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The Landau Problem and non-Classicality in Symplectic Carroll Symmetry

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Abstract. Carroll's group is shown as a group of transformations in a 5-dimensional space ($\mathcal{C}$) obtained from the embedding of the Euclidean space into a (4,1)-de Sitter space. Three of the five dimensions of $\mathcal{C}$ are related to $\mathcal{R}^3$, and the other two to mass and time. A covariant formulation of Caroll's group is established in phase space. The Landau problem was studied. Finally, the negative parameter of the Wigner function is calculated.

1. Introduction

The study of physical systems in different spacetime symmetries has been a subject of great interest in modern theoretical physics. One such symmetry, known as the Carrollian symmetry, provides a unique framework for understanding the behavior of particles and fields in a specific spacetime setting [1, 2]. In recent years, the Carroll group has received increasing attention, especially in the context of strings [3, 4, 5, 6, 7, 8, 9]. There is an intriguing duality between these two limits which, in the context of Covariant Galilean formalism, was, to the author's knowledge, first highlighted by Saradzhev [10] where he coined it as non-Galilean transformation and found that these transformations give us $E' = E$ and $P'_5 \neq P_5$, but was only fully realized for the first time by Petronilo *et al.* [11], where a physical interpretation of P_5 was given. In a recent paper [12] it was noted that made the following redefinitions $C_i \to K_i$, where C_i and K_i are the Carrolian and Galilean boost respectively, and $P_4 \to -M$, $P_5 \to -H$ we recover the extended Galilei group, as the groups are isomorphic, but it's worth noting that these redefinitions completely change physical meaning of the group one is working as showed by [10]. The Carrollian symmetry is associated with a 5-dimensional de Sitter spacetime and is characterized by a distinct set of commutation rules that govern the algebraic structure of the theory. This symmetry has been a topic of growing interest due to its potential applications in various areas of physics, including relativistic quantum mechanics.

In this work, we explore the formalism of quantum mechanics in phase space within the context of Carrollian symmetry. We investigate the representations of spin-0 and spin-1/2 particles, derive their corresponding equations of motion, and examine the properties of their wave functions. Additionally, we study the effects of electromagnetic interactions on these particles within the Carrollian framework. One of the key highlights of our study is the calculation of the Wigner function for electrons in an external field under Carrollian symmetry. We analyze the Wigner function's behavior for different energy levels and delve into the concept of non-classicality by introducing a negativity parameter. In summary, this paper aims to provide a comprehensive exploration of quantum mechanics in phase space under the Carrollian symmetry. We present theoretical developments, analytical solutions, and numerical results that shed light on the intriguing properties of particles and fields in this unique spacetime framework. The order in which this work will be presented is as follows. In Sec. 2 the construction of the Carrollian covariance is presented. Sec. 3, a symplectic structure is constructed in the Carrollian covariance formalism, and using the commutation relations, the scalar equation is constructed in the light cone of five dimensions in phase space. In Sec. 4 we study the Carrollian spin 1/2 particle with an external field and solutions are proposed and discussed, after we calculated the negativity parameter and discussed the physical meaning. The Sec. 5 presents some concluding remarks.

2. The Carrollian Covariance

The following commutation rules describe the algebra associated with the formalism defined in the light cone of a 5-dimensional de Sitter spacetime:

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\sigma}M_{\nu\rho}),\\
&[P_\mu, M_{\rho\sigma}] = -i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho),\\
&[P_\mu, P_\sigma] = 0,
\end{aligned} \tag{1}$$

where $M_{\nu\sigma}$ are the generators of homogeneous transformations, P_μ of the non-homogeneous and $g_{i,j} = 1$, with $i = j = 1, 2, 3$ and $g_{45} = g_{54} = -1$. It is known that the Lie algebra of the extended Carroll group in $\mathcal{R}^3 \times \mathcal{R}$ is a subalgebra of this algebra, with J_i, as generators of rotations, C_i of the pure Carroll boosts, and P_μ spatial and temporal translations, being P_4, in this context, a Casimir invariant associated with the energy, $P_4 = -EI$, where I is the identity matrix [11]. The Casimir invariants of this algebra are

$$I_1 = p^\mu p_\mu, \tag{2a}$$
$$I_2 = p_4, \tag{2b}$$
$$I_3 = W_{4\mu}W^4_\mu, \tag{2c}$$

where $W^{\mu\nu}$ is the 5-dimensional Pauli-Lubanski matrix. It should be noted that although p_5 cannot be interpreted as an invariant mass, it nonetheless carries mass information [11].

For the scalar representation we take the invariants I_1 (2a) and I_2 (2b) and apply to a function ψ, and using the correspondence relation $p^\mu = -i\partial^\mu$ we have

$$\left\{ \begin{array}{l} \partial_\mu\partial^\mu\Psi = k^2\Psi \\ \partial_4\Psi = -iE\Psi \end{array} \right. , \tag{3}$$

where k and E are constants. These results reflect local properties. Then, the equation of motion reads $\partial_i^2\Psi + 2iE\partial_s\Psi = k^2\Psi$. Making a compactification in the s coordinate, such that $\varphi(\mathbf{x}, s) = \varphi(\mathbf{x}, s + 2\pi R)$, one can write a normalized solution as

$$\Psi(\mathbf{x}, s) = \frac{1}{\sqrt{2\pi R}} \sum_n \varphi_n(x) e^{-ins/R}. \tag{4}$$

Substituting solution (4) into Eq. (2), we get

$$\partial_i^2 \varphi_n(x) + 2\frac{n}{R} E\varphi_n(x) = k^2 \varphi_n(x). \tag{5}$$

or

$$\frac{R}{2n}\partial_i^2 \varphi_n(x) + E\varphi_n(x) = \frac{R}{2n} k^2 \varphi_n(x), \qquad \text{for } n \neq 0. \tag{6}$$

Making $k = \sqrt{\frac{2n\, m_o}{R}}$, we have $\partial_i^2\varphi_n(x) + 2\frac{n}{R}E\varphi_n(x) = \frac{2n\,m_o}{R}\varphi_n(x)$. or $\frac{R}{2n}\partial_i^2\varphi_n(x) + E\varphi_n(x) = m_o\varphi_n(x)$, for $n \neq 0$.

Letting $R \to 0$ we get $\partial_i^2\varphi_o(x) = 0$, for $n = 0$ and $(E - m_o)\varphi_n = 0$, for $n \neq 0$. Thus a Carroll scalar field with a dimensional reduction has no dynamics. Note that, $(E - m_o)\varphi_n = 0$ was deduced in many papers for the usual four-dimensional Carroll group [3, 4]. Therefore, the extended 5-dimensional Carroll group in null coordinates, when a dimensional reduction is applied, is reduced to the usual Carroll group for scalar fields.

3. Representation of Quantum Mechanics in Phase Space

In order to establish a connection between the Hilbert space denoted as $\mathcal{H}$ and the phase space denoted as Γ, the set of square-integrable functions of complex value, $\phi(q, p)$, is considered in Γ [13].

The following operators are defined to construct a representation of Carroll algebra in $\mathcal{H}(\Gamma)$:

$$\widehat{P}^\mu = p^\mu\star = p^\mu - \frac{i}{2}\partial_{q_\mu}, \tag{7a}$$

$$\widehat{Q}^\mu = q^\mu\star = q^\mu + \frac{i}{2}\partial_{p_\mu}, \tag{7b}$$

where $\star$ is the Moyal product and we set $\hbar = 1$.

3.1. Scalar representation

Utilizing the Casimir invariant $I_1 = \widehat{P}^\mu\widehat{P}_\mu$ and applying to Ψ, we obtain:

$$\widehat{P}_\mu\widehat{P}^\mu\Psi = k^2\Psi, \tag{8}$$

with $\widehat{P}_4 = -E$, or, explicitly

$$\left(p^2 - i\mathbf{p}\cdot\boldsymbol{\nabla} - \frac{1}{4}\nabla^2 - k^2\right)\Psi = \left(p_4 - \frac{i}{2}\partial_t\right)\left(p_5 - \frac{i}{2}\partial_5\right)\Psi,$$

a solution for this equation is $\Psi = e^{-2i[(p_5+m\alpha)q_5+(p_4+E)t]}\varphi(q,p)$. Therefore, we have

$$\frac{1}{2m\alpha}\left(p^2 - i\mathbf{p}\cdot\boldsymbol{\nabla} - \frac{1}{4}\boldsymbol{\nabla}^2\right)\varphi = \left(E + \frac{k^2}{2m\alpha}\right)\varphi,$$

with α a coefficient that depends on the reference frame [11], this is the Carrollian spin 0 equation in phase space, with the kinetic energy term of $\frac{k^2}{2m\alpha}$ added, that we can always use as the energy zero point. The association between this representation and Wigner formalism is established by $f_w(q,p) = \Psi(q,p) \star \Psi^\dagger(q,p)$, where $f_w(q,p)$ is the Wigner function, that satisfies the 5-dimensional Carrollian covariant Liouville-von Neumann equation in phase space as expressed by

$$p_\mu \partial_{q_\mu} f_w(q,p) = 0. \tag{9}$$

3.2. spinorial representation
In the context of Carrollian covariant formalism, the Lévy-Leblond equation takes on a similar form of the Dirac equation.

$$\gamma^\mu \left(p_\mu - \frac{i}{2}\partial_\mu \right) \Psi(p,q) = k\Psi(p,q) \tag{10}$$

In the case of the Lévy-Leblond equation, the association with the Wigner function is given by $f_w = \Psi \star \overline{\Psi}$, where $\bar{\Psi} = \zeta \Psi^\dagger$, with $\zeta = -\frac{i}{\sqrt{2}}\{\gamma^4 + \gamma^5\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, with each component satisfying Eq. (9).

4. Solution of the LL Equation with Electromagnetic Interactions

The expression that characterizes the behavior of a electric charged spin 1/2 particle within the Carrollian covariant phase space is analogous as presented in Galilean covariant formalism [13]:

$$\gamma^\mu \left(\widehat{P}_\mu - e\widehat{A}_\mu \right) \Psi = 0, \tag{11}$$

where A^μ is the 5-potential of the Carrollian electromagnetism [3].

Letting $\Psi = \left[\gamma^\nu \big(\widehat{P}_\nu - e\widehat{A}_\nu\big)\right]\psi$, we have

$$\left[\gamma^\mu\gamma^\nu \big(\widehat{P}_\mu - e\widehat{A}_\mu\big)\big(\widehat{P}_\nu - e\widehat{A}_\nu\big)\right]\psi = 0, \tag{12}$$

and making the following substitution $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \sigma^{\mu\nu}$, with $\sigma^{\mu\nu} = \frac{1}{2}\Big(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\Big) = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, Eq. (12) becomes

$$\left(\widehat{P}^\mu \widehat{P}_\mu - e\left(\widehat{P}^\mu \widehat{A}_\mu + \widehat{A}^\mu \widehat{P}_\mu\right) - e\sigma^{\mu\nu}\left[\widehat{P}_\nu, \widehat{A}_\mu\right] + e^2 \widehat{A}^\mu \widehat{A}_\mu \right)\psi = 0, \tag{13}$$

Defining $\widehat{A}^i$ as $\frac{1}{2}e^{ijk}B_j\widehat{Q}_k$, where $\widehat{Q}_\mu = (q_\mu + \frac{i}{2}\partial_{p^\mu})$, with the constraints $A^4 = A^5 = 0$ and specifying the magnetic field as $\mathbf{B} = (0,0,B)$, while also ensuring the particle's motion is confined to a plane (q_1,q_2) by setting $\widehat{P}_3 = 0$, and letting $\psi = \begin{pmatrix} \Phi(q^\mu,p^\mu) \\ \Theta(q^\mu,p^\mu) \end{pmatrix}$, we possess two independent equations,

$$\begin{aligned} -2\Big(p_4 - \tfrac{i}{2}\partial_t\Big)\Big(p_5 - \tfrac{i}{2}\partial_s\Big)\Phi(q^\mu,p^\mu) + \Bigg(p_1^2 + p_2^2 - \tfrac{1}{4}\Big(\tfrac{\partial^2}{\partial q_1{}^2} + \tfrac{\partial^2}{\partial q_2{}^2}\Big) - eB\Bigg[\tfrac{i}{2}\Big(p_2\partial_{p_1} - p_1\partial_{p_2}\Big) \\ + \tfrac{1}{4}\Big(\tfrac{\partial^2}{\partial_{q_2}\partial_{p_1}} - \tfrac{\partial^2}{\partial_{q_1}\partial_{p_2}}\Big)\Bigg] - i\Big(p_2\partial_{q_2} + p_1\partial_{q_2}\Big) - eB\Bigg[(q_1p_2 - q_2p_1) - \tfrac{i}{2}\Big(q_1\partial_{q_2} - q_2\partial_{q_1}\Big)\Bigg] \\ + \tfrac{e^2B^2}{4}\Bigg[\Big(q_1 + \tfrac{i}{2}\partial_{p_1}\Big)^2 + \Big(q_2 + \tfrac{i}{2}\partial_{p_2}\Big)^2\Bigg] + e\sigma^3 B\Bigg)\Phi(q^\mu,p^\mu) = 0. \end{aligned} \tag{14}$$

Similarly, the equation for Θ is analogous.

Choosing $\Phi(q^\mu,p^\mu)=\varphi(q^i,p^i)\phi(q^4,q^5,p^4,p^5)$. We obtain

$$\left(p_4-\frac{i}{2}\partial_t\right)\left(p_5-\frac{i}{2}\partial_s\right)\phi=\alpha mE\phi, \tag{15a}$$

and

$$\begin{aligned}&\Bigg(p_1^2+p_2^2-\frac{1}{4}\Big(\frac{\partial^2}{\partial {q_1}^2}+\frac{\partial^2}{\partial {q_2}^2}\Big)-eB\left[\frac{i}{2}\Big(p_2\partial_{p_1}-p_1\partial_{p_2}\Big)+\frac{1}{4}\Big(\frac{\partial^2}{\partial_{q_2}\partial_{p_1}}\right.\\ -&\left.\frac{\partial^2}{\partial_{q_1}\partial_{p_2}}\Big)\right]-i\Big(p_2\partial_{q_2}+p_1\partial_{q_2}\Big)-eB\left[(q_1p_2-q_2p_1)-\frac{i}{2}\Big(q_1\partial_{q_2}-q_2\partial_{q_1}\Big)\right]\\ +&\frac{e^2B^2}{4}\left[\Big(q_1+\frac{i}{2}\partial_{p_1}\Big)^2+\Big(q_2+\frac{i}{2}\partial_{p_2}\Big)^2\right]+e\sigma^3B\Bigg)\varphi=2\alpha mE\varphi. \end{aligned}\tag{15b}$$

The solution of Eq. (15a) is $\phi=C_1e^{-2i[(p_5+\alpha m)q_5+(p_4+E)t]}$, where C_1 is a normalization constant.

To solve Eq. (15b) we will make the following change of variables

$$w(q_1,q_2,p_1,p_2)=p_1^2+p_2^2+eB(q_2p_1-q_1p_2)+\frac{e^2B^2}{4}(q_1^2+q_2^2). \tag{16}$$

After a long calculation, it is shown that the imaginary part of this equation is identically null, which gives us

$$w\varphi-e^2B^2\frac{\partial\varphi(w)}{\partial\omega}-e^2B^2w\frac{\partial^2\varphi(w)}{\partial w^2}=(2mE-esB)\varphi(w), \tag{17}$$

with $s\varphi=\sigma^3\varphi$, with $s=\pm1$. Taking $\omega=w/(eB)$, $\alpha=(2mE-seB)/eB$ and defining $f(w)\equiv e^w\phi(\omega)$, thus

$$\omega f''(\omega)+(1-2\omega)f'(\omega)-af(\omega)=0, \tag{18}$$

where $f'(x)=\frac{\partial f}{\partial\omega}$ and $a=(1-\alpha)$. The equation represented as Eq. (18) corresponds to a confluent hypergeometric equation, specifically known as the Kummer equation. The physical solutions are written as $f_n(\omega)=A_nU\left(\frac{1}{2}-\frac{\alpha}{2},1,2\omega\right)$, in this context, the functions $U(a,b,x)$ correspond to Kummer's functions, and the constants are represented as A_n. Nevertheless, it becomes evident that when the parameter $a=-n$, where n is a non-negative integer $(n=0,1,2,\ldots)$, the series $U(a,b,x)$ transforms into a polynomial series in x with a degree not exceeding n. Therefore, we express this as follows: $\alpha-1=2n$, we have the following relation of eigenvalue $E=\omega_c\left(n+\frac{1}{2}+\frac{s}{2}\right)$, with $\omega_c=\frac{eB}{\alpha m}$, $s=\pm1$, and corresponding the following quasi-amplitudes are,

$$\Phi_n=C_1e^{-2i[(p_5+\alpha m)q_5+(p_4+E)t]}\left(A_nU\left(-n,1,\frac{2w}{eB}\right)\exp\left(-\frac{w}{eB}\right)\right). \tag{19}$$

The analogy holds true for Θ. To compute the corresponding Wigner function, f_w, simply perform the Moyal product of ψ_n with its complex conjugate, denoted as $\psi_n\star\bar{\psi}_n$. It is essential to recognize that $w=2\alpha mh$, where $h=\frac{1}{2\alpha m}\left(p_1^2+p_2^2+eB(q_2p_1-q_1p_2)+\frac{e^2B^2}{4}(q_1^2+q_2^2)\right)$. As a result, the wavefunction ψ_0 can be expressed as $\psi_0 = C_0 e^{-2h/\omega_c}$. Hence, the expression for f_w^0 becomes:

$$f_w^0 = C_0 e^{-2h/\omega_c} \star \psi_0 = C_0 e^{-2\hat{h}/\omega_c} \psi_0 = C_0 e^{-2E_0/\omega_c} \psi_0$$

consequently, the ground state Wigner functions for spin particles with values of $1/2$ and $-1/2$ are as follows:

$$f_w^{0^+} = (C_{0^+})^2 \frac{1}{e^2} e^{-(p_1^2+p_2^2+eB(q_2p_1-q_1p_2)+\frac{e^2B^2}{4}(q_1^2+q_2^2))/eB},$$

while

$$f_w^{0^-} = (C_{0^-})^2 e^{-(p_1^2+p_2^2+eB(q_2p_1-q_1p_2)+\frac{e^2B^2}{4}(q_1^2+q_2^2))/eB}.$$

For the general case,

$$f_w^{n^\pm} = (A_n^\pm)\Big(\frac{1}{n!\pi}\Big) e^{-(p_1^2+p_2^2+eB(q_2p_1-q_1p_2))} U\left(-n, 1, \frac{2(p_1^2+p_2^2+eB(q_2p_1-q_1p_2))}{eB}\right). \tag{20}$$

it is worthwhile comparing with the case of Galilean covariance [13], as the solutions are very similar due the fact of the group isomorphism. Due to this fact, the forms of the Wigner function and its properties are strictly the same. For example, the negativity of the Wigner function, which indicates the deviation of the solution from the usual Gaussian form, has the same behavior as that observed in reference [13]. However, it should be noted that Carrollian symmetry is distinct from Galilean symmetry and is therefore explored in this article.

5. Concluding Remarks

We construct the formalism of the quantum mechanics in phase space in the context of Carrollian symmetry and we arrive at the representations of the spin 0 and spin $1/2$ equations. We analyzed the gauge symmetry for spin $1/2$ particles in phase space and show that the minimal coupling, in this case, is obtained by replacing the lagrangian density $p_\mu\star$ by $p_\mu\star - iA_\mu\star$. We also calculate the Wigner function for electrons in an external field. As an important result, we have shown that the for this system Galilean and Carrollian symmetries the Wigner function and its properties are strictly the same.

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