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## Article

# Positive-Energy Dirac Particles and Dark Matter

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**Abstract:** The relativistic positive-energy wave equation proposed by P. Dirac in 1971 is an old but largely forgotten subject. The purpose of this note is to speculate that particles described by this equation (called here Dirac particles) are natural candidates for the dark matter. The reasoning is based on a fact that the internal structure of such particles simply prohibits their interaction with electromagnetic fields (at least with the minimal coupling) which is exactly what is required for dark matter. Dirac particles have quite unusual properties. In particular, they are transformed by an infinite-dimensional representation of the homogeneous Lorentz group, which clearly distinguishes them from all known elementary particles described by finite-dimensional representations and hints to a physics beyond the Standard Model. To clarify the topic, a brief review of the main features of the above-mentioned Dirac equation is given.

**Keywords:** dark matter; infinite-dimensional representation; new Dirac equation

## 1. Introduction

A tremendous number of various astrophysical observations collected during the last century are explained by postulating the existence of invisible (i.e., not interacting with electromagnetic fields) matter. Such a hypothetical substance goes under the name of dark matter, and according to different estimates, it constitutes  $\sim 85\%$  of the total matter content of the universe (see, e.g., reviews [1–3] and references therein). Though dark matter is a predominant ingredient of the universe, its composition remains unknown. A great number of different candidates have been proposed [1–3], but none have of course been confirmed so far.

The purpose of this short note is to suggest a (seemingly unexplored) possibility that dark matter is made up of particles obeying the so-called new Dirac equation proposed by P. Dirac [4] 40 years after his famous (old) equation [5]. Both equations are relativistic wave equations. The usual (old) Dirac equation describes particles with positive energy as well as antiparticles with negative energy and is one of the best known equations in physics. The principal feature of the new equation is that all its solutions have only positive energies (thus, antiparticles are absent or, at least, not described by the same equation). For relativistic equations, such property is unusual. The relativistic invariance implies that  $E^2 = \vec{p}^2 + m^2$ , and in general, there exist solutions with positive as well as negative energies:  $E = \pm \sqrt{\vec{p}^2 + m^2}$ . Though today negative energy solutions simply indicate the existence of antiparticles, the search for relativistic equations with only positive energies continues to persist (see, e.g., [6] and references therein).

It was E. Majorana [7,8] who first constructed a relativistic wave equation with only positive energy solutions. It appears that it becomes feasible provided one considers particles transformed by an infinite-dimensional representation of the homogeneous (i.e., without translations) Lorentz group. Usual relativistic particles (scalars, spinors, vectors, etc.) realise finite-dimensional representations of the Lorentz group, and their equations necessarily have positive and negative energy solutions. The positive-energy Majorana equation formally has form of the usual Dirac equation:

$$(\Gamma^\mu \partial_\mu - m)\Psi(x) = 0 \quad (1)$$



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but with the difference that its eigenfunctions  $\Psi(x)$  have an infinite number of components (labeled by two non-negative integers) and matrix  $\Gamma^\mu$  is an infinite-dimensional matrix derived from the requirement of the relativistic invariance (whose explicit form could be obtained from expressions in Section 4). This equation explicitly describes an infinite number of particles with all integer and half-integer spins and masses decreasing with spin:  $m(j) = m/(j + 1/2)$  [7,8]. Equation (1) should not be confused with another much better-known Majorana equation for self-conjugate particles with spin- $\frac{1}{2}$  [9].

The distinction between finite-dimensional and infinite-dimensional representations of a group is that for the former, the action of group generators mixes a finite number of wave components (e.g., four for the usual spinors) but for the latter, such action requires an infinite number of components. Mathematically, it is well established that the homogenous Lorentz group has both finite and infinite-dimensional representations (see, e.g., [10,11] and references therein). All known elementary particles are classified according to finite-dimensional representations of this group, yet particles associated with its infinite-dimensional representations are not usually taken into account.

In the new equation, Dirac found an elegant way (see below) to select one state from an infinite tower of Majorana solutions. Particles described by the new Dirac equation are also transformed by an infinite-dimensional representation of the Lorentz group, which markedly differentiates them from all known elementary particles. Such particles can be called positive-energy Dirac particles, but for the sake of brevity, in this note, they are referred to as Dirac particles.

Another characteristic feature of these Dirac particles is a nontrivial fact that they cannot interact with electromagnetic fields (at least with the minimal coupling). Such a no-interaction theorem is related with the internal structure of the new Dirac equation and is not imposed by simply postulating that the charge of Dirac particles equals zero. This propriety emerges as a byproduct of Dirac's construction, and in the past has been considered as a serious defect of the new Dirac equation, but it is exactly what makes Dirac particles a natural candidate for dark matter, which by definition is a substance not interacting with electromagnetic fields.

In the literature, there are articles that relate Dirac particles and dark matter (see, e.g., [12], besides others), but in these papers, the term 'Dirac particles' indicates solely spin- $\frac{1}{2}$  particles obeying the usual Dirac equation [5]. Dirac particles discussed here are of a different nature as they satisfy the new Dirac equation [4] and are transformed by an infinite-dimensional representation of the Lorentz group. To the author's knowledge, such particles were not considered previously as candidates for dark matter.

The purpose of the paper is to attract wider attention to this potentially important subject and to serve as a brief introduction to Dirac particle properties. As such issues are largely undeveloped, the discussion below is restricted only to principal features of Dirac particles leaving aside many important questions.

The plan of the paper is the following. In Section 2, the main properties of the new Dirac equation are discussed. Section 3 is devoted to the investigation of the relativistic invariance of this equation and to the meaning of the infinite-dimensional representation of its solutions. By construction, the new Dirac equation is an overdetermined system of equations, and consequently, one can derive additional equations that are indicated in Section 4. Section 5 deals with the no-electromagnetic-interaction property of the new Dirac equation, which is of primary importance for the conjectured relation of Dirac particles with dark matter. In Section 6, the interaction of Dirac particles with gravitational fields is shortly treated. Section 7 gives an overview of the discussed topics. For clarity, certain useful formulas are presented in Appendix A, and for completeness, the tetrad formalism is recalled in Appendix B.

## 2. New Dirac Equation

The usual Dirac equation [5] describes relativistic fermions with spin  $\frac{1}{2}$  and mass  $m$  and is one of the most important and the best-known equations of mathematical physics.

Much less attention has been attracted to another Dirac work [4], written about 40 years after the famous paper [5], in which a different relativistic equation (called the new Dirac equation) was proposed and analysed.

The new equation has the functional form similar to the usual Dirac equation:

$$(\gamma^\mu \partial_\mu - m)\Psi = 0 \quad (2)$$

but with an important difference in that column-vector  $\Psi$  depends on two sets of variables: four space-time coordinates  $x^\mu$  with  $\mu = 0, 1, 2, 3$  and two 'internal' (auxiliary) variables  $q_1$  and  $q_2$ , arranged in the following manner:

$$\Psi(x, q) = Q \Phi(x, q), \quad Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \quad (3)$$

where components  $q_3$  and  $q_4$  are the momenta in the 'internal' space

$$q_3 = -i \frac{\partial}{\partial q_1}, \quad q_4 = -i \frac{\partial}{\partial q_2}. \quad (4)$$

The choice of such column-vector  $Q$  corresponds to the following commutation relations:

$$[q_j, q_k] = i\beta_{jk}, \quad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (5)$$

As for the usual Dirac equation,  $\gamma^\mu$  are  $4 \times 4$  matrices obeying  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$  with the metric tensor of the Minkowski space  $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  (cf. (A1)).

Function  $\Phi(x, q)$  in (3) depends on both sets of coordinates (i.e. on six variables), and it is the only one unknown function. Thus, the new Dirac equation corresponds to an overdetermined system of four equations in one unknown function:

$$\hat{P}_j \Phi(x, q) = 0, \quad \hat{P}_j = (\gamma^\mu \partial_\mu - m)_{jk} q_k. \quad (6)$$

To ensure the consistency of these equations, it is necessary that the commutator of any two  $\hat{P}_j$  equals zero. Using (5) one finds

$$[\hat{P}_j, \hat{P}_k] = ((\gamma^\mu \partial_\mu - m)i\beta(\gamma^{\mu T} \partial_\mu - m))_{jk} \quad (7)$$

where  $\gamma^{\mu T}$  are the transposed gamma-matrices. In the new Dirac equation, these matrices have to be chosen to fulfil the requirement

$$\beta \gamma^{\mu T} = -\gamma^\mu \beta. \quad (8)$$

By construction, matrix  $\beta$  is an antisymmetric matrix (cf. (5)), and this requirement means that  $\beta \gamma^\mu$  are symmetric matrices.

Under these conditions, the above commutator becomes

$$[\hat{P}_j, \hat{P}_k] = i\beta_{jk}(-\eta^{\mu\nu} \partial_\mu \partial_\nu + m^2) \quad (9)$$

which implies that all four equations (6) are consistent iff  $\Phi(x, q)$  as a function of  $x$  obeys the Klein–Gordon equation:

$$(-\eta^{\mu\nu} \partial_\mu \partial_\nu + m^2)\Phi(x, q) = 0. \quad (10)$$

It is this additional relation that differs the new Dirac equation from the Majorana Equation (1) [7] and permits us to select one state from an infinite number of Majorana solutions.

It is plain that (10) is a necessary condition for the existence of a solution of the new Dirac equation. It could be obtained by, e.g., the multiplication of (2) with (3) by  $Q^T \beta (\gamma^\nu \partial_\nu + m)$  and using the fact that  $Q^T \beta Q = 2i$  (see (A4)). The sufficiency of this equation is more subtle. Usually, one refers to the Frobenius theorem (see, e.g., [13] and references therein), which (in the simplest setting) states that in order for a system of linear differential equations to have a solution, it is necessary and sufficient that their commutators (more general, Lie brackets) lie in their span. The important ingredient of this theorem is the fact that commutators cancel second derivatives, which is not valid for the considered problem. Therefore, the statement that (10) is the sufficient condition for the existence of solutions of the new Dirac equation requires additional investigations.

It has been proved in [4] that these equations determine massive relativistic particles (the Dirac particles) with only positive energy and zero spin in the rest frame. In the next publication [14], Dirac showed that when the noncommutativity of different variables is ignored, Dirac particles could be considered as vibrating spherical shells similar to *Zitterbewegung* of particles obeying the usual Dirac equation.

The simplest solution of the new Dirac equation is the plane wave of the following form [4]:

$$\Phi(x, q) = C \exp \left( -\frac{1}{2} a q_1^2 - \frac{1}{2} b q_2^2 + c q_1 q_2 + i p_\mu x^\mu \right) \quad (11)$$

where  $p_0 = \sqrt{\mathbf{p}^2 + m^2} > 0$ ,  $C$  is a normalisation constant, and coefficients  $a, b, c$  are

$$a = \frac{m + i p_1}{p_0 + p_3}, \quad b = \frac{m - i p_1}{p_0 + p_3}, \quad c = \frac{i p_2}{p_0 + p_3}. \quad (12)$$

The positivity of Dirac particles' energy is attested by the fact that negative energy solutions with  $p_0 = -\sqrt{m^2 + \mathbf{p}^2}$  lead to non-normalised functions and have to be excluded.

### 3. Infinite-Dimensional Representation

The relativistic (co)variance of the new Dirac equation means that a linear Lorentz transformation  $x'^\mu = \Lambda^\mu_\nu x^\nu$  ( $\eta^{\mu\nu} = \eta^{\sigma\rho} \Lambda^\mu_\sigma \Lambda^\nu_\rho$ ) of a solution

$$\Phi(\Lambda x, q) = U(\Lambda) \Phi(x, q) \quad (13)$$

is also a solution of the same equations. Here,  $U(\Lambda)$  is an operator acting on  $q$  variables.

The inspection of Equation (3) shows that it will be the case if the following two conditions are fulfilled:

$$U^{-1} Q U = V Q, \quad V \gamma V^{-1} = \gamma \Lambda \quad (14)$$

where matrix  $V \equiv V_{jk}(\Lambda)$  acts on spinor indices. Indeed, under such changes, Equation (3) transforms linearly; thus,

$$\hat{P} \Phi(\Lambda x, q) = U V \hat{P} \Phi(x, q) = 0. \quad (15)$$

The explicit expressions of  $V$  and  $U$  can easily be obtained from the infinitesimal Lorentz transformations

$$\Lambda^\mu_\nu = \delta^\mu_\nu + a^\mu_\nu = \delta^\mu_\nu + \frac{1}{2} a_{cd} (I^{cd})^\mu_\nu \quad (16)$$

where  $4 \times 4$  antisymmetric matrix  $a_{cd}$  describes six parameters of the Lorentz transformation and matrices  $I^{cd}$  are generators of Lorentz transformation

$$(I^{cd})^\mu_\nu = \eta^{c\mu} \delta^d_\nu - \eta^{d\mu} \delta^c_\nu. \quad (17)$$

These generators obey the commutator relations of the Lie algebra of the Lorentz group

$$[I^{cd}, I^{kr}] = \eta^{dk} I^{cr} + \eta^{cr} I^{dk} - \eta^{dr} I^{ck} - \eta^{ck} I^{dr}. \quad (18)$$

It is plain that  $V$  and  $U$  should be of the following form:

$$V = \exp\left(\frac{1}{2}a_{cd}\Sigma^{cd}\right), \quad U = \exp\left(\frac{i}{2}a_{cd}S^{cd}\right). \quad (19)$$

In the first order of  $a$ , unknown  $\Sigma$  and  $S$  are determined from the following commutators:

$$[\Sigma, \gamma] = \gamma I, \quad i[Q, S] = \Sigma Q \quad (20)$$

whose solutions are

$$\Sigma^{cd} = \frac{1}{4}(\gamma^c \gamma^d - \gamma^d \gamma^c), \quad S^{cd} = \frac{1}{2}(\bar{Q} \Sigma^{cd} Q) \quad (21)$$

where  $\bar{Q} = Q^T \beta$ . Because  $\beta \Sigma^T = -\Sigma \beta$  one has

$$\beta V^T = V^{-1} \beta \quad (22)$$

which implies that  $Q' = VQ$  also obeys the commutation relations (5) and  $U^{-1} \bar{Q} U = \bar{Q} V^{-1}$ .

It is well known that  $\Sigma^{cd}$  obey Equation (18), thus realising the four-dimensional (spinor) representation of the Lie algebra of the Lorentz group. As matrices  $\beta \Sigma^{cd}$  are symmetric (cf. (5)), it follows from (A5) that spin operators  $iS^{cd}$  from the above equation obey the same commutator relations (18), i.e., they also realise a representation of the Lie algebra of the Lorentz group.

The explicit form of these operators is presented in (A8). These operators are a distinct feature of the new Dirac equation. When acting on an  $L_2$  function  $\Phi(x, q)$  depended on auxiliary variables  $q_1$  and  $q_2$  (i.e.,  $\int_{-\infty}^{\infty} |\Phi(x, q)|^2 dq < \infty$  for arbitrary  $x$ ), they lead to an infinite-dimensional representation of the homogeneous Lorentz group. A simple way to see it explicitly is to write the function as an infinite series in suitable functions of  $q_1$  and  $q_2$ . A convenient choice is eigenfunctions of the harmonic oscillators  $\hat{H}_1 = \frac{1}{2}(q_3^2 + q_1^2)$  and  $\hat{H}_2 = \frac{1}{2}(q_4^2 + q_2^2)$  (the Hermite functions):  $\hat{H}_i \phi_m(q_i) = (m + \frac{1}{2})\phi_m(q_i)$  (cf. (A19))

$$\Phi(x, q) = \sum_{n,m=0}^{\infty} A_{mn}(x) \phi_m(q_1) \phi_n(q_2). \quad (23)$$

(Due to a symmetry of the new Dirac equation, integers  $m$  and  $n$  are of the same parity.)

Coefficients  $A_{mn}(x)$  form an infinite number of wave function components of Dirac particles. The action of operators  $S^{cd}$  on functions  $\phi_m(q_1)$  and  $\phi_n(q_2)$  induces a transformation of  $A_{mn}$  corresponded to an infinite (and unitary) representation of the homogeneous Lorentz group (see, e.g., [7,8,10,11,15]). Its explicit form can be obtained from Equation (A21). In a sense, an infinite ladder of coefficients  $A_{mn}$  is hidden in the dependence of an eigenfunction on variables  $q_1$  and  $q_2$ , and function  $\Phi(x, q)$  is a concise generating function of these coefficients.

Spin matrices  $\Sigma^{cd}$  equal the commutator of two  $\gamma$ -matrices:  $\Sigma^{cd} = \frac{1}{4}[\gamma^c, \gamma^d]$ . Similarly, spin operators  $S^{cd}$  also can be written as the commutator of two operators (cf. (A5), (A6)):

$$S^{cd} = i[\Gamma^c, \Gamma^d], \quad \Gamma^c = -\frac{1}{4}(\bar{Q} \gamma^c Q). \quad (24)$$

Operators  $S^{cd}$  together with  $\Gamma^c$  form a representation of the group  $SO(3, 2)$  [16,17] (see also Appendix A).

From (13), it follows that

$$\Phi(x, q) = U(\Lambda) \Phi(\Lambda^{-1}x, q). \quad (25)$$

This invariance leads to the conservation of the total momenta [4]  $J^{cd} = M^{cd} - iS^{cd}$ .  $M^{cd} = x^c \partial^d - x^d \partial^c$  are the usual momenta and  $S^{cd}$  represent the spin momenta.

#### 4. Additional Equations

The new Dirac equation (6) is an overdetermined system of equations and any linear combination of these equations is also a valid equation; thus,  $M\hat{P}_j\Phi = 0$  with any  $4 \times 4$  matrix  $M$ . Multiplying this equality by  $\bar{Q}$  and summing over spin indices, one obtains 16 equations quadratic in  $Q$ :  $\bar{Q}M(\gamma^\mu \partial_\mu - m)Q\Phi = 0$ . When  $M = \gamma^5$ , one obtains identical zero and the remaining 15 equations (corresponded to  $M = 1, \gamma^\mu, \Sigma^{\mu\nu}, \gamma^5\gamma^\mu$ , respectively) take the following form [4,18]:

$$(\Gamma^\mu \partial_\mu + \frac{i}{2}m)\Phi = 0, \quad (26)$$

$$(\frac{i}{2}\partial^\mu + S^{\mu\nu}\partial_\nu + m\Gamma^\mu)\Phi = 0, \quad (27)$$

$$(\Gamma^\mu \partial^\nu - \Gamma^\nu \partial^\mu + mS^{\mu\nu})\Phi = 0, \quad (28)$$

$$(e_{\mu\nu\alpha\beta}S^{\nu\alpha}\partial^\beta)\Phi = 0. \quad (29)$$

The first of these equations is (up to a notation) the positive-energy Majorana Equation (1) [7], and the last one coincides (up to a factor) with the Pauli–Lubanski (pseudo)vector, indicating that field  $\Phi$  has zero spin.

The Majorana Equation (26) is a characteristic feature of positive-energy equations. All solutions of such equations, in a way or another, obey this equation. Though the form of the Majorana equation is similar to the usual Dirac Equation (2), they lead to different conclusions. The main point is that, contrary to the usual  $\gamma^0$  matrix, which has positive and negative eigenvalues (see (A1)), operator  $\Gamma^0$  has the form (cf. (A8))

$$\Gamma^0 = \frac{1}{4} \left( (-\partial_{q_1}^2 + q_1^2) + (-\partial_{q_2}^2 + q_2^2) \right). \quad (30)$$

When acting on  $L_2$  functions of  $q_1, q_2$  invariant over inversion, as in (23), it has only positive eigenvalues equal to  $\frac{1}{4}(2j+1)$  with integer  $j \geq 0$ . It is this property that implies that particles described by the Majorana equation have masses decreasing with the spin:  $m(j) = m/(j+1/2)$ .

In Dirac's approach, the above equations are just consequences of Equation (6). It was argued in [18] that Equation (27) may be chosen as the fundamental one, leading to the same plane wave solution as in (11). A more general equation has been considered in that reference:

$$(i\kappa\partial^\mu + S^{\mu\nu}\partial_\nu + m\Gamma^\mu)\Phi = 0 \quad (31)$$

with a parameter  $\kappa$ . It has been shown that this system of equations is consistent only for two values  $\kappa = \frac{1}{2}$  and  $\kappa = 1$ . The first one corresponds to the Dirac Equation (27), and the second one gives a new equation describing particles transforming by an infinite-dimensional representation of the Lorentz group but with spin  $\frac{1}{2}$  in the rest frame. Contrary to the Dirac particles described in this note, such spin- $\frac{1}{2}$  particles permit the interaction with electromagnetic fields.

#### 5. No-Interaction Theorem

The principal goal of the new Dirac equation was the construction of a relativistic equation with only positive energy solutions. But for the relation with dark matter advo-

cated here, the most important is another unusual fact: that Dirac particles cannot interact with electromagnetic fields.

Usually, the interaction of a particle with an electromagnetic field  $A_\mu$  is introduced within the so-called minimal coupling by the substitution  $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu(x)$  into the wave equation. The remarkable propriety of the new Dirac equation is that this standard procedure leads to the inconsistency of resulting equations, which means that that Dirac particles cannot interact with electromagnetic fields (at least with minimal coupling). This fact had been briefly mentioned in the very end of the Dirac paper [4] and has been discussed in detail in [19] (see also [20]). For completeness, that reasoning is presented below.

Let us consider the new Dirac equation with minimal coupling:

$$(\gamma^\mu D_\mu - m)Q\Phi(x, q) = 0, \quad D_\mu = \partial_\mu + ieA_\mu(x). \quad (32)$$

Applying the operator  $(\gamma^\mu D_\mu + m)$  to this equation, one finds

$$\left( (\eta^{\mu\nu} D_\mu D_\nu - m^2) + ie\Sigma^{\mu\nu} F_{\mu\nu} \right) Q\Phi(x, q) = 0 \quad (33)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  and  $\Sigma^{\mu\nu} = \frac{1}{4}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ .

To remove the first term in (33), (which corresponds to the Klein–Gordon equation with the minimal coupling), let us multiply (33) by an antisymmetric matrix  $G$  such that  $\text{Tr}(G\beta) = 0$  and convolute the result with  $Q^T$ . According to (A4), it will cancel the Klein–Gordon term. Besides 16 independent products of the above  $\gamma$  matrices, there are five matrices with such proprieties:  $\beta\gamma^5\gamma^\mu$  and  $\beta\gamma^5$ . Let  $G_j$  be one of these five matrices. After the indicated transformations, Equation (33) takes the form

$$F_{\mu\nu} h_j^{\mu\nu} \Phi = 0, \quad h_j^{\mu\nu} = Q^T G_j \Sigma^{\mu\nu} Q \quad (34)$$

where  $h_j^{\mu\nu}$  are certain operators constructed from ten possible symmetric bilinear combinations of  $q_j$ . Using properties of  $\gamma$ -matrices (cf. (A9) and (A10)), one obtains

$$(\bar{Q}\gamma^5\gamma^\lambda\Sigma^{\mu\nu}Q) = \frac{1}{2}e^{\lambda\mu\nu\rho}\Gamma_\rho, \quad (\bar{Q}\gamma^5\Sigma^{\mu\nu}Q) = \frac{1}{8}e^{\mu\nu\lambda\rho}S_{\lambda\rho} \quad (35)$$

where  $S^{\mu\nu}$  are antisymmetric spin operators introduced in (21), and  $\Gamma^\mu$  is defined by (24).

Therefore, the above equations take the form

$$\tilde{F}_{\mu\nu}\Gamma^\nu\Phi = 0, \quad \tilde{F}_{\mu\nu}S^{\mu\nu}\Phi = 0. \quad (36)$$

Here,  $\tilde{F}_{\mu\nu} = \frac{1}{2}e_{\mu\nu}^{\rho\delta}F_{\rho\delta}$  is the dual electromagnetic tensor.

In [19], it was argued that from (36), it follows that if  $F_{\mu\nu} \neq 0$ , all operators  $\Gamma^\mu$  and  $S^{\mu\nu} = i[\Gamma^\mu, \Gamma^\nu]$  annihilate  $\Phi(x, q)$ , which means that  $\Phi(x, q)$  is independent on  $q$ , and consequently, the new Dirac equation with the minimal coupling of electromagnetic fields has no nontrivial solution. In other words, Dirac particles cannot interact with nonzero electromagnetic fields.

Practically all works on this subject have stressed this ‘drawback’, and many papers have been devoted to generalisations of the new Dirac equation, which would permit the electromagnetic interaction. It appears that it can be carried out only by considerable complications of the equation, e.g., by considering higher-order momentum terms [18,19], or by increasing the number of ‘intrinsic’ variables [21] or by introducing parabosonic constituents [17,22].

The no-electromagnetic-interaction theorem has been considered in the past as a serious disadvantage of the new Dirac equation, and to a great extent, it was responsible for the loss of interest to this subject. But it is exactly what is required for dark matter, which, by definition, is a substance whose characteristic property is that it does not interact with

electromagnetic fields. Dirac particles cannot be connected to electromagnetic fields not by simply postulating that they have zero charge but because their relativistic equation is inconsistent in the presence of the fields. It is this property that makes Dirac particles an ideal and natural candidate for the illusive dark matter. In a sense, merely the existence of particles transforming by an infinite-dimensional representation of the Lorentz group implies the existence of dark matter.

## 6. Gravity and Dirac Particles

The interaction of elementary particles with gravitational fields has attracted a certain attention from the very beginning of wave mechanics [23,24]. Due to the smallness of the gravitational interaction, only very special experiments are possible at the present time (see, e.g., [25]), and such a subject is mostly of a pure theoretical interest. But for Dirac particles, this problem is more fundamental, as they seem to interact directly only with the gravity (as some other dark matter candidates).

A usual way to describe spinors in gravitational fields (i.e., in a curved space) is the tetrad formalism (see, e.g., [23,26]). In a nutshell, it consists to introduce tetrads (vierbeins)  $t_\mu^a(x)$  instead of (symmetric) metric tensor  $g_{\mu\nu}(x)$ , such that

$$g_{\mu\nu}(x) = t_\mu^a(x)t_\nu^b(x)\eta_{ab}, \quad \eta = \text{diag}(-1, 1, 1, 1). \quad (37)$$

For completeness, the main formulas of the tetrad formalism are presented in Appendix B. Using the indicated rules, any given relativistic equation in a flat space (without gravity) can be transformed in a covariant manner to a curved space (with gravitational fields).

However, the new Dirac equation is not just one equation but an overdetermined system of equations, and it is not clear which of the many (almost) equivalent equations in a flat space (cf., (2), (3) and (26)–(29)) have to be converted to a curved space. Different equations may and will lead to different answers. For example, it is claimed in [27] that the use of Equation (31) with  $\kappa = 1$  (i.e., for spin- $\frac{1}{2}$  particles) leads to a consistent theory in gravitational fields.

A natural way of introducing gravitational fields into the new Dirac Equations (2) and (3) consists of transforming these equations themselves to a curved space. Using formulas from Appendix B, it is easy to check that the transformed equations can be written in two equivalent forms:

$$(\hat{\gamma}^\mu Q D_\mu - m Q)\Phi = 0, \quad (\hat{\gamma}^\mu D'_\mu - m)Q\Phi = 0 \quad (38)$$

where

$$D_\mu = \partial_\mu + \frac{i}{2}\omega_{\mu ab}S^{ab}, \quad D'_\mu = \partial_\mu + \frac{1}{2}\omega_{\mu ab}(\Sigma^{ab} + iS^{ab}). \quad (39)$$

As  $\Sigma Q = i[Q, S]$  (cf. (20)), these two expressions are consistent. (For clarity, covariant derivatives acted on different quantities are denoted here by different symbols.)

As above, one can derive many consequences of that equation. In particular, by acting on this equation by  $(\hat{\gamma}^\nu D'_\nu + m)$ , one concludes that

$$\left( Q(g^{\mu\nu} D_\mu D_\nu - m^2) + \Sigma^{\nu\mu} Q[D_\nu, D_\mu] \right) \Phi = 0. \quad (40)$$

Using (A38), it can be rewritten as follows:

$$\left( Q(g^{\mu\nu} D_\mu D_\nu - m^2) + \frac{i}{2}\Sigma^{\mu\nu} Q R_{\nu\mu ab} S^{ab} \right) \Phi = 0. \quad (41)$$

where  $R_{\lambda\rho ab}$  is the Riemann curvature tensor.

This equation has a form similar to (33), and the same arguments can be applied to cancel the first (scalar) term. Therefore, the necessary conditions for the existence of solutions require that the following two expressions are zero (cf. with Equation (36)):

$$e^{\mu\nu\lambda\rho}\Gamma^\nu R_{\lambda\rho ab}S^{ab}\Phi = 0, \quad e^{\mu\nu\lambda\rho}S^{\mu\nu}R_{\lambda\rho ab}S^{ab}\Phi = 0. \quad (42)$$

Contrary to the interaction with electromagnetic fields, these equations may have nontrivial solutions. The point is that  $R_{\lambda\rho ab}S^{ab}\Phi$  is an antisymmetric tensor of the second degree. When the metric is without torsion, there is only one such tensor,  $S_{\lambda\rho}$ . It means that

$$R_{\lambda\rho ab}S^{ab}\Phi = S_{\lambda\rho}\hat{W}\Phi \quad (43)$$

where  $\hat{W}$  is a scalar operator. By convoluting the above equation with  $S^{\lambda\rho}$  and using (A14), one obtains

$$\hat{W}\Phi = -\frac{2}{3}S^{\lambda\rho}R_{\lambda\rho ab}S^{ab}\Phi. \quad (44)$$

Substituting (43) into (42) and using identities (A15) shows that all Equation (42) become automatically zero.

Multiplying (41) by  $\bar{Q}$ , one finds that

$$\left(g^{\mu\nu}D_\mu D_\nu - m^2 + \frac{1}{2}S^{\mu\nu}R_{\nu\mu ab}S^{ab}\right)\Phi = 0. \quad (45)$$

It is plain that the same equation can be obtained directly by the transformation Equation (27) to the curved space and applying the covariant derivative to the result.

The corresponding equation for the usual Dirac equation has been derived in [24], and it has the similar form:

$$\left(g^{\mu\nu}D_\mu D_\nu - m^2 + \frac{1}{2}\Sigma^{\mu\nu}R_{\nu\mu ab}\Sigma^{ab}\right)\Psi = 0 \quad (46)$$

with the difference being that  $\Psi$  is a usual four component spinor, and due to (A24), (A35) and (A36), the term with the Riemann tensor can be reduced to a scalar curvature  $R = g^{\nu\lambda}g^{\mu\rho}R_{\nu\mu\lambda\rho}$

$$\frac{1}{2}\Sigma^{\mu\nu}R_{\nu\mu ab}\Sigma^{ab} = -\frac{1}{4}R. \quad (47)$$

The above arguments demonstrate how the main obstacle to the interaction of Dirac particles with electromagnetic fields can be overcome in gravitational fields, but unfortunately, they do not prove the existence of its solutions. The author is not aware of mathematical results, which could be applied for such overdetermined equations, and further investigations are desirable.

## 7. Conclusions

The relativistic invariance implies that elementary particles are transformed by irreducible representations of the Lorentz group. All known elementary particles correspond to *finite*-dimensional representations. But *infinite*-dimensional representations of the homogeneous Lorentz group do exist, and it is natural to conjecture that particles related with such representations exist as well.

The new Dirac equation describes the simplest of such particles with zero spin in the rest frame. Originally, this equation has been developed as an example of a relativistic wave equation whose all solutions have only positive energies. Similar to all known positive-energy equations, solutions of the new Dirac equation obey also the infinite-dimensional Majorana equation. But the latter has too large a spectrum of possible solutions. To select a single solution, the new Dirac equation is constructed to be not an isolated equation but an overdetermined system of equations such that one of its consequences is the Klein–Gordon

equation with a fixed mass. This construction necessarily requires that all these different equations are consistent (i.e., have the same solutions), which is a nontrivial statement.

A Dirac particle is described by one function depended on four usual space-time coordinates and two additional variables. The later are only dummy variables introduced in such a way that the expansion over them generates an infinite number of particle wave components corresponding to the infinite-dimensional representation of the Lorentz group.

It was proved that the resulting equations have no common solution when electromagnetic fields are introduced into the new Dirac equation within the minimal coupling. It means that Dirac particles cannot, in principle, interact with electromagnetic fields (with the minimal coupling) but it comes as a consequence of their internal structure and has not been imposed *ad hoc*. Consequently, if Dirac particles exist they will automatically have dark matter properties.

The distinctive property of Dirac particles is that they have only positive energy which infers the absence of Dirac antiparticles. Thus, Dirac particles break the CPT theorem but without violating any fundamental law. It signifies that such particles might be a new source of observed matter–antimatter asymmetry.

Dirac particles (as well as other particles transformed by infinite-dimensional representations of the homogeneous Lorentz group) are unusual, interesting, and potentially important objects, which may be a clue to a new physics. Their characteristic properties are peculiar, and to a large extent, counterintuitive. To summarise:

- They have spin zero (in the rest frame) but are not scalars.
- They obey covariant relativistic equations but have only positive energies, and consequently do not have antiparticles.
- Their equations are linear in the momenta but do not permit the minimal coupling to electromagnetic fields.

Unfortunately, Dirac particle investigations are undeveloped. No usual textbooks have mentioned their existence, no standard models incorporate them, and many important questions remain unanswered. At the present stage of the study, it seems that Dirac particles are sterile and can interact directly only with the gravity. Detailed examinations of the interaction of Dirac particles with usual matter through gravitational loops and their production during the Big Bang are essential to give them the status of established elementary particles capable of explaining dark matter.

Such inquiries are not straightforward and require the development of new methods, but the possibility that Dirac particles could dominate the universe clearly stresses the importance of the topic and the necessity to investigate it seriously.

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## Appendix A. Useful Relations

- The choice in [4] is equivalent to the following gamma matrices obeying (8):  $\gamma^0 = \beta$ ,

$$\gamma^1 = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (\text{A1})$$

and

$$\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \quad (\text{A2})$$

where  $\sigma_i$  are the usual  $2 \times 2$  Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A3})$$

- When working with quadratic operators in  $Q$ , the following two formulas are useful.

(a) For an antisymmetric matrix  $G_{jk} = -G_{kj}$  there is an operator identity

$$Q^T G Q \equiv q_j G_{jk} q_k = -\frac{i}{2} \text{Tr}(G \beta). \quad (\text{A4})$$

(b) Let us consider the following transformation of a  $4 \times 4$  matrix  $A$  (see, e.g., [17]):

$$\mathcal{D} : A \rightarrow \mathcal{D}(A) = \frac{i}{2} \bar{Q} A Q, \quad \bar{Q} = Q^T \beta. \quad (\text{A5})$$

If  $\beta A$  and  $\beta B$  are symmetric matrices, then the direct calculations prove that this transformation preserves commutator relations

$$[\mathcal{D}(A), \mathcal{D}(B)] = \mathcal{D}([A, B]). \quad (\text{A6})$$

- The spin operators, which realise the infinite-dimensional representation of the Lorentz group, are defined in (21):  $S^{cd} = \frac{1}{2} \bar{Q} \Sigma^{cd} Q$ . Rewriting them by components gives

$$\begin{aligned} S^{01} &= \frac{1}{4}(q_1^2 - q_3^2 - q_2^2 + q_4^2), & S^{02} &= \frac{1}{2}(q_3 q_4 - q_1 q_2), \\ S^{03} &= \frac{1}{2}(q_1 q_3 + q_4 q_2), & S^{12} &= \frac{1}{2}(q_2 q_3 - q_1 q_4), \\ S^{13} &= \frac{1}{4}(q_2^2 + q_4^2 - q_1^2 - q_3^2), & S^{23} &= \frac{1}{2}(q_1 q_2 + q_3 q_4). \end{aligned} \quad (\text{A7})$$

- The components of vector operator  $\Gamma^c = -\frac{1}{4} \bar{Q} \gamma^c Q$  are

$$\begin{aligned} \Gamma^0 &= \frac{1}{4}(q_1^2 + q_2^2 + q_3^2 + q_4^2), & \Gamma^1 &= \frac{1}{2}(q_2 q_4 - q_1 q_3), \\ \Gamma^2 &= \frac{1}{2}(q_1 q_4 + q_2 q_3), & \Gamma^3 &= \frac{1}{4}(q_1^2 + q_2^2 - q_3^2 - q_4^2). \end{aligned} \quad (\text{A8})$$

- In the calculations, the following identities are also helpful (see, e.g., [17]):

$$\gamma^5 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = e^{\mu\nu}{}_{\rho\sigma} \gamma^\rho \gamma^\sigma \quad (\text{A9})$$

and

$$\frac{1}{2} \gamma^\lambda (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = \eta^{\lambda\mu} \gamma^\nu - \eta^{\lambda\nu} \gamma^\mu + e^{\lambda\mu\nu}{}_{\rho} \gamma^5 \gamma^\rho. \quad (\text{A10})$$

Here,  $e^{\mu\nu\rho\sigma}$  is the completely antisymmetric tensor with  $e^{0123} = 1$ .

- For convenience, the commutators of spin operators  $S$  and  $\Gamma$  are listed below:

$$[S^{cd}, S^{kr}] = i(\eta^{ck} S^{dr} + \eta^{dr} S^{ck} - \eta^{dk} S^{cr} - \eta^{cr} S^{dk}), \quad (\text{A11})$$

$$[\Gamma^c, \Gamma^d] = -i S^{cd}, \quad (\text{A12})$$

$$[S^{cd}, \Gamma^b] = i(\eta^{cb} \Gamma^d - \eta^{db} \Gamma^c). \quad (\text{A13})$$

These commutators are the periphrases of the corresponding relations for the usual  $\gamma$ -matrices.

- The above operators obey a number of identities (see e.g., [18]):

$$S_{ab} S^{ab} = -\frac{3}{2}, \quad \Gamma_a \Gamma^a = \frac{1}{2}, \quad (\text{A14})$$

$$e^{abcd} S_{ab} S_{cd} = 0, \quad e^{abcd} S_{ab} \Gamma_c = 0, \quad (\text{A15})$$

$$\Gamma_a S^{ab} = -\frac{3i}{2} \Gamma^b, \quad S^{ab} \Gamma_b = -\frac{3i}{2} \Gamma^a, \quad (\text{A16})$$

$$S^a{}_b S^{bc} = \frac{1}{2} (\eta^{ac} - 2\Gamma^a \Gamma^c - 3iS^{ac}). \quad (\text{A17})$$

- Let us introduce a  $5 \times 5$  metric tensor  $\eta^{AB} = \text{diag}(-1, 1, 1, 1, 1)$  with  $A, B = 0, \dots, 4$  and denote  $S^{4a} = \Gamma^a$ . Ten antisymmetric operators  $S^{ab}$  and  $S^{4a}$  with  $a, b = 0, \dots, 3$  form a representation of the group  $O(3, 2)$  as the above commutation relations can be rewritten in the form (cf. with Equation (18))

$$[S^{CD}, S^{KR}] = i(\eta^{CK} S^{DR} + \eta^{DR} S^{CK} - \eta^{DK} S^{CR} - \eta^{CR} S^{DK}) \quad (\text{A18})$$

where majuscule letters are from 0 to 4.

- The Hermite functions used in the expansion (23) have the form (see, e.g., [28])

$$\phi_n(q) = \left(2^n n! \sqrt{\pi}\right)^{-1/2} H_n(q) e^{-q^2/2} \quad (\text{A19})$$

where  $H_n(q)$  are the Hermite polynomials. These functions are orthogonal:

$$\int_{-\infty}^{\infty} \phi_m(q) \phi_n(q) dq = \delta_{mn} \quad (\text{A20})$$

and obey the following relations:

$$\begin{aligned} q\phi_n(q) &= \sqrt{\frac{n}{2}} \phi_{n-1}(q) + \sqrt{\frac{n+1}{2}} \phi_{n+1}(q), \\ \phi'_n(q) &= \sqrt{\frac{n}{2}} \phi_{n-1}(q) - \sqrt{\frac{n+1}{2}} \phi_{n+1}(q). \end{aligned} \quad (\text{A21})$$

The use of these identities permits one to rewrite any equation in auxiliary variables  $q_j$  in the form of infinite-dimensional matrices without these variables.

## Appendix B. Tetrad Formalism

A tetrad (vierbein) is, by definition, a set of (usually orthogonal) vectors  $t_a^\mu(x)$  defined in all space points  $x$  which transform the true symmetric metric tensor of a curved space  $g^{\mu\nu}(x)$  locally into the flat Minkowski metric tensor  $\eta = \text{diag}(-1, 1, 1, 1)$  (see, e.g., [26])

$$\eta_{ab} = t_a^\mu(x) t_b^\nu(x) g_{\mu\nu}(x). \quad (\text{A22})$$

Here Latin, (resp., Greek) letters indicate flat (resp., curved) coordinates. The curved-space gamma matrices are determined as follows:

$$\tilde{\gamma}^\mu(x) = t_a^\mu(x) \gamma^a. \quad (\text{A23})$$

These  $4 \times 4$  matrices obey the relation

$$\tilde{\gamma}^\mu(x) \tilde{\gamma}^\nu(x) + \tilde{\gamma}^\nu(x) \tilde{\gamma}^\mu(x) = 2g^{\mu\nu}(x). \quad (\text{A24})$$

For clarity, all quantities multiplied by tetrads will be indicated by the same letter as in the flat space but with a tilde:

$$\tilde{\Gamma}_\mu(x) = t_a^\mu(x)\Gamma^a, \quad \tilde{\Sigma}^{\mu\nu}(x) = t_a^\mu(x)t_b^\nu(x)\Sigma^{ab}, \quad \tilde{S}^{\mu\nu}(x) = t_a^\mu(x)t_b^\nu(x)S^{ab}. \quad (\text{A25})$$

At a fixed point, a (symmetric) metric tensor is determined by ten components, but there exist sixteen tetrads. The remaining six free parameters correspond to six-parameter local Lorentz transformations, which preserve the Minkowski metric tensor.

It is well known that to conserve a local invariance, it is necessary to change the usual derivatives to covariant ones:

$$\partial_\mu \Psi(x) \longrightarrow D_\mu \Psi(x) = (\partial_\mu + C_\mu) \Psi(x) \quad (\text{A26})$$

where  $C_\mu$  is a connection whose explicit form depends on a quantity  $\Psi(x)$  on which the derivative acts. In general, if under the infinitesimal local Lorentz transformations  $x^\mu = x^\mu + a_\nu^\mu(x)x^\nu$  (cf. (16)) a field  $\Psi(x)$  transforms in the following way  $\Psi(x) \rightarrow \Psi(x) + a_{ab}(x)\hat{S}^{ab}\Psi(x)$ , then

$$C_\mu = \omega_{\mu ab} \hat{S}^{ab} \quad (\text{A27})$$

with certain coefficients  $\omega_{\mu ab}$  depending on quantities considered. It is common to denote a covariant derivative by  $D_\mu$ , but a care should be taken when its explicit expression is used in calculations. For vector (and tensor) fields,  $\omega_{\mu ab} = \Gamma_{\mu ab}$ , and it is called the Christoffel symbol. Different cases are indicated in Table A1.

**Table A1.** Covariant derivatives of different quantities.

Contravariant Lorentz vector	$V^\mu \rightarrow V^\mu + a_\lambda^\mu(x)V^\lambda$	$D_\nu V^\mu = \partial_\nu V^\mu + \Gamma_{\nu\lambda}^\mu V^\lambda$
Contravariant tetrad vector	$V^b \rightarrow V^b + a_c^b(x)V^c$	$D_\nu V^b = \partial_\nu V^b + \omega_{\nu c}^b V^c$
Usual Lorentz spinor	$\Psi \rightarrow \Psi + \frac{1}{2}a_{ab}(x)\Sigma^{ab}\Psi$	$D_\nu \Psi = (\partial_\nu + \frac{1}{2}\omega_{\nu ab}\Sigma^{ab})\Psi$
Dirac particle	$\Phi \rightarrow \Phi + \frac{i}{2}a_{ab}(x)S^{ab}\Phi$	$D_\nu \Phi = (\partial_\nu + \frac{i}{2}\omega_{\nu ab}S^{ab})\Phi$

For covariant quantities, the connections change the sign. When tensors are considered, it is necessary to add the corresponding connections for each index. For matrices (and operators) such transformation corresponds to the commutator with the connections.

Covariant derivatives transform in the same way as the corresponding quantities in the flat space. In particular, they have to commute with the rising and lowering indices with the metric tensor. Therefore,

$$D_\nu g_{\mu\lambda} = 0, \quad \partial_\nu g_{\mu\lambda} - \Gamma_{\nu\mu}^\beta g_{\beta\lambda} - \Gamma_{\nu\lambda}^\beta g_{\mu\beta} = 0. \quad (\text{A28})$$

This relation implies that

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\mu\nu}), \quad \Gamma_{\nu\lambda}^\mu = \Gamma_{\lambda\nu}^\mu. \quad (\text{A29})$$

Similarly, it is convenient to require that

$$D_\nu t_\mu^a = 0, \quad \partial_\nu t_\mu^a + \Gamma_{\nu\mu}^\lambda t_\lambda^a + \omega_{\nu c} t_\mu^c = 0 \quad (\text{A30})$$

which leads to

$$\omega_{\nu c}^b = t_c^\mu(-\partial_\nu t_\mu^b + \Gamma_{\nu\mu}^\lambda t_\lambda^b), \quad \omega_{\mu cb} = -\omega_{\mu bc}. \quad (\text{A31})$$

The same answer can be obtained from

$$D_\nu \hat{\gamma}^\mu(x) = 0, \quad \partial_\nu \hat{\gamma}^\lambda(x) + \Gamma_{\nu\lambda}^\mu \hat{\gamma}^\lambda(x) + [g_\nu, \hat{\gamma}^\mu(x)] = 0, \quad g_\mu = \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}. \quad (\text{A32})$$

In the same way,

$$D_\nu \hat{\Gamma}^\mu(x) = 0, \quad \partial_\nu \hat{\Gamma}^\lambda(x) + \Gamma_{\nu\lambda}^\mu \hat{\Gamma}^\lambda(x) + [G_\nu, \hat{\Gamma}^\mu(x)] = 0, \quad G_\mu = \frac{i}{2} \omega_{\mu ab} S^{ab}. \quad (\text{A33})$$

By definition, the commutator of two covariant derivatives acting at a covariant vector is determined by the Riemann curvature tensor  $R_{\nu\mu\rho}{}^\lambda$

$$[D_\mu, D_\nu] A_\rho = R_{\nu\mu\rho}{}^\lambda A_\lambda, \quad R_{\nu\mu\rho}{}^\lambda = \partial_\nu \Gamma_{\mu\rho}^\lambda - \partial_\mu \Gamma_{\nu\rho}^\lambda + \Gamma_{\mu\rho}^\sigma \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\rho}^\sigma \Gamma_{\mu\sigma}^\lambda. \quad (\text{A34})$$

It is known that the Riemann tensor obeys the symmetry relations (see, e.g., [26])

$$R_{\nu\mu\lambda\rho} = -R_{\mu\nu\lambda\rho} = -R_{\nu\mu\rho\lambda} = R_{\lambda\rho\nu\mu} \quad (\text{A35})$$

and the Bianchi identity

$$R_{\nu\mu\lambda\rho} + R_{\nu\rho\mu\lambda} + R_{\nu\lambda\rho\mu} = 0. \quad (\text{A36})$$

As  $D_\nu t_\rho^a = 0$  it follows that  $[D_\mu, D_\nu] t_\rho^a = 0$ , from which one proves the validity of the following identity:

$$\partial_\mu \omega_{\nu b}^a - \partial_\nu \omega_{\mu b}^a + \omega_{\mu c}^a \omega_{\nu b}^c - \omega_{\nu c}^a \omega_{\mu b}^c = -R_{\nu\mu\rho}{}^\lambda t_\lambda^a t_b^\rho. \quad (\text{A37})$$

Using this expression, one obtains

$$[D_\mu, D_\nu] \Phi = \frac{i}{2} R_{\nu\mu ab} S^{ab} \Phi. \quad (\text{A38})$$

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