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The Nekrasov Conjecture for Toric Surfaces

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Abstract The Nekrasov conjecture predicts a relation between the partition function for $N = 2$ supersymmetric Yang–Mills theory and the Seiberg-Witten prepotential. For instantons on $\mathbb{R}^4$, the conjecture was proved, independently and using different methods, by Nekrasov-Okounkov and Nakajima-Yoshioka. We prove a generalized version of the conjecture for instantons on noncompact toric surfaces.

Contents

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1 Introduction

1.1 Background

The Nekrasov conjecture [28] predicts a surprising relation between two seemingly unrelated quantities: the partition function for $N=2$ supersymmetric Yang–Mills theory, defined in terms of instantons on $\mathbb{R}^4$, and the Seiberg-Witten prepotential [33], defined in terms of period integrals of a family of hyperelliptic curves. For gauge group $U(r)$, Nekrasov and Okounkov proved the conjecture for a list of gauge theories (4d pure gauge theory, 4d gauge theory with matter, 5d theory compactified on a circle) [30], Nakajima and Yoshioka proved the conjecture for 4d pure gauge theory [25] and for 5d theory compactified on a circle [26] (see also Göttsche-Nakajima-Yoshioka [18]). Braverman and Etingof studied 4d pure gauge theory with arbitrary gauge groups [3; 4].

In this paper we prove a generalized version of the conjecture for instantons on noncompact toric surfaces. Instantons on toric surfaces have been studied in [29; 17; 18].

In field theory terms, Nekrasov's insight involves a comparison of the infrared and ultraviolet limits of the SUSY gauge theories, as follows. The vacuum expectation value of their observables is not sensitive to the energy scale. In the ultraviolet, the theory is weakly coupled and dominated by instantons; whereas in the infrared, there appears a relation to the prepotential of the effective theory. In this instance, the physical argument is accompanied by completely rigorous mathematical definitions, thus allowing us to prove the conjecture.

1.2 Partition functions for instantons on noncompact toric surfaces

Let $X_0 = X \setminus \ell_\infty$ be an open toric surface that can be compactified to a non-singular projective toric surface X by adding a line at infinity $\ell_\infty \cong \mathbb{P}^1$ with positive self-intersection number, so that $T_t = (\mathbb{C}^*)^2$ acts on X_0 and on X. Let $\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ denote the moduli space of rank r torsion free sheaves over X having Chern classes $c_1 = d$ and $c_2 = n$, and framed over ℓ_∞. Then $\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ is a smooth variety over $\mathbb{C}$, and it admits a $T_t \times T_e$-action with isolated fixed points, where $T_e \cong (\mathbb{C}^*)^r$ is the maximal torus of the complex gauge group $GL(r,\mathbb{C})$ which acts on framings. We define

$$\int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} 1$$

by formally applying the Atiyah-Bott localization formula. The above integral is a rational function in equivariant parameters $\varepsilon_1, \varepsilon_2 \in H^2_{T_t}(\mathrm{pt})$ and $a_1,\dots,a_r \in H^2_{T_e}(\mathrm{pt})$. The Nekrasov partition function for supersymmetric $SU(r)$ instantons on X_0 is defined as

$$Z^{\mathrm{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\mathrm{def}}{=} \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} 1,$$

where Λ is a formal variable. It lies in the ring $\mathbb{Q}(\varepsilon_1,\varepsilon_2,a_1,\dots,a_r)[[\Lambda]]$.

In further generality, given two multiplicative classes A, B we define

$$Z^{\mathrm{inst}}_{X_0,A,B,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\mathrm{def}}{=} \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} A_{\tilde{T}}(T_{\mathfrak{M}}) B_{\tilde{T}}(V),$$

where $T_{\mathfrak{M}}$ is the tangent bundle and V is the natural bundle on $\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ (see Definition 29).

1.3 Seiberg-Witten prepotential

We briefly recall the definition of the Seiberg-Witten prepotential for 4d pure $SU(r)$ gauge theory. Appendix C contains a more detailed discussion and definitions for other gauge theories.

Consider the family of hyperelliptic curves parametrized by Λ and $\vec{u} = (u_2, u_3, \dots, u_r)$:

$$C_{\vec{u}} : \Lambda^r \left(w + \frac{1}{w} \right) = P(z) = z^r + u_2 z^{r-2} + u_3 z^{r-3} + \cdots + u_r.$$

The parameter space for $\vec{u}$ is called the $\vec{u}$-plane. The *Seiberg-Witten differential*

$$dS = \frac{1}{2\pi\sqrt{-1}} z \frac{dw}{w}$$

is a meromorphic differential defined on the total space of this family such that $\left\{ \omega_p \stackrel{\mathrm{def}}{=} \frac{\partial}{\partial u_p}(dS) \mid p = 2,\dots,r \right\}$ is a basis of holomorphic differentials on the genus $(r-1)$ curve $C_{\vec{u}}$. Choose a symplectic basis $\{A_\alpha, B_\beta \mid \alpha,\beta = 2,\dots,r\}$ of $H_1(C_{\vec{u}},\mathbb{Z})$, and define

$$a_\alpha = \int_{A_\alpha} dS, \quad a^D_\beta = 2\pi\sqrt{-1} \int_{B_\beta} dS.$$

Then the 1-form $\sum_{\alpha=2}^{r} a_\alpha^D da_\alpha$ is closed, so there exists a locally defined function, the *Seiberg-Witten prepotential* $\mathscr{F}_0$, such that

$$\sum_{\alpha=2}^{r} a_\alpha^D da_\alpha = d\mathscr{F}_0, \quad \text{i.e.,} \quad a_\alpha^D = \frac{\partial \mathscr{F}_0}{\partial a_\alpha}.$$

The above definitions of dS, a_α, a_α^D are the same as those in [30], but are $\sqrt{-1}$ times the corresponding definitions in [24; 25].

1.4 Nekrasov conjecture

Let q_0, q_1 be the two T_t fixed points in $\ell_\infty \subset X$, and let $u, v \in \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ be the weights of the T_t-action on $(N_{\ell_\infty/X})_{q_0}$, $(N_{\ell_\infty/X})_{q_1}$, respectively, where $N_{\ell_\infty/X}$ is the normal bundle of ℓ_∞ in X. If w is the weight of T_t-action on $T_{q_0}\ell_\infty$ and $k = \ell_\infty \cdot \ell_\infty > 0$, then $v = u - kw$. Define

$$\mathscr{F}^{\text{inst}}_{X_0,A,B,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) \stackrel{\text{def}}{=} -u(u-kw) \log Z^{\text{inst}}_{X_0,A,B,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda).$$

We now state the prototype statement of the conjecture for toric surfaces, which will have 8 incarnations.

Main Theorem *(Nekrasov conjecture for toric surfaces: prototype statement).*

(a) $\mathscr{F}^{\cdots}_{X_0,A,B,d}(\varepsilon_1, \varepsilon_2, \vec{a}, \mathbf{m}; \Lambda)$ *is analytic in* $\varepsilon_1, \varepsilon_2$ *near* $\varepsilon_1 = \varepsilon_2 = 0$.

(b) $\lim_{\varepsilon_1,\varepsilon_2 \to 0} \mathscr{F}^{\cdots}_{X_0,A,B,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = k\mathscr{F}_0^{\cdots}(\vec{a}, \Lambda)$*, where* $\mathscr{F}_0^{\cdots}(\vec{a}, \Lambda)$ *is the* $\cdots$*part of the Seiberg-Witten prepotential of matter case* $A, B, \mathbf{m}$*, and* $k = \ell_\infty \cdot \ell_\infty > 0$ *is the self intersection number of* ℓ_∞.

The 8 cases we prove are

- *Instanton part: Theorem 521.* With the $^{\cdots}$ replaced by $^{\text{inst}}$, we prove the following cases of the conjecture:
  (1) *4d pure gauge theory*: $A = B = 1$, $\mathbf{m} = \emptyset$.
  (2) *4d gauge theory with N_f fundamental matter hypermultiplets*: $A = 1$, $B = (E_{\mathbf{m}})(V)$ is the T_m-equivariant Euler class of $V \otimes M$, where V is the natural bundle over the moduli space, M is the fundamental representation of $U(N_f)$, T_m is the maximal torus of $U(N_f)$, $\mathbf{m} = (m_1, \ldots, m_{N_f})$.
  (3) *4d gauge theory with one adjoint matter hypermultiplet*: $A = E_m(T_{\mathfrak{M}})$ is the equivariant Euler class of the tangent bundle of the moduli space, $B = 1$, $\mathbf{m} = m$.
  (4) *5d gauge theory compactified on a circle*: $A = \hat{A}_\beta(T_{\mathfrak{M}})$ is the $\hat{A}_\beta$ genus of the tangent bundle (the usual $\hat{A}$ genus being the case $\beta = 1$), $B = 1$, $m = \emptyset$ but $\mathscr{F}$ depends on the additional parameter β.
- *Perturbative part: Theorem 68.* With the $^{\cdots}$ replaced by $^{\text{pert}}$, we derive 4 more cases of the conjecture, with the same restrictions as in the first part:
  (1) *4d pure gauge theory.*
  (2) *4d gauge theory with N_f fundamental matter hypermultiplets.*
  (3) *4d gauge theory with one adjoint matter hypermultiplet.*

(4) *5d gauge theory compactified on a circle of circumference* β.

The instanton part follows by localization, from known results in the $\mathbb{C}^2$ case. Indeed, localization calculations yield an expression of the instanton partition function $Z^{\mathrm{inst}}_{X_0,A,B,d}$ over X_0 in terms of contributions from vertices (T_t fixed points in X_0) and from legs (T_t invariant $\mathbb{P}^1$'s in X_0). Each vertex contributes one copy of the instanton partition function of $\mathbb{C}^2$, for which the singularity along $\varepsilon_1 = \varepsilon_2 = 0$ is already known. The contribution from legs does not introduce more poles along $\varepsilon_1 = \varepsilon_2 = 0$. A priori, the tangent weights at all T_t fixed points in X_0 appear in the denominator, but an argument similar to that in [29, Sect. 6.1] shows that these poles mostly cancel out, and we are left with the two normal weights u, $u - kw$ at the T_t fixed points on ℓ_∞. The perturbative part is fairly straightforward.

1.5 Outline of the paper

In Sect. 2, we describe properties of the instanton moduli spaces. In Sect. 3, we study torus actions on these moduli spaces and the fixed point sets. In Sect. 4, we introduce a general instanton partition function depending on two multiplicative classes A, B for noncompact toric surfaces; different choices of A, B give partition functions of different gauge theories. Section 5 contains localization computations on instanton moduli spaces, and the proof of the instanton part of the conjecture. Section 6 contains definitions of the perturbative part of the partition function, and the proof of the perturbative part of the conjecture.

2 Moduli Spaces of Framed Bundles on Surfaces

We work over $\mathbb{C}$. Let X be a non-singular projective surface. Let $\ell_\infty \subset X$ be a smooth divisor. In this section, we introduce moduli spaces of framed bundles on X, and describe basic properties of these moduli spaces, generalizing the discussion in [25, Sect. 2] on the case $X = \mathbb{P}^2$. The framed moduli spaces were constructed in much more general setting by Huybrechts-Lehn [19].

Given a positive integer r, an integer n, and a cohomology class $d \in H^2(X;\mathbb{Z})$, let $\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ be the moduli space which parametrizes isomorphism classes of pairs (E,Φ) such that

(1) E is a torsion free sheaf on X which is locally free in a neighborhood of ℓ_∞.
(2) $\operatorname{rank}(E) = r$, $c_1(E) = d$ and $\int_X c_2(E) = n$.
(3) $\Phi : E|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ is an isomorphism called "framing at infinity".

Note that (1) and (2) imply $\int_{\ell_\infty} d = 0$.

2.1 Dimension of the moduli space

Given a divisor $D \subset X$, let $E(-D) = E \otimes \mathcal{O}_X(-D)$.

Proposition 21. *Suppose that* $\ell_\infty \cdot \ell_\infty > 0$.

(a) *For any* $(E,\Phi) \in \mathfrak{M}_{r,d,n}(X,\ell_\infty)$ *we have* $\operatorname{Ext}^0_{\mathcal{O}_X}(E,E(-\ell_\infty)) = 0$.

(b) *Assume in addition that* $\ell_\infty \cong \mathbb{P}^1$*, then for any* $(E,\Phi) \in \mathfrak{M}_{r,d,n}(X,\ell_\infty)$ *we have*

$$\mathrm{Ext}^0_{\mathscr{O}_X}(E,E(-\ell_\infty)) = \mathrm{Ext}^2_{\mathscr{O}_X}(E,E(-\ell_\infty)) = 0.$$

Remark 22. If X is a non-singular projective surface which contains a smooth divisor $\ell_\infty \cong \mathbb{P}^1$ such that $k = \ell_\infty \cdot \ell_\infty > 0$. Then $T_X\big|_{\ell_\infty} \cong \mathscr{O}_{\mathbb{P}^1}(k) \oplus \mathscr{O}_{\mathbb{P}^1}(2)$, so X is rationally connected, or equivalently, X is a rational surface. The arithmetic genus of X is $p_a(X) = \chi(\mathscr{O}_X) - 1 = 0$.

Proof of Proposition 21. (a) Assuming that $\ell_\infty \cdot \ell_\infty > 0$, we will show that

$$\mathrm{Hom}_{\mathscr{O}_X}(E,E(-\ell_\infty)) = 0.$$

Let s be a section of $\mathscr{O}_X(\ell_\infty)$ such that its zero locus is ℓ_∞. The exact sequence

$$0 \to E(-(m+1)\ell_\infty) \xrightarrow{s\cdot} E(-m\ell_\infty) \to E(-m\ell_\infty) \otimes \mathscr{O}_D \to 0$$

induces a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathscr{O}_X}(E,E(-(m+1)\ell_\infty)) &\to \mathrm{Hom}_{\mathscr{O}_X}(E,E(-m\ell_\infty)) \\ &\to \mathrm{Hom}_{\mathscr{O}_X}(E,E(-m\ell_\infty) \otimes \mathscr{O}_{\ell_\infty}) \\ &\to \mathrm{Ext}^1_{\mathscr{O}_X}(E,E(-(m+1)\ell_\infty) \to \mathrm{Ext}^1_{\mathscr{O}_X}(E,E(-m\ell_\infty)) \to \cdots, \end{aligned}$$

where

$$\mathrm{Hom}_{\mathscr{O}_X}(E,E(-m\ell_\infty) \otimes \mathscr{O}_{\ell_\infty}) \cong H^0(\ell_\infty, \mathscr{O}_X(-m\ell_\infty)|_{\ell_\infty})^{\oplus r^2},$$

since $E|_{\ell_\infty}$ is trivial. Let $k = \ell_\infty \cdot \ell_\infty > 0$. Then

$$H^0(\ell_\infty, \mathscr{O}_X(-m\ell_\infty)|_{\ell_\infty}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-mk)) = 0$$

when $m > 0$. So, for any positive integer m,

$$\mathrm{Hom}_{\mathscr{O}_X}(E,E(-(m+1)\ell_\infty)) \to \mathrm{Hom}_{\mathscr{O}_X}(E,E(-m\ell_\infty))$$

is an isomorphism, and

$$\mathrm{Ext}^1_{\mathscr{O}_X}(E,E(-(m+1)\ell_\infty)) \to \mathrm{Ext}^1_{\mathscr{O}_X}(E,E(-m\ell_\infty))$$

is injective. As a consequence, any element in $\mathrm{Hom}_{\mathscr{O}_X}(E,E(-\ell_\infty))$ restricts to zero in a formal neighborhood of ℓ_∞ in X. So

$$\mathrm{Hom}_{\mathscr{O}_X}(E,E(-\ell_\infty)) = 0.$$

(b) We now assume that $\ell_\infty \cdot \ell_\infty > 0$ and $\ell_\infty \cong \mathbb{P}^1$. By Serre duality, $\mathrm{Ext}^2_{\mathscr{O}_X}(E,E(-\ell_\infty))$ is dual to $\mathrm{Hom}_{\mathscr{O}_X}(E,E(K_X+\ell_\infty))$. We will show that

$$\mathrm{Hom}_{\mathscr{O}_X}(E,E(K_X+\ell_\infty)) = 0.$$

The exact sequence

$$0 \to E(K_X - m\ell_\infty) \xrightarrow{s\cdot} E(K_X + (1-m)\ell_\infty) \to E(K_X + (1-m)\ell_\infty) \otimes \mathscr{O}_D \to 0$$

induces a long exact sequence

$$
\begin{aligned}
0 &\to \mathrm{Hom}_{\mathscr{O}_X}(E, E(K_X - m\ell_\infty)) \to \mathrm{Hom}_{\mathscr{O}_X}(E, E(K_X + (1-m)\ell_\infty)) \\
&\to \mathrm{Hom}_{\mathscr{O}_X}(E, E(K_X + (1-m)\ell_\infty) \otimes \mathscr{O}_{\ell_\infty}) \\
&\to \mathrm{Ext}^1_{\mathscr{O}_X}(E, E(K_X - m\ell_\infty) \to \mathrm{Ext}^1_{\mathscr{O}_X}(E, E(K_X + (1-m)\ell_\infty)) \to \cdots .
\end{aligned}
$$

$E|_{\ell_\infty}$ is trivial and $K_{\ell_\infty} = (K_X + \ell_\infty)|_{\ell_\infty}$, so

$$
\begin{aligned}
\mathrm{Hom}_{\mathscr{O}_X}(E, E(K_X + (1-m)\ell_\infty) \otimes \mathscr{O}_{\ell_\infty}) \cong H^0(\ell_\infty, &\mathscr{O}_{\ell_\infty}(K_{\ell_\infty}) \\
&\otimes \mathscr{O}_X(-m\ell_\infty)|_{\ell_\infty})^{\oplus r^2}.
\end{aligned}
$$

Note that

$$
H^0(\ell_\infty, \mathscr{O}_{\ell_\infty}(K_{\ell_\infty}) \otimes \mathscr{O}_X(-m\ell_\infty)|_{\ell_\infty}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2-mk)) = 0
$$

for all $m \geq 0$. So, for any nonnegative integer m,

$$
\mathrm{Hom}_{\mathscr{O}_X}(E, E(K_X - m\ell_\infty) \to \mathrm{Hom}_{\mathscr{O}_X}(E, E(K_X + (1-m)\ell_\infty))
$$

is an isomorphism, and

$$
\mathrm{Ext}^1_{\mathscr{O}_X}(E, E(K_X - m\ell_\infty)) \to \mathrm{Ext}^1_{\mathscr{O}_X}(E, E(K_X + (1-m)\ell_\infty))
$$

is injective. As a consequence, any element in $\mathrm{Hom}_{\mathscr{O}_X}(E, E(K_X + \ell_\infty))$ restricts to zero in a formal neighborhood of ℓ_∞ in X, and we conclude that

$$
\mathrm{Hom}_{\mathscr{O}_X}(E, E(K_X + \ell_\infty)) = 0.
$$

□

Corollary 23. *Let X be a non-singular projective surface, and let ℓ_∞ be a smooth divisor of X such that $\ell_\infty \cdot \ell_\infty > 0$. Then for any (E, Φ) in $\mathfrak{M}_{r,d,n}(X, \ell_\infty)$,*

$$
\begin{aligned}
&\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathscr{O}_X}(E, E(-\ell_\infty)) - \dim_{\mathbb{C}} \mathrm{Ext}^2_{\mathscr{O}_X}(E, E(-\ell_\infty)) \\
&\quad = 2rn + (1-r)d \cdot d - r^2(p_a(X) + p_a(\ell_\infty)),
\end{aligned}
$$

where $d \cdot d = \int_X d^2$, $p_a(X)$ is the arithmetic genus of X, and $p_a(\ell_\infty)$ is the arithmetic genus of ℓ_∞.

Proof. Let $(E, \Phi) \in \mathfrak{M}_{r,d,n}(X, \ell_\infty)$ be locally free. By Proposition 21 (a),

$$
\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathscr{O}_X}(E, E(-\ell_\infty)) - \dim_{\mathbb{C}} \mathrm{Ext}^2_{\mathscr{O}_X}(E, E(-\ell_\infty)) = -\chi(\mathrm{End}(E) \otimes \mathscr{O}_X(-\ell_\infty)).
$$

Let $\nu \in H^4(X;\mathbb{Z})$ be the Poincaré dual of $[\mathrm{pt}] \in H_0(X;\mathbb{Z})$, and let $e \in H^2(X;\mathbb{Z})$ be the Poincaré dual of $[\ell_\infty] \in H_2(X;\mathbb{Z})$. By Hirzebruch-Riemann-Roch,

$$
\chi(\mathrm{End}(E) \otimes \mathscr{O}_X(-\ell_\infty)) = \deg\left(\mathrm{ch}(\mathrm{End}(E))\mathrm{ch}(\mathscr{O}_X(-\ell_\infty))\mathrm{td}(T_X)\right),
$$

where

$$\begin{aligned}\mathrm{ch}(\mathrm{End}(E)) &= \mathrm{ch}(E)\mathrm{ch}(E^\vee) = r^2 + (r-1)d^2 - 2rn\nu,\\ \mathrm{ch}(\mathscr{O}_X(-\ell_\infty)) &= 1 - e + \frac{e^2}{2} = 1 - e + \frac{k}{2}\nu \text{ for } k = \ell_\infty \cdot \ell_\infty > 0,\\ \mathrm{td}(T_X) &= 1 + \frac{1}{2}c_1(X) + \frac{1}{12}(c_1(X)^2 + c_2(X)).\end{aligned}$$

Let $N_{\ell_\infty/X}$ be the normal bundle of ℓ_∞ in X. Then

$$\int_X ec_1(X) = \int_{\ell_\infty} \big(c_1(\ell_\infty) + c_1(N_{\ell_\infty/X})\big) = 2 - 2p_a(\ell_\infty) + k.$$

Consequently,

$$\begin{aligned}&\deg\big(\mathrm{ch}(\mathrm{End}(E))\mathrm{ch}(\mathscr{O}_X(-\ell_\infty))\mathrm{td}(T_X)\big)\\ &= \int_X \left(\frac{r^2}{12}(c_1(X)^2 + c_2(X)) - \frac{r^2}{2}ec_1(X) + (r-1)d^2 + (\frac{kr^2}{2} - 2rn)\nu\right)\\ &= \frac{r^2}{12}\int_X (c_1(X)^2 + c_2(X)) - \frac{r^2}{2}(k + 2 - 2p_a(\ell_\infty)) + (r-1)\int_X d^2 + \frac{kr^2}{2} - 2rn\\ &= -2rn + (r-1)\int_X d^2 + r^2(p_a(X) + p_a(\ell_\infty)).\end{aligned}$$

□

Corollary 24. *Let X be a non-singular projective rational surface, and let ℓ_∞ be a divisor of X such that $\ell_\infty \cong \mathbb{P}^1$ and $\ell_\infty \cdot \ell_\infty > 0$. Then $\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ is smooth of (complex) dimension*

$$2rn + (1-r)d \cdot d,$$

where $d \cdot d = \int_X d^2$.

Example 25. Let $X = \mathbb{P}^2$, and let

$$\ell_\infty = \{[Z_0, Z_1, Z_2] \in \mathbb{P}^2 \mid Z_0 = 0\} \cong \mathbb{P}^1.$$

Then $\ell_\infty \cdot \ell_\infty = 1 > 0$. The moduli space $\mathfrak{M}_{r,d,n}(\mathbb{P}^2, \ell_\infty)$ is nonempty only if $\int_{\ell_\infty} d = 0$, which implies $d = 0$. By Corollary 24, the moduli space $\mathfrak{M}_{r,0,n}(\mathbb{P}^2, \ell_\infty)$ is smooth of complex dimension $2rn$. (See [25, Cor. 2.2]).

Example 26. Let $X = \mathbb{F}_k \stackrel{\text{def}}{=} \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-k) \oplus \mathscr{O}_{\mathbb{P}^1})$ be the k^{th} Hirzebruch surface, where k is a positive integer. Let

$$\ell_0 = \mathbb{P}(0 \oplus \mathscr{O}_{\mathbb{P}^1}) \cong \mathbb{P}^1, \quad \ell_\infty = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(-k) \oplus 0) \cong \mathbb{P}^1.$$

Then $\ell_0 \cdot \ell_0 = -k < 0$ and $\ell_\infty \cdot \ell_\infty = k > 0$.

The moduli space $\mathfrak{M}_{r,d,n}(\mathbb{F}_k, \ell_\infty)$ is nonempty only if $\int_{\ell_\infty} d = 0$, which implies $d = m\ell_0$ for some $m \in \mathbb{Z}$. By Corollary 24, the moduli space $\mathfrak{M}_{r,m\ell_0,n}(\mathbb{F}_k, \ell_\infty)$ is smooth of complex dimension $2rn + (r-1)km^2$.

Example 27. Let $\ell \subset \mathbb{P}^2$ be a curve of degree 1, and let $p_1, \ldots, p_k$ be k generic points in $\mathbb{P}^2$ which are disjoint from ℓ. Let $\pi : \mathbb{B}_k \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at $p_1, \ldots, p_k$. Let $\ell_\infty = \pi^{-1}(\ell) \cong \mathbb{P}^1$, and let $\ell_i = \pi^{-1}(p_i)$ be the exceptional divisors. Let $e_\infty, e_1, \ldots, e_k \in H^2(\mathbb{B}_k; \mathbb{Z})$ be the Poincaré duals of $[\ell_\infty], [\ell_1], \ldots, [\ell_k]$, respectively. Then

$$H^2(\mathbb{B}_k; \mathbb{Z}) = \mathbb{Z}e_\infty \oplus \mathbb{Z}e_1 \oplus \cdots \mathbb{Z}e_k.$$

The moduli space $\mathfrak{M}_{r,d,n}(\mathbb{B}_k, \ell_\infty)$ is nonempty only if $\int_{\ell_\infty} d = 0$, which implies

$$d = m_1 e_1 + \cdots + m_k e_k, \quad m_i \in \mathbb{Z}.$$

By Corollary 24, the moduli space $\mathfrak{M}_{r,m_1e_1+\cdots+m_ke_k,n}(\mathbb{B}_k, \ell_\infty)$ is smooth of complex dimension

$$2rn + (r-1)(m_1^2 + \cdots + m_k^2).$$

2.2 The natural bundle

In this subsection, X is a non-singular projective rational surface, and ℓ_∞ is a smooth rational curve in X such that $\ell_\infty \cdot \ell_\infty > 0$. The proof of the following proposition is very similar to that of Proposition 21.

Proposition 28. $H^0(X, E(-\ell_\infty)) = H^2(X, E(-\ell_\infty)) = 0.$

Let $\mathscr{E} \to X \times \mathfrak{M}_{r,d,n}(X, \ell_\infty)$ be the universal sheaf. Let $p_1 : X \times \mathfrak{M}_{r,d,n}(X, \ell_\infty) \to X$ and $p_2 : X \times \mathfrak{M}_{r,d,n}(X, \ell_\infty) \to \mathfrak{M}_{r,d,n}(X, \ell_\infty)$ be the projections to the two factors.

Definition 29. *The natural bundle over* $\mathfrak{M}_{r,d,n}(X, \ell_\infty)$ *is*

$$V \stackrel{\text{def}}{=} (R^1 p_2)_*(\mathscr{E} \otimes p_1^*(\mathscr{O}_X(-\ell_\infty))).$$

Corollary 210. V *is a vector bundle of rank*

$$n - \frac{1}{2}(d \cdot d + c_1(X) \cdot d)$$

over $\mathfrak{M}_{r,d,n}(X, \ell_\infty)$.

Proof. We use the notation in the proof of Corollary 24. Let $(E, \Phi) \in \mathfrak{M}_{r,d,n}(X, \ell_\infty)$ be locally free. The rank of V is given by $-\chi(E(-\ell_\infty))$. By Hirzebruch-Riemann-Roch,

$$\chi(E(-\ell_\infty)) = \deg\left(\mathrm{ch}(E)\mathrm{ch}(\mathscr{O}_X(-\ell_\infty))\mathrm{td}(T_X)\right),$$

where

$$\mathrm{ch}(E) = r + d + (\frac{d^2}{2} - n\nu), \quad \mathrm{ch}(\mathscr{O}_X(-\ell_\infty)) = 1 - e + \frac{e^2}{2} = 1 - e + \frac{k}{2}\nu,$$
$$\mathrm{td}(T_X) = 1 + \frac{1}{2}c_1(X) + \frac{1}{12}(c_1(X)^2 + c_2(X)).$$

Consequently,

$$\begin{aligned}
&\deg\left(\mathrm{ch}(E)\mathrm{ch}(\mathscr{O}_X(-\ell_\infty))\mathrm{td}(T_X)\right)\\
&=\int_X\left(\frac{r}{12}(c_1(X)^2+c_2(X))+\frac{1}{2}(d-re)c_1(X)+\frac{d^2}{2}+(\frac{kr}{2}-n)\nu\right)\\
&=\frac{r}{12}\int_X(c_1(X)^2+c_2(X))-\frac{r}{2}(k+2)+\frac{1}{2}\int_X(d^2+c_1(X)d)+\frac{kr}{2}-n\\
&=-n+\frac{1}{2}\int_X(d^2+c_1(X)d)+rp_a(X),
\end{aligned}$$

where $p_a(X)=0$ since X is a rational surface. □

3 Torus Action and Fixed Points

In this section, X is a non-singular projective toric surface. Therefore $T_t \stackrel{\text{def}}{=} (\mathbb{C}^*)^2$ acts on X. We use notation similar to that in [25, Sect. 2, 3].

3.1 Torus action on the surface

We assume that ℓ_∞ is a T_t-invariant $\mathbb{P}^1$ in X, and $\ell_\infty\cdot\ell_\infty=k>0$. Then $X_0=X\setminus\ell_\infty$ is a non-singular, quasi-projective toric surface. Let Γ be a graph such that the vertices of Γ are in one-to-one correspondence with the T_t fixed points in X_0, and two vertices are connected by an edge if and only if the corresponding fixed points are connected by a T_t-invariant $\mathbb{P}^1$. Then Γ is a chain, so $\#V(\Gamma)-\#E(\Gamma)=1$, and

$$\chi(X_0)=\#V(\Gamma)=\chi(X)-2,$$

where $E(\Gamma)$ is the set of edges in Γ and $V(\Gamma)$ is the set of vertices in Γ. Let p_v be the T_t fixed point in X_0 which corresponds to $v\in V(\Gamma)$, and let ℓ_e be the T_t-invariant $\mathbb{P}^1$ which corresponds to $e\in E(\Gamma)$. Any T_t-invariant divisor D in X disjoint from ℓ_∞ is of the form

$$D=\sum_{e\in E(\Gamma)} m_e\ell_e \cong H_2(X_0;\mathbb{Z}),$$

where $m_e\in\mathbb{Z}$.

3.2 Torus action on moduli spaces

Let T_e be the maximal torus of $GL(r,\mathbb{C})$ consisting of diagonal matrices, and let $\tilde{T}=T_t\times T_e$. We define an action of $\tilde{T}$ on $\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ as follows: for $(t_1,t_2)\in T_t$, let F_{t_1,t_2} be the automorphism of X defined by $F_{t_1,t_2}(x)=(t_1,t_2)\cdot x$. Given $\mathscr{E}=\mathrm{diag}(e_1,\ldots,e_r)\in T_e$, let $G_{\mathscr{E}}$ denote the isomorphism of $\mathscr{O}_{\ell_\infty}^{\oplus r}$ given by $(s_1,\ldots,s_r)\mapsto(e_1s_1,\ldots,e_rs_r)$. For $(E,\Phi)\in\mathfrak{M}_{r,d,n}(X,\ell_\infty)$, we define

$$(t_1,t_2,\mathscr{E})\cdot(E,\Phi)=\left((F_{t_1,t_2}^{-1})^*E,\Phi'\right),$$

where Φ' is the composite of homomorphisms

$$(F_{t_1,t_2}^{-1})^* E|_{\ell_\infty} \xrightarrow{(F_{t_1,t_2}^{-1})^*\Phi} (F_{t_1,t_2}^{-1})^* \mathscr{O}_{\ell_\infty}^{\oplus r} \xrightarrow{\phi_{t_1,t_2}} \mathscr{O}_{\ell_\infty}^{\oplus r} \xrightarrow{G_{\mathscr{E}}} \mathscr{O}_{\ell_\infty}^{\oplus r}.$$

Here ϕ_{t_1,t_2} is the homomorphism given by the action.

3.3 Torus fixed points in moduli spaces

The fixed points set $\mathfrak{M}_{r,d,n}(X,\ell_\infty)^{\tilde{T}}$ consists of $(E,\Phi) = (I_1(D_1),\Phi_1)\oplus\cdots\oplus(I_2(D_r),\Phi_r)$ such that

(1) $I_\alpha(D_\alpha)$ is a tensor product $I_\alpha \otimes \mathscr{O}_X(D_\alpha)$, where D_α is a T_t-invariant divisor which does not intersect ℓ_∞, and I_α is the ideal sheaf of a 0-dimensional subscheme Q_α contained in X_0.
(2) Φ_α is an isomorphism from $(I_\alpha)_{\ell_\infty}$ to the α^{th} factor of $\mathscr{O}_{\ell_\infty}^{\oplus r}$.
(3) I_α is fixed by the action of T_t.

The support of Q_α must be contained in $X_0^{T_t}$, the T_t fixed points set of X_0. Thus Q_α is a union of $\{Q_\alpha^v \mid v \in V(\Gamma)\}$, where Q_α^v is a subscheme supported at the T_t-fixed point $p_v \in X_0$. If we take a coordinate system (x,y) around p_v, then the ideal of Q_α^v is generated by monomials $x^i y^j$, So Q_α^v corresponds to a Young diagram Y_α^v.

Therefore the fixed point set is parametrized by $2r$-tuples

$$(\mathbf{D},\mathbf{Y}) = (D_1,\mathscr{Y}_1,\ldots,D_r,\mathscr{Y}_r),$$

where

$$D_\alpha \in \bigoplus_{e\in E(\Gamma)} \mathbb{Z}\ell_e \cong H_2(X_0;\mathbb{Z}), \quad \mathscr{Y}_\alpha = \{Y_\alpha^v \mid v \in V(\Gamma)\},$$

and each Y_α^v is a Young diagram. Let

$$|\mathscr{Y}_\alpha| = \sum_{v\in V(\Gamma)} |Y_\alpha^v|.$$

Let $d^\vee \in H_2(X;\mathbb{Z})$ be the Poincaré dual of $d \in H^2(X;\mathbb{Z})$. Then $\int_{\ell_\infty} d = 0$ implies $d^\vee \in \bigoplus_{e\in E(\Gamma)} \mathbb{Z}[\ell_e]$. The constraints are

$$\sum_\alpha D_\alpha = d^\vee, \tag{1}$$

$$\sum_{\alpha=1}^r |\mathscr{Y}_\alpha| + \sum_{\alpha<\beta} D_\alpha \cdot D_\beta = n. \tag{2}$$

Note that $2r\sum_{\alpha<\beta} D_\alpha \cdot D_\beta + (1-r)d^\vee \cdot d^\vee = -\sum_{\alpha<\beta}(D_\alpha - D_\beta)^2$, so (2) can be rewritten as

$$2r\sum_{\alpha=1}^r |\mathscr{Y}_\alpha| - \sum_{\alpha<\beta}(D_\alpha - D_\beta)^2 = 2rn + (1-r)d\cdot d = \dim_{\mathbb{C}} \mathfrak{M}_{r,d,n}(X,\ell_\infty). \tag{3}$$

4 Gauge Theory Partition Functions

We refer to Appendix B for a brief review of equivariant cohomology and integration of an equivariant cohomology class over a possibly non-compact manifold.

4.1 Equivariant parameters

For $i=1,2$, let $p_i\colon BT_t \cong \mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$ be the projection to the i^{th} factor, and let

$$\varepsilon_i = c_1(p_i^*\mathscr{O}(1)), \quad t_i = \mathrm{ch}_1(p_i^*\mathscr{O}(1)) = e^{\varepsilon_i}.$$

Then $H^*_{T_t}(\mathrm{pt};\mathbb{Q}) = H^*(BT_t;\mathbb{Q}) = \mathbb{Q}[\varepsilon_1,\varepsilon_2]$. Similarly, for $j=1,\dots,r$, let $q_j : BT_e \cong (\mathbb{P}^\infty)^r \to \mathbb{P}^\infty$ be the projection to the j^{th} factor, and let

$$a_j = c_1(q_j^*\mathscr{O}(1)), \quad e_j = \mathrm{ch}_1(q_j^*\mathscr{O}(1)) = e^{a_j}.$$

Then $H^*_{T_e}(\mathrm{pt};\mathbb{Q}) = H^*(BT_e;\mathbb{Q}) = \mathbb{Q}[a_1,\dots,a_r]$. We write $\vec{a} = (a_1,\dots,a_r)$ and $\vec{e} = (e_1,\dots,e_r) = (e^{a_1},\dots,e^{a_r})$.

4.2 Multiplicative classes of the tangent and natural bundles

Recall that a *multiplicative class* c is a characteristic class which satisfies $c(E_1 \oplus E_2) = c(E_1)c(E_2)$. Such a class is determined by a formal power series $f(x)$ satisfying $c(L) = f(c_1(L))$ for a line bundle L and $c(E) = f(x_1)\cdots f(x_r)$, where $x_1,\dots,x_r$ are Chern roots of E.

Let A, B be multiplicative classes associated to formal power series $f(x)$, $g(x)$, respectively. Then

$$\int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} A_{\tilde{T}}(T_{\mathfrak{M}}) B_{\tilde{T}}(V) \in \mathbb{Q}[[\varepsilon_1,\varepsilon_2,\vec{a}]]_{\mathbf{m}} \subset \mathbb{Q}((\varepsilon_1,\varepsilon_2,\vec{a})),$$

where $T_{\mathfrak{M}}$ is the tangent bundle of $\mathfrak{M}_{r,d,n}(X,\ell_\infty)$, V is the natural bundle over $\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ defined in Definition 29, and $\mathbb{Q}[[\varepsilon_1,\varepsilon_2,\vec{a}]]_{\mathbf{m}}$ is the localization of the ring $\mathbb{Q}[[\varepsilon_1,\varepsilon_2,\vec{a}]]$ at the maximal ideal $\mathbf{m}$ generated by $\varepsilon_1,\varepsilon_2,a_1,\dots,a_r$. If $f(x)$ and $g(x)$ are polynomials, then

$$\int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} A_{\tilde{T}}(T_{\mathfrak{M}}) B_{\tilde{T}}(V) \in \mathbb{Q}[\varepsilon_1,\varepsilon_2,\vec{a}]_{\mathbf{m}} \subset \mathbb{Q}(\varepsilon_1,\varepsilon_2,\vec{a}).$$

Let $X_0 = X \setminus \ell_\infty$. Given $d \in \{\gamma \in H^2(X;\mathbb{Z}) \mid \int_{\ell_\infty}\gamma = 0\} \cong H^2_c(X_0;\mathbb{Z})$, let $d^\vee \in H_2(X;\mathbb{Z})$ be its Poincaré dual. (Here H^*_c is the compact cohomology.) Then $d^\vee \in$

$\bigoplus_{e\in E(\Gamma)} \mathbb{Z}\ell_e \cong H_2(X_0;\mathbb{Z})$. We define

$$\begin{aligned}
&Z^{\text{inst}}_{X_0,A,B,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = \sum_{n\geq 0} \Lambda^{\dim_{\mathbb{C}} \mathfrak{M}_{r,d,n}(X,\ell_\infty)} \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} A_{\tilde{T}}(T_{\mathfrak{M}}) B_{\tilde{T}}(V) \\
&= \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} A_{\tilde{T}}(T_{\mathfrak{M}}) B_{\tilde{T}}(V) \\
&= \sum_{\sum D_\alpha = d^\vee} \Lambda^{-\sum_{\alpha<\beta}(D_\alpha - D_\beta)^2} \sum_{\vec{Y}_\alpha} \Lambda^{\sum_\alpha |\vec{Y}_\alpha|} \frac{A_{\tilde{T}}(T_{(\mathbf{D},\mathbf{Y})}\mathfrak{M}_{r,d,n}(X,\ell_\infty)) B_{\tilde{T}}(V_{(\mathbf{D},\mathbf{Y})})}{e_{\tilde{T}}(T_{(\mathbf{D},\mathbf{Y})}\mathfrak{M}_{r,d,n}(X,\ell_\infty))} \\
&= \sum_{\sum D_\alpha = d^\vee} \sum_{\vec{Y}_\alpha} \prod (\Lambda \frac{f(x_i)}{x_i}) \prod g(y_j) \in \mathbb{Q}((\varepsilon_1,\varepsilon_2,\vec{a}))[[\Lambda]],
\end{aligned}$$

where x_i are $\tilde{T}$-equivariant Chern roots of $T_{(\mathbf{D},\mathbf{Y})}\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ and y_j are $\tilde{T}$-equivariant Chern roots of $V_{(\mathbf{D},\mathbf{Y})}$. If $f(x)$, $g(x)$ are polynomials then

$$Z^{\text{inst}}_{X_0,A,B,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \in \mathbb{Q}(\varepsilon_1,\varepsilon_2,\vec{a})[[\Lambda]].$$

Sometimes we allow A and B to depend on extra parameters, then $Z^{\text{inst}}_{X,A,B,d}$ will depend on extra parameters as well.

Introduce variables $\{Q_e \mid e \in E(\Gamma)\}$. Given $d \in H^2_c(X_0;\mathbb{Z})$, define

$$Q^d = \prod_{e\in E(\Gamma)} Q_e^{\int_{\ell_e} d}.$$

We define a generating function

$$\begin{aligned}
&Z^{\text{inst}}_{X_0,A,B}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q) \stackrel{\text{def}}{=} \sum_{d\in H^2_c(X_0;\mathbb{Z})} Q^d Z^{\text{inst}}_{X_0,A,B,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \\
&= \sum_{d\in H^2_c(X_0;\mathbb{Z})} \sum_{n\geq 0} Q^d \Lambda^{(1-r)d\cdot d+2rn} \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} A_{\tilde{T}}(T_{\mathfrak{M}}) B_{\tilde{T}}(V).
\end{aligned}$$

4.3 4d pure gauge theory

Nekrasov instanton partition functions of 4d pure gauge theory are given by

$$\begin{aligned}
Z^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) &\stackrel{\text{def}}{=} \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} 1, \\
Z^{\text{inst}}_{X_0}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q) &\stackrel{\text{def}}{=} \sum_{d\in H^2_c(X_0;\mathbb{Z})} Q^d Z^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda).
\end{aligned}$$

We have

$$\begin{aligned}
Z^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) &= Z^{\text{inst}}_{X_0,A=1,B=1,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda), \\
Z^{\text{inst}}_{X_0}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q) &= Z^{\text{inst}}_{X_0,A=1,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q).
\end{aligned}$$

We define a grading on the ring $\mathbb{Q}((\varepsilon_1,\varepsilon_2,\vec{a}))[[\Lambda]]$ by

$$\deg\Lambda = \deg\varepsilon_1 = \deg\varepsilon_2 = \deg a_\alpha = 2.$$

Then $Z^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \in \mathbb{Q}((\varepsilon_1,\varepsilon_2,\vec{a}))[[\Lambda]]$ is homogeneous of degree 0.

4.4 4d gauge theory with N_f fundamental matter hypermultiplets

Let T_m be the maximal torus of $U(N_f)$. Then $H^*_{T_m}(\mathrm{pt}) \cong \mathbb{Q}[m_1, \dots, m_{N_f}]$. Let M be the fundamental representation of $U(N_f)$, and write $\vec{m} = (m_1, \dots, m_{N_f})$. Let V be the natural vector bundle as in Definition 29; it is a $\tilde{T}$-equivariant vector bundle over $\mathfrak{M}_{r,d,n}(X, \ell_\infty)$.

Nekrasov instanton partition functions of 4d gauge theory with N_f fundamental matter hypermultiplets are given by

$$\begin{aligned} Z^{\mathrm{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{m}; \Lambda) &\stackrel{\mathrm{def}}{=} \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} (c_{top})_{\tilde{T}\times T_m}(V \otimes M) \\ &= \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} \prod_{f=1}^{N_f} (E_{m_f})_{\tilde{T}}(V), \end{aligned}$$

where E_t is the multiplicative class associated to $f(x) = t + x$, so that

$$\begin{aligned} &E_t(V) = t^k + c_1(V)t^{k-1} + \cdots + c_n(V), \quad k = \mathrm{rank}_{\mathbb{C}} V, \\ &Z^{\mathrm{inst}}_{X_0}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{m}; \Lambda, Q) \stackrel{\mathrm{def}}{=} \sum_{d\in H^2_c(X_0;\mathbb{Z})} Q^d Z^{\mathrm{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{m}; \Lambda). \end{aligned}$$

Let $E_{\vec{m}} = \prod_{f=1}^{N_f} E_{m_f}$. Then

$$\begin{aligned} Z^{\mathrm{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{m}; \Lambda) &= Z^{\mathrm{inst}}_{X_0,A=1,B=E_{\vec{m}},d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda), \\ Z^{\mathrm{inst}}_{X_0}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{m}; \Lambda, Q) &= Z^{\mathrm{inst}}_{X_0,A=1,B=E_{\vec{m}}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, Q). \end{aligned}$$

4.5 4d gauge theory with one adjoint matter hypermultiplet

Nekrasov instanton partition functions of 4d gauge theory with one adjoint matter hypermultiplet are given by

$$\begin{aligned} Z^{\mathrm{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \Lambda) &\stackrel{\mathrm{def}}{=} \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} (E_m)_{\tilde{T}}(T\mathfrak{M}), \\ Z^{\mathrm{inst}}_{X_0}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \Lambda, Q) &\stackrel{\mathrm{def}}{=} \sum_{d\in H^2_c(X_0;\mathbb{Z})} Q^d Z^{\mathrm{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \Lambda). \end{aligned}$$

We have

$$\begin{aligned} Z^{\mathrm{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \Lambda) &= Z^{\mathrm{inst}}_{X_0,A=E_m,B=1,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda), \\ Z^{\mathrm{inst}}_{X_0}(\varepsilon_1, \varepsilon_2, \vec{a}, m; \Lambda, Q) &= Z^{\mathrm{inst}}_{X_0,A=E_m,B=1}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, Q). \end{aligned}$$

4.6 5d gauge theory compactified on a circle of circumference β

Let $\widehat{A}_\beta$ be the multiplicative class associated to $f_\beta(x) = \dfrac{\beta x/2}{\sinh(\beta x/2)}$. For a complex vector bundle E, $\widehat{A}_1(E) = \widehat{A}(E)$ is the $\widehat{A}$-genus of E. The index of the Dirac operator on a complex manifold M is given by $\int_M \widehat{A}(T_M)$.

The Nekrasov partition functions of 5d gauge theory compactified on a circle of circumference β are given by

$$
\begin{aligned}
Z^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\beta) &= \Lambda^{(1-r)d\cdot d}\sum_{n\geq 0}\Lambda^{2rn}\int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)}(\widehat{A}_\beta)_{\tilde{T}}(T_{\mathfrak{M}}),\\
Z^{\text{inst},(m)}_{X_0}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q,\beta) &= \sum_{d\in H^2_c(X_0;\mathbb{Z})} Q^d Z^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\beta).
\end{aligned}
$$

We have

$$
\begin{aligned}
Z^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\beta) &= Z^{\text{inst}}_{X_0,A=\widehat{A}_\beta,B=1,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda),\\
Z^{\text{inst}}_{X_0}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q,\beta) &= Z^{\text{inst}}_{X_0,A=\widehat{A}_\beta,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q).
\end{aligned}
$$

Note that $\lim_{\beta\to 0} f_\beta(x) = 1$, so the partition function of 5d gauge theory compactified on a circle of circumference β specializes to the one of 4d pure gauge theory as $\beta \to 0$, that is:

$$
\begin{aligned}
\lim_{\beta\to 0} Z^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\beta) &= Z^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda),\\
\lim_{\beta\to 0} Z^{\text{inst}}_{X_0}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q,\beta) &= Z^{\text{inst}}_{X_0}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q).
\end{aligned}
$$

4.7 Hirzebruch χ_y genus

Let

$$
(\chi_y)_{\tilde{T}}(\mathfrak{M}_{r,d,n}(X,\ell_\infty)) = \sum_{p=0}^{N}(-y)^p\sum_{q=0}^{N}(-1)^q \mathrm{ch}_{\tilde{T}} H^q(\mathfrak{M}_{r,d,n}(X,\ell_\infty),\Lambda^p T^*\mathfrak{M}_{r,d,n}(X,\ell_\infty))
$$

be the $\tilde{T}$-equivariant Hirzebruch χ_y genus, where $N = \dim_{\mathbb{C}}\mathfrak{M}_{r,d,n}(X,\ell_\infty)$. In particular,

$$
(\chi_0)_{\tilde{T}}(\mathfrak{M}_{r,d,n}(X,\ell_\infty)) = \chi_{\tilde{T}}(\mathfrak{M}_{r,d,n}(X,\ell_\infty),\mathscr{O}).
$$

By Grothendieck-Riemann-Roch,

$$
(\chi_y)_{\tilde{T}}(\mathfrak{M}_{r,d,n}(X,\ell_\infty)) = \sum_{p=0}^{N}(-y)^p\int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} \mathrm{td}_{\tilde{T}}(\mathfrak{M})\mathrm{ch}_{\tilde{T}}(\Lambda^p T^*\mathfrak{M}),
$$

where $\mathfrak{M} = \mathfrak{M}_{r,d,n}(X, \ell_\infty)$. Define

$$Z^{\text{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, y) = \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} (\chi_y)_{\tilde{T}}(\mathfrak{M}_{r,d,n}(X, \ell_\infty)),$$
$$Z^{\text{inst}}_{X_0}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, Q, y) = \sum_{d\in H^2_c(X_0;\mathbb{Z})} Q^d Z^{\text{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, y).$$

Then

$$Z^{\text{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, y) = Z^{\text{inst}}_{X_0,A=A_y,B=1,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda),$$
$$Z^{\text{inst}}_{X_0}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, Q, y) = Z^{\text{inst}}_{X_0,A=A_y,B=1}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, Q),$$

where A_y is the multiplicative class associated to

$$f_y(x) = \frac{x(1-ye^{-x})}{1-e^{-x}}.$$

In particular, $f_0(x) = \dfrac{x}{1-e^{-x}}$, $f_1(x) = x$, so $A_0(E) = \text{td}(E)$ and $A_1(E) = e(E)$.

4.8 Elliptic genus

Let $A_{y,q}$ be the multiplicative class associated to

$$y^{-1/2} x \prod_{n\geq 1} \frac{(1-yq^{n-1}e^{-x})(1-y^{-1}q^n e^x)}{(1-q^{n-1}e^{-x})(1-q^n e^x)}.$$

The $\tilde{T}$-equivariant elliptic genus of $\mathfrak{M}$ is given by

$$\chi_{\tilde{T}}(\mathfrak{M}_{r,d,n}(X, \ell_\infty), y, q) = \int_{\mathfrak{M}_{r,d,n}(X,\ell_\infty)} A_{y,q}(T_{\mathfrak{M}}).$$

Define

$$Z^{\text{inst}}_{X_0,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, y, q) \stackrel{\text{def}}{=} Z^{\text{inst}}_{X_0,A=A_{y,q},B=1,d}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$$
$$= \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} \chi_{\tilde{T}}(\mathfrak{M}_{r,d,n}(X, \ell_\infty), y, q),$$
$$Z^{\text{inst}}_{X_0}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, Q, y, q) \stackrel{\text{def}}{=} Z^{\text{inst}}_{X_0,A=A_{y,q},B=1}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, Q)$$
$$= \sum_{d\in H^2_c(X_0,\mathbb{Z})} \sum_{n\geq 0} Q^d \Lambda^{(1-r)d\cdot d+2rn} \chi_{\tilde{T}}(\mathfrak{M}_{r,d,n}(X, \ell_\infty), y, q).$$

5 The Instanton Part

In this section, we calculate the partition functions defined in Sect. 4.

5.1 The tangent bundle: adjoint representation

Let $(E,\Phi)\in\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ be a fixed point of $\tilde{T}$-action corresponding to $(\mathbf{D},\mathbf{Y})=(D_1,\mathscr{Y}_1,\dots,D_r,\mathscr{Y}_r)$. We want to compute

$$\mathrm{ch}_{\tilde{T}}T_{(E,\Phi)}\mathfrak{M}_{r,d,n}(X,\ell_\infty)=\mathrm{ch}_{\tilde{T}}\mathrm{Ext}^1_{\mathscr{O}_X}(E,E(-\ell_\infty))=-\mathrm{ch}_{\tilde{T}}\mathrm{Ext}^*_{\mathscr{O}_X}(E,E(-\ell_\infty)).$$

Recall that $E=I_1(D_1)\oplus\cdots\oplus I_r(D_r)$ (see Sect. 3.3), so

$$\begin{aligned}-\mathrm{ch}_{\tilde{T}}\mathrm{Ext}^*_{\mathscr{O}_X}(E,E(-\ell_\infty)) &= -\sum_{\alpha,\beta}\mathrm{ch}_{\tilde{T}}\mathrm{Ext}^*_{\mathscr{O}_X}(I_\alpha(D_\alpha),I_\beta(D_\beta-\ell_\infty))\\ &= -\sum_{\alpha,\beta}e^{a_\beta-a_\alpha}\mathrm{ch}_{T_t}\mathrm{Ext}^*_{\mathscr{O}_X}(I_\alpha(D_\alpha),I_\beta(D_\beta-\ell_\infty)).\end{aligned}$$

Let

$$\begin{aligned}L_{\alpha,\beta}(t_1,t_2) &= -\mathrm{ch}_{T_t}\mathrm{Ext}^*_{\mathscr{O}_X}(\mathscr{O}_X(D_\alpha),\mathscr{O}_X(D_\beta-\ell_\infty))\\ &= -\chi_{T_t}(X,\mathscr{O}_X(D_\beta-D_\alpha-\ell_\infty)),\\ M_{\alpha,\beta}(t_1,t_2) &= \mathrm{ch}_{T_t}\mathrm{Ext}^*_{\mathscr{O}_X}(\mathscr{O}_X(D_\alpha),\mathscr{O}_X(D_\beta-\ell_\infty))\\ &\quad -\mathrm{ch}_{T_t}\mathrm{Ext}^*_{\mathscr{O}_X}(I_\alpha(D_\alpha),I_\beta(D_\beta-\ell_\infty)).\end{aligned}$$

Then

$$\mathrm{ch}_{\tilde{T}}T_{(E,\Phi)}\mathfrak{M}_{r,d,n}(X,\ell_\infty)=\sum_{\alpha,\beta=1}^{r}e^{a_\beta-a_\alpha}\left(M_{\alpha,\beta}(t_1,t_2)+L_{\alpha,\beta}(t_1,t_2)\right).\qquad(4)$$

So it remains to compute $M_{\alpha,\beta}(t_1,t_2)$ and $L_{\alpha,\beta}(t_1,t_2)$.

5.1.1 $M_{\alpha,\beta}(t_1,t_2)$

Let $\chi^v_{D_\alpha}\in\mathrm{Hom}_{\mathscr{O}_X}(T_t,\mathbb{C}^*)$ be the characters of the T_t-equivariant line bundle $\mathscr{O}_X(D_\alpha)$ at the T_t fixed point $p_v\in X_0$, and let $\chi^v_1,\chi^v_2\in\mathrm{Hom}_{\mathscr{O}_X}(T_t,\mathbb{C}^*)$ be the characters of $T_{p_v}X$. Then $\chi^v_{D_\alpha},\chi^v_1,\chi^v_2$ are monomials in t_1,t_2.

Let $\mathfrak{t}_t$ be the Lie algebra of T_t. Define weights $w^v_{D_\alpha},w^v_1,w^v_2\in\mathrm{Hom}_{\mathscr{O}_X}(\mathfrak{t}_t,\mathbb{C})=\mathfrak{t}_t^\vee$ by

$$e^{w^v_{D_\alpha}}=\chi^v_{D_\alpha},\quad e^{w^v_1}=\chi^v_1,\quad e^{w^v_2}=\chi^v_2.$$

Given a partition (Young diagram) S and a box $s\in S$, let $a_S(s)$ and $l_S(s)$ be the arm-length and leg-length of s (see e.g. [24, Fig. 2]). Given two partitions S, T, let

$$M_{S,T}(t_1,t_2)=\sum_{s\in S}t_1^{-l_T(s)}t_2^{a_S(s)+1}+\sum_{t\in T}t_1^{l_S(t)+1}t_2^{-a_T(t)},\qquad(5)$$

$$N_{S,T}(\varepsilon_1,\varepsilon_2)\stackrel{\mathrm{def}}{=}M_{S,T}(e^{\varepsilon_1},e^{\varepsilon_2})=\sum_{s\in S}e^{-l_T(s)\varepsilon_1+(a_S(s)+1)\varepsilon_2}+\sum_{t\in T}e^{(l_S(t)+1)\varepsilon_1-a_T(t)\varepsilon_2}.\qquad(6)$$

The expression (5) was introduced in [13, Eq. (4.45)]. (See also [12, Lemma 3.2] and [25, Theorem 2.1].)

Proposition 51 (Vertex contribution to the tangent bundle).

$$
\begin{aligned}
M_{\alpha,\beta}(t_1,t_2) &= \sum_{v\in V(\Gamma)} \frac{\chi^v_{D_\beta}(t_1,t_2)}{\chi^v_{D_\alpha}(t_1,t_2)} M_{Y^v_\alpha,Y^v_\beta}(\chi^v_1(t_1,t_2),\chi^v_2(t_1,t_2))\\
&= \sum_{v\in V(\Gamma)} e^{w^v_{D_\beta}-w^v_{D_\alpha}} N_{Y^v_\alpha,Y^v_\beta}(w^v_1,w^v_2),
\end{aligned}
$$

where $w^v_1 = w^v_1(\varepsilon_1,\varepsilon_2)$, $w^v_2 = w^v_2(\varepsilon_1,\varepsilon_2)$, $t_1 = e^{\varepsilon_1}, t_2 = e^{\varepsilon_2}$.

Proof.

$$
\begin{aligned}
M_{\alpha,\beta}(t_1,t_2) = \mathrm{ch}_{T_t}\mathrm{Ext}^*_{\mathscr{O}_X}(\mathscr{O}_X(D_\alpha),\mathscr{O}_X(D_\beta-\ell_\infty))\\
-\mathrm{ch}_{T_t}\mathrm{Ext}^*_{\mathscr{O}_X}(I_\alpha(D_\alpha),I_\beta(D_\beta-\ell_\infty)).
\end{aligned}
\tag{7}
$$

We will compute the two terms on the right-hand side of (7) using the method in [22, Sect. 4]. For $j\geq 0$ and $1\leq \alpha,\beta\leq r$, define

$$
\mathscr{E}^j_{\alpha,\beta} \stackrel{\text{def}}{=} \mathscr{E}xt^j(I_\alpha(D_\alpha),I_\beta(D_\beta-\ell_\infty)).
$$

Then

$$
\mathrm{Ext}^*_{\mathscr{O}_X}(I_\alpha(D_\alpha),I_\beta(D_\beta-\ell_\infty)) = \sum_{i,j\geq 0}(-1)^{i+j}H^i(X,\mathscr{E}^j_{\alpha,\beta}) = \sum_{i,j\geq 0}(-1)^{i+j}\mathfrak{C}^i(X,\mathscr{E}^j_{\alpha,\beta}),
$$

where $\mathfrak{C}^i$ denote the Čech cochain groups. More explicitly, let $\{p_a \mid a=1,\dots,\chi(X)\}$ be the T_t-fixed points in X, where $\chi(X)$ is the Euler characteristic of X. Let U_a be the $\mathbb{C}^2$ coordinate chart with origin at p_a, and let $U_{ab}=U_a\cap U_b$, etc. Then

$$
\sum_{i\geq 0}(-1)^i\mathfrak{C}^i(X,\mathscr{E}^j_{\alpha\beta}) = \bigoplus_a \Gamma(U_a,\mathscr{E}^j_{\alpha\beta}) - \bigoplus_{a,b}\Gamma(U_{ab},\mathscr{E}^j_{\alpha\beta}) + \bigoplus_{a,b,c}\Gamma(U_{abc},\mathscr{E}^j_{\alpha\beta})\cdots.
$$

Note that $I_\alpha|_{U_{a_1\dots a_i}} = \mathscr{O}_X|_{U_{a_1\dots a_i}}$ unless $i=1$ and $p_{a_1}\in X_0$. Define

$$
\mathscr{O}^j_{\alpha\beta} \stackrel{\text{def}}{=} \mathscr{E}xt^j(\mathscr{O}_X(D_\alpha),\mathscr{O}_X(D_\beta-\ell_\infty)).
$$

Then

$$
\begin{aligned}
&\mathrm{Ext}^*_{\mathscr{O}_X}(\mathscr{O}_X(D_\alpha),\mathscr{O}_X(D_\beta-\ell_\infty)) - \mathrm{Ext}^*_{\mathscr{O}_X}(I_\alpha(D_\alpha),I_\alpha(D_\beta-\ell_\infty))\\
&= \bigoplus_{v\in V(\Gamma)}\sum_{j=0}^2(-1)^j\Gamma(U_v,\mathscr{O}^j_{\alpha\beta}) - \bigoplus_{v\in V(\Gamma)}\sum_{j=0}^2(-1)^j\Gamma(U_v,\mathscr{E}^j_{\alpha\beta}),
\end{aligned}
$$

where U_v is the $\mathbb{C}^2$ chart centered at p_v.

Given a partition (Young diagram) Y and a box $x\in Y$, let $a'(x)$ and $l'(x)$ be the arm-colength and leg-colength of x, respectively (see e.g. [24, Sect. 3.1]). Given a partition Y, we define

$$
Q_Y(s_1,s_2) = \sum_{x\in Y} s_1^{l'(x)} s_2^{a'(x)}.
$$

We have

$$\mathrm{ch}_{\tilde{T}} \sum_{j=0}^{2} (-1)^j \Gamma(U_v, \mathscr{O}^j_{\alpha\beta}) - \mathrm{ch}_{\tilde{T}} \sum_{j=0}^{2} (-1)^j \Gamma(U_v, \mathscr{E}^j_{\alpha\beta}) = \frac{\chi^v_{D_\beta}}{\chi^v_{D_\alpha}} \cdot M_{Y^v_\alpha, Y^v_\beta}(\chi^v_1, \chi^v_2),$$

where

$$M_{S,T}(t_1,t_2) = Q_S(t_1,t_2)t_1t_2 + Q_T(t_1^{-1},t_2^{-1}) - Q_S(t_1,t_2)Q_T(t_1^{-1},t_2^{-1})(1-t_1)(1-t_2).$$

We now compare our expression of $M_{Y^v_\alpha, Y^v_\beta}(t_1,t_2)$ with the notation in the proof of [25, Theorem 2.11]. The correspondence is

$$t_1t_2\mathrm{Hom}_{\mathscr{O}_X}(V_\alpha, W_\beta) = Q_{Y^v_\alpha}(t_1,t_2)t_1t_2\ , \quad \mathrm{Hom}_{\mathscr{O}_X}(W_\alpha, V_\beta) = Q_{Y^v_\beta}(t_1^{-1},t_2^{-1})\ ,$$
$$(t_1+t_2-1-t_1t_2)\mathrm{Hom}_{\mathscr{O}_X}(V_\alpha, V_\beta) = -Q_{Y^v_\alpha}(t_1,t_2)Q_{Y^v_\beta}(t_1^{-1},t_2^{-1})(1-t_1)(1-t_2).$$

So $M_{S,T}(t_1,t_2)$ can be rewritten as (5). □

5.1.2 $L_{\alpha,\beta}(t_1,t_2)$

Lemma 52. *If $D_\alpha = D_\beta$, then $L_{\alpha,\beta}(t_1,t_2) = 0$. In particular, $L_{\alpha,\alpha}(t_1,t_2) = 0$.*

Proof.

$$L_{\alpha,\beta}(t_1,t_2) = -\chi_{T_t}(X, \mathscr{O}_X(-\ell_\infty))$$

which can be identified with the tangent space of $\mathfrak{M}_{1,0,0}(X,\ell_\infty)$ at the trivial line bundle $\mathscr{O}_X$. By Proposition 21,

$$H^0(X, \mathscr{O}_X(-\ell_\infty)) = H^2(X, \mathscr{O}_X(-\ell_\infty)) = 0.$$

By Corollary 24 (here $r=1, d=0, n=0$), $H^1(X, \mathscr{O}_X(-\ell_\infty)) = 0$. □

By Proposition 28 and Corollary 210, we have

Lemma 53. *Suppose that $D \cdot \ell_\infty = 0$. Then*

$$H^0(X, \mathscr{O}_X(D-\ell_\infty)) = H^2(X, \mathscr{O}_X(D-\ell_\infty)) = 0,$$

and

$$\dim_{\mathbb{C}} H^1(X, \mathscr{O}_X(D-\ell_\infty)) = -\frac{1}{2}\left(D^2 + c_1(X)\cdot D\right).$$

In particular, for any D such that $D \cdot \ell_\infty = 0$ we have

$$D^2 = -\dim_{\mathbb{C}} H^1(X, \mathscr{O}_X(D-\ell_\infty)) - \dim_{\mathbb{C}} H^1(X, \mathscr{O}_X(-D-\ell_\infty)) \le 0.$$

Notation 54. Let q_0, q_1 be the two T_t fixed points on ℓ_∞. Let w (resp. u) $\in \mathrm{Hom}(T, \mathbb{C}^*)$ be the tangent weight (resp. normal weight) at q_0, i.e., the weight of the T_t-action on $T_{q_0}\ell_\infty$ (resp. $(N_{\ell_\infty/X})_{q_0}$). Then the tangent weight (resp. normal weight) at q_1, i.e., the weight of the T_t-action on $T_{q_1}\ell_\infty$ (resp. $(N_{\ell_\infty/X})_{q_1}$), must be given by $-w$ (resp. $u-kw$), where $k = \ell_\infty \cdot \ell_\infty > 0$.

Note that the normal weights at q_0 and q_1 are the restrictions of the equivariant first Chern class $(c_1)_{T_t}(\mathscr{O}_X(\ell_\infty))$ to the T_t fixed points q_0 and q_1, respectively:

$$(c_1)_{T_t}(\mathscr{O}_X(\ell_\infty))\Big|_{q_0} = u, \quad (c_1)_{T_t}(\mathscr{O}_X(\ell_\infty))\Big|_{q_1} = u - kw.$$

Proposition 55. (Edge contribution to the tangent bundle)

$$L_{\alpha,\beta}(t_1,t_2) = \sum_{v\in V(\Gamma)} \frac{-e^{w^v_{D_\beta}-w^v_{D_\alpha}}}{(1-e^{-w^v_1})(1-e^{-w^v_2})} + \left(\frac{1}{(1-e^{-w})(1-e^{u})} + \frac{1}{(1-e^{w})(1-e^{u-kw})}\right).$$

Proof. Recall that $L_{\alpha,\beta}(t_1,t_2) = -\chi_{T_t}(X, \mathscr{O}_X(D_\beta - D_\alpha - \ell_\infty))$. By Grothendieck-Riemann-Roch,

$$\begin{aligned}
\chi_{T_t}(X, \mathscr{O}_X(D_\beta - D_\alpha - \ell_\infty)) &= \int_X \mathrm{td}_{T_t}(T_X)\mathrm{ch}_{T_t}(\mathscr{O}_X(D_\beta - D_\alpha - \ell_\infty)) \\
&= \sum_{v\in V(\Gamma)} \frac{e^{w^v_{D_\beta}-w^v_{D_\alpha}}}{(1-e^{-w^v_1})(1-e^{-w^v_2})} + \left(\frac{e^{-u}}{(1-e^{-w})(1-e^{-u})} + \frac{e^{-u+kw}}{(1-e^{w})(1-e^{-u+kw})}\right).
\end{aligned}$$

□

Example 56. Let $X = \mathbb{F}_k$, ℓ_0, ℓ_∞ be as in Example 26, with the following T_t-action:

$T_{p_1}\ell_0$	$(N_{\ell_0/X})_{p_1}$	$T_{p_2}\ell_0$	$(N_{\ell_0/X})_{p_2}$	$T_{p_3}\ell_\infty$	$(N_{\ell_\infty/X})_{p_3}$	$T_{p_4}\ell_\infty$	$(N_{\ell_\infty/X})_{p_4}$
ε_1	ε_2	$-\varepsilon_1$	$\varepsilon_2 + k\varepsilon_1$	$-\varepsilon_1$	$-\varepsilon_2 - k\varepsilon_1$	ε_1	$-\varepsilon_2$

Hence, here $w = \varepsilon_1$ and $u = -\varepsilon_2$, and we have $D_\alpha = d_\alpha \ell_0$ for some $d_\alpha \in \mathbb{Z}$. Then

$$\begin{aligned}
L_{\alpha,\beta}(t_1,t_2) &= \frac{-e^{(d_\beta-d_\alpha)\varepsilon_2}}{(1-e^{-\varepsilon_1})(1-e^{-\varepsilon_2})} + \frac{-e^{(d_\beta-d_\alpha)(\varepsilon_2+k\varepsilon_1)}}{(1-e^{\varepsilon_1})(1-e^{-\varepsilon_2-k\varepsilon_1})} \\
&\quad + \frac{1}{(1-e^{-\varepsilon_1})(1-e^{-\varepsilon_2})} + \frac{1}{(1-e^{\varepsilon_1})(1-e^{-\varepsilon_2-k\varepsilon_1})} \\
&= \frac{1-t_2^{d_\beta-d_\alpha}}{(1-t_1^{-1})(1-t_2^{-1})} + \frac{1-(t_1^k t_2)^{d_\beta-d_\alpha}}{(1-t_1)(1-t_1^{-k}t_2^{-1})},
\end{aligned}$$

and we have

$$L_{\alpha,\beta}(t_1,t_2) = \begin{cases} \displaystyle\sum_{j=0}^{d_\alpha-d_\beta-1}\sum_{i=0}^{kj} t_1^{-i}t_2^{-j} & \text{if } d_\alpha > d_\beta, \\ \displaystyle\sum_{j=1}^{d_\beta-d_\alpha}\sum_{i=1}^{kj-1} t_1^{i}t_2^{j} & \text{if } d_\alpha < d_\beta, \\ 0 & \text{if } d_\alpha = d_\beta. \end{cases}$$

5.2 The natural bundle: fundamental representation

Let $(E,\Phi)\in\mathfrak{M}_{r,d,n}(X,\ell_\infty)$ be a fixed point of the $\tilde{T}$-action corresponding to $(\mathbf{D},\mathbf{Y})=(D_1,\mathscr{Y}_1,\dots,D_r,\mathscr{Y}_r)$. We want to compute

$$\mathrm{ch}_{\tilde{T}}V_{(E,\Phi)}=\mathrm{ch}_{\tilde{T}}H^1(X,E(-\ell_\infty))=-\chi_{\tilde{T}}(X,E(-\ell_\infty)).$$

Recall that $E=I_1(D_1)\oplus\cdots\oplus I_r(D_r)$ (see Sect. 3.3), so

$$-\chi_{\tilde{T}}(X,E(-\ell_\infty))=-\sum_\beta\chi_{\tilde{T}}(X,I_\beta(D_\beta-\ell_\infty))=-\sum_\beta e^{a_\beta}\chi_{T_t}(X,I_\beta(D_\beta-\ell_\infty)).$$

Let

$$\begin{aligned}L_\beta(t_1,t_2)&=-\chi_{T_t}(X,\mathscr{O}_X(D_\beta-\ell_\infty)),\\ M_\beta(t_1,t_2)&=\chi_{T_t}(X,\mathscr{O}_X(D_\beta-\ell_\infty))-\chi_{T_t}(X,I_\beta(D_\beta-\ell_\infty)).\end{aligned}$$

Then

$$\mathrm{ch}_{\tilde{T}}V_{(E,\Phi)}=\sum_{\beta=1}^r e^{a_\beta}\left(M_\beta(t_1,t_2)+L_\beta(t_1,t_2)\right). \tag{8}$$

So it remains to compute $M_\beta(t_1,t_2)$ and $L_\beta(t_1,t_2)$.

Let $w^v_{D_\alpha},w^v_1,w^v_2$ be defined as in Sect. 5.1.1. Given a partition S, let

$$M_S(t_1,t_2)=\sum_{s\in S}t_1^{-l'(s)}t_2^{-a'(s)}, \tag{9}$$

$$N_S(\varepsilon_1,\varepsilon_2)\stackrel{\text{def}}{=}M_S(e^{\varepsilon_1},e^{\varepsilon_2})=\sum_{s\in S}e^{-l'(s)\varepsilon_1-a'(s)\varepsilon_2}. \tag{10}$$

Proposition 57. (Vertex contribution to the natural bundle).

$$M_\beta(t_1,t_2)=\sum_{v\in V(\Gamma)}\chi^v_{D_\beta}(t_1,t_2)M_{Y^v_\beta}(\chi^v_1(t_1,t_2),\chi^v_2(t_1,t_2))=\sum_{v\in V(\Gamma)}e^{w^v_{D_\beta}}N_{Y^v_\beta}(w^v_1,w^v_2).$$

Proof. Let $D_\alpha=0$ in Proposition 51. □

Proposition 58. (Edge contribution to the natural bundle).

$$L_\beta(t_1,t_2)=\sum_{v\in V(\Gamma)}\frac{-e^{w^v_{D_\beta}}}{(1-e^{-w^v_1})(1-e^{-w^v_2})}+\left(\frac{1}{(1-e^{-w})(1-e^{u})}+\frac{1}{(1-e^{w})(1-e^{u-kw})}\right).$$

Proof. Let $D_\alpha=0$ in Proposition 55. □

Example 59. Let $X=\mathbb{F}_k$, ℓ_0, ℓ_∞ be as in Example 26, with the T_t-action as in Example 56. Then

$$L_\beta(t_1,t_2)=\begin{cases}\displaystyle\sum_{j=0}^{-d_\beta-1}\sum_{i=0}^{kj}t_1^{-i}t_2^{-j} & \text{if } d_\beta<0,\\ \displaystyle\sum_{j=1}^{d_\beta}\sum_{i=1}^{kj-1}t_1^{i}t_2^{j} & \text{if } d_\beta>0,\\ 0 & \text{if } d_\beta=0.\end{cases}$$

5.3 Formula for instanton partition functions

Given $\vec{Y} = (Y_1, \dots, Y_r)$, where each Y_α is a Young diagram, and a multiplicative class A associated to $f(x)$, define

$$m^{\vec{Y}}_{A,\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a}) \stackrel{\text{def}}{=} \prod_{s\in Y_\alpha} f(a_\beta - a_\alpha - l_{Y_\beta}(s)\varepsilon_1 + (a_{Y_\alpha}(s)+1)\varepsilon_2) \cdot \prod_{t\in Y_\beta} f(a_\beta - a_\alpha + (l_{Y_\alpha}(t)+1)\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2), \quad (11)$$

$$m^{\vec{Y}}_{A,\beta}(\varepsilon_1,\varepsilon_2,\vec{a}) \stackrel{\text{def}}{=} \prod_{t\in Y^\beta} f(a_\beta - l'_{Y_\beta}(t)\varepsilon_1 - a'_{Y_\beta}(t)\varepsilon_2). \quad (12)$$

In particular,

$$m^{\vec{Y}}_{c_{top},\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a}) = \prod_{s\in Y_\alpha} (a_\beta - a_\alpha - l_{Y_\beta}(s)\varepsilon_1 + (a_{Y_\alpha}(s)+1)\varepsilon_2) \cdot \prod_{t\in Y_\beta} (a_\beta - a_\alpha + (l_{Y_\alpha}(t)+1)\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2). \quad (13)$$

Let $Z^{\text{inst}}_{\mathbb{C}^2,A,B} = Z^{\text{inst}}_{\mathbb{C}^2,A,B,0}$, and let $|\vec{Y}| = \sum_{\alpha=1}^r |Y_\alpha|$. In this case, all $D_\beta = 0$, so the leg contribution is zero (see Lemma 52, Lemma 53):

$$L_{\alpha,\beta} = 0, \quad L_\beta = 0.$$

By (4), Proposition 51, (8), Proposition 57, and above definitions (11), (12), (13), we have:

Proposition 510. (Instanton partition functions for $\mathbb{C}^2$)

$$Z^{\text{inst}}_{\mathbb{C}^2,A,B}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = \sum_{\vec{Y}} \Lambda^{2r|\vec{Y}|} \prod_{\alpha,\beta} \frac{m^{\vec{Y}}_{A,\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a})}{m^{\vec{Y}}_{c_{\text{top}},\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a})} \prod_{\beta=1}^r m^{\vec{Y}}_{B,\beta}(\varepsilon_1,\varepsilon_2,\vec{a}).$$

Given $\vec{D} = (D_1, \dots, D_r)$, where each $D_\alpha \in \oplus_{e\in E(\Gamma)} \mathbb{Z}\ell_e \cong H_2(X_0;\mathbb{Z})$, and a multiplicative class A, define

$$l^{\vec{D}}_{A,\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a}) = A_{\tilde{T}} H^1(X, \mathscr{O}_X(D_\beta - D_\alpha - \ell_\infty)). \quad (14)$$

Then $l^{\vec{D}}_{A,\alpha,\beta}(\varepsilon_1,\varepsilon_2;\vec{a}) = 1$ if $D_\alpha = D_\beta$. In particular, $l^{\vec{D}}_{A,\alpha,\alpha}(\varepsilon_1,\varepsilon_2;\vec{a}) = 1$. Let

$$l^{\vec{D}}_{A,\beta}(\varepsilon_1,\varepsilon_2,\vec{a}) = A_{\tilde{T}} H^1(X, \mathscr{O}_X(D_\beta - \ell_\infty)). \quad (15)$$

Let

$$|\vec{D}|^2 = -\frac{1}{2}\sum_{\alpha\neq\beta} (D_\alpha - D_\beta)^2 \geq 0.$$

By Eqs. (4), (8) and Propositions 51, 55, 57, 58, 510, we have the following analogue of the “master formula” in [29, Sect. 6].

Proposition 511. (Master formula for instanton partition functions)

$$Z^{\text{inst}}_{X_0,A,B,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = \sum_{\sum D_\alpha = d} \Lambda^{|\vec{D}|^2} \prod_{\alpha\neq\beta} \frac{l^{\vec{D}}_{A,\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a})}{l^{\vec{D}}_{c_{top},\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a})} \prod_{\beta=1}^{r} l^{\vec{D}}_{B,\beta}(\varepsilon_1,\varepsilon_2,\vec{a}) \cdot \prod_{v\in V(\Gamma)} Z^{\text{inst}}_{\mathbb{C}^2,A,B}(w_1^v,w_2^v,\vec{a}+\vec{D}^v;\Lambda),$$

where $\vec{D}^v = (w^v_{D_1},\ldots,w^v_{D_r})$.

$$Z^{\text{inst}}_{X_0,A,B}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,Q) = \sum_{D_\alpha\in H^2_c(X;\mathbb{Z})} Q^{\sum_\alpha D_\alpha} \Lambda^{|\vec{D}|^2} \prod_{\alpha\neq\beta} \frac{l^{\vec{D}}_{A,\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a})}{l^{\vec{D}}_{c_{top},\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a})} \cdot \prod_{\beta=1}^{r} l^{\vec{D}}_{B,\beta}(\varepsilon_1,\varepsilon_2,\vec{a}) \cdot \prod_{v\in V(\Gamma)} Z^{\text{inst}}_{\mathbb{C}^2,A,B}(w_1^v,w_2^v,\vec{a}+\vec{D}^v;\Lambda).$$

In the rank 1 case, $Z^{\text{inst}}_{X_0,A,B}$ does not depend on $\vec{a}$.

Corollary 512. (Rank 1, $B = 1$ case)

$$Z^{\text{inst}}_{X_0,A,B=1,d}(\varepsilon_1,\varepsilon_2;\Lambda) = \prod_{v\in V(\Gamma)} Z^{\text{inst}}_{\mathbb{C}^2,A,B=1}(w_1^v,w_2^v;\Lambda),$$
$$Z^{\text{inst}}_{X_0,A,B=1}(\varepsilon_1,\varepsilon_2;\Lambda,Q) = \sum_{d\in H^2_c(X;\mathbb{Z})} Q^d \prod_{v\in V(\Gamma)} Z^{\text{inst}}_{\mathbb{C}^2,A,B=1}(w_1^v,w_2^v;\Lambda).$$

Note that Corollary 512 is applicable to the following cases: 4d pure gauge theory (Sect. 4.3), 4d gauge theory with one adjoint matter hypermultiplet (Sect. 4.5), 5d gauge theory compactified on a circle (Sect. 4.6), Hirzebruch genus (Sect. 4.7), elliptic genus (Sect. 4.8).

5.4 Nekrasov conjecture for $\mathbb{C}^2$: instanton part

Definition 513. (Instanton prepotential for $\mathbb{C}^2$). *Define*

$$\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \overset{\text{def}}{=} -\varepsilon_1\varepsilon_2 \log Z^{\text{inst}}_{\mathbb{C}^2,A,B}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda).$$

There are several versions of Nekrasov conjecture which correspond to the following special cases:

(1) 4d pure gauge theory (see Sect. 4.3):

$$\mathscr{F}^{\text{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = \mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A=1,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda).$$

(2) 4d gauge theory with N_f fundamental matter hypermultiplets (see Sect. 4.4):

$$\mathscr{F}^{\text{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{m};\Lambda) = \mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A=1,B=E_{\vec{m}}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda).$$

(3) 4d gauge theory with one adjoint matter hypermultiplet (see Sect. 4.5):

$$\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},m;\Lambda)=\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A=E_m,B=1}(\varepsilon_1,\varepsilon_2,\vec{a},m;\Lambda).$$

(4) 5d gauge theory compactified on a circle of circumference β (see Sect. 4.6):

$$\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\beta)=\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A=\widehat{A}_\beta,B=1}(\varepsilon_1,\varepsilon_2,\vec{a},m;\Lambda).$$

The above definitions of $\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}$ are the same as those in [30]; the definition in case (1) above is the negative of the definition in [24; 25].

In Theorem 514 below, we summarize the various versions of the Nekrasov conjecture proved by Nakajima-Yoshioka [25; 26], Nekrasov-Okounkov [30], Göttsche-Nakajima-Yoshioka [18]. See also Braverman [3] and Braverman-Etingof [4], who consider the case 4d pure gauge theory with arbitrary gauge groups. We refer to Appendix C for the definitions of the corresponding versions of the Seiberg-Witten prepotential in Theorem 514.

Theorem 514. (Nekrasov conjecture for $\mathbb{C}^2$: instanton part)

(1) 4d pure gauge theory [30; 25; 4]:
  (a) $\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$.
  (b) $\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)=\mathscr{F}^{\mathrm{inst}}_0(\vec{a},\Lambda)$, *where* $\mathscr{F}^{\mathrm{inst}}_0(\vec{a},\Lambda)$ *is the instanton part of the Seiberg-Witten prepotential of 4d pure gauge theory.*

(2) 4d gauge theory with N_f fundamental matter hypermultiplets [30]:
  (a) $\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{m};\Lambda)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$.
  (b) $\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{m};\Lambda)=\mathscr{F}^{\mathrm{inst}}_0(\vec{a},\vec{m},\Lambda)$, *where* $\mathscr{F}^{\mathrm{inst}}_0(\vec{a},\vec{m},\Lambda)$ *is the instanton part of the Seiberg-Witten prepotential of 4d gauge theory with* N_f *fundamental matter hypermultiplets.*

(3) 4d gauge theory with one adjoint matter hypermultiplet [30]:
  (a) $\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},m;\Lambda)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$.
  (b) $\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},m;\Lambda)=\mathscr{F}^{\mathrm{inst}}_0(\vec{a},m,\Lambda)$, *where* $\mathscr{F}^{\mathrm{inst}}_0(\vec{a},m,\Lambda)$ *is the instanton part of the Seiberg-Witten prepotential of 4d gauge theory with one adjoint matter hypermultiplet.*

(4) 5d gauge theory compactified on a circle of circumference β [30; 26; 18]:
  (a) $\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\beta)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$.
  (b) $\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\beta)=\mathscr{F}^{\mathrm{inst}}_0(\vec{a},\Lambda,\beta)$, *where* $\mathscr{F}^{\mathrm{inst}}_0(\vec{a},\Lambda,\beta)$ *is the instanton part of the Seiberg-Witten prepotential of 5d gauge theory compactified on a circle of circumference* β.

5.5 Nekrasov conjecture for toric surfaces: instanton part

The expression of the master formula (Proposition 511) contains two parts.

- Leg contribution:

$$\prod_{\alpha\neq\beta}\frac{l^{\vec{\boldsymbol{\beta}}}_{A,\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a})}{l^{\vec{\boldsymbol{\beta}}}_{c_{top},\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{a})}\prod_{\beta=1}^{r}l^{\vec{\boldsymbol{\beta}}}_{\beta}(\varepsilon_1,\varepsilon_2,\vec{a})$$

is analytic in ϵ_1,ϵ_2 near $\epsilon_1,\epsilon_2=0$, and

$$\lim_{\epsilon_1,\epsilon_2\to 0}\prod_{\alpha\neq\beta}\frac{l^{\vec{\mathscr{D}}}_{A,\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a})}{l^{\vec{\mathscr{D}}}_{c_{top},\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a})}\prod_{\beta=1}^{r} l^{\vec{\mathscr{D}}}_{\beta}(\epsilon_1,\epsilon_2,\vec{a})$$
$$=\prod_{\alpha\neq\beta}\left(\frac{f(a_\beta-a_\alpha)}{a_\beta-a_\alpha}\right)^{-\frac{1}{2}((D_\beta-D_\alpha)^2+c_1(X)(D_\beta-D_\alpha))}\prod_{\beta=1}^{r} g(a_\beta)^{-\frac{1}{2}(D_\beta^2+c_1(X)\cdot D_\beta)}.$$

- Vertex contribution:

$$\prod_{v\in V(\Gamma)} Z^{\text{inst}}_{\mathbb{C}^2,A,B}(w_1^v,w_2^v,\vec{a}+\vec{\mathscr{D}}^v;\Lambda)=\exp\left(-\sum_{v\in V(\Gamma)}\frac{\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(w_1^v,w_2^v,\vec{a}+\vec{\mathscr{D}}^v;\Lambda)}{w_1^v w_2^v}\right).$$

Definition 515. *Given $\vec{\mathscr{D}}=(D_1,\dots,D_r)$, where each $D_\alpha\in\bigoplus_{e\in E(\Gamma)}\mathbb{Z}\ell_e=H_2(X_0;\mathbb{Z})$, define*

$$\mathscr{F}^{\text{inst}}_{X_0,A,B,\vec{\mathscr{D}}}(\epsilon_1,\epsilon_2,\vec{a};\Lambda)=\sum_{v\in V(\Gamma)}\frac{\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(w_1^v,w_2^v,\vec{a}+\vec{\mathscr{D}}^v;\Lambda)}{w_1^v w_2^v}$$
$$+\frac{\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(w,u,\vec{a};\Lambda)}{wu}+\frac{\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(-w,u-kw,\vec{a};\Lambda)}{-w(u-kw)}.$$

Lemma 516. *Assume that $\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(\epsilon_1,\epsilon_2,\vec{a};\Lambda)$ is analytic in ϵ_1,ϵ_2 near $\epsilon_1=\epsilon_2=0$. Then $\mathscr{F}^{\text{inst}}_{X_0,A,B,\vec{\mathscr{D}}}(\epsilon_1,\epsilon_2,\vec{a};\Lambda)$ is analytic in ϵ_1,ϵ_2 near $\epsilon_1=\epsilon_2=0$ for all $\vec{\mathscr{D}}$.*

Proof. $\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(\epsilon_1,\epsilon_2,\vec{a};\Lambda)$ is symmetric in ϵ_1,ϵ_2, so it is a function of $s_1=\epsilon_1+\epsilon_2$, $s_2=\epsilon_1\epsilon_2$, $\vec{a}$, and Λ. For fixed $\vec{a},\Lambda$, let

$$g_{A,B}(s_1,s_2,d_1,\dots,d_r,\vec{a};\Lambda)=\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(\epsilon_1,\epsilon_2,a_1+d_1,\dots,a_r+d_r;\Lambda).$$

Then $g_{A,B}(s_1,s_2,d_1,\dots,d_r,\vec{a};\Lambda)$ is analytic in $s_1,s_2,d_1,\dots,d_r$ near $s_1=s_2=d_1=\cdots=d_r=0$, so it has a power series expansion. Let L_α be the T_t-equivariant line bundle $\mathscr{O}_X(D_\alpha)$. Then

$$I_{A,B,\vec{\mathscr{D}}}(\epsilon_1,\epsilon_2,\vec{a};\Lambda)\stackrel{\text{def}}{=}\int_X g_{A,B}((c_1)_{T_t}(T_X),(c_2)_{T_t}(T_X),(c_1)_{T_t}(L_1),\dots,(c_1)_{T_t}(L_r),\vec{a};\Lambda) \tag{16}$$

is analytic in ϵ_1,ϵ_2 near $\epsilon_1=\epsilon_2=0$, and

$$\lim_{\epsilon_1,\epsilon_2\to 0} I_{A,B,\vec{\mathscr{D}}}(\epsilon_1,\epsilon_2;\vec{a},\Lambda)=\int_X g_{A,B}\left(c_1(T_X),c_2(T_X),c_1(L_1),\dots,c_1(L_r),\vec{a};\Lambda\right). \tag{17}$$

The integral $I_{A,B,\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda)$ is computed by the localization formula as follows:

$$\begin{aligned} I_{A,B,\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda) &= \sum_{v\in V(\Gamma)} \frac{\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(w_1^v,w_2^v,\vec{\mathscr{a}}+\vec{\mathscr{D}}^v;\Lambda)}{w_1^v w_2^v} \\ &\quad + \frac{\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(w,u,\vec{\mathscr{a}};\Lambda)}{wu} + \frac{\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(-w,u-kw,\vec{\mathscr{a}};\Lambda)}{-w(u-kw)} \\ &= \mathscr{F}^{\text{inst}}_{X_0,A,B,\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda). \end{aligned}$$

□

Definition 517. *Assume that* $\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$. *Define*

$$F_{X_0,A,B,\vec{\mathscr{D}}}(\vec{\mathscr{a}};\Lambda) \stackrel{\text{def}}{=} \lim_{\varepsilon_1,\varepsilon_2\to 0} \mathscr{F}^{\text{inst}}_{X_0,A,B,\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda),$$

where the limit exists by Lemma 516.

Lemma 518. *If* $\mathscr{F}^{\text{inst}}_{\mathbb{C}^2,A,B}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$*, then*

$$\log\Big(Z^{\text{inst}}_{X_0,A,B,d}(\varepsilon_1,\varepsilon_2;\vec{\mathscr{a}};\Lambda)Z^{\text{inst}}_{\mathbb{C}^2,A,B}(w,u,\vec{\mathscr{a}};\Lambda)Z^{\text{inst}}_{\mathbb{C}^2,A,B}(-w,u-kw,\vec{\mathscr{a}};\Lambda)\Big)$$

is analytic in $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$.

Proof. We have

$$\begin{aligned} &Z^{\text{inst}}_{X_0,A,B,d}(\varepsilon_1,\varepsilon_2;\vec{\mathscr{a}};\Lambda)Z^{\text{inst}}_{\mathbb{C}^2,A,B}(w,u,\vec{\mathscr{a}};\Lambda)Z^{\text{inst}}_{\mathbb{C}^2,A,B}(-w,u-kw,\vec{\mathscr{a}};\Lambda) \\ &\quad = \sum_{\sum D_\alpha=d} \Lambda^{|\vec{\mathscr{D}}|^2} h_{\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda), \end{aligned}$$

where

$$h_{\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda) = \prod_{\alpha\neq\beta} \frac{l^{\vec{\mathscr{D}}}_{A,\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}})}{l^{\vec{\mathscr{D}}}_{c_{top},\alpha,\beta}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}})} \prod_{\beta=1}^{r} l^{\vec{\mathscr{D}}}_{\beta}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}}) \exp\Big(-\mathscr{F}^{\text{inst}}_{X,A,B,\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda)\Big).$$

$h_{\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$, and

$$\begin{aligned} \lim_{\varepsilon_1,\varepsilon_2\to 0} h_{\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda) &= \prod_{\alpha\neq\beta}\left(\frac{f(a_\beta-a_\alpha)}{a_\beta-a_\alpha}\right)^{-\frac{1}{2}((D_\beta-D_\alpha)^2+c_1(X)(D_\beta-D_\alpha))} \\ &\quad \cdot \prod_{\beta=1}^{r} g(a_\beta)^{-\frac{1}{2}(D_\beta^2+c_1(X)\cdot D_\beta)} \exp(-F_{X_0,A,B,\vec{\mathscr{D}}}(\vec{\mathscr{a}};\Lambda)). \end{aligned}$$

Therefore

$$\log\left(\sum_{\sum D_\alpha=d}\Lambda^{|\vec{\mathscr{D}}|^2} h_{\vec{\mathscr{D}}}(\varepsilon_1,\varepsilon_2,\vec{\mathscr{a}};\Lambda)\right)$$

is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$. □

By Lemma 518, the pole of $\log Z^{\mathrm{inst}}_{X_0,A,B,d}$ along $\varepsilon_1=\varepsilon_2=0$ is the same as that of

$$-\log Z^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(w,u,\vec{a};\Lambda)-\log Z^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(-w,u-kw,\vec{a};\Lambda)$$
$$=\frac{\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(w,u,\vec{a};\Lambda)}{wu}+\frac{\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(-w,u-kw,\vec{a};\Lambda)}{-w(u-kw)}.$$

Definition 519. (Logarithm of the instanton part) *Define*

$$\mathscr{F}^{\mathrm{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)=-u(u-kw)\log Z^{\mathrm{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda).$$

Theorem 520. *If $\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$, then*

(a) *$\mathscr{F}^{\mathrm{inst}}_{X_0,A,B,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$,*
(b) $\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{inst}}_{X_0,A,B,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)=k\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda).$

Proof. Let

$$g_k(w,u,\vec{a};\Lambda)=-u(u-kw)\left(\frac{\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(w,u,\vec{a};\Lambda)}{wu}+\frac{\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(-w,u-kw,\vec{a};\Lambda)}{-w(u-kw)}\right).$$

Note that (w,u) and $(\varepsilon_1,\varepsilon_2)$ are related by a coordinate transformation in $GL(2,\mathbb{Z})$. By Lemma 518, it suffices to show that

(a)' $g_k(w,u,\vec{a};\Lambda)$ is analytic in w, u near $w=u=0$,
(b)' $\lim_{w,u\to 0}g_k(w,u,\vec{a};\Lambda)=k\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(\varepsilon_1,\varepsilon_2,\vec{a},\Lambda)$.

We have

$$\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2}(-w,u-kw,\vec{a};\Lambda)-\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(w,u,\vec{a};\Lambda)=wH_k(w,u,\vec{a};\Lambda),$$

where $H_k(w,u,\vec{a};\Lambda)$ is analytic in w,u near $w=u=0$. So

$$g_k(w,u,\vec{a};\Lambda)=k\mathscr{F}^{\mathrm{inst}}_{\mathbb{C}^2,A,B}(w,u,\vec{a},\Lambda)+uH_k(w,u,\vec{a};\Lambda). \tag{18}$$

(a)' and (b)' are immediate consequences of (18). □

Theorem 514 and Theorem 520 imply:

Theorem 521. (Nekrasov conjecture for toric surfaces: instanton part)

(1) 4d pure gauge theory:
 (a) *$\mathscr{F}^{\mathrm{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$.*
 (b) *$\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)=k\mathscr{F}^{\mathrm{inst}}_0(\vec{a},\Lambda)$, where $\mathscr{F}^{\mathrm{inst}}_0(\vec{a},\Lambda)$ is the instanton part of the Seiberg-Witten prepotential of 4d pure gauge theory.*
(2) 4d gauge theory with N_f fundamental matter hypermultiplets:
 (a) *$\mathscr{F}^{\mathrm{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{m};\Lambda)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$.*

(b) $\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathscr{F}^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{m};\Lambda) = k\mathscr{F}^{\text{inst}}_0(\vec{a},\vec{m},\Lambda)$, *where* $\mathscr{F}^{\text{inst}}_0(\vec{a},\vec{m},\Lambda)$ *is the instanton part of the Seiberg-Witten prepotential of 4d gauge theory with* N_f *fundamental matter hypermultiplets.*

(3) 4d gauge theory with one adjoint matter hypermultiplet:

(a) $\mathscr{F}^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a},m;\Lambda)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$.

(b) $\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathscr{F}^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a},m;\Lambda) = k\mathscr{F}^{\text{inst}}_0(\vec{a},m,\Lambda)$, *where* $\mathscr{F}^{\text{inst}}_0(\vec{a},m,\Lambda)$ *is the instanton part of the Seiberg-Witten prepotential of 4d gauge theory with one adjoint matter hypermultiplet.*

(4) 5d gauge theory compactified on a circle of circumference β:

(a) $\mathscr{F}^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\beta)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$.

(b) $\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathscr{F}^{\text{inst}}_{X_0,d}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda,\beta) = k\mathscr{F}^{\text{inst}}_0(\vec{a},\Lambda,\beta)$, *where* $\mathscr{F}^{\text{inst}}_0(\vec{a},\Lambda,\beta)$ *is the instanton part of the Seiberg-Witten prepotential of 5d gauge theory compactified on a circle of circumference* β.

6 The Perturbative Part

In this section we prove the perturbative parts of the conjecture, of which instanton counterparts were proved in Theorem 521. The perturbative part comes from the difference between framed instantons on the compact toric surface X and unframed instantons on the noncompact toric surface X_0, so we must consider the virtual tangent and natural bundles of the moduli space of unframed instantons on X_0. Evaluating the required multiplicative classes at such bundles gives rise to infinite products which need to be regularised. Following [30] we use zeta-function regularization (Definition 63).

6.1 The virtual tangent bundle of $\mathfrak{M}_{r,d,n}(X_0)$

Given $(E,\Phi)\in\mathfrak{M}_{r,d,n}(X,\ell_\infty)$, we may look at $E|_{X_0}$ as representing a point in the moduli space $\mathfrak{M}_{r,d,n}(X_0)$ of unframed instantons on the noncompact surface X_0. We have

$$\begin{aligned}
&\mathrm{ch}_{\tilde{T}} T^{\text{vir}}_{E|_{X_0}} \mathfrak{M}_{r,d,n}(X_0) = -\mathrm{ch}_{\tilde{T}} \mathrm{Ext}^*_{\mathscr{O}_{X_0}}(E|_{X_0},E|_{X_0}) \\
&= \sum_{\alpha,\beta} e^{a_\beta-a_\alpha} \sum_{v\in V(\Gamma)} e^{w^v_{D_\beta}-w^v_{D_\alpha}} \left(N_{Y^v_\alpha,Y^v_\beta}(w^v_1,w^v_2) - \frac{1}{(1-e^{-w^v_1})(1-e^{-w^v_2})}\right) \\
&= \sum_{v\in\Gamma}\sum_{\alpha,\beta} e^{(a_\beta+w^v_{D_\beta})-(a_\alpha+w^v_{D_\alpha})} \left(N_{Y^v_\alpha,Y^v_\beta}(w^v_1,w^v_2) - \frac{1}{(1-e^{-w^v_1})(1-e^{-w^v_2})}\right).
\end{aligned}$$

The perturbative part of the $\tilde{T}$-equivariant Chern character of the tangent bundle is given by

$$\begin{aligned}
\mathrm{ch}_{\tilde{T}} T^{\mathrm{pert}}_{E|_{X_0}} &\stackrel{\mathrm{def}}{=} \mathrm{ch}_{\tilde{T}} T^{\mathrm{vir}}_{E|_{X_0}} \mathfrak{M}_{r,d,n}(X_0) - \mathrm{ch}_{\tilde{T}} T_{(E,\Phi)} \mathfrak{M}_{r,d,n}(X,\ell_\infty) \\
&= -\sum_{\alpha,\beta} e^{a_\beta - a_\alpha} \left(\frac{1}{(1-e^{-w})(1-e^{u})} + \frac{1}{(1-e^{w})(1-e^{u-kw})} \right) \\
&= \frac{-\sum_{\alpha,\beta} e^{a_\beta - a_\alpha}}{(1-e^{u})(1-e^{u-kw})} \left(1 + \sum_{j=1}^{k-1} e^{u-jw} \right).
\end{aligned}$$

Example 61. $X = \mathbb{P}^2, X_0 = \mathbb{C}^2$.

$$\begin{aligned}
\mathrm{ch}_{\tilde{T}} T^{\mathrm{pert}}_{(E,\Phi)} &= -\sum_{\alpha,\beta} e^{a_\beta - a_\alpha} \left(\frac{1}{(1-e^{\varepsilon_2-\varepsilon_1})(1-e^{-\varepsilon_2})} + \frac{1}{(1-e^{\varepsilon_1-\varepsilon_2})(1-e^{-\varepsilon_1})} \right) \\
&= \frac{-\sum_{\alpha,\beta} e^{a_\beta - a_\alpha}}{(1-e^{-\varepsilon_1})(1-e^{-\varepsilon_2})}.
\end{aligned}$$

Let A be a multiplicative class defined by a formal power series $f(x)$. Formally, evaluating A on the tangent bundle produces the following perturbative part:

$$A_{\tilde{T}}(T^{\mathrm{pert}}_{(E,\Phi)}) = \frac{1}{\prod_{i,j=0}^{\infty} f(a_\beta - a_\alpha - iw + ju) \prod_{i,j=0}^{\infty} f(a_\beta - a_\alpha + iw + j(u-kw))}. \tag{19}$$

The infinite product on the right-hand side requires regularization.

6.2 The natural virtual bundle

Given $(E,\Phi) \in \mathfrak{M}_{r,d,n}(X,\ell_\infty)$, once again looking at $E|_{X_0}$ as representing a point in $\mathfrak{M}_{r,d,n}(X_0)$, we have

$$\begin{aligned}
\mathrm{ch}_{\tilde{T}} V^{\mathrm{vir}}_{E|_{X_0}} &= -\chi_{\tilde{T}} \mathrm{Ext}^*_{\mathscr{O}_{X_0}} E \\
&= \sum_\beta e^{a_\beta} \sum_{v \in V(\Gamma)} e^{w^v_{D_\beta}} \left(N_{Y^v_\beta}(w^v_1, w^v_2) - \frac{1}{(1-e^{-w^v_1})(1-e^{-w^v_2})} \right) \\
&= \sum_{v \in \Gamma} \sum_\beta e^{(a_\beta + w^v_{D_\beta})} \left(N_{Y^v_\beta}(w^v_1, w^v_2) - \frac{1}{(1-e^{-w^v_1})(1-e^{-w^v_2})} \right).
\end{aligned}$$

The perturbative part of the $\tilde{T}$-equivariant Chern character of the natural bundle is given by

$$\begin{aligned}
\mathrm{ch}_{\tilde{T}} V^{\mathrm{pert}}_{E|_{X_0}} &\stackrel{\mathrm{def}}{=} \mathrm{ch}_{\tilde{T}} V^{\mathrm{vir}}_{E|_{X_0}} - \mathrm{ch}_{\tilde{T}} V_{(E,\Phi)} \\
&= -\sum_{\alpha,\beta} e^{a_\beta} \left(\frac{1}{(1-e^{-w})(1-e^{u})} + \frac{1}{(1-e^{w})(1-e^{u-kw})} \right) \\
&= \frac{-\sum_\beta e^{a_\beta}}{(1-e^{u})(1-e^{u-kw})} \left(1 + \sum_{j=1}^{k-1} e^{u-jw} \right).
\end{aligned}$$

Example 62. $X = \mathbb{P}^2, X_0 = \mathbb{C}^2$.

$$\begin{aligned}\mathrm{ch}_{\tilde{T}} V^{\mathrm{pert}}_{E|_{X_0}} &= -\sum_\beta e^{a_\beta} \left(\frac{1}{(1-e^{\varepsilon_2-\varepsilon_1})(1-e^{-\varepsilon_2})} + \frac{1}{(1-e^{\varepsilon_1-\varepsilon_2})(1-e^{-\varepsilon_1})} \right) \\ &= \frac{-\sum_\beta e^{a_\beta}}{(1-e^{-\varepsilon_1})(1-e^{-\varepsilon_2})}.\end{aligned}$$

Let B be a multiplicative class defined by a formal power series $g(x)$. Formally, evaluating B on the natural bundle produces the following perturbative part:

$$B_{\tilde{T}}(V^{\mathrm{pert}}_{E|_{X_0}}) = \frac{1}{\prod_{i,j=0}^\infty g(a_\beta - iw + ju) \prod_{i,j=0}^\infty g(a_\beta + iw + j(u-kw))}. \tag{20}$$

The infinite product on the right hand side requires regularization.

6.3 Regularization

Following [30, App. A], we introduce the following functions.

Definition 63. (Zeta-regularization)

$$\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) \stackrel{\text{def}}{=} \frac{d}{ds}\Big|_{s=0} \frac{\Lambda}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(e^{\varepsilon_1 t}-1)(e^{\varepsilon_2 t}-1)}, \tag{21}$$

$$\begin{aligned}\gamma_{\varepsilon_1,\varepsilon_2}(x \mid \beta;\Lambda) \stackrel{\text{def}}{=} &\frac{1}{2\varepsilon_1\varepsilon_2} \left(-\frac{\beta}{6}\left(x + \frac{1}{2}(\varepsilon_1+\varepsilon_2)\right)^3 + x^2 \log(\beta\Lambda)\right) \\ &+ \sum_{n=1}^\infty \frac{1}{n} \frac{e^{-\beta n x}}{(e^{\beta n \varepsilon_1}-1)(e^{\beta n\varepsilon_2}-1)}.\end{aligned} \tag{22}$$

$\exp(\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda))$ is a regularization of the infinite product

$$\prod_{i,j=0}^\infty \frac{\Lambda}{x - i\varepsilon_1 - j\varepsilon_2}.$$

For a very nice explanation of this regularization scheme see [31]. Let $\mathrm{Li}_s(z) = \sum_{n=1}^\infty \frac{z^n}{n^s}$ be the polylogarithm function. The functions $\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$ and $\gamma_{\varepsilon_1,\varepsilon_2}(x \mid \beta;\Lambda)$ satisfy the following properties (see [30, App. A]):

Fact 64. (1) $\varepsilon_1\varepsilon_2\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$ is analytic in ε_1, ε_2 near $\varepsilon_1 = \varepsilon_2 = 0$;
(2) $\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\varepsilon_2\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) = -\frac{1}{2}x^2 \log\frac{x}{\Lambda} + \frac{3}{4}x^2$.

Fact 65. (1) $\varepsilon_1\varepsilon_2\gamma_{\varepsilon_1,\varepsilon_2}(x \mid \beta;\Lambda)$ is analytic in ε_1, ε_2 near $\varepsilon_1 = \varepsilon_2 = 0$;
(2) $\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\varepsilon_2\gamma_{\varepsilon_1,\varepsilon_2}(x \mid \beta;\Lambda) = \frac{x^2}{2}\log(\beta\Lambda) - \frac{\beta}{12}x^3 + \frac{1}{\beta^2}\mathrm{Li}_3(e^{-\beta x})$.

6.4 Nekrasov conjecture: perturbative part

Applying zeta-regularization to (19) and (20), we obtain the following definitions:

Definition 66. (Perturbative part of the partition function)

(1) *4d pure gauge theory:*

$$\mathscr{F}^{\mathrm{pert}}_{X_0,A=1,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\mathrm{def}}{=} u(u-kw)\cdot\left(\sum_{\alpha,\beta}(\gamma_{w,-u}(a_\beta-a_\alpha;\Lambda)+\gamma_{-w,-u+kw}(a_\beta-a_\alpha;\Lambda))\right)$$

$$Z^{\mathrm{pert}}_{X_0,A=1,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\mathrm{def}}{=} \exp\left(\frac{\mathscr{F}^{\mathrm{pert}}_{X_0,A=1,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)}{-u(u-kw)}\right),$$

(2) *4d gauge theory with N_f fundamental matter hypermultiplets:*

$$\begin{aligned}\mathscr{F}^{\mathrm{pert}}_{X_0,A=1,B=E_{\vec{m}}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) &\stackrel{\mathrm{def}}{=} u(u-kw)\cdot\left(\sum_{\alpha,\beta}\left(\gamma_{w,-u}(a_\beta-a_\alpha;\Lambda)+\gamma_{-w,-u+kw}\left(a_\beta-a_\alpha;\Lambda\right)\right.\right.\\ &\left.\left.-\sum_{\beta,f}\left(\gamma_{w,-u}(a_\beta+m_f;\Lambda)+\gamma_{-w,-u+kw}(a_\beta+m_f,\Lambda)\right)\right)\right)\end{aligned}$$

$$Z^{\mathrm{pert}}_{X_0,A=1,B=E_{\vec{m}}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\mathrm{def}}{=} \exp\left(\frac{\mathscr{F}^{\mathrm{pert}}_{X_0,A=1,B=E_{\vec{m}}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)}{-u(u-kw)}\right),$$

(3) *4d gauge theory with one adjoint matter hypermultiplet:*

$$\begin{aligned}\mathscr{F}^{\mathrm{pert}}_{X_0,A=E_m,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) &\stackrel{\mathrm{def}}{=} u(u-kw)\cdot\left(\sum_{\alpha,\beta}\left(\gamma_{w,-u}(a_\beta-a_\alpha;\Lambda)-\gamma_{w,-u}(m+a_\beta-a_\alpha;\Lambda)\right.\right.\\ &\left.\left.+\gamma_{-w,-u+kw}(a_\beta-a_\alpha;\Lambda)-\gamma_{-w,-u+kw}\left(m+a_\beta-a_\alpha;\Lambda\right)\right)\right)\end{aligned}$$

$$Z^{\mathrm{pert}}_{X_0,A=E_m,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\mathrm{def}}{=} \exp\left(\frac{\mathscr{F}^{\mathrm{pert}}_{X_0,A=E_m,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)}{-u(u-kw)}\right),$$

(4) *5d gauge theory compactified at a circle of circumference β:*

$$\mathscr{F}^{\mathrm{pert}}_{X_0,A=\widehat{A}_\beta,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\mathrm{def}}{=} u(u-kw)\sum_{p,q}(\gamma_{w,-u}(a_p-a_q;\beta,\Lambda)+\gamma_{-w,-u+kw}(a_p-a_q;\beta,\Lambda)$$

$$Z^{\mathrm{pert}}_{X_0,A=\widehat{A}_\beta,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\mathrm{def}}{=} \exp\left(\frac{\mathscr{F}^{\mathrm{pert}}_{X_0,A=\widehat{A}_\beta,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)}{-u(u-kw)}\right).$$

Example 67. $X = \mathbb{P}^2, X_0 = \mathbb{C}^2$.

(1) 4d pure gauge theory:

$$\mathscr{F}^{\text{pert}}_{\mathbb{C}^2,A=1,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = \varepsilon_1\varepsilon_2 \sum_{\alpha,\beta} \gamma_{\varepsilon_1,\varepsilon_2}(a_\beta - a_\alpha;\Lambda),$$

(2) 4d gauge theory with N_f fundamental matter hypermultiplets:

$$\begin{aligned}&\mathscr{F}^{\text{pert}}_{\mathbb{C}^2,A=1,B=E_{\vec{m}}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)\\ &= \varepsilon_1\varepsilon_2 \left(\sum_{\alpha,\beta} \gamma_{\varepsilon_1,\varepsilon_2}(a_\beta - a_\alpha;\Lambda) - \sum_{\beta,f} \gamma_{\varepsilon_1,\varepsilon_2}(a_\beta + m_f;\Lambda) \right),\end{aligned}$$

(3) 4d gauge theory with one adjoint matter hypermultiplet:

$$\begin{aligned}&\mathscr{F}^{\text{pert}}_{\mathbb{C}^2,A=E_m,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)\\ &= \varepsilon_1\varepsilon_2 \sum_{\alpha,\beta} \left(\gamma_{\varepsilon_1,\varepsilon_2}(a_\beta - a_\alpha;\Lambda) - \gamma_{\varepsilon_1,\varepsilon_2}\left(m + a_\beta - a_\alpha;\Lambda\right)\right),\end{aligned}$$

(4) 5d gauge theory compactified at a circle of circumference β:

$$\mathscr{F}^{\text{pert}}_{\mathbb{C}^2,A=\widehat{A}_\beta,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = \varepsilon_1\varepsilon_2 \sum_{p,q} \gamma_{\varepsilon_1,\varepsilon_2}(a_p - a_q \mid \beta;\Lambda).$$

Theorem 68. (Nekrasov conjecture: perturbative part)

(1) *4d pure gauge theory:*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathscr{F}^{\text{pert}}_{X_0,A=1,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = k\mathscr{F}^{\text{pert}}_0(\vec{a},\Lambda),$$

where

$$\mathscr{F}^{\text{pert}}_0(\vec{a},\Lambda) = \sum_{\alpha\neq\beta} \left(-\frac{1}{2}(a_\alpha - a_\beta)^2 \log\left(\frac{a_\alpha - a_\beta}{\Lambda}\right) + \frac{3}{4}(a_\alpha - a_\beta)^2 \right)$$

is the perturbative part of the Seiberg-Witten prepotential of 4d pure gauge theory.

(2) *4d gauge theory with N_f fundamental matter hypermultiplets:*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathscr{F}^{\text{pert}}_{X_0,A=1,B=E_{\vec{m}}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = k\mathscr{F}^{\text{pert}}_0(\vec{a},\vec{m},\Lambda),$$

where

$$\begin{aligned}\mathscr{F}^{\text{pert}}_0(\vec{a},\vec{m},\Lambda) = &\sum_{\alpha\neq\beta} \left(-\frac{1}{2}(a_\alpha - a_\beta)^2 \log\left(\frac{a_\alpha - a_\beta}{\Lambda}\right) + \frac{3}{4}(a_\alpha - a_\beta)^2 \right)\\ &+ \sum_{\beta,f} \left(\frac{1}{2}(a_\beta + m_f)^2 \log\left(\frac{a_\beta + m_f}{\Lambda}\right) - \frac{3}{4}(a_\beta + m_f)^2 \right)\end{aligned}$$

is the perturbative part of the Seiberg-Witten prepotential of 4d gauge theory with N_f fundamental matter hypermultiplets.

(3) *4d gauge theory with one adjoint matter hypermultiplet:*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{pert}}_{X_0,A=E_m,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)=k\mathscr{F}^{\mathrm{pert}}_0(\vec{a},m,\Lambda),$$

where

$$\begin{aligned}\mathscr{F}^{\mathrm{pert}}_0(\vec{a},m,\Lambda)&=\sum_{\alpha\neq\beta}\left(-\frac{1}{2}(a_\alpha-a_\beta)^2\log\left(\frac{a_\alpha-a_\beta}{\Lambda}\right)+\frac{3}{4}(a_\alpha-a_\beta)^2\right.\\&\quad\left.+\frac{1}{2}(a_\alpha-a_\beta+m)^2\log\left(\frac{a_\alpha-a_\beta+m}{\Lambda}\right)-\frac{3}{4}(a_\alpha-a_\beta+m)^2\right)\Bigg)\\&=\sum_{\alpha\neq\beta}\left(-\frac{1}{2}(a_\alpha-a_\beta)^2\log\left(\frac{a_\alpha-a_\beta}{\Lambda}\right)+\frac{1}{2}(a_\alpha-a_\beta+m)^2\right.\\&\quad\left.\times\log\left(\frac{a_\alpha-a_\beta+m}{\Lambda}\right)-\frac{3m^2}{4}\right)\end{aligned}$$

is the perturbative part of the Seiberg-Witten prepotential of 4d gauge theory with one adjoint matter hypermultiplets.

(4) *5d gauge theory compactified at a circle of circumference β:*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\mathscr{F}^{\mathrm{pert}}_{X_0,A=\widehat{A}_\beta,B=1}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)=k\mathscr{F}^{\mathrm{pert}}_0(\vec{a},\Lambda,\beta),$$

where

$$\mathscr{F}^{\mathrm{pert}}_0(\vec{a},\Lambda,\beta)=\sum_{p\neq q}\left(\frac{1}{2}(a_p-a_q)^2\log(\beta\Lambda)-\frac{\beta}{12}(a_p-a_q)^3+\frac{1}{\beta^2}\mathrm{Li}_3(e^{-\beta(a_p-a_q)})\right)$$

is the perturbative part of the Seiberg-Witten prepotential of 5d gauge theory compactified on a circle.

Proof. We prove (1), (2), (3). The proof of (4) is similar, except that we use Fact 65 instead Fact 64.

Define

$$f_k(u,w,x;\Lambda)=u(u-kw)(\gamma_{w,-u}(x;\Lambda)+\gamma_{-w,u+kw}(x;\Lambda)).$$

By Definition 66 (definition of $\mathscr{F}^{\mathrm{pert}}$), it suffices to show that

$$\lim_{u,w\to 0}f_k(u,w,x;\Lambda)=k\left(-\frac{1}{2}x^2\log\frac{x}{\Lambda}+\frac{3}{4}x^2\right).$$

Let $g(\varepsilon_1,\varepsilon_2,x;\Lambda)=\varepsilon_1\varepsilon_2\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)$. Then by Fact 64,

(i) $g(\varepsilon_1,\varepsilon_2,x;\Lambda)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$,

(ii) $\lim_{\varepsilon_1,\varepsilon_2\to 0}g(\varepsilon_1,\varepsilon_2,x;\Lambda)=-\frac{1}{2}x^2\log\frac{x}{\Lambda}+\frac{3}{4}x^2.$

By (i), we have

$$g(-w,-u+kw,x;\Lambda)-g(w,-u,x;\Lambda)=wh_k(u,w,x;\Lambda),$$

where $h_k(u,w,x;\Lambda)$ is analytic in w,u near $w=u=0$. We have

$$\begin{aligned} f_k(u,w,x;\Lambda) &= u(u-kw)\left(\frac{g(w,-u,x;\Lambda)}{w(-u)}+\frac{g(-w,-u+kw;\Lambda)}{-w(-u+kw)}\right) \\ &= kg(w,-u,x;\Lambda)+uh_k(u,w,x;\Lambda). \end{aligned}$$

Therefore

$$\lim_{u,w\to 0} f_k(u,w,x;\Lambda) = k \lim_{\varepsilon_1,\varepsilon_2\to 0} g(\varepsilon_1,\varepsilon_2,x;\Lambda) = k\left(-\frac{1}{2}x^2\log\frac{x}{\Lambda}+\frac{3}{4}x^2\right).$$

□

Appendix A: Kobayashi–Hitchin Correspondence and Existence of Instantons

In this section we recall some results relating instantons in pure gauge theory to holomorphic bundles. The Kobayashi–Hitchin correspondence predicts an equivalence between instantons and holomorphic bundles in various settings, see [21]. For an $SU(n)$ bundle E over compact Kähler surface X this correspondence was proved by Donaldson [9]: The moduli space of irreducible anti-self-dual connections on E is naturally identified with the set of equivalence classes of stable holomorphic $SL(n,\mathbb{C})$ bundles which are topologically equivalent to E (see [11] Corollary 6.1.6 for a proof of the rank 2 case). Note that here stability is taken with respect to the Kähler class. Under this correspondence the topological charge of the instanton corresponds to the second Chern number of the bundle.

To obtain a Kobayashi–Hitchin correspondence over a non-compact Kähler manifold (X,ω) one must impose some conditions on the behaviour of holomorphic bundles at infinity. The instanton charge is obtained by integration of the curvature of the connection over X, and the mildest constraint that guarantees finiteness of this integral is to demand that the curvature decays as $1/r^2$.

For a manifold X that can be compactified to $\bar{X}=X\cup D$ by adding a smooth divisor D with positive normal bundle, Bando [1] defined a notion on $U(r)$ flatness and proved the following: There is a correspondence between the moduli space of Hermitian– Einstein holomorphic vector bundles on (X,ω) whose curvature decays faster than $1/r^2$ with trivial holonomy at infinity and the moduli space of holomorphic vector bundles $\bar{X}$ whose restriction to D are $U(r)-$flat.

Alternatively, one can study non-compact Kobayashi-Hitchin correspondence between instantons and framed bundles, that is, holomorphic bundles that are trivialized at infinity. See Donaldson [10] for the first non-compact instance of the correspondence, namely instantons on $\mathbb{C}^2$; then King [14] for instantons on the blow-up of $\mathbb{C}^2$; and Gasparim–Köppe–Majumdar [16] for instantons on $Z_k := \mathrm{Tot}\mathscr{O}_{\mathbb{P}^1}(-k)$.

We remark that these correspondences refer to classical instantons, and corresponding non-compactified moduli spaces of holomorphic vector bundles (i.e.

locally free sheaves) having $c_1 = 0$, whereas in the supersymmetric case the vocabulary instanton moduli refers to the much more general notion of (partially) compactified moduli spaces of torsion free sheaves. In particular, existence of instantons with a prescribed charge in supersymmetric gauge theories can be obtained simply by considering non-locally free sheaves. Thus, existence results for supersymmetric instantons contrast with existence of classical instantons, cf. [16] Theorem 6.8, which says that the minimal local charge of a nontrivial $SU(2)$-instanton on Z_k is $k-1$.

Appendix B: Equivariant Cohomology

Let ET be a contractible space on which $T = (\mathbb{C}^*)^k$ acts freely, and let $BT = ET/T$. (For example, $ET = (\mathbb{C}^\infty - \{0\})^k$ and $BT = (\mathbb{P}^\infty)^k$.) Then $ET \to BT$ is a universal principal T-bundle.

Suppose that $T = (\mathbb{C}^*)^k$ acts on an m-dimensional complex manifold M. The T-equivariant cohomology of M is defined to be

$$H_T^*(M;\mathbb{Q}) \stackrel{\text{def}}{=} H^*(M_T;\mathbb{Q}),$$

where $M_T = M \times_T ET$. There is a fibration $M_T \to BT = ET/T$ with fiber M. Let $i_M : M \to M_T$ be the inclusion of fiber. This induces a ring homomorphism

$$i_M^* : H_T^*(M;\mathbb{Q}) \to H^*(M;\mathbb{Q}).$$

In particular, when M is a point, the map

$$i_{\text{pt}}^* : H_T^*(\text{pt};\mathbb{Q}) \cong \mathbb{Q}[u_1,\dots,u_k] \to H^*(\text{pt};\mathbb{Q}) \cong \mathbb{Q}$$

is given by $p(u_1,\dots,u_k) \mapsto p(0,...,0)$, where $u_1,\dots,u_k \in H_T^2(\text{pt};\mathbb{Q})$.

B.1: Integral

Now suppose that M is compact. Then integration along the fiber gives $\mathbb{Q}$-linear maps

$$\int_M : H^*(M;\mathbb{Q}) \to H^*(\text{pt};\mathbb{Q}), \tag{23}$$

$$\int_M : H_T^*(M;\mathbb{Q}) = H^*(M_T;\mathbb{Q}) \to H_T^*(\text{pt};\mathbb{Q}) = H^*(BT;\mathbb{Q}), \tag{24}$$

such that

(i) $\int_M \alpha = 0$ if $\alpha \in H^q(M;\mathbb{Q})$, $q < 2m$.
(ii) $\int_M \alpha \in H^0(\text{pt}) \cong \mathbb{Q}$ if $\alpha \in H^{2m}(M;\mathbb{Q})$.
(iii) $\int_M \alpha = 0$ if $\alpha \in H_T^q(M;\mathbb{Q})$, $q < 2m$.
(iv) $\int_M \alpha \in H_T^{q-2m}(\text{pt};\mathbb{Q})$ if $\alpha \in H_T^q(M;\mathbb{Q})$, $q \geq 2m$. Note that $H_T^{q-2m}(\text{pt};\mathbb{Q}) = 0$ when q is odd, and $H_T^{q-2m}(\text{pt};\mathbb{Q})$ consists of homogeneous polynomials in $u_1,\dots,u_k$ of degree $q/2-m$ when q is even.
(v) $i_{\text{pt}}^* \int_M \alpha = \int_M i_M^* \alpha \in H^0(\text{pt};\mathbb{Q}) \cong \mathbb{Q}$ for $\alpha \in H_T^*(M;\mathbb{Q})$.

B.2: Localization

Let M^T denote the set of T-fixed points in M. Suppose that each connected component of M^T is a compact complex submanifold of M, so that M^T has a normal bundle N which is a complex vector bundle. Note that N might have different ranks on different connected components of M^T. T acts on M^T trivially, so $(M^T)_T = M^T \times BT$ and

$$H_T^*(M^T;\mathbb{Q}) \cong H^*(M^T;\mathbb{Q}) \otimes_{\mathbb{Q}} H_T(\mathrm{pt};\mathbb{Q}).$$

The T-equivariant Euler class $e_T(N) \in H_T^*(M^T;\mathbb{Q})$ is invertible in

$$H^*(M^T;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[u_1,\dots,u_k]_{\mathbf{m}},$$

where $\mathbb{Q}[u_1,\dots,u_k]_{\mathbf{m}}$ is the localization of the ring $\mathbb{Q}[u_1,\dots,u_k]$ at the maximal ideal $\mathbf{m}$ generated by $u_1,...,u_k$. The Atiyah-Bott localization formula says

$$\int_M \alpha = \int_{M^T} \frac{i^*\alpha}{e_T(N)}, \tag{25}$$

where $\alpha \in H_T^*(M;\mathbb{Q})$, and $i^* : H_T^*(M;\mathbb{Q}) \to H_T^*(M^T;\mathbb{Q})$ is induced by the inclusion $i : M^T \to M$. In particular, if M^T consists of isolated points $p_1,\dots,p_N$, then

$$\int_M \alpha = \sum_{j=1}^N \frac{i_{p_j}^*\alpha}{e_T(T_{p_i}M)}, \tag{26}$$

where $i_{p_j}^* : H_T^*(M;\mathbb{Q}) \to H_T^*(p_j;\mathbb{Q}) \cong \mathbb{Q}[u_1,\dots,u_k]$ is induced by the inclusion $i_{p_j} : p_j \to M$.

Now suppose that M is non-compact. Then (23) and (24) are not defined. However, when M^T is compact, we may *define* (24) by the right hand side of (25). Now (i), (ii), (v) are irrelevant, and (iii), (iv) do not hold: given $\alpha \in H_T^q(M;\mathbb{Q})$, we have $\int_M \alpha = 0$ if q is odd, and $\int_M \alpha$ is a rational function in $u_1,\dots,u_k$ homogeneous of degree $q/2 - m$ (the degree can be negative).

Example B.1. Let $T_t = (\mathbb{C}^*)^2$ act on $\mathbb{P}^2$ by $(t_1,t_2)\cdot[Z_0,Z_1,Z_2] = [Z_0,t_1Z_1,t_2Z_2]$. We have $H_{T_t}^*(\mathrm{pt};\mathbb{Q}) = \mathbb{Q}[\varepsilon_1,\varepsilon_2]$,

$$\int_{\mathbb{P}^2} 1 = \frac{1}{\varepsilon_1\varepsilon_2} + \frac{1}{(-\varepsilon_1)(-\varepsilon_1+\varepsilon_2)} + \frac{1}{(-\varepsilon_2)(\varepsilon_1-\varepsilon_2)} = 0,$$
$$\int_{\mathbb{C}^2} 1 = \frac{1}{\varepsilon_1\varepsilon_2}.$$

B.3: Characteristic classes

Let c be a characteristic class for complex vector bundles. Given a T-equivariant complex vector bundle V over M, $V_T = V \times_T ET$ is a vector bundle over $M_T = M \times_T ET$. The T-equivariant characteristic class c_T is defined by

$$c_T(E) \stackrel{\text{def}}{=} c(E_T) \in H^*(M_T;\mathbb{Q}) = H_T^*(M;\mathbb{Q}).$$

Appendix C: Seiberg-Witten Prepotential

We present a brief description of the Seiberg–Witten prepotential, which is described in detail in the seminal work [33], where Seiberg and Witten gave an exact solution to $N = 2$ supersymmetric Yang–Mills in 4 dimensions with group $SU(2)$. For more details see also [24 and 7]. For gauge theory with matter see [8 and 6]. The subject of 5d gauge theories compactified on a circle and the corresponding Seiberg-Witten curves were introduced in [27].

C.1: $SU(2)$ case

The constraints of $N = 2$ SUSY imply that the quantum moduli space is the same as the classical one as an algebraic variety. Basic quantities are then the coordinates u of the moduli space and the electric charge a, which in the classical theory are related simply by $u = a^2/2$; in the quantum theory this relation holds approximately for $u \to \infty$ by asymptotic freedom, but for finite u the relation is much more intricate and encodes fundamental geometric and physical information. The description of the theory via the low energy effective Lagrangian presents measurable quantities as functions of the coordinates u of the moduli space, and in particular the electric charge $a = a(u)$. Moreover, Seiberg [32] shows that the magic of supersymmetry allows the effective Lagrangian to be expressed in terms of a single locally defined meromorphic function: the prepotential $\mathscr{F}_0$; all remaining quantities in the theory are expressible as functions of $\mathscr{F}_0$ and a. An appropriate incarnation of Montonen–Olive duality accounts for the appearance of the dual variable

$$a^D = \frac{d\mathscr{F}_0}{da},$$

whose physical meaning is of the dual, that is, magnetic charge. The defining relations giving

$$\tau = \frac{da^D}{da}, \quad \tau^D = \frac{d(-a)}{da^D},$$

which imply that the duality transformation is $\tau^D = -\tau(a)^{-1}$ and specializes to the Montonen–Olive transformation $g^D = g^{-1}$ when the phase angle $\theta = 0$, but not otherwise. The moduli space then acquires expressions for a Kähler metric

$$ds^2 = Im(\tau da d\bar{a})$$

with Kähler potential $\frac{d\mathscr{F}_0}{da}\bar{a}$, where τ is the matrix of periods

$$\tau = \frac{d^2\mathscr{F}_0}{da^2} = \frac{da^D}{da}.$$

For $SU(2)$ the low-energy effective values of this coupling are given by $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, where θ is is defined only modulo $2\pi\mathbb{Z}$; consequently τ is defined only modulo $\mathbb{Z}$ and there is a second transformation fixing a and taking $\tau \mapsto \tau + 1$.

Since $\tau = \frac{da^D}{da}$, it follows that $a^D \mapsto a^D + a$. This pair of transformations acts as multiplication on the 2−vector (a^D, a) by the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and fractional-linearly on τ, thus generating an $SL(2,\mathbb{Z})$ action. The upshot is that what lives intrinsically over a point u in the moduli space is not the electric charge $a(u)$ but the unimodular lattice $\mathbb{Z}a(u) + \mathbb{Z}a^D(u)$ of all electric and magnetic charges. As u varies we obtain a $\mathbb{Z}^2$ local system V over the moduli space, which Seiberg and Witten showed to have as simple as possible behaviour, thus having only 3 singularities at ± 1 and ∞. Fixing a section of V determines the prepotential up to a constant. From a careful analysis of the monodromies at the singular points, it follows that the local system itself can be identified with the fiber cohomology of the elliptic curve

$$E_u \colon y^2 = (x+1)(x-1)(x-u).$$

The complexification $V_{\mathbb{C}}$ can be globally trivialized in terms of a holomorphic 1-form $\lambda_1 = \frac{dx}{y}$ and a residueless meromorphic form $\lambda_2 = \frac{xdx}{y}$. One then chooses a homology basis consisting of a loop γ around the branch points $1, -1$ and a loop γ^D around $1, u$; and using such a basis, the correct geometric solution for the period is

$$\tau_u = \frac{\oint_{\gamma^D} \lambda_1}{\oint_{\gamma} \lambda_1}.$$

In this solution, a and a^D appear as the periods of γ and γ^D of the meromorphic 1-form

$$\lambda = \frac{ydx}{x^2-1} = \frac{(x-u)dx}{y} = \lambda_2 - u\lambda_1.$$

C.2: Higher rank case

The Seiberg-Witten solution is sometimes presented in reverse order, starting directly with the family of curves parametrized by u as we just described. For instance, the solution for the group $SU(r)$ then appears as follows. Let ϕ be an $SU(r)$ gauge field. Then

$$\det(xI - \phi) = x^r + U_2 x^{r-2} - U_3 x^{r-3} + \cdots + (-1)^r U_r,$$

where U_k is the elementary symmetric polynomial of the eigenvalues of ϕ, with $U_1 = 0$ because ϕ takes values in $SU(r)$. These are gauge invariant operators, so their vacuum expectation values $u_k = \langle U_k \rangle$ serve as coordinates of the classical moduli space. These are the coordinates on the $\mathcal{U}$-space: $u_2, \ldots, u_r$, which generalises the so-called u-plane in the $SU(2)$ case.

In case of added matter, then the duality transformations take a different form, e.g. adding N_f fundamental matter hypermultiplets, the duality transformation becomes:

$$\begin{pmatrix} a^D \\ a \end{pmatrix} \mapsto R \begin{pmatrix} a^D \\ a \end{pmatrix} + \sum_{i=1}^{N_f} m_i \begin{pmatrix} n_i^D \\ n_i \end{pmatrix},$$

where $R \in Sp(2(r-1), \mathbb{Z})$, the m_i are the masses of the N_f particles added, and n_i, n_i^D are integral $r \times r$ matrices. Correspondingly, on the total space of the family of curves, there are then N_f divisors $\mathscr{D}_i$ along which the meromorphic differential λ acquires a pole with constant residue $\frac{m_i}{2\pi\sqrt{-1}}$. Here again the charges a, a^D can be recovered as the periods of λ over γ and γ^D.

We now describe the Seiberg-Witten prepotential in various gauge theories with gauge group $SU(r)$, starting directly with the Seiberg–Witten curves. Consider the family of curves parametrized by Λ, $\boldsymbol{u} = (u_2, \ldots, u_r)$, and possibly some extra parameters, in the following cases:

(1) *4d pure gauge theory* (see e.g. [30, (4.5)]):

$$C_{\boldsymbol{u}} : \Lambda^r \left(w + \frac{1}{w} \right) = P(z) = z^r + u_2 z^{r-2} + \cdots + u_r.$$

(2) *4d gauge theory with N_f fundamental matter hypermultiplets* (see e.g. [28, (1.10)]):

$$C_{\boldsymbol{u},\boldsymbol{m}} : w + \frac{\Lambda^{2r-N_f} Q(z)}{w} = P(z), \quad Q(z) = \prod_{f=1}^{N_f} (z + m_f).$$

(3) *4d gauge theory with adjoint matter hypermultiplets* (see e.g. [30, (6.32)]): in this case the SW curve is the spectral curve of the elliptic Calogero–Moser system,

$$C_{\boldsymbol{u},m} : \mathrm{Det}_{l,n}(L(\varpi) - z) = 0,$$

where

$$\begin{aligned} L_{l,n}(\varpi) &= \delta_{ln} \left(p_n + \frac{m}{2\pi\sqrt{-1}} \log(\theta_{11}(\varpi))' \right) \\ &\quad + \frac{m}{2\pi\sqrt{-1}} (1 - \delta_{ln}) \frac{\theta_{11}(\varpi + q_l - q_n) \theta_{11}'(0)}{\theta_{11}(\varpi) \theta_{11}(q_l - q_n)}, \\ \theta_{11}(\varpi; \tau) &= \sum_{n \in \mathbb{Z}} e^{\pi\sqrt{-1}\tau(n+\frac{1}{2})^2 + 2\pi\sqrt{-1}(\varpi + \frac{1}{2})(n+\frac{1}{2})}. \end{aligned}$$

(4) *5d gauge theory compactified at a circle of circumference β* (see e.g. [30, (7.19)]):

$$C_{\boldsymbol{u},\beta} : (\beta\Lambda)^r \left(w + \frac{1}{w}\right) = X^{-r/2} P(X), \quad X = e^{\beta z}.$$

The *Seiberg-Witten differential* is

$$dS = \frac{1}{2\pi\sqrt{-1}} z \frac{dw}{w} = \frac{1}{2\pi\sqrt{-1}} \frac{zP'(z)dz}{y}.$$

Let $\{A_\alpha, B_\beta \mid \alpha, \beta = 2, \ldots, r\}$ be a symplectic basis of $H_1(C_{\mathbf{u}}, \mathbb{Z})$. Define functions a_α, a_β^D on the $\mathbf{u}$-plane by

$$a_\alpha = \oint_{A_\alpha} dS, \quad a_\alpha^D = 2\pi\sqrt{-1} \oint_{B_\beta} dS.$$

Then

$$\omega_p = \frac{1}{2\pi\sqrt{-1}} \frac{z^{r-p}dz}{y}, \quad p = 2, \ldots, r$$

form a basis of holomorphic differentials on $C_{\mathbf{u}}$. The period matrix $\tau = (\tau_{\alpha\beta})$ is given by

$$\tau_{\alpha\beta} = \frac{1}{2\pi\sqrt{-1}} \frac{\partial a_\alpha^D}{\partial a_\beta}.$$

Note that a change of symplectic basis corresponds to an element in $Sp(2(r-1), \mathbb{Z})$, the group of duality acting on the period matrix $\tau = (\tau_{\alpha\beta})$. In the $SU(2)$ or $U(2)$ cases, we have $r = 2$, so the group of duality is $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$ and the SW curve is an elliptic curve.

The *Seiberg-Witten prepotential* is a locally defined function satisfying

$$a_\alpha^D = \frac{\partial \mathscr{F}_0}{\partial a_\alpha}.$$

Therefore the Seiberg-Witten prepotential and the period matrix are related by

$$\tau_{\alpha\beta} = \frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 \mathscr{F}_0}{\partial a_\alpha \partial a_\beta}.$$

The full Seiberg–Witten prepotential is expressed as a sum

$$\mathscr{F}_0 = \mathscr{F}_0^{\text{pert}} + \mathscr{F}_0^{\text{inst}},$$

where $\mathscr{F}_0^{\text{pert}}$ is the *perturbative part* and $\mathscr{F}_0^{\text{inst}}$ is the *instanton part*. The explicit expressions of the perturbative parts $\mathscr{F}_0^{\text{pert}}$ of the SW prepotentials in gauge theories (1), (2), (3), (4) on the previous page are given in (1), (2), (3), (4) of Theorem 68, respectively; they have logarithm singularities along $\Lambda = 0$. The instanton part $\mathscr{F}_0^{\text{inst}}$ of the SW prepotential is a power series in Λ^{2r}:

$$\mathscr{F}_0^{\text{inst}} = O(\Lambda^{2r}) = f_1\Lambda^{2r} + f_2\Lambda^{4r} + \cdots + f_n\Lambda^{2nr} + \cdots.$$

The coefficient f_n coming from the n-instanton moduli space is called the n^{th} *instanton correction* to the prepotential.

For further details we refer to [8; 18; 27; 30], and [24, Sect. 2].

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