Can Charged Stars be Singularity-Free?

Sudan Hansraj, Chevarra Hansraj, Abdelghani Errehymy, and Lushen Moodly*

While the configuration of a static spherically symmetric distribution of perfect fluid in an electric field has received extensive attention in the literature, only a few works have investigated the conformally flat case. The model exhibits a remarkable property being regular or singularity free is proposed. This is unexpected given that Coulombic repulsion opposes the collapse of a star to the center generally making the center unreachable. The conformally flat geometry thus enables the approach to the center of a charged star as shown in our model. Presently there is no known reported exact model that satisfies all the elementary requirements for physical acceptability. Some reported models that have claimed to be physically reliable have been shown to be deficient. The conditions for the existence of physically relevant models and show that exact solutions satisfying the conditions exist is investigated. Then a model satisfying all the requirements with isotropic particle pressure graphically for a carefully chosen parameter space is exhibited. The model admits a barotropic equation of state. The interior spacetime is successfully matched to the exterior Reissner-Nordstrom metric. The causality criterion as well as the adiabatic stability lower bound of Chandrasekar are shown to hold. The energy conditions are also valid within the stellar radius. Other interesting models are considered.

1. Introduction

Algebraically special solutions of Einstein's equations have attracted considerable attention over the last few decades. Petrov[1,2] introduced a scheme to classify spacetimes according to the alignment of the principal null vectors that arise from studying the eigenspaces of the Weyl conformal tensor. Six classes of metrics are distinguishable with the most special being Type O or the conformally flat case. Spacetimes that are conformally flat have been extensively studied in the context of relativistic astrophysics and cosmology. Mathematically, the condition for this to occur is the vanishing of the Weyl conformal curvature tensor thereby introducing an algebraic constraint on the gravitational field. Physically this isolates regions of the spacetime manifold where local effects due to the fluid congruence only govern its evolution and the effects of distant objects is negligible. For isolated stellar systems this is a reasonable constraint. In standard Einstein gravity all conformally flat isotropic stars have been found. They are either the Schwarzschild interior spacetimes[3] in the case of no expansion or the Stephani stars[4] when expansion is present. Since the Einstein field equations for neutral isotropic spheres comprise a system of three partial differential equations in four unknowns, there is latitude in selecting one of the geometrical or dynamical variables or to specify a relationship such as an equation of state to close the system. The latter direction, barring some trivial cases, almost always leads to an intractable system of differential equations requiring numerical treatments with approximation constraints. The problem may be drastically simplified by admitting anisotropic stresses. Alternatively, a geometric restriction such as requiring conformal flatness closes the system and the unique solutions have been found for static and nonstatic spacetimes as mentioned already.

When electric charge is introduced into the problem with a perfect fluid, the number of equations increase to four and the number of unknowns to six. Now there are two possible prescriptions to be made. From a survey of the literature it is evident that only sporadic attempts have been made to impose conformal flatness as a geometrical constraint for a charged star. The question is indeed interesting given that in charged spheres Coulombic repulsion combines with the hydrostatic pressure to oppose the gravitational collapse of the object to a point singularity. For this reason it is not surprising that the approach to the center of a spherical distribution is impossible as shown in ref. [5] in general. An irremovable central singularity is always present. In other words charged spherical distributions are generally rings or shells possibly surrounding other physically acceptable matter such as a neutral fluid. However, it is intriguing that if it is demanded that the spacetime manifold is conformally flat then regular distributions are possible and the approach to the center is possible. We show that such distributions exist such that all standard physical demands are satisfied. The dynamical and geometrical quantities are all well behaved at the stellar center.

S. Hansraj, C. Hansraj, A. Errehymy, L. Moodly
Astrophysics Research Center
School of Mathematics
Statistics and Computer Science
University of KwaZulu-Natal
Private Bag X54001, Durban 4000, South Africa
E-mail: hansrajs@ukzn.ac.za

The ORCID identification number(s) for the author(s) of this article can be found under https://doi.org/10.1002/prop.202300180

DOI: 10.1002/prop.202300180

The charged star has had a long history from the discovery of the exterior solution by Reissner[6] and Nordstrom[7] in 1918. In fact Kaluza[8] and Klein[9] introduced the idea of a five dimension in gravity to explain the role of the electromagnetic field tensor. The new proposal now had 15 components compared to 10 in four dimensions and four of these were attributed to the electromagnetic field, one to a scalar field called a scalaron and the remaining 10 were the usual Einstein components of neutral spheres. Moreover, in the study of black holes which ostensibly originate from collapsing stars, the 'no hair' theorem[10–13] asserts that the only three defining characteristic of a black hole are its mass, angular momentum and charge. Consequently there are good grounds to interrogate charged stars in gravitational physics. Exact solutions of the Einstein–Maxwell field equations are crucial in modeling compact objects such as white dwarfs, neutron stars and quark stars. Evidently observational data has been shown to be discordant with standard neutron star models[14] thus prompting further investigations along these lines. By generating exact solutions using the Einstein–Maxwell field equations, the issue of gravitational collapse of a spherically symmetric distribution can be studied.[15,16] The works of Ivanov[17] and Sharma et al.[18] describe the effect of the electric field on the luminosity, redshift and masses of a star.

Conformal flatness has been prescribed to produce a realistic anisotropic model for a compact star[19] with electric charge. A study of conformally flat spherically symmetric spacetimes in different coordinate systems has also been carried out by Grøn and Johannesen [20]. A polytropic equation of state was used to produce a conformally flat model in ref. [21] assuming anisotropic pressure stresses. Other solutions with static conformal motions for anisotropic charged fluids were generated in refs. [22, 23]. Melfo and Rago[24] tested the conformal flatness case on an anisotropic but non-static charged fluid sphere. It was noted that the solutions favor static configurations. When imposing isotropic pressure, non-static solutions were generated and inconsistent models resulted. Herrera et al.[25] attempted to find conformally flat, interior solutions to the Einstein equations for anisotropic fluids. The focus was to compare these with similar models that did not include the vanishing of the Weyl tensor in order to observe its effect on stellar models. Ivanov[17] treated conformal flatness and prescribed the potential λ which led to prescribing ρ effectively. The solutions were examined analytically and then compared to a real pulsar.

In this paper we first derive the Einstein–Maxwell equations of motion for a charged spherical distribution of isotropic matter in Section 2. In Section 3, we review some of the basic constraints imposed on charged star models for physical applicability. Thereafter in Section 4 we invoke the conformal flatness criterion and reduce the field equations to a single master equation requiring one more prescription. In Section 5, we prove rigorously that the physical conditions permit the existence of physically viable conformally flat models provided certain restrictions are met. We demonstrate explicitly a physically reasonable model satisfying the elementary constraints in Section 6. Thereafter in Section 7, we review a known conformally flat metric due to Wang[26] and show it to be deficient physically. In Section 8, we consider some well known metric ansatze based on historically well-behaved models to find new classes of exact solutions. None of these models were able to satisfy all the physical requirements and were consequently dismissed.

2. Einstein–Maxwell Field Equations

The line element

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

where the functions $\nu(r)$ and $\lambda(r)$ are gravitational potentials, may be used to study the interior of a spherically symmetric charged star. Further a co-moving fluid four–velocity field $u^a = e^{-\nu}\delta_0{}^a$ is assumed.

The non-vanishing Weyl conformal tensor components may be expressed as

$$\begin{aligned} C_{r\theta r\theta} &= \frac{1}{\sin^2\theta}C_{r\phi r\phi} = \frac{r^2}{2e^{2\nu}}C_{rtrt} = -\frac{e^{2\lambda}}{2r^2\sin^2\theta}C_{\theta\phi\theta\phi} \\ &= -e^{2(\lambda-\nu)}C_{\theta t\theta t} = -\frac{e^{2(\lambda-\nu)}}{\sin^2\theta}C_{\phi t\phi t} = f(r) \end{aligned} \tag{2}$$

where

$$f(r) = \frac{1}{6}\left(r\left(\lambda' - \nu'\right) - e^{2\lambda} + 1 + r^2\left(\nu'' + \nu'^2 - \nu'\lambda'\right)\right). \tag{3}$$

Following[27] the Einstein–Maxwell field equations governing the stellar structure in the presence of an electric field, is given by the system

$$\left[r\left(1 - e^{-2\lambda}\right)\right]' = r^2\rho + \frac{1}{2}r^2E^2 \tag{4}$$

$$-\left(1 - e^{-2\lambda}\right) + 2\nu' re^{-2\lambda} = pr^2 - \frac{1}{2}r^2E^2 \tag{5}$$

$$e^{-2\lambda}\left[r(\nu' - \lambda') + r^2\left(\nu'' + \nu'^2 - \nu'\lambda'\right)\right] = pr^2 + \frac{1}{2}E^2r^2 \tag{6}$$

$$\sigma^2 = \frac{4e^{-2\lambda}}{r^2}\left(r^2E' + E\right)^2 \tag{7}$$

where refers to differentiation with respect to the variable r. E is the electrostatic field intensity and σ is the proper charge density. Note that the above system has four equations in six unknowns so that two conditions will need to be specified for a unique model. The conservation laws $T^{ab}{}_{;b} = 0$ generate the condition

$$p' + (\rho + p)\nu' = \frac{E}{r^2}\left[r^2E\right]' \tag{8}$$

also known as the equation of hydrodynamical equilibrium. It may be observed that (8) includes four physical quantities hence has limited use in exact solutions since we have only two choices available.

A coordinate transformation due to Buchdahl[28] defines x and two metric functions $y(x)$ and $Z(x)$ as follows, $x = Cr^2$, $Z(x) =$

$e^{-2\lambda(r)}$, $y^2(x) = e^{2\nu(r)}$ convert (4) to (7) to the following equivalent form

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C} \tag{9}$$

$$\frac{Z-1}{x} + \frac{4Z\dot{y}}{y} = \frac{p}{C} - \frac{E^2}{2C} \tag{10}$$

$$4x^2Z\ddot{y} + 2x^2\dot{Z}\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2x}{C}\right)y = 0 \tag{11}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2 \tag{12}$$

where dots represent differentiation with respect to x. This version has the distinct advantage that the equation of pressure isotropy (11) is now linear in one of the potentials y and exponential functions have been eliminated. This is bound to greatly assist in the process of locating exact solutions.

The exterior gravitational field for a static, spherically symmetric charged distribution is governed by the Reissner–Nordstrom[6,7] solution. The Reissner–Nordstrom exterior line element has the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2$$
$$+r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{13}$$

where M and Q are associated with the mass and charge of the sphere respectively as measured by an observer at spatial infinity. For the Reissner–Nordstrom solution (13) the radial electric field is described by

$$E = \frac{Q}{r^2} \tag{14}$$

outside a charged star and consequently the proper charge density is $\sigma = 0$. Upon setting $Q = 0$ in (13) the exterior Schwarzschild[3] solution is regained.

3. Elementary Conditions for Physical Admissibility

Customarily the following elementary conditions are imposed for solutions of the Einstein-Maxwell system to be physically reasonable.

(a) The metric functions $e^{2\nu}$ and $e^{2\lambda}$ should be positive and non–singular everywhere in the interior of the star.
(b) Positivity and finiteness of pressure and energy density everywhere in the interior of the star including the origin and boundary: $0 \leq p < \infty \quad 0 < \rho < \infty$
(c) The pressure and energy density are usually monotonic decreasing functions of the coordinate r from the center of the distribution and the pressure vanishes at the boundary $r = R$: $\frac{dp}{dr} \leq 0 \quad \frac{d\rho}{dr} \leq 0 \quad p(R) = 0$
(d) The interior line element should be matched smoothly to the exterior Reissner–Nordstrom[6] line element at the boundary: $e^{2\nu(R)} = e^{-2\lambda(R)} = 1 - \frac{2M}{R} + \frac{Q^2}{R^2}$
(e) The electric field intensities E of the exterior and interior should also coincide at the boundary: $E(R) = \frac{Q}{R^2}$.
(f) The speed of sound should remain subluminal everywhere in the interior: $0 \leq \frac{dp}{d\rho} \leq 1$
(g) The following energy conditions should be satisfied: (i) Weak energy condition (WEC): $\rho - p > 0$; Strong energy condition (SEC): $\rho + p > 0$ and Dominant energy condition (DEC): $\rho + 3p > 0$
(h) The Chandrasekhar[29] criterion for adiabatic stability demands the constraint: $(\frac{\rho+p}{p})\frac{dp}{d\rho} > \frac{4}{3}$.

Admittedly although the task of finding exact solutions for charged fluid spheres is not difficult the caveat arises when the physical conditions have to be met. Note that there are other conditions imposed on charged spheres however they are obtained under very specific conditions[30–32] and we shall not examine them in this work.

4. Conformally Flat Charged Spheres

In order to solve the Einstein-Maxwell field equations, geometric restrictions based on physical considerations are customarily imposed. Algebraically special solutions have been intensively studied in the area of general relativity. The Petrov[1] scheme of classifying metrics depends on the eigenspaces of the conformal curvature (or Weyl) tensor and six independent cases arise. Conformal flatness is the most special case (type O) and requires the vanishing of the Weyl tensor (3). This constrains the metric potentials to obey the relation

$$4x^2Z\ddot{y} + 2x^2\dot{Z}\dot{y} - \left(\dot{Z}x - Z + 1\right)y = 0 \tag{15}$$

which is to be used in conjunction with the field equations. Note that in the absence of charge, (15) and pressure isotropy suggests that $\dot{Z}x - Z + 1 = 0$ or $Z = 1 + kx$ which is exactly the Schwarzschild interior spacetime solution in the static case. Conformal flatness depends solely on the geometry or metric hence charge does not feature in (15). It is also known that if expansion is permitted then the unique conformally flat metrics are the Stephani stars.[4]

With the aid of (15), Equations (9) to (12) reduce to the system

$$\frac{\rho}{C} = -3\dot{Z} \tag{16}$$

$$\frac{p}{C} = \frac{\dot{Z}y + 4Z\dot{y}}{y} \tag{17}$$

$$\frac{E^2}{C} = 2\left(\frac{\dot{Z}x - Z + 1}{x}\right) \tag{18}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2 \tag{19}$$

$$y = A\sqrt{x}\cosh\left(\frac{1}{2}\int\frac{dx}{x\sqrt{Z}} + B\right) \tag{20}$$

where (20) is the integrated version of (15).[25] This system constitutes a set of five independent equations in six unknowns governing the behavior of charged conformally flat perfect fluids. One of the components may be specified initially to yield the remaining five. Alternatively, some of the variables may be connected on physical grounds such as an equation of state ($p = \alpha\rho$) or $\sigma = \pm\rho$ as demanded for stable equilibrium by De and Raychaudhuri.[33]

5. Existence of Physically Reasonable Solutions

Before embarking on a search for exact solutions for conformally flat charged spacetimes, it is prudent to analyze whether solutions satisfying the elementary physical conditions exist. In other words, there should be no impediment inherent in the field equations that rules out physically viable models. At the outset, it is clear that $Z(x) = e^{-2\lambda(r)} > 0$ as well as $y(x) = e^{\nu(r)} > 0$. Note that without loss of generality we assume $C > 0$. Then the density equation (16) suggests that $\dot{Z} < 0$, which implies the metric potential Z is always a monotonically decreasing function. It is also important to note that the conclusion $\dot{Z}y < 0$ follows and will be useful later. Positivity of the pressure demands that $-4Z\dot{y} < \dot{Z}y$ as a further restriction on the metric functions. By using the result $\dot{Z}y < 0$, we conclude $Z\dot{y} > 0$ and hence $\dot{y} > 0$ since $Z > 0$. Thus the temporal metric potentials y must be monotonically increasing functions. Additionally, since $\dot{Z} < 0$ and $\dot{y} > 0$ we also deduce $\dot{Z}\dot{y} < 0$, which will be a useful result in the work to follow. Now turning our attention to the weak energy condition

$$\frac{\rho - p}{C} = -3\dot{Z} - \frac{\dot{Z}y + 4Z\dot{y}}{y} > 0$$

the inequality

$$(Z\dot{y}) < 0 \qquad \text{or} \qquad \dot{Z}y < -Z\dot{y}$$

results, which is consistent with the result (V). Similarly the strong energy condition

$$\frac{\rho + p}{C} = -3\dot{Z} + \frac{\dot{Z}y + 4Z\dot{y}}{y} > 0$$

yields $\dot{Z}y < 2Z\dot{y}$ which also harmonizes with (V). The dominant energy condition

$$\frac{\rho + 3p}{C} = -3\dot{Z} + 3\left(\frac{\dot{Z}y + 4Z\dot{y}}{y}\right) > 0$$

leads to $Z\dot{y} > 0$ which is not in violation of any of the restrictions above. At this stage it is clear that spacetimes satisfying the energy conditions certainly exist.

Next consider the sound speed. Requiring a causal sound speed $0 < \frac{dp}{d\rho} < 1$ generates the condition

$$-1 < \frac{Z\ddot{y}}{\ddot{Z}y} + \frac{\dot{Z}\dot{y}}{\ddot{Z}y} - \frac{Z\dot{y}^2}{\ddot{Z}y^2} < -\frac{1}{4} \tag{21}$$

on the metric potentials. Recall that thus far we have established that for a positive definite energy density and pressure satisfying the weak, strong and dominant energy conditions we have $\dot{Z} < 0$, $\dot{y} > 0$, $0 < Z < 1$, $y > 0$. Presently, there are no means available to constrain the second derivatives $\ddot{Z}$ and $\ddot{y}$ required in (21). For this reason we consider in turn all four possible signatures for $\ddot{Z}$ and $\ddot{y}$. For convenience we introduce the naming

$$h(x) = \frac{Z\ddot{y}}{\ddot{Z}y} + \frac{\dot{Z}\dot{y}}{\ddot{Z}y} - \frac{Z\dot{y}^2}{\ddot{Z}y^2} \tag{22}$$

and we note from (21) that $h(x) < 0$.

- **Case 1**: $\ddot{Z} > 0, \ddot{y} > 0$
  In this case $\frac{Z\ddot{y}}{\ddot{Z}y} > 0$, $\frac{\dot{Z}\dot{y}}{\ddot{Z}y} < 0$ and $\frac{Z\dot{y}^2}{\ddot{Z}y^2} > 0$ so it follows from the negativity of $h(x)$ that

  $$Z\ddot{y} < Z\dot{y}^2 - \dot{Z}\dot{y} \tag{23}$$

  Since both the left hand side and right hand side of (23) are positive, we have the constraint

  $$0 < \ddot{y} < \dot{y}^2 - \frac{\dot{Z}\dot{y}}{Z} \tag{24}$$

  for a causal solution. Hence a realistic solution exists provided (23) is satisfied.
- **Case 2**: $\ddot{Z} < 0, \ddot{y} > 0$
  Now $h(x) < 0$ requires

  $$\frac{Z\ddot{y}}{\ddot{Z}y} + \frac{\dot{Z}\dot{y}}{\ddot{Z}y} < \frac{Z\dot{y}^2}{\ddot{Z}y^2} \tag{25}$$

  But under the present hypothesis $\frac{Z\ddot{y}}{\ddot{Z}y} < 0$, $\frac{\dot{Z}\dot{y}}{\ddot{Z}y} > 0$ and $\frac{Z\dot{y}^2}{\ddot{Z}y^2} < 0$. Multiplying (25) by $\ddot{Z} < 0$ reverses the inequality

  $$\frac{Z\ddot{y}}{y} + \frac{\dot{Z}\dot{y}}{y} > \frac{Z\dot{y}^2}{y^2} \tag{26}$$

  and since the right hand side > 0, it follows that

  $$Z\ddot{y} + \dot{Z}\dot{y} > 0 \tag{27}$$

  that is

  $$\frac{d}{dx}(Z\dot{y}) > 0 \Rightarrow \dot{y} > \frac{k}{Z} \tag{28}$$

  where $k < 0$ is a constant because $\dot{y} < 0$. This suggests that a solution satisfying (28), $\ddot{Z} < 0, \ddot{y} > 0$ will be physically reasonable.
- **Case 3**: $\ddot{Z} < 0, \ddot{y} < 0$
  In this case $\frac{Z\ddot{y}}{\ddot{Z}y} > 0$, $\frac{\dot{Z}\dot{y}}{\ddot{Z}y} > 0$ and $\frac{Z\dot{y}^2}{\ddot{Z}y^2} < 0$ so that $h(x)$ is the sum of three positive quantities and is thus positive which is a contradiction since $h(x) < 0$. Therefore, solutions satisfying $\ddot{Z} < 0$ and $\ddot{y} < 0$ are acausal and consequently not physically feasible.
- **Case 4**: $\ddot{Z} > 0, \ddot{y} < 0$
  Under these conditions we have $\frac{Z\ddot{y}}{\ddot{Z}y} > 0$, $\frac{\dot{Z}\dot{y}}{\ddot{Z}y} < 0$ and $\frac{Z\dot{y}^2}{\ddot{Z}y^2} > 0$. This indicates that $h(x) < 0$ for all functions $\ddot{Z} > 0$ and $\ddot{y} < 0$. So a physically realistic model may be generated under these circumstances.

Turning attention to the electric field intensity formula (18) it is clear that the right hand side should be positive. This leads to

$\frac{1-Z}{x} > -\dot{Z}$. But $\dot{Z} < 0$ so it follows that $Z < 1$. That is a monotonically decreasing function with upper bound of one should be sought in finding a physically reasonable model. Finally, from $\dot{Z}x - Z + 1 > 0$ then by Gronwall's Theorem[34]

$$Z > 1 + kx \tag{29}$$

and by (16) we see that $k < 0$. These impose boundaries on our choice for Z. To summarize, this analysis has demonstrated that an exact solution that satisfies the energy conditions and the causality requirements does indeed exist. In what is to follow, we pursue such solutions mindful of the constraints we have established here.

6. A Physically Viable Model

The choice $Z = \frac{1}{(x+1)^2}$ for the spatially directed gravitational potential is inspired by a model due to Ivanov[19] and immediately meets the requirement of monotonic decrease of Z from the central position $x = 0$. However, in the Ivanov paper the author assumed the form $Z = (x-1)^2$ and reduced the problem to the anisotropic case thus opening up a second choice of variables. In our case this choice closes the model as the system of differential equations is now fully determined. Substituting $Z = (x+1)^{-2}$ in (15) generates the potential

$$y = Ae^{\frac{x}{2}} + Bxe^{-\frac{x}{2}} \tag{30}$$

where A and B are constants of integration that can be settled by matching of the interior and exterior geometries across a surface of vanishing pressure.

With this combination of metric potentials, the thermodynamical quantities density ρ and pressure p are given by

$$\frac{\rho}{C} = \frac{6}{(x+1)^3} \tag{31}$$

$$\frac{p}{C} = \frac{2Ae^x x - 2B\left(x^2-2\right)}{(x+1)^3(Ae^x + Bx)} \tag{32}$$

while the expressions

$$\frac{dp}{d\rho} = \frac{A^2e^{2x}(2x-1) - 2ABe^x x_1 + B^2 x_2}{9(Ae^x + Bx)^2} \tag{33}$$

$$\left(\frac{\rho+p}{p}\right)\frac{dp}{d\rho} = \frac{(Ae^x(x+3) - B(x^2-3x-2))(A^2e^{2x}(2x-1) - 2ABe^x x_1 + B^2x_2)}{9(Ae^x+Bx)^2(Ae^x x - B(x^2-2))} \tag{34}$$

where we have put $x_1 = (x^3 - 2x - 4)$ and $x_2 = (-2x^3 + x^2 + 8x + 2)$ will be useful to study the sound speed and the Chandrasekar adiabatic stability criterion respectively. The electric field intensity E and the proper charge density σ evaluate to

$$\frac{E^2}{C} = \frac{2x(x+3)}{(x+1)^3} \tag{35}$$

$$\frac{\sigma^2}{C} = \frac{2\left(x^2+4x+9\right)^2}{(x+1)^7(x+3)} \tag{36}$$

for this geometry. The active gravitational mass may be computed with the formula $\int_0^r \rho v^2 dv$ and is given by

$$m = \frac{3}{2}\left(\frac{(x-1)\sqrt{x}}{(x+1)^2} + \tan^{-1}\left(\sqrt{x}\right)\right) \tag{37}$$

in our coordinates. The energy conditions are studied with the help of the expressions

$$\frac{\rho - p}{C} = \frac{2B\left(x^2+3x-2\right) - 2Ae^x(x-3)}{(x+1)^3(Ae^x+Bx)} \tag{38}$$

$$\frac{\rho + p}{C} = \frac{2Ae^x(x+3) + B\left(-2x^2+6x+4\right)}{(x+1)^3(Ae^x+Bx)} \tag{39}$$

$$\frac{\rho + 3p}{C} = \frac{6Ae^x - 6B(x-2)}{(x+1)^2(Ae^x+Bx)} \tag{40}$$

and these are all expected to be positive and continuous within the stellar radius.

To finalize the model, it remains to do the matching of the interior and exterior spacetimes and to obtain the integration constants in terms of the mass M, radius R and charge Q of the star. The vanishing boundary pressure $p(R) = 0$ generates the equation

$$Ae^{CR^2}CR^2 - B(C^2R^4 - 2) = 0 \tag{41}$$

while the matching of the g_{00} components of the interior and exterior metric tensors yields the condition

$$(Ae^{CR^2/2} + Be^{-CR^2/2})^2 = 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \tag{42}$$

through the Reissner-Nordstrom solution. Solving (41) and (42) simultaneously gives

$$A = \frac{\pm e^{-\frac{1}{2}(CR^2)}\sqrt{R^2 - 2MR + Q^2}}{2(C^2R^4 - 1)}\left(\frac{2 \mp C^2R^4}{R}\right) \tag{43}$$

$$B = \mp\frac{CRe^{\frac{CR^2}{2}}\sqrt{R^2 - 2MR + Q^2}}{(C^2R^4 - 1)} \tag{44}$$

for the two integration constants in terms of M, Q, R and C. Equating the interior and exterior electric field intensities $\frac{E^2}{C}$ with $E = \frac{Q}{R^2}$ provides the value of C in the form

$$C = -\frac{1}{R^2} + \frac{V}{R(2 - Q^2R^2)} + \frac{2}{R^3V} \tag{45}$$

where we have put

$$V = \sqrt[3]{2(-Q^4R^4 + 4Q^2R^2 + QR\sqrt{(Q^2R^2-2)^3} - 4)}.$$

Clearly on substituting (45) in (43) and (44) the values of A and B emerge only in terms of M, R and Q. Having expressed all con-

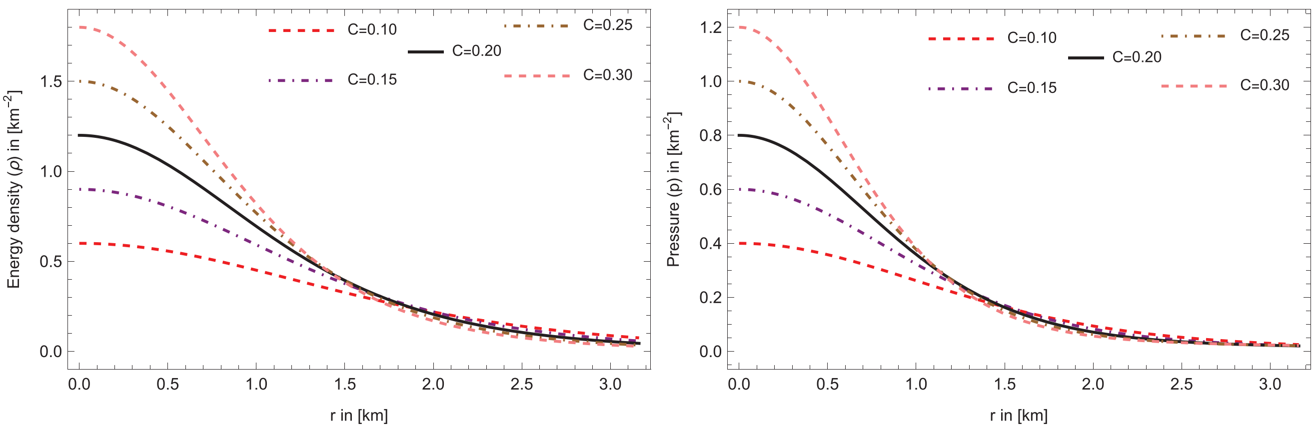

Figure 1. Density (ρ) with pressure (p) versus the radial coordinate r for different values of C with $A = 0.085$ and $B = 0.085$.

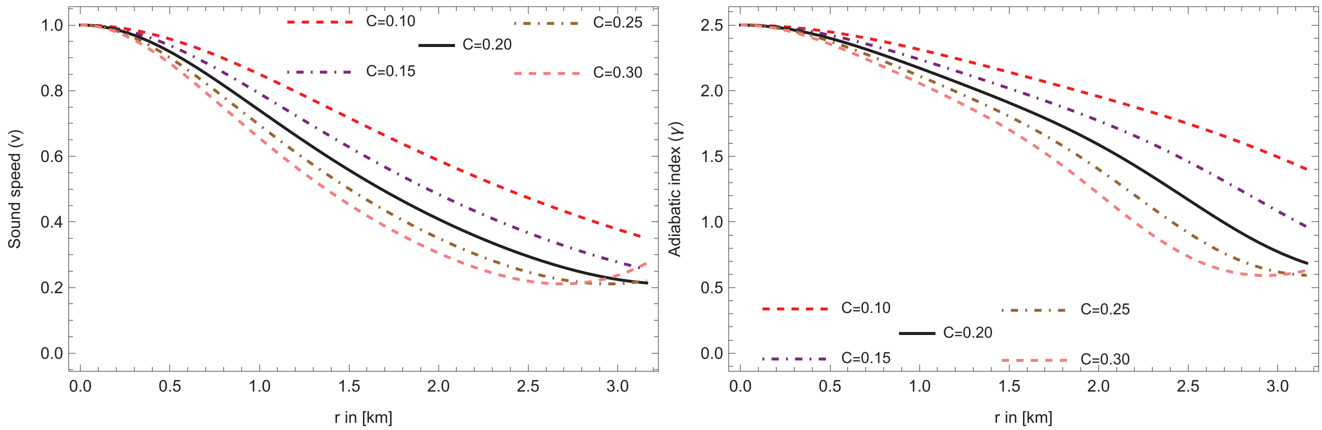

Figure 2. Sound speed ($\frac{dp}{d\rho}$) with Chandrasekhar adiabatic stability index (γ) versus the radial coordinate r for different values of C with $A = 0.085$ and $B = 0.085$.

stants in terms of the mass, radius and charge of the sphere completes the matching of the interior and exterior spacetime manifolds.

We now analyze graphical plots of the dynamical quantities. These plots have been constructed using parameter values $A = 0.085$, $B = 0.085$ and a range of values for C as indicated in the plot. Note that this parameter space has been determined through a process of fine-tuning and is by no means unique. However, we have found at least one viable parameter space such that all the physical requirements hold.

From the plots, in **Figure 1** it can be observed that the density and isotropic particle pressure are both monotonically decreasing functions, which is expected. The pressure vanishes for a finite radius thus establishing the boundary of the spherical distribution of charge. The left pane of **Figure 2** confirms that the sound speed squared index is less than one within the vanishing pressure surface radius which verifies that the model is causal for the choice of parameter space. The right pane of Figure 2 provides an indication of the behavior of the adiabatic stability parameter γ which is expected to be $\frac{4}{3}$. This appears to be the case for most of the sphere however there appears to be a slight drop below the value $4/3$ close to the boundary. This could be an indication that the Chandrasekhar bound is influenced by the presence of charge and may be in need of revision in this case. **Figure 3** indicates that the electric field intensity as well as the proper charge density are well behaved within the interior and there are no discontinuities or other pathologies present. For our choice of parameters, it can also be checked from **Figure 4** that all the energy conditions are satisfied. **Figure 5** depicts the behavior of the equation of state index $\frac{p}{\rho}$ as well as the behavior of the pressure change against the density variation. Both panes indicate smooth well behaved functional forms that is expected of realistic stars. Note that it is a particularly pleasing feature of the model that an explicit barotropic equation of state $p = p(\rho)$ is realizable. With the help of (31) we get $x = \sqrt[3]{\frac{6C}{\rho}} - 1$ which may be substituted in (32) to give p in terms of ρ only.

7. Some Known Conformally Flat Charged Models

Although no systematic algorithmic treatment of isotropic conformally flat static charged stars has been undertaken, there does exist isolated conformally flat solutions to the Einstein–Maxwell equations reported in the literature using ad hoc

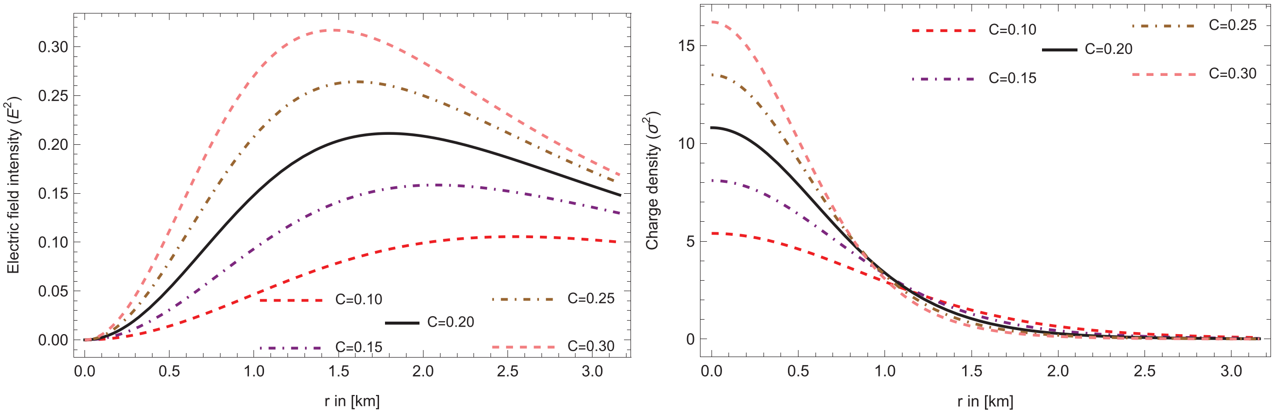

Figure 3. Electric field intensity (E^2) with charge density (σ^2) versus the radial coordinate r for different values of C with $A = 0.085$ and $B = 0.085$.

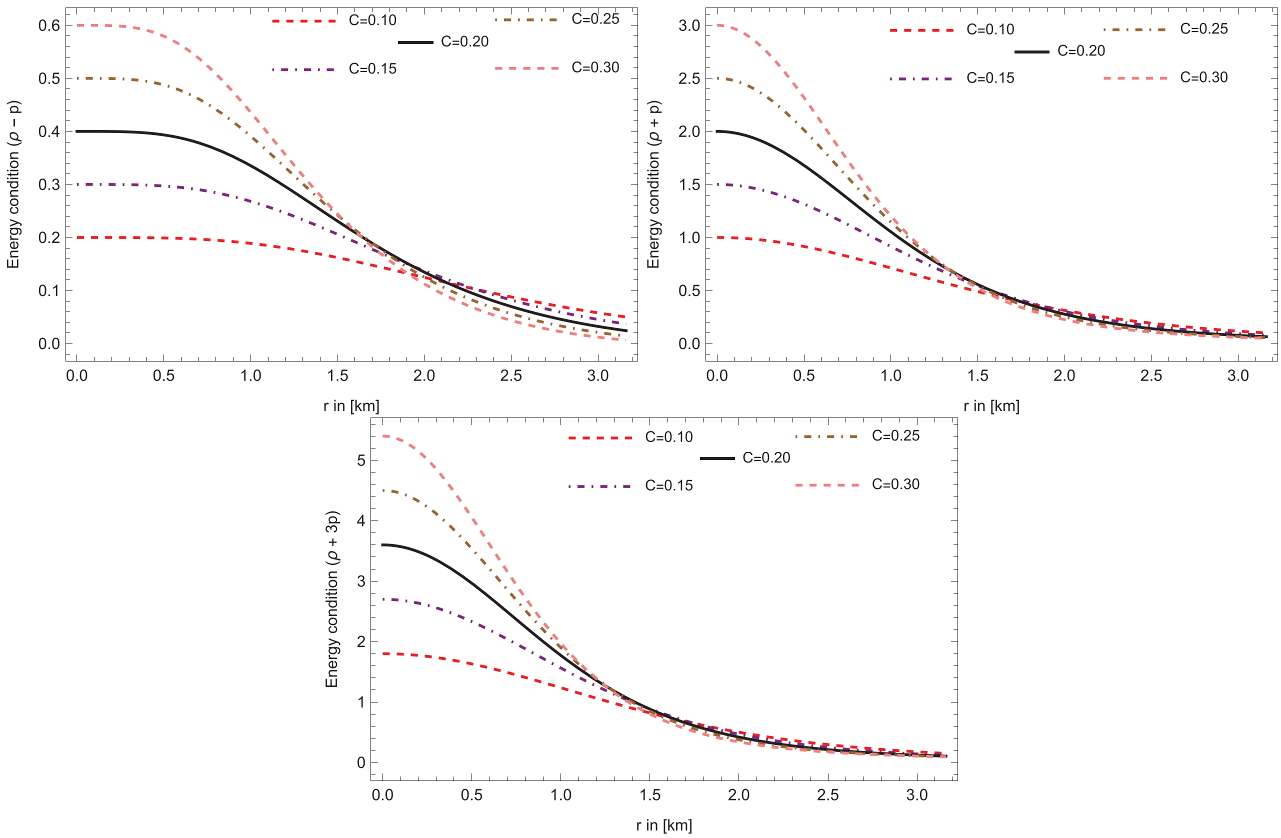

Figure 4. Energy conditions $(\rho - p)$, $(\rho + p)$ and $(\rho + 3p)$ versus the radial coordinate r for different values of C with $A = 0.085$ and $B = 0.085$.

prescriptions. Banerjee and Santos[35] worked on anisotropic models for a charged dust sphere and Shi-Chang[36] obtained some interior isotropic solutions for a charged stable static sphere. These solutions however, are not free from singularities and do not satisfy all the energy conditions.

Wang Xingxiang[26] claimed to present an ostensibly physically reasonable solution of the conformally flat Einstein–Maxwell field equations by prescribing the form of the density as

$$\frac{\rho}{C} = \frac{6n(1-x)}{(1+(n-1)x)^3} \tag{46}$$

for an arbitrary parameter n. Note that this is analogous to prescribing $\dot{Z}$ by virtue of Equation (16) and we obtain

$$\dot{Z} = \frac{-2n(1-x)}{(1+(n-1)x)^3} \tag{47}$$

which in turn can be integrated to give

$$Z = \frac{3n(-2nx + n + 2x - 2)}{2(n-1)^2((n-1)x+1)^2} \tag{48}$$

as the spatial metric potential. At this point Wang's solution $Z = \frac{(1-x)^2}{(1+(n-1)x)^3}$ is incorrect. However, this metric potential may be

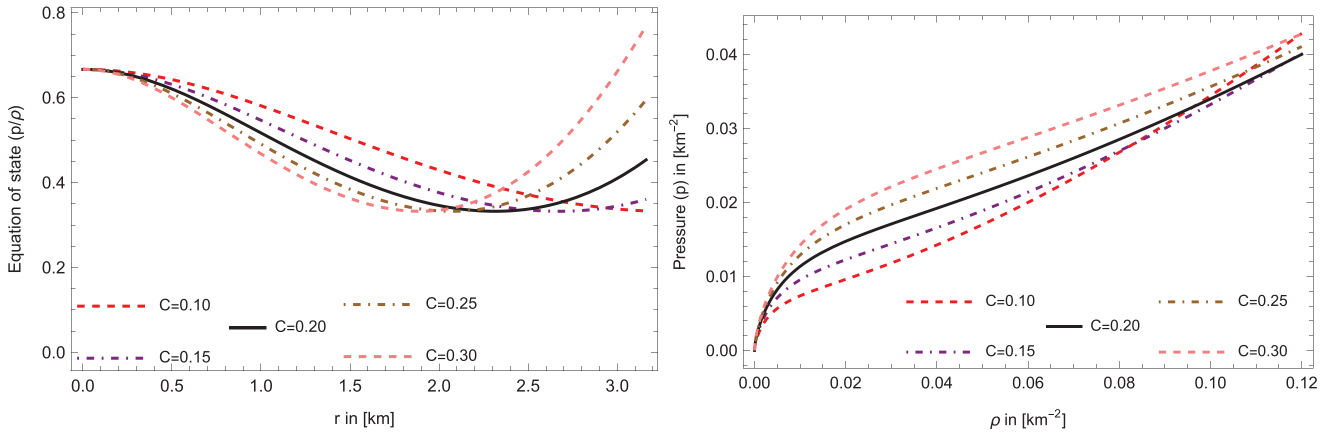

Figure 5. Equation of state (p/ρ) versus the radial coordinate r, and relation between pressure (p) and energy density (ρ) for different values of C with $A = 0.085$ and $B = 0.085$.

postulated and then the rest of Wang's model can be rescued. In other words his pressure profile would end up as different. This impacts on the physical analysis and so the density and quantities depending on it such as the speed of sound will be affected. When either form of Z is utilized to complete the model, it is found that the basic physical conditions fail. In fact Wang does not study the stability situation such as the causality and adiabatic parameter of Chandrasekar. Hence the model is of only limited use and its claim of being a physically reasonable conformally flat charged static fluid sphere may be questioned. Consequently there appears to be no physically valid conformally flat charged spheres reported in the literature.

8. Other Physically Interesting Cases

From (15)–(19) it can be noted that to close the system any one of the six variables may be prescribed and then the system solved for the remaining five. Since the pressure and charge density equations contain two variables on the right hand side, it is not feasible to prescribe one of these. It therefore remains to prescribe one of the density ρ, electric field intensity E, the metric potentials Z or y. We now consider some historically important cases of interest.

8.1. Equations of State

Amongst the more important questions to ask in stellar modeling is whether the a priori imposition of an equation of state leads to exact models of conformally flat charged spheres. Consider the linear barotropic equation of state $p = \alpha\rho$ where α is a real number and $0 < \alpha < 1$ in order not to violate causality. With this prescription the relationship

$$y = C_1 Z^{-\frac{1+3\alpha}{4}} \tag{49}$$

where C_1 is an integration constant, arises with the help of (16) and (17). Inserting (49) into (15) gives the differential equation

$$3\left(3\alpha^2 + 4\alpha + 1\right)x^2\dot{Z}^2 - 4Z\left((3\alpha + 1)x^2\ddot{Z} + x\dot{Z} + 1\right) + 4Z^2 = 0 \tag{50}$$

which is second order but nonlinear. The general solution of (50) is not available. However, some special cases for α are of interest. The case $\alpha = 1$ corresponds to a stiff fluid such that the sound speed and light speed are equal. The case $\alpha = \frac{1}{3}$ is related to incoherent radiation. Unfortunately neither of these cases permit the integration of (50). In the instance where $\alpha > 1$ this is known as ultrarelativistic fluid and causality is violated. Setting $\alpha = -1$ in a cosmological setting pertains to dark energy and in this context equation (50) integrates as

$$Z = 1 + a\sqrt{x} + bx = 1 + ar + br^2 \tag{51}$$

where a and b are constants. With this combination of y and Z, the pressure and energy density turn out to be $p = -\rho = 3C(\frac{a}{2\sqrt{x}} + b)$. Hence we have the equation of state $\rho + p = 0$ which is identified with dark energy. In the event that $\alpha < -1$ then the spacetime regime is termed phantom. To facilitate the accelerated expansion of the universe the requirement is $\alpha < -\frac{1}{3}$. For the special case $\alpha = -\frac{1}{3}$ the solution to (50) is $Z = 1 + ax$ which is the Schwarzschild interior potential however the remaining potential will have the form $y =$ constant. This in turn implies a constant density and pressure and is in fact the defective Einstein universe model. No other cases for α yielded a closed form solution to (50). This is a common occurrence when invoking an equation of state at the outset in stellar modeling.

8.2. Isothermal Conformally Flat Sphere

The prescription $Z =$ a constant is a necessary and sufficient condition for isothermal behavior of static neutral spheres both in Einstein gravity[37] and its generalization Lovelock theory.[38] Isothermal behavior is characterized by an inverse square law fall–off of the density and pressure in Einstein four–dimensional gravity, as is the case in Newtonian physics. However, this case is forbidden as inserting $Z = k$ into (16) results in $\rho = 0$ which is not acceptable for a stellar interior. Hence no conformally flat charged isotropic spheres exhibit isothermal behavior. This can also be seen directly. Requiring an inverse square law fall off of density is tantamount to setting $\rho = H/x$ for some constant H

in (16). This gives $Z = -\frac{A}{3C}\log Kx$ where K is an integration constant. The potential y may be established via (20). While (20) integrates it may be noted that this combination of Z and y does not cause the pressure p to vary inversely to the square of the radius according to Equation (17).

8.3. The Vaidya–Tikekar Superdense Star Ansatz

Vaidya and Tikekar[39] constructed models of superdense static stars with the metric ansatz equivalent to $Z = \frac{1+ax}{1+bx}$, for a and b being real constants. With this form, Equation (15) simplifies to

$$4(ax+1)(bx+1)\ddot{y} + (a-b) + 2(a-b)\dot{y} + b(a-b)y = 0 \tag{52}$$

with general solution

$$y = C_2\big(a(a-b)(bx+1)\big)^{3/2}\,{}_2F_1\left(1-\frac{\sqrt{b}}{2\sqrt{a}}, \frac{\sqrt{b}}{2\sqrt{a}}+1; \frac{5}{2}; \frac{a(bx+1)}{a-b}\right) + C_1\,{}_2F_1\left(-\frac{\sqrt{b}}{2\sqrt{a}}-\frac{1}{2}, \frac{1}{2}\left(\frac{\sqrt{b}}{\sqrt{a}}-1\right); -\frac{1}{2}; \frac{a(bx+1)}{a-b}\right) \tag{53}$$

where ${}_2F_1$ is the hypergeometric function and C_1 and C_2 are constants of integration. The solution (53) is not suitable for modeling astrophysical objects in its current form. We are interested in special cases of (53) that reduce to elementary functions.

Interestingly when b is an integral multiple of a, solutions in terms of elementary functions result. Consider the case where $b = 9a$. The solution for this case has the form

$$y = C_1\left(27a^2x^2 + 18ax - 1\right) + C_2\sqrt{ax+1}(9ax+1)^{3/2} \tag{54}$$

which is a smooth singularity-free curve. The dynamical quantities accordingly have the form

$$\frac{\rho}{C} = \frac{24a}{(9ax+1)^2} \tag{55}$$

$$\frac{p}{C} = \frac{8aC_1\left(243a^3x^3 + 324a^2x^2 + 99ax + 10\right)\sqrt{ax+1}}{C_1\left(27a^2x^2+18ax-1\right)\sqrt{ax+1}(9ax+1)^2 + C_2(ax+1)(9ax+1)^{7/2}} + \frac{24C_2\left(27a^3x^3 + 48a^2x^2 + 23ax + 2\right)\sqrt{9ax+1}}{C_1\left(27a^2x^2+18ax-1\right)\sqrt{ax+1}(9ax+1)^2 + C_2(ax+1)(9ax+1)^{7/2}} \tag{56}$$

$$\frac{E^2}{C} = \frac{144a^2x}{(9ax+1)^2} \tag{57}$$

$$\frac{\sigma^2}{C} = \frac{1296a^2(ax+1)(3ax+1)^2}{(9ax+1)^5}. \tag{58}$$

Observe that the model above is completely free of singularities at the center of the stellar distribution. Despite this positive feature, empirical testing could not generate a suitable parameter space such that all the basic requirements for physical acceptability could be met.

8.4. The Finch–Skea Potential

The special case of Vaidya–Tikekar where $a = 0$, $b = 1$ corresponds to the Finch–Skea[40] metric which is known to have pleasing physical properties. Equation (52) yields a solution in elementary functions. The complete solution for this case has the form

$$y = c_1\left(\sqrt{x+1}\cosh\left(\sqrt{x+1}\right) - \sinh\left(\sqrt{x+1}\right)\right) + c_2\left(\cosh\left(\sqrt{x+1}\right) - \sqrt{x+1}\sinh\left(\sqrt{x+1}\right)\right) \tag{59}$$

and bears a strong resemblance to the Finch–Skea static neutral star metric with trigonometric functions exchanged for hyperbolic functions. Putting $u = \cosh\sqrt{x+1}$ and $v = \sinh\sqrt{x+1}$, the dynamical quantities have the form

$$\frac{\rho}{C} = \frac{3}{(x+1)^2} \tag{60}$$

$$\frac{p}{C} = \frac{\sqrt{x+1}(v-\beta u) + (2x+3)(\beta v - u)}{(x+1)^2\left(\sqrt{x+1}(\beta u - v) + (u - \beta v)\right)} \tag{61}$$

$$\frac{E^2}{C} = \frac{2x}{(x+1)^2} \tag{62}$$

$$\frac{\sigma^2}{C} = \frac{2(x+3)^2}{(x+1)^5} \tag{63}$$

for a conformally flat star with Finch–Skea potential. Note we have introduced $\beta = \frac{C_1}{C_2}$ following Finch and Skea. The solution also admits an equation of state

$$p = \frac{\rho\left[\left(2\beta\sqrt{\frac{3}{\rho}} + \beta + \sqrt[4]{\frac{3}{\rho}}\right)\sinh\left(\sqrt[4]{\frac{3}{\rho}}\right) - \left(\beta\sqrt[4]{\frac{3}{\rho}} + 2\sqrt{\frac{3}{\rho}} + 1\right)\cosh\left(\sqrt[4]{\frac{3}{\rho}}\right)\right]}{3\left(\beta\sqrt[4]{\frac{3}{\rho}} + 1\right)\cosh\left(\sqrt[4]{\frac{3}{\rho}}\right) - 3(\beta+1)\sinh\left(\sqrt[4]{\frac{3}{\rho}}\right)} \tag{64}$$

expressing the pressure as a function of the density.

To search for a viable parameter space a common strategy is to examine the physical conditions at the stellar center. The central pressure at $x = 0$

$$\left(\frac{p}{C}\right)_0 = \frac{e^2(\beta-1)}{\beta+1} - 2 \tag{65}$$

giving the constraint $\beta < -1$ or $\beta > 1.7$. The expressions for the energy conditions are given by

$$\rho - p = \frac{2\left(\sqrt{x+1}(2u - 2\beta v) + (x+1)(\beta u - v)\right)}{(x+1)^{5/2}\left(\sqrt{x+1}(\beta u - v) + (u - \beta v)\right)} \tag{66}$$

$$\rho + p = \frac{4\left(\sqrt{x+1}\left(\frac{1}{2}u - \frac{1}{2}\beta v\right) + (x+1)(\beta u - v)\right)}{(x+1)^{5/2}\left(\sqrt{x+1}(\beta u - v) + (u - \beta v)\right)} \tag{67}$$

$$\rho + 3p = \frac{6(\beta u - v)}{(x+1)^{3/2}\left(\beta\left(u\sqrt{x+1} - v\right) + \left(u - v\sqrt{x+1}\right)\right)} \tag{68}$$

and at the center ($x = 0$) we have

$$(\rho - p)_0 = \frac{e^2(1-\beta)}{\beta+1} + 5 \tag{69}$$

$$(\rho + p)_0 = \frac{e^2(\beta-1)}{\beta+1} + 1 \tag{70}$$

$$(\rho + 3p)_0 = \frac{e^2(\beta-1)}{\beta+1} - 2 \tag{71}$$

and these are all expected to be positive, giving the constrains $1.7 < \beta < 5.2$. The speed of sound squared index is given by

$$\begin{aligned}\frac{dp}{d\rho} = & \frac{\beta^2\left(2u^2(x+1)^{3/2} + uv(x+1) - 2v^2\sqrt{x+1}\right)}{6\sqrt{x+1}\left(\beta\left(v - u\sqrt{x+1}\right) + \left(v\sqrt{x+1} - u\right)\right)^2} \\ & - \frac{\beta\left(u^2(x+1) + 4uvx\sqrt{x+1} + v^2(x+1)\right)}{6\sqrt{x+1}\left(\beta\left(v - u\sqrt{x+1}\right) + \left(v\sqrt{x+1} - u\right)\right)^2} \\ & + \frac{\left(-2u^2\sqrt{x+1} + uv(x+1) + 2v^2(x+1)^{3/2}\right)}{6\sqrt{x+1}\left(\beta\left(v - u\sqrt{x+1}\right) + \left(v\sqrt{x+1} - u\right)\right)^2}\end{aligned} \tag{72}$$

and at the center

$$\left(\frac{dp}{d\rho}\right)_0 = \frac{1}{24}\left(\frac{e^4(\beta-1)^2}{(\beta+1)^2} - 13\right) \tag{73}$$

which constrains β to $\beta < -1 \cup -1 < \beta < 0.34 \cup \beta > 2.9$ and finally the adiabatic stability index of Chandrasekhar γ may be expressed as

$$\begin{aligned}&\left(\frac{\rho+p}{p}\right)\frac{dp}{d\rho} \\ &= \left(\sqrt{x+1}(u - \beta v) + 2(x+1)(\beta u - v)\right)(x+1)(u - \beta v)(v - u\beta) \\ &\times\left(\sqrt{x+1}\left(\beta^2\left(u^2(x+1) - v^2\right) - \left(u^2 - v^2(x+1)\right) - 2\beta uvx\right)\right)/ \\ &\left(3\sqrt{x+1}\left(\sqrt{x+1}(\beta u - v) + u - \beta v\right)^2\right) \\ &\times\left(\sqrt{x+1}(\beta v - u) + (x+1)(\beta u - v)\right)\end{aligned} \tag{74}$$

and at the center

$$\gamma_0 = \frac{1}{24}\left(\frac{e^4(\beta-1)^2}{(\beta+1)^2} + \frac{3e^2(\beta-1)}{\beta+1} - \frac{27(\beta+1)}{e^2(\beta-1) - 2(\beta+1)} - 7\right). \tag{75}$$

Requiring $\gamma > \frac{4}{3}$ restricts β to the intervals $\beta < -1 \cup -1 < \beta < -0.02 \cup 1.4 < \beta < 1.7 \cup \beta > 6.81$. Finally, combining all these constraints on β, it can be seen that no β value exists such all the required conditions are satisfied. Consequently no Finch-Skea physically reasonable model of conformally flat charged fluid exists. At least one of the physical demands fails at the stellar center.

8.5. Conformally Flat Charged Dust ($p = 0$) Sphere

Charged dust is the simplest matter field to consider. Here, Coulombic repulsion opposes the gravitational field to prevent collapse to a point singularity. This case was analyzed by Hansraj et al.[4] in general without conformal flatness. Substituting $p = 0$ in (17) we obtain,

$$y = C_1 Z^{-\frac{1}{4}} \tag{76}$$

for the temporal potential, where C_1 is a constant of integration. Inserting (76) into (15), the differential equation

$$3x^2\dot{Z}^2 - 4Z\left(x\left(\dot{Z} + x\ddot{Z}\right) + 1\right) + 4Z^2 = 0 \tag{77}$$

results. The general solution of (77) is given by

$$Z = \frac{C_2}{256x^2}\left(\left(C_3 - C_1 x\right)^2 - 64x^2\right)^2 \tag{78}$$

where C_2 and C_3 are constants of integration. This is the unique solution for all conformally flat charged dust spheres. We also list the other physical quantities, with $(C_1)^2 = K_1$ and $(C_2)^2 = K_2$,

$$\frac{\rho}{C} = -\frac{3C_3\left(\left(C_1 - 8\right)x - C_2\right)\left(\left(C_1 + 8\right)x - C_2\right)\left(\left(K_1 - 64\right)x^2 - K_2\right)}{128x^3} \tag{79}$$

$$\frac{E^2}{C} = \frac{\begin{array}{c}C_3\left(C_1^4x^4 + 128K_2x^2 - 2K_1\left(3K_2x^2 + 64x^4\right)\right. \\ \left. + 8C_1C_2^3x - 3C_2^4 + 4096x^4\right)\end{array}}{128x^3} + \frac{2}{x} \tag{80}$$

$$\begin{aligned}\frac{\sigma^2}{C} = & C_3\left(\left(C_2 - C_1x\right)2 - 64x^2\right)^2\left(32768x^6\right)^{-1} \\ & \times\left(C_3\left(3\left(K_1 - 64\right)^2x^4 + \left(128 - 6K_1\right)K_2x^2 + 3C_2^4\right) + 256x^2\right)^2 \\ & \times\left[C_3\left(C_1x - 8x - C_2\right)\left(C_1x + 8x - C_2\right)\right]^{-1} \\ & \times\left[\left(\left(K_1 - 64\right)x^2 + 2C_1C_2x - 3K_2\right) + 256x^2\right]^{-1}\end{aligned} \tag{81}$$

The model above has a singularity at the origin as expected. As there is no boundary as such we do not study this solution in more detail.

9. Discussion

The gravitational field equations governing the behavior of charged static fluid spheres in general relativity in a conformally flat background spacetime have been studied systematically. As the system is under-determined one more choice has to be made of one of the geometrical or physical quantities to find an exact solution and then to construct the complete model. At the outset we reviewed the basic requirements for a distribution of charged perfect fluid to be physically viable. We then analyzed the equations in general to determine whether any conflict arises in the physical conditions. It was found that physically relevant solutions are possible if conformal flatness was imposed on charged spheres. A simple postulate for one of the metric potentials generated a model that was found to satisfy all the physical requirements for a specified parameter space. The model was causal and in agreement with the Chandrasekar adiabatic stability criterion. Moreover all the energy conditions were met. A surface of vanishing pressure exists to denote the boundary of the compact object. The remarkable property of the model was the absence of a singularity at the stellar center. In other words the center was reachable. Ostensibly the curvature due to the conformally flat background allowed for the Coulombic repulsion to be negated near the center. Some models earlier claimed to be physically satisfactory were shown to be deficient. A study was made of some interesting cases in the context of charge and conformal flatness such as an equation of state, isothermal fluids, the Vaidya-Tikekar superdense star, the Finch-Skea star and finally charged dust spheres. Of course the field equations for conformally flat charged spheres admit many more solutions however, we have succeeded in presenting a model that comports with the elementary requirements for physical plausibility.

Acknowledgements

S.H. thanks the National Research Foundation of South Africa for support under the competitive programme for rated researchers (Grant number : 138012). C.H. thanks the Oppenheimer Memorial Trust for the award of a postdoctoral fellowship. A.E. thanks the National Research Foundation (NRF) of South Africa for the award of a postdoctoral fellowship. L.M. thanks the National Research Foundation of South Africa for the award of a doctoral bursary.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Keywords

charged stars, conformally flat, Einstein-Maxwell, electric field, Reissner-Nordstrom

Received: August 20, 2023

[1] A. Z. Petrov, *Uch. Zapiski Kazan. Gos. Univ.* **1954**, *114*, 55. English translation A. Z. Petrov *Gen. Relativ. Gravit.***2000**, *32*, 1665.
[2] A. A. Coley, *Classical Quantum Gravity* **2004**, *21*, 35.
[3] K. Schwarzschild, *Sitzungsber Preuss. Akad. Wiss. Phys. Math.* **1916a**, *K1*, 424.
[4] S. Hansraj, S. D. Maharaj, S. Mlaba, N. Qwabe, *J. Math. Phys.* **2017**, *58*, 052501.
[5] S. Hansraj, S. D. Maharaj, S. Mlaba, N. Qwabe, *J. Math. Phys.* **2017**, *58*, 052501.
[6] H. Reissner, *Ann. Phys. Lpz.* **1916**, *50*, 106.
[7] G. Nordstrom, *Proc. Ned. Akad. Wet.* **1918**, *26*, 1201.
[8] T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.)* **1921**, *1921*, 966 (in German).
[9] O. Klein, *Z. Phys. A (in German)* **1926**, *37*, 895.
[10] R. Ruffini, J. A. Wheeler, *Phys. Today* **1971**, *24*, 30.
[11] W. Israel, *Phys. Rev.* **1967**, *164*, 1776; *Comm. Math. Phys.* **1968**, *8*, 245.
[12] B. Carter, *Phys. Rev. Lett.* **1971**, *26*, 331.
[13] R. B. Wald, *Phys. Rev. Lett.* **1971**, *26*, 1653.
[14] V. Pandharipande, D. Pines, R. A. Smith, *Astrophys. J.* **1976**, *208*, 550.
[15] C. R. Ghezzi, *Phys. Rev. D* **2005**, *72*, 104017.
[16] S. Chakrabarti, R. Goswami, S. D. Maharaj, N. Banerjee, *Gen. Rel. Grav.* **2018**, *50*, 148.
[17] B. V. Ivanov, *Phys. Rev. D* **2002**, *65*, 10411.
[18] R. Sharma, S. Mukherjee, S. D. Maharaj, *Gen. Rel. Grav.* **2001**, *33*, 999.
[19] B. V. Ivanov, *Eur. Phys. J. C* **2018**, *78*, 332.
[20] Ø. Grøn, S. Johannesen, *Eur. Phys. J. Plus* **2013**, *128*, 92.
[21] L. Herrera, A. Di Prisco, W. Barreto, J. Ospino, *Gen. Rel. Grav.* **2014**, *46*, 1827.
[22] A. M. Manjonjo, S. D. Maharaj, S. Moopanar, *Eur. Phys. J. Plus* **2017**, *132*, 62.
[23] A. M. Manjonjo, S. D. Maharaj, S. Moopanar, *Class. Quantum Grav.* **2018**, *35*, 045015.
[24] A. Melfo, H. Rago, *Astrophys. Space Sci.* **1992**, *193*, 9.
[25] L. Herrera, *J. Math. Phys.* **2001**, *42*, 2129.
[26] W. Xingxiang, *J Math. Phys.* **1987**, *28*, 2697.
[27] S. Hansraj, S. D. Maharaj, *Int. J. Mod. Phys. D* **2006**, *15*, 1311.
[28] H. A. Buchdahl, *Phys. Rev. D* **1959**, *116*, 1027.
[29] S. Chandrasekhar, *Astrophys. J.* **1964**, *140*, 417.
[30] C. G. Böhmer, T. Harko, *Gen Rel. Grav.* **2007**, *39*, 757.
[31] G. K. Andréasson, *J. Math. Phys.* **2009**, *300*, A69.
[32] F. C. Cooperstock, V. de la Cruz, *Gen. Rel. Grav.* **1978**, *9*, 835.
[33] U. K. De, A. K. Raychaudhuri, *Proc. R. Soc. Ser. A* **1968**, *303*, 97.
[34] T. Gronwall, *Ann. of Math.* **1919**, *20*, 292.
[35] A. Banerjee, N. O. Santos, *J Math. Phys.* **1981**, *22*, 824.
[36] Z. Shi-Chang, *Gen Rel. Grav.* **1983**, *15*, 293.
[37] W. C. Saslaw, S. D. Maharaj, N. K. Dadhich, *Astrophys. J.* **1996**, *471*, 571.
[38] N. Dadhich, S. Hansraj, S. D. Maharaj, *Phys. Rev. D* **2016**, *93*, 044072.
[39] P. C. Vaidya, R. Tikekar, *J. Astrophys. Astron.* **1982**, *3*, 325.
[40] M. R. Finch, J. E. F. Skea, *Class. Quant. Grav.* **1989**, *6*, 467.