# Aspects of non-supersymmetric string theory
# (非超対称なストリング理論の諸側面)

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# Abstract

We investigate some properties of non-supersymmetric string theory which are expected to be applied to phenomenology. We consider the non-supersymmetric string models constructed by orbifolding the toroidal models by $\mathbb{Z}_2$ freely acting twists. The heterotic string models are mainly focused on in this thesis.

We begin with identifying target space duality groups (T-duality groups) in the non-supersymmetric string models by noting that the Narain lattice is split into two subsets in the process of the construction. It is shown that the T-duality groups are congruence subgroups of level 2 in $O(d_L, d_R, \mathbb{Z})$, which are regarded as automorphisms of the two subsets obtained by splitting the Narain lattice. We also point out that the transitions among the non-supersymmetric string models can be induced by acting elements of $O(d_L, d_R, \mathbb{Z})$ which are not included in the congruence subgroup.

Secondly, we study the massless spectra in the nine-dimensional non-supersymmetric heterotic models which depend on the Wilson line and the radius. In particular, we restrict our attention to the unwinding string states and figure out patterns of the gauge symmetry enhancement and massless states in the untwisted and twisted sectors.

We then evaluate the cosmological constant in a particular class of the non-supersymmetric string models and show that the exponential suppression of the cosmological constant can occur if there is Bose-Fermi degeneracy at the massless level. This extremely small cosmological constant is preferable to make a realistic scenario from non-supersymmeric string theory to low-energy physics. We find some configurations of the Wilson line that yield the exponentially suppressed cosmological constant.

Finally, we analyze stability of the Wilson line moduli from the one-loop effective potential. We conclude that the global minima correspond to the maximal symmetry enhancements which lead to the negative cosmological constant. Some of the Wilson lines that realize the suppression of the cosmological constant correspond to the saddle points.

This thesis is based on a series of our work [1–5].

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# Contents

# 1 Introduction

String theory is a promising candidate for a unified description of all the fundamental interactions including gravity. One of the interesting features of string theory is that the dimension of spacetime is required to be ten for superstrings or twenty-six for bosonic strings. This fact is apparently inconsistent with our universe being four-dimensional. However, there are some ways to rescue this situation. The most familiar one is compactifications; six spatial dimensions are supposed to form a compact space of which the volume is sufficiently small so that we cannot observe. Although there is no general principle to determine which compact spaces should be chosen, supersymmetry plays a key role in indicating a policy of the compactifications. It is known that some issues of the Standard Model (e.g. unnaturalness) could be explained by introducing supersymmetry. It is hence favorable to adopt Calabi-Yau manifolds [6] or orbifolds [7, 8] in top-down approaches from string theory so that some supersymmetry is preserved in four-dimensional effective theories. In particular, the compact spaces that lead to $\mathcal{N} = 1$ supersymmetric theories in four dimensions are phenomenologically preferred. For this reason, a lot of top-down scenarios have been considered under the assumption that supersymmetry is completely broken at a somewhat low energy scale. According to recent accelerator experiments, however, supersymmetry has not been found at the multi TeV scale. Based on this fact, it is worth adopting the viewpoint that supersymmetry is already broken at a very high energy scale, e.g., the Planck/string scale, putting the benefits of low-energy supersymmetry aside.

The existence of non-supersymmetric string theories has already been known since the mid-1980's [9–12]. Although most of them have a tachyon in the free-spectra, there were found few tachyon-free models, e.g., the $SO(16) \times SO(16)$ heterotic model even in ten-dimensions [9, 10]. More tachyon-free string models without supersymmetry in general dimensions have been constructed [13, 14] and used for directly building realistic models such as the Standard Models (see e.g. [15, 16]) or exploring the early universe and the cosmic history (see e.g. [17–19]). However, there are some issues in non-supersymmetric string phenomenology even though the models are tachyon-free. One of the serious ones is the vacuum instability; in usual, the vacuum energies (cosmological constants) without supersymmetry are rather large, and the dilaton tadpoles, which are proportional to the vacuum energies, induce the instability [20, 21]. For instance, in the $SO(16) \times SO(16)$ heterotic models six-dimensionally toroidal compactified, the value of the one-loop cosmological constant is order of $M_s^4$, where $M_s$ is the string scale. To address this difficulty, some non-supersymmetric string models with vanishing or small cosmological constants have been proposed (see e.g. [22–33]). In this thesis, in particular, we focus on the interpolating models in which the cosmological constant can be exponentially suppressed. The interpolating models are constructed by a stringy version of the Scherk-Scwharz compactifications [34–38], and hence supersymmetry is broken at the scale of the inverse volume of the internal directions accompanied with the Scherk-Scwharz mechanism. In Ref [39, 40], it is shown that in nine-dimensional interpolating heterotic models, as the (dimensionless) radius $R$ of a circle goes to large, the leading contribution of the one-loop cosmological constant can be evaluated as follows:

$$\Lambda^{(9)}(R) \sim (n_F - n_B)\xi R^{-9} + \mathcal{O}(e^{-R}), \tag{1.1}$$

where $\xi$ is a constant which we will calculate in this thesis, and $n_F$ ($n_B$) is the degrees of freedom of massless fermionic (bosonic) states. Eq. (1.1) implies that the cosmological constants are exponentially suppressed in interpolating models in which Bose-Fermi degeneracy at massless level is realized. Such models, which are often called super no-scale models [41–43], can avoid the problem of instability even without supersymmetry. In fact, the string models with supersymmetry breaking by the Scherk-Schwarz mechanism has been attracting a lot of attention in the context of non-supersymmetric heterotic string phenomenology [42–59].

Duality is a key feature in studies of theoretical physics. In string theory, there are two well-known dualities: S-duality (strong-weak coupling duality) [60, 61] and T-duality (target space duality) [62–65]. A familiar example of S-duality is that between the heterotic $SO(32)$ superstring theory and the type I superstring theory. Although we will mainly focus on the heterotic models in this thesis, the similar analysis in the type I theory has been discussed in [66–71]. As for T-duality, one can see the typical example in bosonic string theory compactified on a circle; strings on $S^1$ with a radius $R$ are equivalent to those on $S^1$ with a radius $\alpha'/R$, where $\alpha'$ is the squared length of a string. This equivalence of two circles cannot be found in theory of point particles, and hence T-duality is a characteristic property in string theory. Such unique symmetries allow us to consider non-geometric backgrounds [72–82] (e.g., asymmetric orbifolds or T-folds), and furthermore field theories that have the T-duality as a manifest symmetry have been proposed [83–85]. Most interest of T-duality has been devoted to compact spaces that preserve some supersymmtry. For instance, it is known that the T-duality group of closed strings compactified on a torus in which supersymmetry are maximally preserved is given as a $O(d_L, d_R, \mathbb{Z})$, where $d_L$ ($d_R$) denotes the degrees of freedom of the left (right) movers propagating in the compact space. In particular, on a two-dimensional torus, T-duality group includes modular symmetries, and the supersymmetric effective field theories that are invariant under a modular transformation have been constructed in [86]. In the context of flavor physics, non-Abelian discrete symmetries (e.g. the modular symmetry), which can be regarded as (parts of) T-dualities, are considered as a candidate for an origin of the flavor symmetry in the Standard Model [87–94]. It has however not been known what kind of structure the T-duality group in the non-supersymmetric models has. In this thesis, we will identify the T-duality groups of the string models in which supersymmetry is completely broken by the Scherk-Schwarz mechanism.

This thesis is organized as follows. In section 2, we review the procedure of the construction of the non-supersymmetric models of which we will studied some properties in the subsequent sections. We also give the partition functions and concrete examples, some of which correspond to interpolating models. In section 3, we identify the T-duality groups of the non-supersymmetric string models by noting the construction introduced in section 2. We also point out that the transitions among the non-supersymmetric models are induced by acting elements of $O(d_L, d_R, \mathbb{Z})$ which are not in the T-duality group. In section 5, we study patterns of the symmetry enhancement which depends on configurations of the Wilson line. As shown in (1.1), the leading behavior of the cosmological constant is controlled only by the massless spectrum. So, in order to find the exponentially suppressed cosmological constants, it is important to understand possible enhancements of the gauge symmetry and massless fermions. In section 6, we devote ourselves to analyze stability of the Wilson line moduli, focusing on the region where supersymmetry is asymptotically restored. In section 7, we summarize this thesis and show some future directions.

# 2 Review of non-supersymmetric string models

In this section, we review the construction of the non-supersymmetric models which were originally proposed in ten-dimensions by Dixon and Harvey in [9], and generalized to arbitrary dimensions by Ginsparg and Vafa in [11].

## 2.1 Construction

The construction of the non-supersymmetric models is done by $\mathbb{Z}_2$ freely acting orbifolding. Note that any moduli cannot be fixed in this construction and hence we can start with string models compactified on any tori. In the toroidal compactifications, it is known that modular invariance requires that the internal momenta live in an even-self dual lattice with Lorentzian signature $(d_L, d_R)$, which is called a Narain lattice [95, 96]. The spectra in the toroidal models are made of the following pairings of the spacetime representations and the internal momentum lattices:

$$\text{Type IIB (IIA) strings:} \ \left(v\bar{v}, s\bar{s}\ (s\bar{c}), v\bar{s}\ (v\bar{c}), s\bar{v}; \Gamma^{d,d}\right), \tag{2.1}$$

$$\text{Heterotic strings:} \ \left(\bar{v}, \bar{s}; \Gamma^{16+d,d}\right), \tag{2.2}$$

where $\Gamma^{d_L,d_R}$ denotes the Narain lattice and $(o, v, s, c)$ represents the conjugacy classes of $SO(8)$ (see appendix A). The $\mathbb{Z}_2$ generator which leads to the non-supersymmetric model is given by $(-1)^F\alpha$, where $F$ is the spacetime fermion number ($F = F_L F_R$ for type II models, $F = F_R$ for heterotic models) and $\alpha$ is a shift of order 2 in the Narain lattice. Note that the state with an internal momentum $P$ yields an eigenvalue $e^{2\pi i P\cdot\delta}$ under $\alpha$, where $\delta$ is a shift-vector such that $2\delta \in \Gamma^{d_L,d_R}$ and the inner product is taken by $\eta = diag\,(\mathbf{1}_{d_L}, -\mathbf{1}_{d_R})$. It is convenient to decompose the Narain lattice $\Gamma^{d_L,d_R}$ into $\Gamma_+^{d_L,d_R}$ and $\Gamma_-^{d_L,d_R}$ depending on the inner products with $\delta$ being even or odd:

$$\Gamma_+^{d_L,d_R} = \left\{ P \in \Gamma^{d_L,d_R} \,\middle|\, \delta \cdot P \in \mathbb{Z} \right\}, \qquad \Gamma_-^{d_L,d_R} = \left\{ P \in \Gamma^{d_L,d_R} \,\middle|\, \delta \cdot P \in \mathbb{Z} + \frac{1}{2} \right\}. \tag{2.3}$$

Then, after modding out by $(-1)^F\alpha$, the spectrum in the untwisted sectors is expressed by the following pairings:

$$\text{Type IIB (IIA) strings:} \ \left(v\bar{v},\ s\bar{s}\ (s\bar{c}); \Gamma_+^{d,d};\right), \left(v\bar{s}\ (v\bar{c}),\ s\bar{v}; \Gamma_-^{d,d}\right), \tag{2.4}$$

$$\text{Heterotic strings:} \ \left(\bar{v}; \Gamma_+^{16+d,d}\right), \left(\bar{s}; \Gamma_-^{16+d,d}\right). \tag{2.5}$$

Modular invariance of the one-loop partition function requires that $\delta^2$ be an integer[1] and the twisted sectors be added. In the twisted sectors, the internal momenta are shifted by $\delta$ and the spacetime representations are identified by the $S$-transformations of the $SO(8)$ characters[2]. For $\delta^2$ odd, the pairings of states in the twisted sectors are

$$\text{Type IIB (IIA) strings:}\ \left(o\bar{o},\ c\bar{c}\ (c\bar{s});\Gamma_-^{d,d}+\delta\right),\left(o\bar{c}\ (o\bar{s}),\ c\bar{o};\Gamma_+^{d,d}+\delta\right), \tag{2.6}$$

$$\text{Heterotic strings:}\ \left(\bar{o};\Gamma_+^{16+d,d}+\delta\right),\left(\bar{c};\Gamma_-^{16+d,d}+\delta\right), \tag{2.7}$$

and for $\delta^2$ even,

$$\text{Type IIB (IIA) strings:}\ \left(o\bar{o},\ c\bar{c}\ (c\bar{s});\Gamma_+^{d,d}+\delta\right),\left(o\bar{c}\ (o\bar{s}),\ c\bar{o};\Gamma_-^{d,d}+\delta\right), \tag{2.8}$$

$$\text{Heterotic strings:}\ \left(\bar{o};\Gamma_-^{16+d,d}+\delta\right),\left(\bar{c};\Gamma_+^{16+d,d}+\delta\right), \tag{2.9}$$

This dependence of the twisted sectors on $\delta^2$ even or odd comes from the requirement of the left-right level-matching condition. We will see the detail in the next subsection and appendix B.

## 2.2 Partition function

Possible Narain lattices are characterized by a set of parameters $\lambda^a$ called moduli. We can introduce the generalized vierbein $\tilde{\mathcal{E}}(\lambda^a)$ of the Narain lattice, which is expressed as a $(d_L+d_R)\times(d_L+d_R)$ matrix. In order for the Narain lattice to be even and self-dual, the Narain metric, which is defined as $J=\tilde{\mathcal{E}}\eta\tilde{\mathcal{E}}^t$, must be an integer matrix with signature $(d_L,d_R)$ of which diagonal components are even and determinant is $\pm 1$. Then, an element $P$ of the Narain lattice is written as

$$P=Z\tilde{\mathcal{E}}(\lambda^a), \tag{2.10}$$

where $Z$ is a $(d_L+d_R)$-dimensional row vector with integer components. Note that the inner product of two elements $P_1=Z_1\tilde{\mathcal{E}}$ and $P_2=Z_2\tilde{\mathcal{E}}$ is independent of the moduli $\lambda^a$:

$$P_1\cdot P_2=Z_1\tilde{\mathcal{E}}(\lambda^a)\eta\tilde{\mathcal{E}}^t(\lambda^a)Z_2^t=Z_1JZ_2^t. \tag{2.11}$$

[1]See appendix B for details.

[2]We will see the details in the next subsection and appendix A .

The one-loop partition function of the toroidal string models with maximal supersymmetry preserved can be written as

$$Z^{T^d}(\lambda^a) = Z_B^{(8-d)} Z_F Z_{\Gamma^{d_L,d_R}}(\lambda^a). \tag{2.12}$$

with the individual contributions defined as follows:

$$Z_B^{(8-d)} = \tau_2^{-\frac{8-d}{2}} (\eta\bar{\eta})^{-(8-d)}, \tag{2.13}$$

$$Z_F = \begin{cases} (V_8 - S_8)\left(\bar{V}_8 - \bar{S}_8\right) \text{ or } (V_8 - S_8)\left(\bar{V}_8 - \bar{C}_8\right) & \text{(type IIB or type IIA strings)} \\ \bar{V}_8 - \bar{S}_8 & \text{(heterotic strings)} \end{cases}, \tag{2.14}$$

$$Z_{\Gamma^{d_L,d_R}} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2}, \quad \text{with } \begin{cases} d_L = d_R = d & \text{(type II strings)} \\ d_L - 16 = d_R = d & \text{(heterotic strings)} \end{cases}, \tag{2.15}$$

where $q = e^{2\pi i\tau}$. Here $\eta(\tau)$ is the Dedekind eta function and $(O_8, V_8, S_8, C_8)$ denotes a set of the $SO(8)$ characters (see appendix A). Note that the partition function is invariant under the rotations $O(d_L,\mathbb{R}) \times O(d_R,\mathbb{R})$ which act on the left- and right-moving momenta individually. Namely, two generalized vierbeins $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}u$ give the same toroidal model if $u \in O(d_L,\mathbb{R}) \times O(d_R,\mathbb{R})$.

As mentioned in the previous subsection, the non-supersymmetric model with the moduli $\lambda^a$ is constructed by orbifolding the toroidal model with the moduli $\lambda^a$ by a $\mathbb{Z}_2$ twist $(-1)^F\alpha$. Since $2\delta$ is in $\Gamma^{d_L,d_R}$, the shift-vector $\delta$ is expressed as

$$\delta = \frac{1}{2}\hat{Z}\tilde{\mathcal{E}}(\lambda^a), \tag{2.16}$$

for a certain integer vector $\hat{Z} \in \mathbb{Z}^{d_L+d_R}$. Recalling that $\delta^2$ is required to be an integer, $\hat{Z}$ must satisfy

$$\hat{Z}J\hat{Z}^t = 0 \ (\text{mod } 4). \tag{2.17}$$

From the definition (2.3) of $\Gamma_{\pm}^{d_L,d_R}$, we find that two choices $\hat{Z}$ and $\hat{Z}'$ give the same splitting of the Narain lattice into $\Gamma_{+}^{d_L,d_R}$ and $\Gamma_{-}^{d_L,d_R}$ if

$$\hat{Z} = \hat{Z}' \ (\text{mod } 2). \tag{2.18}$$

Thus, the choices of $\hat{Z}$ are constrained by (2.17), and the non-supersymmetric models are classified by the inequivalent choices of $\hat{Z}$ with the equivalent relation given by (2.18). We can choose $\hat{Z}$ such that the components take 0 or 1 without loss of generality. It is of course possible to choose $\hat{Z} = \left(0^{d_L}, 0^{d_R}\right)$. But, this choice corresponds to the toroidal compactification, and we will exclude this choice from our consideration in this thesis. It is convenient to denote the shift-vector and the partition function labeled by $\hat{Z}$ such as $\delta_{(\hat{Z})}$ and $Z^{\not{SUSY}}_{(\hat{Z})}$, for the choice of $\hat{Z}$ to be clear. Let us now write down the partition function $Z^{\not{SUSY}}_{(\hat{Z})}$. From (2.4) and (2.5), the contributions from the untwisted sectors are

$$\text{Type IIB strings: } Z_B^{(8-d)} \left\{ \left(V_8\bar{V}_8 + S_8\bar{S}_8\right) Z_{\Gamma_+^{d,d}}(\lambda^a) - \left(V_8\bar{S}_8 + S_8\bar{V}_8\right) Z_{\Gamma_-^{d,d}}(\lambda^a) \right\}, \tag{2.19}$$

$$\text{Heterotic strings: } Z_B^{(8-d)} \left\{ \bar{V}_8 Z_{\Gamma_+^{16+d,d}}(\lambda^a) - \bar{S}_8 Z_{\Gamma_-^{16+d,d}}(\lambda^a) \right\}, \tag{2.20}$$

where $Z_{\Gamma_\pm^{16+d,d}}$ is defined as

$$Z_{\Gamma_\pm^{d_L,d_R}} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma_\pm^{d_L,d_R}} q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} \frac{1 \pm e^{2\pi i\delta\cdot P}}{2} q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}p_R^2}. \tag{2.21}$$

The contributions from the twisted sectors can be obtained by the modular covariance of the partition traces. Note that (2.19) and (2.20) are rewritten as

$$\text{Type IIB strings: } \frac{1}{2} Z_B^{(8-d)} \left\{ \left(V_8 - S_8\right)\left(\bar{V}_8 - \bar{S}_8\right) Z_{\Gamma^{d,d}} + \left(V_8 + S_8\right)\left(\bar{V}_8 + \bar{S}_8\right) Z^{(0,\delta)}_{\Gamma^{d,d}} \right\}, \tag{2.22}$$

$$\text{Heterotic strings: } \frac{1}{2} Z_B^{(8-d)} \left\{ \left(\bar{V}_8 - \bar{S}_8\right) Z_{\Gamma^{16+d,d}} + \left(\bar{V}_8 + \bar{S}_8\right) Z^{(0,\delta)}_{\Gamma^{16+d,d}} \right\}, \tag{2.23}$$

where we omit the argument $\lambda^a$ and define $Z^{(0,\delta)}_{\Gamma^{d_L,d_R}}$ as

$$Z^{(0,\delta)}_{\Gamma^{d_L,d_R}} = Z_{\Gamma_+^{d_L,d_R}} - Z_{\Gamma_-^{d_L,d_R}} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} e^{2\pi i\delta\cdot P} q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2}. \tag{2.24}$$

One can notice that $Z^{(0,\delta)}_{\Gamma^{d_L,d_R}}$ corresponds to the partition trace over the internal momentum space in which the $\mathbb{Z}_2$ operator $\alpha$ is inserted in the time-direction of the world-sheet. Then we can obtain the partition traces of the twisted sectors by performing the $S$-transformation for the second terms in (2.22) and (2.23). By using the formula

$$\sum_{P\in\Gamma^{d_L,d_R}} \delta\left(P' - P\right) = \sum_{P''\in\Gamma^{d_L,d_R}} \exp\left(2\pi i P'\cdot P''\right), \tag{2.25}$$

we find[3]

$$S : \begin{cases} (V_8+S_8)\left(\bar{V}_8+\bar{S}_8\right) Z^{(0,\delta)}_{\Gamma^{d,d}} \to (O_8-C_8)\left(\bar{O}_8-\bar{C}_8\right) Z^{(\delta,0)}_{\Gamma^{d,d}} & \text{(type IIB strings)} \\ \left(\bar{V}_8+\bar{S}_8\right) Z^{(0,\delta)}_{\Gamma^{16+d,d}} \to \left(\bar{O}_8-\bar{C}_8\right) Z^{(\delta,0)}_{\Gamma^{16+d,d}} & \text{(heterotic strings)} \end{cases}, \qquad (2.26)$$

where $Z^{(\delta,0)}_{\Gamma^{d_L,d_R}}$ is defined as

$$Z^{(\delta,0)}_{\Gamma^{d_L,d_R}} = Z_{\Gamma^{d_L,d_R}+\delta} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} q^{\frac{1}{2}(P_L+\delta_L)^2}\bar{q}^{\frac{1}{2}(P_R+\delta_R)^2}. \qquad (2.27)$$

The action of $(-1)^F\alpha$ on the twisted sectors can be determined by requiring that the partition function be invariant under $\tau \to \tau+1$. The following partition traces must be added in order for the partition function to be modular invariant:

$$\text{Type IIB strings: } \mp (O_8+C_8)\left(\bar{O}_8+\bar{C}_8\right) Z^{(\delta,\delta)}_{\Gamma^{d,d}}, \qquad (2.28)$$

$$\text{Heterotic strings: } \pm \left(\bar{O}_8+\bar{C}_8\right) Z^{(\delta,\delta)}_{\Gamma^{16+d,d}}, \qquad (2.29)$$

where $Z^{(\delta,\delta)}_{\Gamma^{d_L,d_R}}$ is defined as

$$Z^{(\delta,\delta)}_{\Gamma^{d_L,d_R}} = Z_{\Gamma_+^{d_L,d_R}+\delta} - Z_{\Gamma_-^{d_L,d_R}+\delta} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} e^{2\pi i\delta\cdot P} q^{\frac{1}{2}(P_L+\delta_L)^2}\bar{q}^{\frac{1}{2}(P_R+\delta_R)^2}. \qquad (2.30)$$

The upper and lower signs of the prefactor in (2.28) and (2.29) are applied for $\delta^2$ odd and $\delta^2$ even respectively, which is required for the invariance under the $T$-transformation (see appendix B for details). As a result, the full partition function of the non-supersymmetric models are written as follows; for type IIB strings

$$\begin{aligned} Z^{\not{SUSY}}_{(\hat{Z})}(\lambda^a) = Z_B^{(8-d)} \Big\{ & \left(V_8\bar{V}_8+S_8\bar{S}_8\right) Z_{\Gamma_+^{d,d}}(\lambda^a) - \left(V_8\bar{S}_8+S_8\bar{V}_8\right) Z_{\Gamma_-^{d,d}}(\lambda^a) \\ & + \left(O_8\bar{O}_8+C_8\bar{C}_8\right) Z_{\Gamma_\mp^{d,d}+\delta}(\lambda^a) - \left(O_8\bar{C}_8+C_8\bar{O}_8\right) Z_{\Gamma_\pm^{d,d}+\delta}(\lambda^a) \Big\}, \end{aligned} \qquad (2.31)$$

and for heterotic strings

$$\begin{aligned} Z^{\not{SUSY}}_{(\hat{Z})}(\lambda^a) = Z_B^{(8-d)} \Big\{ & \bar{V}_8 Z_{\Gamma_+^{16+d,d}}(\lambda^a) - \bar{S}_8 Z_{\Gamma_-^{16+d,d}}(\lambda^a) \\ & + \bar{O}_8 Z_{\Gamma_\pm^{16+d,d}+\delta}(\lambda^a) - \bar{C}_8 Z_{\Gamma_\mp^{16+d,d}+\delta}(\lambda^a) \Big\}. \end{aligned} \qquad (2.32)$$

These partition functions reproduce the free spectra of the non-supersymmetric strings given in the previous subsection.

[3]See appandix A for the $S$-transformation laws of the $SO(8)$ characters.

## 2.3 Ten-dimensional models

As the simplest example, in this subsection, we see the ten-dimensional non-supersymmetric models (i.e. $d = 0$). In the type II models, there is no internal direction, and hence we have one possibility of the $\mathbb{Z}_2$ generator, i.e. $(-1)^F$. Orbifolding by this $\mathbb{Z}_2$ twist gives the well-known ten-dimensional non-supersymmetric models: the type 0B model and the type 0A model, both of which are tachyonic. The partition functions of the type 0B and type 0A models are

$$Z^{(0\mathrm{B})} = Z_B^{(8)} \left( O_8\bar{O}_8 + V_8\bar{V}_8 + S_8\bar{S}_8 + C_8\bar{C}_8 \right), \tag{2.33}$$

$$Z^{(0\mathrm{A})} = Z_B^{(8)} \left( O_8\bar{O}_8 + V_8\bar{V}_8 + S_8\bar{C}_8 + C_8\bar{S}_8 \right). \tag{2.34}$$

In the heterotic models, there are sixteen chiral left-moving bosons of which the momenta live in an even self-dual 16-dimensional lattice with the Euclidean signature, which we denote as $\Gamma^{16}$. It is known that such an even self-dual Euclidean lattice can be realized only if the dimension is the multiple of eight, and there are the two possibilities in sixteen dimensions: the $E_8 \times E_8$ root lattice and the $Spin(32)/\mathbb{Z}_2$ root lattice[4]. Choosing an element $\hat{\pi}$ of $\Gamma^{16}$, we get a shift-vector $\delta = \hat{\pi}/2$. As mentioned in the previous subsection, two choices $\hat{\pi}$ and $\hat{\pi}'$ give the same non-supersymmetric model if $\hat{\pi} = \hat{\pi}' + 2\pi_0$ for ${}^\exists\pi_0 \in \Gamma^{16}$. Furthermore, with $d = 0$ there are no continuous parameters to couple to quantum numbers such as winding numbers, and hence $\hat{\pi}'$ is in the equivalent choice to $\hat{\pi}$ if $\hat{\pi} = \hat{\pi}'$ up to permutations of the components. Considering these constraints for inequivalent choices of $\hat{\pi}$, there are four possible shift-vectors in the case with $\Gamma^{16}$ being the $Spin(32)/\mathbb{Z}_2$ root lattice, which are shown in Table 1. The gauge symmetries and the spectra in the non-supersymmetric models of course depend on the choices of $\hat{\pi}$. With the $E_8 \times E_8$ root lattice, we have three possibilities of the shift-vectors, and Table 2 shows them and the corresponding gauge groups.

From (2.32), the partition functions of the ten-dimensional non-supersymmetric models are written as

$$Z_{(\hat{\pi})}^{\not{SUSY}} = Z_B^{(8)} \left\{ \bar{V}_8 Z_{\Gamma_+^{16}} - \bar{S}_8 Z_{\Gamma_-^{16}} + \bar{O}_8 Z_{\Gamma_\pm^{16}+\delta} - \bar{C}_8 Z_{\Gamma_\mp^{16}+\delta} \right\}, \tag{2.35}$$

where $\Gamma_\pm^{16}$ is defined as

$$\Gamma_+^{16} = \left\{ \pi \in \Gamma^{16} \;\middle|\; \hat{\pi}\cdot\pi \in 2\mathbb{Z} \right\}, \qquad \Gamma_-^{16} = \left\{ \pi \in \Gamma^{16} \;\middle|\; \hat{\pi}\cdot\pi \in 2\mathbb{Z}+1 \right\}. \tag{2.36}$$

[4] We review these lattices in appendix A

| $\delta=\frac{\hat{\pi}}{2}$ | $(1,0^{15})$ | $\left(\left(\frac{1}{2}\right)^4,0^{12}\right)$ | $\left(\left(\frac{1}{4}\right)^{16}\right)$ | $\left(\left(\frac{1}{2}\right)^8,0^8\right)$ |
|---|---|---|---|---|
| gauge sym. | $SO(32)$ | $SO(24)\times SO(8)$ | $SU(16)\times U(1)$ | $SO(16)\times SO(16)$ |

Table 1: The shift-vectors to construct 10D non-supersymmetric heterotic models with the $Spin(32)/\mathbb{Z}_2$ lattice and the gauge symmetries in the models.

| $\delta=\frac{\hat{\pi}}{2}$ | $(1,0^7;0^8)$ | $\left(\left(\frac{1}{2}\right)^2,0^6;\left(\frac{1}{2}\right)^2,0^6\right)$ | $(1,0^7;1,0^7)$ |
|---|---|---|---|
| gauge sym. | $SO(16)\times E_8$ | $(E_7\times SU(2))^2$ | $SO(16)\times SO(16)$ |

Table 2: The shift-vectors to construct 10D non-supersymmetric heterotic models with the $E_8\times E_8$ lattice and the gauge symmetries in the models.

From the partition function (2.35), one can check whether the spectrum includes physical tachyonic states or not. From the expansions of the $SO(8)$ characters and the Dedekind eta function, a tachyonic state can appear only from the pairing $\left(\bar{o};\Gamma^{16}_{\pm}+\delta\right)$ (see appendix A). In this sector, the right-moving excitation starts from the level $-\frac{1}{2}$, while the tower of the left-moving states is made by the Hamiltonian $-1+\frac{1}{2}\left(\pi+\delta\right)^2+N_L$ where $N_L$ runs over non-negative integers, and $\pi\in\Gamma^{16}_{+}$ or $\pi\in\Gamma^{16}_{-}$ depending on $\delta^2$ odd or even. The left-moving vacuum is thus at $-1+\delta^2$ since $2\delta\in\Gamma^{16}$ and $\delta\notin\Gamma^{16}$. Note that the shift-vector can be chosen such that $\delta^2$ is 1 or 2, as shown in Tables 1 and 2. Thus, except for the $SO(16)\times SO(16)$ heterotic models, all the 10D non-supersymmetric models, in which $\delta^2=1$, have physical tachyonic states in the free spectra. In other words, only the $SO(16)\times SO(16)$ heterotic models with $\delta^2=2$ are tachyon-free in ten dimensions.

The non-supersymmetric heterotic models shown in Table 1 and Table 2 were constructed in the bosonic formulation by Dixon and Harvey [9], just as we have done above. One can obtain the same non-supersymmetric models in the fermionic formulation by introducing a discrete torsion [10].

## 2.4 Nine-dimensional models

The type II models compactified on a circle have the internal momenta in $\Gamma^{1,1}$:

$$P=(P_L,P_R)=\frac{1}{\sqrt{2}}\left(nR^{-1}+mR,nR^{-1}-mR\right),\quad m,n\in\mathbb{Z},\tag{2.37}$$

where $R$ is the radius of the circle normalized to be dimensionless by using the string length scale. In the type II models with $d = 1$ there are two inequivalent choices of $\hat{Z} = (\hat{m}, \hat{n})$: $(1, 0)$ and $(0, 1)$. For the choice $(1, 0)$ ($(0, 1)$), $n$ ($m$) is even in $\Gamma_{+}^{1,1}$ while odd in $\Gamma_{-}^{1,1}$ since $\delta \cdot P = n$ ($\delta \cdot P = m$). Then the partition function of the type IIB model with $\hat{Z} = (1, 0)$ is

$$Z_{(1,0)}^{\cancel{SUSY}} = Z_B^{(7)} \left\{ \left(V_8\bar{V}_8 + S_8\bar{S}_8\right) \Lambda_{(1,0)}\left[0|0\right] - \left(V_8\bar{S}_8 + S_8\bar{V}_8\right) \Lambda_{(1,0)}\left[0|1\right] \right. \\ \left. + \left(O_8\bar{O}_8 + C_8\bar{C}_8\right) \Lambda_{(1,0)}\left[1|0\right] - \left(O_8\bar{C}_8 + C_8\bar{O}_8\right) \Lambda_{(1,0)}\left[1|1\right] \right\}, \tag{2.38}$$

where $\Lambda_{(1,0)}\left[\alpha|\beta\right]$ is defined as

$$\Lambda_{(1,0)}\left[\alpha|\beta\right] = \left(\eta\bar{\eta}\right)^{-1} \sum_{m \in \mathbb{Z}+\frac{\alpha}{2}} \sum_{n \in 2\mathbb{Z}+\beta} q^{\frac{1}{4}\left(nR^{-1}+mR\right)^2} \bar{q}^{\frac{1}{4}\left(nR^{-1}-mR\right)^2}. \tag{2.39}$$

Noting that the states with $m \neq 0$ become very massive as $R \to \infty$, one can find

$$\Lambda_{(1,0)}\left[\alpha|\beta\right] \xrightarrow{R\to\infty} \begin{cases} \frac{R}{2}\tau_2^{-\frac{1}{2}}\left(\eta\bar{\eta}\right)^{-1} & \text{for } \alpha = 0 \\ 0 & \text{for } \alpha \neq 0 \end{cases}. \tag{2.40}$$

In the limit $R \to 0$, the states with $n = 0$ only contribute and the behavior of $\Lambda_{(1,0)}\left[\alpha|\beta\right]$ is

$$\Lambda_{(1,0)}\left[\alpha|\beta\right] \xrightarrow{R\to 0} \begin{cases} R^{-1}\tau_2^{-\frac{1}{2}}\left(\eta\bar{\eta}\right)^{-1} & \text{for } \beta = 0 \\ 0 & \text{for } \beta \neq 0 \end{cases}. \tag{2.41}$$

Therefore, from the partition function (2.38), the type IIB model with $\hat{Z} = (1, 0)$ produces the 10D type IIB model and the 10D type 0A model in the endpoint limits $R \to \infty$ and $R \to 0$ respectively. Note that the chirality of the right-moving states is flipped in the limit $R \to 0$ since we perform T-dual to open up the compactified dimension. In the same way, one can check that the type IIB model with $\hat{Z} = (0, 1)$ produces the 10D type 0B model and the 10D type IIA model in the endpoint limits.

In the heterotic models with $d = 1$, the left- and right-moving internal momenta $P_L = (\ell_L, p_L)$ and $P_R = p_R$ are written as

$$\ell_L = \pi - mA, \tag{2.42a}$$

$$p_L = \frac{1}{\sqrt{2}R}\left[\pi \cdot A + m\left(R^2 - \frac{1}{2}|A|^2\right) + n\right], \tag{2.42b}$$

$$p_R = \frac{1}{\sqrt{2}R}\left[\pi \cdot A - m\left(R^2 + \frac{1}{2}|A|^2\right) + n\right], \tag{2.42c}$$

where $A = A_{9I}$ is a Wilson line. Here we take the Narain metric as

$$J = \begin{pmatrix} g_{16} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.43}$$

where $g_{16}$ is a Cartan metric of $\Gamma^{16}$ which is defined as $g_{16} = \alpha_{16}\alpha_{16}^t$ with $\alpha_{16}$ being a set of the basis of $\Gamma^{16}$. Note that the condition (2.17) for $\hat{Z} = (\hat{q}, \hat{m}, \hat{n}) \in \mathbb{Z}^{16+2d}$ is rewritten as

$$|\hat{\pi}|^2 + 2\hat{m}\hat{n} = 0 \mod 4, \qquad (\hat{\pi} = \hat{q}\alpha_{16}). \tag{2.44}$$

We then classify the 9D non-supersymmetric heterotic models into the following four classes depending on the choice of $(\hat{m}, \hat{n})$;

(1) $|\hat{\pi}|^2 = 0 \pmod 4$, $\quad (\hat{m}, \hat{n}) = (0, 0)$;

With this choice, the splitting of the Narain lattice is

$$\Gamma^{17,1}_{\pm} = \left\{ Z\tilde{\mathcal{E}} \,\middle|\, (\pi, m, n) \in \Gamma^{16}_{\pm} \times \mathbb{Z} \times \mathbb{Z} \right\}, \tag{2.45}$$

where $\Gamma^{16}_{\pm}$ is defined as in (2.36) by using $\hat{\pi}$. In the twisted sector, the momenta live in

$$\Gamma^{17,1}_{\pm} + \delta = \left\{ Z\tilde{\mathcal{E}} \,\middle|\, (\pi, m, n) \in \left( \Gamma^{16}_{\pm} + \frac{\hat{\pi}}{2} \right) \times \mathbb{Z} \times \mathbb{Z} \right\}. \tag{2.46}$$

The non-supersymmetric models in this class correspond to the circle compactification of the 10D non-supersymmetric heterotic models which are shown in Table 1 and Table 2. To see this, let us study the behaviors in the endpoint limits. Since the states with $m = 0$ ($n = 0$) only contribute as $R \to \infty$ ($R \to 0$), we find

$$Z_{\Gamma^{17,1}_{\pm}} \to \frac{R}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}_{\pm}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \to \frac{R}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}_{\pm}+\frac{\hat{\pi}}{2}}, \qquad (R \to \infty), \tag{2.47}$$

$$Z_{\Gamma^{17,1}_{\pm}} \to \frac{1}{R\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}_{\pm}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \to \frac{1}{R\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}_{\pm}+\frac{\hat{\pi}}{2}}, \qquad (R \to 0). \tag{2.48}$$

Thus, one can see from (2.35) that both of the endpoint limits in this class give the 10D non-supersymmetric model with $\hat{\pi}$.

(2) $|\hat{\pi}|^2 = 0 \pmod 4$, $\quad (\hat{m}, \hat{n}) = (1, 0)$;

In this class, $\Gamma_{\pm}^{17,1}$ and $\Gamma_{\pm}^{17,1}+\delta$ are expressed as

$$\Gamma_{\pm}^{17,1} = \left\{ Z\tilde{\mathcal{E}} \,\middle|\, (\pi, m, n) \in \left(\Gamma_{\pm}^{16} \times \mathbb{Z} \times 2\mathbb{Z}\right) + \left(\Gamma_{\mp}^{16} \times \mathbb{Z} \times (2\mathbb{Z}+1)\right) \right\}. \tag{2.49}$$

$$\Gamma_{\pm}^{17,1}+\delta = \left\{ Z\tilde{\mathcal{E}} \,\middle|\, (\pi, m, n) \in \left(\left(\Gamma_{\pm}^{16} + \frac{\hat{\pi}}{2}\right) \times \left(\mathbb{Z}+\frac{1}{2}\right) \times 2\mathbb{Z}\right) + \left(\left(\Gamma_{\mp}^{16} + \frac{\hat{\pi}}{2}\right) \times \left(\mathbb{Z}+\frac{1}{2}\right) \times (2\mathbb{Z}+1)\right) \right\}. \tag{2.50}$$

In the endpoint limits, $Z_{\Gamma_{\pm}^{17,1}}$ and $Z_{\Gamma_{\pm}^{17,1}+\delta}$ behave as

$$Z_{\Gamma_{\pm}^{17,1}} \to \frac{R}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}}, \quad Z_{\Gamma_{\pm}^{17,1}+\delta} \to 0, \qquad (R\to\infty), \tag{2.51}$$

$$Z_{\Gamma_{\pm}^{17,1}} \to \frac{1}{R\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma_{\pm}^{16}}, \quad Z_{\Gamma_{\pm}^{17,1}+\delta} \to \frac{1}{R\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma_{\pm}^{16}+\frac{\hat{\pi}}{2}}, \qquad (R\to 0). \tag{2.52}$$

Note that the states with $m=0$ do not live in $\Gamma_{\pm}^{17,1}+\delta$ and hence $Z_{\Gamma_{\pm}^{17,1}+\delta}$ is vanishing as $R\to\infty$. The model in this class reproduces the 10D supersymmetric heterotic model while give the 10D non-supersymmetric model with $\hat{\pi}$. In section 5 and 6, we will focus on the heterotic models in this class and discuss the cosmological constant and the moduli stabilization.

(3) $|\hat{\pi}|^2 = 0 \pmod 4$, $\quad (\hat{m},\hat{n}) = (0,1)$;
In this class, $\Gamma_{\pm}^{17,1}$ and $\Gamma_{\pm}^{17,1}+\delta$ are expressed as

$$\Gamma_{\pm}^{17,1} = \left\{ Z\tilde{\mathcal{E}} \,\middle|\, (\pi, m, n) \in \left(\Gamma_{\pm}^{16} \times 2\mathbb{Z} \times \mathbb{Z}\right) + \left(\Gamma_{\mp}^{16} \times (2\mathbb{Z}+1) \times \mathbb{Z}\right) \right\}, \tag{2.53}$$

$$\Gamma_{\pm}^{17,1}+\delta = \left\{ Z\tilde{\mathcal{E}} \,\middle|\, (\pi, m, n) \in \left(\left(\Gamma_{\pm}^{16} + \frac{\hat{\pi}}{2}\right) \times 2\mathbb{Z} \times \left(\mathbb{Z}+\frac{1}{2}\right)\right) + \left(\left(\Gamma_{\mp}^{16} + \frac{\hat{\pi}}{2}\right) \times (2\mathbb{Z}+1) \times \left(\mathbb{Z}+\frac{1}{2}\right)\right) \right\}. \tag{2.54}$$

The behaviors of $Z_{\Gamma_{\pm}^{17,1}}$ and $Z_{\Gamma_{\pm}^{17,1}+\delta}$ in the endpoint limits are

$$Z_{\Gamma_{\pm}^{17,1}} \to \frac{R}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma_{\pm}^{16}}, \quad Z_{\Gamma_{\pm}^{17,1}+\delta} \to \frac{R}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma_{\pm}^{16}+\frac{\hat{\pi}}{2}}, \qquad (R\to\infty), \tag{2.55}$$

$$Z_{\Gamma_{\pm}^{17,1}} \to \frac{1}{R\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}}, \quad Z_{\Gamma_{\pm}^{17,1}+\delta} \to 0, \qquad (R\to 0). \tag{2.56}$$

The model in this class gives the 10D non-supersymmetric model with $\hat{\pi}$ and the 10D supersymmetric heterotic model in the endpoint limits $R \to \infty$ and $R \to 0$ respectively. The models in class (2) and class (3) are called interpolating models since they interpolate between two different higher-dimensional string vacua.

(4) $|\hat{\pi}|^2 = 2 \pmod 4$, $\quad (\hat{m}, \hat{n}) = (1,1)$;
In this class, $\Gamma^{17,1}_{\pm}$ and $\Gamma^{17,1}_{\pm} + \delta$ are written as

$$\Gamma^{17,1}_{\pm} = \left\{ Z\tilde{\mathcal{E}} \,\middle|\, (\pi, m, n) \in \left(\Gamma^{16}_{\pm} \times \Gamma^{(1)}_{g}\right) + \left(\Gamma^{16}_{\mp} \times \Gamma^{(1)}_{v}\right) \right\}, \tag{2.57}$$

$$\Gamma^{17,1}_{\pm} + \delta = \left\{ Z\tilde{\mathcal{E}} \,\middle|\, (\pi, m, n) \in \left(\left(\Gamma^{16}_{\pm} + \frac{\hat{\pi}}{2}\right) \times \Gamma^{(1)}_{s}\right) + \left(\left(\Gamma^{16}_{\mp} + \frac{\hat{\pi}}{2}\right) \times \Gamma^{(1)}_{c}\right) \right\}, \tag{2.58}$$

where $\Gamma^{(n)}_g$, $\Gamma^{(n)}_v$, $\Gamma^{(n)}_s$ and $\Gamma^{(n)}_c$ are the conjugacy classes of $SO(2n)$ (see appendix A). The behaviors of $Z_{\Gamma^{17,1}_{\pm}}$ and $Z_{\Gamma^{17,1}_{\pm}+\delta}$ in the endpoint limits are

$$Z_{\Gamma^{17,1}_{\pm}} \to \frac{R}{\sqrt{\tau_2}} \left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma^{16}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \to 0, \qquad (R \to \infty), \tag{2.59}$$

$$Z_{\Gamma^{17,1}_{\pm}} \to \frac{1}{R\sqrt{\tau_2}} \left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma^{16}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \to 0, \qquad (R \to 0). \tag{2.60}$$

We find then that supersymmetry is asymptotically restored in both of the endpoint limits, although broken at finite values of $R$. Note that if the models with $A = 0$ in this class have the gauge symmetries with rank 16 that cannot be realized in the 10D non-supersymmetric models as shown in Table 1 and Table 2 since $\hat{\pi}$ must be chosen such that $|\hat{\pi}|^2 = 2$ mod 4.

# 3 Target space duality

From now, we assume $d \geq 1$ in order to discuss the T-duality groups. Modular invariance requires that the Narain lattice be an even self-dual with Lorentzian signature $(d_L, d_R)$. Picking up a Narain lattice with a generalized vierbein $\tilde{\mathcal{E}}_0$, it is known that all the even self-dual lattices with the same Narain metric $J = \tilde{\mathcal{E}}_0 \eta \tilde{\mathcal{E}}_0^t$ can be obtained by acting $\mathcal{E}(\lambda^a) \in O(d_L, d_R, \mathbb{R})$ on $\tilde{\mathcal{E}}_0$ from the left side, where the Lorentz boost $O(d_L, d_R, \mathbb{R})$ is defined in terms of the Narain metric $J$: $\mathcal{E} J \mathcal{E}^t = J$. The deformed Narain lattices have the generalized vierbein $\tilde{\mathcal{E}}(\lambda^a) = \mathcal{E}(\lambda^a)\tilde{\mathcal{E}}_0$. As seen in the previous section, the deformation by $\mathcal{E}(\lambda^a)$ that satisfies $\mathcal{E}\tilde{\mathcal{E}}_0 = \tilde{\mathcal{E}}_0 u$ for $u \in O(d_L, \mathbb{R}) \times O(d_R, \mathbb{R})$ does not change the partition function. So, the moduli space of the toroidal models is locally isomorphic to $O(d_L, d_R, \mathbb{R})/O(d_L, \mathbb{R}) \times O(d_R, \mathbb{R})$ [95, 96]. However, we should notice that the discrete subgroup $O(d_L, d_R, \mathbb{Z}) \subset O(d_L, d_R, \mathbb{R})$ acts on the Narain lattice as an automorphism and keep the toroidal model unchanged. Namely, two moduli $\lambda^a$ and $\lambda'^a$ give the same toroidal model if

$$\mathcal{E}(\lambda'^a)\tilde{\mathcal{E}}_0 = g\mathcal{E}(\lambda^a)\tilde{\mathcal{E}}_0 u \tag{3.1}$$

holds for $u \in O(d_L, \mathbb{R}) \times O(d_R, \mathbb{R})$ and $g \in O(d_L, d_R, \mathbb{Z})$. Therefore, the space of inequivalent Narain lattices is given as $O(d_L, d_R, \mathbb{Z}) \backslash O(d_L, d_R, \mathbb{R})/O(d_L, \mathbb{R}) \times O(d_R, \mathbb{R})$. The discrete subgroup $O(d_L, d_R, \mathbb{Z})$ is called a T-duality group of the toroidal models.

The main goal of this section is to identify T-duality groups of the non-supersymmetric models constructed in the previous section. The question is whether $\lambda^a$ and $\lambda'^a$ give the equivalent non-supersymmetric model whenever they satisfy (3.1), i.e., whether the following proposition is true for any $g \in O(d_L, d_R, \mathbb{Z})$:

$$Z^{T^d}(\lambda^a) = Z^{T^d}(\lambda'^a) \implies Z^{\not{SUSY}}_{(\hat{Z})}(\lambda^a) \stackrel{?}{=} Z^{\not{SUSY}}_{(\hat{Z})}(\lambda'^a). \tag{3.2}$$

Recalling that the partition function $Z^{\not{SUSY}}_{(\hat{Z})}(\lambda^a)$ is obtained from $Z^{T^d}(\lambda^a)$ by splitting the Narain lattice by $\delta(\lambda^a)$, we can easily see that (3.2) does not always hold for any $g \in O(d_L, d_R, \mathbb{Z})$. In order for $Z^{\not{SUSY}}_{(\hat{Z})}$ to be unchanged under the discrete deformations, $g$ must maintain the inner products of any $P \in \Gamma^{d_L, d_R}$ with $\delta$ mod 1:

$$\delta \cdot P = \delta \cdot P' \pmod 1 \quad \text{for any } P \in \Gamma^{d_L, d_R}, \tag{3.3}$$

where $P'$ is the corresponding element of the Narain lattice deformed by $g$. Inserting $P = Z\tilde{\mathcal{E}}$, $P' = Zg\tilde{\mathcal{E}}$ and $\delta = \frac{1}{2}\hat{Z}\tilde{\mathcal{E}}$ into (3.3), we find

$$\hat{Z} = \hat{Z}g \pmod 2. \tag{3.4}$$

For a choice of $\hat{Z}$, let us define a discrete group $D_{(\hat{Z})}(d_L, d_R)$ as

$$D_{(\hat{Z})}(d_L, d_R) = \left\{ g \in O(d_L, d_R, \mathbb{Z}) \,\middle|\, \hat{Z}g = \hat{Z} \ (\text{mod } 2) \right\}. \tag{3.5}$$

Then, $D_{(\hat{Z})}(d_L, d_R)$ corresponds to the T-duality group of the non-supersymmetric model with $\hat{Z}$. Obviously, $D_{(\hat{Z})}(d_L, d_R)$ is a subgroup of $O(d_L, d_R, \mathbb{Z})$ since if $g_1$ and $g_2$ are elements of $D_{(\hat{Z})}(d_L, d_R)$, then the product $g_1 g_2$ is also in $D_{(\hat{Z})}(d_L, d_R)$:

$$\hat{Z} g_1 g_2 \overset{\text{mod } 2}{=} \hat{Z} g_2 \overset{\text{mod } 2}{=} \hat{Z}. \tag{3.6}$$

One can furthermore show that the principal congruence subgroup of level 2 of $O(d_L, d_R, \mathbb{Z})$, which is defined as

$$\Gamma(2) = \left\{ g \in O(d_L, d_R, \mathbb{Z}) \,\middle|\, (g)_{AB} \overset{\text{mod } 2}{=} 1 \text{ for } A = B, \quad (g)_{AB} \overset{\text{mod } 2}{=} 0 \text{ for } A \neq B \right\}, \tag{3.7}$$

is a subgroup of $D_{(\hat{Z})}(d_L, d_R)$. The T-duality group $D_{(\hat{Z})}(d_L, d_R)$ is thus a congruence subgroup of $O(d_L, d_R, \mathbb{Z})$.

We can understand the above result from a different point of view. Let $\lambda'^a$ denote the moduli that are related to $\lambda^a$ by $g$ as in (3.1). The shift-vector $\delta(\lambda'^a)$ with $\hat{Z}$ can be then expressed in terms of $\lambda^a$ by using $g$:

$$\delta(\lambda'^a) = \frac{1}{2} \hat{Z} \tilde{\mathcal{E}}(\lambda'^a) = \frac{1}{2} \hat{Z} g \tilde{\mathcal{E}}(\lambda^a) u, \tag{3.8}$$

where $u \in O(d_L, \mathbb{R}) \times O(d_R, \mathbb{R})$. Recalling that $Z^{T^d}(\lambda^a)$ is invariant under $u$, one can find from (3.8) that the shift-vector $\delta(\lambda'^a)$ with $\hat{Z}$ is equivalent to $\delta(\lambda^a)$ with $\hat{Z}g$. Using $Z^{T^d}(\lambda'^a) = Z^{T^d}(\lambda^a)$, we get

$$Z^{\not{SUSY}}_{(\hat{Z})}(\lambda'^a) = Z^{\not{SUSY}}_{(\hat{Z}g)}(\lambda^a). \tag{3.9}$$

Therefore, in order for the proposition (3.2) to be true, it is required that $\hat{Z}g$ be in an equivalent choice to $\hat{Z}$. From (2.18), a T-duality element $g$ in the non-supersymmetric model with $\hat{Z}$ must satisfy $\hat{Z}g = \hat{Z}$ mod 2. Then we obtain $D_{(\hat{Z})}(d_L, d_R)$ defined in (3.5) as the T-duality group of the non-supersymmetric model with $\hat{Z}$. Eq. (3.9) also implies that acting $g \in O(d_L, d_R, \mathbb{Z})$ not in $D_{(\hat{Z})}(d_L, d_R)$ on the non-supersymmetric model with $\hat{Z}$ gives another non-supersymmetric model with $\hat{Z}g$. Therefore $g \in O(d_L, d_R, \mathbb{Z})$, in general, induces the transitions among the non-supersymmetric models, and the models of which the T-duality groups include $g$ correspond to the fixed points of the transitions.

## 3.1 T-duality in type II models

In the type II models $d$-dimensionally compactified, there are $d \times d$ moduli: a metric $G = ee^t$ of the compactification lattice, an anti-symmetric two-form $B$. Note that these moduli are described by a $d \times d$ matrix $E = G + B$ called a background matrix. The standard choice of a Narain metric in $\Gamma^{d,d}$ is

$$J = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}. \tag{3.10}$$

An element of the Narain lattice in the type II models is then expressed as

$$P = Z\tilde{\mathcal{E}}(e,B) = Z\mathcal{E}(e,B)\tilde{\mathcal{E}}_0, \tag{3.11}$$

where

$$\mathcal{E}(e,B) = \begin{pmatrix} e & Be^{-t} \\ 0 & e^{-t} \end{pmatrix}, \quad \tilde{\mathcal{E}}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_d & -\mathbf{1}_d \\ \mathbf{1}_d & \mathbf{1}_d \end{pmatrix}. \tag{3.12}$$

One can check $\tilde{\mathcal{E}}_0 \eta \tilde{\mathcal{E}}_0^t = J$ and $\mathcal{E} J \mathcal{E}^t = J$ so that $\tilde{\mathcal{E}} \eta \tilde{\mathcal{E}}^t = J$. Using the background matrix $E$, $P = (P_L, P_R)$ is written as

$$P = \frac{1}{\sqrt{2}} \left( n + mE, n - mE^t \right) e^{-t}, \tag{3.13}$$

where $Z = (m,n) = \left( m^1, \cdots, m^d, n_1, \cdots, n_d \right)$. One can easily check that (3.11) with $d = 1$ agrees with (2.37). The free spectrum of a string is given by the Hamiltonian

$$H \sim \frac{1}{2} \left( P_L^2 + P_R^2 \right) = \frac{1}{2} Z \mathcal{M}(E) Z^t, \tag{3.14}$$

where the part of the oscillators is omitted and a $2d \times 2d$ matrix $\mathcal{M}$ is defined as

$$\mathcal{M}(E) = \mathcal{E}(e,B)\mathcal{E}^t(e,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{3.15}$$

The T-duality group $O(d,d,\mathbb{Z})$ of the toroidal models acts on $P$ as follows:

$$P \to P' = Zg\mathcal{E}(e,B)\tilde{\mathcal{E}}_0, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d,\mathbb{Z}), \tag{3.16}$$

where $a,\ b,\ c,\ d$ are $d \times d$ integer matrices that satisfy the following relations:

$$a^t c + c^t a = 0, \quad b^t d + d^t b = 0, \quad a^t d + c^t b = \mathbf{1}_d. \tag{3.17}$$

We see that the $O(d,d,\mathbb{Z})$ transformation is an automorphism of the free spectrum since the Hamiltonian $H$ transforms to $H' = Zg\mathcal{M}(E)g^t Z^t$ which gives another point in the same space of states. The transformation (3.16) can be interpreted as acting on the background matrix $E$ as

$$E \to E' = (aE + b)\,(cE + d)^{-1}\,. \tag{3.18}$$

Let us apply the above discussion about the T-duality group of the non-supersymmetric models to the type II models. The non-supersymmetric models are classified by the possible choices of $\hat{Z} = (\hat{m}, \hat{n})$ with each slot taking 0 or 1 and satisfying $\hat{m}\hat{n}^t = 0 \bmod 2$. The T-duality group (3.5) of the non-supersymmetric type II model with $\hat{Z}$ is

$$D_{(\hat{Z})}(d,d) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d,\mathbb{Z}) \middle| \; (\hat{m}a + \hat{n}c, \hat{m}b + \hat{n}d) = (\hat{m}, \hat{n}) \pmod 2 \right\}. \tag{3.19}$$

### 3.1.1 Specific elements of $D_{(\hat{Z})}(d,d)$

Let us focus on well-known elements of $O(d,d,\mathbb{Z})$ and identify which elements survive in the non-supersymmetric models. We here introduce the following four types of T-duality elements:

- Basis change of the compactification lattice:

$$g_e(K) = \begin{pmatrix} K & 0 \\ 0 & K^{-t} \end{pmatrix}, \qquad (K \in GL\,(d,\mathbb{Z}))\,. \tag{3.20}$$

  In order for $g_e(K)$ to be in $D_{(\hat{Z})}(d,d)$, $K$ needs to satisfy $(\hat{m}K, \hat{n}K^{-t}) = (\hat{m}, \hat{n}) \bmod 2$.

- Integer theta-parameter shift of $B$-field:

$$g_B(\Theta) = \begin{pmatrix} \mathbf{1}_d & \Theta \\ 0 & \mathbf{1}_d \end{pmatrix}, \qquad \left(\Theta^t = -\Theta \in M_{d\times d}(\mathbb{Z})\right). \tag{3.21}$$

  From (3.19), the non-supersymmetric model with $\hat{Z} = (\hat{m}, \hat{n})$ is invariant under the shifts $B_{ij} \to B_{ij} + \Theta_{ij}$ with shift parameters satisfying $\hat{m}\Theta = 0 \bmod 2$.

- Factorized duality and inversion:

$$g_{D_i} = \begin{pmatrix} \mathbf{1}_d - e_i & e_i \\ e_i & \mathbf{1}_d - e_i \end{pmatrix}, \tag{3.22}$$

where $e_i$ is a $d \times d$ matrix whose components are zero, except for the $ii$ one taking 1. The condition for $g_{D_i}$ to be in $D_{(\hat{Z})}(d,d)$ is

$$(\hat{m}, \hat{n}) = (\hat{m} - \hat{m}e_i + \hat{n}e_i, \hat{n} - \hat{n}e_i + \hat{m}e_i) \pmod 2. \tag{3.23}$$

Thus, the non-supersymmetric model with $\hat{Z} = (\hat{m}, \hat{n})$ satisfying $\hat{m}^i = \hat{n}_i$ is invariant under the $i$-th factorized duality $g_{D_i}$. The inversion $g_D$ of the background matrix $E$, which is generated by the products of the factorized dualities,

$$g_D = \prod_{i=1}^{d} g_{D_i} = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \tag{3.24}$$

is a symmetry only in the non-supersymmetric model with $\hat{Z} = \left(1^d, 1^d\right)$.

- Integer theta-parameter shift of dual $B$-field:

$$g_{\tilde{B}}(\tilde{\Theta}) = g_D g_B(\tilde{\Theta}) g_D = \begin{pmatrix} \mathbf{1}_d & 0 \\ \tilde{\Theta} & \mathbf{1}_d \end{pmatrix}, \qquad \left(\tilde{\Theta}^t = -\tilde{\Theta} \in M_{d\times d}(\mathbb{Z})\right). \tag{3.25}$$

The non-supersymmetric model with $\hat{Z} = (\hat{m}, \hat{n})$ is invariant under the shifts with parameters satisfying $\hat{n}\tilde{\Theta} = 0$ mod 2.

The first two elements are called geometric ones. Indeed one can check $\mathcal{E}(e, B) = g_e(e) g_B(B)$, and hence any generalized vierbeins are obtained by starting from $\mathcal{E}(\mathbf{1}_d, 0) = \mathbf{1}_{2d}$ and acting $g_e$ and $g_B$. On the other hand, $g_{D_i}$, $g_D$ and $g_{\tilde{B}}$ are known as non-geometric elements.

The simplest example is the $d = 1$ case which we have introduced in subsection 2.4 and in which there are two possibilities of $\hat{Z}$, i.e., $(1, 0)$ and $(0, 1)$. There is only one non-trivial element in $O(1, 1, \mathbb{Z})$, that is, the factorized duality $g_{D_1}$. But, the factorized duality does not survive in both of the non-supersymmetric models since neither of the choices of $\hat{Z}$ satisfies $\hat{m}^1 = \hat{n}_1$ mod 2. Rather than that, acting $g_{D_1}$ on either of the models induces the transition to the other model. As seen in subsection 2.4, the 9D non-supersymmetric models produce the different 10D models in the limits $R \to \infty$ and $R \to 0$, and $g_{D_1}$ is interpreted as the interchange of the two 10D endpoint models.

| $g_\rho(\gamma_T)$ | $g_\rho(\gamma_S)$ | $g_R$ | $g_{S^2}$ | $g_{D_1}$ |
|---|---|---|---|---|
| $g_{D_2} g_\tau(\gamma_T) g_{D_2}$ | $g_{D_2} g_\tau(\gamma_S) g_{D_2}$ | $g_{D_2} g_W g_{D_2} g_W$ | $g_R g_\tau(\gamma_S)$ | $g_{S^2} g_{D_2} g_{S^2}$ |

Table 3: The elements $g_\tau(\gamma_T)$, $g_\tau(\gamma_S)$, $g_W$ and $g_{D_2}$ generate the T-duality group in the toroidal model. This table lists the products of the generators which give $g_\rho(\gamma)$, $g_R$, $g_{S^2}$ and $g_{D_1}$.

### 3.1.2 Type II models with $d = 2$

One of the simple and interesting examples is the $d = 2$ case. We can change the basis of the moduli space such that the T-duality group $O(2, 2, \mathbb{Z})$ is decomposed into $PSL(2, \mathbb{Z}) \times PSL(2, \mathbb{Z})$. To do this, we define two complex parameters $\tau$ and $\rho$ by combining the four real parameters $G_{11},\ G_{22},\ G_{12},\ B_{12}$ as follows:

$$\tau = \tau_1 + i\tau_2 = \frac{G_{12}}{G_{22}} + i\frac{\sqrt{G}}{G_{22}}, \tag{3.26a}$$

$$\rho = \rho_1 + i\rho_2 = B_{12} + i\sqrt{G}, \tag{3.26b}$$

where $G = G_{11}G_{22} - G_{12}^2$. Then we get the two complex momenta:

$$|P_L| = \frac{1}{\sqrt{2\tau_2\rho_2}} \left|(n_1 - \tau n_2) - \rho\,(m_2 + \tau m_1)\right|, \tag{3.27a}$$

$$|P_R| = \frac{1}{\sqrt{2\tau_2\rho_2}} \left|(n_1 - \tau n_2) - \bar{\rho}\,(m_2 + \tau m_1)\right|. \tag{3.27b}$$

Here, we have given only the absolute values of the momenta because there are the $O(2, \mathbb{R}) \times O(2, \mathbb{R})$ symmetry which is isomorphic to $U(1) \times U(1)$.

In the toroidal models, one can find two modular symmetries which act on the complex structure $\tau$ and the Kähler structure $\rho$ individually.

$$g_\tau(\gamma): \ \ (\tau, \rho) \to \left(\frac{a\tau + b}{c\tau + d}, \rho\right), \tag{3.28}$$

$$g_\rho(\gamma): \ \ (\tau, \rho) \to \left(\tau, \frac{a\rho + b}{c\rho + d}\right), \tag{3.29}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$. Besides the above modular groups, there are some specific duality elements. One of them is the interchange of the complex and Kähler structures,

$$g_{D_2}: \ \ (\tau, \rho) \to (\rho, \tau), \tag{3.30}$$

which corresponds to the factorized duality for the $X^2$-direction. The factorized duality for the $X^1$-direction is realized by the following transformation:

$$g_{D_1}: \ (\tau, \rho) \to \left(-\frac{1}{\rho}, -\frac{1}{\tau}\right). \tag{3.31}$$

The interchange of the basis $X^1 \leftrightarrow X^2$ is given by

$$g_{S^2}: \ (\tau, \rho) \to \left(\frac{1}{\bar{\tau}}, -\bar{\rho}\right). \tag{3.32}$$

The others are the reflection $X_2 \to -X_2$ and the world sheet parity $P_L \leftrightarrow P_R$, which are respectively expressed as the following transformations:

$$g_R: \ (\tau, \rho) \to (-\bar{\tau}, -\bar{\rho}), \qquad g_W: \ (\tau, \rho) \to (\tau, -\bar{\rho}). \tag{3.33}$$

Not all the elements we present above are independent. In fact, we can pick up the four elements $g_\tau(\gamma_T)$, $g_\tau(\gamma_S)$, $g_W$ and $g_{D_2}$ as a minimum set of the generators. Here $\gamma_T$ and $\gamma_S$ are matrices generating a modular group:

$$\gamma_T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \gamma_S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.34}$$

The other elements are obtained by the combinations of the generators. For instance, the modular group (3.29) which acts on $\rho$ is generated by

$$g_\rho(\gamma_T) = g_{D_2} g_\tau(\gamma_T) g_{D_2}, \qquad g_\rho(\gamma_S) = g_{D_2} g_\tau(\gamma_S) g_{D_2}. \tag{3.35}$$

The $\mathbb{Z}_2$ elements $g_R$, $g_{S^2}$ and $g_{D_1}$ can be also expressed as the products of the generators, as shown in Table 3.

The above transformations of $(\tau, \rho)$ can be regarded as those of $Z = (m, n)$. Under $g_\tau(\gamma)$, $g_{D_2}$ and $g_W$, for instance, $Z$ transforms as

$$g_\tau(\gamma): \ Z \to Z M_\tau(\gamma), \tag{3.36}$$

$$g_{D_2}: \ Z \to Z M_{D_2}, \tag{3.37}$$

$$g_W: \ Z \to Z M_W, \tag{3.38}$$

where $M_\tau(\gamma)$, $M_{D_2}$ and $M_W$ are $4 \times 4$ matrices defined as

$$M_\tau(\gamma) = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-t} \end{pmatrix}, \qquad M_{D_2} = \begin{pmatrix} e_1 & e_2 \\ e_2 & e_1 \end{pmatrix}, \qquad M_W = \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}, \tag{3.39}$$

| $(\hat{m},\hat{n})$ | $g_\tau(\gamma)$ | $g_\rho(\gamma)$ | $g_{D_2}$ | $g_{D_1}$ | $g_{S^2}$ |
|---|---|---|---|---|---|
| $(1,0,0,0)$ | $\gamma\in\Gamma^1(2)$ | $\gamma\in\Gamma^1(2)$ | $g_{D_2}$ | — | — |
| $(0,1,0,0)$ | $\gamma\in\Gamma_1(2)$ | $\gamma\in\Gamma^1(2)$ | — | $g_{D_1}$ | — |
| $(0,0,1,0)$ | $\gamma\in\Gamma_1(2)$ | $\gamma\in\Gamma_1(2)$ | $g_{D_2}$ | — | — |
| $(0,0,0,1)$ | $\gamma\in\Gamma^1(2)$ | $\gamma\in\Gamma_1(2)$ | — | $g_{D_1}$ | — |
| $(1,0,0,1)$ | $\gamma\in\Gamma^1(2)$ | $\gamma\in\Gamma_\vartheta$ | — | — | — |
| $(0,1,1,0)$ | $\gamma\in\Gamma_1(2)$ | $\gamma\in\Gamma_\vartheta$ | — | — | — |
| $(1,1,0,0)$ | $\gamma\in\Gamma_\vartheta$ | $\gamma\in\Gamma^1(2)$ | — | — | $g_{S^2}$ |
| $(0,0,1,1)$ | $\gamma\in\Gamma_\vartheta$ | $\gamma\in\Gamma_1(2)$ | — | — | $g_{S^2}$ |
| $(1,1,1,1)$ | $\gamma\in\Gamma_\vartheta$ | $\gamma\in\Gamma_\vartheta$ | $g_{D_2}$ | $g_{D_1}$ | $g_{S^2}$ |

Table 4: The elements of $D_{(\hat{Z})}(2,2)$ which depend on the choice of $\hat{Z}$ are shown.

where $e_1 = diag(1,0)$ and $e_2 = diag(0,1)$. The representation matrices of the other elements are expressed as the products of $M_\tau(\gamma)$, $M_W$ and $M_{D_2}$. As shown in Table 3, for instance, the representation matrices of $g_\rho(\gamma)$ and $g_R$ are given by

$$M_\rho(\gamma) = M_{D_2} M_\tau(\gamma) M_{D_2}, \qquad M_R = M_{D_2} M_W M_{D_2} M_W. \tag{3.40}$$

Let us study the T-duality group $D_{(\hat{Z})}(2,2)$ of the non-supersymmetric model with $\hat{Z}$ on the basis given in (3.26). There are nine possible choices of $\hat{Z}$ with $d=2$: $\hat{Z} = \big(\underline{1,0},0,0\big)$, $\big(0,0,\underline{1,0}\big)$, $(1,0,0,1)$, $(0,1,1,0)$, $(1,1,0,0)$, $(0,0,1,1)$, $(1,1,1,1)$. Here the underline indicates the permutation of the components. We can identify the elements of $D_{(\hat{Z})}(2,2)$ by acting the representation matrix of $g$ on $\hat{Z}$ and checking whether the congruence condition (3.4) is satisfied or not. For the modular group (3.28), $g_\tau(\gamma)$ is in $D_{(\hat{Z})}(2,2)$ if $\gamma$ satisfies

$$(\hat{m},\hat{n}) = \big(\hat{m}\gamma, \hat{n}\gamma^{-t}\big) \pmod 2. \tag{3.41}$$

The other elements of $D_{(\hat{Z})}(2,2)$ can be identified in the same way by using the corresponding representation matrices. Note that the reflection $g_R$ and the world-sheet parity $g_W$ are in $D_{(\hat{Z})}(2,2)$ whatever the choice of $\hat{Z}$ is since the representation matrices are diagonal. The specific elements of $D_{(\hat{Z})}(2,2)$ are shown in Table 4. Here $\Gamma_1(n)$ and $\Gamma^1(n)$ are the Hecke

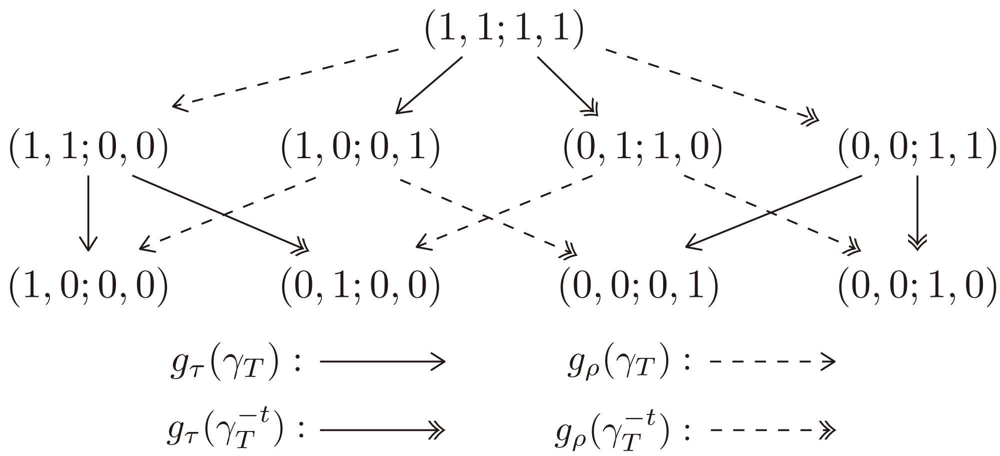


Fig. 1: An example of the transitions among the non-supersymmetric type II models with $d = 2$.

congruence subgroups of the modular group

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \,\middle|\, a, d = 1,\ c = 0 \ (\mathrm{mod}\ n) \right\}, \tag{3.42}$$

$$\Gamma^1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \,\middle|\, a, d = 1,\ b = 0 \ (\mathrm{mod}\ n) \right\}, \tag{3.43}$$

and $\Gamma_\vartheta$ is the theta subgroup

$$\Gamma_\vartheta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \,\middle|\, ac = 0,\ bd = 0 \ (\mathrm{mod}\ 2) \right\}. \tag{3.44}$$

At the end of this example, let us see the transitions among the non-supersymmetric models induced by acting $g$. Focusing on $\gamma_T$ and $\gamma_T^{-t}$, we notice

$$\gamma_T \notin \Gamma^1(2), \qquad \gamma_T^{-t} \notin \Gamma_1(2), \qquad \gamma_T, \gamma_T^{-t} \notin \Gamma_\vartheta. \tag{3.45}$$

For example, starting from the model with $\hat{Z} = (1, 1, 1, 1)$, we can obtain all of the other models by acting on $g_\tau(\gamma_T)$, $g_\tau(\gamma_T^{-t})$, $g_\rho(\gamma_T)$ or $g_\rho(\gamma_T^{-t})$ successively and appropriately (see Fig. 1).

## 3.2 T-duality in the heterotic models

In the heterotic models $d$-dimensionally toroidal compactified, there are $(16+d)\times d$ moduli: a metric $G = ee^t$ of the compactification lattice, an anti-symmetric two-form $B$ and Wilson lines $A$. As in (2.43), we can choose a Narain metric as

$$J = \begin{pmatrix} g_{16} & 0 & 0 \\ 0 & 0 & \mathbf{1}_d \\ 0 & \mathbf{1}_d & 0 \end{pmatrix}, \quad (g_{16} = \alpha_{16}\alpha_{16}^t). \tag{3.46}$$

An internal momentum $P \in \Gamma^{16+d,d}$ is then expressed as

$$P = Z\mathcal{E}(e,B,A)\tilde{\mathcal{E}}_0, \tag{3.47}$$

where $Z = (q,m,n) \in \mathbb{Z}^{16+2d}$, and an element $\mathcal{E}$ of $O(16+d,d,\mathbb{R})$ and the initial generalized vierbein $\tilde{\mathcal{E}}_0$ are given by

$$\mathcal{E}(e,B,A) = \begin{pmatrix} \mathbf{1}_{16} & 0 & \alpha_{16}A^te^{-t} \\ -A\alpha_{16}^{-1} & e & -C^te^{-t} \\ 0 & 0 & e^{-t} \end{pmatrix}, \quad \tilde{\mathcal{E}}_0 = \begin{pmatrix} \alpha_{16} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\mathbf{1}_d & -\frac{1}{\sqrt{2}}\mathbf{1}_d \\ 0 & \frac{1}{\sqrt{2}}\mathbf{1}_d & \frac{1}{\sqrt{2}}\mathbf{1}_d \end{pmatrix}, \quad \left(C = B + \frac{1}{2}AA^t\right). \tag{3.48}$$

One can check that $\tilde{\mathcal{E}}_0$ and $\mathcal{E}$ satisfy $\tilde{\mathcal{E}}_0\eta\tilde{\mathcal{E}}_0^t = J$, $\mathcal{E}J\mathcal{E}^t = J$ and the inner product is independent of the moduli:

$$P_1 \cdot P_2 = P_1\eta P_2^t = Z_1JZ_2^t = \pi_1\pi_2^t + m_1n_2^t + n_1m_2^t. \tag{3.49}$$

Writing down $P = (\ell_L, p_R, p_L)$ explicitly, we get

$$\ell_L = \pi - mA, \tag{3.50a}$$

$$p_L = \frac{1}{\sqrt{2}}\left[\pi A^t + m\left(G - C^t\right) + n\right]e^{-t}, \tag{3.50b}$$

$$p_R = \frac{1}{\sqrt{2}}\left[\pi A^t - m\left(G + C^t\right) + n\right]e^{-t}, \tag{3.50c}$$

where $\pi = q\alpha_{16}$ lives in $\Gamma^{16}$. One can obtain (2.42) from (3.50) with $d=1$. The T-duality element $g \in O(16+d,d,\mathbb{Z})$ of the toroidal models acts on $P$ as

$$P \to P' = Zg\mathcal{E}(e,B,A)\tilde{\mathcal{E}}_0, \tag{3.51}$$

where $g$ is a $(16+2d)\times(16+2d)$ integer matrix that satisfies $gJg^t = J$.

Choosing a certain set of integers $\hat{Z} = (\hat{q}, \hat{m}, \hat{n})$ that satisfies

$$|\hat{\pi}|^2 + 2\hat{m}\hat{n}^t = 0 \pmod 4, \tag{3.52}$$

where $\hat{\pi} = \hat{q}\alpha_{16}$ and $|\hat{\pi}|^2 = \hat{\pi}\hat{\pi}^t$, the shift vector $\delta$ is expressed as

$$\delta = \frac{1}{2}\hat{Z}\mathcal{E}(e, B, A)\tilde{\mathcal{E}}_0. \tag{3.53}$$

The T-duality group of the non-supersymmetric heterotic model with $\hat{Z}$ is

$$D_{(\hat{Z})}(16+d,d) = \left\{ g \in O(16+d,d,\mathbb{Z}) \,\middle|\, \hat{Z} = \hat{Z}g \pmod 2 \right\}. \tag{3.54}$$

### 3.2.1 Specific elements of $D_{(\hat{Z})}(16+d,d)$

Let us see specific elements of $O(16+d,d,\mathbb{Z})$ and identify the congruence conditions which the elements of $D_{(\hat{Z})}(16+d,d)$ must satisfy.

- Basis change of the compactification lattice:

$$g_e(K) = \begin{pmatrix} \mathbf{1}_{16} & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K^{-t} \end{pmatrix}, \qquad (K \in GL(d,\mathbb{Z})). \tag{3.55}$$

  The elements $g_K$ of $D_{(\hat{Z})}(16+d,d)$ must satisfy $(\hat{m}K, \hat{n}K^{-t}) = (\hat{m}, \hat{n})$ mod 2.

- Basis change of the gauge lattice:

$$g_{\alpha_{16}}(W) = \begin{pmatrix} \alpha_{16}W\alpha_{16}^{-1} & 0 & 0 \\ 0 & \mathbf{1}_d & 0 \\ 0 & 0 & \mathbf{1}_d \end{pmatrix}, \qquad (W \in O(16,\mathbb{Z})). \tag{3.56}$$

  By acting $g_{\alpha_{16}}(W)$ on $\mathcal{E}$, one can check that the Wilson lines transform as $A \to AW^t$ while $G$ and $B$ are unchanged. Acting $g_{\alpha_{16}}(W)$ on $Z$ leads to a change of the basis of $\Gamma^{16}$ as $\pi \to \pi W$ accompanied with the $O(16+d)\times O(d)$ rotation $(\ell_L, p_L, p_R) \to (\ell_L W^t, p_L, p_R)$. The condition for $g_{\alpha^{16}}(W)$ to be in $D_{(\hat{Z})}(16+d,d)$ is $\hat{\pi}W = \hat{\pi} + 2\pi_0$ for $^\exists\pi_0 \in \Gamma^{16}$.

- Integer theta-parameter shift of $B$-field:

$$g_B(\Theta) = \begin{pmatrix} \mathbf{1}_{16} & 0 & 0 \\ 0 & \mathbf{1}_d & \Theta \\ 0 & 0 & \mathbf{1}_d \end{pmatrix}, \qquad \left(\Theta^t = -\Theta \in M_{d\times d}(\mathbb{Z})\right). \tag{3.57}$$

If a shift parameter $\Theta$ satisfies $\hat{m}\Theta = 0$ mod 2 then $g_B(\Theta)$ is an element of $D_{(\hat{Z})}(16 + d, d)$.

- Wilson line shift:

$$g_A(a) = \begin{pmatrix} \mathbf{1}_{16} & 0 & g_{16}a^t \\ -a & \mathbf{1}_d & -\frac{1}{2}ag_{16}a^t \\ 0 & 0 & \mathbf{1}_d \end{pmatrix}, \qquad (a \in M_{d\times 16}(\mathbb{Z}))\,. \tag{3.58}$$

Under $g_A(a)$, the Wilson lines $A$ and the two-form $B$ are shifted as

$$A \to A + \pi_a, \qquad B \to B + \frac{1}{2}\left(A\pi_a^t - \pi_a A^t\right), \tag{3.59}$$

where $\pi_a = a\alpha_{16}$. The elements $g_A(a)$ of $D_{(\hat{Z})}(16 + d, d)$ must satisfy both of the following conditions:

$$\hat{m}a = 0 \ (\text{mod } 2), \quad \left(\hat{\pi} - \frac{1}{2}\hat{m}\pi_a\right)\pi_a^t = 0 \ (\text{mod } 2). \tag{3.60}$$

- Factorized duality and inversion:

$$g_{D_i} = \begin{pmatrix} \mathbf{1}_{16} & 0 & 0 \\ 0 & \mathbf{1}_d - e_i & e_i \\ 0 & e_i & \mathbf{1}_d - e_i \end{pmatrix}, \tag{3.61}$$

The non-supersymmetric models with $\hat{Z}$ satisfying $\hat{m}^i = \hat{n}_i$ have the $i$-th factorized duality symmetry $g_{D_i}$. The inversion $g_D$, which is expressed as

$$g_D = \prod_{i=1}^{d} g_{D_i} = \begin{pmatrix} \mathbf{1}_{16} & 0 & 0 \\ 0 & 0 & \mathbf{1}_d \\ 0 & \mathbf{1}_d & 0 \end{pmatrix}, \tag{3.62}$$

is an element of $D_{(\hat{Z})}(16 + d, d)$ with $\hat{Z}$ satisfying $\hat{m} = \hat{n}$ for all directions.

- Integer theta-parameter shift of dual $B$-field:

$$g_{\tilde{B}}(\tilde{\Theta}) = g_D g_B(\tilde{\Theta}) g_D = \begin{pmatrix} \mathbf{1}_{16} & 0 & 0 \\ 0 & \mathbf{1}_d & 0 \\ 0 & \tilde{\Theta} & \mathbf{1}_d \end{pmatrix}, \qquad \left(\tilde{\Theta}^t = -\tilde{\Theta} \in M_{d\times d}(\mathbb{Z})\right). \tag{3.63}$$

If a shift parameter $\tilde{\Theta}$ satisfies $\hat{n}\tilde{\Theta} = 0$ mod 2, then $g_{\tilde{B}}(\tilde{\Theta})$ is an element of $D_{(\hat{Z})}(16+d, d)$.

- dual Wilson line shift:

$$g_{\tilde{A}}(\tilde{a}) = g_D g_A(\tilde{a}) g_D = \begin{pmatrix} \mathbf{1}_{16} & g_{16}\tilde{a}^t & 0 \\ 0 & \mathbf{1}_d & 0 \\ -\tilde{a} & -\frac{1}{2}\tilde{a}g_{16}\tilde{a}^t & \mathbf{1}_d \end{pmatrix}, \qquad (\tilde{a} \in M_{d\times 16}(\mathbb{Z}))\,. \tag{3.64}$$

The elements $g_{\tilde{A}}(\tilde{a})$ of $D_{(\hat{Z})}(16+d, d)$ must satisfy both of the following conditions:

$$\hat{n}\tilde{a} = 0 \text{ (mod 2)}, \quad \left(\hat{\pi} - \frac{1}{2}\hat{n}\pi_{\tilde{a}}\right)\pi_{\tilde{a}}^t = 0 \text{ (mod 2)}. \tag{3.65}$$

The first four elements are geometric ones and the last three elements are non-geometric ones. Indeed, one can check $\mathcal{E}(e, B, A) = g_A(A\alpha_{16}^{-1}) g_B(B) g_e(e)$ from (3.48) as in the type II models.

### 3.2.2 Heterotic models with $d = 1$

Unlike in the type II models, there are a lot of T-duality elements in the heterotic models even if $d = 1$. In subsection 2.4, we classified the 9D non-supersymmetric models into the four classes by the possible choices of $(\hat{m}, \hat{n})$. With $d = 1$ there is no anti-symmetric two form $B$, and then no degrees of freedom to make $g_B$ and $g_{\tilde{B}}$. For the basis change of the compactification lattice, $K$ can only be $\pm 1$, and hence $g_e(K)$ is in the T-duality group $D_{(\hat{Z})}(17, 1)$ for any choices of $\hat{Z}$. The basis change of the gauge lattice $g_{\alpha_{16}}(W)$ acts only on $\hat{\pi}$ and does not change $\hat{m}$ and $\hat{n}$. So, $g_{\alpha_{16}}(W)$ cannot induce the transitions among the non-supersymmetric models in the different classes. Let us now study the (dual) Wilson line shift $g_A(a)$ ($g_{\tilde{A}}(\tilde{a})$) and the inversion $g_D$ for each of the classes.

- class (1): $|\hat{\pi}|^2 = 0$ (mod 4), $\quad (\hat{m}, \hat{n}) = (0, 0)$

  The Wilson line shift parameter $a$ must satisfy the condition (3.60). In this class,

| | class (1) | class (2) | class (3) | class (4) |
|---|---|---|---|---|
| $g_A(a)$ | $\pi_a \in \Gamma^{16}_+(\hat{\pi})$ | $\pi_a \in 2\Gamma^{16}$ | $\pi_a \in \Gamma^{16}_+(\hat{\pi})$ | $\pi_a \in 2\Gamma^{16}$ |
| $g_{\tilde{A}}(\tilde{a})$ | $\pi_{\tilde{a}} \in \Gamma^{16}_+(\hat{\pi})$ | $\pi_{\tilde{a}} \in \Gamma^{16}_+(\hat{\pi})$ | $\pi_{\tilde{a}} \in 2\Gamma^{16}$ | $\pi_{\tilde{a}} \in 2\Gamma^{16}$ |
| $g_D$ | $g_D$ | — | — | $g_D$ |

Table 5: The (dual) Wilson line shift and the inversion in $D_{(\hat{Z})}(17,1)$.

the first condition in (3.60) is always satisfied, and the second one requires $\pi_a \in \Gamma^{16}_+$. For the dual Wilson line shift parameter $\tilde{a}$, we obtain the same requirement $\pi_{\tilde{a}} \in \Gamma^{16}_+$ from (3.65). The non-supersymmetric models in this class obviously have the inversion duality as $\hat{m} = \hat{n}$.

- class (2): $|\hat{\pi}|^2 = 0 \pmod 4$, $\quad (\hat{m}, \hat{n}) = (1, 0)$
  With this choice of $\hat{Z}$, the condition (3.60) for the Wilson line shift parameter means $\pi_a \in 2\Gamma^{16}$. Note that $a$ satisfying the first condition in (3.60) is sufficient for the second one. On the other hand, for the dual Wilson line shift, (3.65) indicates $\pi_{\tilde{a}} \in \Gamma^{16}_+$, which comes from the second condition. The inversion cannot be a duality in this class as $\hat{m} \neq \hat{n}$.

- class (3): $|\hat{\pi}|^2 = 0 \pmod 4$, $\quad (\hat{m}, \hat{n}) = (0, 1)$
  The situation in this class is the same in class (2) with the interchange of $\hat{m}$ and $\hat{n}$. Then, the shift parameters $a$ and $\tilde{a}$ must satisfy $\pi_a \in \Gamma^{16}_+$ and $\pi_{\tilde{a}} \in 2\Gamma^{16}$, and $g_D$ is not in $D_{(\hat{Z})}(17,1)$.

- class (4): $|\hat{\pi}|^2 = 2 \pmod 4$, $\quad (\hat{m}, \hat{n}) = (1, 1)$
  The conditions (3.60) is the same as in class (2) which requires $\pi_a \in 2\Gamma^{16}$, while (3.65) is the same as in class (3) which requires $\pi_{\tilde{a}} \in 2\Gamma^{16}$. The non-supersymmetric models in this class are clearly invariant under $g_D$.

Note that class (1) and class (4) having the inversion duality is consistent with both of the endpoint limits being the same, as we have seen in subsection 2.4. Table 5 summarizes the (dual) Wilson line shift dualities and the inversion duality in each of the four classes. Let us next see the transitions among the 9D non-supersymmetric heterotic models induced by elements of $O(17, 1, \mathbb{Z})$. As mentioned before, $g_e$, $g_B$ and $g_{\alpha_{16}}$ cannot realize the transitions among the different classes. So, we focus on the three elements $g_A(a)$, $g_{\tilde{A}}(\tilde{a})$ and $g_D$. To see

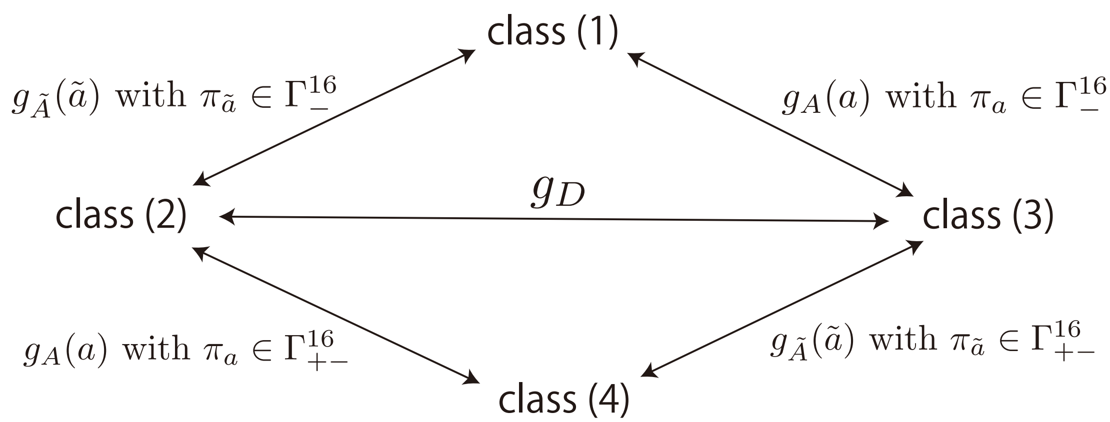


Fig. 2: An example of the transitions among the non-supersymmetric heterotic models with $d = 1$.

the transitions induced by the (dual) Wilson line shifts, we introduce $\Gamma^{16}_{+-}(\hat{\pi})$ defined as the following subset of $\Gamma^{16}$:

$$\Gamma^{16}_{+-}(\hat{\pi}) = \left\{\pi \in \Gamma^{16}_{+} \,\middle|\, |\pi|^2 = 2 \pmod 4\right\} + \left\{\pi \in \Gamma^{16}_{-} \,\middle|\, |\pi|^2 = 0 \pmod 4\right\}. \qquad (3.66)$$

Note that $\hat{\pi} \cdot \pi - |\pi|^2/2$ is always odd for $\pi \in \Gamma^{16}_{+-}(\hat{\pi})$. One can see that the Wilson line shifts $g_A(a)$ with $\pi_a \in \Gamma^{16}_{+}$ induce the transitions between class (1) and class (3), while those with $\pi_a \in \Gamma^{16}_{+-}$ induce the transitions between class (2) and class (4) occur. On the other hand, for the dual Wilson line shifts $g_{\tilde{A}}(\tilde{a})$, the transitions between class (1) and class (2) are realized by $\pi_{\tilde{a}} \in \Gamma^{16}_{+}$, while one can obtain the transitions between class (3) and class (4) by acting $\pi_{\tilde{a}} \in \Gamma^{16}_{+-}$ . The transitions between class (2) and class (3) are realized by the inversion $g_D$. Fig. 2 shows the transitions among the different classes induced by $g_A(a)$, $g_{\tilde{A}}(\tilde{a})$ and $g_D$.

# 4 Massless spectrum and symmetry enhancement

In the remaining sections, we focus on the 9D non-supersymmetric heterotic models which are classified into the four classes in subsection 2.4. We assume henceforth that $\hat{Z}$ is chosen such that the gauge twist is non-trivial (i.e. $\hat{\pi} \notin 2\Gamma^{16}$), which means that neither $\Gamma^{16}_{+}$ nor $\Gamma^{16}_{-}$ are empty.

In this section, we consider the massless spectra in the non-supersymmetric heterotic strings and clarify patterns of the gauge symmetry enhancement depending on $R$ and $A$[5]. In particular, we will pay much attention to the region in the moduli space where $R$ is approaching either of the endpoint models. This analysis can be used for studying the cosmological constant and the stability of the Wilson line moduli, which we will discuss in the subsequent sections.

## 4.1 Untwisted sector

In heterotic models one-dimensionally compactified, the left- and right-moving mass formulae in the untwisted sectors are given by

$$M_L^2 = |\ell_L|^2 + p_L^2 + 2\,(N_L - 1)\,, \tag{4.1a}$$

$$M_R^2 = p_R^2 + 2\,(N_R - a_R)\,, \tag{4.1b}$$

where $a_R = 1/2$ for NS-sector and $a_R = 0$ for R-sector. We have the two possibilities to get the massless states. One of them comes from the states that satisfy

$$N_L = 1, \quad N_R = a_R, \quad \pi = n = 0. \tag{4.2}$$

We call a set of the massless states satisfying (4.2) sector 1. In NS-sector, sector 1 is consist of a gravity multiplet (a graviton, an antisymmetric two-form and dilaton) and gauge bosons of $U(1)^{16}_L \times U(1)_l \times U(1)_r$. Here, we denote $U(1)^{16}_L$ as an Abelien gauge group which comes from the excitations by $\alpha^I_{-1}\tilde{b}^\mu_{-1/2}$, while $U(1)_l \times U(1)_r$ comes from those by $\alpha^9_{-1}\tilde{b}^\mu_{-1/2}$ and $\alpha^\mu_{-1}\tilde{b}^9_{-1/2}$, where $\alpha_{-n}$ and $\tilde{b}_{-s}$ are the ocsillation modes of left-moving bosons and right-moving fermions in NS-sector, and $I$ and $\mu$ denote the sixteen internal indices and the spacetime indices. We

[5]From the viewpoint of the Higgs mechanism, it is appropriate to express symmetry "breaking" rather than symmetry "enhancement". But, in this thesis, we regard the Abelian gauge group as a reference point and interpret the non- Abelian gauge groups as being enhanced at special points in the moduli space.

obtain their fermionic superpartners from R-sector if supersymmetry is preserved. Note that the conditions (4.2) are independent of the moduli, and hence there are always the massless states in sector 1 at any points in the moduli space.

The other possibility, which we call sector 2, arises from the states that satisfy

$$N_L = 0, \quad N_R = a_R, \quad |\ell_L|^2 + p_L^2 = 2, \quad p_R^2 = 0. \tag{4.3}$$

Inserting (2.42) into the last two conditions, we find

$$n = m\left(R^2 + \frac{1}{2}|A|^2\right) - \pi \cdot A, \quad |\pi - mA|^2 + 2m^2R^2 = 2. \tag{4.4}$$

We should note that the conditions (4.4) can be written as

$$m = n\left(\tilde{R}^2 + \frac{1}{2}|\tilde{A}|^2\right) - \pi \cdot \tilde{A}, \quad \left|\pi - n\tilde{A}\right|^2 + 2n^2\tilde{R}^2 = 2, \tag{4.5}$$

where $\tilde{R}$ and $\tilde{A}$ are the dual radius and the dual Wilson line:

$$\tilde{R} = \frac{R}{R^2 + \frac{1}{2}|A|^2}, \quad \tilde{A} = -\frac{A}{R^2 + \frac{1}{2}|A|^2}. \tag{4.6}$$

In fact, one can check that acting the inversion $g_D$ on the generalized vierbein (3.48) with $d = 1$ gives the transformations $R \to \tilde{R}$ and $A \to \tilde{A}$ accompanied with an appropriate $O(17,\mathbb{R}) \times O(1,\mathbb{R})$ rotation. The massless states in sector 2 correspond to the gauge bosons with non-zero roots of a semisimple group, and hence the gauge symmetry is enhanced if massless states in sector 2 exist. We can get the massless states in sector 2 only when $A$ and $R$ satisfy (4.4) for $n$, $m$ and $\pi$. So, the gauge symmetry is broken to $U(1)_L^{16} \times U(1)_l \times U(1)_r$ at generic points in the moduli space.

Let us first focus on the enhancement $U(1)_l \to SU(2)$. For simplicity, we assume that the Wilson line $A$ takes a generic value so that the conditions (4.4) can be satisfied only for the states with $\pi = 0$, for which (4.4) is written as

$$n = m\left(R^2 + \frac{1}{2}|A|^2\right), \quad 2mn = 2. \tag{4.7}$$

These conditions lead to $m = n = \pm 1$ and $R^2 + \frac{1}{2}|A|^2 = 1$. The latter implies $R = \tilde{R}$ and $A = -\tilde{A}$, that is, the fixed points under the inversion $g_D$. Focusing on the structures of $\Gamma_+^{17,1}$ in each class, we find that spacetime vectors with $\pi = 0$ and $m = n = \pm 1$ exist only in class (1) and class (4). Namely, the enhancement $U(1)_l \to SU(2)$ can occur at the $g_D$-fixed points in the moduli space of the non-supersymmetric models in class (1) and class (4). This reflects the result in Table 5 which shows that only class (1) and class (4) are invariant under the inversion $g_D$.

Let us next study the symmetry enhancements of $U(1)_L^{16}$. We henceforth focus on only the states with $m = 0$ or $n = 0$ since we are interested in the region near either of the endpoint models where $R$ is large or small. In particular, we will focus on the region with supersymmetry being asymptotically restored in the subsequent sections, which is possible in class (2), class (3) and class (4) as seen in subsection 2.4. The condition (4.4) for the states with $m = 0$ is

$$n = -\pi \cdot A, \qquad |\pi|^2 = 2. \tag{4.8}$$

Note that the second condition implies that the massless states correspond to nonzero roots of simisimple subgroup $g' \subset g$ with $g$ being $SO(32)$ or $E_8 \times E_8$. For the states with $n = 0$, it is useful to adopt the dual description (4.6). From (4.5),

$$m = -\pi \cdot \tilde{A}, \qquad |\pi|^2 = 2. \tag{4.9}$$

In the rest of this subsection, we will pay our attention to the states with $m = 0$. The same discussion can be done for the states with $n = 0$ since the condition (4.9) is the same form as (4.8). We shall study not only the case of the non-supersymmetric models but also of the toroidal model, in order to clarify the difference between them. So, let us first focus on the toroidal model in which $n \in \mathbb{Z}$, $m \in \mathbb{Z}$ and $\pi \in \Gamma^{16}$ for both NS- and R-sector. Let $\Delta_{g'}$ denote a set of the nonzero roots of a semisimple subgroup $g' \subset g$. In particular, for $g' = g$,

$$\Delta_{SO(32)} = \left\{ \left( \underline{\pm, \pm, 0^{14}} \right) \right\}, \tag{4.10}$$

$$\begin{aligned} \Delta_{E_8\times E_8} = &\left\{ \left( \underline{\pm, \pm, 0^6}; 0^8 \right), \frac{1}{2} \left( \underline{\pm, \pm, \pm, \pm, \pm, \pm, \pm, \pm}_{+}; 0^8 \right) \right\} \\ &+ \left\{ \left( 0^8; \underline{\pm, \pm, 0^6} \right), \frac{1}{2} \left( 0^8; \underline{\pm, \pm, \pm, \pm, \pm, \pm, \pm, \pm}_{+} \right) \right\}, \end{aligned} \tag{4.11}$$

where the underline indicates permutations of the components and the subscript + denotes the number of + is even. The condition (4.8) means that a non-Abelian part of the gauge group is $g'$ if $A = A^{(g')}$ with $A^{(g')}$ satisfying the following conditions[6]:

$$\begin{cases} \pi \cdot A^{(g')} \in \boldsymbol{Z} & \text{for } \pi \in \Delta_{g'} \\ \pi \cdot A^{(g')} \notin \boldsymbol{Z} & \text{for } \pi \in \Delta_g \backslash \Delta_{g'} \end{cases}. \tag{4.12}$$

[6]The rank of $g'$ is not always 16. It is of course possible that some of $U(1)_L$'s remain being abelian.

For $A^{(g)}$, in particular, the above conditions mean $A^{(g)} \in \Gamma_g^*$ where $\Gamma_g^*$ is the weight lattice of $g$, i.e.,

$$\Gamma_g^* = \begin{cases} \Gamma_w^{(16)} = \Gamma_g^{(16)} + \Gamma_v^{(16)} + \Gamma_s^{(16)} + \Gamma_c^{(16)} & \text{for } g = SO(32) \\ \Gamma^{16} = \left(\Gamma_v^{(8)} + \Gamma_s^{(8)}\right) \times \left(\Gamma_v^{(8)} + \Gamma_s^{(8)}\right) & \text{for } g = E_8 \times E_8 \end{cases} . \tag{4.13}$$

From (2.42) with $m = 0$, one can see that in the $Spin(32)/\mathbb{Z}_2$ model, the sector with $m = 0$ is invariant under the shift $A \to A + \pi_a$ with $\pi_a \in \Gamma_g^*$. So, we can choose $A^{(g)} = (0^{16})$ by using the Wilson line shift dualities. Note that for the full space of states, as seen in (3.59), the shift parameter $\pi_a$ must be an element of $\Gamma_{16}$. Thus, in the $E_8 \times E_8$ model, in which $\Gamma^{16} = \Gamma_g^*$, there is no difference of the Wilson line shift between for the full spectrum and for the sector with $m = 0$. Including the sector with $m \neq 0$ in the $Spin(32)/\mathbb{Z}_2$ model, the gauge group can be enhanced to a larger one than $SO(32)$ (e.g. $SO(34)$) when the Wilson line is in $\Gamma_v^{(16)}$ or $\Gamma_c^{(16)}$ and $R$ takes particular values (see e.g. [96–98]). We can of course find massless spinors transforming in the same representation of the gauge group as the bosonic states from R-sector since supersymmetry is preserved under toroidal compactifications.

In the non-supersymmetric models, the situation is different since the Narain lattice is split into $\Gamma_+^{17,1}$ and $\Gamma_-^{17,1}$, and bosonic states live in $\Gamma_+^{17,1}$ while fermionic states live in $\Gamma_-^{17,1}$. The momentum with $\pi = m = n = 0$ must be in $\Gamma_+^{17,1}$ independent of the choice of $\hat{Z}$. So, there are no fermionic massless states in sector 1. Let us study below the condition (4.4) for sector 2 in each of the four classes:

- class (1);

  In this class, $n$ is an integer for both $\Gamma_+^{17,1}$ and $\Gamma_-^{17,1}$, and the condition for giving rise to massless states in sector 2 is the same form as in (4.12). We should however remember that $\Gamma^{16}$ is split into $\Gamma_+^{16}$ and $\Gamma_-^{16}$ depending on the right-moving states being in NS-sector or R-sector. Thus, the gauge symmetries in the non-supersymmetric models with $A = A^{(g')}$ are not necessarily enhanced to $g'$, unlike in the toroidal models. Let $\Delta_g^+$ and $\Delta_g^-$ denote subsets of $\pi \in \Gamma_+^{16}$ and $\pi \in \Gamma_-^{16}$ respectively that satisfy $|\pi|^2 = 2$. In the non-supersymmetric models in class (1), then, the gauge group realized by the Wilson line $A^{(g')}$ have the following nonzero roots,

$$\Delta_{g'}^+ = \left\{ \pi \in \Delta_g^+ \,\middle|\, \pi \cdot A^{(g')} \in \mathbb{Z} \right\}, \tag{4.14}$$

and the representation in which massless spniors transform is given by

$$\Delta_{g'}^{-} = \left\{ \pi \in \Delta_g^{-} \,\middle|\, \pi \cdot A^{(g')} \in \mathbb{Z} \right\}. \tag{4.15}$$

The gauge symmetry in the non-supersymmetric model in class (1) is at least broken to the group of which the nonzero roots consist of $\Delta_g^+$. There are two possibilities of the Wilson lines which enhance $U(1)_L^{16}$ to the maximal gauge group. One of them is $A = A^{(g)} \in \Gamma_g^*$, which also yields massless spinors transforming in the representation given by $\Delta_g^-$ unless $\hat{\pi}$ is chosen such that $\Delta_g = \Delta_g^+$. Note that $\Delta_g = \Delta_g^+$ implies that $\Delta_g^-$ is empty and fermions cannot be massless, and recall that in construction of the 10D non-supersymmetric heterotic models, such choice of $\hat{\pi}$ is possible in the $Spin(32)/\mathbb{Z}_2$ lattice, which gives the non-supersymmetric $SO(32)$ model (see Table 1). The other possibility of the Wilson line with the maximal enhancement is $A = \frac{\hat{\pi}}{2}$, which leads to $\Delta_{g'}^+ = \Delta_g^+$, but at the same time makes $\Delta_{g'}^-$ empty. Namely, with $A = \frac{\hat{\pi}}{2}$, the gauge group $U(1)_L^{16}$ is maximally enhanced, while all spinors become massive.

Note that there are states with $m = 0$ in twisted sector, which live in $\Gamma_{\pm}^{17,1} + \delta$ given by (2.46). We will discuss the twisted sectors in the next subsection.

- class (2);
  In class (2), $\Gamma_{\pm}^{17,1}$ is given by (2.49), and hence the condition (4.8) implies for massless vectors,

$$\pi \cdot A_{(2)} \in 2\mathbb{Z} \ \text{ for } \pi \in \Delta_g^+ \ \text{ and/or } \ \pi \cdot A_{(2)} \in 2\mathbb{Z}+1 \ \text{ for } \pi \in \Delta_g^-, \tag{4.16}$$

while for massless spinors,

$$\pi \cdot A_{(2)} \in 2\mathbb{Z} \ \text{ for } \pi \in \Delta_g^- \ \text{ and/or } \ \pi \cdot A_{(2)} \in 2\mathbb{Z}+1 \ \text{ for } \pi \in \Delta_g^+, \tag{4.17}$$

where we denote as $A_{(2)}$ the Wilson line in the non-supersymmetric models in class (2). Let $A_{(2)}^{g'}$ and $A_{(T)}^{(g')}$ denote the Wilson lines that realize the enhancement to $g'$ in class (2) and in the toroidal models respectively. Noting that $A_{(T)}^{(g')}$ satisfies (4.12), we find from (4.16) that $A_{(2)}^{(g')}$ can be expressed in terms of $A_{(T)}^{(g')}$ as follows:

$$A_{(2)}^{(g')} = 2A_{(T)}^{(g')} + \hat{\pi}, \tag{4.18}$$

In particular, the gauge group of this class is enhanced to $g$ if $A_{(2)} = \hat{\pi}$ up to the shift by $2\Gamma_g^*$. Note that the shift parameter must be doubled in class (2), as we have seen in Table 5.

By using (4.18), one can find that the conditions (4.16) and (4.17) are respectively expressed in terms of $A_{(T)}$ as follows:

$$A_{(T)} \cdot \pi \in \mathbb{Z} \text{ for } \pi \in \Delta_{g'} \quad \text{(for massless vectors)}, \tag{4.19}$$

$$A_{(T)} \cdot \pi \in \mathbb{Z} + \frac{1}{2} \text{ for } \pi \in \Delta_{g'} \quad \text{(for massless spinors)}. \tag{4.20}$$

Note that these conditions do not depend on $\hat{\pi}$. Thus, by using $A_{(T)}$ but not $A_{(2)}$, we can identify the massless spectrum in class (2) without specifying the choice of $\hat{\pi}$. But, one should note that this argument is valid only for the sector with $m = 0$, i.e., in the region with supersymmetry restoration.

In this class, there is no state with $m = 0$ in the twisted sectors because of the shift by $\frac{1}{2}$ in $\Gamma^{17,1}_{\pm} + \delta$.

- class (3);
  Focusing on the states with $m = 0$, we find from (2.53) that the spectrum in class (3) is the same as in class(1). So, the analysis for identifying the massless spectrum in sector 2 can be done in the same way as in class (1). Note that the region where the states with $m = 0$ only contribute corresponds to $R$ approaching the 10D non-supersymmetric endpoint models.

- class (4);
  In class (4) in which $\Gamma^{17,1}_{\pm}$ is given by (2.57), the spectrum with $m = 0$ agrees with that in class (2), and (4.8) leads to the conditions (4.16) and (4.17). So, we can use the relation (4.18) for identifying the massless spectra.

Although we have focused on the states with $m = 0$, the similar results as above are obtained for the states with $n = 0$ by using the dual descriptions (4.6) of the moduli. For class (1) and class (2) with $n = 0$, the situation is the same as above if one replaces $A$ to $\tilde{A}$. The spectrum in class (2) with $n = 0$ in the dual description is the same as in class (3) with $m = 0$ in the normal description. So, the above analysis in class(2) (in class (3)) can be applied for identifying the massless states with $n = 0$ in class (3) (in class (2)). Recall that the transitions between class (2) and class (3) are induced by the inversion $g_D$, as shown in Fig. 2.

## 4.2 Twisted sector

Let us keep focusing on the states with $m = 0$. As mentioned in the previous subsection, there are no twisted states with $m = 0$ in class (2) and class (4). So, we only consider the twisted sectors in class (1) and class (3). In class (1), supersymmetry cannot be restored in any region of the moduli space. In class (3), the twisted states with $n \in \mathbb{Z} + 1/2$ do not contribute in the region with supersymmetry restoration, or rather they become significant in the region with $R$ approaching the non-supersymmetric endpoint. Thus, the analysis we will do in this subsection cannot be used in the subsequent sections, but we believe that it is worth figuring out the massless twisted states.

We can read off massless states in the twisted sectors from the partition function. From the expansions (A.24), only conjugate spinors with $p_R+\delta_R = 0$ and scalars with $(p_R+\delta_R)^2 = 1$ can be massless. With $m = 0$, we get

$$\left(\pi + \frac{\hat{\pi}}{2}\right) \cdot A = \begin{cases} -\left(n + \frac{\hat{n}}{2}\right) & \text{for conjugate spinors} \\ -\left(n + \frac{\hat{n}}{2}\right) \pm \sqrt{2}R & \text{for scalars} \end{cases}, \tag{4.21}$$

where $n \in \mathbb{Z}$ and $\pi \in \Gamma^{16}_{\pm}$. Recall that the upper (lower) sign of $\Gamma^{16}_{\pm}$ in the twisted sectors is applied to conjugate spinors with $\hat{\pi}^2/4$ even (odd) or to scalars with $\hat{\pi}^2/4$ odd (even). The condition (4.23) requires that $R$ be a special value for scalars to be massless. For left-moving states to be massless, the momenta must satisfy $(P_L + \delta_L)^2 = 2^7$. From (3.49), with $m = 0$,

$$(P_L + \delta_L)^2 - (p_R + \delta_R)^2 = \left|\pi + \frac{\hat{\pi}}{2}\right|^2, \tag{4.22}$$

and the condition for the left-moving momentum is

$$\left|\pi + \frac{\hat{\pi}}{2}\right|^2 = \begin{cases} 2 & \text{for conjugate spinor} \\ 1 & \text{for scalar} \end{cases}. \tag{4.23}$$

Let us define $\Delta_g^{\pm,c}$ and $\Delta_g^{\pm,o}$ as

$$\Delta_g^{\pm,c} = \left\{\pi + \frac{\hat{\pi}}{2} \in \Gamma^{16}_{\pm} + \frac{\hat{\pi}}{2} \,\middle|\, \left|\pi + \frac{\hat{\pi}}{2}\right|^2 = 2\right\}, \quad \Delta_g^{\pm,o} = \left\{\pi + \frac{\hat{\pi}}{2} \in \Gamma^{16}_{\pm} + \frac{\hat{\pi}}{2} \,\middle|\, \left|\pi + \frac{\hat{\pi}}{2}\right|^2 = 1\right\}, \tag{4.24}$$

[7] The assumption that the gauge twist is non-trivial excludes the possibility of $(P_L + \delta_L)^2 = 0$.

Note that conjugate spinors and scalars have $\pi + \frac{\hat{\pi}}{2} \in \Delta_g^{-,c}$ and $\pi + \frac{\hat{\pi}}{2} \in \Delta_g^{+,o}$ respectively for $\left|\frac{\hat{\pi}}{2}\right|^2$ odd, while have $\pi + \frac{\hat{\pi}}{2} \in \Delta_g^{+,c}$ and $\pi + \frac{\hat{\pi}}{2} \in \Delta_g^{-,o}$ respectively for $\left|\frac{\hat{\pi}}{2}\right|^2$ even. As seen in subsection 2.3, we can choose $\hat{\pi}$ such that $\left|\frac{\hat{\pi}}{2}\right|^2$ is 1 or 2, and the lower bound of $\left|\pi + \frac{\hat{\pi}}{2}\right|^2$ with non-trivial gauge twist of order 2 is $\left|\frac{\hat{\pi}}{2}\right|^2$. Then, for the choice $\hat{\pi}$ with $\left|\frac{\hat{\pi}}{2}\right|^2 = 2$, which correspond to the $SO(16) \times SO(16)$ non-supersymmetric endpoint model, $\Delta_g^{\pm,o}$ is empty, and hence there is no massless scalar with $m = 0$ in the twisted sector. In fact, the massless scalars with (4.21) and (4.23) are caused by the tachyonic states in the 10D non-supersymmetric models acquiring the mass due to the compactification. The condition (4.23) also implies that massless conjugate spinors and the massless scalars in the twisted sectors are not gauge singlets.

We should note that unlike in the untwisted sectors, not all elements in $\Delta_g^{\pm,c}$ necessarily satisfy (4.23) with $A \in \Gamma_g^*$. Rather than that, the condition (4.23) for conjugate spinors holds for any elements in $\Delta_g^{\pm,c}$ if $A \in \Gamma_+^{16}$ for class (1) and $A \in \Gamma_-^{16}$ for class (3). This fact implies that the Wilson line is invariant under the shift by $\pi_a \in \Gamma_+^{16}$, as we have seen in subsection 2.4 (see Table 5). In addition, in order to get massless scalars in class (1) (class (2)), from the condition (4.21), we find that $A \in \Gamma_+^{16}$ requires $\sqrt{2}R \in \mathbb{Z}$ ($\sqrt{2}R \in \mathbb{Z} + 1/2$), while $A \in \Gamma_-^{16}$ requires $\sqrt{2}R \in \mathbb{Z} + 1/2$ ($\sqrt{2}R \in \mathbb{Z}$).

As in the untwisted sectors, the analysis we have performed above can be used for identifying the massless states with $n = 0$ in the twisted sectors in class (1) and class (2) by using the dual descriptions $\tilde{A}$ and $\tilde{R}$.

### 4.3 Example 1: class (1) with the $Spin(32)/\mathbb{Z}_2$ lattice

In this and the following subsections, we will give examples of the Wilson lines and identify the corresponding massless spectra in class (1). We will keep restricting our attention to the states with $m = 0$ for which the massless conditions are given by (4.8) for the untwisted sectors, while given by (4.21) and (4.23) for the twisted sectors. Although we will only focus on class (1), the massless spectra in class (3) can be obtained in a similar way; the difference only appears in the twisted sectors; $n$ in (4.21) is shifted by 1/2. The discussion of massless states in class (2) will be given in the next section when we explore the possibility of suppression of the cosmological constant. The study in class (2) is in fact easier than in class (1) and class (3) because we can use the relation (4.18), and then the massless spectrum with $m = 0$ can be identified without the information of $\hat{\pi}$.

In this subsection, we consider the non-supersymmetric models in class (1) with the $Spin(32)/\mathbb{Z}_2$ root lattice. For simplicity, we only pay our attention to the following two types of the Wilson lines which satisfy $2A \in \Gamma_g^*$:

$$A = \left(0^p, \left(\frac{1}{2}\right)^q\right) \qquad (p+q=16), \tag{4.25a}$$

$$A = \left(\left(\frac{1}{4}\right)^{16}\right). \tag{4.25b}$$

Recall that $\Gamma_g^* = \Gamma_g^{(16)} + \Gamma_v^{(16)} + \Gamma_s^{(16)} + \Gamma_c^{(16)}$, and (4.25a) with $p$ even (odd) satisfies $2A \in \Gamma_g^{(16)}$ ($2A \in \Gamma_v^{(16)}$) while (4.25b) satisfies $2A \in \Gamma_s^{(16)}$.

As shown in Table 1, there are the four choices of $\hat{\pi}$ in class (1). Let us study massless states in the untwisted and twisted sectors with $m = 0$ for each of the choices.

#### 4.3.1 $SO(32)$ model: $\frac{\hat{\pi}}{2} = (1, 0^{15})$

With this choice of $\hat{\pi}$, one can see that $\Delta_g^+ = \Delta_{SO(32)}$ and $\Delta_g^-$ is empty. Thus, the gauge symmetry is $SO(32)$ if $A \in \Gamma_g^*$, whereas there is no massless fermion in the untwisted sectors whatever configuration the Wilson line takes. Noting $\left|\frac{\hat{\pi}}{2}\right|^2 = 1$, in the twisted sectors, massless conjugate spinors and massless scalars live in $\Delta_g^{-,c}$ and $\Delta_g^{+,o}$ respectively if they exist. However, since $\Gamma_-^{16} = \Gamma_s^{(16)}$, in which $|\pi|^2 \geq 4$ for any elements, $\Delta_g^{-,c}$ is empty, and hence there is no massless conjugate spinor in the twisted sectors. We obtain $\Delta_g^{+,o}$ as

$$\Delta_g^{+,o} = \left\{ \left(\underline{\pm, 0^{15}}\right) \right\}, \tag{4.26}$$

and there are massless scalars transforming in a fundamental representation of the $SO(32)$ if $A \in \Gamma_+^{16}$ and $\sqrt{2}R \in \mathbb{Z}$ or $A \in \Gamma_-^{16}$ and $\sqrt{2}R \in \mathbb{Z} + 1/2$.

Let us consider the massless spectrum with the Wilson line (4.25a). In $\Delta_g^+$, the following $\pi$'s satisfy $\pi \cdot A \in \mathbb{Z}$:

$$\Delta_{g'}^+ = \left\{ \left(\underline{\pm, \pm, 0^{p-2}}, 0^q\right), \left(0^p \underline{\pm, \pm, 0^{q-2}}\right) \right\}. \tag{4.27}$$

Then, $U(1)_L^{16}$ is enhanced to $SO(2p) \times SO(2q)$. The massless scalars in the twisted sectors exist if the radius $R$ takes a special values. Applying (4.25a) to (4.21), we find that the massless scalars have

$$\pi + \frac{\hat{\pi}}{2} = \begin{cases} \left(\underline{\pm, 0^{p-1}}, 0^q\right) & \text{if } \sqrt{2}R \in \mathbb{Z} \\ \left(0^p, \underline{\pm, 0^{q-1}}\right) & \text{if } \sqrt{2}R \in \mathbb{Z} + 1/2 \end{cases}. \tag{4.28}$$

There are then massless scalars transforming in the fundamental representation of $SO(2p)$ (or $SO(2q)$) if $\sqrt{2}R$ is an integer (or a half-integer).

We now turn to the Wilson line (4.25b). The nonzero roots of $SU(16)$ in $\Delta_g^+ = \Delta_{SO(32)}$ only satisfy $\pi \cdot A \in \mathbb{Z}$:

$$\pi = \left(\underline{+, -, 0^{14}}\right). \tag{4.29}$$

As for massless scalars in the twisted sectors, all elements in $\Delta_g^{+,o}$ satisfy (4.21) if $\sqrt{2}R = \pm\frac{1}{4}$ mod 1. After all, with the Wilson line (4.25b), there are gauge bosons of $SU(16) \times U(1)$, and charged scalars transforming in $\mathbf{16} \oplus \overline{\mathbf{16}}$ of $SU(16)$ which can be massless only if $\sqrt{2}R \in \mathbb{Z} + 1/4$ or $\sqrt{2}R \in \mathbb{Z} - 1/4$.

As mentioned above, in this model, all fermions are massive at any points in the moduli space.

### 4.3.2 $SO(24) \times SO(8)$ model: $\frac{\hat{\pi}}{2} = \left(0^{12}, \left(\frac{1}{2}\right)^4\right)$

This $\hat{\pi}$ splits $\Delta_{SO(32)}$ into $\Delta_g^+$ and $\Delta_g^-$ as follows:

$$\Delta_g^+ = \Delta_{SO(24)\times SO(8)} = \left\{\left(\underline{\pm, \pm, 0^{10}}, 0^4\right), \left(0^{12}, \underline{\pm, \pm, 0^2}\right)\right\}, \quad \Delta_g^- = \left\{\left(\underline{\pm, 0^{11}}, \underline{\pm, 0^3}\right)\right\}. \tag{4.30}$$

These correspond to the nonzero roots of $SO(24) \times SO(8)$ and a bi-fundamental representation of $SO(24) \times SO(8)$ respectively. As for the twisted sectors, $\Delta_g^{-,c}$ and $\Delta_g^{+,o}$ are given as

$$\Delta_g^{-,c} = \left\{\left(\underline{\pm, 0^{11}}, \underline{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}}_{-}\right)\right\}, \quad \Delta_g^{+,o} = \left\{\left(0^{12}, \underline{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}}_{+}\right)\right\}, \tag{4.31}$$

which correspond to $(\mathbf{24}, \mathbf{8}_-)$ and $(\mathbf{1}, \mathbf{8}_+)$ of the $SO(24) \times SO(8)$ respectively. Here the underline with the subscript $+$ $(-)$ means permutations with the number of $+1/2$ being even (odd).

We now consider the Wilson line (4.25a) with $p \leq 12$. The subsets $\Delta_{g'}^+$ and $\Delta_{g'}^-$ which satisfy $\pi \cdot A \in \mathbb{Z}$ are

$$\Delta_{g'}^+ = \left\{\left(\underline{\pm, \pm, 0^{p-2}}, 0^q\right), \left(0^p, \underline{\pm, \pm, 0^{10-p}}, 0^4\right), \left(0^{12}, \underline{\pm, \pm, 0^2}\right)\right\}, \tag{4.32}$$

$$\Delta_{g'}^- = \left\{\left(0^p, \underline{\pm, 0^{11-p}}, \underline{\pm, 0^3}\right)\right\}, \tag{4.33}$$

which lead to the symmetry enhancement $U(1)_L^{16} \to SO(2p) \times SO(24-2p) \times SO(8)$ and $(\mathbf{1}, \mathbf{24-2}\boldsymbol{p}, \mathbf{8})$ of the $SO(2p) \times SO(24-2p) \times SO(8)$ respectively. In $\Delta_g^{-,c}$, we find that the following elements satisfy $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}$:

$$\pi+\frac{\hat{\pi}}{2}=\left(0^p,\underline{\pm,0^{11-p}},\underline{\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}}_{-}\right). \tag{4.34}$$

Then, conjugate spinors in the twisted sectors transform in $(\mathbf{1},\mathbf{24-2}\boldsymbol{p},\mathbf{8}_{-})$ of the $SO(2p)\times SO(24-2p)\times SO(4)$. For scalars in the twisted sectors, $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}$ holds for all elements in $\Delta_g^{+,o}$, and hence the scalars transform in $\mathbf{8}_{+}$ of the $SO(8)$ if $\sqrt{2}R\in\mathbb{Z}$.

In the case of the Wilson line (4.25a) with $p>12$ $(q<4)$, we get

$$\Delta_{g'}^{+}=\left\{\left(\underline{\pm,\pm,0^{10}},0^4\right),\left(0^{12},\underline{\pm,\pm,0^{2-q}},0^q\right),\left(0^p,\underline{\pm,\pm,0^{q-2}}\right)\right\}, \tag{4.35}$$

$$\Delta_{g'}^{-}=\left\{\left(\underline{\pm,0^{11}},\underline{\pm,0^{3-q}},0^q\right)\right\}, \tag{4.36}$$

and then the enhanced gauge symmetry from $U(1)_L^{16}$ is $SO(24)\times SO(8-2q)\times SO(2q)$ and massless spinors transform in $(\mathbf{24},\mathbf{8-2}\boldsymbol{q},\mathbf{1})$ of the $SO(24)\times SO(8-2q)\times SO(2q)$. As for the twisted sectors, any elements in $\Delta_g^{-,c}$ do not satisfy $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}$, and there is no massless cojugate spinor. Assuming $\sqrt{2}R\in\mathbb{Z}$ or $\sqrt{2}R\in\mathbb{Z}+1/2$, we can find massless scalars in the twisted sectors only if $p=14$ or 16. With $p=16$ which means $A\in\Gamma_{+}^{16}$, we have already mentioned this case; massless scalars transform in $(\mathbf{1},\mathbf{8}_{+})$ of the $SO(24)\times SO(8)$ are obtained if $\sqrt{2}R\in\mathbb{Z}$. With $p=14$, scalars that have the following elements in $\Delta_g^{+,o}$ become massless, depending on values $R$ takes:

$$\pi+\frac{\hat{\pi}}{2}=\begin{cases}\left(0^{12},\underline{\pm\frac{1}{2},\pm\frac{1}{2}}_{+},\underline{\pm\frac{1}{2},\pm\frac{1}{2}}_{+}\right) & \text{if } \sqrt{2}R\in\mathbb{Z}\\ \left(0^{12},\underline{\pm\frac{1}{2},\pm\frac{1}{2}}_{-},\underline{\pm\frac{1}{2},\pm\frac{1}{2}}_{-}\right) & \text{if } \sqrt{2}R\in\mathbb{Z}+\frac{1}{2}\end{cases}. \tag{4.37}$$

Then, we find the massless scalars transforming in $(\mathbf{1},\mathbf{2}_{\pm},\mathbf{2}_{\pm})$ of $SO(24)\times SO(4)\times SO(4)$ where the chiralities of the spinors of the two $SO(4)$'s depend on whether $\sqrt{2}R\in\mathbb{Z}$ or $\sqrt{2}R\in\mathbb{Z}+1/2$.

Let us next consider the Wilson line (4.25b). In the untwisted sectors, massless vactors and massless spinors respectively live in

$$\Delta_{g'}^{+}=\left\{\left(\underline{+,-,0^{10}},0^4\right),\left(0^{12},\underline{+,-,0^2}\right)\right\},\quad \Delta_{g'}^{-}=\left\{\pm\left(\underline{+,0^{11}},\underline{-,0^3}\right)\right\}. \tag{4.38}$$

Then $U(1)_L^{16}$ is enhanced to $SU(12)\times SU(4)\times U(1)^2$, and the massless spinors transform in $\left(\mathbf{12},\overline{\mathbf{4}}\right)\oplus\left(\overline{\mathbf{12}},\mathbf{4}\right)$ of the $SU(12)\times SU(4)$. As for conjugate spinors in the twisted sectors, the elements in $\Delta_g^{-,c}$ that satisfy $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}$ are

$$\pi+\frac{\hat{\pi}}{2}=\pm\left(\underline{+,0^{11}},\underline{+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\right), \tag{4.39}$$

which leads to the $U(1)$ charged conjugate spinors transforming in $(\mathbf{12},\mathbf{4}) \oplus \left(\overline{\mathbf{12}},\overline{\mathbf{4}}\right)$ of the $SU(12)\times SU(4)$. Note that the last four components in (4.39) are decomposed as follows:

$$\left(\underline{+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\right) = \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) + \left(\underline{+,0^3}\right). \tag{4.40}$$

As for scalars in the twisted sectors, the following elements in $\Delta_g^{+,o}$ satisfy $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A \in \mathbb{Z}$:

$$\pi + \frac{\hat{\pi}}{2} = \left(0^{12}, \underline{+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\right), \tag{4.41}$$

while those satisfying $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A \in \mathbb{Z}+1/2$ are

$$\pi + \frac{\hat{\pi}}{2} = \pm\left(0^{12}, \underline{+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2}}\right). \tag{4.42}$$

Then the massless scalars transform in $\mathbf{6}$ of the $SU(4)$ if $\sqrt{2}R\in\mathbb{Z}$, while have the $U(1)$ charge $\pm1$ if $\sqrt{2}R\in\mathbb{Z}+1/2$.

### 4.3.3 $SU(16)\times U(1)$ model: $\frac{\hat{\pi}}{2} = \left(\left(\frac{1}{4}\right)^{16}\right)$

With this choice of $\hat{\pi}$, $\Delta_g^+$ and $\Delta_g^-$ are

$$\Delta_g^+ = \left\{\left(\underline{+,-,0^{14}}\right)\right\}, \quad \Delta_g^- = \left\{\pm\left(\underline{+,+,0^{14}}\right)\right\}, \tag{4.43}$$

which respectively correspond to the nonzero roots of $SU(16)$ and the representation $\mathbf{120}\oplus\overline{\mathbf{120}}$ of the $SU(16)$. As $\left|\frac{\hat{\pi}}{2}\right|^2$ is odd, we need $\Delta_g^{-,c}$ and $\Delta_g^{+,o}$ in order to clarify massless states in the twisted sectors:

$$\Delta_g^{-,c} = \left\{\pm\left(\underline{\left(\frac{3}{4}\right)^2,\left(-\frac{1}{4}\right)^{14}}\right)\right\}, \quad \Delta_g^{+,o} = \left\{\pm\left(\left(\frac{1}{4}\right)^{16}\right)\right\}. \tag{4.44}$$

In the twisted sectors, then, we have $U(1)$ charged massless cojugate spinors transforming in $\mathbf{120}\oplus\overline{\mathbf{120}}$ of the $SU(16)$, and massless scalars with the $U(1)$ charges $\pm1$ if $A\in\Gamma_+^{16}$ and $\sqrt{2}R\in\mathbb{Z}$.

Let us consider the massless spectrum with the Wilson line (4.25a). Imposing the condition $\pi\cdot A\in\mathbb{Z}$ into (4.43), we get

$$\Delta_{g'}^+ = \left\{\left(\underline{+,-,0^{p-2}},0^q\right),\left(0^p,\underline{+,-,0^{q-2}}\right)\right\}, \quad \Delta_{g'}^- = \left\{\pm\left(\underline{+,+,0^{p-2}},0^q\right),\pm\left(0^p,\underline{+,+,0^{q-2}}\right)\right\}, \tag{4.45}$$

which lead to the enhancement $U(1)_L^{16} \to SU(p) \times SU(q) \times U(1)^2$ and the representation $\left({}_{\boldsymbol{p}}\mathbf{C_2} \oplus \overline{{}_{\boldsymbol{p}}\mathbf{C_2}}, \mathbf{1}\right) \oplus \left(\mathbf{1}, {}_{\boldsymbol{q}}\mathbf{C_2} \oplus \overline{{}_{\boldsymbol{q}}\mathbf{C_2}}\right)$ of the $SU(p) \times SU(q)$. As for conjugate spinors in the twisted sectors, noting that the elements in $\Delta_g^{-,c}$ can be expressed as

$$\pm \left(\underline{\left(\frac{3}{4}\right)^2, \left(-\frac{1}{4}\right)^{14}}\right) = \pm \left(\left(-\frac{1}{4}\right)^{16}\right) \pm \left(\underline{+, +, 0^{14}}\right), \tag{4.46}$$

we see that $\left(\pi + \frac{\hat{\pi}}{2}\right) \cdot A \in \mathbb{Z}$ holds for some elements in $\Delta_g^{-,c}$ if $p$ is a multiple of 4. If $p = 0$ or $p = 8$, we get

$$\pi + \frac{\hat{\pi}}{2} = \pm \left(\left(-\frac{1}{4}\right)^{16}\right) \pm \begin{cases} \left(\underline{+, +, 0^{p-2}}, 0^q\right) \\ \left(0^p, \underline{+, +, 0^{q-2}}\right) \end{cases} . \tag{4.47}$$

which lead to the $U(1)$ charged massless conjugate spinors transforming in $({}_{\boldsymbol{p}}\mathbf{C_2}, \mathbf{1}) \oplus (\mathbf{1}, {}_{\boldsymbol{q}}\mathbf{C_2})$ of the $SU(p) \times SU(q)$ and its conjugate representation. If $p = 4$ or $p = 12$, we find

$$\pi + \frac{\hat{\pi}}{2} = \pm \left(\left(-\frac{1}{4}\right)^{16}\right) \pm \left(\underline{+, 0^{p-1}}, \underline{+, 0^{q-1}}\right). \tag{4.48}$$

which lead to the $U(1)$ charged massless conjugate spinors transforming in $(\boldsymbol{p}, \boldsymbol{q})$ of the $SU(p) \times SU(q)$ and its conjugate representation. From $\Delta_g^{+,o}$ in (4.44), we obtain the $U(1)$ charged massless scalars if $\sqrt{2}R \in \mathbb{Z}$ with $p = 0, 8$ or $\sqrt{2}R \in \mathbb{Z} + 1/2$ with $p = 4, 12$.

Turning to the Wilson line (4.25b), we notice that $\Delta_{g'}^{+} = \Delta_g^{+}$ and $\Delta_{g'}^{-}$ is empty since $A = \frac{\hat{\pi}}{2}$. Then, $U(1)_L^{16}$ is enhanced to $SU(16) \times U(1)$, while all spinors in the untwisted sectors are massive. As for the twisted sectors, we find from (4.44) that there are no massless conjugate spinors, and the $U(1)$ charged massless scalars exist if $\sqrt{2}R \in \mathbb{Z}$.

#### 4.3.4 $SO(16) \times SO(16)$ model: $\frac{\hat{\pi}}{2} = \left(0^8, \left(\frac{1}{2}\right)^8\right)$

With this choice of $\hat{\pi}$, $\Delta_g^{+}$ and $\Delta_g^{-}$ are

$$\Delta_g^{+} = \Delta_{SO(16)\times SO(16)} = \left\{\left(\underline{\pm, \pm, 0^6}, 0^8\right), \left(0^8, \underline{\pm, \pm, 0^6}\right)\right\}, \quad \Delta_g^{-} = \left\{\left(\underline{\pm, 0^7}, \underline{\pm, 0^7}\right)\right\}, \tag{4.49}$$

and hence $U(1)_L^{16}$ is enhanced to $SO(16) \times SO(16)$ and massless spinors transform in $(\mathbf{16}, \mathbf{16})$ of the $SO(16) \times SO(16)$ if $A \in \Gamma_g^*$. Noting $\left|\frac{\hat{\pi}}{2}\right|^2 = 2$, we need $\Delta_g^{+,c}$ and $\Delta_g^{-,o}$, but not $\Delta_g^{-,c}$ and $\Delta_g^{+,o}$, in order to identify massless states in the twisted sectors. As mentioned in the previous subsection, $\Delta_g^{-,o}$ is empty for $\hat{\pi}$ with $\left|\frac{\hat{\pi}}{2}\right|^2 = 2$, while $\Delta_g^{+,c}$ is given as

$$\Delta_g^{+,c} = \Delta_{\mathbf{128}_+} \oplus \Delta_{\mathbf{128}_+} = \left\{ \left( \underline{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}}_+, 0^8 \right), \left( 0^8, \underline{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}}_+ \right) \right\}. \tag{4.50}$$

The massless conjugate spinors then transform in $(\mathbf{128}_+, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{128}_+)$ of the $SO(16) \times SO(16)$ if $A \in \Gamma_+^{16}$.

Let us first consider the Wilson line (4.25a) with assuming $p \leq 8$. Then, we obtain $\Delta_{g'}^+$ and $\Delta_{g'}^-$ as

$$\Delta_{g'}^+ = \left\{ \left( \underline{\pm, \pm, 0^{p-2}}, 0^{16-p} \right), \left( 0^p, \underline{\pm, \pm, 0^{6-p}}, 0^{16} \right), \left( 0^8, \underline{\pm, \pm, 0^6} \right) \right\}, \tag{4.51}$$

$$\Delta_{g'}^- = \left\{ \left( 0^p \underline{\pm, 0^{7-p}}, \underline{\pm, 0^7} \right) \right\}, \tag{4.52}$$

which lead to the enhancement $U(1)_L^{16} \to SO(2p) \times SO(16-2p) \times SO(16)$ and the massless spinors transformin in $(\mathbf{1}, \mathbf{16-2}\boldsymbol{p}, \mathbf{16})$ of the $SO(2p) \times SO(16-2p) \times SO(16)$. As for conjugate spinors in the twisted sectors, we find that the following elements in $\Delta_{g'}^{+,c}$ satisfy $\left(\pi + \frac{\hat{\pi}}{2}\right) \cdot A \in \mathbb{Z}$ if $p$ is even:

$$\pi + \frac{\hat{\pi}}{2} = \left( \underline{\left(\pm\frac{1}{2}\right)^p}_\pm, \underline{\left(\pm\frac{1}{2}\right)^{8-p}}_\pm, 0^8 \right), \left( 0^8, \underline{\left(\pm\frac{1}{2}\right)^8} \right), \tag{4.53}$$

where the chirality of the spinors of the $SO(2p) \times SO(16-2p)$ is $+$ for $p = 4, 8$ while $-$ for $p = 2, 6$. Then, the massless conjugate spinors transform in $\left(\mathbf{2}_\pm^{\boldsymbol{p}-\mathbf{1}}, \mathbf{2}_\pm^{\mathbf{7}-\boldsymbol{p}}, \mathbf{1}\right) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{128}_+)$ of the $SO(2p) \times SO(16-2p) \times SO(16)$. For the Wilson line (4.25a) with $p > 8$, we obtain the same result as above but we need the replacement $p \to q$.

We now pay our attention to the Wilson line (4.25b). With this Wilson line, $\Delta_{g'}^+$ and $\Delta_{g'}^-$ are

$$\Delta_{g'}^+ = \left\{ \left( \underline{+, -, 0^6}, 0^8 \right), \left( 0^8, \underline{+, -, 0^6} \right) \right\}, \tag{4.54}$$

$$\Delta_{g'}^- = \left\{ \pm \left( \underline{+, 0^7}, \underline{-, 0^7} \right) \right\}, \tag{4.55}$$

and then $U(1)_L^{16}$ is enhanced to $SU(8) \times SU(8) \times U(1)^2$ and massless spinors transform in $\left(\mathbf{8}, \overline{\mathbf{8}}\right) \oplus \left(\overline{\mathbf{8}}, \mathbf{8}\right)$ of the $SU(8) \times SU(8)$. As for conjugate spinors in the twisted sectors, the elements in $\Delta_g^{+,c}$ satisfying $\left(\pi + \frac{\hat{\pi}}{2}\right) \cdot A \in \mathbb{Z}$ are

$$\pi + \frac{\hat{\pi}}{2} = \left( \pm\left(\frac{1}{2}\right)^8, 0^8 \right), \left( \underline{\left(+\frac{1}{2}\right)^4, \left(-\frac{1}{2}\right)^4}, 0^8 \right), \left( 0^8, \pm\left(\frac{1}{2}\right)^8 \right), \left( 0^8, \underline{\left(+\frac{1}{2}\right)^4, \left(-\frac{1}{2}\right)^4} \right). \tag{4.56}$$

These give massless conjugate spinors transforming in $(\mathbf{70},\mathbf{1})\oplus(\mathbf{1},\mathbf{70})$ of the $SU(8)\times SU(8)$ and four massless conjugate spinors with the charges $\left(\pm\sqrt{2},0\right)\oplus\left(0,\pm\sqrt{2}\right)$ of the $U(1)\times U(1)$.

## 4.4 Example 2: class (1) with the $E_8\times E_8$ lattice

We now consider the $E_8\times E_8$ root lattice. For simplicity, we assume that the last eight components of the Wilson lines vanish: $A=(A_1;0^8)$. Moreover, we restrict our attention to $A_1$ that satisfied $2A_1\in\Gamma_g^*$. Namely, we focus on the following two types of configurations:

$$A_1=\left(0^{p_1},\left(\frac{1}{2}\right)^{q_1}\right)\qquad(p_1+q_2=8), \tag{4.57a}$$

$$A_1=\left(\left(\frac{1}{4}\right)^{8}\right), \tag{4.57b}$$

where $p_1$ is supposed to be even so that $2A_1\in\Gamma_g^*=\Gamma_g^{(16)}+\Gamma_s^{(16)}$. Note that $\Delta_{E_8\times E_8}$, which is defined in (4.11), can be decomposed as

$$\Delta_{E_8\times E_8}=\Delta_{E_8}\oplus\Delta_{E_8}=\left(\Delta_{SO(16)}+\Delta_{\mathbf{128}_+}\right)\oplus\left(\Delta_{SO(16)}+\Delta_{\mathbf{128}_+}\right), \tag{4.58}$$

where $\Delta_{SO(16)}$ and $\Delta_{\mathbf{128}_+}$ are given as

$$\Delta_{SO(16)}=\left\{\left(\underline{\pm,\pm,0^6}\right)\right\},\qquad\Delta_{\mathbf{128}_+}=\left\{\frac{1}{2}\left(\underline{\pm,\pm,\pm,\pm,\pm,\pm,\pm,\pm}_+\right)\right\}. \tag{4.59}$$

We can also decompose $\Delta_g^+$ and $\Delta_g^-$ into the two parts:

$$\Delta_g^+=\Delta_{g_1}^+\oplus\Delta_{g_2}^+,\qquad\Delta_g^-=\Delta_{g_1}^-\oplus\Delta_{g_2}^-. \tag{4.60}$$

Here we denote $\frac{\hat{\pi}}{2}=\left(\frac{\hat{\pi}_1}{2};\frac{\hat{\pi}_2}{2}\right)$ with $\hat{\pi}_i$ $(i=1,2)$ being in the $E_8$ root lattice, and $\Delta_{g_i}^{\pm}$ is defined as

$$\Delta_{g_i}^+=\left\{\pi_i\in\Delta_{E_8}\,|\hat{\pi}_i\cdot\pi_i\in2\mathbb{Z}\right\},\qquad\Delta_{g_i}^-=\left\{\pi_i\in\Delta_{E_8}\,|\hat{\pi}_i\cdot\pi_i\in2\mathbb{Z}+1\right\}. \tag{4.61}$$

Since the Wilson line is assumed to be expressed as $A=(A_1;0^8)$, $\Delta_{g'}^+$ and $\Delta_{g'}^-$ which are defined in(4.14) and (4.15) are written as

$$\Delta_{g'}^+=\Delta_{g_1'}^+\oplus\Delta_{g_2}^+,\qquad\Delta_g^-=\Delta_{g_1'}^-\oplus\Delta_{g_2}^-. \tag{4.62}$$

where $\Delta_{g_1'}^{\pm}$ is give as

$$\Delta_{g_1'}^+=\left\{\pi_1\in\Delta_{g_1}^+\,|\pi_1\cdot A_1\in\mathbb{Z}\right\},\qquad\Delta_{g_1'}^-=\left\{\pi_1\in\Delta_{g_1}^-\,|\pi_1\cdot A_1\in\mathbb{Z}\right\}. \tag{4.63}$$

Namely, we only need to focus on the first eight components of $\pi \in \Delta_{E_8\times E_8}$ and identify $\Delta^{\pm}_{g'_1}$ in order to see massless states in the untwisted sectors.

As shown in Table 2, there are the three inequivalent choices of $\hat{\pi}$ in the $E_8 \times E_8$ root lattice. Let us study the massless spectrum in each of the three non-supersymmetric models in class (1).

### 4.4.1 $SO(16) \times E_8$ model: $\frac{\hat{\pi}}{2} = (1, 0^7; 0^8)$

This choice of $\hat{\pi}$ splits $\Delta_{E_8\times E_8}$ into $\Delta^+_g$ and $\Delta^-_g$ as follows:

$$\Delta^+_g = \Delta_{SO(16)} \oplus \Delta_{E_8}, \qquad \Delta^-_g = \Delta_{\mathbf{128}_+} \oplus \left\{ \left(0^8\right) \right\}. \tag{4.64}$$

In particular, one should note $\Delta^+_{g_1} = \Delta_{SO(16)}$ and $\Delta^-_{g_1} = \Delta_{\mathbf{128}_+}$. Then, if $A \in \Gamma^*_g$, we obtain gauge bosons transforming in an adjoint representation of $SO(16)\times E_8$ and massless spinors transforming in $(\mathbf{128}_+; \mathbf{1})$ of the $SO(16) \times E_8$. In order to see massless states in the twisted sectors, $\Delta^{-,c}_g$ and $\Delta^{+,o}_g$ should be clarified:

$$\Delta^{-,c}_g = \Delta_{\mathbf{128}_-} \oplus \left\{ \left(0^8\right) \right\} = \left\{ \frac{1}{2} \left( \underline{\pm, \pm, \pm, \pm, \pm, \pm, \pm, \pm}_{\,-} ; 0^8 \right) \right\}, \qquad \Delta^{+,o}_g = \left\{ \left( \underline{\pm, 0^7}; 0^8 \right) \right\}. \tag{4.65}$$

In the twisted sectors, then, there are massless conjugate spinors transforming in $(\mathbf{128}_-; \mathbf{1})$ of the $SO(16)\times E_8$ and massless scalars transforming in $(\mathbf{16}; \mathbf{1})$ of the $SO(16)\times E_8$ if $A \in \Gamma^{16}_+$ and $\sqrt{2}R \in \mathbb{Z}$.

Let us see the massless spectra which are realized by the non-trivial Wilson lines. As mentioned above, it is sufficient to consider the inner products of the first eight components in $\pi \in \Delta^{\pm}_g$ with $A_1$. If $A_1$ is given as in (4.57a), then $\Delta^{\pm}_{g'_1}$ is

$$\Delta^+_{g'_1} = \left\{ \left( \underline{\pm, \pm, 0^{p-2}}, 0^q \right), \left( 0^p, \underline{\pm, \pm, 0^{q-2}} \right) \right\}, \qquad \Delta^-_{g'_1} = \left\{ \left( \underline{\left(\pm\frac{1}{2}\right)^p}_{\pm}, \underline{\left(\pm\frac{1}{2}\right)^q}_{\pm} \right) \right\}, \tag{4.66}$$

where the chirality $+$ in $\Delta^-_{g'_1}$ is applied for $p = 0, 4, 8$ while $-$ is for $p = 2, 6$. So, the enhanced gauge symmetry from $U(1)^{16}_L$ is $SO(2p) \times SO(2q) \times E_8$ and massless spinors transform in $\left(\mathbf{2}^{\mathbf{p-1}}_{\pm}, \mathbf{2}^{\mathbf{q-1}}_{\pm}; \mathbf{1}\right)$ of the $SO(2p) \times SO(2q) \times E_8$. As for the twisted sectors, imposing the condition (4.23) into $\Delta^{-.c}_g$ given in (4.65), we find

$$\pi + \frac{\hat{\pi}}{2} = \begin{cases} \left( \underline{\left(\pm\frac{1}{2}\right)^p}_{-}, \underline{\left(\pm\frac{1}{2}\right)^q}_{+}; 0^8 \right) & \text{for } p = 4, 8 \\ \left( \underline{\left(\pm\frac{1}{2}\right)^p}_{+}, \underline{\left(\pm\frac{1}{2}\right)^q}_{-}; 0^8 \right) & \text{for } p = 2, 6 \end{cases}, \tag{4.67}$$

and for $p=0$ there is no element in $\Delta_g^{-,c}$ satisfying $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}$. We obtain massless conjugate spinors transforming in $\left(\mathbf{2}_{\mp}^{\boldsymbol{p-1}},\mathbf{2}_{\pm}^{\boldsymbol{q-1}};\mathbf{1}\right)$ of the $SO(2p)\times SO(2q)\times E_8$ with the upper signs for $p=4,8$ and the lower signs for $p=2,6$. For $p=0$, which gives the same massless states in the untwisted sectors as with $p=8$, there is no massless conjugate spinor in the twisted sectors. Note that both of the Wilson lines (4.57a) with $p=8$ and $p=0$ are elements of $\Gamma^{16}$, but the former lives in $\Gamma^{16}_+$ while the latter in $\Gamma^{16}_-$. As regards scalars in the twisted sectors, the elements in $\Delta_g^{+,o}$ given in (4.65) that satisfy $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}$ are

$$\pi+\frac{\hat{\pi}}{2}=\left(\underline{\pm,0^{p-1}},0^q;0^8\right), \tag{4.68}$$

while satisfying $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}+1/2$ are

$$\pi+\frac{\hat{\pi}}{2}=\left(0^q,\underline{\pm,0^{q-1}};0^8\right). \tag{4.69}$$

Then, massless scalars transform in $(\mathbf{2}\boldsymbol{p},\mathbf{1};\mathbf{1})$ of the $SO(2p)\times SO(2q)\times E_8$ if $\sqrt{2}R\in\mathbb{Z}$, while transform in $(\mathbf{1},\mathbf{2}\boldsymbol{q};\mathbf{1})$ of the $SO(2p)\times SO(2q)\times E_8$ if $\sqrt{2}R\in\mathbb{Z}+1/2$.

We now turn to the Wilson line (4.57b). The condition $\pi\cdot A\in\mathbb{Z}$ leads to the following $\Delta_{g'}^+$ and $\Delta_{g'}^-$:

$$\Delta_{g'}^+=\left\{\left(\underline{+,-,0^6}\right)\right\}\oplus\Delta_{E_8},\qquad \Delta_{g'}^-=\left\{\left(\pm\left(\frac{1}{2}\right)^8;0^8\right),\left(\underline{\left(+\frac{1}{2}\right)^4,\left(-\frac{1}{2}\right)^4};0^8\right)\right\}. \tag{4.70}$$

Then, $U(1)_L^{16}$ is enhanced to $SU(8)\times U(1)\times E_8$, and we obtain massless spinors transforming in $(\mathbf{70};\mathbf{1})$ of the $SU(8)\times E_8$ and two $U(1)$ charged massless spinors. There are no elements in $\Delta_g^{-,c}$ that satisfy $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}$, and hence conjugate spinors in the twisted sectors cannot be massless with this Wilson line. One can notice that all the elements in $\Delta_g^{+,o}$ satisfy $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}+1/4$ or $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}-1/4$. We then obtain massless scalars transforming in $\mathbf{8}\oplus\overline{\mathbf{8}}$ of the $SU(8)$ if $\sqrt{2}R\in\mathbb{Z}\pm 1/4$.

### 4.4.2 $(E_7\times SU(2))^2$ model: $\frac{\hat{\pi}}{2}=\left(0^6,\left(\frac{1}{2}\right)^2;0^6,\left(\frac{1}{2}\right)^2\right)$

With this choice of $\hat{\pi}$, $\Delta_g^+$ and $\Delta_g^-$ are given as

$$\Delta_g^+=\Delta_{E_7\times SU(2)}\oplus\Delta_{E_7\times SU(2)},\qquad \Delta_g^-=\Delta_{(\mathbf{56},\mathbf{2})}\oplus\Delta_{(\mathbf{56},\mathbf{2})}, \tag{4.71}$$

where $\Delta_{E_7\times SU(2)}$ and $\Delta_{(\mathbf{56},\mathbf{2})}$ are defined as

$$\Delta_{E_7\times SU(2)} = \left\{ \left(\underline{\pm,\pm,0^4},0^2\right), \left(0^6,\underline{\pm,\pm}\right), \left(\underline{\left(\pm\frac{1}{2}\right)^6_-}, \underline{\left(\pm\frac{1}{2}\right)^2_-}\right) \right\}, \tag{4.72}$$

$$\Delta_{(\mathbf{56},\mathbf{2})} = \left\{ \left(\underline{\pm,0^5},\underline{\pm,0}\right), \left(\underline{\left(\pm\frac{1}{2}\right)^6_+}, \underline{\left(\pm\frac{1}{2}\right)^2_+}\right) \right\}. \tag{4.73}$$

Then, $\Delta_g^+$ gives the nonzero roots of $E_7\times SU(2)\times E_7\times SU(2)$ and $\Delta_g^-$ corresponds to $(\mathbf{56},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})$ of the $E_7\times SU(2)\times E_7\times SU(2)$. Looking at the twisted sectors, $\Delta_g^{-,c}$ and $\Delta_g^{+,o}$ are given as

$$\Delta_g^{-,c} = \Delta_{(\mathbf{56},\mathbf{1})}\oplus\left\{\frac{\hat\pi_2}{2}\right\} + \left\{\frac{\hat\pi_1}{2}\right\}\oplus\Delta_{(\mathbf{56},\mathbf{1})}, \tag{4.74}$$

$$\Delta_g^{+,o} = \Delta_{(\mathbf{1},\mathbf{2})}\oplus\left\{\frac{\hat\pi_2}{2}\right\} + \left\{\frac{\hat\pi_1}{2}\right\}\oplus\Delta_{(\mathbf{1},\mathbf{2})}, \tag{4.75}$$

where $\frac{\hat\pi_1}{2}=\frac{\hat\pi_2}{2}=\left(0^6,\left(\frac{1}{2}\right)^2\right)$, and we define $\Delta_{(\mathbf{56},\mathbf{1})}$ and $\Delta_{(\mathbf{1},\mathbf{2})}$ as

$$\Delta_{(\mathbf{56},\mathbf{1})} = \left\{ \left(\underline{\pm,0^5}, \underline{\left(\pm\frac{1}{2}\right)^2_-}\right), \left(\underline{\left(\pm\frac{1}{2}\right)^6_+},0^2\right) \right\}, \quad \Delta_{(\mathbf{1},\mathbf{2})} = \left\{ \left(0^6, \underline{\left(\pm\frac{1}{2}\right)^2_+}\right) \right\}. \tag{4.76}$$

Thus, there are massless conjugate spinors transforming in $(\mathbf{56},\mathbf{1};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{1})$ of the $E_7\times SU(2)\times E_7\times SU(2)$ and massless scalars transforming in $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})$ of the $SU(2)\times SU(2)$ if $A\in\Gamma_+^{16}$ and $\sqrt{2}R\in\mathbb{Z}$.

Let us turn on the non-trivial Wilson lines given in (4.57a) or (4.57b). In this model, we study each of the five Wilson lines (i.e. (4.57a) with $p=6,4,2,0$ and (4.57b)) individually.

- $A_1=\left(0^6,\left(\frac{1}{2}\right)^2\right)$
  We notice $A_1=\frac{\hat\pi_1}{2}$, and hence $\Delta_{g_1'}^+=\Delta_{E_7\times SU(2)}$ while $\Delta_{g_1'}^-$ is empty. Thus, the gauge symmetry remains to be enhanced to $E_7\times SU(2)\times E_7\times SU(2)$, although spinors transforming in $(\mathbf{56},\mathbf{2};\mathbf{1},\mathbf{1})$ of the $E_7\times SU(2)\times E_7\times SU(2)$ are massive. We also find that there are no elements that satisfy $\left(\pi+\frac{\hat\pi}{2}\right)\cdot A\in\mathbb{Z}$ in the second sets in (4.74) because $\left(\frac{\hat\pi_1}{2}\right)\cdot A_1=\frac{1}{2}$ and $A=(A_1;0^8)$. So, only conjugate spinors in the twisted sectors with $\pi+\frac{\hat\pi}{2}\in\Delta_{(\mathbf{56},\mathbf{1})}\oplus\left\{\frac{\hat\pi_2}{2}\right\}$ can be massless. Note that this statement is true for all the Wilson lines we consider in this subsection. For any elements in

$\Delta_{(\mathbf{56},\mathbf{1})} \oplus \left\{\frac{\hat{\pi}_2}{2}\right\}$, the inner products with $A$ are

$$\left(\pi + \frac{\hat{\pi}}{2}\right) \cdot A = \left(\pi_1 + \frac{\hat{\pi}_1}{2}\right) \cdot A_1 = \left(\pi_1 + \frac{\hat{\pi}_1}{2}\right) \cdot \frac{\hat{\pi}_1}{2} \in \mathbb{Z}, \qquad ((\pi_1, 0^8) \in \Gamma_-^{16}). \tag{4.77}$$

Then, massless conjugate spinors transform in $(\mathbf{56}, \mathbf{1}; \mathbf{1}, \mathbf{1})$ of the $E_7 \times SU(2) \times E_7 \times SU(2)$. We also get massless scalars transforming in $(\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{2})$ of the $E_7 \times SU(2) \times E_7 \times SU(2)$ if $\sqrt{2}R \in \mathbb{Z} + 1/2$.

- $A_1 = \left(0^4, \left(\frac{1}{2}\right)^4\right)$
With this Wilson line, $\Delta^+_{g'_1}$ and $\Delta^-_{g'_1}$ are

$$\Delta^+_{g'_1} = \left\{ \left(\underline{\pm, \pm, 0^2}, 0^4\right), \left(0^2, \underline{\pm, \pm}, 0^2\right), \left(0^6, \underline{\pm, \pm}\right), \left(\underline{\left(\pm\frac{1}{2}\right)^4}_+, \underline{\left(\pm\frac{1}{2}\right)^2}_-, \underline{\left(\pm\frac{1}{2}\right)^2}_-\right) \right\}, \tag{4.78}$$

$$\Delta^-_{g'_1} = \left\{ \left(0^4, \underline{\pm, 0}, \underline{\pm, 0}\right), \left(\underline{\left(\pm\frac{1}{2}\right)^4}_+, \underline{\left(\pm\frac{1}{2}\right)^2}_+, \underline{\left(\pm\frac{1}{2}\right)^2}_+\right) \right\}, \tag{4.79}$$

which lead to the $SO(12) \times SU(2) \times SU(2)$ gauge group and massless spinors transforming in $(\mathbf{12}, \mathbf{2}, \mathbf{2})$ of the $SO(12) \times SU(2) \times SU(2)$. As for conjugate spinors in the twisted sectors, the following elements $\pi_1 + \frac{\hat{\pi}_1}{2} \in \Delta_{(\mathbf{56},\mathbf{1})}$ satisfy $\left(\pi_1 + \frac{\hat{\pi}_1}{2}\right) \cdot A_1 \in \mathbb{Z}$:

$$\pi_1 + \frac{\hat{\pi}_1}{2} = \left(\underline{\pm, 0^3}, 0^2, \underline{\left(\pm\frac{1}{2}\right)^2}_-\right), \left(\underline{\left(\pm\frac{1}{2}\right)^4}_-, \underline{\left(\pm\frac{1}{2}\right)^2}_-, 0^2\right), \tag{4.80}$$

which correspond to $(\mathbf{32}, \mathbf{1}, \mathbf{1})$ of the $SO(12) \times SU(2) \times SU(2)$. We obtain massless scalars transforming in $(\mathbf{1}, \mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{2})$ of the $SO(12) \times SU(2) \times SU(2) \times E_7 \times SU(2)$ if $\sqrt{2}R \in \mathbb{Z} + 1/2$.

- $A_1 = \left(0^2, \left(\frac{1}{2}\right)^6\right)$
One can check that $\Delta^+_{g'_1}$ with $A_1 = \left(0^2, \left(\frac{1}{2}\right)^6\right)$ is the same as with $A_1 = \left(0^4, \left(\frac{1}{2}\right)^4\right)$ up to the permutations, and hence $U(1)^8_L$ is enhanced to $SO(12) \times SU(2) \times SU(2)$. On the other hand, $\Delta^-_{g'_1}$ with this Wilson line is

$$\Delta^-_{g'_1} = \left\{ \left(0^2, \underline{\pm, 0^3}, \underline{\pm, 0}\right), \left(\underline{\left(\pm\frac{1}{2}\right)^2}_-, \underline{\left(\pm\frac{1}{2}\right)^4}_-, \underline{\left(\pm\frac{1}{2}\right)^2}_+\right) \right\}, \tag{4.81}$$

which corresponds to $(\mathbf{32},\mathbf{1},\mathbf{2})$ of the $SO(12)\times SU(2)\times SU(2)$. As for conjugate spinors in the twisted sectors, the following elements $\pi_1+\frac{\hat{\pi}_1}{2}\in\Delta_{(\mathbf{56},\mathbf{1})}$ satisfy $\left(\pi_1+\frac{\hat{\pi}_1}{2}\right)\cdot A_1\in\mathbb{Z}$:

$$\pi_1+\frac{\hat{\pi}_1}{2}=\left(\underline{\pm,0,0^4},\underline{\left(\pm\frac{1}{2}\right)^2}_{-}\right),\left(\underline{\left(\pm\frac{1}{2}\right)^2}_{+},\underline{\left(\pm\frac{1}{2}\right)^4}_{+},0^2\right), \tag{4.82}$$

which give $(\mathbf{12},\mathbf{2},\mathbf{1})$ of the $SO(12)\times SU(2)\times SU(2)$. Whereas, all the elements in $\Delta_{(\mathbf{1},\mathbf{2})}$ satisfy $\left(\pi_1+\frac{\hat{\pi}_1}{2}\right)\cdot A_1\in\mathbb{Z}+1/2$. In summary, in the twisted sectors, there are massless conjugate spinors transforming in $(\mathbf{12},\mathbf{2},\mathbf{1})$ of the $SO(12)\times SU(2)\times SU(2)$, and scalars transforming in $(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})$ of the $SO(12)\times SU(2)\times SU(2)\times E_7\times SU(2)$ become massless if $\sqrt{2}R\in\mathbb{Z}+1/2$.

- $A_1=\left(\left(\frac{1}{2}\right)^8\right)$
  As $A=(A_1,0^8)\in\Gamma_g^*$, massless states in the untwisted sectors are the same as in the case with $A_1=(0^8)$; the enhanced gauge symmetry from $U(1)_L^{16}$ is $E_7\times SU(2)\times E_7\times SU(2)$ and massless spinors transform in $(\mathbf{56},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})$ of the $E_7\times SU(2)\times E_7\times SU(2)$. There are however no elements in $\Delta_g^{-,c}$ satisfying $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}$ because $A\in\Gamma_-^{16}$. Thus, conjugate spinors in the twisted sectors cannot be massless. We obtain massless scalars transforming in $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})$ of the $E_7\times SU(2)\times E_7\times SU(2)$ if $\sqrt{2}R\in\mathbb{Z}+1/2$.

- $A_1=\left(\left(\frac{1}{4}\right)^8\right)$
  This Wilson line gives $\Delta_{g_1'}^+$ and $\Delta_{g_1'}^-$ as

$$\Delta_{g_1'}^+=\left\{\left(\underline{+,-,0^4},0^2\right),\left(0^6,\underline{+,-}\right),\left(\underline{\left(+\frac{1}{2}\right)^3,\left(-\frac{1}{2}\right)^3},\underline{\left(\pm\frac{1}{2}\right)^2}_{-}\right)\right\}, \tag{4.83}$$

$$\Delta_{g_1'}^-=\left\{\pm\left(\underline{+,0^5},\underline{-,0}\right),\pm\left(\underline{\left(+\frac{1}{2}\right)^2,\left(-\frac{1}{2}\right)^4},\left(\frac{1}{2}\right)^2\right),\pm\left(\left(\frac{1}{2}\right)^8\right)\right\}. \tag{4.84}$$

One can find that $\Delta_{g_1'}^+$ corresponds to a set of the nonzero roots of $E_6$, and $\Delta_{g_1'}^-$ yields $\mathbf{27}\oplus\overline{\mathbf{27}}$ of the $E_6$ and $U(1)$ charges $\pm1$. In the untwisted sectors, thus, there are gauge bosons of $E_6\times U(1)_1\times U(1)_2$ and massless spinors transforming in $\left(\mathbf{27},-1/\sqrt{2},0\right)\oplus\left(\overline{\mathbf{27}},1/\sqrt{2},0\right)\oplus\left(\mathbf{1},\pm1/\sqrt{2},\pm\sqrt{2}\right)$ of the $E_6\times U(1)_1\times U(1)_2$[8]. Let us move on to the twisted sectors. There are no elements in $\Delta_g^{-,c}$ satisfying $\left(\pi+\frac{\hat{\pi}}{2}\right)\cdot A\in\mathbb{Z}$, and hence

[8]The unbolded letters indicate $U(1)$ charges.

conjugate spinors cannot massless. Whereas, we can obtain massless scalars by tuning $R$ to an appropriate value. If $\sqrt{2}R \in \mathbb{Z} + 1/2$, there are massless scalars transforming in $(\mathbf{1};\mathbf{1},\mathbf{2})$ of the $E_6 \times E_7 \times SU(2)$. If $\sqrt{2}R \in \mathbb{Z} + 1/4$ or $\sqrt{2}R \in \mathbb{Z} - 1/4$, there are massless scalars with charges $\pm 1/\sqrt{2}$ of the $U(1)_1$.

### 4.4.3 $SO(16) \times SO(16)$ model: $\frac{\hat{\pi}}{2} = (1, 0^7; 1, 0^7)$

The splitting of $\Delta_{E_8 \times E_8}$ by this choice of $\hat{\pi}$ is

$$\Delta_g^+ = \Delta_{SO(16)} \oplus \Delta_{SO(16)}, \qquad \Delta_g^- = \Delta_{\mathbf{128}_+} \oplus \Delta_{\mathbf{128}_+}. \tag{4.85}$$

With the Wilson lines $A \in \Gamma_g^*$, thus, we obtain gauge bosons of $SO(16) \times SO(16)$ and massless spinors transforming in $(\mathbf{128}_+, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{128}_+)$ of the $SO(16) \times SO(16)$. Note that in the twisted sectors, scalars cannot be massless and conjugate spinors live in $\Delta_g^{+,c}$ (but not $\Delta_g^{-,c}$) since $\left|\frac{\hat{\pi}}{2}\right|^2 = 2$. If $A \in \Gamma_+^{16}$, then there are massless conjugate spinors transforming in $(\mathbf{16}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{16})$ of the $SO(16) \times SO(16)$ as $\Delta_g^{+,c}$ is given as

$$\Delta_g^{+,c} = \Delta_{\mathbf{16}} \oplus \left\{\frac{\hat{\pi}_2}{2}\right\} + \left\{\frac{\hat{\pi}_1}{2}\right\} \oplus \Delta_{\mathbf{16}}, \tag{4.86}$$

where $\frac{\hat{\pi}_1}{2} = \frac{\hat{\pi}_2}{2} = (1, 0^7)$ and

$$\Delta_{\mathbf{16}} = \left\{ \left( \underline{\pm, 0^7} \right) \right\}. \tag{4.87}$$

Let us study the massless spectra with the non-trivial Wilson lines given in (4.57a) and (4.57b). Note that $\Delta_{g_1}^{\pm}$ is the same as in the $SO(16) \times E_8$ model. Thus, the massless vectors and spinors in the untwisted sectors respectively live in $\Delta_{g_1'}^+ \oplus \Delta_{SO(16)}$ and $\Delta_{g_1'}^- \oplus \Delta_{\mathbf{128}_+}$ where $\Delta_{g_1'}^+$ and $\Delta_{g_1'}^-$ are obtained in subsection 4.4.1. We have already known the massless states in the untwisted sectors. So, in this subsection, we only need to take care of conjugate spinors in the twisted sectors.

We now consider the Wilson line (4.57a). If $p = 0$ (i.e. $A = \left( \left(\frac{1}{2}\right)^8, 0^8 \right) \in \Gamma_-^{16}$), then it is clear that any elements in $\Delta_g^{+,c}$ do not satisfy $\left(\pi + \frac{\hat{\pi}}{2}\right) \cdot A \in \mathbb{Z}$, and massless conjugate spinors do not exist. If $p \neq 0$, we find the following elements in $\Delta_g^{+,c}$ that satisfy $\left(\pi + \frac{\hat{\pi}}{2}\right) \cdot A \in \mathbb{Z}$:

$$\pi + \frac{\hat{\pi}}{2} = \left(\underline{\pm, 0^{p-1}}, 0^q; 1, 0^7\right), \left(0^p, \underline{\pm, 0^{q-1}}; 1, 0^7\right), \left(1, 0^7; \underline{\pm, 0^7}\right). \tag{4.88}$$

Then, we obtain massless conjugate spinors transforming in $(\mathbf{2p}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2q}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{16})$ of the $SO(2p) \times SO(2q) \times SO(16)$.

For the Wilson line (4.57b), there are no elements in $\Delta_g^{+,c}$ satisfying $\left(\pi + \frac{\hat{\pi}}{2}\right) \cdot A \in \mathbb{Z}$, and all conjugate spinors in the twisted sectors are massive.

The massless spectra in class (1) which we identified in subsections 4.3 and 4.4 are summarized in appendix C.

# 5 Cosmological constant

In this subsection, we evaluate the one-loop cosmological constant (vacuum energy) and show that it is exponentially suppressed in the region where supersymmetry is restored if there is a Bose-Fermi degeneracy in the massless level, as seen in (1.1). We only focus on the non-supersymmetric models in class (2), but the same discussion can be done in class (3) and class (4) by adopting the dual description of the moduli. Note that restoration of supersymmetry occurs as $R \to \infty$ in the case of class (2).

## 5.1 Exponential suppression of cosmological constant

From (2.32) and (2.49), the partition function in class (2) is written as

$$Z^{\not{SUSY}}_{(\hat{\pi},1,0)}(R, A_{(2)}) = Z_B^{(7)} \left\{ \bar{V}_8 \left( \Lambda^{(+)}[0|0|0] + \Lambda^{(-)}[0|1|0] \right) - \bar{S}_8 \left( \Lambda^{(+)}[0|1|0] + \Lambda^{(-)}[0|0|0] \right) \right.$$
$$\left. + \bar{O}_8 \left( \Lambda^{(\pm)}[1|0|1] + \Lambda^{(\mp)}[1|1|1] \right) - \bar{C}_8 \left( \Lambda^{(\pm)}[1|1|1] + \Lambda^{(\mp)}[1|0|1] \right) \right\}, \tag{5.1}$$

where $\Lambda^{(\pm)}[\alpha|\beta|\gamma]$ is defined as

$$\Lambda^{(\pm)}[\alpha|\beta|\gamma] = \eta^{-16} (\eta\bar{\eta})^{-1} \sum_{\pi \in \Gamma^{16}_{\pm} + \frac{\gamma}{2}\hat{\pi}} \sum_{m \in \mathbb{Z} + \frac{\alpha}{2}} \sum_{n \in 2\mathbb{Z} + \beta} q^{\frac{1}{2}\left(\ell_L^2 + p_L^2\right)} \bar{q}^{\frac{1}{2} p_R^2}, \tag{5.2}$$

with $(\ell_L, p_L, p_R)$ given in (2.42). Note that the lower signs of $\Lambda^{(\pm)}$ and $\Lambda^{(\mp)}$ in the twisted sectors are adopted only when the non-supersymmetric endpoint model is the $SO(16) \times SO(16)$ model.

The cosmological constant is defined as the integral of the partition function over the fundamental domain of the modular group:

$$\Lambda = -\frac{1}{2} \left(4\pi^2 \alpha'\right)^{-\frac{9}{2}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z^{\not{SUSY}}_{(\hat{\pi},1,0)}, \tag{5.3}$$

where the fundamental domain $\mathcal{F}$ is

$$\mathcal{F} = \left\{ \tau = \tau_1 + i\tau_2 \in \boldsymbol{C} \;\middle|\; -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2},\ |\tau| \geq 1 \right\}. \tag{5.4}$$

For our convenience, we decompose $\mathcal{F}$ into two pieces $\mathcal{F}_{\geq 1} = \mathcal{F}|_{\tau_2 \geq 1}$ and $\mathcal{F}_{<1} = \mathcal{F}|_{\tau_2 < 1}$. The contributions from the states with $m \neq 0$ are exponentially suppressed as $R$ grows larger.

Thus, we can ignore the last two terms in the partition function (5.1). Using the Jacobi's abstruse identity (A.14), we get

$$\Lambda \sim -\frac{1}{2}\left(4\pi^2\alpha'\right)^{-\frac{9}{2}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}Z_B^{(7)}\bar{V}_8\sum_{\epsilon=\pm}\epsilon\left(\Lambda^{(\epsilon)}[0|0|0]-\Lambda^{(\epsilon)}[0|1|0]\right). \tag{5.5}$$

Focusing on the internal momenta with $m=0$, we find

$$\begin{aligned}\Lambda^{(\epsilon)}[0|0|0]-\Lambda^{(\epsilon)}[0|1|0] &\sim \eta^{-17}\bar{\eta}^{-1}\sum_{\pi\in\Gamma^{16}_\epsilon}q^{\frac{|\pi|^2}{2}}\sum_{n\in\mathbb{Z}}\left(e^{-\frac{4\pi\tau_2}{R^2}\left(n+\frac{\pi\cdot A_{(2)}}{2}\right)^2}-e^{-\frac{4\pi\tau_2}{R^2}\left(n+\frac{\pi\cdot A_{(2)}+1}{2}\right)^2}\right)\\ &=\eta^{-17}\bar{\eta}^{-1}\sum_{\pi\in\Gamma^{16}_\epsilon}q^{\frac{|\pi|^2}{2}}\left(\vartheta\begin{bmatrix}\frac{\pi\cdot A_{(2)}}{2}\\0\end{bmatrix}\left(0,\frac{4i\tau_2}{R^2}\right)-\vartheta\begin{bmatrix}\frac{\pi\cdot A_{(2)}+1}{2}\\0\end{bmatrix}\left(0,\frac{4i\tau_2}{R^2}\right)\right).\end{aligned} \tag{5.6}$$

One should not confuse the two different uses of $\pi$: one is for the ratio of a circle's circumference, while the other is for a element of $\Gamma^{16}$. By using the $S$-transformation low (A.16) of the theta function, we see that (5.6) can be written as

$$\frac{2R}{\sqrt{\tau_2}}\eta^{-17}\bar{\eta}^{-1}\sum_{\pi\in\Gamma^{16}_\epsilon}q^{\frac{|\pi|^2}{2}}\sum_{n\geq 1}\cos\left[\pi(2n-1)\left(\pi\cdot A_{(2)}\right)\right]\exp\left[-\frac{\pi(2n-1)^2}{4\tau_2}R^2\right]. \tag{5.7}$$

The cosmological constant can be then expressed as

$$\Lambda \sim -\frac{R}{\left(4\pi^2\alpha'\right)^{\frac{9}{2}}}\sum_{n\geq 1}\sum_{\epsilon=\pm}\sum_{M_+,M_-}\epsilon a^{(\epsilon)}_{M_+,M_-}\cos\left[\pi(2n-1)\left(\pi\cdot A_{(2)}\right)\right]\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^6}e^{2\pi i\tau_1 M_-}e^{-\pi\left(2\tau_2 M_+ +\frac{(2n-1)^2}{4\tau_2}R^2\right)}, \tag{5.8}$$

where we expanded the contributions from $\pi$ and the oscillators as follows:

$$\sum_{\pi\in\Gamma^{16}_\epsilon}q^{\frac{|\pi|^2}{2}}\eta^{-24}\bar{\eta}^{-8}\bar{V}_8=\sum_{M_-,M_+}a^{(\epsilon)}_{M_+,M_-}e^{2\pi i\tau_1 M_-}e^{-2\pi\tau_2 M_+}. \tag{5.9}$$

Let us evaluate the integral over $\mathcal{F}$ by decomposing into $\mathcal{F}_{<1}$ and $\mathcal{F}_{\geq 1}$. For the integration over $\mathcal{F}_{<1}$, the domain itself is finite and the integrand itself is singularity free. We can thus easily bound the integration as follows:

$$\int_{\mathcal{F}_{<1}}\frac{d^2\tau}{\tau_2^6}e^{2\pi i\tau_1 M_-}e^{-2\pi\tau_2 M_+}e^{-\frac{\pi(2n-1)^2}{4\tau_2}R^2}<e^{-2\pi M_+}e^{-\frac{\pi(2n-1)^2}{4}R^2}\int_{\mathcal{F}_{<1}}\frac{d^2\tau}{\tau_2^6}e^{2\pi i\tau_1 M_-}. \tag{5.10}$$

So, the contribution from the integration over $\mathcal{F}_{<1}$ is suppressed at least by the factor $e^{-\pi R^2/4}$ as $R$ goes to infinity. As for $\mathcal{F}_{\geq 1}$, the domain is infinite and it is non-trivial to give a finite contribution. By using the inequality on the arithmetic-geometric mean, however, we can determine the upper bound of the $\tau_2$-dependent factor:

$$\exp\left[-\pi\left(2\tau_2 M_+ + \frac{(2n-1)^2}{4\tau_2}R^2\right)\right] \leq e^{-\sqrt{2}\pi(2n-1)\sqrt{M_+}R}. \tag{5.11}$$

This bound is $\tau_2$ independent and together with $\tau_2^{-6}$, it can be integrated over $\mathcal{F}_{\geq 1}$, giving a finite prefactor. Note that we can carry out the integration over $\tau_1$ and $\tau_2$ individually, and the former gives $\delta_{M_-,0}$. Under the level-matching condition $M_- = 0$, $M_+$ cannot be negative, and hence the contribution from the integration over $\mathcal{F}_{\geq 1}$ is suppressed at least by the factor $e^{\sqrt{2}\pi\sqrt{M_+}R}$ unless $M_+ = 0$. Therefore, the leading contribution comes from the integration of the terms with $M_+ = 0$ over $\mathcal{F}_{\geq 1}$:

$$\begin{aligned}\Lambda &\sim -\frac{R}{(4\pi^2\alpha')^{\frac{9}{2}}}\sum_{n\geq 1}\sum_{\epsilon=\pm}\sum_{M_+,M_-=0}\epsilon a^{(\epsilon)}_{M_+,M_-}\cos\left[\pi(2n-1)\left(\pi\cdot A_{(2)}\right)\right]\int_1^\infty \frac{d\tau_2}{\tau_2^6}e^{-\frac{\pi(2n-1)^2}{4\tau_2}R^2} \\ &\sim -\frac{48}{\pi^{14}\left(\sqrt{\alpha'}R\right)^9}\sum_{n\geq 1}(2n-1)^{-10}\sum_{\epsilon=\pm}\sum_{M_+,M_-=0}\epsilon a^{(\epsilon)}_{M_+,M_-}\cos\left[\pi(2n-1)\left(\pi\cdot A_{(2)}\right)\right], \end{aligned}\tag{5.12}$$

where we perform the $\tau_2$-integration and omit the exponentially suppressed terms. Let us show that this leading contribution (5.12) is proportional to $n_F - n_B$ with $n_F$ and $n_B$ being the degrees of freedom of massless fermions and massless bosons respectively if $A_{(2)} \in \Gamma_g^*$. There are two possibilities of $M_+ = M_- = 0$. One is with $\pi = 0$ which corresponds to sector 1 we introduced in the previous section and which has $8 \times 24$ degrees of freedom. The other is with $|\pi|^2 = 2$ which correspond to sector 2. Then, the cosmological constant (5.12) is

$$\Lambda \sim -\frac{48}{\pi^{14}\left(\sqrt{\alpha'}R\right)^9}\sum_{n\geq 1}(2n-1)^{-10}8\left\{24 + \sum_{\epsilon=\pm}\sum_{\pi\in\Delta_g^\epsilon}\epsilon\cos\left[\pi(2n-1)\left(\pi\cdot A_{(2)}\right)\right]\right\}. \tag{5.13}$$

Let us assume that $A_{(2)}$ satisfies $\pi \cdot A_{(2)} \in \mathbb{Z}$ for all elements $\pi$ of $\Delta_g$, i.e., $A \in \Gamma_g^*$. Under this assumption, the second term in the parentheses in (5.13) is independent of $n$, and one can notice

$$\epsilon\cos\left[\pi(2n-1)\left(\pi\cdot A_{(2)}\right)\right] = \begin{cases} +1 & \text{for } A_{(2)} \text{ with (4.16)} \\ -1 & \text{for } A_{(2)} \text{ with (4.17)} \end{cases}. \tag{5.14}$$

Recalling that the massless conditions for vectors and spinors in sector 2 are given by (4.16) and (4.17) respectively, this factor assigns +1 to massless vectors and -1 to massless spinors.

Including the contribution from sector 1, up to the exponentially suppressed terms, the cosmological constant is expressed as

$$\Lambda \sim \frac{48}{\pi^{14}\left(\sqrt{\alpha'}R\right)^9} 2^{-10}\zeta\left(10, \frac{1}{2}\right)(n_F - n_B)\,, \tag{5.15}$$

where $n_F$ and $n_B$ are the degrees of freedom of massless bosons and fermions, and $\zeta(s,a)$ is the Hurwitz zeta function:

$$\zeta(s,a) = \sum_{n\geq 0}(n+a)^{-s}\,. \tag{5.16}$$

We have therefore shown that the cosmological constant is exponentially suppressed as $R \to \infty$ if there exists the Bose-Fermi degeneracy at the massless level.

Note that the Wilson line was above assumed to satisfy $\pi \cdot A_{(2)} \in \mathbb{Z}$ for any $\pi \in \Delta_g$ so that all the cosine factors multiplied by $\epsilon$ in (5.13) give $+1$ or $-1$, which is respectively assigned to a massless boson and a massless fermion. So, it seems that the expression (5.15) of the cosmological constant is valid only when $A_{(2)} \in \Delta_g^*$. However, we can relax this assumption to $2A_{(2)} \in \Delta_g^*$. This is because, under the new assumption, there might exist $\pi \in \Delta_g$ with $\pi \cdot A_{(2)} \in \mathbb{Z} + 1/2$ which do not give massless states, but such $\pi$'s do not contribute to the leading term of the cosmological constant, and hence the non-vanishing contributions come only from $\pi$'s with $\pi \cdot A_{(2)} \in \mathbb{Z}$. Then, it can be interpreted that the cosmological constant is proportional to $n_F - n_B$ up to the exponentially suppressed terms if the Wilson line satisfies $2A_{(2)} \in \Delta_g^*$.

At the end of this subsection, we should point out that the leading term (5.15) does not depend on the choice $\hat{\pi}$ (i.e. splitting of $\Gamma^{16}$ into $\Gamma^{16}_+$ and $\Gamma^{16}_-$). Recall that the Wilson line $A_{(2)}$ in class (2) is related to that in toroidal models by (4.18). Inserting (4.18) into (5.13),

$$\Lambda \sim -\frac{48}{\pi^{14}\left(\sqrt{\alpha'}R\right)^9}\sum_{n\geq 1}(2n-1)^{-10} 8\left\{24 + \sum_{\pi\in\Delta_g}\cos\left[2\pi(2n-1)\left(\pi\cdot A_{(T)}\right)\right]\right\}. \tag{5.17}$$

The cosine factor gives $+1$ and $-1$ for $\pi \in \Delta_g$ with $\pi \cdot A_{(T)} \in \mathbb{Z}$ and $\pi \cdot A_{(T)} \in \mathbb{Z} + 1/2$ respectively, while vanishes for $\pi$ with $\pi \cdot A_{(T)} \in \mathbb{Z} + 1/4$. Then, the cosmological constants of the non-supersymmetric models (interpolating models) in class (2) do not depend on the non-supersymmetric endpoint models as long as we only focus on the region with supersymmetry being restored. We will adopt the description by $A_{(T)}$ but not $A_{(2)}$ in the later discussions in which the symmetry enhancement and the moduli stability in class (2) are mainly considered.

In [44], the subleading contributions to the cosmological constant have been derived, but we do not consider the exponentially suppressed contributions in this thesis. We can evaluate the cosmological constants in class (3) and class (4) in the same way as we have done above. But one should note that supersymmetry is restored as $R \to 0$ in class (3), and needs to replace $(A, R)$ to $(\tilde{A}, \tilde{R})$ in (5.15). In class (4), supersymmetry is restored in both of the endpoint limits, and (5.15) is valid for both of the normal and dual descriptions of the moduli.

## 5.2 Exponential suppression with Wilson line

In this subsection, we identify the massless spectra in class (2) as we have done in subsection 4.3 and 4.4. In particular, we devote our attention to searching for the massless spectra with $n_F = n_B$ which realize the exponentially suppressed cosmological constants. Unlike in class (1), we do not need to specify the choice of $\hat{\pi}$ in order to figure out the massless spectrum with $m = 0$ due to the relation (4.18) between $A_{(2)}$ and $A_{(T)}$. The information of $\hat{\pi}$ is needed if one would like to know the corresponding Wilson line $A_{(2)}$ in the non-supersymmetric models. We thus use the massless conditions (4.19) and (4.20) (but not (4.16) and (4.17)), and restrict our attention to the Wilson lines that satisfy $4A_{(T)} \in \Gamma_g^*$ so that the expression (5.15) can be used.

We here define $\Delta_{g'}^{(B)}$ and $\Delta_{g'}^{(F)}$ as

$$\Delta_{g'}^{(B)} = \left\{\pi \in \Delta_g \,\middle|\, \pi \cdot A_{(T)} \in \mathbb{Z}\right\}, \qquad \Delta_{g'}^{(F)} = \left\{\pi \in \Delta_g \,\middle|\, \pi \cdot A_{(T)} \in \mathbb{Z} + \frac{1}{2}\right\}. \tag{5.18}$$

Note that $\Delta_{g'}^{(B)}$ is a set of the nonzero roots of the gauge group that is enhanced by $A_{(T)}$, and $\Delta_{g'}^{(F)}$ yields the representation of massless spinors. In order to realize $n_F = n_B$, the Wilson line must give a massless spectrum that satisfies

$$|\Delta_{g'}^{(F)}| - |\Delta_{g'}^{(B)}| = 24, \tag{5.19}$$

where $|\Delta|$ indicates the number of elements in a set $\Delta$. Note that 24 is the degrees of freedom of the left-moving massless states in sector 1.

### 5.2.1 Supersymmetric $Spin(32)/\mathbb{Z}_2$ endpoint model

Let us first consider the following configuration of the Wilson line:

$$A_{(T)} = \left(0^p, \left(\frac{1}{2}\right)^q, \left(\frac{1}{4}\right)^r\right) \qquad (p+q+r=16). \tag{5.20}$$

Note that this Wilson line satisfies $4A_{(T)} \in \Gamma_g^{(16)}$ or $4A_{(T)} \in \Gamma_v^{(16)}$. Recalling $\Delta_g$ given by (4.10), we find

$$\Delta_{g'}^{(B)} = \left\{\left(\underline{\pm,\pm,0^{p-2}},0^{q+r}\right), \left(0^p,\underline{\pm,\pm,0^{q-2}},0^r\right), \left(0^{p+q},\underline{-,+,0^{r-2}}\right)\right\}, \tag{5.21}$$
$$\Delta_{g'}^{(F)} = \left\{\left(\underline{\pm,0^{p-1}},\underline{\pm,0^{q-1}},0^r\right), \pm\left(0^{p+q}\underline{+,+,0^{r-2}}\right)\right\}. \tag{5.22}$$

Then, with the Wilson line (5.20), there are gauge bosons transforming in the adjoint representation of $SO(2p)\times SO(2q)\times SU(r)$ and massless spinors transforming in $(\boldsymbol{2p},\boldsymbol{2q},\boldsymbol{1})\oplus\left(\boldsymbol{1},\boldsymbol{1},\boldsymbol{\frac{r(r-1)}{2}}\right)\oplus\left(\boldsymbol{1},\boldsymbol{1},\overline{\boldsymbol{\frac{r(r-1)}{2}}}\right)$ of the $SO(2p)\times SO(2q)\times SU(r)$. Then, the numbers of elements in $\Delta_{g'}^{(B)}$ and $\Delta_{g'}^{(F)}$ are

$$|\Delta_{g'}^{(B)}| = 2p(p-1)+2q(q-1)+r(r-1), \tag{5.23}$$
$$|\Delta_{g'}^{(F)}| = 4pq+r(r-1). \tag{5.24}$$

and (5.19) is written as

$$p+q-(p-q)^2 = 12. \tag{5.25}$$

With $p+q+r=16$, the solutions of (5.25) are

$$(p,q,r) = (\underline{7,9},0), (\underline{6,7},3), (6,6,4). \tag{5.26}$$

The cosmological constant is therefore exponentially suppressed with the gauge symmetry enhancements from $U(1)_L^{16}$ to $SO(18)\times SO(14)$ or $SO(14)\times SO(12)\times SU(3)\times U(1)$ or $SO(12)\times SO(12)\times SU(4)\times U(1)$.

The other interesting configuration of the Wilson line is that satisfying $4A_{(T)}\in\Gamma_s^{(16)}$ or $4A_{(T)}\in\Gamma_c^{(16)}$. So, let us consider the following Wilson line:

$$A_{(T)} = \left(\left(\frac{1}{8}\right)^s, \left(\frac{3}{8}\right)^t\right) \qquad (s+t=16). \tag{5.27}$$

Note that $4A_{(T)} \in \Gamma_s^{(16)}$ for $t$ even, while $4A_{(T)} \in \Gamma_c^{(16)}$ for $t$ odd. With this $A_{(T)}$, we obtain $\Delta_{g'}^{(B)}$ and $\Delta_{g'}^{(F)}$ as

$$\Delta_{g'}^{(B)} = \left\{ \left( \underline{+,-,0^{s-2}}, 0^t \right), \left( 0^s, \underline{+,-,0^{t-2}} \right) \right\}, \tag{5.28}$$

$$\Delta_{g'}^{(F)} = \left\{ \pm \left( \underline{+,0^{s-1}}, \underline{+,0^{t-1}} \right) \right\}, \tag{5.29}$$

which lead to the symmetry enhancement $U(1)_L^{16} \to SU(s) \times SU(t) \times U(1)^2$ and massless spinors transforming in $(\boldsymbol{s}, \boldsymbol{t}) \oplus (\overline{\boldsymbol{s}}, \overline{\boldsymbol{t}})$ of the $SU(s) \times SU(t)$, and then $|\Delta_{g'}^{(B)}|$ and $|\Delta_{g'}^{(F)}|$ are

$$|\Delta_{g'}^{(B)}| = s(s-1) + t(t-1), \qquad |\Delta_{g'}^{(F)}| = 2st. \tag{5.30}$$

Inserting (5.30) into (5.19),

$$-(s-t)^2 + 16 = 24. \tag{5.31}$$

Clearly, there is no solution of (5.31). Rather than that, (5.31) implies that the cosmological constant with the Wilson line (5.27) is always negative. Thus, we cannot obtain the exponentially suppressed cosmological constant with the Wilson line (4.25b).

### 5.2.2 Supersymmetric $E_8 \times E_8$ endpoint model

As in the case with the $Spin(32)/\mathbb{Z}_2$ endpoint model, we consider the Wilson line $A_{(T)} = (A_1; A_2)$ satisfying $4A_{(T)} \in \Gamma^{16}$, that is, $4A_i \in \Gamma_g^{(8)}$ or $4A_i \in \Gamma_s^{(8)}$ for $i = 1, 2$. As in subsection 4.4, $\Delta_{g'}^{(B)}$ and $\Delta_{g'}^{(F)}$ can be expressed as the direct sums of the two sets:

$$\Delta_{g'}^{(B)} = \Delta_{g'_1}^{(B)} \oplus \Delta_{g'_2}^{(B)}, \qquad \Delta_{g'}^{(F)} = \Delta_{g'_1}^{(F)} \oplus \Delta_{g'_2}^{(F)}, \tag{5.32}$$

where $\Delta_{g'_i}^{(B)}$ and $\Delta_{g'_i}^{(F)}$ are defined as

$$\Delta_{g'_i}^{(B)} = \left\{ \pi_i \in \Delta_{E_8} \left| \pi_i \cdot A_i \in \mathbb{Z} \right. \right\}, \qquad \Delta_{g'_i}^{(F)} = \left\{ \pi_i \in \Delta_{E_8} \left| \pi_i \cdot A_i \in \mathbb{Z} + \frac{1}{2} \right. \right\}. \tag{5.33}$$

So, it is sufficient to see only the first eight components of the Wilson line and identify $\Delta_{g'_1}^{(B)}$ and $\Delta_{g'_1}^{(F)}$. The counterparts $\Delta_{g'_2}^{(B)}$ and $\Delta_{g'_2}^{(F)}$ can be obtained in the same way. We then focus on the following configurations of $A_1$:

$$A_1 = \left( 0^{p_1}, \left( \frac{1}{2} \right)^{q_1}, \left( \frac{1}{4} \right)^{r_1} \right), \tag{5.34a}$$

$$A_1 = \left( \left( \frac{1}{8} \right)^8 \right), \tag{5.34b}$$

where $p_1 + q_1 + r_1 = 8$ and $r_1$ is even so that $4A_1 \in \Gamma_g^{(8)}$. The results of the massless spectra that the above Wilson lines realize are shown in Table 6. We here give the procedure of identification of the massless spectra only for $A_1$ given in (5.34a) with $r_1 = 0$ and $r_1 = 8$. The results for the other Wilson lines can be obtained in the same way we will present below.

Recall that $\Delta_{E_8}$ is decomposed into $\Delta_{SO(16)}$ and $\Delta_{\mathbf{128}_+}$. Restricting our attention to the elements in $\Delta_{SO(16)}$, the following $\pi_1$'s satisfy $\pi_1 \cdot A_1 \in \mathbb{Z}$ for $A_1 = \left(0^{p_1}, \left(\frac{1}{2}\right)^{q_1}\right)$:

$$\pi_1 = \left(\underline{\pm, \pm, 0^{p_1-2}}, 0^{q_1}\right), \left(0^{p_1}, \underline{\pm, \pm, 0^{q_1-2}}\right). \tag{5.35}$$

Whereas, $\pi_1 \in \Delta_{SO(16)}$ that satisfy $\pi_1 \cdot A_1 \in \mathbb{Z} + 1/2$ are

$$\pi_1 = \left(\underline{\pm, 0^{p_1-2}}, \underline{\pm, 0^{q_1-2}}\right). \tag{5.36}$$

One can check that the elements $\pi_1 \in \Delta_{\mathbf{128}_+}$ satisfy $\pi_1 \cdot A_1 \in \mathbb{Z}$ or $\pi_1 \cdot A_1 \in \mathbb{Z} + 1/2$ only if $p_1$ is even. Thus, for $A_1$ with $p_1$ odd, $\Delta^{(B)}_{g_1'}$ gives the nonzero roots of $SO(2p_1) \times SO(2q_1)$, and $\Delta^{(F)}_{g_1'}$ gives $(\mathbf{2p_1}, \mathbf{2q_1})$ of the $SO(2p_1) \times SO(2q_1)$.

Let us consider the cases with $p_1$ even. If $p_1 = 0$ or $p_1 = 8$, which means $A_1$ is in the $E_8$ root lattice, then $\Delta^{(B)}_{g_1'} = \Delta_{E_8}$ and $\Delta^{(F)}_{g_1'}$ is empty. If $p_1 = 2$, we have the following $\pi_1 \in \Delta_{\mathbf{128}_+}$ satisfying $\pi_1 \cdot A_1 \in \mathbb{Z}$:

$$\pi_1 = \frac{1}{2}\left(\underline{\pm, \pm}_-, \underline{\pm, \pm, \pm, \pm, \pm, \pm}_-\right), \tag{5.37}$$

Accompanied with (5.35) with $p_1 = 2$, we get the nonzero roots of $SU(2) \times E_7$. As for massless spinors, $\pi_1 \in \Delta_{\mathbf{128}_+}$ satisfying $\pi_1 \cdot A_1 \in \mathbb{Z} + 1/2$ are

$$\pi_1 = \frac{1}{2}\left(\underline{\pm, \pm}_+, \underline{\pm, \pm, \pm, \pm, \pm, \pm}_+\right). \tag{5.38}$$

Including (5.36) with $p_1 = 2$, then, we obtain $(\mathbf{2}, \mathbf{56})$ of the $SU(2) \times E_7$. The case with $p_1 = 6$ leads to the same $\Delta^{(B)}_{g_1'}$ and $\Delta^{(F)}_{g_1'}$ as with $p_1 = 2$ up to the permutations of the components. If $p_1 = 4$, subsets of $\Delta_{\mathbf{128}_+}$ in which the elements satisfy $\pi_1 \cdot A_1 \in \mathbb{Z}$ and $\pi_1 \cdot A_1 \in \mathbb{Z} + 1/2$ are respectively

$$\pi_1 = \frac{1}{2}\left(\underline{\pm, \pm, \pm, \pm}_+, \underline{\pm, \pm, \pm, \pm}_+\right), \tag{5.39}$$

$$\pi_1 = \frac{1}{2}\left(\underline{\pm, \pm, \pm, \pm}_-, \underline{\pm, \pm, \pm, \pm}_-\right). \tag{5.40}$$

Accompanied with the nonzero roots of $SO(8) \times SO(8)$ given by (5.35) with $p_1 = 4$, (5.39) which corresponds to $(\mathbf{8}_+, \mathbf{8}_+)$ of $SO(8) \times SO(8)$ leads to the nonzero roots of $SO(16)$. Meanwhile, (5.40) and (5.36) with $p_1 = 4$ yield $\mathbf{128}$ of $SO(16)$.

If $A_1 = \left(\left(\frac{1}{4}\right)^8\right)$, the following $\pi_1 \in \Delta_{SO(16)}$ satisfy $\pi_1 \cdot A_1 \in \mathbb{Z}$ and $\pi_1 \cdot A_1 \in \mathbb{Z} + 1/2$:

$$\pi_1 = \begin{cases} \left(\underline{+,-,0^6}\right) & \text{for } \pi_1 \cdot A_1 \in \mathbb{Z} \\ \pm\left(\underline{+,+,0^6}\right) & \text{for } \pi_1 \cdot A_1 \in \mathbb{Z} + \frac{1}{2} \end{cases} . \tag{5.41}$$

For $\pi_1 \in \Delta_{\mathbf{128}_+}$, we find the following elements satisfying $\pi_1 \cdot A_1 \in \mathbb{Z}$ or $\pi_1 \cdot A_1 \in \mathbb{Z} + 1/2$:

$$\pi_1 = \begin{cases} \pm\left(\left(\frac{1}{2}\right)^8\right), \frac{1}{2}\left(\underline{(+)^4, (-)^4}\right) & \text{for } \pi_1 \cdot A_1 \in \mathbb{Z} \\ \pm\frac{1}{2}\left(\underline{(+)^2, (-)^6}\right) & \text{for } \pi_1 \cdot A_1 \in \mathbb{Z} + \frac{1}{2} \end{cases} . \tag{5.42}$$

Putting (5.41) and (5.42) together, we obtain the nonzero roots of $E_7 \times SU(2)$ for $\pi_1 \cdot A_1 \in \mathbb{Z}$ and $(\mathbf{2}, \mathbf{56})$ of the $E_7 \times SU(2)$ for $\pi_1 \cdot A_1 \in \mathbb{Z} + 1/2$.

The one sides of the massless spectra with $A_1 = \left(0^{p_1}, \left(\frac{1}{2}\right)^{q_1}\right)$ and $A_1 = \left(\left(\frac{1}{4}\right)^8\right)$ which we have just seen above are shown in the first six rows and in Table 6. Note that $A_1 = \left(0^{p_1}, \left(\frac{1}{2}\right)^{q_1}\right)$ with $p_1$ even and $A_1 = \left(\left(\frac{1}{4}\right)^8\right)$ satisfy $2A_1 \in \Gamma_g^{(8)}$ or $2A_1 \in \Gamma_s^{(8)}$, and hence $|\Delta_{g_1'}^{(B)}| + |\Delta_{g_1'}^{(F)}| = |\Delta_{E_8}|$ since all the elements in $\Delta_{E_8}$ satisfy either $\pi_1 \cdot A_1 \in \mathbb{Z}$ or $\pi_1 \cdot A_1 \in \mathbb{Z} + 1/2$.

We now search for the possibility of suppression of the cosmological constant. We show $|\Delta_{g_1'}^{(F)}| - |\Delta_{g_1'}^{(B)}|$ for each of $A_1$ for which $4A_1 \in \left(\Gamma_g^{(8)} + \Gamma_s^{(8)}\right)$ holds in the fourth column in Table 6. In order to find out the massless spectra which lead to the exponentially suppressed cosmological constant, we need to get the combination in the fourth column in Table 6 such that the sum is 24. There are two such combinations: 16 + 8 and 12 + 12. Then, the cosmological constant is exponentially suppressed when $U(1)_L^{16}$ is enhanced to $SO(16) \times SO(10) \times SO(6)$ or $SU(8) \times SU(2) \times SU(8) \times SU(2)$.

| $A_1$ | Gauge bosons | Massless spinors | $\lvert\Delta^{(F)}_{g'_1}\rvert - \lvert\Delta^{(B)}_{g'_1}\rvert$ |
|---|---|---|---|
| $(0^8)\,,\left(\left(\frac{1}{2}\right)^8\right)$ | $E_8$ | — | $-240$ |
| $\left(0^7,\frac{1}{2}\right),\ \left(0,\left(\frac{1}{2}\right)^7\right)$ | $SO(14)\times U(1)$ | $(\mathbf{14},\pm1)$ | $-56$ |
| $\left(0^6,\left(\frac{1}{2}\right)^2\right),\left(0^2,\left(\frac{1}{2}\right)^6\right)$ | $E_7\times SU(2)$ | $(\mathbf{56},\mathbf{2})$ | $-16$ |
| $\left(0^5,\left(\frac{1}{2}\right)^3\right),\ \left(0^3,\left(\frac{1}{2}\right)^5\right)$ | $SO(10)\times SO(6)$ | $(\mathbf{10},\mathbf{6})$ | $+8$ |
| $\left(0^4,\left(\frac{1}{2}\right)^4\right)$ | $SO(16)$ | $\mathbf{128}$ | $+16$ |
| $\left(\left(\frac{1}{4}\right)^8\right)$ | $E_7\times SU(2)$ | $(\mathbf{56},\mathbf{2})$ | $-16$ |
| $\left(0^6,\left(\frac{1}{4}\right)^2\right),\left(\left(\frac{1}{2}\right)^6,\left(\frac{1}{4}\right)^2\right)$ | $E_7\times U(1)$ | $\left(\mathbf{1},\pm\sqrt{2}\right)$ | $-124$ |
| $\left(0^5,\frac{1}{2},\left(\frac{1}{4}\right)^2\right),\left(0,\left(\frac{1}{2}\right)^5,\left(\frac{1}{4}\right)^2\right)$ | $E_6\times SU(2)\times U(1)$ | $\left(\mathbf{27},\mathbf{1},\pm2/\sqrt{6}\right)$ | $-20$ |
| $\left(0^4,\left(\frac{1}{2}\right)^2,\left(\frac{1}{4}\right)^2\right),\left(0^2,\left(\frac{1}{2}\right)^4,\left(\frac{1}{4}\right)^2\right)$ | $SO(12)\times SU(2)\times U(1)$ | $(\mathbf{32},\mathbf{2},0)\oplus\left(\mathbf{1},\mathbf{1},\pm\sqrt{2}\right)$ | $+4$ |
| $\left(0^3,\left(\frac{1}{2}\right)^3,\left(\frac{1}{4}\right)^2\right)$ | $SU(8)\times SU(2)$ | $(\mathbf{70},\mathbf{1})$ | $+12$ |
| $\left(0^4,\left(\frac{1}{4}\right)^4\right),\left(\left(\frac{1}{2}\right)^4,\left(\frac{1}{4}\right)^4\right)$ | $SO(14)\times U(1)$ | $(\mathbf{14},\pm1)$ | $-56$ |
| $\left(0^3,\frac{1}{2},\left(\frac{1}{4}\right)^2\right),\left(0,\left(\frac{1}{2}\right)^3,\left(\frac{1}{4}\right)^4\right)$ | $SU(8)\times U(1)$ | $\left(\mathbf{28},\pm1/\sqrt{2}\right)$ | $0$ |
| $\left(0^2,\left(\frac{1}{2}\right)^2,\left(\frac{1}{4}\right)^4\right)$ | $SO(10)\times SO(6)$ | $(\mathbf{10},\mathbf{6})$ | $+8$ |
| $\left(0^2,\left(\frac{1}{4}\right)^6\right),\left(\left(\frac{1}{2}\right)^2,\left(\frac{1}{4}\right)^6\right)$ | $E_6\times SU(2)\times U(1)$ | $\left(\mathbf{27},\mathbf{1},\pm2/\sqrt{6}\right)$ | $-20$ |
| $\left(0,\frac{1}{2},\left(\frac{1}{4}\right)^6\right)$ | $SO(12)\times SU(2)\times U(1)$ | $(\mathbf{32},\mathbf{2},0)\oplus\left(\mathbf{1},\mathbf{1},\pm\sqrt{2}\right)$ | $+4$ |
| $\left(\left(\frac{1}{8}\right)^8\right)$ | $E_7\times U(1)$ | $\left(\mathbf{1},\pm\sqrt{2}\right)$ | $-124$ |

Table 6: This table shows the massless spectra and $|\Delta^{(F)}_{g'_1}| - |\Delta^{(B)}_{g'_1}|$ depending on the configurations of the Wilson line in class (2) with the $E_8$ root lattice. The unbolded letters indicate $U(1)$ charges.

# 6 Moduli stability

The cosmological constants in the non-supersymmetric models, which can be regarded as the effective potential of the moduli, do not vanish because of supersymmetry breaking, and hence some of the moduli can be stabilized. In this section, we analyze stability of the Wilson line moduli by using the cosmological constant we calculated in the previous section. We use the expression (5.17) but not (5.15) so that the stability analysis does not depend on the choice of the non-supersymmetric endpoint model of the interpolation. In this section, we denote $A_{(T)}$ as $A$, omitting the subscription.

## 6.1 Supersymmetric $Spin(32)/\mathbb{Z}_2$ endpoint model

We first consider the interpolating models with the supersymmetric endpoint model being the $Spin(32)/\mathbb{Z}_2$ model. Inserting $\Delta_g = \Delta_{so(32)}$ into (5.17), the Wilson line dependent part can be written as

$$\begin{aligned}
&-\sum_{n\geq 1}(2n-1)^{-10}\sum_{\pi\in\Delta_{SO(32)}}\cos\left[2\pi(2n-1)\left(\pi\cdot A\right)\right] \\
&= -2\sum_{n\geq 1}(2n-1)^{-10}\sum_{I>J}\left(\cos\left[2\pi(2n-1)\left(A^I+A^J\right)\right]+\cos\left[2\pi(2n-1)\left(A^I-A^J\right)\right]\right) \\
&= -4\sum_{n\geq 1}(2n-1)^{-10}\sum_{I>J}\cos\left[2\pi(2n-1)A^I\right]\cos\left[2\pi(2n-1)A^J\right], \qquad (6.1)
\end{aligned}$$

where $I, J = 1, \cdots, 16$ indicate the indices in the six-teen internal dimensions, and we have omitted the positive prefactor which is not important in the stability analysis. Then, the first derivative of the cosmological constant is

$$\frac{\partial\Lambda}{\partial A^I} \sim 8\pi\sum_{n\geq 1}(2n-1)^{-9}\sin[2\pi(2n-1)A^I]\sum_{J\neq I}\cos\left[2\pi(2n-1)A^J\right]. \qquad (6.2)$$

For simplicity, we only consider the Wilson lines given in (5.20) for which $4A \in \Gamma_g^{(16)}$ or $4A \in \Gamma_v^{(16)}$ holds. Inserting (5.20) into (6.2), we find

$$\frac{\partial\Lambda}{\partial A^I} \sim \begin{cases} 0 & (I = 1, \cdots, p+q) \\ 8\pi\sum_{n\geq 1}(2n-1)^{-9}(-1)^{n+1}\,(p-q) & (I = p+q+1, \cdots, 16) \end{cases}. \qquad (6.3)$$

We thus find the two types of the critical points which satisfy either of the following two conditions:

$$\text{(i)}:\ r = 0,\ p + q = 16,$$
$$\text{(ii)}:\ r \geq 1,\ p = q,\ p + q + r = 16.$$

Note that the critical points (i) satisfy $2A \in \Gamma_g^*$.

The next step of the stability analysis is to calculate the Hessian matrix, that is, to evaluate the second derivative. From the first derivative (6.2), we get

$$\frac{\partial \Lambda}{\partial A^I \partial A^J} \sim \begin{cases} -16\pi^2 \sum_{n\geq 1}(2n-1)^{-8} \sin[2\pi(2n-1)A^I] \sin[2\pi(2n-1)A^I] & (I \neq J) \\ 16\pi^2 \sum_{n\geq 1}(2n-1)^{-8} \cos\left[2\pi(2n-1)A^I\right] \sum_{J\neq I} \cos\left[2\pi(2n-1)A^J\right] & (I = J) \end{cases}. \tag{6.4}$$

For the critical points (i), all the off-diagonal components of the Hessian matrices vanish, and the diagonal components are, up to the prefactor,

$$\cos\left[2\pi(2n-1)A^I\right] \sum_{J\neq I} \cos\left[2\pi(2n-1)A^J\right] = \begin{cases} p - q - 1 & (I = 1, \cdots, p) \\ -(p - q + 1) & (I = p+1, \cdots, 16) \end{cases}. \tag{6.5}$$

For the Hessian matrix to be positive definite, therefore, $(p, q)$ must be $(16, 0)$ or $(0, 16)$. Note that these Wilson lines imply $A \in \Gamma_g^*$, and $U(1)_L^{16}$ is enhanced to $SO(32)$ while there are no massless spinors, as we have shown in the previous section. One can also find from (6.5) that the Hessian matrix is negative definite only when $p = q = 8$, which means that the points with the enhanced gauge symmetry $SO(16) \times SO(16)$ correspond to the maxima of the effective potential.

As for the critical points (ii), the Hessian matrices can be expressed as a block diagonal matrix as follows:

$$\frac{\partial \Lambda}{\partial A^I \partial A^J} \sim 16\pi^2 \sum_{n\geq 1}(2n-1)^{-8} \left(\begin{array}{c|c} H_{1/2} & 0 \\ \hline 0 & H_{1/4} \end{array}\right), \tag{6.6}$$

where $H_{1/2}$ is a $(p+q) \times (p+q)$ diagonal matrix with the diagonal components given as in (6.5), and $H_{1/4}$ is a $r \times r$ off-diagonal matrix with the off-diagonal components given by $-1$. The Hessian matrix (6.6) clearly have at least one negative leading principle minor and cannot be positive definite. As a result, the critical points (ii) cannot be minima.

To summarize, in the non-supersymmetric models in class (2) with the $Spin(32)/\mathbb{Z}_2$ lattice, the Wilson lines are stabilized such that $U(1)_L^{16}$ is maximally enhanced to $SO(32)$. Note that these Wilson lines correspond to global minima since the cosine factor in (5.17) gives $+1$ for all $\pi \in \Delta_{SO(32)}$, which means that $\Lambda$ takes a minimum value.

As we revealed in the previous section, the cosmological constant is exponentially suppressed when $(p,q,r) = (\underline{7,9},0), (\underline{6,7},3), (6,6,4)$. The points with $(p,q,r) = (\underline{7,9},0)$ and $(p,q,r) = (6,6,4)$ respectively satisfy the conditions (i) and (ii). Thus, the exponential suppression with the symmetry enhancements $U(1)_L^{16} \to SO(18)\times SO(14)$ and $U(1)_L^{16} \to SO(12)\times SO(12)\times SU(3)\times U(1)$ occurs at the saddle points.

## 6.2 Supersymmetric $E_8 \times E_8$ endpoint model

We next consider the interpolating models with the supersymmetric $E_8\times E_8$ endpoint model. The sum over $\pi \in \Delta_{E_8\times E_8}$ can be decomposed into the two copies over $\pi \in \Delta_{E_8}$. So, it is sufficient to pay our attention to the following contributions in the cosmological constant (5.17):

$$
\begin{aligned}
&-\sum_{n\geq 1}(2n-1)^{-10}\sum_{\pi\in\Delta_{E_8}\oplus\{0^8\}}\cos\left[2\pi(2n-1)\left(\pi\cdot A\right)\right] \\
&= -\sum_{n\geq 1}(2n-1)^{-10}\left(\sum_{\pi_1\in\Delta_{SO(16)}}+\sum_{\pi_1\in\Delta_{\mathbf{128}_+}}\right)\cos\left[2\pi(2n-1)\left(\pi_1\cdot A_1\right)\right]. \qquad (6.7)
\end{aligned}
$$

The sum over $\pi_1 \in \Delta_{SO(16)}$ can be expressed as in (6.1):

$$
\begin{aligned}
&-\sum_{n\geq 1}(2n-1)^{-10}\sum_{\pi_1\in\Delta_{SO(16)}}\cos\left[2\pi(2n-1)\left(\pi_1\cdot A_1\right)\right] \\
&= -2\sum_{n\geq 1}(2n-1)^{-10}\sum_{I_1>J_1}\left(\cos\left[2\pi(2n-1)\left(A^{I_1}+A^{J_1}\right)\right]+\cos\left[2\pi(2n-1)\left(A^{I_1}-A^{J_1}\right)\right]\right) \\
&= -4\sum_{n\geq 1}(2n-1)^{-10}\sum_{I_1>J_1}\cos\left[2\pi(2n-1)A^{I_1}\right]\cos\left[2\pi(2n-1)A^{J_1}\right], \qquad (6.8)
\end{aligned}
$$

where $I_1, J_1 = 1, \cdots, 8$ indicate the first eight components of $A$ and $\pi$. In order to evaluate the sum over $\pi_1 \in \Delta_{\mathbf{128}_+}$, we can use the following trigonometric identity:

$$\cos(x_1 + \cdots + x_8) = \prod_{I=1}^{8} \cos(x_I) - \prod_{I=1}^{2} \sin(x_I) \prod_{I=3}^{8} \cos(x_I) + \prod_{I=1}^{4} \sin(x_I) \prod_{I=5}^{8} \cos(x_I) - \prod_{I=1}^{6} \sin(x_I) \prod_{I=7}^{8} \cos(x_I) + \prod_{I=1}^{8} \sin(x_I). \quad (6.9)$$

Then, we find

$$- \sum_{n \geq 1} (2n-1)^{-10} \sum_{\pi_1 \in \Delta_{\mathbf{128}_+}} \cos\left[2\pi(2n-1)(\pi_1 \cdot A_1)\right] = -128 \sum_{n \geq 1} (2n-1)^{-10} \left( \prod_{I_1=1}^{8} \cos\left[\pi(2n-1)A^{I_1}\right] + \prod_{I_1=1}^{8} \sin\left[\pi(2n-1)A^{I_1}\right] \right). \quad (6.10)$$

Note that the second, third and fourth terms in (6.9) are canceled by summing up $\pi_1 \in \Delta_{\mathbf{128}_+}$. For instance, $\pi_1 = \frac{1}{2}\left(\underline{(\pm)^4}_+, \underline{(\pm)^4}_+\right)$ and $\pi_1 = \frac{1}{2}\left(\underline{(\pm)^4}_-, \underline{(\pm)^4}_-\right)$ give the different sign of the third terms and have the same degrees of freedom. As a result, (a half part of) the cosmological constant (6.7) can be expressed as

$$- \sum_{n \geq 1} (2n-1)^{-10} \left\{ 4 \sum_{I_1 > J_1} \cos\left[2\pi(2n-1)A^{I_1}\right] \cos\left[2\pi(2n-1)A^{J_1}\right] + 128 \left( \prod_{I_1=1}^{8} \cos\left[\pi(2n-1)A^{I_1}\right] + \prod_{I_1=1}^{8} \sin\left[\pi(2n-1)A^{I_1}\right] \right) \right\}. \quad (6.11)$$

The first derivative is then given as

$$\frac{\partial \Lambda}{\partial A^{I_1}} \sim 8\pi \sum_{n \geq 1} (2n-1)^{-9} \left\{ \sin[2\pi(2n-1)A^{I_1}] \sum_{J_1 \neq I_1} \cos\left[2\pi(2n-1)A^{J_1}\right] + \mathcal{V}_n^{(1)I_1} \right\}, \quad (6.12)$$

where $\mathcal{V}_n^{(1)I_1}$ is defined as

$$\mathcal{V}_n^{(1)I_1} = 16 \left( \sin\left[\pi(2n-1)A^{I_1}\right] \prod_{J_1 \neq I_1} \cos\left[\pi(2n-1)A^{J_1}\right] - \cos\left[\pi(2n-1)A^{I_1}\right] \prod_{J_1 \neq I_1} \sin\left[\pi(2n-1)A^{J_1}\right] \right). \quad (6.13)$$

As in the $Spin(32)/\mathbb{Z}_2$ case, we only focus on the Wilson lines that satisfy $4A_1 \in \Gamma_g^{(8)}$:

$$A^{I_1} = \left(0^{p_1}, \left(\frac{1}{2}\right)^{q_1}, \left(\frac{1}{4}\right)^{r_1}\right), \qquad (p_1 + q_1 + r_1 = 8,\ r_1 \in 2\mathbb{Z}). \quad (6.14)$$

Inserting (6.14) to (6.12), we find for $I_1 = 0, \cdots, p_1$

$$\frac{\partial\Lambda}{\partial A^{I_1}} \sim \begin{cases} 0 & \text{for } p_1 \neq 1 \\ 128\pi\left(\frac{1}{\sqrt{2}}\right)^{r_1} \sum_{n\geq 1}(2n-1)^{-9}(-1)^{n+1} & \text{for } p_1 = 1 \end{cases}, \tag{6.15}$$

and for $I_1 = p_1+1, \cdots, p_1+q_1$,

$$\frac{\partial\Lambda}{\partial A^{I_1}} \sim \begin{cases} 0 & \text{for } q_1 \neq 1 \\ 128\pi\left(\frac{1}{\sqrt{2}}\right)^{r_1} \sum_{n\geq 1}(2n-1)^{-9}(-1)^{n} & \text{for } q_1 = 1 \end{cases}. \tag{6.16}$$

As for $I_1 = p_1+q_1+1, \cdots, 8$, noting that $r_1$ is even, we see that the first derivative is

$$\frac{\partial\Lambda}{\partial A^{I_1}} \sim 8\pi \sum_{n\geq 1}(2n-1)^{-9}\left\{(-1)^{n+1}(p_1-q_1) + \mathcal{V}_n^{(1)I_1}\big|_{A^{I_1}=1/4}\right\} \tag{6.17}$$

where $\mathcal{V}_n^{(1)I_1}\Big|_{A^{I_1}=1/4}$ is given as

$$\mathcal{V}_n^{(1)I_1}\big|_{A^{I_1}=1/4} = \begin{cases} 0 & \text{for } p_1\neq 0,\ q_1\neq 0 \text{ or } p_1=q_1=0 \\ 16\left(\frac{1}{\sqrt{2}}\right)^{r_1}(-1)^n & \text{for } p_1=0,\ q_1\neq 0 \\ 16\left(\frac{1}{\sqrt{2}}\right)^{r_1}(-1)^{n+1} & \text{for } p_1\neq 0,\ q_1=0 \end{cases}. \tag{6.18}$$

From (6.15), (6.16) and (6.17), we find two types of the critical points which satisfy one of the following conditions:

$$\text{(i): } r_1=0,\ p_1\neq 1,\ q_1\neq 1,$$
$$\text{(ii): } r_1\geq 2,\ p_1\neq 1,\ q_1\neq 1,\ p_1=q_1.$$

We now evaluate the Hessian matrix. The off-diagonal components are

$$\frac{\partial\Lambda}{\partial A^{I_1}\partial A^{J_1}} \sim -16\pi^2\sum_{n\geq 1}(2n-1)^{-8}\left\{\sin[2\pi(2n-1)A^{I_1}]\sin[2\pi(2n-1)A^{J_1}] + \mathcal{V}_n^{(2)I_1J_1}\right\}, \tag{6.19}$$

where we define $\mathcal{V}_n^{(2)I_1J_1}$ as

$$\begin{aligned}\mathcal{V}_n^{(2)I_1J_1} =& 8\sin[\pi(2n-1)A^{I_1}]\sin[\pi(2n-1)A^{J_1}]\prod_{K_1\neq I_1,J_1}\cos[\pi(2n-1)A^{K_1}] \\ &+ 8\cos[\pi(2n-1)A^{I_1}]\cos[\pi(2n-1)A^{J_1}]\prod_{K_1\neq I_1,J_1}\sin[\pi(2n-1)A^{K_1}].\end{aligned} \tag{6.20}$$

The diagonal components are

$$\frac{\partial \Lambda}{\partial A^{I_1} \partial A^{I_1}} \sim 16\pi^2 \sum_{n\geq 1} (2n-1)^{-8} \left\{ \cos[2\pi(2n-1)A^{I_1}] \sum_{J_1 \neq I_1} \cos[2\pi(2n-1)A^{J_1}] + \mathcal{V}_n^{(2)} \right\}, \tag{6.21}$$

where $\mathcal{V}_n^{(2)}$ is defined as

$$\mathcal{V}_n^{(2)} = 8 \prod_{I_1=1}^{8} \cos[\pi(2n-1)A^{I_1}] + 8 \prod_{I_1=1}^{8} \sin[\pi(2n-1)A^{I_1}]. \tag{6.22}$$

Let us first calculate the Hessian matrix of the critical points (i). Obviously, the first term in (6.19) vanishes for any $I_1$, $J_1$ if $r_1 = 0$. One can also see that $\mathcal{V}_n^{(2)I_1 J_1} = 0$ for any $I_1$, $J_1$ unless $(p_1, q_1) = (2,6), (6,2)$. As for the diagonal components, the first term in (6.21) is given as the same form as in (6.5) with $(p,q)$ replaced by $(p_1, q_1)$, and $\mathcal{V}_n^{(2)}$ is

$$\mathcal{V}_n^{(2)} = \begin{cases} 8 & \text{for } (p_1, q_1) = (8,0), (0,8) \\ 0 & \text{for the others} \end{cases}. \tag{6.23}$$

Then, the critical points (i) with $(p_1, q_1) = (8,0), (0,8)$, at which $U(1)_L^8$ is enhanced to $E_8$, give the positive definite Hessian matrix. If $(p_1, q_1) = (2,6)$, then $\mathcal{V}_n^{(2)12} = \mathcal{V}_n^{(2)21} = 8$ and the other components of $\mathcal{V}_n^{(2)I_1 J_1}$ vanish, and then we obtain

$$\frac{\partial \Lambda}{\partial A^{I_1} \partial A^{J_1}} \sim 16\pi^2 \sum_{n\geq 1} (2n-1)^{-8} \begin{pmatrix} -5 & -8 & 0 & \cdots & 0 \\ -8 & -5 & 0 & \cdots & 0 \\ 0 & 0 & 3 & & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & 3 \end{pmatrix}. \tag{6.24}$$

This matrix is clearly neither positive definite nor negative definite, and hence the critical point (i) with $(p_1, q_1) = (2,6)$ corresponds to a saddle point of the effective potential. In the same way, one can check that the critical point (i) with $(p_1, q_1) = (6,2)$ is also a saddle point. One can also see that the critical point (i) with $p_1 = q_1 = 4$ gives the negative definite Hessian matrix, and hence the points with the enhanced symmetry $SO(16) \times SO(16)$ correspond to the maxima of the potential, as we have obtained the same result in the $Spin(32)/\mathbb{Z}_2$ case.

We now turn to the critical points (ii). If $p_1 = q_1 \neq 0$, then the diagonal components (6.21) are

$$\frac{\partial \Lambda}{\partial A^{I_1}\partial A^{I_1}} \sim 16\pi^2 \sum_{n\geq 1}(2n-1)^{-8} \times \begin{cases} -1 & (I_1 = 1, \cdots, 2p_1) \\ 0 & (I_1 = 2p_1, \cdots, 8) \end{cases} . \tag{6.25}$$

If $p_1 = q_1 = 0$, then all the diagonal components are $+1$ (except the prefactor). As for the off-diagonal components, the first term in (6.19) vanishes unless both $A^{I_1}$ and $A^{J_1}$ are $1/4$, for which it gives $+1$ (except the prefactor). If $p_1 = q_1 \geq 3$, then $\mathcal{V}_n^{(2)I_1J_1} = 0$ for any $I_1$, $J_1$, and the Hessian matrix is

$$\frac{\partial \Lambda}{\partial A^I \partial A^J} \sim 16\pi^2 \sum_{n\geq 1}(2n-1)^{-8} \left(\begin{array}{c|c} -\mathbf{1}_{2p\times 2p} & 0 \\ \hline 0 & H_{1/4} \end{array}\right), \tag{6.26}$$

where $H_{1/4}$ is a $r_1 \times r_1$ off-diagonal matrix with all the off-diagonal components being $-1$. This matrix is clearly not positive definite. If $p_1 = q_1 = 2$, then $\mathcal{V}_n^{(2)12} = \mathcal{V}_n^{(2)34} = 2$, and the Hessian matrix is

$$\frac{\partial \Lambda}{\partial A^{I_1}\partial A^{J_1}} \sim 16\pi^2 \sum_{n\geq 1}(2n-1)^{-8} \left(\begin{array}{c|c|c} H_{1/2}^{(2,2)} & 0 & 0 \\ \hline 0 & H_{1/2}^{(2,2)} & 0 \\ \hline 0 & 0 & H_{1/4} \end{array}\right), \quad \text{with } H_{1/2}^{(2,2)} = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}, \tag{6.27}$$

which is not positive definite. If $p_1 = q_1 = 0$, then $\mathcal{V}_n^{(2)I_1J_1} = \mathcal{V}_n^{(2)I_1J_1} = 1$ for any $I_1$, $J_1$, and the Hessian matrix is

$$\frac{\partial \Lambda}{\partial A^{I_1}\partial A^{J_1}} \sim 16\pi^2 \sum_{n\geq 1}(2n-1)^{-8} \begin{pmatrix} 1 & -2 & \cdots & -2 \\ -2 & \ddots & & \vdots \\ \vdots & & \ddots & -2 \\ -2 & \cdots & -2 & 1 \end{pmatrix}. \tag{6.28}$$

which is not positive definite.

We can perform the same analysis for the last eight components $A^{I_2}$ of the Wilson line. To summarize, as in the $Spin(32)/\mathbb{Z}_2$ case, the minima of the effective potential correspond to the points with $A \in \Gamma^{16}$ at which the maximal enhancement $U(1)_L^{16} \to E_8 \times E_8$ is realized.

We revealed in the previous section that the cosmological constant is exponentially suppressed when the Wilson line takes one of the following configurations[9]:

$$A = \begin{cases} \left(0^4, \left(\frac{1}{2}\right)^4; 0^5, \left(\frac{1}{2}\right)^3\right) \\ \left(0^4, \left(\frac{1}{2}\right)^4; 0^2, \left(\frac{1}{2}\right)^2, \left(\frac{1}{4}\right)^4\right) \\ \left(0^3, \left(\frac{1}{2}\right)^3, \left(\frac{1}{4}\right)^2; 0^3, \left(\frac{1}{2}\right)^3, \left(\frac{1}{4}\right)^2\right) \end{cases} . \tag{6.29}$$

All these Wilson lines satisfy either of the conditions (i) or (ii) for the critical points. Therefore, the exponential suppression of the cosmological constant occurs at the saddle points.

[9] Of course, these configurations should be considered up to appropriate permutations.

# 7 Summary and outlook

In this thesis, we consider the non-supersymmetric string models constructed by orbifolds with a $\mathbb{Z}_2$ freely acting twist, in which supersymmetry is completely broken due to the stringy Scherk-Schwarz mechanism. One of the important features in the non-supersymmetric string models is that the Narain lattice is split into $\Gamma_+^{d_L,d_R}$ and $\Gamma_-^{d_L,d_R}$, and spacetime bosons and fermions live in different subsets $\Gamma_+^{d_L,d_R}$ and $\Gamma_-^{d_L,d_R}$ respectively in the untwisted sectors. We reviewed construction of the non-supersymmetric models in section 2. In particular, we classified the 9D heterotic models into the four classes by the choice of a $\mathbb{Z}_2$ freely acting twist (more precisely, the choice of $\hat{Z} \in \mathbb{Z}^{d_L} \times \mathbb{Z}^{d_R}$).

We first discussed the T-duality group in the non-supersymmetric string models. A T-duality element in the non-supersymmetric models must be an automorphism of $\Gamma_+^{d_L,d_R}$ and $\Gamma_-^{d_L,d_R}$ but not $\Gamma^{d_L,d_R}$, and hence the T-duality group is restricted to a congruence subgroup of order 2 in $O(d_L, d_R, \mathbb{Z})$, which depends on the choice of $\hat{Z}$. As concrete examples, we clarified which elements of $O(d_L, d_R, \mathbb{Z})$ still survive in the non-supersymmetric type II models with $d = 2$ and heterotic models with $d = 1$. Moreover, we noted that the transitions among the non-supersymmetric string models with different choices of shift-vectors can be induced by acting an element of $O(d_L, d_R, \mathbb{Z})$ that is not in the congruence subgroup. We also gave the examples of the transitions in the type II models with $d = 2$ and the heterotic models with $d = 1$ (see Fig. 1 and Fig. 2).

Secondly, we studied the possible massless spectra of the 9D non-supersymmetric heterotic strings at various points in the moduli space. We gave the massless conditions for states which transform as not gauge singlets. In particular, we focused on the unwinding states (i.e. with $m = 0$) and clarified the massless conditions for both of the untwisted and twisted sectors in each of the four classes of the 9D heterotic models. As concrete examples, we revealed patterns of the gauge symmetry enhancements and the representations in which massless states transform, in class (1). Furthermore, we pointed out that the Wilson line in class (2) is related to that in the toroidal models. This relation allows us to figure out the massless spectra in class (2) easily since the massless conditions for unwinding states do not depend on the choice of 10D non-supersymmetric endpoint models.

Thirdly, we evaluated the cosmological constant of the non-supersymmetric models in class (2). We showed that the leading contribution of the cosmological constant is proportional to $n_F - n_B$, where $n_F$ and $n_B$ are the degrees of freedom of the massless fermions and bosons respectively, up to the exponentially suppressed terms in the region with supersymmetry asymptotically being restored. We also found some configurations of the Wilson line that realize the massless spectra with $n_F = n_B$, that is, the exponentially suppressed cosmological constant. In the models with the supersymmetric $Spin(32)/\mathbb{Z}_2$ endpoint model, the suppression occurs when the Wilson line leads to the enhancement from $U(1)_L^{16}$ to $SO(18) \times SO(14)$ or $SO(14) \times SO(12) \times SU(3) \times U(1)$ or $SO(12) \times SO(12) \times SU(4) \times U(1)$. If the supersymmetric endpoint is the $E_8 \times E_8$ one, the Wilson lines that induce the enhancements to $SO(16) \times SO(10) \times SO(6)$ or $SU(8) \times SU(2) \times SU(8) \times SU(2)$ realize the suppression of the cosmological constant.

Finally, we analyzed stability of the Wilson line moduli in class (2) from the one-loop effective potential in the region where supersymmetry is restored. We have shown that the Wilson line is stabilized at the points where $U(1)_L^{16}$ is maximally enhanced. We did not find any local minima, and the points at which the cosmological constant is exponentially suppressed correspond to the saddle points.

There are some issues that we have to overcome and something worth investigating. We would like to here present some possible future directions:

- In this thesis, we restrict our attention to a particular class of non-supersymmetric string models which are constructed by orbifolding with freely acting $\mathbb{Z}_2$ twists. It is interesting to generalize the non-supersymmetric models by $\mathbb{Z}_N$ orbifolding and consider various patterns of supersymmetry breaking, e.g. $\mathcal{N} = 2 \to 0$ or $\mathcal{N} = 1 \to 0$. In particular, it is worth exploring asymmetric orbifolds or non-geometric backgrounds with the stringy Scherk-Schwarz mechanism since we have already revealed the T-duality elements, including non-geometric ones, in the non-supersymmetric string models.

- It is in this thesis shown that the T-duality groups of the non-supersymmetric string models are congruence subgroups of $O(d_L, d_R, \mathbb{Z})$. In the context of flavor physics, recently, the non-Abelian discrete groups have been used for explaining the origin of the flavor structure of the Standard Model. In particular, congruence subgroups of $PSL(2, \mathbb{Z})$ are frequently in the spotlight. Then, the T-duality groups we obtained in this thesis are expected to be applied to the exploration of the flavor structure, accompanied with the scenario that supersymmetry is broken at a very high-energy scale.

- In this thesis, we conclude that the moduli are unstable when the cosmological constant is zero or positive up to the exponentially suppressed terms, and the stable moduli correspond to anti-de Sitter vacua. This is not a desirable situation for making realistic models from non-supersymmetric string theory. We would like to stabilize the moduli at the same time as suppressing the cosmological constant. To realize that, it is worth calculating the higher loop corrections and clarifying the effects that they have on the effective potential. One of the other directions is to include the non-perturbative effects which are anticipated to uplift the effective potential and make a de-Sitter vacuum as in [99]. Thus, it is interesting to investigate the type I models, which are related to the heterotic ones by S-duality, in order to extract the information of the non-perturbative corrections.

# A Lattices and characters

Irreducible representations of $SO(2n)$ can be classified into the four conjugacy classes:

- The trivial conjugacy class (the root lattice):

$$\Gamma_g^{(n)} = \left\{ (n_1, \cdots, n_n) \,\middle|\, n_i \in \mathbb{Z},\ \sum_{i=1}^{n} n_i \in 2\mathbb{Z} \right\}. \tag{A.1}$$

- The vector conjugacy class:

$$\Gamma_v^{(n)} = \left\{ (n_1, \cdots, n_n) \,\middle|\, n_i \in \mathbb{Z},\ \sum_{i=1}^{n} n_i \in 2\mathbb{Z}+1 \right\}. \tag{A.2}$$

- The spinor conjugacy class:

$$\Gamma_s^{(n)} = \left\{ \left(n_1 + \frac{1}{2}, \cdots, n_n + \frac{1}{2}\right) \,\middle|\, n_i \in \mathbb{Z},\ \sum_{i=1}^{n} n_i \in 2\mathbb{Z} \right\}. \tag{A.3}$$

- The conjugate spinor conjugacy class:

$$\Gamma_c^{(n)} = \left\{ \left(n_1 + \frac{1}{2}, \cdots, n_n + \frac{1}{2}\right) \,\middle|\, n_i \in \mathbb{Z},\ \sum_{i=1}^{n} n_i \in 2\mathbb{Z}+1 \right\}. \tag{A.4}$$

The weight lattice of $SO(2n)$, which is dual to $\Gamma_g^{(n)}$, is given by the sum of the four conjugacy classes:

$$\Gamma_w^{(n)} = \Gamma_g^{(n)} + \Gamma_v^{(n)} + \Gamma_s^{(n)} + \Gamma_c^{(n)}. \tag{A.5}$$

Modular invariance of the partition functions of the 10D supersymmetric heterotic string models requires that the internal momenta should live in an even self-dual Euclidean lattice. In 16-dimensions, only two such lattices exist. One of them is the root lattice of $E_8 \times E_8$,

$$\Gamma^{16} = \left(\Gamma_g^{(8)} + \Gamma_s^{(8)}\right) \times \left(\Gamma_g^{(8)} + \Gamma_s^{(8)}\right), \tag{A.6}$$

and the other is that of $Spin(32)/\mathbb{Z}_2$ which is expressed as the sum of the trivial and spinor conjugacy classes of $SO(32)$:

$$\Gamma^{16} = \Gamma_g^{(16)} + \Gamma_s^{(16)}. \tag{A.7}$$

The $SO(2n)$ characters of the corresponding conjugacy classes are defined as

$$O_{2n} = \frac{1}{\eta^n} \sum_{\pi \in \Gamma_g^{(n)}} q^{\frac{1}{2}|\pi|^2} = \frac{1}{2\eta^n} \left( \vartheta^n \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0,\tau) + \vartheta^n \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (0,\tau) \right), \tag{A.8}$$

$$V_{2n} = \frac{1}{\eta^n} \sum_{\pi \in \Gamma_v^{(n)}} q^{\frac{1}{2}|\pi|^2} = \frac{1}{2\eta^n} \left( \vartheta^n \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0,\tau) - \vartheta^n \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (0,\tau) \right), \tag{A.9}$$

$$S_{2n} = \frac{1}{\eta^n} \sum_{\pi \in \Gamma_s^{(n)}} q^{\frac{1}{2}|\pi|^2} = \frac{1}{2\eta^n} \left( \vartheta^n \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (0,\tau) + \vartheta^n \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (0,\tau) \right), \tag{A.10}$$

$$C_{2n} = \frac{1}{\eta^n} \sum_{\pi \in \Gamma_c^{(n)}} q^{\frac{1}{2}|\pi|^2} = \frac{1}{2\eta^n} \left( \vartheta^n \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (0,\tau) - \vartheta^n \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (0,\tau) \right), \tag{A.11}$$

where the Dedekind eta function and the theta function with characteristics are defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \tag{A.12}$$

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z,\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i (n+\alpha)^2 \tau + 2\pi i (n+\alpha)(z+\beta)\right). \tag{A.13}$$

It is known that the $SO(8)$ characters satisfy the Jacobi's abstruse identity

$$V_8 - S_8 = 0. \tag{A.14}$$

In order to check modular invariance of the partition functions, it is useful to reveal how the $SO(2n)$ characters transform under $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$. The eta function and the theta function satisfy the following identities:

$$\eta(\tau+1) = e^{\frac{\pi i}{12}} \eta(\tau), \qquad \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0,\tau+1) = e^{-\pi i \alpha(\alpha-1)} \vartheta \begin{bmatrix} \alpha \\ \alpha+\beta-\frac{1}{2} \end{bmatrix} (0,\tau), \tag{A.15}$$

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2} \eta(\tau), \qquad \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(0, -\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{2\pi i \alpha\beta} \vartheta \begin{bmatrix} \beta \\ -\alpha \end{bmatrix} (0,\tau). \tag{A.16}$$

Then we find the transformation laws of the $SO(2n)$ characters:

$$T : (O_{2n}, V_{2n}, S_{2n}, C_{2n}) \to (O_{2n}, V_{2n}, S_{2n}, C_{2n})\, \mathcal{T}_{2n}, \tag{A.17}$$

$$S : (O_{2n}, V_{2n}, S_{2n}, C_{2n}) \to (O_{2n}, V_{2n}, S_{2n}, C_{2n})\, \mathcal{S}_{2n}, \tag{A.18}$$

(A.19)

where $\mathcal{T}_{2n}$ and $\mathcal{S}_{2n}$ are $4 \times 4$ matrices defined as follows:

$$\mathcal{T}_{2n} = \begin{pmatrix} e^{-\frac{n\pi i}{12}} & 0 & 0 & 0 \\ 0 & -e^{-\frac{n\pi i}{12}} & 0 & 0 \\ 0 & 0 & e^{\frac{n\pi i}{6}} & 0 \\ 0 & 0 & 0 & e^{\frac{n\pi i}{6}} \end{pmatrix}, \quad \mathcal{S}_{2n} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^n & -i^n \\ 1 & -1 & -i^n & i^n \end{pmatrix}. \tag{A.20}$$

We here expand the products $\eta^{-8}(O_{2n}, V_{2n}, S_{2n}, C_{2n})$, which appear in the partition functions. This expansions are useful for studying the free spectra of strings. Note that $\eta^{-8}$ is the contribution from the oscillators of eight world-sheet bosons while the $SO(2n)$ characters are the contributions from $n$ compactified world-sheet bosons or $2n$ world-sheet Majorana-Weyl fermions.

$$\eta^{-8} O_{2n} \sim q^{-\frac{8+n}{24}} \left( 1 + \left( 8 + \frac{2n(2n-1)}{2} \right) q + \mathcal{O}(q^2) \right), \tag{A.21}$$

$$\eta^{-8} V_{2n} \sim q^{-\frac{8+n}{24} + \frac{1}{2}} \left( 2n + \mathcal{O}(q) \right), \tag{A.22}$$

$$\eta^{-8} S_{2n} = \eta^{-8} S_{2n} \sim q^{-\frac{8-2n}{24}} \left( 2^{n-1} + \mathcal{O}(q) \right). \tag{A.23}$$

In particular, for $n = 4$, which appears in the type II models or in the right-moving parts of the heterotic models,

$$\eta^{-8} O_8 \sim q^{-\frac{1}{2}} \left( 1 + 36q + \mathcal{O}(q^2) \right), \tag{A.24a}$$

$$\eta^{-8} V_8 \sim 8 + \mathcal{O}(q), \tag{A.24b}$$

$$\eta^{-8} S_8 = \eta^{-8} C_8 \sim 8 + \mathcal{O}(q). \tag{A.24c}$$

# B Invariance under the $T$-transformation

In this appendix, we check the invariance of the partition functions (2.31) and (2.32) under $\tau \to \tau + 1$. In particular, we show that the $SO(8)$ characters and $Z_{\Gamma_{\pm}^{d_L,d_R}+\delta}$ in the twisted sectors must be combined appropriately. From (A.15) and (A.17), we find that under the $T$-transformation, the product of the $SO(8)$ character and $\eta^8$ has an eigenvalue $-1$ only when the character is of the trivial conjugacy class:

$$T : \eta^{-8} \left( O_8, V_8, S_8, C_8 \right) \to \eta^{-8} \left( -O_8, V_8, S_8, C_8 \right). \tag{B.1}$$

Then, the untwisted sectors, which include neither $O_8$ nor $\bar{O}_8$, are obviously invariant since $P$ is in an even lattice. In the twisted sectors, the momenta are shifted by $\delta$, and then we get the following phase from $Z_{\Gamma_{\pm}^{d_L,d_R}+\delta}$ under $\tau \to \tau + 1$, excluding the phase which comes from $\eta^{-d_L} \bar{\eta}^{-d_R}$:

$$e^{\pi i \left( P^2 + \delta^2 + 2P\cdot\delta \right)}. \tag{B.2}$$

If $\delta^2$ is even and $P \in \Gamma_{+}^{d_L d_R}$, or $\delta^2$ is odd and $P \in \Gamma_{-}^{d_L d_R}$, then the phase (B.2) is $+1$. Thus, $Z_{\Gamma_{+}^{d_L d_R}+\delta}$ with $\delta^2$ even or $Z_{\Gamma_{-}^{d_L d_R}+\delta}$ with $\delta^2$ odd must be accompanied with $\left( O_8 \bar{O}_8 + C_8 \bar{C}_8 \right)$ in the type IIB case and with $\bar{C}_8$ in the heterotic case. On the other hand, the phase (B.2) is $-1$ if $\delta^2$ is even and $P \in \Gamma_{-}^{d_L d_R}$, or $\delta^2$ is odd and $P \in \Gamma_{+}^{d_L d_R}$. Thus, $Z_{\Gamma_{-}^{d_L d_R}+\delta}$ with $\delta^2$ even or $Z_{\Gamma_{+}^{d_L d_R}+\delta}$ with $\delta^2$ odd must be accompanied with $\left( O_8 \bar{C}_8 + C_8 \bar{O}_8 \right)$ in the type IIB case and with $\bar{O}_8$ in the heterotic case. As a result, we see that the partition functions (2.31) and (2.32) have the appropriate combinations of the $SO(8)$ characters and $Z_{\Gamma_{\pm}^{d_L,d_R}+\delta}$, which make the partition functions invariant under the shift $\tau \to \tau + 1$.

# C Massless spectra in class (1)

In this appendix, we summarize the massless spectra with $m = 0$ in class (1) which we revealed in subsections 4.3 and 4.4. We only denote $U(1)$ charges for states which trivially transform under non-Abelian gauge groups in the tables below. Recall that all the Wilson lines we considered in subsections 4.3 and 4.4 satisfy $2A \in \Gamma_g^*$.

## C.1 The $Spin(32)/\mathbb{Z}_2$ lattice

| | Untwisted sectors | | Twisted sectors | |
|---|---|---|---|---|
| $A$ | Gauge sym. | Spinors | Co-spinors | Scalars |
| $(0^{16})$ | $SO(32)$ | — | — | $\mathbf{32}$ $(\sqrt{2}R \in \mathbb{Z})$ |
| $\left(\left(\frac{1}{2}\right)^{16}\right)$ | $SO(32)$ | — | — | $\mathbf{32}$ $(\sqrt{2}R \in \mathbb{Z} + \frac{1}{2})$ |
| $\left(0^p, \left(\frac{1}{2}\right)^q\right)$ | $SO(2p) \times SO(2q)$ | — | — | $(\mathbf{2p}, \mathbf{1})$ $(\sqrt{2}R \in \mathbb{Z})$ <br> $(\mathbf{1}, \mathbf{2q})$ $(\sqrt{2}R \in \mathbb{Z} + \frac{1}{2})$ |
| $\left(\left(\frac{1}{4}\right)^{16}\right)$ | $SU(16) \times U(1)$ | — | — | $\mathbf{16} \oplus \overline{\mathbf{16}}$ $(\sqrt{2}R \in \mathbb{Z} \pm \frac{1}{4})$ |

Table 7: $SO(32)$ model: $\frac{\hat{\pi}}{2} = (1, 0^{15})$.

| | Untwisted sectors | | Twisted sectors | |
|---|---|---|---|---|
| $A$ | Gauge sym. | Spinors | Co-spinors | Scalars |
| $(0^8)$, $\left(\left(\frac{1}{2}\right)^{16}\right)$ | $SO(24) \times SO(8)$ | $(\mathbf{24}, \mathbf{8})$ | $(\mathbf{24}, \mathbf{8}_-)$ | $(\mathbf{1}, \mathbf{8}_+)$ $(\sqrt{2}R \in \mathbb{Z})$ |
| $\left(0^p, \left(\frac{1}{2}\right)^q\right)$ <br> $(p \leq 12)$ | $SO(2p) \times SO(24-2p)$ <br> $\times SO(8)$ | $(\mathbf{1}, \mathbf{24-2p}, \mathbf{8})$ | $(\mathbf{1}, \mathbf{24-2p}, \mathbf{8}_-)$ | $(\mathbf{1}, \mathbf{1}, \mathbf{8}_+)$ $(\sqrt{2}R \in \mathbb{Z})$ |
| $\left(0^p, \left(\frac{1}{2}\right)^q\right)$ <br> $(p = 13, 15)$ | $SO(24)$ <br> $\times SO(8-2q) \times SO(2q)$ | $(\mathbf{24}, \mathbf{8-2q}, \mathbf{1})$ | — | — |
| $\left(0^{14}, \left(\frac{1}{2}\right)^2\right)$ | $SO(24)$ <br> $\times SO(4) \times SO(4)$ | $(\mathbf{24}, \mathbf{4}, \mathbf{1})$ | — | $(\mathbf{1}, \mathbf{2}_+, \mathbf{2}_+)$ $(\sqrt{2}R \in \mathbb{Z})$ <br> $(\mathbf{1}, \mathbf{2}_-, \mathbf{2}_-)$ $(\sqrt{2}R \in \mathbb{Z} + \frac{1}{2})$ |
| $\left(\left(\frac{1}{4}\right)^{16}\right)$ | $SU(12) \times SU(4) \times U(1)^2$ | $(\mathbf{12}, \overline{\mathbf{4}}) \oplus (\overline{\mathbf{12}}, \mathbf{4})$ | $(\mathbf{12}, \mathbf{4}) \oplus (\overline{\mathbf{12}}, \overline{\mathbf{4}})$ | $(\mathbf{1}, \mathbf{6})$ $(\sqrt{2}R \in \mathbb{Z})$ <br> $(0, \pm 1)$ $(\sqrt{2}R \in \mathbb{Z} + \frac{1}{2})$ |

Table 8: $SO(24) \times SO(8)$ model: $\frac{\hat{\pi}}{2} = \left(0^{12}, \left(\frac{1}{2}\right)^4\right)$.

| $A$ | Untwisted sectors | | Twisted sectors | |
|---|---|---|---|---|
| | Gauge sym. | Spinors | Co-spinors | Scalars |
| $(0^{16})\,,\left(\left(\frac{1}{2}\right)^{16}\right)$ | $SU(16)\times U(1)$ | $\mathbf{120}\oplus\overline{\mathbf{120}}$ | $\mathbf{120}\oplus\overline{\mathbf{120}}$ | $\pm 1$ $(\sqrt{2}R\in\mathbb{Z})$ |
| $\left(0^p,\left(\frac{1}{2}\right)^q\right)$<br>$(p\notin 4\mathbb{Z})$ | $SU(p)\times SU(q)\times U(1)^2$ | $\left({}_{\boldsymbol{p}}\mathbf{C_2}\oplus\overline{{}_{\boldsymbol{p}}\mathbf{C_2}},\mathbf{1}\right)$<br>$\oplus\left(\mathbf{1},{}_{\boldsymbol{q}}\mathbf{C_2}\oplus\overline{{}_{\boldsymbol{q}}\mathbf{C_2}}\right)$ | — | $\pm\left(1,1+\frac{p}{8}\right)$ $(\sqrt{2}R\in\mathbb{Z}\pm\frac{p}{8})$ |
| $\left(0^p,\left(\frac{1}{2}\right)^q\right)$<br>$(p=8)$ | $SU(p)\times SU(q)\times U(1)^2$ | $\left({}_{\boldsymbol{p}}\mathbf{C_2}\oplus\overline{{}_{\boldsymbol{p}}\mathbf{C_2}},\mathbf{1}\right)$<br>$\oplus\left(\mathbf{1},{}_{\boldsymbol{q}}\mathbf{C_2}\oplus\overline{{}_{\boldsymbol{q}}\mathbf{C_2}}\right)$ | $\left({}_{\boldsymbol{p}}\mathbf{C_2}\oplus\overline{{}_{\boldsymbol{p}}\mathbf{C_2}},\mathbf{1}\right)$<br>$\oplus\left(\mathbf{1},{}_{\boldsymbol{q}}\mathbf{C_2}\oplus\overline{{}_{\boldsymbol{q}}\mathbf{C_2}}\right)$ | $\pm\left(1,1+\frac{p}{8}\right)$ $(\sqrt{2}R\in\mathbb{Z})$ |
| $\left(0^p,\left(\frac{1}{2}\right)^q\right)$<br>$(p=4,12)$ | $SU(p)\times SU(q)\times U(1)^2$ | $\left({}_{\boldsymbol{p}}\mathbf{C_2}\oplus\overline{{}_{\boldsymbol{p}}\mathbf{C_2}},\mathbf{1}\right)$<br>$\oplus\left(\mathbf{1},{}_{\boldsymbol{q}}\mathbf{C_2}\oplus\overline{{}_{\boldsymbol{q}}\mathbf{C_2}}\right)$ | $(\boldsymbol{p},\boldsymbol{q})\oplus(\overline{\boldsymbol{p}},\overline{\boldsymbol{q}})$ | $\pm\left(1,1+\frac{p}{8}\right)$ $(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$ |
| $\left(\left(\frac{1}{4}\right)^{16}\right)$ | $SU(16)\times U(1)$ | — | — | $\pm\left(1,1+\frac{p}{8}\right)$ $(\sqrt{2}R\in\mathbb{Z})$ |

Table 9: $SU(16)\times U(1)$ model: $\frac{\hat{\pi}}{2}=\left(\left(\frac{1}{4}\right)^{16}\right)$.

| $A$ | Untwisted sectors | | Twisted sectors | |
|---|---|---|---|---|
| | Gauge sym. | Spinors | Co-spinors | Scalars |
| $(0^{16})\,,\left(\left(\frac{1}{2}\right)^{16}\right)$ | $SO(16)\times SO(16)$ | $(\mathbf{16},\mathbf{16})$ | $(\mathbf{128}_+,\mathbf{1})\oplus(\mathbf{1},\mathbf{128}_+)$ | — |
| $\left(0^p,\left(\frac{1}{2}\right)^q\right)$<br>$(p\leq 8,\ p\text{ odd})$ | $SO(2p)\times SO(16-2p)\times SO(16)$ | $(\mathbf{1},\mathbf{16-2}\boldsymbol{p},\mathbf{16})$ | — | — |
| $\left(0^p,\left(\frac{1}{2}\right)^q\right)$<br>$(p=4,8)$ | $SO(2p)\times SO(16-2p)\times SO(16)$ | $(\mathbf{1},\mathbf{16-2}\boldsymbol{p},\mathbf{16})$ | $\left(\mathbf{2}_+^{\boldsymbol{p}-\mathbf{1}},\mathbf{2}_+^{\mathbf{7}-\boldsymbol{p}},\mathbf{1}\right)$<br>$\oplus(\mathbf{1},\mathbf{1},\mathbf{128}_+)$ | — |
| $\left(0^p,\left(\frac{1}{2}\right)^q\right)$<br>$(p=2,6)$ | $SO(2p)\times SO(16-2p)\times SO(16)$ | $(\mathbf{1},\mathbf{16-2}\boldsymbol{p},\mathbf{16})$ | $\left(\mathbf{2}_-^{\boldsymbol{p}-\mathbf{1}},\mathbf{2}_-^{\mathbf{7}-\boldsymbol{p}},\mathbf{1}\right)$<br>$\oplus(\mathbf{1},\mathbf{1},\mathbf{128}_+)$ | — |
| $\left(0^p,\left(\frac{1}{2}\right)^q\right)$<br>$(p>8,\ p\text{ odd})$ | $SO(16)\times SO(2q)\times SO(16-2q)$ | $(\mathbf{16},\mathbf{1},\mathbf{16-2}\boldsymbol{q})$ | — | — |
| $\left(0^p,\left(\frac{1}{2}\right)^q\right)$<br>$(p=12)$ | $SO(16)\times SO(2q)\times SO(16-2q)$ | $(\mathbf{16},\mathbf{1},\mathbf{16-2}\boldsymbol{q})$ | $\left(\mathbf{1},\mathbf{2}_+^{\boldsymbol{q}-\mathbf{1}},\mathbf{2}_+^{\mathbf{7}-\boldsymbol{q}}\right)$<br>$\oplus(\mathbf{128}_+,\mathbf{1},\mathbf{1})$ | — |
| $\left(0^p,\left(\frac{1}{2}\right)^q\right)$<br>$(p=10,14)$ | $SO(16)\times SO(2q)\times SO(16-2q)$ | $(\mathbf{16},\mathbf{1},\mathbf{16-2}\boldsymbol{q})$ | $\left(\mathbf{1},\mathbf{2}_-^{\boldsymbol{q}-\mathbf{1}},\mathbf{2}_-^{\mathbf{7}-\boldsymbol{q}}\right)$<br>$\oplus(\mathbf{128}_+,\mathbf{1},\mathbf{1})$ | — |
| $\left(\left(\frac{1}{4}\right)^{16}\right)$ | $SU(8)\times SU(8)\times U(1)^2$ | $\left(\mathbf{8},\overline{\mathbf{8}}\right)\oplus\left(,\overline{\mathbf{8}},\mathbf{8}\right)$ | $(\mathbf{70},\mathbf{1})\oplus(\mathbf{1},\mathbf{70})\,,$<br>$\left(\pm\sqrt{2},0\right)\oplus\left(0,\pm\sqrt{2}\right)$ | — |

Table 10: $SO(16)\times SO(16)$ model: $\frac{\hat{\pi}}{2}=\left(0^8,\left(\frac{1}{2}\right)^8\right)$.

## C.2 The $E_8 \times E_8$ root lattice

| $A$ | Untwisted sectors: Gauge sym. | Untwisted sectors: Spinors | Twisted sectors: Co-spinors | Twisted sectors: Scalars |
|---|---|---|---|---|
| $(0^8;0^8)$ | $SO(16)\times E_8$ | $(\mathbf{128}_+;\mathbf{1})$ | $(\mathbf{128}_-;\mathbf{1})$ | $(\mathbf{16};\mathbf{1})$ $(\sqrt{2}R\in\mathbb{Z})$ |
| $\left(\left(\frac{1}{2}\right)^8;0^8\right)$ | $SO(16)\times E_8$ | $(\mathbf{128}_+;\mathbf{1})$ | — | $(\mathbf{16};\mathbf{1})$ $(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$ |
| $\left(0^2,\left(\frac{1}{2}\right)^6;0^8\right)$ | $SO(4)\times SO(12)\times E_8$ | $(\mathbf{2}_-,\mathbf{32}_-;\mathbf{1})$ | $(\mathbf{2}_+,\mathbf{32}_+;\mathbf{1})$ | $(\mathbf{4},\mathbf{1};\mathbf{1})$ $(\sqrt{2}R\in\mathbb{Z})$<br>$(\mathbf{1},\mathbf{12};\mathbf{1})$ $(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$ |
| $\left(0^4,\left(\frac{1}{2}\right)^4;0^8\right)$ | $SO(8)\times SO(8)\times E_8$ | $(\mathbf{8}_+,\mathbf{8}_+;\mathbf{1})$ | $(\mathbf{8}_-,\mathbf{8}_-;\mathbf{1})$ | $(\mathbf{8},\mathbf{1};\mathbf{1})$ $(\sqrt{2}R\in\mathbb{Z})$<br>$(\mathbf{1},\mathbf{8};\mathbf{1})$ $(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$ |
| $\left(0^6,\left(\frac{1}{2}\right)^2;0^8\right)$ | $SO(12)\times SO(4)\times E_8$ | $(\mathbf{32}_-,\mathbf{2}_-;\mathbf{1})$ | $(\mathbf{32}_+,\mathbf{2}_+;\mathbf{1})$ | $(\mathbf{12},\mathbf{1};\mathbf{1})$ $(\sqrt{2}R\in\mathbb{Z})$<br>$(\mathbf{1},\mathbf{4};\mathbf{1})$ $(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$ |
| $\left(\left(\frac{1}{4}\right)^8;0^8\right)$ | $SU(8)\times U(1)\times E_8$ | $(\mathbf{70};\mathbf{1})$, $\pm\sqrt{2}$ | — | $\left(\mathbf{8}\oplus\overline{\mathbf{8}};\mathbf{1}\right)$ $(\sqrt{2}R\in\mathbb{Z}\pm\frac{1}{4})$ |

Table 11: $SO(16)\times E_8$ model: $\frac{\hat{\pi}}{2}=(1,0^7;0^8)$.

| $A$ | Untwisted sectors: Gauge sym. | Untwisted sectors: Spinors | Twisted sectors: Co-spinors | Twisted sectors: Scalars |
|---|---|---|---|---|
| $(0^8;0^8)$ | $E_7\times SU(2)\times E_7\times SU(2)$ | $(\mathbf{56},\mathbf{2};\mathbf{1},\mathbf{1})$<br>$\oplus(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})$ | $(\mathbf{56},\mathbf{1};\mathbf{1},\mathbf{1})$<br>$\oplus(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{1})$ | $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})$<br>$(\sqrt{2}R\in\mathbb{Z})$ |
| $\left(\left(\frac{1}{2}\right)^8;0^8\right)$ | $E_7\times SU(2)\times E_7\times SU(2)$ | $(\mathbf{56},\mathbf{2};\mathbf{1},\mathbf{1})$<br>$\oplus(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})$ | — | $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})$<br>$(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$ |
| $\left(0^2,\left(\frac{1}{2}\right)^6;0^8\right)$ | $E_7\times SU(2)\times E_7\times SU(2)$ | $(\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})$ | $(\mathbf{56},\mathbf{1};\mathbf{1},\mathbf{1})$ | $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})$<br>$(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$ |
| $\left(0^4,\left(\frac{1}{2}\right)^4;0^8\right)$ | $SO(12)\times SU(2)\times SU(2)$<br>$\times E_7\times SU(2)$ | $(\mathbf{12},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$<br>$\oplus(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})$ | $(\mathbf{32},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$ | $(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})$<br>$(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$ |
| $\left(0^6,\left(\frac{1}{2}\right)^2;0^8\right)$ | $SO(12)\times SU(2)\times SU(2)$<br>$\times E_7\times SU(2)$ | $(\mathbf{32},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})$<br>$\oplus(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{56},\mathbf{2})$ | $(\mathbf{12},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1})$ | $(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2})$<br>$(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$ |
| $\left(\left(\frac{1}{4}\right)^8;0^8\right)$ | $E_6\times U(1)\times U(1)$<br>$\times E_7\times SU(2)$ | $\left(\mathbf{27}\oplus\overline{\mathbf{27}};\mathbf{1},\mathbf{1}\right)$<br>$\oplus(\mathbf{1};\mathbf{56},\mathbf{2})$,<br>$\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$ | — | $(\mathbf{1};\mathbf{1},\mathbf{2})$ $(\sqrt{2}R\in\mathbb{Z}+\frac{1}{2})$<br>$\left(\pm\frac{1}{\sqrt{2}},0\right)$ $(\sqrt{2}R\in\mathbb{Z}\pm\frac{1}{4})$ |

Table 12: $(E_7\times SU(2))^2$ model: $\frac{\hat{\pi}}{2}=\left(0^6,\left(\frac{1}{2}\right)^2;0^6,\left(\frac{1}{2}\right)^2\right)$.

| $A$ | Untwisted sectors | | Twisted sectors | |
|---|---|---|---|---|
| | Gauge sym. | Spinors | Co-spinors | Scalars |
| $(0^8;0^8)$ | $SO(16)\times SO(16)$ | $(\mathbf{128}_+;\mathbf{1})\oplus(\mathbf{1};\mathbf{128}_+)$ | $(\mathbf{16};\mathbf{1})\oplus(\mathbf{1};\mathbf{16})$ | — |
| $\left(\left(\frac{1}{2}\right)^8;0^8\right)$ | $SO(16)\times SO(16)$ | $(\mathbf{128}_+;\mathbf{1})\oplus(\mathbf{1};\mathbf{128}_+)$ | $(\mathbf{1};\mathbf{16})$ | — |
| $\left(0^2,\left(\frac{1}{2}\right)^6;0^8\right)$ | $SO(4)\times SO(12)\times SO(16)$ | $(\mathbf{2}_-,\mathbf{32}_-;\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1};\mathbf{128}_+)$ | $(\mathbf{4},\mathbf{1};\mathbf{1})\oplus(\mathbf{1},\mathbf{12};\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1};\mathbf{16})$ | — |
| $\left(0^4,\left(\frac{1}{2}\right)^4;0^8\right)$ | $SO(8)\times SO(8)\times SO(16)$ | $(\mathbf{8}_+,\mathbf{8}_+;\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1};\mathbf{128}_+)$ | $(\mathbf{8},\mathbf{1};\mathbf{1})\oplus(\mathbf{1},\mathbf{8};\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1};\mathbf{16})$ | — |
| $\left(0^6,\left(\frac{1}{2}\right)^2;0^8\right)$ | $SO(12)\times SO(4)\times SO(16)$ | $(\mathbf{32}_-,\mathbf{2}_-;\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1};\mathbf{128}_+)$ | $(\mathbf{12},\mathbf{1};\mathbf{1})\oplus(\mathbf{1},\mathbf{4};\mathbf{1})$ $\oplus(\mathbf{1},\mathbf{1};\mathbf{16})$ | — |
| $\left(\left(\frac{1}{4}\right)^8;0^8\right)$ | $SU(8)\times U(1)\times SO(16)$ | $(\mathbf{70};\mathbf{1})\oplus(\mathbf{1};\mathbf{128}_+)$, $\pm\sqrt{2}$ | — | — |

Table 13: $SO(16)\times SO(16)$ model: $\frac{\hat{\pi}}{2}=(1,0^7;1,0^7)$.

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