks. There it is usually asserted that the Higgs phase is a phase of spontaneously broken gauge symmetry, which contrasts with, *e.g.*, QCD, where the gauge symmetry is unbroken. In fact, such assertions about spontaneously broken *versus* unbroken gauge symmetry can be a little misleading, in view of a theorem by Elitzur [5]:

Elitzur's Theorem

Local gauge symmetries do not break spontaneously. In the absence of gauge fixing, the vacuum expectation value (VEV) of a Higgs field is zero, regardless of the shape of the Higgs potential.

This theorem seems to say that spontaneously broken gauge symmetry is a myth. That is not quite true, however, because one can always fix to some gauge, *e.g.* Landau or Coulomb gauge, having some residual gauge symmetry which is a global subgroup of the local gauge group. Such global symmetries *can* break spontaneously, and they raise the possibility that the confinement phase can be characterized as the unbroken phase of some global gauge symmetry.

Two concrete proposals along these lines are the Kugo–Ojima criterion, and the Coulomb confinement criterion, formulated in covariant and Coulomb gauges, respectively.

2.1. The Kugo–Ojima criterion

Landau gauge does not fix the gauge completely. There is still a remaining global symmetry: if $A_\mu(x)$ satisfies the gauge condition $\partial_\mu A_\mu = 0$, then so does the gauge-transformed configuration $A' = g \circ A$, where $g(x, t) = g$ is spacetime-independent. The set of all such transformations is a global SU(2) subgroup, called a residual symmetry, of the local SU(2) invariance. There is, in addition, a residual invariance under spatially inhomogeneous transformations, which can be expressed as a power series in the coupling g

$$\begin{aligned} g(x) &= \exp \left[i A^a(\epsilon; x) \frac{1}{2} \sigma_a \right], \\ A^a(\epsilon; x) &= \epsilon_\mu^a x^\mu - g \frac{1}{\partial^2} (A_\mu \times \epsilon_\mu)^a + O(g^2). \end{aligned} \quad (2.2)$$

This is still a global symmetry, in the sense that these inhomogeneous transformations depend on only a finite number of parameters.

These residual symmetries are an important component of a confinement criterion put forward by Kugo and Ojima [6]. Consider a non-Abelian gauge theory quantized in some covariant (*e.g.* Landau) gauge, with $c^a(x)$, $\bar{c}^a(x)$ the ghost and anti-ghost fields, respectively. Kugo and Ojima introduce a function $u^{ab}(p^2)$ defined by

$$u^{ab}(p^2) \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) = \int d^4x e^{ip(x-y)} \langle 0 | T \left[D_\mu c^a(x) g(A_\nu \times \bar{c})^b(y) \right] | 0 \rangle. \quad (2.3)$$

They then show that the expectation value of a (suitably defined) color charge operator vanishes in any physical state, *i.e.*

$$\langle \text{phys} | Q^a | \text{phys} \rangle = 0 \quad (2.4)$$

providing the following conditions are satisfied:

- Remnant symmetry under $g(x) = g$ is unbroken;
- The condition $u^{ab}(0) = -\delta^{ab}$ is satisfied.

It turns out that the second condition implies that the spatially inhomogeneous global gauge symmetry defined by (2.3) is also unbroken, *cf.* Hata [7] and Kugo [8]. This means that the Kugo–Ojima criterion depends on the unbroken realization of all remnant gauge symmetries in Landau gauge, which would also imply that the expectation value of the Higgs field vanishes.

2.2. The Coulomb criterion

Here the idea is to show that the Coulomb self-energy of a static color charge is infinite, even when the ultraviolet contribution is cut off with, *e.g.*, a lattice regulator, and to further show that the Coulomb potential between static quarks is confining, *i.e.* rises indefinitely with quark separation. Both of these requirements are satisfied if the remnant gauge symmetry that exists in Coulomb gauge is not spontaneously broken, as first pointed out by Marinari *et al.* [9] (see also Ref. [10]).

If $A_\mu(x, t)$ satisfies the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$, then so does the gauge-transformed configuration $g \circ A$ where the gauge transformation $g(\mathbf{x}, t) = g(t)$ depends only on time, but not the space coordinates. Now let us consider the time-like Wilson line²

$$L(\mathbf{x}, T) = P \exp \left[i \int_0^T dt A_0(\mathbf{x}, t) \right]. \quad (2.5)$$

² This is an open line, *not* a Polyakov line closed by periodicity. The time T is arbitrary, and is not the time extension of a periodic volume.

This quantity transforms under the residual gauge symmetry, and if the residual invariance with respect to spatially global ($\mathbf{x}$ -independent) transformations at times $t = 0$ and $t = T$ is unbroken, then it follows that $\langle \text{Tr } L \rangle = 0$. This immediately implies that the Coulomb self-energy of an isolated color charge in an infinite volume is infinite, for the following reason: Let Ψ_0 denote the vacuum state in Coulomb gauge. Then a physical state containing a single static quark at point x is

$$\Psi_q^a = q^a(x) \Psi_0, \quad (2.6)$$

where q is a heavy quark operator. When the quark is so massive that fluctuations in spatial position can be neglected, the Euclidean amplitude for time propagation is

$$G(T) = \langle \Psi_q^a | e^{-(H-E_0)T} | \Psi_q^a \rangle \\ \propto \langle \text{Tr} [L(\mathbf{x}, T)] \rangle. \quad (2.7)$$

Now if the Coulomb energy of the isolated quark is infinite, then $G(T) = 0$ for arbitrarily small $T \neq 0$, which is clearly equivalent to $\langle \text{Tr } L \rangle = 0$. In this way, the unbroken residual symmetry in Coulomb gauge implies an infinite Coulomb self-energy for static charges.

In a very similar way, one can show (by considering states $q^a(\mathbf{x})q^{\dagger a}(\mathbf{y})\Psi_0$) that the instantaneous Coulomb potential between static sources, separated by a distance $R = |\mathbf{x} - \mathbf{y}|$ is

$$V_{\text{coul}}(R) = - \lim_{T \rightarrow 0} \frac{d}{dT} \log \left[\text{Tr} \left[L(\mathbf{x}, T) L^\dagger(\mathbf{y}, T) \right] \right] + \text{constant}. \quad (2.8)$$

It is clear that if $\langle \text{Tr } L \rangle \neq 0$, then $V_{\text{coul}}(R)$ flattens out as $R \rightarrow \infty$. The conclusion is that the infinite Coulomb self-energy of a confining charge, and the infinitely rising Coulomb potential of static charges, both require that the remnant gauge symmetry in Coulomb gauge is unbroken.

2.3. The ambiguity of spontaneously broken gauge symmetry

The Kugo–Ojima criterion and the Coulomb criterion both make perfect sense in real QCD, with light quark fields, where the static quark potential goes flat asymptotically, and both depend on the unbroken realization of some remnant gauge symmetry. So is that what we mean by confinement? I think there are at least two problems with this idea. First, the residual symmetries associated with Coulomb and Landau gauges break in different places (so which is the relevant breaking?), and secondly, these symmetries break in the absence of any other abrupt change in the physical state.

Fig. 2 shows the situation for SU(2) gauge-Higgs model of Eq. (2.1); the data points indicate the locations of spontaneous breaking of the Landau gauge remnant symmetry, and the Coulomb gauge remnant symmetry. Along the crossover line at $\beta > 2$ these transitions seem to coincide, but at $\beta < 2$ the symmetry-breaking transitions occur at different values of γ . Moreover, these transitions occur in regions where the Osterwalder–Seiler theorem assures us that there is no physical transition whatever in the gauge-Higgs system. We must, therefore, conclude that both the Kugo–Ojima and Coulomb confinement conditions are misleading, because each predicts the occurrence of a non-existent phase transition, and the two predictions are not even in agreement with each other. In fact, what the figure indicates is that the whole notion of “spontaneously broken” gauge symmetry is ambiguous, because (in view of the Elitzur theorem) one really has to specify which global subgroup of the gauge group is under consideration, and we see that the breakings of such subgroups are not universal, but can occur at different places in the phase diagram.

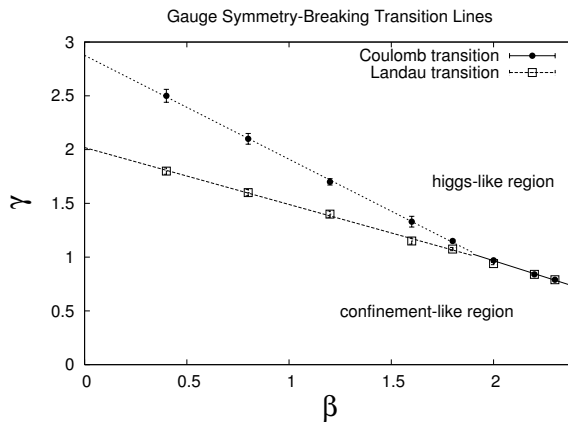


Fig. 2. The location of remnant global gauge symmetry breaking in Landau and Coulomb gauges, in the $\beta - \gamma$ coupling plane. From Caudy and Greensite [11].

2.4. Dual superconductivity

The idea that the Yang–Mills vacuum could be thought of as a dual superconductor, in which the roles of the electric and magnetic fields are interchanged, was put forward independently by 't Hooft and Mandelstam in the mid-1970s. In an ordinary Type II superconductor, magnetic lines of force are squeezed into flux tubes by the Meissner effect, and if there were magnetic monopoles and antimonopoles present in the material, then a magnetic flux tube would form between a monopole–antimonopole pair,

and the corresponding potential would grow linearly with monopole separation. A relativistic system of this kind is described by an Abelian gauge-Higgs model, and what is usually called the Higgs phase corresponds to the superconducting (or massive) phase, which is separated from the massless Coulomb phase by a transition³. In the “dual superconductor” proposal, it is electric, rather than magnetic field lines which are compressed into flux tubes, and electric, rather than magnetic, charges which are confined by the dual Meissner effect.

This proposal is essentially Abelian in character. In a compact U(1) gauge theory there is a magnetic current

$$j_\mu^M = \partial^\nu \tilde{F}_{\mu\nu} = \frac{1}{2} \partial^\nu \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \quad (2.9)$$

whose conservation is a consequence of the dual U(1) symmetry, just as conservation of the electric current is a consequence, via Noether’s theorem, of the usual U(1) gauge symmetry. The idea is that spontaneous breaking of (a global subgroup of) this dual gauge symmetry leads to confinement via the dual Meissner effect. The question is how to detect the spontaneous breaking of a dual global gauge symmetry, *i.e.* what is the appropriate order parameter?

An order parameter for the dual symmetry breaking has been put forward by di Giacomo *et al.* [12]. The order parameter is the expectation value of a monopole creation operator, denoted μ , which does not commute with the magnetic charge (spatial integral of j_0^M), and therefore $\langle \mu \rangle = 0$ in the unbroken, non-confining phase, and $\langle \mu \rangle \neq 0$ in the broken, confined phase.

The monopole operator inserts a monopole field A^M centered at a point $\mathbf{x}$

$$\mu(\mathbf{x})|A_i\rangle = |A_i + A_i^M\rangle \quad (2.10)$$

and this is accomplished by

$$\mu(\mathbf{x}) = \exp\left[i \int d^3y A_i^M(y) E_i(y)\right] \quad (2.11)$$

assuming a U(1) gauge group. In a non-Abelian theory, it is necessary to choose an Abelian subgroup of the gauge group, in order to define the μ operator. There are, of course, an infinite number of inequivalent ways to do that in an SU(N) gauge theory, all of which correspond to picking a so-called Abelian-projection gauge which leaves a residual U(1) ^{$N-1$} gauge invariance. The μ operator for an Abelian theory is defined with respect to this Abelian subgroup, and it has been argued [14] that the choice of Abelian projection gauge makes no difference.

³ Note that in the non-Abelian gauge-Higgs theory, both the Higgs-like region and the confinement-like region are massive.

In practice, one computes numerically the logarithmic derivative

$$\rho = \frac{\partial}{\partial \beta} \log \langle \mu \rangle = -\beta^{-1} [\langle S \rangle_S - \langle S \rangle_M] \quad (2.12)$$

since this can be computed in terms of the expectation value of the action S with $(\langle S \rangle_M)$ and without $(\langle S \rangle_S)$ a monopole insertion. It can be shown analytically that $\langle \mu \rangle = 1$ at $\beta = 0$, which supplies the necessary boundary condition. Then a large negative peak in ρ at some critical value $\beta = \beta_c$ of the lattice coupling, whose depth increases with lattice volume, is the signal that $\langle \mu \rangle = 0$, and dual superconductivity disappears, for $\beta > \beta_c$. In fact, it has been shown in case after case that $\rho \rightarrow -\infty$, and therefore $\langle \mu \rangle \rightarrow 0$, precisely at the high-temperature deconfinement transition. But although the ability of this parameter to pick out the deconfinement transition can be counted as a success, one can still ask about the behavior of ρ near other types of transitions that occur at zero temperature, or in theories (such as pure SU(2) gauge theory, and SU(2) gauge-Higgs theory) in which there is no transition at all.

Unfortunately, numerical simulations have found [15] that the $\mu \rightarrow 0$ transition is found at zero temperature in a number of gauge theories, such as

1. SU(5) pure gauge theory, which has a 1st-order transition;
2. SU(2) pure gauge theory, which has no transition;
3. mixed fundamental-adjoint SU(2) gauge theory, with a line of first-order transitions;
4. SU(2) gauge-Higgs theory defined by (2.1);
5. G(2) gauge theory, which has a 1st-order transition (*cf.* [16]).

As an example, two $\langle \mu \rangle \rightarrow 0$ transitions in gauge-Higgs theory, one at a point on the crossover line, and one at point where there is no thermodynamic activity at all, are displayed in Fig. 3. In these, and in all of the cases listed, the transition to the unbroken phase of dual gauge symmetry may or may not be accompanied by a thermodynamic transition of some sort, but they are not associated with a transition to a deconfined phase. This means that, as with the Kugo–Ojima and Coulomb confinement criteria, the broken or unbroken realization of a gauge symmetry (in this case a dual gauge symmetry) is not a reliable indicator of the presence or absence of the confinement phase.

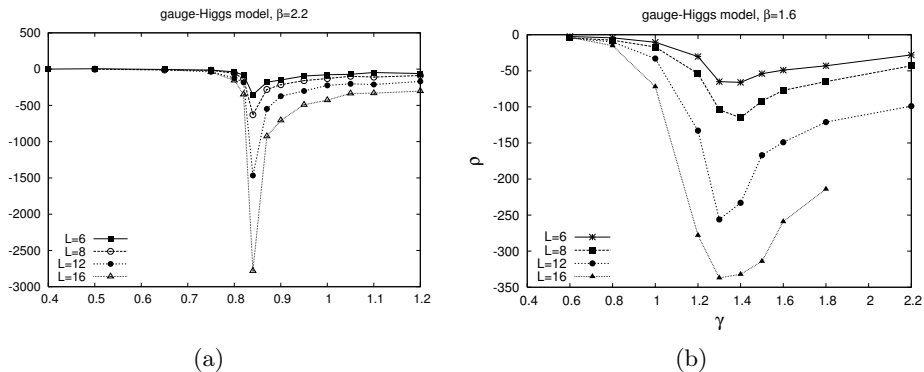


Fig. 3. ρ versus β in the SU(2) gauge-Higgs (Fradkin–Shenker) model of Eq. (2.1). (a) At $\beta = 2.2$ a very sharp negative peak in ρ is found at the thermodynamic crossover point at $\gamma = 0.84$. (b) At $\beta = 1.6$, there is not even a crossover, but nevertheless there is a broad negative peak in ρ centered at $\gamma \approx 1.3$, and growing deeper with volume. From Ref. [15].

2.5. Magnetic disorder and center symmetry

The evidence suggests that broken *versus* unbroken gauge symmetry, dual or otherwise, is not a very meaningful distinction so far as confinement is concerned, and if by the term “confinement” one means the existence of a color-neutral spectrum, then this term must also be applied to the non-Abelian Higgs phase.

On the other hand, when comparing QCD with a Higgs theory in the “Higgs-like” region of couplings, there is clearly a qualitative difference in the dynamics of the two theories. In QCD there is flux tube formation and a linear potential, at least in some range of distances, and a hadron spectrum which lies on linear Regge trajectories. The gauge-Higgs theory in the Higgs-like region exhibits none of these phenomena, and there is only a Yukawa potential between static charges. If we focus on such things as flux tube formation and the linear potential, then I think it better to describe confinement as a phase of *magnetic disorder*; *i.e.* a phase characterized by the existence of vacuum fluctuations which are strong enough to induce an area law falloff in Wilson loops at arbitrarily large scales. The vacuum state of a gauge-Higgs theory has this property in the $\gamma \rightarrow 0$ limit, while QCD would have this property in the $m_q \rightarrow \infty$ infinite quark mass limit. What is interesting is that in these two limits, where the static quark potential rises linearly out to infinity as $R \rightarrow \infty$, the Lagrangian acquires an unbroken global symmetry which is known as *center symmetry*. When this symmetry is broken, either

1. spontaneously; which can happen at high temperatures (the deconfinement transition), or through the addition of matter fields in the adjoint representation of the gauge group,
2. explicitly, through the introduction (as in QCD) of matter fields in the fundamental representation of the gauge group,
3. or the (non-trivial) symmetry does not exist in the first place, as in the case of the $G(2)$ gauge group,

then magnetic disorder is lost. In fact, all of the “traditional” conditions for confinement, namely

A. area law for large Wilson loops,

$$W(C) = \left\langle P \exp \left[i \oint_C dx^\mu A_\mu \right] \right\rangle \sim \exp[-\sigma \text{Area}(C)], \quad (2.13)$$

B. vanishing Polyakov lines

$$P(\vec{x}) = \left\langle P \exp \left[i \int_0^{L_t} dt A_0(\vec{x}, t) \right] \right\rangle = 0, \quad (2.14)$$

C. perimeter law falloff for large 't Hooft loops

$$B(C) \sim \exp[-\mu \text{Perimeter}(C)], \quad (2.15)$$

D. vanishing of the center vortex free energy, in the large volume limit, according to

$$F_v = c L_z L_t \exp[-\sigma' L_x L_y], \quad (2.16)$$

where the vortex is a sheet of quantized magnetic flux in the z - t plane, in a finite $L_x \times L_y \times L_z \times L_t$ volume

require unbroken center symmetry. None of these conditions are satisfied if global center symmetry is broken spontaneously (deconfinement) or explicitly (quarks) or is trivial ($G(2)$ gauge group).

At this point, we need to recall a little group theory. The center subgroup of any group is the set of all group elements that commute with the full group. For an $SU(N)$ group, the center subgroup consists of the elements proportional to the identity matrix

$$Z_N = \left\{ z_n = e^{2\pi i n/N} I_N, n = 0, 1, \dots, N-1 \right\}. \quad (2.17)$$

Suppose $M[g]$ is an irreducible representation of the group element g . Then there is a fixed power k , known as the “ N -ality” of the representation, such that

$$M[z_n g] = (z_n)^k M[g]. \quad (2.18)$$

The charge of a gluon lies in the adjoint color representation, which has zero N -ality. Gluons binding to another particle can reduce the dimensionality of the color representation of the bound state, but not the N -ality of the particle’s original color representation. What this implies is that if we have a heavy quark–antiquark pair, then as the quarks move apart it may become energetically favorable to pair-produce gluons which bind with each of the quarks, reducing the dimensionality of the effective quark color charge representation, but not its N -ality. Ultimately the gluons screen the quark charges down to the lowest dimensional representation of the original N -ality. This fact has a profound conclusion, which bears directly on the type of vacuum fluctuations which must be dominant at large scales: Asymptotically, the string tension of a quark–antiquark pair can only depend on the N -ality of the quark color charge representation.

Center symmetry on the lattice is invariance of the action with respect to transformations that are gauge-equivalent to the global transformations

$$U_0(\mathbf{x}, t_0) \rightarrow z U_0(\mathbf{x}, t_0). \quad (2.19)$$

These transformations are carried out at one timeslice $t = t_0$ but all $\mathbf{x}$, and z is any element of the center of the gauge group. It is easy to see that this transformation does not change the value of plaquettes or Wilson loops (where every z is paired with a z^{-1}), but such a transformation does transform any Polyakov line $P \rightarrow zP$. This means that the VEV of the Polyakov line vanishes if and only if center symmetry is unbroken, and the self-energy of any isolated color charged particle, in a representation of non-zero N -ality, is infinite. Likewise, if center symmetry is completely broken, then any charge can be screened to zero N -ality, and large Wilson loops can only fall off with a perimeter law. Thus, if we take the word “confinement” to mean magnetic disorder, then we may say that confinement is the phase of unbroken center symmetry⁴.

It is natural to ask, if the center is so important, then why we do not see gluons in the asymptotic spectrum, given that the N -ality of gluons vanishes, and they are therefore insensitive to center transformations. The answer is that gluons disappear from the spectrum by the same process that forbids an object the size of, *e.g.*, an atomic nucleus, to have an electric charge greater

⁴ This terminology comes with a price: we must then describe real QCD as a “confinement-like” theory, with magnetic disorder only up to some color-screening length scale.

than some critical value Q_C . Suppose we create, by some collision process, two such objects with charges $Q_C + 1$ and $-Q_C - 1$, respectively. Then the electric field energy surrounding these objects is such that it becomes energetically favorable for the vacuum to pair-produce an electron positron pair, with the electron binding to the positively charged object, and the positron binding to the negatively charged object. The end result is that the effective charge of the two objects is reduced to the critical value. The process is similar for any two objects having adjoint color charge in a non-Abelian gauge theory. As the objects separate, the energy due to the color electric field increases, until at some point it becomes energetically favorable for the vacuum to pair-produce gluons, which bind to each of the adjoint sources and completely neutralize their color charge. If the adjoint sources are themselves gluons, then as they fly apart from one another we end up with two glueballs. Since the process involves screening the adjoint charge with the charge of pair-produced gluons, I prefer to refer to this process as “color screening” rather than confinement.

3. Some features of the confining force

In theories with unbroken, non-trivial center symmetry, charges with non-zero N -ality cannot be completely screened by gluons, as already mentioned. These theories — mainly pure SU(2) and SU(3) but also higher N to some extent — have been studied extensively via lattice Monte Carlo simulations. The first striking feature is the linearity of the quark–antiquark potential, for quarks in the fundamental representation of the gauge group, at large separations. A sample calculation in SU(3) pure gauge theory is displayed in Fig. 4. In this figure, and in Fig. 5, $r_0 \approx 0.5$ fm is the Sommer scale. If we inquire how things change when the quark and the antiquark are in color representations beyond the fundamental, then the answer is that there are (at least) three distance regimes. At small quark–antiquark separations, where weak-coupling perturbation theory makes sense, the static potential goes as $-1/R$ times the running coupling. This is followed by an intermediate distance regime, where the quark–antiquark potential rises linearly, up to the onset of color-screening. Beyond this is an asymptotic regime, where the string tension depends only on the N -ality of the quark color representations.

In the intermediate distance regime (as in the perturbative regime) it is found that the static quark potential is proportional to the quadratic Casimir C_r of the quark color charge representation r , *i.e.*

$$V_r(R) = \frac{C_r}{C_F} V_F(R), \quad (3.1)$$

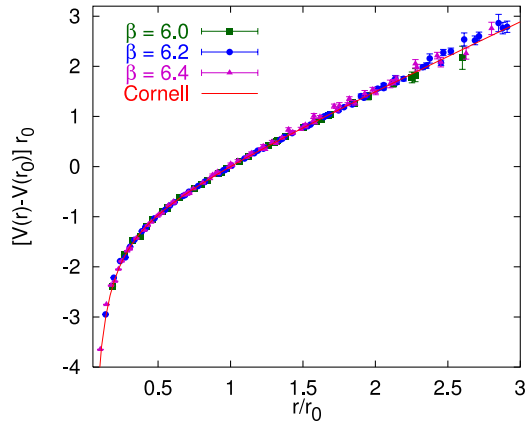


Fig. 4. The static quark potential in SU(3) lattice gauge theory, normalized to $V(r_0) = 0$. From Bali [17].

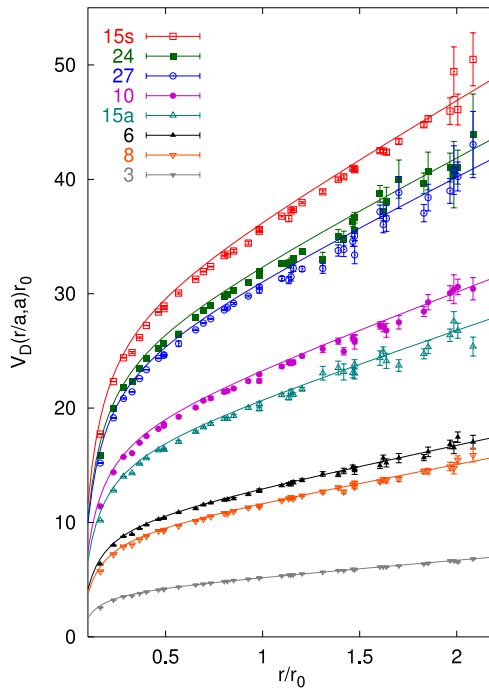


Fig. 5. Numerical evidence for Casimir scaling in SU(3) lattice gauge theory. The solid lines are obtained from a fit of the potential in fundamental representation, multiplied by a ratio of quadratic Casimirs C_r/C_F . From Bali [18].

where $V_F(R)$ is the static potential of quarks in the fundamental representation. This behavior is known as “Casimir scaling”, and numerical evidence of the effect is shown in Fig. 5. It should be noted that the calculation on which this figure is based uses a method which creates metastable flux tubes, which are then allowed to propagate for a relatively short Euclidean time interval. This procedure is insensitive to the string-breaking process, and hence one can only calculate the string tension of the metastable states. To observe string-breaking (*i.e.* color screening by gluons) for adjoint representation heavy quark states, using only rectangular Wilson $R \times T$ Wilson loops is numerically difficult, but was achieved by de Forcrand and Kratochvila [19] using a noise reduction technique due to Lüscher and Weisz [21]. The result of their calculation in SU(2) lattice gauge theory is shown in Fig. 6; the adjoint string breaks at about ten lattice spacing, corresponding (at the given value of the lattice coupling) to 1.25 fm in physical units.

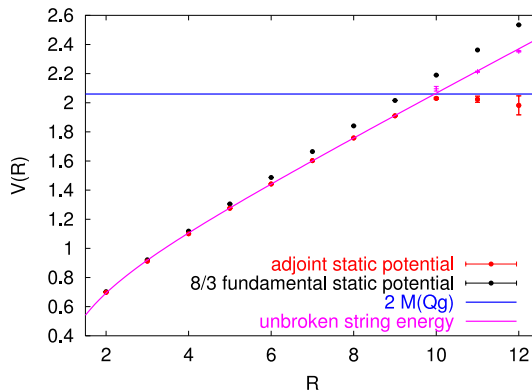


Fig. 6. The adjoint and $\frac{8}{3}$ fundamental static potentials $V(R)$ versus R , in $D = 3$ dimensional SU(2) lattice gauge at $\beta = 6.0$. The horizontal line at 2.06(1) represents twice the energy of a gluelump. From de Forcrand and Kratochvila [19].

A final striking feature is that the QCD flux tube appears to share certain properties of the Nambu string. In string theory, the energy of the string with fixed ends a distance R apart includes a linear piece, σR , where σ depends on the string tension, but also, in d spacetime dimensions, includes a universal contribution due to the fluctuations of the string

$$V_L(R) = \frac{c(R)}{R}, \quad \text{where} \quad c(R) = -\frac{(d-2)}{24}. \quad (3.2)$$

Note that this term is independent of the string tension. The subscript L refers to Lüscher, who suggested that such a term would also appear in the static quark potential. Numerical simulations performed by Lüscher and Weisz [21] and others indicate that such a universal $-1/R$ contribution,

with the coefficient expected from bosonic string theory, is indeed present (*cf.* Fig. 7, which displays values of $c(R)$ obtained via numerical simulation).

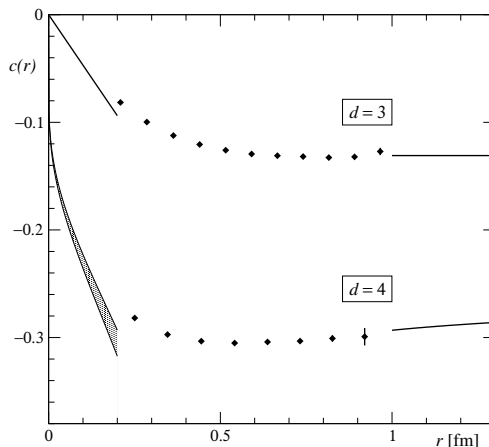


Fig. 7. Coefficient of the Lüscher term $c(r)$ in $D = 3$ and $D = 4$ dimensions. The Nambu–Goto string action predicts $c = -0.262$ for $D = 4$ and $c = -0.131$ for $D = 3$. From Lüscher and Weisz, Ref. [21].

Judging from Figs. 4–6, it appears that if we set the scale from an asymptotic string tension (extracted from the hadronic Regge slope) of $(440 \text{ MeV})^2$, then the perturbative region persists up to roughly 0.25 fm, followed by the Casimir scaling region up to 1.25 fm, and all distances beyond that are in the N -ality regime, where the string tension depends, not on the quadratic Casimir, but only on the N -ality. Because color screening is a non-planar, $1/N^2$ -suppressed process, we expect that the transition between Casimir scaling and the N -ality regime will run off to infinity for $SU(N)$ gauge theories, as $N \rightarrow \infty$.

There is still the question, in the N -ality regime, of precisely how the string tension depends on the N -ality k of the quark color charges. This dependence is known as “ k -string scaling”, and at present there are two proposals on the table:

$$\frac{\sigma(k)}{\sigma_F} = \begin{cases} \frac{\sin(\pi k/N)}{\sin(\pi/N)} & \text{sine law scaling,} \\ \frac{k(N-k)}{N-1} & \text{Casimir } k\text{-string scaling.} \end{cases} \quad (3.3)$$

At present the data seems to lie between these two proposals, slightly above the Casimir prediction [20].

So the bar is set rather high: an explanation of confinement will have to account for asymptotic linearity of the potential, Casimir scaling in the intermediate region, N -ality dependence asymptotically, and the presence of the Lüscher term in the potential. At present, theories of how confinement comes about fall into a few broad categories:

- I. Dominance of the functional integral by a special class of confining field configurations. There is a number of competing (or possibly complementary) proposals for what this special class might consist of; suggestions include center vortices, monopoles, and calorons.
- II. Ladder exchange. It has been argued that the confining force is generated by ladder exchange processes involving the dressed gluon propagator and dressed vertices. There are variations of this idea in Coulomb gauge (the Gribov–Zwanziger scenario), and in covariant gauges, treated via the Dyson–Schwinger equations.
- III. Vacuum wavefunctionals. It is an old idea, which is still under active investigation, that confinement might be manifest from the form of the wavefunctional of the ground state, and the problem is therefore to determine that wavefunctional.
- IV. AdS/CFT. In string theory there is a kind of equivalence (a “duality”) between $N = 4$ super Yang–Mills theory (a non-confining theory) and string theory in the product space of five-dimensional anti-deSitter space with a five-dimensional hypersphere. The theories are equivalent in the sense that there is a correspondence such that correlators in the field theory can be calculated from observables in the string theory. Calculations have mainly been carried out in the limit of strong coupling, and an infinite number of colors. There have been efforts to extend this approach to non-conformal theories, which are more like QCD.

In the next sections I will briefly describe some of these promising ideas. AdS/CFT, however, demands some background in string theory, and is beyond the scope of these lectures.

4. Center vortices

The center vortex theory of confinement, introduced by 't Hooft [22] in 1978, has two motivations. First, as already emphasized, the asymptotic string tension depends only the N -ality of static charges, and if we are describing confinement in terms of some special class of field configurations, then this property must somehow be manifest in that special class. Secondly,

all of the unambiguous order parameters for confinement indicate that confinement is a phase of unbroken center symmetry. The only scenario I am aware of, which explains N -ality dependence solely in terms of vacuum field configurations, is the center vortex mechanism.

A center vortex is a loop of quantized magnetic flux which sweeps out a sheet (of finite thickness), as it propagates in time. If we imagine a Wilson loop holonomy

$$U(C) = P \exp \left[i \oint_C dx^\mu A_\mu \right] \quad (4.1)$$

running around a loop C , then the creation of a center vortex topologically linked to the loop C , with linking number 1, multiplies the loop holonomy by a center element, *i.e.*

$$U(C) \rightarrow zU(C), \quad \text{where } z \in Z_N. \quad (4.2)$$

The exterior field of a center vortex is a close analogy to the exterior of a solenoid in ordinary electrodynamics, where there is a non-vanishing vector potential which cannot be completely removed by a gauge transformation, but this potential does not give rise to any finite field strength. In the vortex theory, the area-law falloff of a large Wilson loop $W(C) = \langle \text{Tr}[U(C)] \rangle$ is due to random fluctuations in the number of vortices linked to the loop. The fact that the asymptotic string tension then depends only on N -ality follows automatically from the fact that vortices affect large loops only by multiplication by a center element.

There is a *lot* of numerical evidence in favor of this picture, based on methods, developed in 1997–98, for locating vortices in lattice configurations. This evidence was reviewed in quite some detail in my review article [23] on this subject, but briefly the successes are as follows:

1. Vortex linking number is correlated with the phase of the Wilson loop;
2. Vortices by themselves give about the right string tension;
3. Plaquette action is high on vortex surfaces;
4. The vortex density scales according to asymptotic freedom;
5. When vortices are removed from Monte Carlo-generated lattice configurations, both the string tension and chiral symmetry breaking disappears;
6. Vortex thickness agrees with independent estimates based on adjoint string breaking, and measurements of the vortex free energy;
7. The string tension of space-like Wilson loops at high temperatures comes from vortices which are closed in the periodic time direction.

It may be useful to illustrate this last point in 2+1 dimensions, where vortices are line-like, rather than surface-like objects. Fig. 8 shows a pair of Polyakov lines with a vortex between them; the vortex is running in a spacelike direction. High temperature corresponds to a small extension in the periodic time-like direction, and when this extension becomes smaller than the thickness of the vortex, the vortex is “squeezed”. This means that its free energy increases (*cf.* [25]), and vortices stop percolating in the space-like directions. As a result, vortices cease to disorder Polyakov lines. On the other hand, thick vortices can still propagate perfectly well in the periodic time direction. Such vortices are not squeezed at all (Fig. 9) by a small time extension, their free energies remain negligible at high temperature, and they continue to furnish an area law to space-like Wilson loops. For a more quantitative discussion, *cf.* [26].

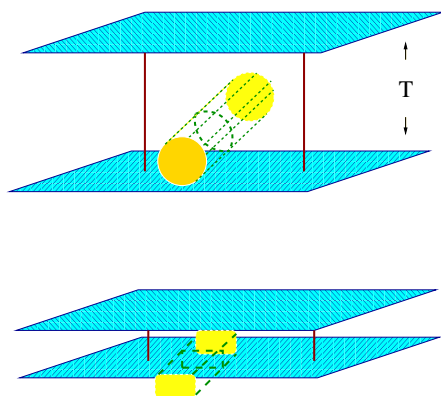


Fig. 8. Vortices running in the space-like directions disorder Polyakov lines. When the time extension T is smaller than the diameter of the vortex (high temperature case), then space-like vortices are “squeezed” and cease to percolate.

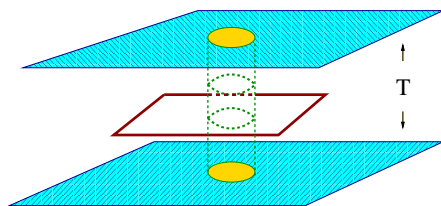


Fig. 9. Vortices running in the time-like direction disorder space-like Wilson loops. The vortex cross-section is not constrained by a small extension in the time direction.

Although the vortex picture accounts very well for fact that asymptotic string tensions can depend only on N -ality, there is still the question of how to reconcile the existence of Casimir scaling at intermediate distances with the vortex scenario. Here the finite thickness of vortices is crucial, because vortices only multiply a Wilson loop by a center element if vortex does not overlap the loop itself; *i.e.* if all of the vortex color magnetic flux passes through the minimal area of the loop, and none is outside the loop.

A simple model (*cf.* [24]) may illustrate the effect of a finite thickness. Consider the projection of a vortex onto the plane of a planar Wilson loop; we will refer to this projection as a vortex “domain” in the plane. In this model, it is assumed that the effect of a domain (2D cross-section of a vortex) on a planar Wilson loop holonomy is to multiply the holonomy by a group element

$$G(\alpha^n, S) = S \exp \left[i \vec{\alpha}^n \cdot \vec{H} \right] S^\dagger, \quad (4.3)$$

where the $\{H_i\}$ are generators of the Cartan subalgebra, S is a random group element, $\vec{\alpha}^n$ depends on the location of the domain relative to the loop, and n indicates the domain type. If the domain lies entirely within the planar area enclosed by the loop, then

$$\exp \left[i \vec{\alpha}^n \cdot \vec{H} \right] = z_n I, \quad (4.4)$$

where

$$z_n = e^{2\pi i n/N} \in Z_N \quad (4.5)$$

and I is the unit matrix. At the other extreme, if the domain is entirely outside the planar area enclosed by the loop, then

$$\exp \left[i \vec{\alpha}^n \cdot \vec{H} \right] = I. \quad (4.6)$$

For a Wilson loop in representation r , the average contribution from a domain will be

$$\begin{aligned} \bar{G}_r[\alpha^n] I_{d_r} &= \int dS \, S \exp \left[i \vec{\alpha}^n \cdot \vec{H} \right] S^\dagger \\ &= \frac{1}{d_r} \chi_r \left[\exp \left[i \vec{\alpha}^n \cdot \vec{H} \right] \right] I_{d_r}, \end{aligned} \quad (4.7)$$

where d_r is the dimension of representation r , I_{d_r} is the unit matrix, and χ_r is the group character in representation r .

Consider, *e.g.*, SU(2) lattice gauge theory, choosing $H = L_3$. The center subgroup is Z_2 , and there are two types of domains, corresponding to $z_0 = 1$ and $z_1 = -1$. Let f_1 represent the probability that the midpoint of a z_1

domain is located at any given plaquette in the plane of the loop, with f_0 the corresponding probability for a $z_0 = 1$ domain. Let us also assume that the probabilities of finding domains of either type centered at any two plaquettes x and y are independent. Then

$$\begin{aligned} W_j(C) &\approx \prod_x \left\{ (1 - f_1 - f_0) + f_1 \bar{G}_j [\alpha_C^1(x)] + f_0 \bar{G}_j [\alpha_C^0(x)] \right\} W_j^{\text{pert}}(C) \\ &= \exp \left[\sum_x \log \left\{ (1 - f_1 - f_0) + f_1 \bar{G}_j [\alpha_C^1(x)] \right. \right. \\ &\quad \left. \left. + f_0 \bar{G}_j [\alpha_C^0(x)] \right\} \right] W_j^{\text{pert}}(C), \end{aligned} \quad (4.8)$$

where the product and sum over x runs over all plaquette positions in the plane of the loop C , and $\alpha_C^n(x)$ depends on the position of the vortex midpoint x relative to the location of loop C . The expression $W_j^{\text{pert}}(C)$ contains the short-distance, perturbative contribution to $W_j(C)$; this will just have a perimeter-law falloff.

The question is what to use as an ansatz for $\alpha_R^n(x)$. The ansatz of Ref. [24] was motivated by the idea that the magnetic flux in the interior of vortex domains fluctuates almost independently, in subregions of extension l , apart from the restriction that the total flux results in a center element. For the SU(2) gauge group, this leads to

$$\begin{aligned} \left(\alpha_C^1(x) \right)^2 &= \frac{A_v}{2\mu} \left[\frac{A}{A_v} - \frac{A^2}{A_v^2} \right] + \left(2\pi \frac{A}{A_v} \right)^2, \\ \left(\alpha_C^0(x) \right)^2 &= \frac{A'_v}{2\mu} \left[\frac{A}{A'_v} - \frac{A^2}{A'^2_v} \right], \end{aligned} \quad (4.9)$$

where A_v, A'_v are the cross-sectional areas of the $n = 1$ and $n = 0$ domains, respectively, and A is the area of the domain which is contained within the interior of the minimal area of the loop. The result is a vortex-induced potential which is proportional to the quadratic Casimir of the representation at small R (compared to the vortex thickness), and depends only on the N -ality at large R . A sample calculation for the $j = \frac{1}{2}, 1, \frac{3}{2}$ representations is shown in Fig. 10. Details of the calculation are found in Ref. [24].

In G(2) gauge theory there would be only one type of vortex domain, corresponding to the single element of the trivial center group. Therefore, in this theory, one would expect all color representations to give rise to Casimir scaling in the intermediate distance regime, and flatten out asymptotically. Casimir scaling in this theory has in fact been observed, *cf.* Ref. [27].

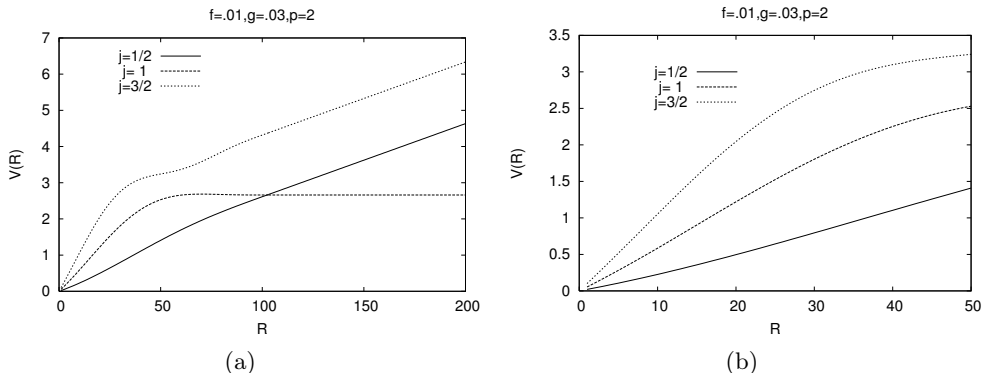


Fig. 10. The static potential for $j = \frac{1}{2}, 1, \frac{3}{2}$ static sources, for vortex width = 100, in the distance range: (a) $R \in [0, 200]$; and (b) $R \in [0, 50]$. From Ref. [24].

5. Monopoles, vortices, and calorons

Lattice U(1) gauge theory has monopole as well as photon excitations. A Dirac monopole on a three dimensional lattice (or a time-slice of a four dimensional lattice) can be visualized as shown in Fig. 11. If we write the product of links around a plaquette p as $U(p) = \exp[i\theta(p)]$, then a Dirac string corresponds to a line piercing a set of plaquettes with $\theta(p) = 2\pi$, and has no cost in action. The line ends at the center of some cube, where the monopole itself is located, and the 2π magnetic flux, which entered through the invisible Dirac line, exits through the six plaquettes forming the cube. The position of the Dirac line can be changed via U(1) gauge transformations, but the monopole position is gauge invariant.

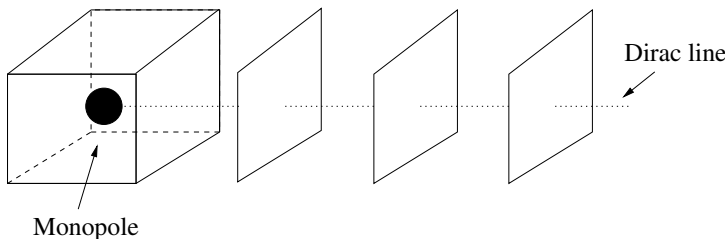


Fig. 11. In compact QED₃, a Dirac line is a line piercing the middle of plaquettes with flux $\Phi_B = 2\pi/e$. The line is bounded at one end by a Dirac monopole, and at the other by an antimonopole.

Polyakov, in a classic calculation [28], showed that lattice U(1) gauge theory (also known as “compact QED”) could be expressed as a monopole

Coulomb gas, with a partition function

$$Z_{\text{mon}} = \sum_{m(r)=-\infty}^{\infty} \exp \left[-\frac{2\pi^2}{g^2 a} \sum_{r,r'} m(r') G(r-r') m(r) \right], \quad (5.1)$$

where $m(r)$ is the monopole charge at site r , and $G(r)$ is the lattice propagator in $D = 3$ dimensions, going like $1/r$ at large separations. The Wilson loop can be thought of as a loop of current, interacting with the monopole background

$$\oint_C dr_\mu A_\mu(r) = \int_{S(C)} dS_\mu(r) H_\mu(r) = \int d^3r \eta_{S(C)}(r) m(r), \quad (5.2)$$

where

$$\eta_{S(C)}(r) = -\frac{1}{2} \frac{\partial}{\partial r_\mu} \int_{S(C)} dS_\mu(r') \frac{1}{|r-r'|}. \quad (5.3)$$

Everything can be calculated explicitly in $D = 3$ dimensions, and the result is that for a Wilson loop associated with n units of electric charge [28, 30]

$$\langle U_n(C) \rangle \approx \exp \left[-n\sigma \text{area}(C) \right], \quad (5.4)$$

where string tension σ is a calculable function of coupling β .

A very rough image of what's going on is this: A Wilson loop can be thought of as a current loop which itself is the source of a magnetic field. In a monopole plasma, the free magnetic charges, *i.e.* monopoles and antimonopoles, will tend to line up along the minimal area of the loop, and screen out the magnetic field that would have been generated by the Wilson (current) loop source (*cf.* Fig. 12). The area law falloff for Wilson loops is associated with the screening sheet of monopoles and antimonopoles along the minimal loop area.

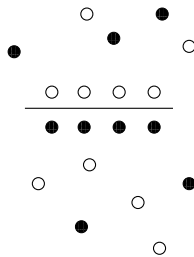


Fig. 12. Monopoles and antimonopoles near the minimal surface of a Wilson loop.

The monopole confinement mechanism can also be related [29] to the 't Hooft–Mandelstam theory of dual superconductivity. A relativistic version of a superconductor is provided by the Abelian Higgs model, described by the action

$$S = - \int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |(\partial_\mu + ieA_\mu\phi)|^2 + \frac{\lambda}{4} (\phi\phi^* - v^2)^2 \right). \quad (5.5)$$

This theory has both a massive and a Coulomb phase. The massive phase is the superconducting phase, and in this phase magnetic flux is collimated into Nielsen–Olesen vortices, which are the analog of Abrikosov vortices in an ordinary type II superconductor. Magnetic charges, in this phase, are confined by a linear potential, which is the energy stored in the Nielsen–Olesen flux tube running between the positive and negative magnetic charges.

According to the dual superconductor idea, the Higgs field is magnetically, rather than electrically charged, and couples to a dual “photon” field. Electric, rather than magnetic fields are squeezed into flux tubes, and electric, rather than magnetic, charges are confined.

Since this picture is essentially Abelian, its application to a non-Abelian gauge theory involves the selection of an Abelian subgroup of the non-Abelian group, generated by a Cartan subalgebra. The subgroup is identified either via a Higgs field in the adjoint representation, or by the imposition of an “Abelian projection” gauge, which leaves the subgroup unfixed. For an $SU(N)$ gauge theory, gauge-fixing to an Abelian projection gauge leaves a residual $U(1)^{N-1}$ Abelian gauge symmetry. The most common such gauge is the “maximal Abelian gauge”, which makes the gauge fields as Abelian as possible. In lattice $SU(2)$ gauge theory, the gauge condition is to maximize the quantity

$$R = \sum_x \sum_{\mu=1}^4 \text{Tr} \left[U_\mu(x) \sigma_3 U_\mu^\dagger(x) \sigma_3 \right]. \quad (5.6)$$

In lattice simulations, the $SU(2)$ link variables are gauge-fixed to maximal Abelian gauge, and then projected to the $U(1)$ subgroup by dropping the off-diagonal elements of the link variables (which are $SU(2)$ matrices), and rescaling the links to restore unitarity. The location of monopole currents can then be identified from the Abelian projected lattice.

Monopole confinement mechanisms, at least in their simplest versions, have difficulties with N -ality for Wilson loops constructed in the Abelian subgroup. For example, in the $SU(2) \rightarrow U(1)$ case, consider the double-charged Wilson loop

$$W_2[C] = \left\langle \exp \left[2i \oint dx^\mu \frac{1}{2} A_\mu^3 \right] \right\rangle. \quad (5.7)$$

On the lattice, where we may write $U_\mu^{ab}(x) = \exp[i\theta_\mu(x)\sigma_3]$ for the Abelian-projected links, the double-charged loop is

$$W_2[C] = \left\langle \prod_{(x,\mu) \in C} \exp[2i\theta_\mu(x)] \right\rangle. \quad (5.8)$$

This loop has an area law in the monopole Coulomb gas and dual superconductor pictures. But in fact, the loop can be screened by gluons which are charged with respect to the U(1) subgroup, and the asymptotic string tension of such loops must vanish asymptotically. What this means is that the distribution of U(1) flux *cannot* be that of a monopole Coulomb gas, at least on large scales. The spatial distribution of monopole field strength must be altered, in such a way that $W_2[C]$ follows an asymptotic perimeter-law falloff. This can be achieved, in 't Hooft's Abelian projection picture, if the Abelian monopole worldlines, and monopole fields, lie on vortex sheets.

The way this works is illustrated in Figs. 13–15. Let us consider a vortex at some fixed time, where the vortex appears as a tube of color magnetic flux. In the absence of gauge fixing, the field strength pointing along the vortex is oriented in a random direction in color space. For the SU(2) gauge group, upon fixing to maximal Abelian gauge, the field strength tends to line up in the $\pm\sigma_3$ direction in color space, but there will still be transition regions where the field rotates in color space from the $+\sigma_3$ to the $-\sigma_3$ direction. If we then project the configuration to the U(1) subgroup, then the center vortex appears as a monopole–antimonopole chain, with collimated flux running from monopole to neighboring antimonopole. This is the picture which has emerged from lattice simulations (*cf.* [31]), and the collimation of magnetic flux explains why, *e.g.*, a double-charged Abelian loop must have a perimeter law, rather than an area law asymptotically.

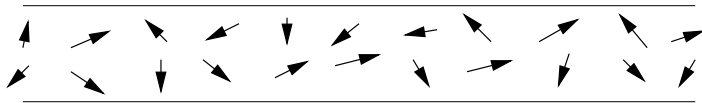


Fig. 13. Vortex field strength before gauge fixing. The arrows indicate direction in color space.

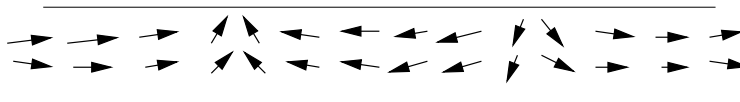


Fig. 14. Vortex field strength after maximal Abelian gauge fixing. Vortex strength is mainly in the $\pm\sigma_3$ direction.

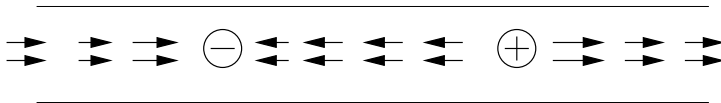


Fig. 15. Vortex field after Abelian projection.

5.1. Calorons

The most recent version of the monopole Coulomb gas mechanism, which has emerged in the last few years, is a picture of confinement based on the caloron configurations introduced by Kraan and van Baal [32] and Lee and Lu [33]. Calorons are instanton configurations at finite temperature (represented, on a Euclidean volume, by a finite extent in the time direction), and they have the important feature that Polyakov lines, away from the center of the vortex, can take on values that are not in the center of the gauge group. In particular, one type of caloron is associated with a vanishing trace for the Polyakov lines.

What is particularly interesting about calorons is they have monopole constituents which may, depending on the type of caloron, be distant from one another⁵. In fact, the positions of these constituents are set by the parameters (“moduli”) of the solution. The constituents can be widely separated; in fact they can be placed anywhere, depending on the choice of moduli. An example for the SU(2) group is shown in Fig. 16, which displays the action density of two SU(2) caloron solutions in a timeslice at $t = 0$. Because the caloron is an instanton solution, one might think that the dyon constituents appear and then disappear in a certain time interval. In fact, this is true for small dyon separations. When the dyons are widely separated, however, they persist throughout the entire periodic time interval.

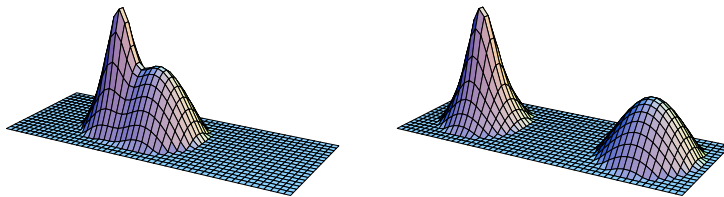


Fig. 16. Action density of an SU(2) caloron in a timeslice $t = 0$, with eigenvalues of the Polyakov line holonomy $\mu_2 = -\mu_1 = 0.125$. The parameter ρ determines the separation of the constituent dyons; the figure on the left is at $\rho = 0.8$, and the figure on the right has $\rho = 1.6$. From van Baal [34].

⁵ The monopoles in this theory are also referred to as “dyons” because they are electrically, as well as magnetically, charged.

The relevance for confinement is the possibility that at low temperatures the monopole constituents of calorons are widely separated, and form the sort of monopole Coulomb gas which is known, from Polyakov's work, to lead to confinement. A model calculation, due to Diakonov and Petrov [35], gives just this result, with sine-law scaling of the k -string tensions, and an ensemble of calorons with vanishing Polyakov lines. Their calculation involves a particular suggestion for the appropriate measure of the caloron collective coordinates.

The caloron proposal can, however, be critiqued on a number of grounds. In the first place, since the confinement mechanism is essentially Abelian, and closely analogous to that of a monopole Coulomb gas, there is no obvious reason why, in an appropriate gauge where the constituent monopole fields are asymptotically Abelian, a double-charged Abelian loop would not end up with an area law, in violation of color screening and N -ality. In $G(2)$ gauge theory the Diakonov–Petrov calculation also leads to the vanishing of Polyakov lines, in contradiction to the requirement of color screening in that theory⁶. Another point is that if we take seriously the idea that the distribution of Polyakov loop holonomies are peaked at zero trace, as in the dyon gas calculations, this would then imply that the expectation value of Polyakov lines in the adjoint representation are negative. That prediction appears to contradict the existing numerical evidence, which indicates that adjoint Polyakov lines have a positive expectation value below the deconfinement transition [36].

6. Gribov horizon and Dyson–Schwinger equations

The BRST quantization presented in most textbooks is problematic at the non-perturbative level due to a theorem by Neuberger [37]. Let Q be any gauge or BRST invariant operator, $c, \bar{c}$ the ghost and antighost fields, and S_{gf} the gauge-fixing part of the action. Then, according to Neuberger's theorem, the expression for the vacuum expectation value of Q is ambiguous when formulated non-perturbatively on the lattice, *i.e.*

$$\langle Q \rangle = \frac{\int DUDcD\bar{c}Q[U]e^{-(S+S_{gf})}}{\int DUDcD\bar{c}e^{-(S+S_{gf})}} = \frac{0}{0}. \quad (6.1)$$

⁶ Since it takes three gluons to screen a quark in $G(2)$ gauge theory, the “gluelump” state is so massive that Polyakov line expectation values are expected to be tiny. It is not surprising that it is difficult to extract a non-zero value by numerical simulation; this is difficult (below the deconfinement temperature) even for adjoint Polyakov lines in $SU(2)$ gauge theory, where only one gluon is required to screen a static adjoint charge. But unless all current ideas about string-breaking are mistaken, the Polyakov line expectation value in $G(2)$ gauge theory must have a non-zero, albeit very small value, and this is *qualitatively* different from a zero value.

The underlying reason for this ambiguity is that in covariant gauges and Coulomb gauge the gauge condition does not completely fix the gauge, even disregarding the residual global symmetries mentioned previously. The point is that there are many gauge copies (known as “Gribov copies”, unrelated by the residual symmetry, all of which satisfy the gauge condition. In a lattice regularization it turns out that there are even numbers of Gribov copies on a gauge orbit, half having a positive sign for the Faddeev–Popov (F–P) determinant, and half negative, so that the sum over all copies in both the numerator and denominator of (6.1) vanishes.

Because of the indefinite sign of the F–P determinant, it was suggested by Gribov that the functional integral should be restricted to be inside the first Gribov horizon, which is the manifold in configuration space where the lowest non-trivial eigenvalue of the F–P operator becomes zero. The interior of the region, where all non-trivial eigenvalues are positive, is known as the Gribov region. In fact, in order to count the contribution of each gauge orbit only once, the functional integration should really be restricted to a subvolume of the Gribov region, known as the “Fundamental Modular Region”, where the norm of the gauge field vector potential is minimized. Gribov and Zwanziger argued that most of the volume of configuration space within the Gribov region (and the fundamental modular region) is concentrated near the first Gribov horizon, where the F–P operator develops a zero non-trivial eigenvalue⁷.

6.1. Coulomb confinement

The connection of the Gribov horizon to confinement is most direct in Coulomb gauge, because the Coulomb potential in that gauge depends explicitly on the inverse F–P operator M^{-1}

$$V_{\text{Coul}}(|x - y|) \propto - \left\langle (M^{-1})_{xz}^{ab} (-\nabla_z^2) (M^{-1})_{zy}^{ba} \right\rangle, \quad (6.2)$$

where

$$M = -\nabla \cdot D \quad (6.3)$$

is the F–P operator, and D is the covariant derivative in the adjoint representation of the gauge group. At the first Gribov horizon, M develops a non-trivial zero eigenvalue. Then if

1. there is a concentration of small eigenvalues for configurations near the horizon; and
2. the configurations which dominate the functional integration lie very close to the horizon;

⁷ The trivial zero eigenvalues are associated with the global residual gauge symmetry.

then this could lead to an enhancement of the inverse F–P operator, leading to a confining Coulomb potential. In fact, something like this *must* be true, because it can be shown rigorously that the Coulomb potential is an upper bound on the static quark potential [38] , *i.e.*

$$V(R) \leq V_{\text{coul}}(R) \quad (6.4)$$

which means that Coulomb confinement is a necessary (although not sufficient) condition for confinement.

Let

$$M^{ab} \phi_n^b = \lambda_n \phi_n^a \quad (6.5)$$

be the eigenvalue equation for the F–P operator, and let $\rho(\lambda)$ be the eigenvalue density. Then the Coulomb self energy of a static charge (with some appropriate, *e.g.* lattice, ultraviolet cutoff) is given by [39]

$$\begin{aligned} E_{\text{self}} &\propto \left\langle (M^{-1})_{xz}^{ab} (-\nabla_z^2) (M^{-1})_{zx}^{ba} \right\rangle \\ &= \int_0^{\lambda_{\text{max}}} d\lambda \left\langle \rho(\lambda) \frac{(\phi_\lambda | (-\nabla^2) | \phi_\lambda)}{\lambda^2} \right\rangle \end{aligned} \quad (6.6)$$

so that the Coulomb confinement criterion, which says that the Coulomb self energy of an isolated color charge is infinite, is

$$\lim_{\lambda \rightarrow 0} \left\langle \rho(\lambda) \frac{(\phi_\lambda | (-\nabla^2) | \phi_\lambda)}{\lambda} \right\rangle > 0 \quad (6.7)$$

and this condition has been verified numerically [39].

An easy way to compute the Coulomb potential on the lattice was suggested in Ref. [40]. Define, in Coulomb gauge, the physical state

$$\Psi_{q\bar{q}} = \bar{q}^a(0) q^a(R) \Psi_0, \quad (6.8)$$

where Ψ_0 is the ground state. Then, defining also

$$G(R, T) = \langle \Psi_{q\bar{q}} | e^{-(H-E_0)T} | \Psi_{q\bar{q}} \rangle \quad (6.9)$$

we have the excitation energy

$$\begin{aligned} V_C(R) &= \frac{\langle \Psi_{q\bar{q}} | H - E_0 | \Psi_{q\bar{q}} \rangle}{\langle \Psi_{q\bar{q}} | \Psi_{q\bar{q}} \rangle} \\ &= - \lim_{T \rightarrow 0} \frac{d}{dT} \log[G(R, T)]. \end{aligned} \quad (6.10)$$

This energy includes both an R -dependent Coulomb interaction energy, plus R -independent self-energies. It is not hard to see that for heavy quarks

$$G(R, T) = \left\langle \text{Tr} \left[L^\dagger(0, T) L(R, T) \right] \right\rangle, \quad (6.11)$$

where $L(\mathbf{x}, T)$ is a time-like Wilson line

$$L(\mathbf{x}, T) = P \exp \left[i \int_0^T dt A_0(\mathbf{x}, t) \right]. \quad (6.12)$$

Note that $L(\mathbf{x}, T)$ is not a Polyakov line holonomy, since T is taken to be smaller than the extent of the lattice in the time direction.

On the lattice, these lines are products of time-like link variables, and time derivatives are replaced by finite differences. The upshot, on the lattice, is that the Coulomb potential can be extracted from the correlator of time-like link variables

$$V_C(R) = -\log \left\langle \text{Tr} \left[U_0^\dagger(\mathbf{x}, t) U_0(\mathbf{x} + R, t) \right] \right\rangle \quad (6.13)$$

and the result of this calculation, for SU(2) lattice gauge theory at lattice coupling $\beta = 2.5$, is displayed in Fig. 17. Here the upper line is the Coulomb energy, which is clearly rising linearly, and the lower line is the result when center vortices are removed from the lattice configurations by the de Forcrand–D’Elia procedure [41]. The Coulomb string tension obtained in this way is found to be about three times as large as usual asymptotic string tension, and it vanishes when vortices are removed.

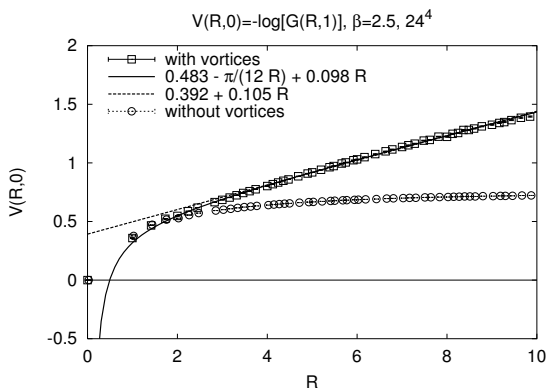


Fig. 17. $V(R, 0)$ at $\beta = 2.5$, which is equal, up to a constant self-energy term, to the Coulomb potential. The solid (dashed) line is a fit to a linear potential with (without) the Lüscher term. The result from vortex-removed lattices is also shown. From Ref. [40].

6.2. Dyson–Schwinger equations

The Dyson–Schwinger approach to non-Abelian gauge theories has been under very active development in the last decade; the hope is that the Dyson–Schwinger equations (DSEs) for n -point quark–gluon Green’s functions may be soluble in certain kinematic regimes (for a review, *cf.* [42]).

Dyson–Schwinger equations for the ghost propagator and ghost-gluon vertex are displayed diagrammatically in Figs. 18, 19. In these figures the filled circles along lines denote dressed propagators, and the large filled circles at vertices denote the full vertex (the “ -1 ” indicates the inverse propagator).

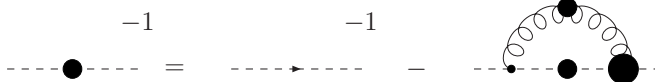


Fig. 18. Diagrammatic representation of the Dyson–Schwinger equation for the ghost propagator. Lines with (without) filled circles represent dressed (bare) propagators, the dashed lines are ghost propagators (or inverse propagators), and the wavy line is a dressed gluon propagator. The loop diagram contains one bare and one dressed ghost-gluon vertex. From Fischer [42].

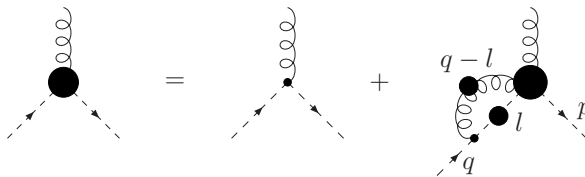


Fig. 19. Diagrammatic representation of the Dyson–Schwinger equation for the ghost-gluon vertex, with the same conventions as the previous figure. From Fischer [42].

Writing, for the full ghost and gluon propagators

$$\begin{aligned} D_{\mu\nu}(p) &= \frac{Z(p^2)}{p^2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right), \\ D_{\text{ghost}}(p) &= -\frac{G(p^2)}{p^2} \end{aligned} \quad (6.14)$$

it has been shown that the relevant Dyson–Schwinger equations have a solution known as the “scaling solution” [43], in which the ghost and gluon dressing functions Z and G , respectively, have the behavior

$$Z(p^2) \sim (p^2)^{2\kappa} \quad \text{and} \quad G(p^2) \sim (p^2)^{-\kappa} \quad (6.15)$$

with

$$\kappa = 0.595353. \quad (6.16)$$

Thus the gluon propagator is less singular (tending, in fact, to zero), and the ghost propagator more singular than the perturbative behavior, as $p^2 \rightarrow 0$. These results tie in very nicely with both the Zwanziger horizon conditions, which predict that the gluon propagator vanishes in the infrared, and the Kugo–Ojima criterion, which predicts that $G(p^2) \rightarrow \infty$ as $p^2 \rightarrow 0$.

But although the gluon propagator vanishes in the infrared, according to the scaling solution, in one-particle exchange this dressed propagator is accompanied by a dressed vertex which has singular infrared behavior. The upshot is that in the ladder expansion of the four-quark one-particle irreducible Green's function shown in Fig. 20, the propagator and vertex combine to give a $1/p^4$ ladder exchange, leading to a linear potential between the quark and antiquark.

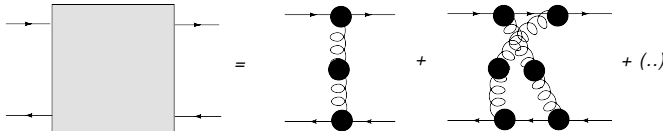


Fig. 20. The four quark one-particle irreducible Green's function. From Alkofer *et al.* [44].

All of this seems very promising. Unfortunately, the prediction of a vanishing gluon propagator and singular ghost dressing function has not been born out by large volume lattice simulations, carried out Cucchieri and Mendes [45], and by the Berlin group [46]. It has since been shown [47–49] that the scaling solution is a rather special solution of the Dyson–Schwinger equations, and there also exists a more generic “decoupling” solution in which the gluon propagator tends to a constant in the infrared, and the ghost dressing function is non-singular. This decoupling solution appears to be more compatible with the lattice Monte Carlo data.

7. Vacuum wavefunctionals

In recent years there have been renewed efforts to solve for the ground state of the Yang–Mills Schrödinger equation $H\Psi_0 = E_0\Psi_0$, to see if anything can be learned about the origin of confinement and the mass gap.

There are several approaches. In Coulomb gauge, Szczepaniak and co-workers [50], and Reinhardt and co-workers [51], have put forward a Gaussian ansatz for the vacuum wavefunctional

$$\Psi_0 = \exp \left[- \int d^3x d^3y A_i^a(x) K_{xy}^{ij} A_j^a(y) \right] \quad (7.1)$$

with a field-independent kernel K , determined by minimizing $\langle H \rangle$. While this approach has been pushed quite far, it is not clear how a vacuum of this type gives rise to an area-law falloff for Wilson loops⁸.

A quite different approach has been pursued by Karabali, Kim, and Nair [52], and Leigh *et al.* [53], in $D = 2 + 1$ dimensions. The idea is to change variables from the gauge potential A_k^a in temporal gauge, to new, gauge-invariant variables. The tradeoff is the replacement of local gauge invariance with a local invariance under holomorphic transformations.

Karabali, Kim, and Nair begin by defining a complexified gauge potential

$$A = A_1 + A_2 \quad \text{and} \quad z = x_1 - x_2 \quad (7.2)$$

and define a new complex, matrix-valued variable $M \in SL(2, C)$ such that

$$\begin{aligned} A &= -\frac{\partial M}{\partial z} M^{-1}, \\ H &= M^\dagger M, \\ J &= \frac{C_A}{\pi} \frac{\partial H}{\partial z} H^{-1}. \end{aligned} \quad (7.3)$$

In terms of these gauge invariant variables, the Hamiltonian becomes

$$H_{\text{KKN}} = T + V, \quad (7.4)$$

where T is derived from the E^2 term in the standard Hamiltonian

$$T = m \left(\int d^2 u \, J^a(u) \frac{\delta}{\delta J^a(x)} + \delta^2 u \int d^2 v \, \Omega_{ab}(u, v) \frac{\delta}{\delta J^a(u)} \frac{\delta}{\delta J^b(v)} \right), \quad (7.5)$$

where

$$\Omega_{ab}(u, v) = \frac{C_A}{\pi^2} \frac{\delta_{ab}}{(u - v)^2} - i f_{abc} \frac{J^c(v)}{\pi(u - v)} \quad (7.6)$$

and

$$V = \frac{1}{2g^2} \int B^2(x) = \frac{\pi}{mC_A} \int d^2 x \, \bar{\partial} J^a \bar{\partial} J^a, \quad (7.7)$$

where, in this case,

$$m = \frac{g^2 C_A}{2\pi} \quad (7.8)$$

and C_A is the quadratic Casimir of the adjoint representation. It is important to note that although the theory is now expressed in terms of gauge-invariant variables J , a new local invariance — *holomorphic invariance* — has appeared in the problem. This is the transformation

$$M(z, \bar{z}) \rightarrow M(z, \bar{z}) h^\dagger(\bar{z}), \quad M^\dagger(z, \bar{z}) \rightarrow h(z) M^\dagger(z, \bar{z}) \quad (7.9)$$

⁸ In very recent work Szczepaniak has proposed a modified version of the ansatz, which inserts monopole configurations to overcome this problem.

under which

$$\begin{aligned} H(z, \bar{z}) &\rightarrow h(z)H(z, \bar{z})h^\dagger(\bar{z}), \\ J &\rightarrow hJh^{-1} + \frac{C_A}{\pi}\partial h h^{-1}. \end{aligned} \quad (7.10)$$

It can be seen that J transforms under a holomorphic transformation $h(z)$ much like a gauge field vector potential under a gauge transformation. The Hamiltonian H_{KKN} is invariant under these local transformations, and it is crucial that the eigenstates of H_{KKN} are also invariant. In effect, by going to the new variables one trades gauge invariance (in temporal gauge formulation) for holomorphic invariance.

In these new variables, it is possible to carry out a systematic strong-coupling expansion. Karabali, Kim and Nair were able to sum up all terms bilinear in J , with the result

$$\Psi_0[J] = \exp \left[-\frac{2\pi^2}{g^2 C_A^2} \int d^2x d^2y \bar{\partial} J^a(x) \left(\frac{1}{\sqrt{-\nabla^2 + m^2} + m} \right)_{xy} \bar{\partial} J^a(y) \right]. \quad (7.11)$$

In terms of the usual variables, this wavefunctional becomes

$$\Psi_0[A] = \exp \left[-\frac{1}{2g^2} \int d^2x d^2y B^a(x) \left(\frac{1}{\sqrt{-\nabla^2 + m^2} + m} \right)_{xy} B^a(y) \right]. \quad (7.12)$$

If we disregard ∇^2 , then this ground state becomes

$$\Psi_0[A] = \exp \left[-\frac{1}{4mg^2} \int d^2x B^a(x) B^a(x) \right] \quad (7.13)$$

which can be used to calculate the string tension of space-like Wilson loops. The result is

$$\sigma = \frac{g^4}{8\pi} (N^2 - 1) \quad (7.14)$$

which in fact agrees quite closely with the known lattice results in 2+1 dimensions.

The difficulty here, in my opinion, is that the wavefunctional $\Psi_0[J]$ is not holomorphic invariant, and the equivalent form $\Psi_0[A]$ of Eq. (7.12) is not gauge-invariant, and therefore not a physical state. It is not so clear, then, whether it is allowable to use such states to extract a string tension, but one can still ask whether trilinear and higher contributions in J would improve the situation regarding gauge and holomorphic invariance. I will return to this issue shortly.

A third approach, again in 2+1 dimensions, has been followed by Olejnik and myself in temporal gauge [54]. In temporal gauge, the problem is to find the ground state of the Hamiltonian operator

$$H = \int d^2x \left\{ -\frac{1}{2} \frac{\delta^2}{\delta A_k^a(x)^2} + \frac{1}{2} B^a(x)^2 \right\}, \quad (7.15)$$

where $B = F_{12}$ subject to the Gauss Law constraint

$$\left(\delta^{ac} \partial_k + g \epsilon^{abc} A_k^b \right) \frac{\delta}{\delta A_k^c} \Psi = 0 \quad (7.16)$$

which is equivalent to requiring the invariance of $\Psi[A]$ under infinitesimal gauge transformations. In the free-field, $g^2 = 0$ limit, it is easy to solve for the ground state, which is

$$\psi_0[A] = \exp \left[- \int d^2x d^2y B^a(x) \left(\frac{\delta^{ab}}{\sqrt{-\nabla^2}} \right)_{xy} B^b(y) \right]. \quad (7.17)$$

One can also solve for the ground state in a quite different limit, where we throw away all degrees of freedom except the zero mode of the A -field. In that case the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2} \int d^2x \left[\partial_t A_k \cdot \partial_t A_k - g^2 (A_1 \times A_2) \cdot (A_1 \times A_2) \right] \\ &= \frac{1}{2} V \left[\partial_t A_k \cdot \partial_t A_k - g^2 (A_1 \times A_2) \cdot (A_1 \times A_2) \right] \end{aligned} \quad (7.18)$$

with corresponding Hamiltonian

$$H = -\frac{1}{2} \frac{1}{V} \frac{\partial^2}{\partial A_k^a \partial A_k^a} + \frac{1}{2} g^2 V (A_1 \times A_2) \cdot (A_1 \times A_2), \quad (7.19)$$

where V is the volume of 2-space. In this case it can be shown that the ground state wavefunction, to leading order in V , is

$$\psi_0 = \exp \left[-\frac{1}{2} g V \frac{(A_1 \times A_2) \cdot (A_1 \times A_2)}{\sqrt{|A_1|^2 + |A_2|^2}} \right]. \quad (7.20)$$

Now we observe that the following wavefunctional

$$\Psi_0[A] = \exp \left[-\frac{1}{2} \int d^2x d^2y B^a(x) \left(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)_{xy}^{ab} B^b(y) \right] \quad (7.21)$$

agrees with both soluble limits of the Yang–Mills Schrodinger equation, and also (since it is gauge invariant) satisfies the Gauss Law constraint. Here $-D^2$ is minus the covariant Laplacian, λ_0 is the lowest eigenvalue of $-D^2$, and m^2 is a parameter which vanishes as $g \rightarrow 0$. If we consider the zero-mode strong-field limit of the covariant derivative

$$(-D^2)_{xy}^{ab} = g^2 \delta(x-y) \left[(A_1^2 + A_2^2) \delta^{ab} - A_1^a A_1^b - A_2^a A_2^b \right], \quad (7.22)$$

where m^2 is negligible, then we find, after some algebra, that

$$\begin{aligned} \Psi_0[A] &= \exp \left[- \int d^2x d^2y B^a(x) \left(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)_{xy}^{ab} B^b(y) \right] \\ &\Rightarrow \exp \left[- \frac{1}{2} g V \frac{(A_1 \times A_2) \cdot (A_1 \times A_2)}{\sqrt{A_1^2 + A_2^2}} \right] \end{aligned} \quad (7.23)$$

which is the ground state of the zero-mode Hamiltonian.

7.1. Dimensional reduction

A long time ago it was suggested [55] that at large distance scales, the vacuum state of pure Yang–Mills theory has the following form

$$\Psi_0[A] \approx \Psi_0^{\text{eff}}[A] = \exp \left[- \frac{1}{2} \mu \int d^3x \text{Tr} [F_{ij}^2] \right]. \quad (7.24)$$

This vacuum state has the property of “dimensional reduction”: The computation of a large space-like Wilson loop in $d+1$ dimensions reduces to the calculation of a Wilson loop in Yang–Mills theory in d Euclidean dimensions. Suppose $\Psi_0^{(3)}$ is the ground state of the 3+1 dimensional theory, and $\Psi_0^{(2)}$ is the ground state of the 2+1 dimensional theory. If these ground states both have the dimensional reduction form, and $W(C)$ is a large, planar Wilson loop, then

$$\begin{aligned} W(C) &= \langle \text{Tr}[U(C)] \rangle^{D=4} = \left\langle \Psi_0^{(3)} \left| \text{Tr}[U(C)] \right| \Psi_0^{(3)} \right\rangle \\ &\approx \langle \text{Tr}[U(C)] \rangle^{D=3} = \left\langle \Psi_0^{(2)} \left| \text{Tr}[U(C)] \right| \Psi_0^{(2)} \right\rangle \\ &\approx \langle \text{Tr}[U(C)] \rangle^{D=2}. \end{aligned} \quad (7.25)$$

In $D = 2$ dimensions the Wilson loop can be calculated analytically, and we know there is an area law falloff, with Casimir scaling of the string tensions.

Let us now introduce a mode number cutoff in the proposed ground state wavefunctional (7.21). Expand $B(x)$ in eigenmodes of the covariant Laplacian, *i.e.*

$$\begin{aligned} (-D^2)^{ab} \phi_n^b(x) &= \lambda_n \phi_n^a(x), \\ B^a(x) &= \sum_{n=0}^{\infty} b_n \phi_n^a(x), \\ B^{a,\text{slow}}(x) &= \sum_{n=0}^{n_{\text{max}}} b_n \phi_n^a(x). \end{aligned} \quad (7.26)$$

The cutoff mode sum defines the “slowly varying” B -field. Choosing n_{max} such that $\lambda_{n_{\text{max}}} - \lambda_0 \ll m^2$, we have

$$\begin{aligned} &\int d^2x d^2y B^{a,\text{slow}}(x) \left(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)_{xy}^{ab} B^{b,\text{slow}}(y) \\ &\approx \frac{1}{m} \int d^2x B^{a,\text{slow}}(x) B^{a,\text{slow}}(x). \end{aligned} \quad (7.27)$$

This gives us a dimensional reduction wavefunctional, since the right-hand side is simply the Euclidean action of a two-dimensional gauge theory, and the string tension can be computed, in terms of m , analytically:

$$\sigma = \frac{3}{16} m g^2 \quad (7.28)$$

or, in terms of the lattice coupling $\beta = 4/g^2$, $\sigma = 3m/(4\beta)$. If we turn this around, and write $m = 4\beta\sigma/3$, then we have a complete proposal for the Yang–Mills vacuum wavefunctional, in $2 + 1$ dimensions.

7.2. Numerical tests

We would like to test this proposal by calculating, *e.g.*, the mass gap of the theory, or the Coulomb potential. To get the mass gap, we need to compute the connected correlator

$$G(x - y) = \langle (B^a B^a)_x (B^b B^b)_y \rangle - \langle (B^a B^a)_x \rangle^2 \quad (7.29)$$

in the probability distribution

$$P[A] = |\Psi_0[A]|^2 = \exp \left[- \int d^2x d^2y B^a(x) K_{xy}^{ab}[A] B^b(y) \right], \quad (7.30)$$

where

$$K_{xy}^{ab}[A] = \left(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)_{xy}^{ab}. \quad (7.31)$$

Numerically, this looks hopeless! Not only is the kernel $K_{xy}^{ab}[A]$ non-local, it is not even known explicitly for arbitrary $A_i^a(x)$, which would seem to rule out any lattice Monte Carlo approach. But suppose, after eliminating the variance along gauge orbits by a gauge choice, that $K[A]$ has very little variation among thermalized configurations. Then things are more promising.

Define

$$P[A; K[A']] = \exp \left[- \int d^2x d^2y B^a(x) K_{xy}^{ab}[A'] B^b(y) \right], \quad (7.32)$$

where B is computed from A , not A' , and $P[A] = P[A, K[A]]$. Then, assuming the variance of K is small,

$$\begin{aligned} P[A] &\approx P[A, \langle K \rangle] \\ &= P \left[A, \int DA' K[A'] P[A'] \right] \\ &\approx \int DA' P[A, K[A']] P[A']. \end{aligned} \quad (7.33)$$

We can then solve this equation iteratively

$$\begin{aligned} P^{(1)}[A] &= P[A; K[0]], \\ P^{(n+1)}[A] &= \int DA' P[A; K[A']] P^{(n)}[A']. \end{aligned} \quad (7.34)$$

The procedure is to work in an axial $A_1 = 0$ gauge, introduce a lattice regularization, and change integration variables from A_2 to B . Initially, set also $A_2 = 0$. Then

1. Set $A'_2 = 0$.
2. $P[A; K[A']]$ is Gaussian in B . Diagonalize $K_{xy}^{ab}[A']$, and generate a new B -field stochastically.
3. Given B , calculate A_2 in the $A_1 = 0$ gauge, and compute observables.
4. Return to step 1, repeat as many times as necessary.

Observables of interest include the eigenvalue spectrum $\{\lambda_n\}$ of the covariant Laplacian, and the connected field strength correlator

$$\langle B^2(x) B^2(y) \rangle_{\text{conn}} \propto G(x - y), \quad (7.35)$$

where

$$G(x-y) = \left\langle (K^{-1})_{xy}^{ab} (K^{-1})_{yx}^{ba} \right\rangle,$$

$$K^{-1} = \sqrt{-D^2 - \lambda_0 + m^2}. \quad (7.36)$$

From $G(R)$ we can extract the mass gap.

We can also compute these same observables on two-dimensional slices of lattices generated in a $D = 3$ dimensional lattice Monte Carlo simulation. This is essentially simulating the ground state of the transfer matrix of the Euclidean theory. The results obtained from two-dimensional “MC” lattices, generated by ordinary lattice Monte Carlo, can be compared with the results obtained from “recursion” lattices, generated by steps described above. The recursion lattices are generated from the probability distribution $P[U] = \Psi_0^2[U]$, where $\Psi_0[U]$ is the latticized version of Eq. (7.21).

Fig. 21 is a plot of the eigenvalue *versus* mode number of the

- zero-field operator $-\nabla^2 + m^2$; and the
- covariant operator $-D^2 - \lambda_0 + m^2$

calculated on 10 recursion lattices at $\beta = 18$ on a 50×50 lattice area. Note that these values are not averaged; the values for each lattice are plotted, and (almost) fall on top of one another for each mode number. This means that there is very little variance in the spectrum of $-D^2 - \lambda_0$ from one lattice to the next.

Fig. 22 plots the data for the observable $G(R)$ in Eq. (7.36). The data is obtained from ten recursion lattices, and ten MC lattices. The recursion and MC values obviously agree to high accuracy. Note the tiny values of $G(R)$ obtained at the larger R values, and the smoothness of the data. This clearly implies a near-absence of fluctuations in the K^{-1} operator, among thermalized lattices, which is a check of the assumption underlying our simulation procedure. The mass gap is obtained by fitting the data for $G(R)$ to extract the exponential falloff. Define

$$G(R) = \delta^{ab} \delta^{ba} \left[\left(\sqrt{-\nabla^2 + (\mathcal{M}/2)^2} \right)_{xy} \right]^2$$

$$= \frac{3}{4\pi^2} \left(1 + \frac{1}{2} \mathcal{M} R \right)^2 \frac{e^{-\mathcal{M} R}}{R^6}. \quad (7.37)$$

Fig. 23 is a plot of the recursion lattice data for $G(R)$, compared to $G_0(R)$ with a best-fit value for $\mathcal{M}$. This value is our estimate for the mass gap.

Fig. 24 shows our values for the mass gap $\mathcal{M}$ (0^+ glueball mass) *versus* coupling β , extracted from the recursion lattice data. The points labeled “expt” are the lattice Monte Carlo results for the 0^+ glueball mass, reported

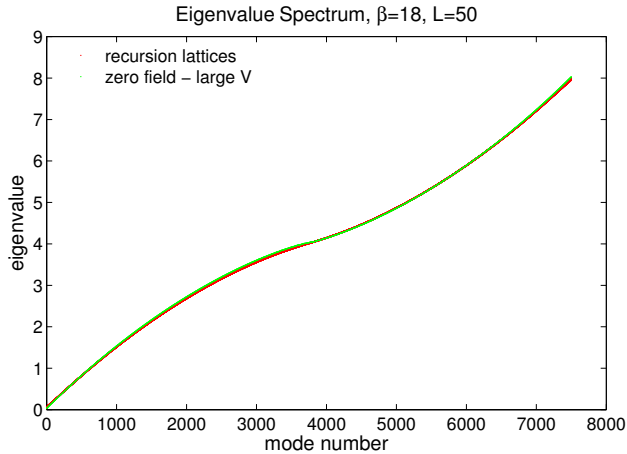


Fig. 21. Ten sets of eigenvalue spectra of the operator $-D^2 - \lambda_0 + m^2$, at $\beta = 18$, from ten independent 50×50 recursion lattices. Also plotted, but indistinguishable from the other spectra, is the rescaled spectrum of the large-volume zero-field operator $-\nabla^2 + m^2$.

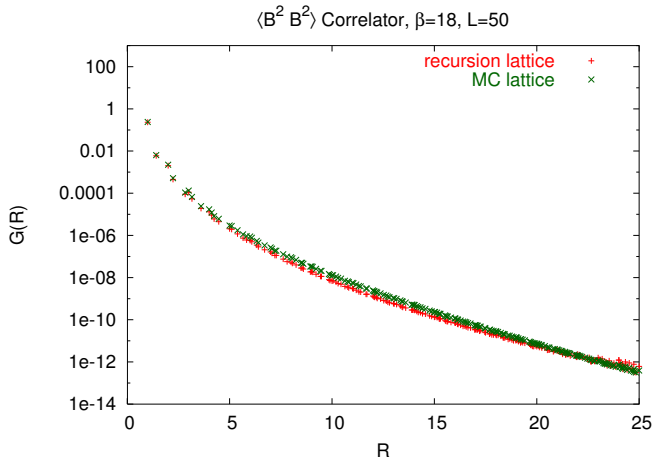


Fig. 22. The correlator $G(R)$ computed (i) on two-dimensional lattice configurations generated from the vacuum wavefunctional by the method described in the text; and (ii) on constant-time slices of three-dimensional lattice configurations generated by the usual lattice Monte Carlo method. Lattices generated by the first method are denoted “recursion”, and by the second as “MC”. In each case, the lattice extension is 50 sites at $\beta = 18$.

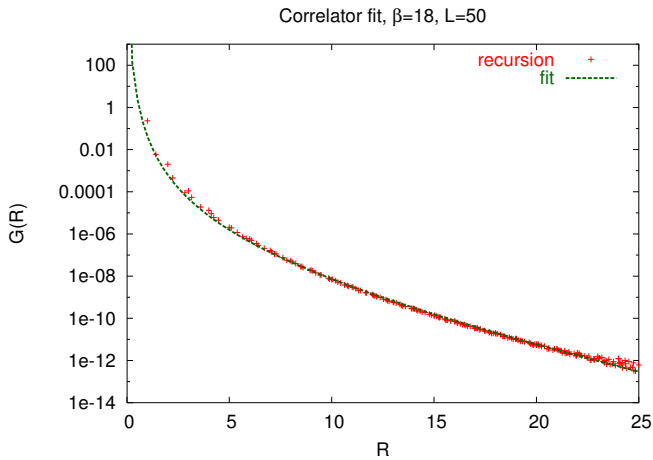


Fig. 23. Best fit (dashed line) of the recursion lattice data for $G(R)$ by the analytic form given in Eq. (7.37).

by Meyer and Teper in Ref. [56]. It appears that, given the asymptotic string tension as input, we can determine fairly the mass gap fairly accurately from our vacuum wavefunctional.

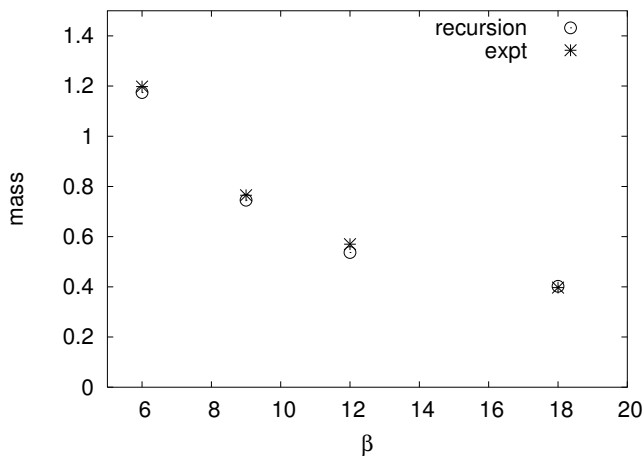


Fig. 24. Mass gaps extracted from recursion lattices at various lattice couplings, compared to the 0^+ glueball masses in 2+1 dimensions obtained in Ref. [56] (denoted “expt”) via standard lattice Monte Carlo methods. Error-bars are smaller than the symbol sizes.

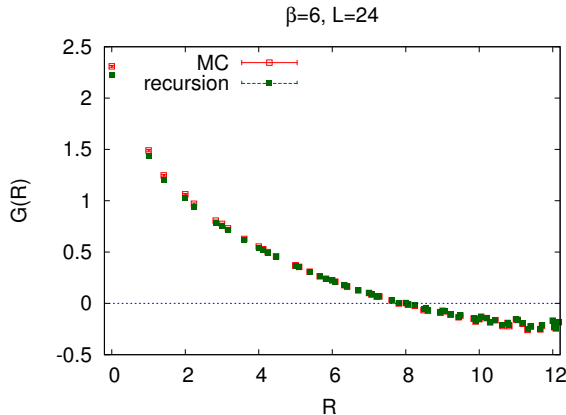


Fig. 25. The Coulomb ghost propagator evaluated on both recursion and MC lattices, at lattice coupling $\beta = 6$.

Another observable that we have looked at is the Coulomb gauge ghost propagator. This is evaluated by transforming each lattice to Coulomb gauge, and then evaluating

$$G_{\text{ghost}}(R) = \langle (M^{-1})_{\mathbf{x}\mathbf{y}}^{aa} \rangle, \quad (7.38)$$

where $R = |\mathbf{x} - \mathbf{y}|$. The result obtained on MC and recursion lattices is shown in Fig. 25. Once again, the points essentially overlap.

We can also compute the Coulomb potential, Eq. (6.2). It turns out that this observable (which involves two powers of the inverse F-P operator) is highly sensitive to “exceptional” configurations with unusually low values of λ_0 , and this sensitivity leads to very large statistical fluctuations. In order to compare MC and recursion lattice values, which is our main aim, we have imposed cuts on the data, throwing away the rare configurations with very low λ_0 . The result, at $\beta = 6$ and lattice extension $L = 24$, is shown in Fig. 26, where we have excluded lattices giving rise to $V(0) < -10, -20, -100, -200$. Agreement between MC and recursion lattices is still good, although the error-bars grow systematically as the cutoff is removed.

At this point it is interesting to return to the Karabali–Kim–Nair proposal for the ground state. Although Eq. (7.12) is not a physical state, it was pointed in Ref. [52] that terms which are higher order than bilinear in J could sum up to change the ∇^2 operator into a covariant Laplacian D^2 . A state of that form is then amenable to our methods of simulation, and it is found that the error in the string tension is at least 50%, and may even become infinite in the continuum limit [54]. The reason is that the $-D^2$ is positive definite and (unlike $-\nabla^2$) its lowest eigenvalue cannot be approximated by zero, even when calculating observables at large scales.

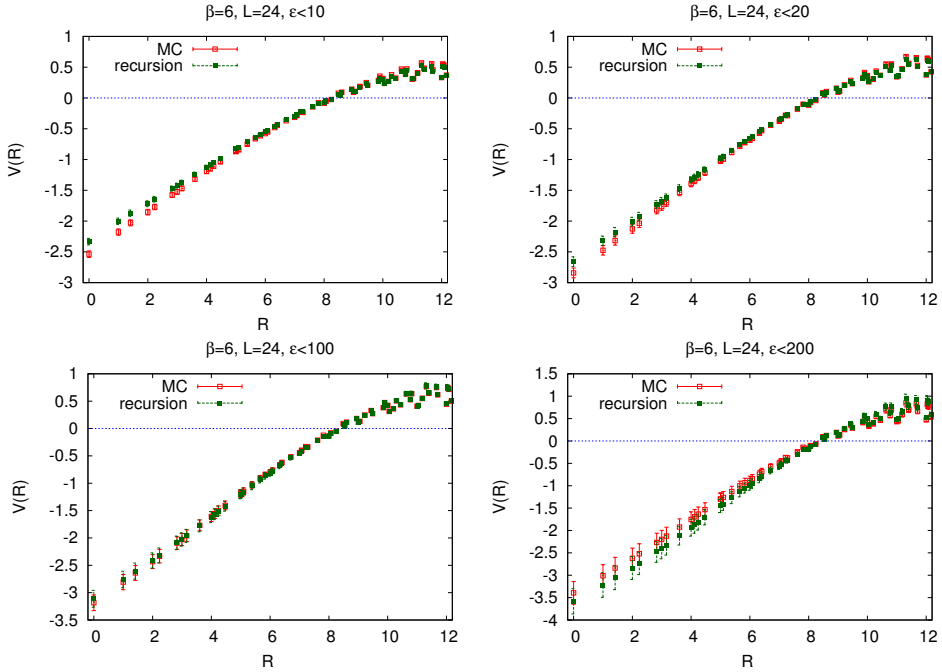


Fig. 26. Coulomb potential evaluated from data sets with cuts at $V(0) < -10$, -20 , -100 , -200 , respectively, at $\beta = 6$ and lattice extension $L = 24$. Results are shown for both MC and recursion lattices.

8. Constituent gluons and the gluon chain model

We have seen that the color Coulomb potential is linear in pure $SU(2)$ gauge theory, but there are (at least) two serious objections to claiming that the Coulomb potential explains confinement. First, the Coulomb string tension seems to be about three times larger than the asymptotic string tension of the static quark potential. Secondly, a Coulombic electric field, which depends on the charge distribution ρ in this way:

$$\vec{E}^L = g \nabla \frac{1}{\nabla \cdot D} \rho \quad (8.1)$$

will result in long-range Coulomb dipole fields, and long-range van der Waals forces among hadrons. This problem is really generic to any model of confinement based on ladder diagrams or (dressed) one-gluon exchange, as found, *e.g.*, in the Dyson–Schwinger approach.

The underlying problem is that Coulomb confinement, while providing a linear potential, does not involve the collimation of the color-electric field into a flux tube. This raises the question, given the existence of a linear Coulomb potential, of how a flux tube forms in Coulomb gauge.

Let us recall that the Coulomb potential is simply the R -dependent part of the energy $V_C(R)$ of the quark–antiquark physical state

$$\Psi_{q\bar{q}} = \bar{q}^a(0)q^a(R)\Psi_0. \quad (8.2)$$

But there is no reason that this state should be the minimal energy state out of all states containing a static quark–antiquark pair of separation R , and it should be possible to construct lower-energy states with the help of additional gluon operators, creating “constituent” gluons. Schematically, we are looking for lower energy states of the form

$$\Psi'_{q\bar{q}} = \bar{q}^a(0)\left\{c_0 + c_1 A^{ab} + c_2 A^{ac}A^{cb} + \dots\right\}q^b(R)\Psi_0. \quad (8.3)$$

The idea that a quark–antiquark pair, as they separate, pull out a chain of gluons between them (Fig. 27), is known as the “gluon-chain model” (*cf.* [57] and references therein, and Ref. [58]). In its original form, it was supposed that as the quark and antiquark separate, the field energy increases faster than linearly, and at some point it is energetically favorable to insert a gluon between the quark–antiquark charges to reduce the effective charge separation. In fact it is now known that the Coulomb energy grows only linearly, but it may still be the case that physical states with widely separated constituent gluons are lower in energy than the corresponding states with no constituent gluons.

One of the motivations of the gluon chain model is that a gluon chain can be regarded as a time-slice of a high-order planar Feynman diagram, as indicated in Fig. 28. The gluon chain also has string-like properties (*e.g.* a Lüscher term) due to fluctuations in the position of its gluon constituents, it features Casimir scaling at large N , and also has the right N -ality properties due to string-breaking [57]. For example, at large N there are two chains between heavy sources in the adjoint representation, giving rise to twice the string tension as in the fundamental representation, which is the correct Casimir ratio at large N . Interaction between the two chains is $1/N^2$ suppressed, but at finite N the chains can interact and rearrange themselves (Fig. 29) into two “gluelumps”, having negligible energy dependence on the quark separation R .

In Euclidean lattice gauge theory, the transfer matrix $\mathcal{T} = \exp[-Ha]$ is the Euclidean time version of the Minkowski space time evolution operator, where a is lattice spacing in the time direction. It is useful to define the

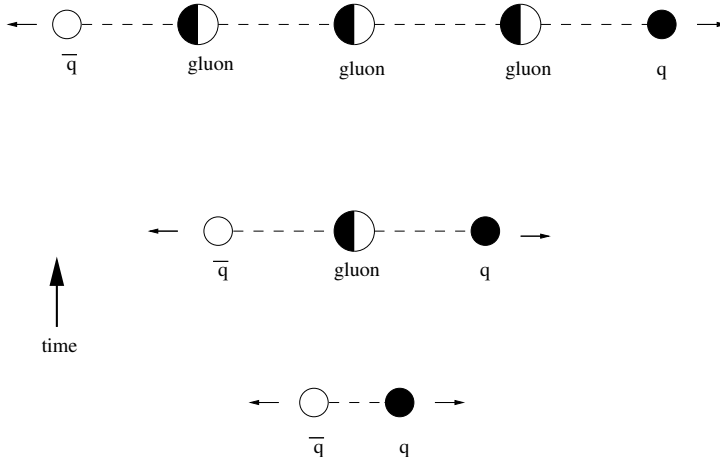


Fig. 27. Formation of a gluon chain, as a quark–antiquark pair separates.

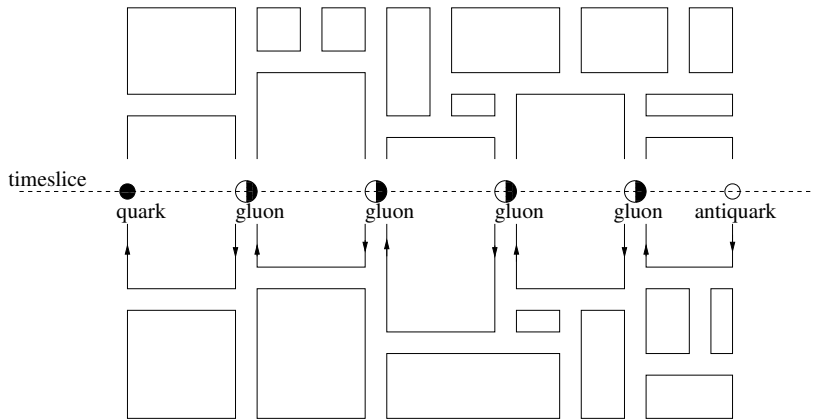


Fig. 28. The gluon chain as a time slice of a planar diagram (shown here in double-line notation). A solid hemisphere indicates a quark color index, open hemisphere an antiquark color index.

rescaled transfer matrix

$$T = \exp[-(H - E_0)a], \quad (8.4)$$

where E_0 is the vacuum energy.

To compute the static quark potential of a quark–antiquark pair separated by a distance R , one would ideally diagonalize the transfer matrix in the infinite-dimensional subspace of states which contain a single massive quark, and a single massive antiquark, located at sites $\mathbf{x}$ and $\mathbf{y}$ with $R = |\mathbf{x} - \mathbf{y}|$. The minimal energy eigenstate of the transfer matrix, in this

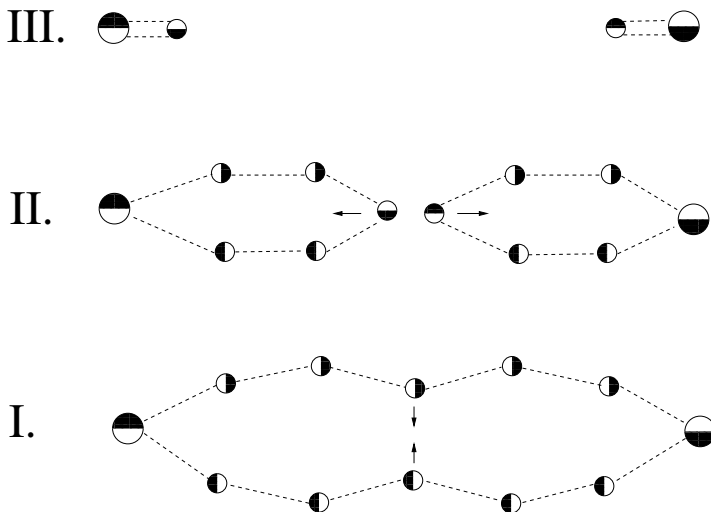


Fig. 29. Adjoint string-breaking in the gluon chain model. Two gluons in separate chains (I) scatter by a contact interaction, resulting in the re-arrangement of color indices indicated in II. This corresponds to chains starting and ending on the same heavy source. The chains then contract down to smaller “gluelumps” (III).

subspace, is the state with the largest eigenvalue $\lambda_{\max}$ of T , and the energy of the state in lattice units is given by

$$V(R) = -\log(\lambda_{\max}) . \quad (8.5)$$

In practice, we seek to diagonalize T in the subspace spanned by a finite number of $q\bar{q}$ states of the form

$$|k\rangle = \psi^a(\mathbf{x}) Q_k^{ab} \psi^b(\mathbf{y}) |\Psi_0\rangle \quad k = 1, 2, \dots, M , \quad (8.6)$$

where the Q_k operators are functionals of the lattice link variables. In general the set of $|k\rangle$ states are not orthonormal, but they can be used to construct an orthonormal set $\{\phi_k\}$ of states via the Gram–Schmidt orthogonalization procedure, and then the matrix elements of the rescaled transfer matrix

$$T_{ij} \equiv \langle \phi_i | T | \phi_j \rangle \quad (8.7)$$

are computed between these orthonormal states. In practice, it is only nec-

essary to compute, via lattice Monte Carlo, the quantities

$$\begin{aligned} O_{mn} &= \langle m|n \rangle \\ &= \left\langle \frac{1}{2} \text{Tr}[Q_m^\dagger(t)Q_n(t)] \right\rangle, \\ t_{mn} &= \langle m|T|n \rangle, \\ &= \left\langle \frac{1}{2} \text{Tr}[Q_m^\dagger(t+1)U_0^\dagger(\mathbf{x},t)Q_n(t)U_0(\mathbf{y},t)] \right\rangle. \end{aligned} \quad (8.8)$$

The matrix elements T_{ij} can be derived from the $\{O_{mn}, t_{mn}\}$, as explained in Ref. [59].

To construct the Q_k constituent gluon operators, define the lattice vector potential (in a lattice gauge fixed to Coulomb gauge)

$$A_k(\mathbf{x}, t) = \frac{1}{2i} \left(U_k(\mathbf{x}, t) - U_k^\dagger(\mathbf{x}, t) \right). \quad (8.9)$$

These potentials are Fourier-transformed, and we suppress the high-momentum components in directions transverse to direction j of the line joining a $q\bar{q}$ pair, *i.e.*

$$\begin{aligned} A_i(\mathbf{k}, t) &\rightarrow \exp \left[-\rho (\mathbf{k}^2 - k_j^2) \right] A_i(\mathbf{k}, t) \\ &\rightarrow \exp \left[-\rho \mathbf{k}_\perp^2 \right] A_i(\mathbf{k}, t), \end{aligned} \quad (8.10)$$

where ρ is a variational parameter. Transform this vector potential back to position space, and denote the resulting “transverse-smoothed” operator as $A_i(\mathbf{x}, t, j)$. This is the A -field operator which is smeared in directions transverse to the direction j . We also define

$$B_i(\mathbf{x}, t) = 1 - \frac{1}{2} \text{Tr}[U_i(\mathbf{x}, t)] \quad (8.11)$$

which is transverse-smoothed in the same way. It is also convenient to define for $i \neq j$, the average

$$\begin{aligned} \bar{A}_i(\mathbf{x}, t, j) &= \frac{1}{2} (A_i(\mathbf{x}, t, j) + A_i(\mathbf{x} - \mathbf{e}_i, t, j)), \\ \bar{B}_i(\mathbf{x}, t, j) &= \frac{1}{2} (B_i(\mathbf{x}, t, j) + B_i(\mathbf{x} - \mathbf{e}_i, t, j)) \end{aligned} \quad (8.12)$$

derived from two links in the i direction which touch the line, in the j direction, running from quark to antiquark.

Then, for an antiquark at site $\mathbf{x}_0$ and a quark at site $\mathbf{x}_0 + R\mathbf{e}_j$, Ref. [59] adopts the following set of Q_k

$$\begin{aligned}
 Q_1(t) &= \mathbb{1}_2, \\
 Q_2(t) &= \sum_{n=0}^{R-1} A_j(\mathbf{x}_0 + n\mathbf{e}_j, t, j), \\
 Q_3(t) &= \sum_{n=-2}^{R+1} \sum_{n'=n}^{R+1} A_j(\mathbf{x}_0 + n\mathbf{e}_j, t, j) A_j(\mathbf{x}_0 + n'\mathbf{e}_j, t, j), \\
 Q_4(t) &= \sum_{n=-2}^{R+2} \sum_{n'=n}^{R+2} \sum_{i \neq j} \bar{A}_i(\mathbf{x}_0 + n\mathbf{e}_j, t, j) \bar{A}_i(\mathbf{x}_0 + n'\mathbf{e}_j, t, j), \\
 Q_5(t) &= \sum_{n=0}^{R-1} B_j(\mathbf{x}_0 + n\mathbf{e}_j, t, 1) \mathbb{1}_2, \\
 Q_6(t) &= \sum_{n=0}^{R-1} \sum_{i \neq j} \bar{B}_i(\mathbf{x}_0 + n\mathbf{e}_j, t, j) \mathbb{1}_2.
 \end{aligned} \tag{8.13}$$

Q_1 is the zero constituent gluon operator, Q_2 is the one gluon operator (one power of A), and the Q_{3-6} are two-gluon operators, containing two powers of the A -field. The corresponding set of states $\{|k\rangle\}$ are not orthogonal, but this is taken care of by the Gram–Schmidt orthogonalization procedure.

For each choice of ρ there is a transfer matrix, and the strategy is to choose the particular value of ρ , at each R , which maximizes the largest eigenvalue $\lambda_{\max}$ of T_{ij} . Denote the corresponding eigenstate

$$|\psi(R)\rangle = \sum_{k=1}^6 a_k(R) |\phi_k\rangle. \tag{8.14}$$

Then a_1^2 is the fraction of the norm of $\psi(R)$ due to the zero (constituent) gluon component $|\phi_1\rangle = |1\rangle$; a_2^2 is the fraction of the norm $\psi(R)$ from the one-gluon component $|\phi_2\rangle \propto |2\rangle$; and $1 - a_1^2 - a_2^2$ is the fraction of the norm due to the remaining two-gluon component states. The energy of this state is

$$V_{\text{chain}}(R) = -\log(\lambda_{\max}) \tag{8.15}$$

which can be compared to the Coulomb (self+interaction) energy

$$V_C(R) = -\log(T_{11}). \tag{8.16}$$

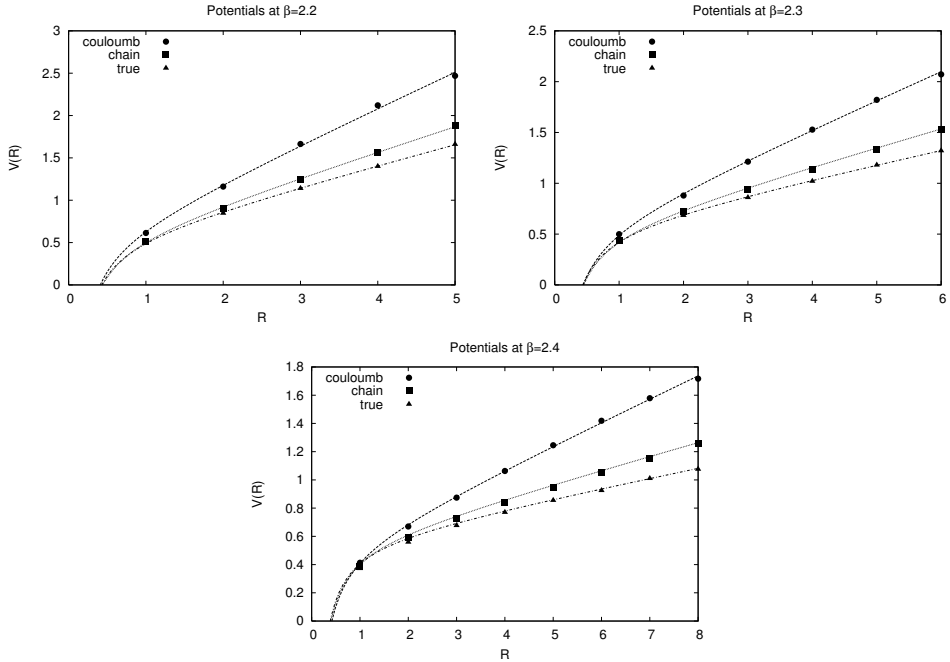


Fig. 30. The color Coulomb potential $V_{\text{Coul}}(R)$, the “gluon-chain” potential $V_{\text{chain}}(R)$ derived from the variational state, and the static quark potential $V_{\text{true}}(R)$ extracted from “fat-link” Wilson loops. Results are shown at lattice couplings $\beta = 2.2, 2.3, 2.4$. Continuous lines are from a fit of data points to Eq. (23).

The resulting potentials, for SU(2) lattice gauge theory at values of $\beta = 2.2, 2.3, 2.4$ are shown in Fig. 30, where the static quark potential, computed by standard methods, is also displayed. Two features worth emphasizing in these figures is that $V_{\text{chain}}(R)$ remains linear, and its slope is much closer to that of the asymptotic string tension, as compared to the Coulomb string tension. Fig. 31 is a comparison of the zero, one, and two constituent-gluon content of the minimal energy state at each R , where by “content” is meant the fraction of the norm of the minimal energy state. The zero gluon content obviously dominates at small $q\bar{q}$ separation. This is of course expected, because placing a gluon between the quark and antiquark comes with a price in kinetic energy. Around $R \approx 1$ fm, however, the zero and one-gluon contributions to the minimal energy state are about equal. It is important to note also that the gluon content *versus* R in physical units is almost coupling independent, which serves as a check of the whole procedure.

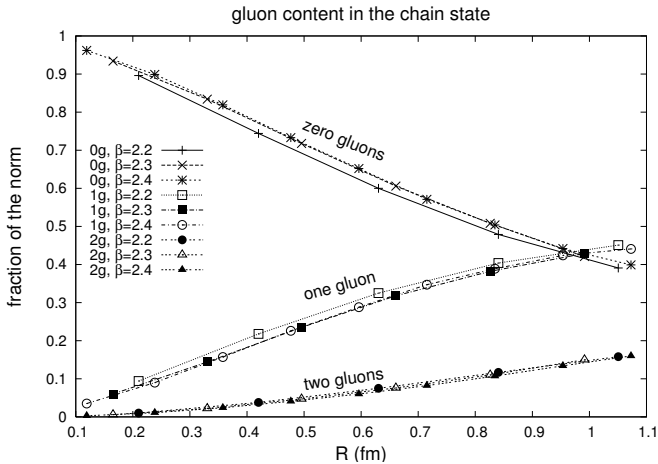


Fig. 31. Zero, one, and two-gluon content (fraction of the norm of the variational state) *versus* quark separation R in fermis, at $\beta = 2.2, 2.3, 2.4$.

As already mentioned, the color Coulomb field is not expected to be collimated into a flux tube. This means that there should be strong sensitivity to lattice volume, on a lattice of spatial extension L , for quark–antiquark separations close to $R = L/2$. The reason is that for separations of that size, the finite volume cuts off a region where the field energy is still significant. If, instead, the field energy were collimated into a flux tube of diameter d ,

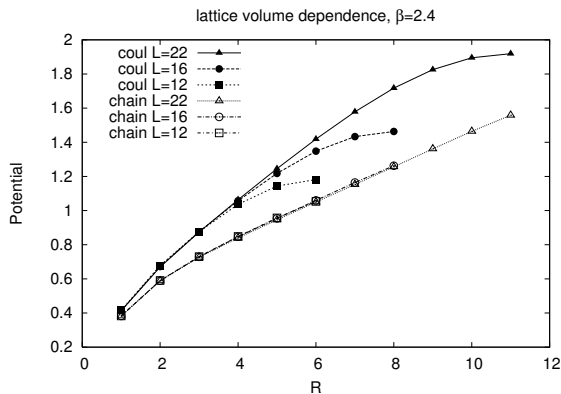


Fig. 32. Sensitivity of the Coulomb potential $V_{\text{Coul}}(R)$ (solid symbols), and insensitivity of the chain potential $V_{\text{chain}}(R)$ (open symbols), to lattice volume. Data is for the gauge coupling $\beta = 2.4$, and lattice volumes $L^4 = 12^4, 16^4, 22^4$. Quark–antiquark separation R is in lattice units.

then so long as $L > d$, there would not be a similar sensitivity to the finite volume. Fig. 32 shows the Coulomb energy $V_C(R)$, and the multi-gluon variational state energy $V_{\text{chain}}(R)$ computed at $\beta = 2.4$, at a variety of lattice sizes. The Coulomb energy shows a strong sensitivity to lattice extension, with the potential flattening out as $R \rightarrow L/2$. In contrast, $V_{\text{chain}}(R)$ appears to be rather insensitive to the lattice size. This suggests that the chain state has no long-range dipole field, or at least that the long-range field is greatly suppressed relative to the color dipole field of the zero-gluon state. It is possible that, in the multi-gluon state, we are beginning to see the formation of a color electric flux tube.

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