THE STATISTICS OF PARTICLES IN LOCAL QUANTUM THEORIES

S. Doplicher
Istituto Matematico, Università
di Roma, Italy

Introduction

This is a review talk on joint work with R.Haag and J.E. Roberts (see [1,2,3] and references given there to previous works).

Let us start with the question: why is there an alternative between the Bose and the Fermi type of particle statistics?

This brings in succession two other questions: what are the possible statistics compatible with general principles; and: how is formulated the concept of statistics in terms of general principles.

The usual description in terms of field operators cannot be entirely satisfactory since fields are not observable in general, (e.g. as soon as they do not commute with one another at spacelike distances) and also because you have to introduce from the outset into the formalism the type of statistics appearing in your theory, by assuming specific commutation relations at spacelike distances.

These comments apply also to the superselection structure of a theory. For the sake of strong interaction physics with short range forces, that structure is customarily embodied into the field formalism requiring that the exact internal symmetries of the theory are described by a compact group, the gauge group of the theory. The choice of this group is in practice suggested by the empirical patterns of elementary particles and resonances. This group acts locally on fields but leaves the observables uneffected since they are by definition gauge invariant.

Now first principles ought to be formulated precisely in terms of observables — if you start from there, no special commutation property nor gauge invariance is built in explicitly.

Our input is the algebra of all local observable $\mathfrak{A}$ acting on the Hilbert space $\mathcal{H}_0$ they generate on the vacuum state vector Ω: i.e. the vacuum superselection sector alone is given.

Equivalently you might think as given the abstract C*-algebra $\mathfrak{A}$ and a pure state ω_0 on $\mathfrak{A}$, the vacuum expectation functional.

This means: $\mathfrak{A}$ is an irreducible C*-algebra acting on $\mathcal{H}_0$.

The main postulate is locality. To each nice bounded region $\mathcal{O}$ in space-time (double cones) a subalgebra $\mathfrak{A}(\mathcal{O})$ of $\mathfrak{A}$ is assigned, in a way which preserves inclusions; by definition $\mathfrak{A}(\mathcal{O})$ is generated by the local observables which can be measured within the space-time limitations of $\mathcal{O}$. The closure of the union of all $\mathfrak{A}(\mathcal{O})$'s is $\mathfrak{A}$.

Einstein causality together with Quantum Mechanics say that observables affiliated with spacelike separated double cones commute. This locality postulate in turn defines what is meant by "local observables". So it is natural to strengthen it by the so called duality requirement, so that the postulate becomes:

(i) $\mathfrak{A}(\mathcal{O})$ is the set of all bounded operators on $\mathcal{H}_0$ commuting with every observable spacelike located to the double cone $\mathcal{O}$.

In other words: $B \in \mathfrak{A}(\mathcal{O})$ iff for each double cone in the spacelike complement $\mathcal{O}'$ of $\mathcal{O}$ and each $A \in \mathfrak{A}(\mathcal{O}_1)$ we have

$$AB = BA .$$

By recent work of Bisognano and Wichmann the duality requirement appears necessary for the existence of a suitable underlying Wightman field theory.

Another requirement of this nature, which is actually used in a very limited way, is the additivity assumption saying that any small region in space time contains sufficiently many local observables to build up any local algebra: if $\mathcal{O}_1 \cup \dots \cup \mathcal{O}_n \supset \mathcal{O}$ (all double cones) then

(i') $$\{ \mathfrak{A}(\mathcal{O}_1) \cup \dots \cup \mathfrak{A}(\mathcal{O}_n) \}'' \supset \mathfrak{A}(\mathcal{O}).$$

A detailed analysis of the possible superselection structures can be done on the basis of postulate (i). The statistics can be defined as a property of each superselection sector and classified in a simple way.

To relate this to properties of one particle states and scattering states one needs in full the general assumptions of relativistic quantum theories, namely

(ii) relativistic covariance of the local algebras
(iii) the vacuum is Poincaré invariant and is a ground state in $\mathcal{H}_0$.

As mentioned this is not needed for most of the general analysis; to be quite careful, an algebraic consequence of (i'), (ii), (iii) is freely used as a technical assumption, when needed [1].

Superselection Sectors and Statistics.

Let $\mathcal{S}_0$ be the set of vector states on $\mathfrak{A}$ coming from $\mathcal{H}_0$ and $\mathcal{S}(\mathfrak{A})$ the set of all states on $\mathfrak{A}$. Since $\mathfrak{A}$ is irreducible on $\mathcal{H}_0$, $\mathcal{S}_0$ is a collection of pure states among which the superposition principle holds, and forms the vacuum superselection sector.

How does the structure of $\mathfrak{A}$ determine the collection of all superselection sectors?

We want a universal recipee to select from $\mathcal{S}(\mathfrak{A})$ the subset of "elementary" states which in the end will carry each a finite number of elementary "charges".

The main criterion is that $\omega \in \mathcal{S}_r$ should describe a deviation from the vacuum state which becomes negligible in far away regions of space. In general:

$$\mathcal{S}_0 \subsetneq \mathcal{S}_r \subsetneq \mathcal{S}(\mathfrak{A}) .$$

The superselection sectors are then the coherence classes of pure states in $\mathcal{S}_r$:

$$\text{superselection sectors} = \mathcal{S}_r \cap \text{Pure states of } \mathfrak{A}/\cong$$

where for pure states ω_1, ω_2 on $\mathfrak{A}$, $\omega_1 \cong \omega_2$ means equivalently a. or b.:

a. $\pi_{\omega_1} \cong \pi_{\omega_2}$;

b. there is $B_{12} \in \mathfrak{A}$ s.t., for all $A \in \mathfrak{A}$,

$$\omega_2(A) = \omega_1(B_{12}^* A B_{12})$$

i.e. ω_2 results from ω_1 by a physical operation B_{12}, which typically creates pairs but no single charge; a. and b. being equivalent by the theorem of Kadison.

As you see the criterion to select $\mathcal{S}_r$ is of central importance. By making the foregoing precise, one can prove that, in theories describing only short range forces, $\mathcal{S}_r$ is the vector states of the representations of a special type, the localized morphisms. [1].

These are * isomorphisms ρ of $\mathfrak{A}$ into $\mathfrak{A}$ each localized in some double cone $\mathcal{O}_\rho$, i.e.:

$$\rho(A) = A \quad \text{if} \quad A \in \mathfrak{A}(\mathcal{O}) \text{ and } \mathcal{O} \text{ is a double cone spacelike to } \mathcal{O}_\rho .$$

Let Δ be the collection of those localized morphisms which up to a unitary equivalence can be localized everywhere (a less stringent condition suffices, see [1]). This initial result implies

1. each sector contains a <u>strictly localized state</u> ω_ρ: here $\rho \in \Delta$ and if $A \in \mathfrak{A}$

$$\omega_\rho(A) = \omega_0 \circ \rho(A) \equiv (\Omega, \rho(A)\Omega)$$

so that $\omega_\rho = \omega_0$ on observables spacelike to $\mathcal{O}_\rho$ (in electrodynamics this would imply absence of electric charge by the law of Gauss).

2. if ω_1, ω_2 are such states, $\omega_i = \omega_0 \circ \rho_i$, localized in mutually spacelike double cones, we can define an <u>exact product state</u>

(1) $$\omega_1 \times \omega_2 = \omega_0 \circ \rho_1 \rho_2$$

i.e. $\omega_1 \times \omega_2 = \omega_2 \times \omega_1$ and

$$\omega_1 \times \omega_2(A) = \omega_1(A)$$

if A is an observable localized in a double cone spacelike to the localization region $\mathcal{O}_{\rho_2}$ of ω_2, and conversely.

Then we have the Theorem: if ω_1, ω_2 are strictly localized states in the same sector, and $\rho \in \Delta$ is any morphism in the class of that sector,

(2) $$\omega_1 \times \omega_2 \text{ is } \underline{\text{pure}} \text{ iff } \rho(\mathfrak{A}) = \mathfrak{A}.$$

Now the product (1) composes the "charges" of the two sectors; such "charges" are labelled by the equivalence class of irreducible $\rho \in \Delta$. Allowing mixtures, as seems appropriate by (2), we can see that the equivalence classes of our representations are just the elements of the quotient

(3) $$\Delta/\mathcal{J}$$

of the <u>semigroup</u> Δ modulo <u>inner</u> localized automorphisms. The generalized charges (3) form a <u>commutative</u> semigroup; those corresponding to the localized <u>automorphisms</u> Γ(i.e. $\rho \in \Gamma$ if $\rho \in \Delta$ and $\rho(\mathfrak{A}) = \mathfrak{A}$) form an abelian <u>group</u> $\Gamma/\mathcal{J}$ in $\Delta/\mathcal{J}$, the group of "simple" charges.

The generalized charges $\Delta/\mathcal{J}$ of a theory are found so far by classifying a subset of the states $\mathcal{S}(\mathfrak{A})$ of $\mathfrak{A}$. It is however possible to construct them at least in principle from the vacuum sector: if ω_ξ is a strictly localized state with charge ξ and $\mathcal{O}_n$ is a sequence of double cones moving off to infinity in a spacelike direction, there is a sequence $\omega_n \in \mathcal{S}_0$ of states with no overall charge s.t.

(4) $\omega_n(A) = \omega_\xi(A)$ if A is an observable associated to any double cone specelike to $\mathcal{O}_n$;

$\omega_n(A) \to \omega_\xi(A)$, $A \in \mathfrak{A}$.

Thus ω_ξ is the limit of states with the same charge ξ in the localization region of ω_ξ and with a compensating charge in $\mathcal{O}_n$. If ξ is a simple charge i.e. $\xi \in \Gamma/\mathcal{J}$, the compensating charge is ξ^{-1}. If all charges are simple, the dual G of the discrete abelian group $\Gamma/\mathcal{J}$ is the gauge group of the theory and is compact <u>abelian</u>. In theories with non abelian gauge groups there are conversely non simple charges as seen by the physical example of single nucleon states in a fully SU_2 invariant theory: if ω_1, ω_2 are such states the product state $\omega_1 \times \omega_2$ is a <u>mixture</u> of singlet and triplet (compare (2)).

Simple sectors can also characterized by the fact that the duality condition holds in the associated representations of $\mathfrak{A}$.

A deeper characterization is: the simple sectors obey an ordinary (Bose or Fermi) statistics; the other sectors obey a parastatistics.

This brings us to our second question: what is the statistics of a sector?

The product operation (1) allows us to define quite generally such a notion without referring to one particle states.

Let ξ be a sector and $\rho \in \Delta$ a representative.

Let $\omega_1,\dots,\omega_n$ be mutually spacelike localized states of the same charge ξ . The product state

(5) $\omega_1 \times \omega_2 \times \cdots \times \omega_n$

is symmetric; it is a vector state in the representation ρ^n. Now $\rho(\mathfrak{A})$ is irreducible but $\rho^n(\mathfrak{A})$ is reducible for $n \geq 2$ unless ξ is simple. In that case there are several linearly independent state vectors for (5) and they can change from one another under permutation of the factors. Indeed let $\Psi_1,\cdots,\Psi_n$ be state vectors for $\omega_1,\dots,\omega_n$ resp. in the representation ρ; Ψ_i is unique up to a phase and given by

$$\Psi_i = U_i^* \Omega$$

where U_i is a local unitary in $\mathfrak{A}$. To the product (5) corresponds a product

(6) $\Psi_1 \times \Psi_2 \times \ldots \times \Psi_n = (U_1^* \times U_2^* \times \ldots \times U_n^*)\Omega$

where $U_1 \times U_2 = U_1\rho(U_2)$ is an associative composition law between all pairs of intertwining operators (see [1]).

One gets easily from this the permutation behaviour

(7) $\Psi_{p^{-1}(1)} \times \ldots \times \Psi_{p^{-1}(n)} = \varepsilon_\rho^{(n)}(p)\ \Psi_1 \times \cdots \times \Psi_n$

where $\varepsilon_\rho^{(n)}(p)$ is a unitary in $\mathfrak{A}$ commuting with $\rho^n(\mathfrak{A})$ and p is an element of the permutation group $\mathbb{P}(n)$. The $\varepsilon_\rho^{(n)}(p)$ could depend upon the choice of $\omega_1, \ldots, \omega_n$; however:

<u>Theorem</u>. $\rho \in \mathbb{P}(n) \rightarrow \varepsilon_\rho^{(n)}(p)$ is a representation of $\mathbb{P}(n)$ depending upon ρ alone; its equivalence class $\varepsilon_\xi^{(n)}$ depends upon the sector ξ alone.

This result makes it possible and natural to define the <u>statistics of the sector</u> ξ to be the sequence $\{\varepsilon_\xi^{(n)};\ n=1,2,\ldots\}$. Each $\varepsilon_\xi^{(n)}$ is specified by a set of irreducible representations each with infinite multiplicity.

Which are the possible statistics? A priori, continuously many. However

<u>Theorem</u>. To each sector ξ is associated a number $\lambda(\xi)$, the statistics parameter of ξ, with values inverse integer or zero, which determines entirely the statistics of ξ as follows:

if $\lambda(\xi) = 0$, $\varepsilon_\xi^{(n)}$ contains all representations of $\mathbb{P}(n)$;
if $|\lambda(\xi)| = 1/d(\xi)$, $\varepsilon_\xi^{(n)}$ contains all representations of $\mathbb{P}(n)$ with at most $d(\xi)$ antisimmetrizations resp. symmetrizations for $\lambda(\xi) > 0$ resp. $\lambda(\xi) < 0$.

We see here that ξ obeys the ordinary statistics if ξ is a simple sector; the converse holds too i.e. ξ is simple iff $d(\xi) = 1$.

We say that ξ is a finite sector if $\lambda(\xi) \neq 0$ and call $|\lambda(\xi)|^{-1} = d(\xi)$ the order of the parastatistics; ξ is paraBose resp. paraFermi if $\lambda(\xi) > 0$ resp. $\lambda(\xi) < 0$.

Let Σ be the smallest set in $\Delta/\mathcal{J}$ containing the finite sectors, their products and their subrepresentations (but not all direct sums!). Then:

<u>Theorem</u>. Each $\xi \in \Sigma$ is a finite sum of irreducible elements of Σ; to each $\xi \in \Sigma$ there is a statistics parameter $\lambda(\xi)$ as above and

$$\xi \in \Sigma \to \lambda(\xi)^{-1} \in \mathbb{Z}$$

is a homomorphism of the commutative semigroup Σ into the multiplicative semigroup $\mathbb{Z}$. If $\xi \in \Sigma$ and $\xi = \xi_1 \oplus \xi_2$ then

$$d(\xi) = d(\xi_1) + d(\xi_2)$$

(8)

$$\text{sign } \lambda(\xi) = \text{sign } \lambda(\xi_1) \; .$$

It was mentioned before that each state in a sector $\xi \in \Delta/y$ is the limit of bilocalized states carrying the charge ξ and a compensating charge which makes them neutral (i.e. in the vacuum sector). If ξ is finite this compensating charge can be found by the the opposite process on such states, of removing ξ far away to spacelike infinity.

The resulting state belongs to the conjugate sector characterized by:

Theorem. Let ξ be a finite sector; there is one and only one sector $\bar{\xi}$ such that

(9) $\qquad \xi \cdot \bar{\xi} \simeq$ vacuum representation $\oplus \cdots$;

the sectors $\xi, \bar{\xi}$ have then the same statistics

$$\lambda(\xi) = \lambda(\bar{\xi})$$

and the vacuum charge appears only once in (9).

Note that this charge conjugation of superselection quantum numbers exists, as the statistics of the sectors and its classification, solely on the grounds of the locality postulate (i), without use of space-time covariance principles.

Note also the analogy between the structure of Σ and that of the dual object of a compact gauge group.

We call also attention on the fact that, in theories of short range forces, by the results of this section superselection charges appearing in compounds must also appear isolated, with a compatible statistics in the sense of the preceding theorems.

Gauge groups and parastatistics

Statistics has been analyzed irrespectively of field commutation properties, actually in a formulation where no field at all is given to transfer charge quantum numbers.

If however you do deal with a theory specified by a field algebra $\mathcal{F}$ and a gauge group G acting on it so that the observables are precisely the fixed points

$$(10) \qquad \mathfrak{A} = \mathcal{F} \cap \mathcal{U}(G)'$$

then you can relate the superselection structure of $\mathfrak{A}$ and the statistics intrinsically defined by it to properties of G and of the field algebra $\mathcal{F}$.

Firstly there is a one to one correspondence from the set of all classes of irreducible representations of G into the irreducible elements of Σ; let

$$(11) \qquad u \in \hat{G} \to \xi_u \in \Sigma$$

be this correspondence; we have the theorem

$$(12) \qquad \dim u = d(\xi_u)$$

that is parastatistics appear necessarily as soon as the exact symmetry group is not abelian.

Actually the relations (11) and (12) are particular cases of a full correspondence between the dual structure of the gauge group G and the intertwining operators between the representations in Σ (see [2]).

This indicates that a compact gauge group should always be associated with finite sectors, although the solution of this problem is not compelling for the physical interpretation of a theory: the "dual gauge structure" is determined by the observables and gives all the informations usually derived from the gauge group itself: reduction of products, "Clebsh-Gordan coefficients" etc. (see [2,3]).

About the theorem expressed by (12) we stress the fact that parastatistics might appear also if all fields obey ordinary commutation or anticommutation properties; the only relevant hypothesis is that fields should commute with observables at spacelike distances. In turn the statistics of the sector ξ_u determines partially the commutation properties of fields transforming like $\bar{u}$ (i.e. carrying the charge ξ_u): if ψ, ψ' are irreducible tensor field operators of type u, $\bar{u}$ respectively which are spacelike located, then necessarily

$$\int_G \alpha_g(\psi\psi' \mp \psi'\psi)dg = 0$$

where $\mp = -\mathrm{sign}(\lambda(\xi_u))$.

Statistics and Particles

If you assume full Lorentz invariance, axioms (i), (ii), (iii) above, we have that:

a. the spectrum condition $P_\mu P^\mu \geq 0$ follows from convariance in all finite sectors.

b. if there is a one particle state $[m,j,\xi]$ in the finite sector ξ, s.t. there are finitely many descrete irreducible representations of the Poincaré group with mass m in the sector ξ, then there are antiparticle states $[m,j,\bar{\xi}]$ and the multiplicities are the same:

$$\nu([m,j,\xi]) = \nu([m,j,\bar{\xi}])$$

c. in the previous hypothesis,

$$(-1)^{2j} = \mathrm{sign}(\lambda(\xi)).$$

This expresses the generalized connection of spin and statistics; note that here none of these concepts is mediated by algebraic or covariance properties of fields [2].

If particles $[m,j,\xi]$ appear isolated from the continuum in the sector ξ, by additivity of the spectrum [2] there is necessarily a mass gap in the vacuum sector and scattering theory can be developed. Indeed the exact product operation we defined between strictly localized state vectors can be used to deduce, by the usual limiting procedure, an asymptotic product between one particle state vectors [2]. The resulting scattering states $\Psi_1 \overset{ex}{\times} \cdots \overset{ex}{\times} \Psi_n$ are state vectors in a representation given by a localized morphism.

Since scattering states describe assembly of asymptotically free particles we can ask what is the statistics of these particles. For this sake we study n identical particles of "charge" ξ each in a state specified by a vector Ψ_i and a common reference morphism ρ. Then $\Psi_1 \overset{ex}{\times} \cdots \overset{ex}{\times} \Psi_n$ is a state vector of ρ^n and we find:

$$(13) \qquad \Psi_{p^{-1}(1)} \overset{ex}{\times} \cdots \overset{ex}{\times} \Psi_{p^{-1}(n)} = \varepsilon_\rho^{(n)}(p)\Psi_1 \overset{ex}{\times} \cdots \overset{ex}{\times} \Psi_n$$

namely the statistics of the sector ξ defined in terms of strictly localized states coincides with that of the asymptotically free particles of charge ξ.

The state vectors (13) have natural tensor product metric properties and can be used to calculate transition probabilities for scattering processes

$$w_{\alpha\beta} = \langle \Psi_{\alpha_1}^{in} \times \cdots \times \Psi_{\alpha_n}^{in} , E_\beta^{out} \Psi_{\alpha_1}^{in} \times \cdots \times \Psi_{\alpha_n}^{in} \rangle$$

where E_β^{out} is the appropriate support projection for the outgoing state vector $\Psi_{\beta_1}^{out} \times \cdots \times \Psi_{\beta_m}^{out}$ in the weak closure of the algebra $\mathfrak{A}$ of quasilocal observables, describing the possible final configurations contained in the (generally non pure) outgoing state.

Problems

Of the many open problems related to the subject, I mention only three. The following statements should become theorems under same natural additional postulate besides (i) to (iii) above.

1. Each localized isomorphism of $\mathfrak{A}$ into itself is in Δ.
2. Each $\rho \in \Delta$ is covariant under Poincaré transformations (from the covering group).
3. Each irreducible element in $\Delta/\mathcal{J}$ has finite statistics;

i.e. no sector with infinite statistics occurs; see also [2, Appendix]. I hope that the relevance of these problems is clear from the content of this talk.

References

[1] S. Doplicher, R. Haag, J.E. Roberts: "Local Observables and Particle Statistics" I, Commun. Math. Phys. 23, 199 (1971).

[2] S. Doplicher, R. Haag, J.E. Roberts: "Local Observables and Particle Statistics" II, Commun. Math. Phys. 35, 49 (1974).

[3] S. Doplicher, J.E. Roberts: "Fields, Statistics and non Abelian Gauge Groups" Commun. Math. Phys. 28, 331 (1972).

Discussion

Kamefuchi (Question): Have you got anything to say about the CPT theorem within your formalism?

Doplicher (Answer): Starting with local algebras generated by locally commuting or anticommuting bounded field operators, Henri Epstein proved in 1967 that, if you assume asymptotic completeness, the S-matrix is CPT invariant.

You can formulate asymptotic completeness in our scattering theory, and this assumption should imply quite in the same way CPT invariance of the S matrix.