Non-commutative gauge theory on fuzzy $\mathbb{C}P^2$

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Abstract

Gauge theory on fuzzy $\mathbb{C}P^2$ can be defined as a multi-matrix model, which consistently combines a UV cutoff with the classical symmetries of $\mathbb{C}P^2$. The degrees of freedom are 8 hermitian matrices of finite size, 4 of which are tangential gauge fields and 4 are auxiliary variables. The model depends on a noncommutativity parameter $\frac{1}{N}$, and reduces to the usual $U(n)$ Yang-Mills action on the 4-dimensional classical $\mathbb{C}P^2$ in the limit $N \to \infty$. The quantization of the model is defined in terms of a path integral, which is manifestly finite.

1. Introduction

Fuzzy spaces are a nice class of noncommutative spaces with finite-dimensional algebras of "functions", and the same symmetries as their classical counterparts. This means that field theory on fuzzy spaces is regularized, but compatible with a geometrical symmetry group unlike lattice field theory. A large family of such spaces is given by the quantization of (co)adjoint orbits $\mathcal{O}$ of a Lie group in terms of certain finite matrix algebras $\mathcal{O}_N$. They are labeled by a a noncommutativity parameter $\frac{1}{N}$, and the classical space is recovered in the large N limit. The simplest example is the fuzzy sphere S^2_N, which has been studied in great detail; see e.g. [1, 2, 3, 4, 5, 6] and references therein.

The simplest 4-dimensional fuzzy spaces are $S^2_N \times S^2_N$ and $\mathbb{C}P^2_N$. While the former is technically easier to handle, $\mathbb{C}P^2_N$ (see e.g. [7,8,9,10]) has an 8-dimensional symmetry group $SU(3)$, which is larger than that of $S^2_N \times S^2_N$ or e.g. that of non-commutative tori. This leads to the hope that $\mathbb{C}P^2_N$ should be most suitable for analytical studies, once the appropriate tools are developed.

In the present notes we give a brief review of the definition of gauge theory on the 4-dimensional fuzzy space $\mathbb{C}P^2_N$ given in [11]. The definition is given in terms of certain multi-matrix models, generalizing the approach of [4, 6]. The basic requirement is that it should reduce to the usual $U(n)$

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Yang-Mills gauge theory on classical $\mathbb{C}P^2$ in the commutative limit, but it should also be simple and promise advantages over the commutative case. The model is similar to certain matrix model which arise in string theory, notably the IKKT matrix model [12] and effective actions for certain D-branes [13]. These models lead to a picture where the space (a "brane") arises dynamically as solution of a matrix action, and can be interpreted as submanifold of a higher-dimensional space. The gauge fields are then fluctuations of the tangential coordinates, while the transversal coordinates become scalar fields on the brane. All matrices are finite-dimensional for fuzzy spaces.

The formulation of gauge theory as multi-matrix model has at least 2 notable features, which are not present in the classical case: First, it leads to a very simple picture of nontrivial gauge sectors such as monopoles, which arise as nontrivial solutions in the matrix configuration space. This was noted in [14] and further explored in [6] for the fuzzy sphere. The concepts of fiber bundles are not required but arise automatically, in an intrinsically noncommutative way. Second, the matrix-model formulation allows a nonperturbative quantization in terms of a finite "path" integral, which in the case of $U(n)$ Yang-Mills on S^2_N can be carried out explicitly in the large N limit [6]. We want to see if these features can be extended to $\mathbb{C}P^2_N$. It turns out that one can indeed find monopole and (generalized) instanton solutions on $\mathbb{C}P^2_N$, generalizing the approach of [6].

The guiding principle in this construction of gauge theory is to find an action in terms of "covariant coordinates" C_a, which has a unique "vacuum" solution $C_a = \xi_a$ which defines the space, i.e. fuzzy $\mathbb{C}P^2$. Furthermore, the fluctuations $C_a = \xi_a + A_a$ should describe the gauge field with the correct classical limit. The nontrivial part is to make sure that the "transversal" fluctuations become very massive scalars and decouple. We can then test the model by looking for non-trivial solutions. Using a purely group-theoretical ansatz, we find indeed such solutions corresponding to both monopoles (which do exist on $\mathbb{C}P^2$) and certain instantons. The latter should be viewed as connections on quantized rank 2 bundles over $\mathbb{C}P^2$ with non-trivial first and second Chern class. The fact that all these monopole (and instanton) solutions arise in the same configuration space is a remarkable simplification over the commutative case, and provides further support for this approach.

The quantization of this gauge theory is straightforward in principle, in terms of a "path integral" which is convergent. As opposed to the 2-dimensional case [6], it can no longer be performed analytically.

2. Classical $\mathbb{C}P^2$

For our purpose, the most useful description of $\mathbb{C}P^2$ is as a (co)adjoint orbit in $su(3)$,

$$\mathbb{C}P^2 = \{gtg^{-1}; \quad g \in SU(3)\} \tag{1}$$

for a suitable $t \in su(3)$ which will be determined below. Such a conjugacy classes is invariant under the adjoint action of $SU(3)$. In fact, $\mathbb{C}P^2$ can be viewed as a homogeneous space:

$$\mathbb{C}P^2 \cong SU(3)/SU(2) \times U(1) \tag{2}$$

where $SU(2) \times U(1)$ is the stabilizer of

$$t = \tau_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

Here τ_a are the "conjugated" Gell-mann matrices[1] of $su(3)$, which satisfy

$$\begin{aligned} tr(\tau_a \tau_b) &= 2\delta_{ab}, \\ \tau_a \tau_b &= \frac{2}{3}\delta_{ab} + \frac{1}{2}(i f_{ab}{}^c + d_{ab}{}^c)\tau_c. \end{aligned} \tag{4}$$

One can use (2) to derive the decomposition of the space of functions on $\mathbb{C}P^2$ into harmonics i.e. irreps under the adjoint action of $SU(3)$ [7, 15],

$$\mathcal{C}^\infty(\mathbb{C}P^2) = \mathop{\oplus}_{p=0}^{\infty} V_{(p,p)}. \tag{5}$$

Here $V_{(n,m)}$ denotes the irrep of $su(3)$ with highest weight $n\Lambda_1 + m\Lambda_2$, and Λ_i are the fundamental weights of $su(3)$.

It is convenient to work with an over-complete set of 8 global coordinates defined by the embedding $\mathbb{C}P^2$ in the Lie algebra $su(3) \cong \mathbb{R}^8$. We can then write any element $X \in \mathbb{C}P^2$ as

$$\mathbb{C}P^2 = \{X = x_a \tau_a = g^{-1} t g;\ \ t = \tau_8,\ g \in SU(3)\}. \tag{6}$$

It is characterized by the characteristic (matrix) equation

$$XX = \frac{1}{\sqrt{3}}\, X + \frac{2}{3} \tag{7}$$

which is easy to check for $X = \tau_8$. In component notation, this implies

$$g^{ab} x_a x_b = 1, \tag{8}$$

$$d_c^{ab} x_a x_b = \frac{2}{\sqrt{3}}\, x_c. \tag{9}$$

One can show that (8) is a consequence of (9). Observe that the matrix

$$P = \frac{1}{\sqrt{3}}(X + \frac{1}{\sqrt{3}}) \tag{10}$$

satisfies

$$P^2 = P, \qquad Tr(P) = 1 \tag{11}$$

as a consequence of (7), hence P is a projector of rank 1. Such projectors are equivalent via $P = |z_i\rangle\langle z_i|$ to complex lines in $\mathbb{C}^3$, which leads to the more familiar definition of $\mathbb{C}P^2$ as $\mathbb{C}^3/\mathbb{C}^*$. An arbitrary radius R can be introduced by rescaling $x_a \to x_a R$.

[1] We follow the conventions in [11] using τ_a instead of the standard Gell-mann matrices

Some geometry. The symmetry group $SU(3)$ contains both "rotations" as well as "translations" in the intuitive sense. The generators L_a act on an element $X = x_a\tau_a \in \mathbb{C}P^2$ as

$$L_a X = [\tau_a, X] = i f_{abc}\, x_b \tau_c. \tag{12}$$

In terms of the coordinate functions on the embedding space $\mathbb{R}^8$, this can be realized as differential operator

$$L_a = \frac{i}{2} f_{abc}(x_b\partial_c - x_c\partial_b). \tag{13}$$

Now we can identify the rotations: consider the *"north pole"*

$$X_{np} = \tau_8 = x_a\tau_a \quad \in \mathbb{C}P^2 \qquad \text{with } \; x_a = \delta_{a,8}.$$

The rotation subgroup is its stabilizer subalgebra $\mathfrak{r} \cong su(2)\times u(1) \subset su(3)$ generated by the "rotation" generators

$$\mathfrak{r} = \{\tau_1, \tau_2, \tau_3, \tau_8\} \tag{14}$$

resp. the corresponding[2] $L_{\mathfrak{r}}$. It is clearly a subalgebra of the Euclidean rotation algebra $so(4) = su(2)_L \times su(2)_R$. The translations of X_{np} are generated by the "translation generators"

$$\mathfrak{t} = \{\tau_4, \tau_5, \tau_6, \tau_7\}. \tag{15}$$

$\mathbb{C}P^2$ is a symplectic (even Kähler) space. The symplectic form is given by[3]

$$\eta = \frac{1}{2\sqrt{3}R} f_{abc} x_a dx_b dx_c \tag{16}$$

which is clearly invariant under $SU(3)$. Here η is normalized such that $\langle \eta, \eta\rangle = 2$ where $\langle , \rangle$ is the obvious inner product for forms. The volume form is then given by $dV = \frac{1}{2}\eta^2$. In particular, note that η is *selfdual*: By $su(3)$ invariance it suffices to check this at the north pole $x_a = \delta_{a,8}$. There f_{ab8} is manifestly selfdual, and so is $\eta f(x)$ for any function $f(x)$. In fact any self-dual 2-from on $\mathbb{C}P^2$ can be written in this form.

3. Fuzzy $\mathbb{C}P^2_N$.

In general, any (co)adjoint orbit (1) on a compact semi-simple Lie group G can be quantized in terms of simple matrix algebras $End_{\mathbb{C}}(V_N)$, where V_N are suitable representations of G. The appropriate representations V_N can be identified by matching the spaces of harmonics, i.e. the decomposition

[2] we will sometimes denote the indices $1, 2, 3, 8$ with $\mathfrak{r}$, etc

[3] to see that it is closed, note that $d\eta \propto f_{abc} dx_a dx_b dx_c = 0$ on $\mathbb{C}P^2$

into irreps under the symmetry group G as in (5). For $\mathbb{C}P^2$, the correct harmonics are reproduced for[4] $V_N = V_{N\Lambda_2} = V_{(0,N)}$, which is the irrep of $su(3)$ with highest weight $N\Lambda_2$. Then the space of "functions" on fuzzy $\mathbb{C}P^2$ decomposes as

$$\mathbb{C}P^2_N := End(V_{(0,N)}) = V_{(0,N)} \otimes V^*_{(0,N)} = \oplus_{n=0}^N V_{(n,n)}. \tag{17}$$

under the (adjoint) action of $SU(3)$. This matches the decomposition (5) of functions on $\mathbb{C}P^2$ up to the cutoff. To identify the fuzzy coordinate functions, we denote with ξ_a the representation of an ON-basis of generators of $su(3)$ acting on $V_{N\Lambda_2}$, with dimension $D_N = (N+1)(N+2)/2$. It is easy to show that they satisfy the relations

$$i f_c^{ab} \xi_a \xi_b = -3\, \xi_c, \qquad [\xi_a, \xi_b] = \frac{i}{2} f_c^{ab} \xi^c \tag{18}$$

$$g^{ab} \xi_a \xi_b = \frac{1}{3} N^2 + N, \tag{19}$$

$$d_c^{ab} \xi_a \xi_b = (\frac{2N}{3} + 1)\, \xi_c. \tag{20}$$

One then defines the rescaled variables $x_i = (x_1, ... x_8)$ of $\mathbb{C}P^2_N$ as

$$x_a = \Lambda_N\, \xi_a, \qquad \Lambda_N = R \frac{1}{\sqrt{\frac{1}{3}N^2 + N}} \tag{21}$$

which satisfy [8]

$$i f_c^{ab} x_a x_b = -3\Lambda_N\, x_c = -3 \frac{R}{\sqrt{\frac{1}{3}N^2 + N}}\, x_c, \tag{22}$$

$$g^{ab} x_a x_b = R^2, \tag{23}$$

$$d_c^{ab} x_a x_b = R\, \frac{2N/3 + 1}{\sqrt{\frac{1}{3}N^2 + N}}\, x_c. \tag{24}$$

They reduce to (8), (9) in the large N limit. Here R is an arbitrary radius, which will usually be 1 here. Hence the algebra of functions on fuzzy $\mathbb{C}P^2_N$ is simply $Mat(D_N, \mathbb{C})$.

The decomposition (17) of functions into harmonics defines an embedding of the spaces

$$\mathbb{C}P^2_N \hookrightarrow \mathbb{C}P^2_{N+1} \hookrightarrow ... \hookrightarrow \mathcal{C}^\infty(\mathbb{C}P^2) \tag{25}$$

[4]alternatively one could use $V_{N\Lambda_1} = V_{(N,0)}$, which gives an equivalent algebra but a different embedding of $\mathbb{C}P^2 \subset \mathbb{R}^8$.

by matching the harmonics of $su(3)$. This allows to measure the "difference" between fuzzy and classical functions using the operator norm resp. supremum norm, and statements like $f \in \mathbb{C}P^2_N \to f \in \mathcal{C}^\infty(\mathbb{C}P^2)$ as $N \to \infty$ are understood in this sense throughout this paper.

Additional structure. We can easily identify a "north pole" in the fuzzy case[5]. Indeed ξ_8 and ξ_3 can be simultaneously diagonalized, and the highest weight state $|N\Lambda_2\rangle$ of $V_{(0,N)}$ has eigenvalues $\xi_8|N\Lambda_2\rangle = \frac{N}{\sqrt{3}}\,|N\Lambda_2\rangle$ and $\xi_3|N\Lambda_2\rangle = 0$. This is the unique vector in $V_{(0,N)}$ with this maximal eigenvalue of ξ_8. It is therefore natural to identify the delta-function on the north pole with the projector $|N\Lambda_2\rangle\langle N\Lambda_2|$, i.e. to consider $\langle N\Lambda_2|f(x)|N\Lambda_2\rangle$ as value of $f(x) \in \mathbb{C}P^2_N$ "at the north pole". For large N, the eigenvalue of x_8 approaches R as it should.

The "angular momentum" operators (generators of $SU(3)$) become now inner,

$$L_a f(x) = [\xi_a, f], \tag{26}$$

because then $L_a x_b = [\xi_a, x_b] = \frac{i}{2} f_{abc} x_c$, as classically. The integral on $\mathbb{C}P^2_N$ is defined by the suitably normalized trace,

$$\int f(x) = \frac{1}{D_N} Tr(f) \tag{27}$$

and is invariant under $SU(3)$.

4. Multi-Matrix Models for Yang-Mills on fuzzy $\mathbb{C}P^2$

4.1. Degrees of freedom and field strength

Our basic degrees of freedom are 8 hermitian matrices $C_a \in Mat(D_N, \mathbb{C})$ transforming in the adjoint of $su(3)$, which are naturally arranged as a single $3D_N \times 3D_N$ matrix

$$C = C_a \tau^a + C_0 1\!\!1 \tag{28}$$

where $C_0 = 0$ in much of the following. The size D_N of these matrices will be relaxed later. We want to find a multi-matrix model in terms these C_a, which for large N reduces to (euclidean) Yang-Mills gauge theory on $\mathbb{C}P^2$. The idea is to interpret the C_a as suitably rescaled "covariant coordinates" [16] on fuzzy $\mathbb{C}P^2_N$, with the gauge transformation

$$C_a \to U^{-1} C_a U \tag{29}$$

for unitary matrices U of the same size. The C_a can also be interpreted as components of a one-form if desired [11]. Following the approach of [6], we look for an action which has the "vacuum" solution

$$C_a = \xi_a \tag{30}$$

[5] for $V_{(N,0)}$ there would be a south pole

corresponding to $\mathbb{C}P^2_N$, and forces C_a to be at least approximately the corresponding representation $V_{N\Lambda_2}$ of $su(3)$. Then the fluctuations

$$C_a = \xi_a + A_a \tag{31}$$

are small, and describe the gauge fields. By inspection, these gauge fields A_a transform as

$$\delta A_a = i[\xi_a + A_a, \Lambda] = iL_a\Lambda + i[A_a, \Lambda] \tag{32}$$

for $U = e^{i\Lambda}$, which is the appropriate formula for a gauge transformation. Since the C_a resp. ξ_a correspond to "global" coordinates in the embedding space $\mathbb{R}^8$, we can hope that nontrivial solutions such as instantons can also be described in this way.

A suitable definition for the field strength is then given by

$$F_{ab} = i[C_a, C_b] + \frac{1}{2}\, f_{abc}C_c = i(L_aA_b - L_bA_a + [A_a, A_b]) + \frac{1}{2}\, f_{abc}A_c. \tag{33}$$

We will also need

$$\begin{aligned} F_a &= if_{abc}C_bC_c + 3C_a = \tfrac{1}{2}f_{abc}F_{ab}, \\ D_a &= d_c^{ab}C_aC_b - (\tfrac{2N}{3} + 1)\, C_c. \end{aligned} \tag{34}$$

Under gauge transformations, the field strength transforms as

$$F_{ab} \to U^{-1}F_{ab}U. \tag{35}$$

F can also be interpreted as 2-form

$$F = dA + AA \tag{36}$$

if one considers the fields C_a as one-forms $C = C_a\theta_a = \Theta + A$, using the differential calculus introduced in [11]. Furthermore, one can show that F_{ab} is (approximately) tangential if C_a satisfies (approximately) the constraints of $\mathbb{C}P^2$. Assuming that A_a tend to well-defined functions on $\mathbb{C}P^2$ in the large N limit, this implies that F_{ab} are the components of the usual field strength 2-form in the commutative (large N) limit. This justifies the above definition of F_{ab}, and it is a matter of taste whether one works with the components or with forms.

4.2. Constraints

In order to describe fuzzy $\mathbb{C}P^2$, the fields C_a should satisfy at least approximately the constraints (19), (20) of $\mathbb{C}P^2_N$,

$$D_a = 0, \tag{37}$$

$$g_{ab}C_aC_b = \frac{1}{3}N^2 + N \tag{38}$$

which are gauge invariant. These constraints ensure that C_a can be interpreted as describing a ("dynamical" or fluctuating) $\mathbb{C}P^2_N$. However, notice that they are not independent: (9) implies (8) at least in the commutative limit. These constraints are analyzed in considerable detail in [11] in the non-commutative case.

4.3. The Yang-Mills action

Assume that the C_a satisfy the constraints (37) (and therefore also (38)) of $\mathbb{C}P^2_N$ exactly or approximately. This implies that F_{ab} is tangential in the commutative limit, as shown in [11]. Then one can define the "Yang-Mills" action as

$$S_{YM} = \frac{1}{g} \int F_{ab} F_{ab} = \frac{1}{g D_N}\, Tr(-[C_a, C_b]^2 + 2i f_{abc} C_a C_b C_c + 3 C_a C_a). \quad (39)$$

It reduces to the classical Yang-Mills action on $\mathbb{C}P^2$, because only the tangential indices contribute in the commutative limit. The corresponding equation of motion is

$$2[C_b, F_{ab}] - i F_a = 0 \quad (40)$$

We now have to impose the constraints (37), (38) either exactly or approximately, and there are several possibilities how to proceed. Imposing both of them exactly seems too restrictive; recall that they are not independent even classically. One can hence either

1. consider all 8 fields C_a as dynamical and add something like
$$S_D = \frac{1}{g D_N}\, Tr\left(\mu_1 (dCC - (\frac{2N}{3} + 1) C)^2 + \mu_2 (C \cdot C - (\frac{N^2}{3} + N))^2\right) \quad (41)$$
to the action. This will impose the constraint dynamically for suitable $\mu_1 > 0$ and $\mu_2 \geq 0$, by giving the 4 transversal fields a large mass $m \to \infty$. Or,
2. impose the constraint $D = dCC - (\frac{2N}{3} + 1)C = 0$ exactly, or a slightly modified version.

In the second approach, it is not clear whether there are sufficiently many solutions of $D = 0$ in the noncommutative case to admit 4 tangential gauge fields. This concern could be circumvented by modifying the constraint, which is discussed in [11]. However we have not been able to find instanton-like solutions in this case (which may just be a technical problem). Therefore we concentrate on the first approach here, where we do find topologically nontrivial instanton solutions. It offers the additional possibility to give physical meaning to the non-tangential degrees of freedom.

Therefore our action is

$$\boxed{S = S_{YM} + S_D.} \quad (42)$$

It is shown in [11] that this reproduces the classical Yang-Mills action on $\mathbb{C}P^2$ in the large N limit provided

$$\mu_1 = o(\frac{1}{N}), \quad \text{and} \quad \mu_2 \leq o(\frac{1}{N^3}). \quad (43)$$

Here $o(\frac{1}{N})$ stands for a function which scales exactly like $\frac{1}{N}$. These constraints on the scaling of $\mu_{1,2}$ ensure that the vacuum which defines the geometry of $\mathbb{C}P^2_N$ is stable (note that the geometry is determined dynamically in noncommutative gauge theories!), and the monopole-and instanton solutions which will be discussed below survive. Imposing e.g. $\mu_1 = 0$ strictly would suppress the instanton solutions, hence in some sense fix the topology of the gauge fields. These issues certainly need further investigations; similarly, one may or may not fix the size of the matrices to be exactly D_N, which also has some influence on the existence of certain nontrivial solutions. Our choices are such that the conventional Yang-Mills theories emerge in the large N limit. These issues are discussed in more detail in [11].

We proceed to find the "vacuum", i.e. the minimum of the action. Assume first that the size of the matrices is D_N. Then the absolute minima of the action are given by solutions of $F_{ab} = 0$ and $D_a = 0$, which means that C_a is a representation of $su(3)$ with $D_a = 0$. The latter implies that the only allowed irreps are $V_{N\Lambda_2}$ or the trivial representation. Ignoring the latter (it has a smaller "phase space" of fluctuations), the vacuum solution is therefore

$$(C_{vac})_a = \xi_a \tag{44}$$

in a suitable basis. These arguments go through if we allow the size of the matrices C_a to be somewhat bigger that D_N, say

$$C_a \in Mat(D_N + N, \mathbb{C}) \tag{45}$$

(anything much smaller that $2D_N$ will do), which is needed to accomodate all the nontrivial solutions found below. Any configuration with finite action is therefore close to (44), and can hence be written as

$$C_a = \xi_a + A_a \tag{46}$$

in a suitable basis, with "small" A_a. This justifies the assumptions made in the beginning of Section 4.3.

It is quite straightforward to include scalar fields in this construction. Assume that we have an additional complex scalar field ϕ. Without gauge coupling, a natural action would be $\int ([\xi_a, \phi])^\dagger [\xi_a, \phi] = -\int \phi^\dagger \Delta\phi$. If we assume that ϕ is charged and transforms under gauge transformations as

$$\phi \to U\phi, \tag{47}$$

then a natural gauge-invariant action would be

$$S[\phi] = \int (C_a\phi - \phi\xi_a)^\dagger (C_a\phi - \phi\xi_a). \tag{48}$$

This reduces to $\int (D_a\phi)^\dagger D_a\phi$ where $D_a = [\xi_a, .] + A_a$. Fermions are much more difficult to handle since $\mathbb{C}P^2$ is a spin^c manifold but not spin, and a fully satisfactory treatment in the fuzzy case is still lacking; see [8] for a possible approach.

4.4. Decoupling of auxiliary variables

As discussed above, we impose the constraints of $\mathbb{C}P_N^2$ by adding the term (41) to the action. We will now show that this amounts to giving the 4 transversal fields a large mass $m \to \infty$ which therefore decouple, leaving 4 massless tangential gauge fields (in fact one can put $\mu_2 = 0$, since (38) is not an independent constraint). Note that

$$\begin{aligned} D_c &= d_{abc}\{\xi_a, A_b\} + d_{abc}A_aA_b - (\frac{2N}{3}+1)A_c \\ C_aC_a - (\frac{N^2}{3}+N) &= \xi_aA_a + A_a\xi_a + A_aA_a. \end{aligned} \tag{49}$$

using $C_a = \xi_a + A_a$. Assuming that A_a and $[A,A]$ are "smooth", this gives

$$\frac{D_c}{2N} = \frac{1}{2N}d_{abc}\{\xi_a, A_b\} - \frac{1}{3}\,A_c + o(1/N). \tag{50}$$

To understand the meaning of this, consider the "north pole", where $x_a \approx \delta_{a,8}$. Then $A_{\mathfrak{t}} = A_{4,5,6,7}$ are tangential, and $A_{1,2,3}$ and A_8 are "transversal" with

$$\begin{aligned} \frac{D_{\mathfrak{t}}}{2N} &= o(1/N), \\ \frac{D_{1,2,3}}{2N} &= -A_{1,2,3}\; + o(1/N), \\ \frac{D_8}{2N} &= \frac{1}{3}A_8\; + o(1/N), \\ \frac{1}{2N}(C_aC^a - (\frac{N^2}{3}+N)) &= \frac{1}{\sqrt{3}}A_8 + o(1/N). \end{aligned} \tag{51}$$

This shows explicitly how D_a separates the tangential from the transversal fields. Therefore the term $\mu_1 D_aD_a$ gives the transversal modes $A_{\mathfrak{r}}$ a mass term of order $\mu_1 N^2$, while the tangential modes are affected by terms of order at most μ_1. In particular for $\mu_1 = 1/N$ this is

$$\mu_1 D_aD_a = 4N(A_1^2 + A_2^2 + A_3^2 + \frac{1}{9}A_8^2) \tag{52}$$

(at the north pole), and similar for term with μ_2. This extends to any point on $\mathbb{C}P^2$ by $SU(3)$ covariance. Therefore the action (42) indeed approaches the classical Yang-Mills action for $N \to \infty$.

Additional terms in the action. Based on $su(3)$ invariance, one should also allow other terms such as

$$\int a_1\; C\cdot C + a_2\; fCCC + a_3\; dCCC \tag{53}$$

etc in the action. However, note that such $su(3)$ singlets may *not* be invariant under local $so(4)$ rotations in the commutative limit, i.e. they are not covariant in the usual sense. Nevertheless, they turn out to be harmless:

The terms $dCCC$ and $C \cdot C$ are clearly related to the constraints (37) and (38), hence they are essentially constants. They are in fact covariant in the usual sense, which can be seen using the explicit form of d_{abc}. The term $fCCC$ is less obvious at first sight, since it is not covariant in the usual sense. However, one can show [11] that it essentially reduces to the first Chern number (plus a constraint) in the classical limit, which is topological and does not affect the local physics as long as $a_2 = o(1/N)$. The breaking of covariance is due to the existence of a symplectic form on $\mathbb{C}P^2$, which can be interpreted as some kind of background field.

4.5. Nonabelian case: $U(n)$ Yang-Mills

The generalization to a $U(n)$ gauge theory is straightforward, by considering the same action (42) for larger matrices[6] $C_a \in Mat(n(D_N), \mathbb{C})$. The absolute minima of the action are given (for $\mu_2 = 0$) by solutions of $D_a = 0$ and $F_{ab} = 0$, which means that C_a is a representation of $su(3)$ with $D_a = 0$. The latter implies that the representation decomposes into a direct sum of either $V_{N\Lambda_2}$ or the trivial rep. Ignoring again the trivial representations, C takes the form

$$(C_{vac})_a = \xi_a \, \mathbf{1}_{n\times n}, \tag{54}$$

which is a block matrix consisting of n blocks of the fuzzy $\mathbb{C}P^2$ solutions. Any configuration can then be written as

$$C_a = \xi_a \; + A_{a,\alpha} \, \lambda_\alpha. \tag{55}$$

where $A_{a,\alpha}$ carries an additional $u(n)$ index, λ_α denote the Gell-Mann matrices of $u(n)$, and $\lambda_0 = \mathbf{1}$ is the $n\times n$ unit matrix. The rest of the analysis of the previous sections goes essentially through, in particular the transversal components of $A_{a,\alpha}$ will be very massive and decouple due to S_D. Since all non-tangential components are suppressed, we recover the usual Yang-Mills theory on $\mathbb{C}P^2$ with gauge group $U(n)$ in the commutative limit.

4.6. Quantization

The quantization of these models is straightforward in principle, by a "path integral" over the hermitian matrices

$$Z[J] = \int dC_a e^{-(S_{YM}+S_D+TrC_aJ_a)} \tag{56}$$

Note that there is no need to fix the gauge unless one wants to do perturbation theory, since the gauge orbit is compact. The gauge-fixing terms required for perturbation theory will not be discussed here. One can show

[6] see [11] for some subtleties: one should replace D_N by e.g. $D_N + N$

that the above path integral is well-defined and finite for any fixed N provided $\mu_1 > 0$ and $\mu_2 \geq 0$. The nontrivial question of course remains whether the model is renormalizable, i.e. whether there exists a suitable scaling of the coefficients in the action such that the limit $N \to \infty$ defines a meaningful quantum field theory on fuzzy $\mathbb{C}P^2$.

5. Topologically nontrivial solutions

In [11], some explicit solutions of the equations of motion with finite action are found, which in the classical (large N) limit become topologically nontrivial solutions such as monopoles and instantons. The idea is to identify the solutions of the gauge theory with certain irreps of the symmetry group $SU(3)$. Recalling that the "vacuum" solution $C_a = \xi_a$ of our action $S_{YM} + S_D$ is obtained as irrep $V_{(0,N)}$, it is natural to consider other representations with highest weight Λ close to $(0, N)$. It seems plausible that they should give rise to nontrivial saddle-points of the action. This idea essentially works, with some modifications which are necessary in the nonabelian case. In particular this allows to find explicitly the monopole or $U(1)$ instanton solutions (which exist on $\mathbb{C}P^2$ since the second cohomology is nontrivial), and also a particular $U(2)$ instanton solution with nontrivial first and second Chern number. The corresponding gauge fields are computed explicitly in [11] using the Gelfand-Tsetlin basis. The value of the action is

$$S_{YM} = \frac{1}{g} \int F_{ab} F_{ab} = \frac{1}{g} \int 12 m^2 \tag{57}$$

for the monopole solutions in the large N limit, and

$$S_{YM} = \frac{1}{g} \int F_{ab} F_{ab} = \frac{1}{g} \int tr(1 - m + m^2). \tag{58}$$

for the $SU(2)$ instanton. Here m is an arbitrary integer which coincides with the first Chern number for the $U(1)$ bundle, and is related to the total Chern character for the $U(2)$ solution by

$$ch = 2 + c_1 + \frac{1}{2}(c_1 \wedge c_1 - 2c_2) = 2 + (2m - 1)\omega + (m^2 - m - \frac{1}{2})\omega^2, \tag{59}$$

where ω is the generator of $H^2(\mathbb{C}P^2, \mathbb{Z})$. The instanton hence contains also a monopole part, and is in particular not self-dual.

6. Discussion and outlook

The main merits of this type of gauge theories on fuzzy spaces are 1) they are completely "regularized" so that that the quantization is well-defined and finite, and 2) topologically nontrivial configurations arise simply as solutions of the matrix equations of motion. In particular, we do not have to sum over disconnected topological sectors; they are included in the "path" integral over all matrices. One may hope that the formulation as matrix model will provide useful new insights to 4-dimensional gauge theory.

There are many open issues which deserve further investigations. One is the inclusion of fermions, which is nontrivial due to the fact that $\mathbb{C}P^2$ has no spin but spin^c structure. There are several papers where this is investigated, but the appropriate coupling to a gauge field in our formulation is not clear. Another open problem is to find "localized" instantons and their moduli space. This is complicated by the apparent lack of a Hodge-star (with correct classical limit) on $\mathbb{C}P^2_N$. In particular, our instantons contain a nontrivial $U(1)$ sector, and are neither selfdual nor anti-selfdual. The $U(1)$ monopole part seems to be related to the spin^c structure on $\mathbb{C}P^2$, and may be important for the coupling to fermions. Finally, it would be very desirable to get some insight into the large N behavior of the quantized model.

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