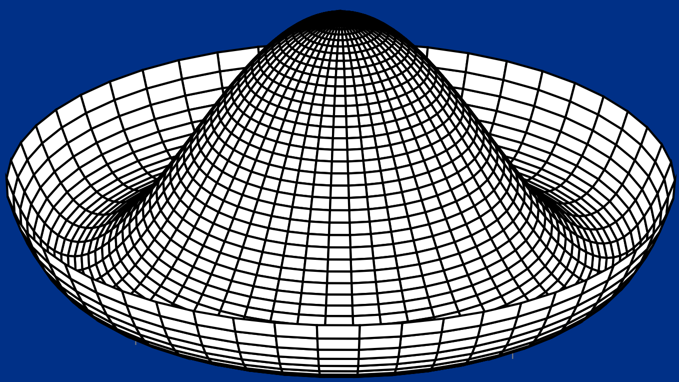
Perturbative and holographic study of symmetry breaking in non-relativistic theories



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Perturbative and holographic study of symmetry breaking in non-relativistic theories

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Physique Théorique et Mathématique
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Abstract

This thesis lies in the framework of the spontaneous symmetry breaking mechanism. When such a mechanism occurs, Goldstone's theorem predicts the existence of massless modes, called Nambu-Goldstone modes (NG modes). The current knowledge on NG modes is classified following the types of symmetries involved in the considered breaking pattern. Spacetime symmetries are the ones for which most of the analysis remains to be done. From a perturbative approach, we separately and concomitantly study the breaking of dilatation symmetry and of spatial translation symmetry. It allows us to comment on the present-day conjectures concerning the counting of NG modes associated to breaking patterns involving spacetime symmetries. Moreover, we get closer to standard laboratory conditions by investigating the situation in presence of a chemical potential. The considered Landau-Ginzburg's like models constitute plausible effective field theories to describe superfluids. The higher derivative terms required to spontaneously break translations lead to emergent subsystem symmetries. A connection between NG modes and fractonic modes, i.e. excitations with reduced mobility, is then made.

Non-relativistic systems are less constrained by the symmetries compared to Lorentz invariant systems which make the former more general. Even for non-spacetime symmetries, some uncertainties on the physics of NG modes remain when dealing with non-relativistic models. One of them is the critical dimension of Minkowski spacetime under which no spontaneous symmetry breaking can occur. This dimension has been conjectured and we propose an explicit computation in order to attest this conjecture. However, through a holographic analysis, we discuss some way out for large N field theories.

All along the dissertation, concrete future research perspectives on the above-mentioned discussions are provided.

Key words: Spontaneous symmetry breaking, Goldstone physics, effective field theories, bottom-up holography.

Résumé

Cette thèse porte sur le mécanisme de la brisure spontanée de symétrie. Lorsqu'un tel mécanisme se produit, le théorème de Goldstone prédit l'existence de modes non-massifs, appelés modes de Nambu-Goldstone (modes NG). Nos connaissances actuelles sur les modes NG sont classifiées suivant le type de symétries impliquées dans le motif de brisure considéré. Les symétries d'espace-temps sont les symétries pour lesquelles la majorité de l'analyse reste encore à faire. A travers une approche perturbative, nous étudions séparément et de façon concomitante la brisure de la symétrie de dilatation et la brisure de la symétrie de translation spatiale. Cela nous permet de commenter les conjectures actuelles portant sur le comptage des modes NG associés au motif de brisure faisant intervenir des symétries d'espace-temps. De plus, nous nous rapprochons des conditions expérimentales en investiguant la situation en présence d'un potentiel chimique. Les modèles de type Landau-Ginzburg considérés constituent des théories effectives plausibles pour la description de superfluides. Les termes de dérivées supérieures nécessaires à la brisure des translations mènent à l'émergence de symétries de sous-systèmes. Un lien entre les modes NG et les modes fractoniques, c-à-d. des excitations à mobilité réduite, est alors établi.

Les systèmes non-relativistes sont moins contraints par les symétries comparés aux systèmes invariants de Lorentz, ce qui rend les premiers plus généraux. Ainsi, même pour des symétries qui ne sont pas d'espace-temps, certaines incertitudes sur la physique des modes NG subsistent lorsque nous considérons des modèles non-relativistes. L'une d'entre elles est la dimension critique de l'espace-temps de Minkowski en-dessous de laquelle aucune brisure spontanée de symétrie ne peut se produire. Cette dimension a été conjecturée et nous proposons un calcul explicite en vue de valider cette conjecture. Cependant, à travers une analyse holographique, nous discutons d'une échappatoire concernant les théories de champs à grand N.

Tout au long de ce travail, de futures perspectives concrètes de recherche portant sur les discussions mentionnées ci-dessus sont proposées.

Mots clés : brisure spontanée de symétrie, physique de Goldstone, théories effectives, dualité holographique.

Acknowledgments

I would like to express my warmest gratitude to Riccardo Argurio, my advisor, for his full support during this thesis. I wish to deeply thanks Riccardo for his advices, for his kindness and for his sense of humour. Working with him was extremely pleasant. He has introduced me to the rich and exciting domain of Goldstone physics. The balance between formal developments and applied science we recover in this field of research matches my scientific interest. The diversity of subjects associated to the mechanism of spontaneous symmetry breaking permitted me to acquire, on a regular basis, new knowledge in various areas of science, ranging from mathematics to condensed matter physics. It kept me motivated during all these Ph.D. years. For this purpose, I am genuinely grateful to Riccardo. I am eager to keep trying to understand the subtleties of my field of research and I hope that, in the future, we will have other occasions to exchange and to collaborate.

The original part of this thesis would not have been possible without all the people I collaborated with: Carlos Hoyos, Daniele Musso, Antoine Pasternak, Jelle Hartong, Andrea Marzolla, Gaston Giribet, J. Anibal Sierra-Garcia and Andrea Mezzalira. Working with them has been a rewarding experience. In particular, I have truly appreciated the informal discussions I had with Daniele, Antoine and Andrea (Marzolla). It definitely helped me to acquire intuition on certain aspects of physics and to overcome some computational issues. I am indebted to Daniele who permitted me to become more independent as a researcher. Indeed, not only he had the kindness to include me in a solo project he was working on at that time but also, he introduced me in a new collaboration with Daniel Areán, Jewel K. Ghosh and Ignacio Salazar Landea. I acknowledge Daniele for his trust.

I am thankful to Glenn Barnich, Tomas Brauner, Thomas Hambye and Francisco Peña-Benitez for accepting to be part of my thesis jury. Their questions and their comments helped me to improve my knowledge of the field and it added new perspectives to my research. I will be pleased to pursue these discussions on any occasions.

I would be remiss in not mentioning that the exchanges with Ankit Aggarwal and Adrien Druart permitted me to clarify some technical details on the coset construction presented in the first part of this thesis.

Working in a nice and friendly environment has played a major role in my fulfilment. Consequently, I would like to extend my sincere thanks to the members of the "theoretical and mathematical physics" group, both the researchers and the administrative staff. I am also thankful to the people with whom I shared teaching duties.

In these acknowledgments I made the choice to remain at the academic level and to use a rather formal phrasing. However, I would like to emphasise that I am forever grateful to my family and my friends to be by my side and to make my journey through the years so joyful. Mastering new knowledge is what drives me; therefore, I am sure that I will not forget anyone by stating that *I sincerely thank all those who have ever taught me something.*

List of Publications

The original work presented in this thesis is based on the selected papers 1, 2 and 4 in the list below, while some of the highlights of paper 3 are as well commented. This choice is motivated by our desire to discuss self-consistent research projects with some diversity in the types of questionings addressed.

Let us emphasise that the concept of "first author" is non-existing in this research area, the authors are systematically listed in alphabetic order.

Scientific papers:

1. R. Argurio, C. Hoyos, D. Musso, and D. Naegels, "Fractons in effective field theories for spontaneously broken translations," *Phys. Rev. D* **104** no. 10, (2021) 105001, `arXiv:2107.03073 [hep-th]`.
2. R. Argurio, C. Hoyos, D. Musso, and D. Naegels, "Gapped dilatons in scale invariant superfluids," *Phys. Rev. D* **102** no. 7, (2020) 076011, `arXiv:2006.11047 [hep-th]`.
3. D. Musso and D. Naegels, "Independent Goldstone modes for translations and shift symmetry from a real modulated scalar," *Phys. Rev. D* **101** no. 4, (2020) 045016, `arXiv:1907.04069 [hep-th]`.
4. R. Argurio, D. Naegels, and A. Pasternak, "Are there Goldstone bosons in $d \leqslant z+1$?," *Phys. Rev. D* **100** no. 6, (2019) 066002, `arXiv:1903.11417 [hep-th]`.
5. R. Argurio, J. Hartong, A. Marzolla, and D. Naegels, "Symmetry breaking in holographic theories with Lifshitz scaling," *JHEP* **02** (2018) 053, `arXiv:1709.08383 [hep-th]`.
6. R. Argurio, G. Giribet, A. Marzolla, D. Naegels, and J. A. Sierra-Garcia, "Holographic Ward identities for symmetry breaking in two dimensions," *JHEP* **04** (2017) 007, `arXiv:1612.00771 [hep-th]`.
7. R. Argurio, A. Marzolla, A. Mezzalira, and D. Naegels, "Note on holographic non-relativistic Goldstone bosons," *Phys. Rev. D* **92** no. 6, (2015) 066009, `arXiv:1507.00211 [hep-th]`.

Scientific proceedings:

1. D. Naegels, "Goldstone Boson Physics and Effective Field Theories," *PoS* **Modave2021** (2022) 004, `arXiv:2110.14504 [hep-th]` .

Note to the reader

This thesis is structured in four parts. Part I is an introduction to the area of interest, namely Goldstone physics. A state of the art is provided at the end of this introductory part. The original component of the dissertation are Part II and Part III. Finally, Part IV is focusing on the conclusion of the thesis and on the potential research perspectives.

Units

We will use the natural units: $\hbar = c = k_{\mathrm{B}} = 1$, where c, $\hbar$ and k_{B} are, respectively, the speed of light, the Planck constant and the Boltzmann constant.

It implies that [energy] = [mass] = [temperature] = $[\text{space}]^{-1}$ = $[\text{time}]^{-1}$. We do not specify the energy unit. In particular, small and large limits of dimensionful parameters intervening in a given theory will always be with respect to other scales present in the model.

For the holographic computations, in fixed geometry, we will set the radius of the considered curved spacetime to one. When the geometry is dynamical, we will rather set the gravitational coupling between the gravitational field and matter to one.

Conventions

For quantum field theory computations, concerning the metric, we are going to use the mostly minus signature

$$(+, -, -, \ldots) \ ,$$

i.e. for Parts I and II as well as half of Part III, while for the holographic computations, we will consider the mostly plus signature

$$(-, +, +, \ldots) \ ,$$

i.e. for the other half of Part III and for Part IV.

We use Greek indices for the Minkowski spacetime components such that μ runs over $0, 1, 2, 3, \ldots$ or $t, x, y, z, \ldots$. The lowercase Roman indices i, j, k correspond to the spatial components such that they run over $1, 2, 3, , \ldots$ or $x, y, z, , \ldots$. In holography, the bulk $d+1$ coordinates will be labelled either with capital Roman indices or with the lowercase Roman indices m, n, o. If not mentioned otherwise, Einstein's summation convention is considered, i.e. indices that are repeated are summed over.

For d-vectors, we may also write: $x^\mu = x = (t, \mathbf{x}) = (t, \vec{x})$.

Eventually, we chose the following convention for the Fourier transform:

$$f(x) = \int \frac{d^d k}{(2\pi)^d} \, \tilde{f}(k) \, e^{-ik_\mu x^\mu} \ .$$

For the mostly minus signature of the metric, we schematically have $\partial_t \to -i\,\omega$ and $\partial_i \to i\,k_i$.

Acronyms

Here are listed the most recurrent acronyms in this dissertation:

AdS: Anti-de Sitter space.

CFT: Conformal Field Theory.

EFT: Effective Field Theory.

EOM: Equation(s) Of Motion.

IHC: Inverse Higgs Constraint.

IR: Infra Red.

NG mode: Nambu-Goldstone mode.

QFT: Quantum Field Theory.

RG flow: Renormalisation Group flow.

SSB: Spontaneous Symmetry Breaking.

UV: Ultra Violet.

VEV: Vacuum Expectation Value.

Contents

Introduction

This dissertation lies in the context of Goldstone physics. This area of physics revolves around the Goldstone theorem which, briefly stated, ensures that there is at least one massless mode, called the Nambu-Goldstone mode[1] (NG mode), in the spectrum of the theory when a spontaneous symmetry breaking (SSB) occurs [3, 4]. When the dynamics of a considered physical system is invariant under a transformation, we call such transformation a symmetry. An SSB is the situation where a theory possesses a symmetry, but the vacuum around which we perform the perturbation analysis/the quantisation does not have such symmetry – the state of the system is not invariant when acting the symmetry on it [5]. More precisely, Goldstone's theorem stands for the breaking of global continuous symmetry groups. These are continuous sets of symmetries which are parametrised by continuous numbers (we speak of local continuous symmetry groups when these numbers are promoted to be functions over spacetime). A cartoon visualisation of this concept can be observed in Figure 1 through the Mexican hat potential. These different notions will be revised more formally later. However, we can already understand that Goldstone's theorem provides some information on the spectrum content at low energy, i.e. the infrared region (the IR region). Therefore, combining this information with tools to build Effective Field Theories (EFTs) would provide us an almost complete description of the IR physics. That could be a definition of what Goldstone physics is: a thorough analysis of the Goldstone theorem and of the related results supplemented by/completed with EFT tools.

The main asset of studying Goldstone physics is the universality of the results that we get. Indeed, we can find SSB in particle physics: the Brout-Englert-Higgs mechanism is a good illustration [6, 7]. Furthermore, SSB can be used in statistical physics to describe phase transitions such as superfluidity [8]. Hence, we deduce that the progress in the comprehension of the SSB mechanism will help us to understand the fundamental laws of nature and it can also be useful in applied science. Another major impact of spontaneous symmetry breaking in science is the exactness of the results that we extract. The laws of nature are complicated. Physicists usually do some approximations to study natural phenomena. However, when a spontaneous symmetry breaking occurs, we are sure that the spectrum of the physical system at study will contain exactly gapless excitations and hierarchically small massive modes. This assertion is the statement of the already mentioned Goldstone's theorem.

For a given spontaneous symmetry breaking pattern, the theorem of Goldstone predicts the existence of NG modes but does not predict their exact number. Establishing a counting rule for the symmetry originated gapless excitations is one of the principal research directions of Goldstone physics. Two other dominant lines of investigation are understanding better the criteria under which a system can sustain an SSB and understanding how NG modes interconnect and interact with other physical particles. In this thesis, these three aspects will be discussed.

The counting rule for NG modes depends on the types of the symmetries which are

[1] Historically, Nambu is the one who conjectured the link between symmetry breaking and the mass constraint it implies [1, 2], while it is Goldstone who clarified and proved this conjecture [3, 4].

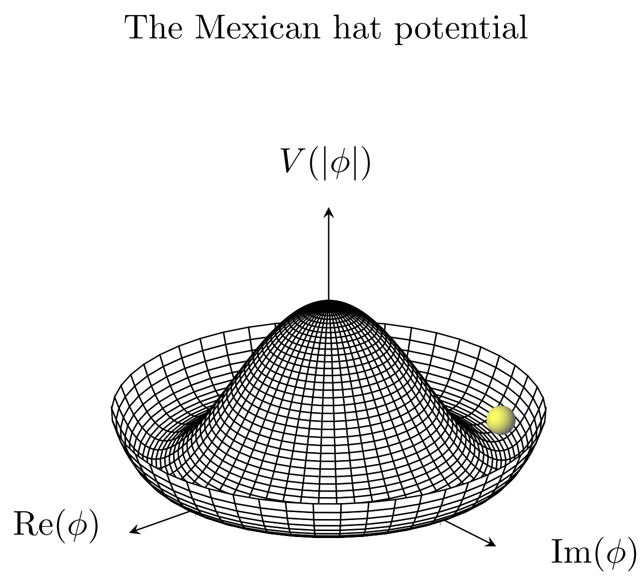

Figure 1: The dynamics of a physical system is mainly dictated by its potential. We observe that the represented potential in the above cartoon has a rotation symmetry since all the directions in the complex plane are equivalent. This means that the theory describing our system possesses this symmetry. However, the stable state into which we place the system (represented by the yellow sphere) selects one specific direction. Hence, all the directions are not anymore equivalent, the rotation symmetry is broken by the state of the system. We say that the symmetry is spontaneously broken.

involved in the given symmetry breaking pattern. Concerning symmetries which act solely on the internal space of the degrees of freedom, namely internal symmetries, a strong counting rule as well as a classification of the NG modes have already been obtained [9, 10]. It is not the case for spacetimes symmetries. These symmetries are both technically and conceptually involved since they act on the internal space of the degrees of freedom but as well on the spacetime coordinates. The number of gapless NG modes can also be affected by the presence of a chemical potential. The latter might provide a gap to some of them [11–13]. To encompass both subtleties, we will build toy models displaying dilatation symmetry breaking and spatial translation symmetry breaking both separately and concomitantly at zero and at finite density. The perturbative analysis of such toy models will offer a consistency check of the current conjectures concerning the counting rules of NG modes for spacetime symmetries. It will also add new materials to the discussion on the effect of a chemical potential on the gaps of the symmetry originated modes. The technicalities of the computations will lead us to introduce new tools to study low energy spectra. Especially, it will be the use of the Ward-Takahashi identities to deduce the symmetry origin of the obtained dispersion relations. Finally, the breaking of spatial translation symmetry requires higher derivative terms in the fundamental Lagrangian. Once we compute the effective action, these higher derivative terms ensure that additional symmetries emerge, called subsystem symmetries. These symmetries lead to supplementary conservation laws at low energy, in particular the conservation of multipole moments. As a consequence, some restriction in the mobility of the excitations is observed. These modes are known in the literature under the name of "fractonic excitations" [14, 15]. Hence, our Goldstone physics oriented toy models open new doors for fractonic model building. The interplay between NG modes and fractonic modes offers a perspective of research which will be useful to better understand both types of excitations.

Theoretical physics considers both Lorentz relativistic theories and non-relativistic theories, the former in the perspective of fundamental physics and the latter in the perspective of macroscopic physics. Non-relativistic theories are more general because they are not constrained by Lorentz symmetry. Therefore, establishing generic results for such theories is arduous. Determining the criteria under which a spontaneous symmetry breaking can be initiated in a physical system is an involved process, even for internal symmetries. Nevertheless, for relativistic theories, it is known that the critical spacetime dimension under which no spontaneous symmetry breaking can occur at quantum level is two [16]. In the literature, a critical dimension has been conjectured for non-relativistic systems [17]. To some extends, we will verify the validity of this conjecture through an explicit computation in quantum field theory. Although, large N field theories offer a counter-example to the conjecture. Such a counter-example will be found thanks to a holographic study. A memory of the peculiar spacetime dimension will however remain through the need of a specific holographic renormalisation.

The dissertation is organised in four parts. The first one is an introduction to Goldstone physics where the notions we discussed so far are formally defined and explained. The concluding chapter of Part I is the state of the art summarising the key features of Goldstone physics necessary for the rest of the thesis. Part II is dedicated to the study of the breaking of spacetime symmetries. More specifically, a perturbative approach of the spontaneous symmetry breaking of dilatation and spatial translation is provided. Because it leads to a connection with fracton physics, a journalistic overview of the latter area of physics is presented. In Part III, the spontaneous symmetry breaking of internal symmetries in non-relativistic systems at lower spacetime dimension is examined. Among other things, holography will be needed. This tool will be explained. Eventually, Part IV focuses on a specific future research project and contains the conclusion of the thesis. It should be mentioned that, all along the dissertation, outlooks will be proposed.

Part I

An introduction to Goldstone physics

Preamble Part I

This part of the dissertation has for purpose to define the physical notions and quantities we will use in this thesis and to give a global picture of what Goldstone physics is. In particular, it will permit to clearly state what are the current challenges of this area of science and so, to give a meaning to the computations we will perform in the next parts of the thesis dedicated to the original contribution of the author and his collaborators to some of the open problems. This introduction to Goldstone physics is meant as a pedagogical approach which favours intuitive reasoning over technical details. We nevertheless keep a certain level of formal developments in order to provide a concrete idea of what is behind the general principles and also, to introduce the computational tools.

These introductory notes are structured as follow. We will begin by motivating the subject. Afterwards, the formalism we will use will be settled. It should be seen as a way to set the conventions and the definitions rather than in an axiomatic goal. Then, Goldstone's theorem will be stated and proved (without any claim of full mathematical rigour). From a discussion on the NG modes, some of their properties will emerge: they are massless, they are weakly coupled in the IR and they transform non-linearly under the spontaneously broken symmetries. These properties will formally be displayed through the construction of a generic EFT for NG modes. This, by using the coset construction formalism, which will itself be introduced. Furthermore, the EFT approach will allow us to acquire additional knowledge compared to the prediction of Goldstone's theorem, namely, we will obtain a classification and a counting rule for the NG modes. This counting rule will be adapted for the situation at finite density. Furthermore, the established classification will allow for a discussion on no-go theorems concerning the occurrence of spontaneous symmetry breaking in lower dimensional spacetimes. The range of validity of these results will be detailed. In particular, spacetime symmetries will need a specific treatment and an entire chapter will be dedicated to them. The abstract discussions and developments will be illustrated in the appendices by two concrete examples in statistical physics: ferromagnetism and superfluidity. Finally, a state of the art of Goldstone physics will be provided.

The five main references that were used to write this part of the dissertation are [5, 9, 10, 18, 19]. Let us mention that the subject of spontaneous symmetry breaking is vast and many articles appeared in the past decades. The bibliography of this work is not meant to be exhaustive; it focuses on the references the author is the most familiar with. We apologise for any unintentional omissions of relevant papers or reviews.

We draw the reader's attention to the fact that chapters 1, 2, 3, 4 and appendix A are slightly edited from the proceeding the author of this thesis published following the lecture he gave at the XVII Modave summer school in mathematical physics [20].

Chapter 1

Motivations

To motivate the study of Goldstone physics we need first to recap the well-known assets of symmetries. This, in order to put into perspective the interesting aspects of spontaneous symmetry breaking.

It is intuitive that knowing a symmetry of an object (geometric figures, mathematical equations etc. left invariant after a given transformation) permits to ease the description and the manipulation of the considered object. In physics, this idea has been formalised through the Noether theorems which establish a connection between symmetries and conserved quantities. A noteworthy point is that these conserved quantities are exactly conserved no matter how complex the dynamics is. Hence, symmetries offer exact (i.e. non-perturbative) results. Furthermore, symmetries rely on the mathematical description of physics, it is thus not specific to a given physical scenario[1]. When a concept applies to several physical phenomena, we say that this concept is universal. This is the case of symmetries. Finally, the symmetries permit to constrain the shape of a Lagrangian when we do model building. The importance of symmetries in physics can be heuristically shown by noticing that symmetries are one of the current paradigms of modern physics: special relativity has for cornerstone Lorentz symmetry, general relativity is based on diffeomorphism invariance and the standard model is constructed on the notion of gauge symmetries.

Paradoxically, physics is richer when symmetries are spontaneously broken. At first, we could think that we lose universality but this is not the case since many physical phenomena take place around a pre-existing background which breaks spontaneously the fundamental symmetries. For example, the crystal structure in solid state physics is usually taken as granted and it breaks the Galilean group (spatial translations and rotations as well as the Galilean boosts). Another example could be the quark condensate; at low energy the quarks are not free and form bound states. This condensation breaks the chiral symmetry. An additional possible false idea we could have when SSB occurs is that we lose the exactness of the results related to symmetries. This can be denied thanks to Goldstone's theorem where the massless aspect of the NG modes is exact. Furthermore, we can mention that even in the case of an explicit symmetry breaking, if we are able to write down the source generating the breaking, the Ward-Takahashi identities can be generalised and still be exact (see for example [21, 22]). This can be relevant for pseudo NG modes, cf. Section 4.3. Finally, the spontaneously broken symmetries still constrain the shape of the Lagrangian. However, it is less easy to see because these symmetries are now "hidden". We will see that the formal explanation is that they are non-linearly realised instead of being linearly realised,[2] which makes the invariance of the Lagrangian

[1] An example is that we use energy conservation in any area of physics.

[2] In physics, in general, the usual symmetries ($U(1)$, $SO(3)$, $SU(N)\ldots$) are realised through matrices acting on fields which makes their action intrinsically linear – we then speak of the representations of the groups. Because non-linearity will play a major role in Goldstone physics, we will keep the generic

less obvious. Gathering all these observations, we can (with a bit of exaggeration) say that spontaneous symmetry breaking gives us the IR matter content of a given physical phenomenon through Goldstone's theorem and constrains the shape of the associated EFT. In other words, it completely settles the effective field theory. Therefore, while symmetries are giving partial information on the dynamics through the conserved quantities, spontaneous symmetry breaking provides all the dynamics at low energy. It is in this sense that physics is richer in the case of SSB. Of course, that is hasty said, it should not be taken literally but more as a guideline which motivates the study of Goldstone physics.

Since the last assertion is the main motivation of Goldstone physics, it could be interesting to have an additional viewpoint on it – in order to double check the consistency of this statement. It can be done through the concept of the renormalisation group flow (the RG flow). In the UV (at high energies) each physical phenomenon is described by one theory. When we follow the RG flow toward the IR, the irrelevant operators become progressively suppressed. We thus remain with a handful numbers of theories – the number of parameters is now limited – which are constrained by the symmetries. Indeed, the RG flow modifies the theories consistently with the symmetries (if there are no anomalies). Hence, in the IR, one theory is describing several physical phenomena. The effective field theories are therefore universal and symmetrically constrained. The different physical phenomena are discriminated by the different interpretations we give to the parameters (the mass, the compression modulus etc.) and by the numerical values of these parameters[3]. Furthermore, when we go to low energy, the system tends to condense (e.g. liquid to solid phase transition at low temperature, quark condensate at low energy, Bose-Einstein condensation at low temperature). The condensate will spontaneously break some symmetries. We can thus apply Goldstone's theorem to get information on the IR spectrum content. This leads to the rough idea that IR physics is universally described by Goldstone physics. As mentioned earlier, it should be understood more as an argument motivating the subject rather than a strong statement.

Until now, we used abstract ideas to justify the universality of spontaneous symmetry breaking and Goldstone physics. We will close this chapter by stating some concrete examples where such concepts are found (it is not an exhaustive list).

- High energy physics: light mesons physics [18], composite Higgs model [23], Higgs mechanism [24].
- Statistical physics: phase transitions (ferromagnetism [18] ...), transport phenomena (superfluidity [8] ...), condensed matter (crystal structures ...) [25].
- Astrophysics: stellar superfluids (e.g. in neutron stars) [8]

Of course, these domains are interconnected and it is one of the reasons why spontaneous symmetry breaking occurs in many areas of physics. For example, spontaneous symmetry breaking intervenes in the study of neutron stars because the latter have the right

nomenclature of "linear realisation" rather than of "representation".

[3] An EFT is the most general theory we can write which is invariant under a given symmetry group and that is written as an expansion in energy (usually a power series in derivatives of the fields). The experimental precision (or just the desired precision) tells us where to truncate this series. From the UV theory or from experiments we are thus able to determine the remaining parameters and our EFT becomes predictive at low energy.

thermodynamic conditions to sustain a superfluid phase. Some of the superfluid phase transitions correspond to a Bose-Einstein condensation which can be described by the spontaneous symmetry breaking of a $U(1)$ symmetry [8]. Thus, with the single example of superfluid, SSB intervenes in phase transition physics, in transport phenomena study and in astrophysics.

Chapter 2

Setting the formalism

Before starting to discuss and to compute physical quantities related to spontaneous symmetry breaking, we will recap the necessary prerequisites as well as establish the framework we are going to work with. This is the purpose of this chapter. Let us mention that we do claim a scientific approach (i.e. justify every step and keep an appropriate rigour level). However, we do not claim a full mathematical rigour and we do not pretend to have an axiomatic approach of physics. This chapter should be understood as a way to refresh our memory on standard notions (no claim of originality) and to establish the vocabulary as well as the definitions.

We are going to work with physical quantum field theories (QFT) defined on Minkowski spacetime of dimension $d \geqslant 2$. Using fields as degrees of freedom is consistent with the infrared limit because this limit is equivalent to probe for phenomena occurring at large distances and so, there is not much loss of generality by considering the continuous limit (e.g. crystal structures at large spatial scale compared to their lattice spacing can be seen as continuous) [18]. Furthermore, in addition to the time direction permitting to define the notion of energy, we need at least one spatial direction to be able to have the notion of momentum (necessary for the concept of mass/gap). The term "physical" is deliberately vague[1] but it should include at least:

- A notion of locality: following the theorems or the results we will consider, there will be specific restrictions on locality. For example, while building effective field theories, we will only consider interactions between fields evaluated at the same spacetime position. On the contrary, Goldstone's theorem is robust with respect to a partial relaxation of locality. It is valid up to interactions with a finite range in space.

- Stability: we want our vacuum as well as the fluctuations around it to be stable, i.e. to remain finite through time.

- Consistency with a possible Poincaré-relativistic UV completion: Goldstone physics encompasses phenomenological descriptions of macroscopic systems. These phenomenological theories could be non-relativistic, however, we know that at the fundamental level, physics is relativistic. Therefore, any non-relativistic theory in the IR should, at higher energy, come from the spontaneous symmetry breaking of

[1] The idea of physical theories is intrinsically vague since this notion evolves with our understanding of nature. A naive example is that at a moment of history we thought that time was absolute but with special relativity, we learned that it was not correct. So, what a physical theory is evolves through science history. A more relevant example for us is that in the early seventies Coleman stated that for a relativistic field theory in two spacetime dimensions, no spontaneous symmetry breaking can occur [16]. This theorem can be evaded if we consider strictly large N theories [26–28]. These theories could be thought of as purely exotic and non-physical. However, thanks to the holographic duality postulated in the nineties [29–31], it appears that they could be linked to consistent physical gravitational theories.

Poincaré symmetry (as it is suggested in [32] for example). We will not check explicitly this last requirement concerning a possible relativistic completion, but we should keep in mind this physical constraint. Notice that in this dissertation, when we mention relativity it is with respect to Poincaré symmetry (Poincaré-Lorentz relativity).

It is important to notice that the considered QFTs are not necessary Lorentz invariant. Since the paradigm among high energy physicists is that QFT has been introduced to build quantum Lorentz invariant theories, let us take some time to justify why we can be interested in non-relativistic field theories. We have that macroscopic systems correspond usually to some fluctuations around a given condensate (e.g. in solid state physics where the crystal structure is given for granted). The centre of mass of this condensate corresponds therefore to a preferential frame (the rest frame) which is opposed to the paradigm of relativity: fundamentally, the laws of physics are the same in any (inertial) frame. Furthermore, the thermodynamic state of the system is given through a computation involving the probability weight $e^{-\beta H}$ where H is the Hamiltonian. The interplay between the Hamiltonian and the Lorentz group is non-trivial, thus, thermodynamic states tend to break Lorentz invariance. Finally, the macroscopic systems are important for this thesis because Goldstone's theorem does apply to them as well.

Even if we just mentioned the importance of statistical field theory, we are going to work with QFTs at zero temperature and, if not mentioned otherwise, at zero chemical potential. Of course, switching on temperature is part of the research area of Goldstone physic. But switching on temperature is a technically non-trivial process since in the imaginary time formalism the time is periodic. And if we want to study the dynamics, we need to go to real time formalism where time is an oriented curve in the time complex plan (or we should proceed to an analytic continuation of the correlators obtained in the imaginary time formalism) [33]. This constitutes a domain of research on its own. Besides some brief comments, Goldstone physics at finite temperature will not be analysed in this dissertation. It should however be mentioned that, doing zero temperature QFT does not prevent us to compute thermal quantities at low temperature. An example of this assertion can be found in Appendix A.

2.1 What do we mean by symmetry ?

2.1.1 At classical level

In field theory, a symmetry is a transformation applied on the fields which leaves the equations of motion (EOM) unchanged. An equivalent formulation is that under such a transformation, a solution of the EOM remains a solution. Mathematically, a transformation on the fields is defined as

$$\begin{cases} x^\mu & \to x'^\mu = x'^\mu(x) \ , \\ \phi^i(x) & \to \phi'^i(x') = F^i\left[x, \phi(x)\right] \ , \end{cases} \tag{2.1.1}$$

where ϕ is a generic field and the index i refers to its possible multi-component nature, F^i is a function, finally, the prime index represents the transformed object.

In this work we will do a small misnomer by defining “a symmetry” as a transformation applied on the fields which leaves the action of the field theory unchanged:

$$S[\phi'] = S[\phi] \ . \tag{2.1.2}$$

This small misappropriation of the term symmetry is consistent in the sense that a symmetry of the action implies a symmetry of the EOM. Furthermore, we do not lose much generality because most of the important symmetries (and the mostly used ones) in physics are the ones which can be seen at the level of the action.

If we consider several transformations of the type (2.1.1), we can combine them through the law of function composition and get another symmetry by “chain reaction”. It is therefore possible to define an internal associative product. The identical transformation is trivially a symmetry. Finally, physical interesting transformations are predominantly invertible[2] (e.g. rotations, phase-shifts, translations ...). Hence, the symmetries of a theory form a group. It is common to denote this set of transformations as a realisation of the usual groups: $\mathbb{Z}_2$, $U(1)$, $SO(3)$, $SU(N)$ .... We will of course come back to it later, but Goldstone's theorem only applies when such groups are continuous. We will from now on focus on continuous groups. If we consider a continuous set of transformations (2.1.1) parametrised by α^a, we can write an infinitesimal expression

$$\begin{cases} x^\mu & \to \ x'^\mu = x^\mu + \alpha^a \, \xi^\mu_a(x) \ , \\ \phi^i(x) & \to \ \phi'^i(x) = \phi^i(x) + \alpha^a \, \delta_a \phi^i(x) \ . \end{cases} \tag{2.1.3}$$

These transformations correspond to the realisation of the continuous connected part to the identity ($\alpha = 0$) of the symmetry group. Therefore, α^a parametrises the Lie algebra of the continuous group and we can define the representation of the generators G_a by the infinitesimal action on the fields

$$\alpha^a G_a \phi^i(x) \equiv \delta_\alpha \phi^i(x) \equiv \phi'^i(x) - \phi^i(x) \ . \tag{2.1.4}$$

Let us mention that we will slowly start to stop to do the distinction between the generators and the realisation of the generators. The context should make it clear which case is considered.

We recall that through Noether's first theorem, it exists a one-to-one relation between the set of symmetry generators G_a associated to constant parameters over spacetime α^a (i.e. the generators of global symmetries) and the set of conserved currents $j^\mu_a(x)$, satisfying $\partial_\mu j^\mu_a = 0$ on-shell[3] – i.e. when the EOM are satisfied. To construct the conserved current associated to G_a, we consider an invariant theory under the action of G_a. The Lagrangian $\mathcal{L}(\phi, \partial\phi)$ of the theory can transform up to a global derivative under the transformation

$$\delta_a \phi^i \equiv G_a \phi^i \ , \tag{2.1.5}$$

[2] A symmetry transformation could be somehow interpreted as a change of frame. Physically, nothing prevents us to go back to the original frame.

[3] To be more precise, a gauge symmetry is a symmetry (2.1.4) where the parameter α^a is an arbitrary function of spacetime. The global symmetries are the set of symmetries (2.1.4) where we identify the transformations (2.1.4) which are equal on-shell up to a gauge transformation. Noether's first theorem establishes a one-to-one correspondence between the set of global symmetries and the set of classes of conserved currents $[j^\mu_a]$, where $j^\mu_a \sim j^\mu_a + \partial_\nu R^{\nu\mu}$ with R a generic anti-symetric tensor. A more rigorous statement can be found in [34].

such that we express the global derivative through K_a^ν:

$$\delta_a \mathcal{L} \equiv \partial_\mu K_a^\mu \ . \tag{2.1.6}$$

The conserved current is then given by[4] [5,35]

$$j_a^\nu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi^i\right)} \delta_a \phi^i - K_a^\nu \ . \tag{2.1.7}$$

From j_a^μ, a conserved quantity can be built

$$Q_a \equiv \int d^{d-1}x \ j_a^0(x) \ , \tag{2.1.8}$$

where $\partial_t Q_a = 0$ on-shell.

2.1.2 At quantum level

Quantum mechanics is described by a complex Hilbert space $\mathcal{H} = \{|\psi\rangle\}$ and by a Hamiltonian $\hat{H}$. Conceptually, a symmetry transformation $|\psi\rangle \to |\psi'\rangle$ can be seen as a change of frame. Changing the frame should not alter the relative results of an experiment. Therefore, a necessary condition that a symmetry transformation should satisfy is

$$\langle\psi| \hat{A} |\psi\rangle = \langle\psi'| \hat{A}' |\psi'\rangle \ , \tag{2.1.9}$$

where $\hat{A}$ is an observable and the prime represents its transformation [36]. Let us mention that the transformations (2.1.9) can be seen as the quantum version of the canonical transformations in classical mechanics. Wigner theorem states that for (2.1.9) to be fulfilled, a symmetry transformation acting on $\mathcal{H}$ should be either unitary and linear or antiunitary and antilinear [36]. As it will become clear later, in Goldstone physics we are interested in the part of the symmetry group which is continuously connected to the identity. Said otherwise, we are interested in the symmetry transformations which can be parametrised by the Lie algebra. Identity is a unitary operator and the switch between unitary and antiunitary is discontinuous, thus, we have that our considered transformations are unitary. Hence, the symmetry transformations we are interested in, realising a given symmetry group, can be written as

$$|\psi\rangle \to e^{i\alpha^a \hat{Q}_a} |\psi\rangle \ , \tag{2.1.10}$$

where the realisation of the generators is Hermitian

$$\hat{Q}_a^\dagger = \hat{Q}_a \ . \tag{2.1.11}$$

From (2.1.9) and (2.1.10), we have

$$\hat{A} \to e^{i\alpha^a \hat{Q}_a} \hat{A}\, e^{-i\alpha^a \hat{Q}_a} \ . \tag{2.1.12}$$

[4] Let us mention that defining the conserved current with an opposite sign would not change the conservation property.

To define what a symmetry is, we will use a pragmatic approach. From the classical discussion we know that a symmetry is associated to a conserved quantity. Therefore, we are going to say that a continuous group of transformations $|\psi\rangle \to |\psi'\rangle$ realising on $\mathcal{H}$ one of the usual continuous groups ($SU(2)$, $O(N)$ etc.) is a symmetry group if the generators $\hat{Q}_a$ satisfy:

$$\frac{d\hat{Q}_a}{dt} \equiv \partial_t \hat{Q}_a + [\hat{Q}_a, \hat{H}] = 0 \ . \tag{2.1.13}$$

A final comment is that, from the canonical quantisation, the operatorisation of the conserved charges of a field theory corresponds to the realisation of the generators at quantum level

$$[i\hat{Q}_a, \hat{\phi}^i] = \delta_a \hat{\phi}^i \equiv G_a \hat{\phi}^i \ , \tag{2.1.14}$$

if $[\delta_a \hat{\phi}^i, \hat{\phi}^j] = 0$ [5].

A bit more of details

The phase space of a quantum theory corresponds to the projective space of $\mathcal{H}$ [36]. To say it more simply, a state of the system is a ray $\mathcal{R}$ of $\mathcal{H}$ (this because global phases are not observable). Let us consider a continuous set $\{T\}$ of transformations $T : \mathcal{R} \to \mathcal{R}'$ which has a group structure (through function composition) that realises one of the usual continuous groups ($SU(2)$, $O(N)$ etc.). From a transformation T of our given set, we may define a transformation $U(T)$ acting on the Hilbert space, $U(T) : |\psi\rangle \to |\psi'\rangle$. The product law induced on $\{U(T)\}$ is defined up to a global phase (cf. the projective nature of the phase space VS. the vector space $\mathcal{H}$). Therefore, the representation of a given symmetry group on the phase space corresponds to a projective representation on the Hilbert space. It can be shown that a central charge c_{ab} might appear in the realisation of the Lie algebra on $\mathcal{H}$

$$[\hat{Q}_a, \hat{Q}_b] \sim f_{ab}{}^c \hat{Q}_c + c_{ab} \ , \tag{2.1.15}$$

where $f_{ab}{}^c$ are the structure constants [36]. Let us mention that to pursue with the manipulation of usual representations on $\mathcal{H}$, a possible trick is to consider a central extension of the symmetry group we want to realise on $\mathcal{H}$.

We should mention that the appearance of a central charge can already be retrieved at the classical level in some Lie algebras formed by the conserved charges and the Poisson bracket (e.g. the Galilean symmetry algebra and its associated central extension, the Bargmann algebra).

2.1.3 Ward-Takahashi identities

The quantum theory coming from the canonical quantisation of fields is usually not explicitly expressed in terms of its Hilbert space and its Hamiltonian. A quantum field theory is entirely described by its asymptotic spectrum (the spectrum at infinite times), which by assumption corresponds to the free (renormalised) spectrum, and by its correlators, which illustrate how the system evolves from $t \to -\infty$ to $t \to +\infty$. In this framework, the quantum conservation laws (2.1.13) correspond to the Ward-Takahashi identities. With the path integral formalism, it can be shown that when the classical field theory has the symmetry (2.1.3) – $S[\phi'] = S[\phi]$ – and that the functional integration measure has this symmetry as well ($\mathcal{D}\phi' = \mathcal{D}\phi$), then the following identities are satisfied [37]:

$$\begin{aligned}&\frac{\partial}{\partial x^\mu}\langle +\infty, 0| T\left\{\hat{j}^\mu_a(x)\prod_{k=1}^{m}\hat{\phi}^{i_k}(x_k)\right\}|-\infty, 0\rangle = \\ &(-i)\sum_{k=1}^{m}\delta(x-x_k)\langle +\infty, 0| T\left\{\hat{\phi}^{i_1}(x_1)\ldots\delta_a\hat{\phi}^{i_k}(x_k)\ldots\hat{\phi}^{i_m}(x_m)\right\}|-\infty, 0\rangle \ ,\end{aligned} \tag{2.1.16}$$

where $\hat{j}^\mu_a$ is the operator obtained from the canonical quantisation of (2.1.7) and $|0\rangle$ is the asymptotic vacuum state (we consider the normalised correlators: $\langle +\infty, 0|-\infty, 0\rangle \equiv 1$). These identities are the Ward-Takahashi identities. We say that the quantum field theory has the symmetry (2.1.3) when the associated Ward-Takahashi identities (2.1.16) are satisfied.

Let us notice that if we take $m = 0$ in (2.1.16) we recover the quantum version of the classical conservation law:

$$\frac{\partial}{\partial x^\mu}\langle +\infty, 0| T\left\{\hat{j}^\mu_a(x)\right\}|-\infty, 0\rangle = 0 \ . \tag{2.1.17}$$

The left-hand side of the identities (2.1.16) are in general non-zero because the derivative operator is infinitesimally non-local and so, the presence of an operator $\hat{\phi}^i$ near $\hat{j}^\mu_a$ might interfere with (2.1.17). This is the reason why each term on the right-hand side of (2.1.16) are weighted by the factor $\delta(x - x_k)$. We label such terms as contact terms.

In the case where $\mathcal{D}\phi' \neq \mathcal{D}\phi$ (and $S[\phi'] = S[\phi]$), the symmetry of the classical theory will not be recovered at quantum level. We call such situation an anomaly. It can be interpreted as an explicit breaking of the symmetry due to the quantisation. It is a rich domain of QFT [38], however, since we are interested in the spontaneous breaking of symmetries we will not study such scenarios in this work.

2.1.4 Symmetries through the quantum effective action

From the discussion on symmetries we had so far, it seems that at classical level the symmetries of the theory are defined through the action while at quantum level they are defined through the conservation laws. It therefore looks like that we have two separate treatments to study the symmetries, one for the classical study and one for the quantum study. In order to draw an analogy with the classical case, we can use the quantum effective action $\Gamma[\phi]$.

The quantum effective action encompasses the quantum corrections and by doing so, it permits to effectively describe the QFT as a classical field theory [35, 39]. Indeed, the stationary point of $\Gamma[\phi]$ is the one-point correlator of the QFT in the same way that the stationary point of $S[\phi]$ is the on-shell field. The connected n-point correlators of the QFT are obtained by summing on all the possible connected tree graphs where the Feynman rules are taken from $\Gamma[\phi]$, alike the tree graphs computed from $S[\phi]$ which are the QFT results in the classical limit ($\hbar \to 0$). In particular, the connected propagator of the QFT is given by:

$$\langle +\infty, 0| T\left\{\hat{\phi}(x)\hat{\phi}(y)\right\}|-\infty, 0\rangle_c = i\left(\left.\frac{\delta^2\Gamma[\phi]}{\delta\phi(x)\delta\phi(y)}\right|_{\phi_0}\right)^{-1} \ , \tag{2.1.18}$$

where the label c stand for "connected" and ϕ_0 is such that

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi_0} = 0 \ , \text{ i.e. } \phi_0(x) = \langle +\infty, 0|\, T\left\{\hat{\phi}(x)\right\} |-\infty, 0\rangle \ . \tag{2.1.19}$$

As we will see, the expression (2.1.18) will be used to define the notion of mass and will be important for the perturbative proof of Goldstone's theorem.

When $S[\phi]$ is invariant under (2.1.3) and there are no anomalies, the quantum effective action $\Gamma[\phi]$ is invariant under the transformation [39]

$$\phi^i(x) \to \phi'^i(x) = \phi^i(x) + \alpha^a \, \langle +\infty, 0|\, T\left\{\delta_a\hat{\phi}^i(x)\right\} |-\infty, 0\rangle_{J_\phi} \ , \tag{2.1.20}$$

where $\langle\ldots\rangle_{J_\phi}$ is the correlator in presence of the source J_ϕ which is defined as the source imposing

$$\langle +\infty, 0|\, T\left\{\hat{\phi}^i(x)\right\} |-\infty, 0\rangle_{J_\phi} = \phi^i(x) \ . \tag{2.1.21}$$

In particular, if we take (2.1.3) to be an affine transformation

$$\phi^i(x) \to \phi'^i(x) = \phi^i(x) + \alpha^a \left(s_a(x) + \int d^dy \, t^i_{\ j}(x,y)\, \phi^j(y) \right) \ , \tag{2.1.22}$$

we can observe that, by using (2.1.21), (2.1.20) is the same expression as (2.1.22).

Hence, in the same way $S[\phi'] = S[\phi]$ defines a symmetry of the classical field theory, $\Gamma[\phi'] = \Gamma[\phi]$ defines a symmetry of the QFT. This analogy can be pushed forward to QFTs at finite density and/or finite temperature. For thermal field theories expressed in the real time formalism, we can as well define a thermal effective action $\Gamma_\beta[\phi]$ which generates the thermal connected correlators and which defines the notion of symmetry through its invariance under certain transformations $\phi \to \phi'$ [40].

For the Section 2.2 and beyond, we will not denote with a circumflex accent anymore the quantum operators. We will as well shorten the writing $\langle +\infty, 0|\, T\{\ldots\} |-\infty, 0\rangle$ to $\langle\ldots\rangle$.

2.2 Different classifications of the symmetries

Goldstone physics depends heavily on symmetries, we can then naturally be convinced that some of the results we will state in the following chapters rely on the nature of the considered symmetries. One way to characterise the symmetries is through the mathematical properties of the symmetry group. As we will see, the symmetry group being compact or not will play a major role. We then qualify the symmetries to be compact or non-compact. Other possible criteria on which to classify the symmetries could be the way the symmetries act on the fields and on spacetime. We list here the main important classifications:

- Local symmetries are such that the parameters α^a in (2.1.3) are functions of spacetime. Otherwise, we speak of global symmetries.
- Spacetime symmetries are symmetries which act non-trivially on spacetime. Consequently, a non-spacetime symmetry will have $x'^\mu = x^\mu$ in (2.1.1). The typical examples of spacetime symmetries are translations, rotations, boosts etc.

- Internal symmetries are the ones where the generators commute with the Poincaré algebra. $U(1)$, $SO(2)$, $SU(N)$ ... are typical internal symmetries. Internal symmetries can be thought as non-spacetime symmetries. But the converse is false, a counter-example is the spatial linear shift symmetry $\phi(x) \to \phi(x) + \alpha_i\, x^i$, where α_i is a constant parameter, which is a non-internal symmetry (it does not commute with translations for example) and it is also a non-spacetime symmetry.

- Uniform symmetries are the ones where F^i in (2.1.1) does not depend explicitly on x^μ. For example, $SO(N)$ is a uniform symmetry. The spatial linear shift symmetry is a case where F^i does depend explicitly on x^μ, it is therefore not a uniform symmetry. An equivalent definition for uniform symmetries is when the realisation of the generators does not depend on spacetime coordinates. For instance, the generators of translations and rotations acting on a scalar field are respectively

$$P_i = -\partial_i \ , \ L_{ij} = x_i \partial_j - x_j \partial_i \ . \tag{2.2.1}$$

  We notice that translations are uniform symmetries while rotations are not. Let us mention that translations are the only spacetime symmetries which are uniform.

- Compact symmetries are the ones which realise compact groups.

In QFT, we mostly encounter internal symmetries and spacetime symmetries, where spacetime symmetries are more subtle since they act both on the internal space of fields and on the manifold on which the fields are defined on. The systematic recurrence of these two radical different types of symmetries is the reason why Goldstone physics literature has the tendency to classify the studied symmetries following being internal or spacetime. However, these two distinct sets of symmetries do not cover all the possible global continuous symmetries, the ones – as it will be seen – we are interested in Goldstone physics. A cartoon visualisation on how the different classifications of symmetries interconnect is proposed in Figure 2.1. From this cartoon, a more natural analysis seems to classify the symmetries following being uniform or non-uniform. By doing so, we are covering all the possible cases. The paper [41] argues that it would indeed be a more appropriate classification. In particular, they show that the known results in Goldstone physics for internal symmetries extend to uniform symmetries. In this dissertation, we will keep the historical approach by first presenting the internal case in Chapter 4 and then the spacetime situation in Chapter 5. We will explicitly comment on the non-uniform symmetries and indeed, from the abstract point of view, there is no major reasons to not do the analysis based on the uniform VS. non-uniform symmetries. Nevertheless, in practice, for example when building effective field theories, to make a distinction between spacetime symmetries and non-uniform symmetries in general can be relevant because the former act on spacetime coordinates, we therefore need to be vigilant on how we deal with the integration measure or with the derivative operators to keep the theory symmetric invariant.

2.3 Spontaneous symmetry breaking

Spontaneous symmetry breaking (SSB) is the phenomenon in which a stable state of the system transforms non-trivially under certain symmetries of the theory. These symmetries are then said to be spontaneously broken and the state is called the broken state [5].

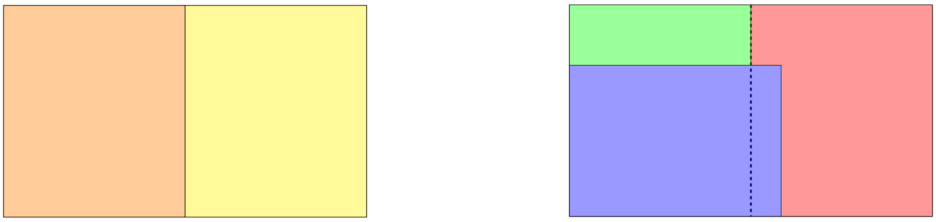

Figure 2.1: These two cartoons correspond to Venn diagrams where the frame rectangle corresponds to all the global continuous symmetries. On the left cartoon, the orange part are the non-uniform symmetries and the yellow part are the uniform symmetries. We see that these two sets cover all the global continuous symmetries with no overlap. On the right cartoon, the dashed line shows the previous separation between non-uniform symmetries and uniform symmetries. The blue part is the spacetime symmetries where the overlapping with the uniform symmetries are the translations. The red part are the internal symmetries which are entirely included in the set of uniform symmetries. We can observe that the union of the blue part and the red part does not cover all the frame rectangle. The remaining green part are the non-uniform symmetries which are not spacetime symmetries, e.g. polynomial shift symmetries.

2.3.1 At classical level

In classical field theory, the state of the system is characterised by one of the solutions of the EOM of the fields. We will call this particular solution the background, it can also be referred to as the vacuum. It is a stable solution if it remains finite along its evolution through spacetime and if small perturbations around it remain small along their dynamical evolution. It is customary to look for such stable background among the solutions which minimise the energy, at least corresponding to a local minimum. This could be intuitively understood from point-like classical mechanics where the conservative forces act in the opposite direction to the gradient of the potential. Hence, being originally at a minimum of the potential, we have that the forces tend to bring back the system to its original state. Furthermore, by minimising the kinetic energy, we ensure that the system has not enough inertia to pass a potential hill. Otherwise, the system could go from one potential minimum to another.

2.3.2 At quantum level

At the quantum level, for a given symmetry, the vacuum state $|0\rangle$ of the system breaks spontaneously this symmetry if

$$e^{i\alpha Q} |0\rangle \neq |0\rangle \text{ up to a global phase.} \tag{2.3.1}$$

However, this naive definition of SSB might not be well settled because the non-trivial action of the broken generator Q on our vacuum might lead to an ill defined state, i.e. a state with an infinite norm [42, 43]. Indeed, if $|0\rangle$ is homogeneous (i.e. an eigenstate of

P_μ) and Q is uniform – in addition to be Hermitian – then

$$\begin{aligned} ||Q\left|0\right\rangle||^2 &= \left\langle 0\right| Q^\dagger Q \left|0\right\rangle = \left\langle 0\right| QQ \left|0\right\rangle && (2.3.2)\\ &= \int d^{d-1}x \left\langle 0\right| j^0(x) Q \left|0\right\rangle && (2.3.3)\\ &= \left\langle 0\right| j^0(0) Q \left|0\right\rangle \int d^{d-1}x \ , && (2.3.4) \end{aligned}$$

which, in infinite volume, could tend to infinity if $Q\left|0\right\rangle \neq 0$. Notice that the symmetry being uniform, we have been able to express $j^0(x)$ as a translation in spacetime of $j^0(0)$,

$$j^0(x) = e^{ix^\mu P_\mu} j^0(0) e^{-ix^\mu P_\mu} \ . \qquad (2.3.5)$$

By still using the uniform aspect of the symmetry, we have that $[Q, P_\mu] = 0$ because Q is either internal or a spacetime translation generator (we look to the case without central charges). Then, considering our vacuum as being homogeneous (we are not currently looking to the breaking of spacetime symmetries) and choosing it as the zero-energy (we are not considering gravity, only the relative energy among states is physical), we have $P_\mu\left|0\right\rangle = 0$. Combining these observations allowed us to go from (2.3.3) to (2.3.4).

A more formal definition of spontaneous symmetry breaking is then used to evade this possible inconsistency. We will say that a state $\left|\psi\right\rangle$ breaks the symmetry generated by Q if there exists any field Φ, called the interpolaing field, such that [10]:

$$\left\langle\psi\right| [Q, \Phi(x)] \left|\psi\right\rangle \neq 0 \ . \qquad (2.3.6)$$

If no such operator Φ exists, the state is symmetric. An argument to use a local field Φ to define SSB is that we are working in infinite volume or more generally in the thermodynamic limit (cf. the coming section about singular limits). It is thus more convenient to be able to probe locally if the SSB occured rather than to perform a global analysis on the full ket state [44].

We can convince ourselves that both definitions (2.3.1) and (2.3.6) are consistent with each other. Indeed, if we had $Q\left|\psi\right\rangle = \lambda\left|\psi\right\rangle$, where λ is a real constant since Q is Hermitian, then

$$\begin{aligned} \left\langle\psi\right| [\Phi, Q] \left|\psi\right\rangle &= \left\langle\psi\right| \Phi Q \left|\psi\right\rangle - \left\langle\psi\right| Q\Phi \left|\psi\right\rangle && (2.3.7)\\ &= \lambda \left\langle\psi\right| \Phi \left|\psi\right\rangle - \lambda^* \left\langle\psi\right| \Phi \left|\psi\right\rangle && (2.3.8)\\ &= 0 \ \forall \Phi \ . && (2.3.9) \end{aligned}$$

So, we have that the relation (2.3.6) is a signature of a spontaneous symmetry breaking following the definition (2.3.1).

The notion of stability remains the same as in the classical theory: a small perturbation (e.g. local measurements [5]) of the state should not radically alter the state.

2.3.3 Generalities

In practice, to observe if an SSB occurred or not we define an order parameter $O(x)$. The order parameter should be zero when the symmetry is not broken and should be different from zero when the symmetry is spontaneously broken. Ideally, it should take different values for different broken states and broken states close to each other[5] should correspond to close values of $O(x)$. A possible order parameter is of course the definition of SSB itself (2.3.6). In Quantum Field Theory, it is customary to use the fundamental fields with nontrivial transformation properties under the symmetry as order parameter – $\phi(x)$ on-shell at the classical level and $\langle\phi(x)\rangle$ at the quantum level. We shortly designated this order parameter as the "VEV" (for vacuum expectation value).

Another device than the order parameter which can provide a clue if an SSB occured is the two-point correlation function. Correlation at long range can indeed be a signature of SSB. However, we will not often use this indicator in this dissertation. Therefore, we will further comment on it at the appropriated time (i.e. in Subsection 4.5.1).

Let us finish with a brief vocabulary comment. When we speak about spontaneous symmetry breaking, the use of the term "fundamental theory" can be misleading. Usually in physics, the fundamental theory refers to the fundamental microscopic theory / to the fundamental UV theory. In SSB physics, the fundamental theory is the theory we have prior the spontaneous symmetry breaking, it is therefore not necessarily the "standard model fundamental" UV theory. The term "fundamental theory" is thus used in order to contrast with the perturbation theory obtained from fluctuations around the broken state. From now on, in this dissertation, we will refer to the fundamental theory in the sense of SSB physics.

We close this section on the definition of spontaneous symmetry breaking with an important assertion. When a symmetry group G is spontaneously broken by a VEV ϕ_0 such that a subset $H \subset G$ of the symmetries still leaves ϕ_0 invariant, then H has automatically the structure of a group [24]. Indeed, by using the fact that G is a group (the product is defined and associative and the inverse exists), we have

- Trivially, $e \in H$ where e is the identity.
- $\forall h_1,\, h_2 \in H,\; h_1 h_2 \phi_0 = h_1 \phi_0 = \phi_0$ so, $h_1 h_2 \in H$.
- $\forall h_1 \in H \subset G$, $\exists h_1^{-1} \in G$. Furthermore, $h_1^{-1} h_1 \phi_0 = e\phi_0 \Leftrightarrow h_1^{-1}\phi_0 = \phi_0 \Leftrightarrow h_1^{-1} \in H$.

H is called the unbroken subgroup of G. If both G and H are continuous groups, the generators of H are labelised as the unbroken generators while the remaining ones of G are named the broken generators.

2.4 Singular limits

The way spontaneous symmetry breaking is naively defined might suggest that it is only a pure academical concept. Indeed, we are looking for a stable solution by minimising the energy, and see if there is arbitrariness in the choice of the vacuum due to the symmetries (cf. the example of Figure 1 where there is a set of possible vacua due to the rotational symmetry). But in Nature, any physical system interacts with the external world, at least weakly. These external interactions will make such that one particular state of the system is energetically favourable. Thus, there is no more arbitrariness, no more spontaneity in

[5] From one broken state we can get another one by applying the broken symmetry on it. Since we consider continuous symmetry group, the notion of "broken states close to each other" is understood in the sense of the continuous action of the symmetry group.

the choice of the background. The background is explicitly chosen by the dynamics and so, we have an explicit breaking of the symmetries rather than a spontaneous one.

The wondering can be deeper at quantum level where the notion of SSB might not even exists. Indeed, quantum superposition might allow the system to be in a superposition of broken states which end up as a symmetric state – for example, the system could be in a superposition of all the classical vacua of Figure 1 (the $U(1)$-circle at the bottom of the Mexican hat), this superposition all over the $U(1)$-circle does not choose a specific direction anymore and so, rotation symmetry is re-established. The same argument can hold for thermal physics where the thermal state being an average of microscopic states will be/can be symmetric.

Two additional arguments against the concept of SSB at quantum level can be formulated. The first one is that even if we initially place the system on one specific chosen vacuum, by quantum tunnelling, the system might evolve in a superposition of the degenerate vacua and so, end up in a stable symmetric state. The second additional argument is that in some cases, due to quantum superposition, the symmetric state has a lower energy than the one of the individual broken states. It is therefore favourable for the system to be in the symmetric state. The latter two arguments can be illustrated through an example [39]. Let us consider a classical potential with the $\mathbb{Z}_2$ symmetry and displaying two vacua. The quantisation around the first vacuum provides the state $|0,+\rangle$ and the quantisation around the second one gives the state $|0,-\rangle$. Because of the $\mathbb{Z}_2$ symmetry, we have

$$\langle 0,+| \, H \, |0,+\rangle = \langle 0,-| \, H \, |0,-\rangle \equiv a \ . \tag{2.4.1}$$

Due to quantum tunnelling, we as well have

$$\langle 0,+| \, H \, |0,-\rangle \equiv b \neq 0 \ . \tag{2.4.2}$$

Therefore, up to a global sign /global phase, the symmetric state $(1/\sqrt{n})\,(|0,+\rangle - |0,-\rangle)$, where n has the right value to normalise the state, has the energy $(2/n)\,(a - Re(b))$ which tends to be lower than the one of the individual broken states. In this example, SSB seems to be lost at quantum level.

These questionings on the well defined nature of SSB can be answered through the singular limits [5] (cf. later for the meaning of this expression). If we take the thermodynamic limit, which in zero temperature field theory corresponds to a large spatial volume limit, the system becomes highly sensitive to external perturbations. Thus, a small external perturbation is enough to explicitly select one of the degenerate vacua. This perturbation can be taken as small as we want thanks to the thermodynamic limit. Hence, the influence of the external world on the vacuum selection is not observable. From the point of view of the physicists, the selection of the vacuum is therefore spontaneous! We can see it with the example of the simplest Mexican hat model:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|) - \epsilon \, V_1(\phi) \ , \tag{2.4.3}$$

where ϕ is a complex scalar field, $V(|\phi|)$ is the Mexican hat potential of Figure 1, $V_1(\phi)$ is a perturbation promoting a specific phase of ϕ and ϵ is a small parameter. The conjugate momentum is given by $\pi = \partial\mathcal{L}/\partial(\partial_0\phi) = \partial_0\phi^*$ from which we construct the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \pi\partial_0\phi + \pi^*\partial_0\phi^* - \mathcal{L} = \partial_0\phi\partial_0\phi^* + \partial_i\phi\partial_i\phi^* + V(|\phi|) + \epsilon\, V_1(\phi) \\ \Rightarrow H &= \int_\Omega d^{d-1}x \left[\partial_0\phi\partial_0\phi^* + \partial_i\phi\partial_i\phi^* + V(|\phi|) + \epsilon\, V_1(\phi)\right] \ , \end{aligned} \tag{2.4.4}$$

where Ω is the spatial volume. The minimisation of the energy requires the vacuum ϕ_0 to be static and homogeneous. The norm $|\phi_0|$ is given by the minimisation of $V(|\phi|)$ which geometrically is given by the radius of the circle at the bottom of Figure 1. Finally, the vacuum is entirely settled thanks to the minimisation of $V_1(\phi)$ which fixes the phase of ϕ_0. It should be noticed that the influence of $V_1(\phi)$ on the minimisation of the energy can be finite even if ϵ is small thanks to the large volume limit:

$$H_1[\phi_0] = \lim_{\epsilon\to 0}\lim_{\Omega\to+\infty} \Omega\,\epsilon\, V_1(\phi_0) \neq 0 \ . \tag{2.4.5}$$

We recover the idea that in the thermodynamic limit, the perturbation can be taken so small that it is not observable[6] but yet, it can select a vacuum.

The two limits intervening in (2.4.5) are not commuting. Indeed, taking the $\epsilon \to 0$ limit first corresponds to set ϵ to zero from the beginning, i.e. in (2.4.3). The latter case is strictly equivalent to the unperturbed case which leads to $H_1[\phi_0] = 0$. Considering the thermodynamic limit first permits to make the system sensitive to perturbations. Such non commuting limits are called "singular limits".

The thermodynamic limit permits as well to answer to the problematic of quantum superposition [24]. In order to be in a superposition of states, the system must be able to transit between these different states (for simplicity, let us assume that the latter ones belong to the vector basis diagonalizing the observable we are measuring). Indeed, from the postulate of the reproducibility of the measurement, the state collapses onto the eigensubspace associated with the eigenvalue measured. In our example (2.4.3), even if there is no potential barrier between the potential minima, because we are working in infinite volume, it requires an infinite amount of energy to move from one vacuum to another. In fact, to do such a jump, the field ϕ is now dynamical $\phi(t)$ (but still considered as homogeneous – we assume the transition to be fast and so, imposing a global collapse). Hence, there is a kinetic energy cost which is arbitrarily large

$$E_{kin} \propto \Omega\, \partial_0\phi\partial_0\phi^* \underset{\Omega\to+\infty}{\longrightarrow} +\infty \ . \tag{2.4.6}$$

Thus, a superposition of several fundamental states is not achievable.

The argument from the preceding paragraph could naively be used as well to argue against quantum tunnelling between degenerate vacua: because of the thermodynamic limit, the energy cost to go from one classical vacuum to another is too high and so, the probability of tunnelling is vanishing. However, on the contrary to the collapse due to a measurement, quantum tunnelling does not need to be fast. Hence, the evolution from one vacuum to another does not need to be global, local transitions are possible. Studying quantum tunnelling in QFT requires to find particular solutions of the equations of motion evolving from one classical vacuum to another (in the Euclidean time formalism, we look for bouncing solutions) [24]. This is a non-trivial issue since we are dealing with non-linear differential equations. Hence, there is no general/generic arguments to discuss the effect of quantum tunnelling on SSB besides the fact that the thermodynamic limit avoids global transitions between vacua. Each model should be dealt with case by case.

[6] Let us emphasise that the influence of $V_1(\phi_0)$ on the energy is finite but tiny compared to $V(|\phi_0|)$ which scales proportionally to Ω. So, an observer will not discriminate the case $\epsilon = 0$ from the case $\epsilon \neq 0$ by measuring the energy.

Finally, the thermodynamic limit can solve the issue that, at quantum level, symmetric states might have lower energy than the broken states. In our example of the theory with the $\mathbb{Z}_2$ symmetry, we can argue that quantum tunnelling tends to be suppressed. Hence, b in (2.4.2) is small. The symmetric state is now almost at the same energetic level than the broken states. A small external perturbation is therefore enough to impose one of the broken states to be the selected vacuum state (we recover the notion of singular limits).

From this discussion, we can draw two conclusions:

1. We provided hand-waving arguments that the notion of spontaneous symmetry breaking is well defined and physical. Our discussion does not constitute a formal proof of the later statement, however, it proves that there are at least some physical systems which do indeed present spontaneous symmetry breaking. It is therefore worth to study the SSB mechanism.

2. The lack of formal developments in our discussion displays that it is a highly non-trivial question to know generically if a given system can or cannot sustain an SSB. There are some general theorems on this problematic, we will mention them later on (cf. Section 4.5), however such a question is usually dealt with case by case.

The fact that we are working in Minkowski spacetime and so, in an infinite spatial volume, ensures the singular limits to be satisfied. Hence, in our framework, we evade most of the above-mentioned conceptual problems about SSB.

2.5 Mass and gap

Goldstone's theorem refers to massless particles. The notion of mass is a central idea of Goldstone physics. This is the reason why we will briefly remind here this standard notion. Let us notice that we work on Minkowski spacetime and therefore, we do not do general relativity or QFT on curved spacetime. Hence, we will not encounter the related difficulties to define energy and momentum as well as their conservation. The mass we are going to discuss is the QFT textbook definition.

2.5.1 The notion of mass

In classical field theory, the square of the mass is given by the coefficient of the quadratic no-derivative term in the action ($m_0^2\,\phi\phi$) [24]. In this work we are just concerned if this term is present or not, which will tell us if the associated field is massive or not. For a quantum particle, we use the relativistic definition which is, its mass is its energy in the zero-limit of the spatial-momentum. The classical and quantum definitions are consistent. Indeed, the particle states of a QFT correspond to the asymptotic states of the theory, i.e. the spectrum we get while quantising the free theory. From standard QFT textbooks, quantising a free theory tells us that the energy corresponds to the dispersion relation, which in the free case is of the form $\omega^l = v\,p^n + m_0^2$, where l and n are respectively the number of time-derivative and the number of space-derivative (of the dominant terms), and v is a constant. Sending the momentum p to zero, we see a correspondence between the classical mass and the quantum mass. In particular, when $l = 2$ (it could be the relativistic case $l = n = 2$), we recover the standard idea that the square of the mass is m_0^2.

However, even if the two definitions are consistent, due to the renormalisation, the quantum mass might be different from the classical mass. The classical mass is a bare parameter which might need to be redefined through renormalisation conditions. The physical interpretation is that, in presence of interactions, the effective mass that we observe is different from the bare mass due to the self-energy of the particle: through its propagation, other particles can be created and annihilated which modifies the apparent propagation of our studied particle. The renormalisation condition to define the physical (i.e. the observable) mass is the pole of the connected propagator at low energy [36]. In particular, such mass can then be extracted from the quantum effective action $\Gamma[\phi]$.

Goldstone's theorem is valid at classical level as well as at quantum level. This suggests that the masses (which are zero) of the Nambu-Goldstone modes are symmetry protected during the quantisation (modulo that no anomalies occur and that the SSB is not altered by the quantisation). A specific computation at one loop for the linear sigma model is done in [35] to illustrate this assertion.

2.5.2 The generalisation of mass: the notion of gap

To define the mass, we need the energy and the momentum to be defined. Hence, we need continuous spacetime translation symmetries for our theory. For the explicit computations and proofs, we will perform in this part of the dissertation, we will assume to have such symmetries.

However, it is not uncommon that physical systems do not have continuous spacetime translation symmetries. For example, crystal lattices do have "only" discrete spatial translations. Another example could be open macroscopic systems which do not have time translation symmetry since the external world can at any time modify the value of the conserved quantities (cf. the chemical potentials). For these kinds of examples, it is possible to generalise the notion of mass, we then speak about gaps. In particular, the gap can be defined with respect to the free energy rather than the energy [11, 12] and it can use crystal momentum rather than momentum [25, 45].

In thermal quantum field theory, the apparent gap is the pole of the thermal propagator [33] and it can thus be extracted from the thermal effective action $\Gamma_\beta[\phi]$. The interpretation of this gap is that, at zero chemical potential, the bare mass of the particle propagating in a thermal bath will have quantum corrections coming from its self-energy and thermal corrections coming from its interactions with the thermalised surrounding particles[7].

The last case we will present, and which is necessary for this thesis (in particular for

[7] The renormalisation of a thermal QFT due to UV divergences is the same as the associated QFT at zero temperature, i.e. the counterterms are independent of the temperature [33]. It can be understood by the fact that in the UV, the energy scale is much bigger than the thermal energy. Hence, the field theory in the UV is blind with respect to the temperature. Therefore, defining a quantum mass was strictly necessary to cancel the UV divergences but defining a thermal mass on top of it is for convenience. It should be mentioned that temperature has a significant influence in the IR. As it will be commented later, the IR divergences play a role on the possibility or not to have spontaneous symmetry breaking (see Section 4.5). We therefore understand that temperature will have an impact under which conditions a system can display SSB (cf. Subsection 4.5.3).

Part II), is a sort of in-between situation. It is called the homogeneous breaking. This is when translation symmetry is spontaneously broken following one direction of spacetime but that the perturbative theory around the VEV does not explicitly depend on the broken spacetime coordinate (as we would have expected). This is made possible when there is an additional internal symmetry which is also spontaneously broken in a way that it can compensate the action of the broken translation on the VEV.

Let us clarify this cryptic description with an example. We consider a spacetime translation invariant fundamental Lagrangian $L[\phi, \partial\phi]$ with an additional $U(1)$ symmetry. The fundamental field is considered to be a complex scalar field $\phi(x)$. If the VEV is given by

$$\phi_0(t) = v\, e^{ict} \;, \tag{2.5.1}$$

where v and c are constants, we have that $U(1)$ and time translation symmetries are spontaneously broken, however the transformation

$$t \to t + a \;, \qquad \phi \to e^{-i\,a\,c}\phi \;, \tag{2.5.2}$$

still leaves the background $\phi_0(t)$ invariant. In the Lie algebra language, the generators P_0 and Q (respectively time translation and $U(1)$ generators) are broken but the diagonal direction $P_0 - c\,Q$ is not. We therefore sometimes call it a diagonal breaking. Now, if we perform a fluctuation around $\phi_0(t)$, the Lagrangian for the fluctuations will not depend explicitly on time. This is because

$$L[\phi_0(t_1), \partial\phi_0(t_1)] = L[\phi_0(t_2), \partial\phi_0(t_2)] \;, \quad t_1 \neq t_2 \;. \tag{2.5.3}$$

Indeed, evaluate the fundamental Lagrangian on the background at two different times is equivalent to apply a $U(1)$ symmetry (cf. the shape of $\phi_0(t)$). But the fundamental Lagrangian is invariant under $U(1)$, hence, the value of the fundamental Lagrangian evaluates on the background is the same at any time t. We can thus perform the fluctuations at any time t around $\phi_0(t)$, the perturbed Lagrangian will always be the same. The latter will have no explicit time dependency. As a consequence, we can define energy as usual for the perturbation theory despite that originally time translation is spontaneously broken! From the perspective of the fundamental theory, this energy corresponds to the conserved quantity associated to $P_0 - c\,Q$ rather than P_0. This is why, strictly speaking, the masses we will compute for the perturbations are rather gaps.

This example for the homogenous breaking of time translation has been thoroughly studied in [46], it plays a major role in QFT at finite density as it will be seen in Section 5.3 and in Part II. The discussion holds as well for the homogeneous breaking of space translation [47], a toy model displaying that situation will be studied in Part II.

Finally, even though in this part (i.e. Part I) of the dissertation we will compute assuming spacetime translations invariance, we will mention if the same final results can be obtained by relaxing this hypothesis.

Chapter 3

Goldstone's theorem

Goldstone's theorem has been mainly established in the sixties and has been refined over the following decades. It is Nambu who first conjectured the existence of a relation between symmetries and masses [1, 2][1]. Goldstone improved the conjecture of Nambu by specifying the notion of spontaneous symmetry breaking and by stressing the importance that the broken symmetry should be continuous [3]. Goldstone, Salam and Weinberg provided two general poofs of the conjecture in [4]. Following this publication, several other papers came out in order to clarify under which hypotheses Goldstone's theorem is valid [41, 48–54][2]. Some alternative (formal/axiomatic) proofs and corollaries have also been provided, e.g. [59–62]. These research efforts led to the current statement of Goldstone's theorem.

Theorem 1 (Goldstone's theorem)**.** *Let us consider a physical (field) theory at the quantum level, respectively at the classical level, with a global continuous symmetry group G (realised non-trivially on the theory) such that it is spontaneously broken to a subgroup H different from G ($H \subsetneq G$) and that the notion of gap is well defined. Then, the spectrum of the theory will contain at least one gapless particle, respectively at least one gapless mode. This statement remains true for thermal theories (non-zero temperature and/or non-zero chemical potentials).*

Let us notice how generic G can be: it can be uniform or non-uniform, involving spacetime symmetries or not, being compact or not etc. Furthermore, the theorem is relatively loose concerning the notion of mass (therefore we speak about gaps). The theorem is thus valid for theories defined on crystal lattice, for (thermal) open systems etc. Finally, the locality requirement of the theory is hidden in the "physical" aspect. More explicitly, the interactions should at most have a finite range or an exponential spatial decay (otherwise, the validity of the theorem should be checked case by case) [9, 52–54]. In conclusion, Goldstone's theorem is very general!

There are two main proofs of Goldstone's theorem, one which is using the quantum effective action formalism and one which is established in the Dirac notation of quantum mechanics. The first one is a perturbative proof. The second proof is stricter on the hypotheses than what is mentioned for Theorem 1 but it permits to display straightforwardly the spectral content and it is an exact proof. Let us begin with an intuitive explanation of why Goldstone's theorem holds, we will then develop each of the two proofs. We emphasise that the proofs present here, and in general in this introduction to Goldstone physics,

[1] Let us mention that the second cited paper is in collaboration with Jona-Lasinio. The first cited paper is written by Nambu alone and is older than the second cited one. Furthermore, Nambu being the common thread between the two papers, he is considered as the principal investigator of the conjecture.

[2] Since we are in the historical genesis of Goldstone's theorem, it should be mentioned that, sometimes in the literature, NG modes are labelled as pions. This because historically, NG modes were studied in the framework of particle physics and light mesons analysis (e.g. [2, 55, 56]). For example, [57, 58] which are cornerstone papers in the building of effective theory for NG modes call the latter pions.

are not meant to have an orthodox mathematical rigour, we refer the reader to the cited papers and the references therein for the formal and axiomatic developments.

3.1 Intuitive picture of Goldstone's theorem

Goldstone's theorem and its hypotheses can be understood intuitively. We have that the broken state is degenerate. Indeed, we can get a set of broken states by applying successively the spontaneously broken symmetries on our broken states. Let us call this set of so obtained broken states the coset space[3]. If we do the shortcut that the symmetries of the theory are also the symmetries of the energy, we have that all the broken states of the coset space have the same energy. For simplicity, we do not consider the breaking of spacetime symmetries. Hence, there is no interplay between the kinetic energy and the potential energy while applying the spontaneously broken symmetries on the broken states. So, the broken states of the coset space have the same potential energy. Furthermore, the broken symmetries are continuous, which means that the coset space is continuously connected as well. Therefore, there is no potential hill between the broken states. A possible visualisation is to consider the Mexican hat potential example of Figure 1. The degenerate broken states correspond to the $U(1)$-circle lying at the bottom of the potential[4], we do indeed observe that there is no potential hill between them. Thus, fluctuations around a chosen broken state, in the directions of the broken symmetries, will, at quadratic order, not have potential terms in the perturbation Lagrangian. Hence, such fluctuations are massless. These are precisely the (candidate[5]) NG modes! NG modes correspond to a spacetime modulated action of the spontaneously broken symmetries on the considered background.

This schematic reasoning allowed us to understand why SSB leads to massless modes and in particular, why the continuity of G is crucial to reach masslessness. Concerning the global aspect of G, it permits to evade the Brout-Englert-Higgs mechanism [6, 7, 63–65] which illustrates that the NG modes coming from the spontaneous breaking of local symmetries are absorbed by gauge transformations and are thus unphysical – not observable. This result of the Brout-Englert-Higgs mechanism can be understood. We have seen that NG modes are spacetime modulated fluctuations in the directions of the broken symmetries. Gauge transformations are arbitrary spacetime modulated fluctuations in the directions of the gauge symmetries, and in particular in the directions of the broken gauge symmetries. In physics, we need to fix by hand this arbitrariness, interpreted as a redundancy, otherwise the theory would not be predictable. This because the solutions of the EOM are obtained up to arbitrary spacetime functions, and it would require an infinite number of boundary conditions to univocally determine the solutions – this means an infinite number of measurements. Since the broken gauge redundancies are in the same directions than the fluctuations associated to the corresponding NG modes, we can simply

[3]This nomenclature can be understood by seeing the spontaneously broken symmetries as elements of G/H. The broken states being obtained by the successive action of the spontaneously broken symmetries, they are parametrised by the coset space G/H.

[4]Notice that the $U(1)$-circle is indeed the coset space $G/\{e\}$, where e is the identity, corresponding to the full spontaneous breaking of $U(1)$ symmetry.

[5]It remains to see if these fluctuations are independent from each other. We understand as well that, by definition, the number of NG modes is bounded from above by the number of broken generators.

fix the gauge by setting to zero the NG modes. This picture is formally commented in Subsection 4.2.1 in the coset construction framework.

Let us mention that discussing SSB of gauge symmetries might even be not relevant at all. Indeed, the Elitzur theorem, briefly stated, says that local symmetries cannot spontaneously be broken at quantum level [66]. This assertion can be intuitively understood from the discussion associated to equation (2.4.6). We have seen that no quantum superposition of degenerate vacua was possible because going from one vacuum to the other was equivalent to perform a global symmetry transformation (i.e. a transformation occurring on all the spatial manifold) which due to the large volume limit corresponds to an infinite amount of kinetic energy. In the case of local SSB, the vacua are connected between them through local transformations. Thus, we can go from one to the other with a transformation localised on an infinitesimal subregion of the spatial manifold. The kinetic energy cost is not anymore proportional to the entire volume of the system but only to the volume of this infinitesimal subregion. Hence, the kinetic energy cost is infinitesimally small and quantum superposition over the degenerate vacua is possible. At quantum level, it seems that the state is a symmetric state for local symmetries. We do not have SSB of gauge symmetries at quantum level.

Following this intuitive picture of Goldstone's theorem, we can go even further on collecting information on the properties of the NG modes. We have that the Fourier transform of the spacetime modulated action of the spontaneously broken symmetries tell us how fast these modulations fluctuate through spacetime. If we go in the IR, usually we use the scale of the VEV to determine what low energy means, it is equivalent to look for modulations with small wave vectors k^μ and thus fluctuating with long wavelength. In the zero k^μ limit, the modulations become constant over spacetime. So, in this limit, the modulated action of the spontaneously broken symmetries is nothing else than the regular action of symmetries joining two vacua. Hence, the NG modes do no provide additional energy to the background. This is the signature that they do not interact. We arrive at the conclusion that in the IR, the NG modes are weakly coupled[6]. Finally, it is customary to hear about the NG modes as "NG bosons". This is because many of the practical cases involve only internal spontaneously broken symmetries. Indeed, the action of such symmetries does not mix the Lorentz group representations (the symmetry algebra commutes with Lorentz algebra). So to speak, the algebra of G is spin zero, hence, the fluctuations produced by such elements by acting on the vacuum are scalars (it is formally shown in Subsection 4.2.1). The common example of NG modes with a non-trivial spin are the Goldstinos coming from the spontaneous symmetry breaking of supersymmetry which mix non-trivially the Lorentz representations – in particular it links a boson with a fermion, we thus understand that the fluctuations should have a non-trivial spin.

What to bear in mind

We learned that the definition of an NG candidate is a fluctuation around the background in the direction of one of the spontaneously broken generators. A thorough analysis is still needed to establish if the NG candidates are independent or not, but we know that

[6] Let us stress that we are doing here an intuitive reasoning which is mostly valid for the spontaneous breaking of internal symmetries. However, even for internal symmetries some examples can be found where NG modes are not weakly coupled in the IR. Here are two references (among others) which discuss this particular feature [67, 68].

the related independent degrees of freedom are massless and (most of the time) weakly coupled in the IR, these are the NG modes. Furthermore, the spontaneous breaking of internal symmetries leads to scalar NG modes, we then qualify them as (Nambu) Goldstone bosons.

3.2 Perturbative proof

We follow the proof of [4, 54]. Let us consider a QFT with spacetime translation symmetry and described by a quantum effective action[7] $\Gamma[\phi]$ invariant under the infinitesimal transformation $\delta\phi^i(y) = \alpha^a F^i_a[y,\phi] \equiv F^i[y,\phi]$. The associated symmetry is assumed to be spontaneously broken by the VEV of the theory $\langle\phi(x)\rangle \equiv v(x)$, i.e. $F^i[y,v] \neq 0$ for at least one value of i.

From the symmetry, we have

$$\delta\Gamma[\phi] = \int d^d y \, \frac{\delta\Gamma[\phi]}{\delta\phi^i(y)} \, F^i[y,\phi] = 0 \tag{3.2.1}$$

$$\Rightarrow \int d^d y \, \left.\frac{\delta^2\Gamma[\phi]}{\delta\phi^j(x)\delta\phi^i(y)}\right|_v \, F^i[y,\phi]\big|_v = 0 \ , \tag{3.2.2}$$

where we used (2.1.19) while differentiating and evaluating the result on the VEV. Let us notice that the second derivative of $\Gamma[\phi]$ is proportional to the inverse of the propagator matrix, cf. (2.1.18). We abbreviate the notation of the 2-point correlators as $G_{ji}(x,y)$ and thanks to the spacetime translation invariance, we have $G_{ji}(x,y) = G_{ji}(x-y)$. By multiplying by $e^{ik_\mu x^\mu}$ and integrating on x, we can make appear the Fourier transform of the propagator matrix and we can study its poles:

$$\int d^d y \, d^d x \, e^{ik_\mu x^\mu} G^{-1}_{ji}(x-y) \, F^i[y,\phi]\big|_v = 0 \tag{3.2.3}$$

$$\Leftrightarrow G^{-1}_{ji}(k) \int d^d y \, e^{ik_\mu y^\mu} \, F^i[y,\phi]\big|_v = 0 \ , \tag{3.2.4}$$

where we made the change of variable $x = l + y$. In the limit of $k^\mu \to 0$, the remaining integral on y is non-zero thanks to the hypothesis of SSB and it is finite (we make the physical and customary assumption that $\delta\phi^i(y)$ has a finite support). As a consequence, we conclude that the matrix $G^{-1}_{ji}(k)$ has at least one zero eigenvalue in the limit $k^\mu \to 0$, which means that in a suitable basis, at least one propagator has a pole at $k^\mu \to 0$, thus the spectrum contains at least one massless particle.

If we replace the quantum effective action $\Gamma[\phi]$ by the thermal effective action $\Gamma_\beta[\phi]$, we can reproduce exactly the same reasoning and arrive at the same conclusion. It shows that Goldstone's theorem is indeed valid as well for thermal systems [40]. We can also replace $\Gamma[\phi]$ by the classical action $S[\phi]$. In such a case, $G_{ji}(k)$ is the Fourier transform of the quadratic part of $S[\phi]$ and taking the limit $k^\mu \to 0$ permits to isolate the quadratic part of $S[\phi]$ containing no derivatives. Hence, $G_{ji}(0)$ is the classical mass matrix. We thus arrive at the same conclusion: $G_{ji}(0)$ has at least one zero eigenvalue which means that we have at least one classical massless mode. Goldstone's theorem is also valid at classical level [24, 35].

[7]Hence, we implicitly ask for the axiomatic hypotheses necessary for a QFT to have a well defined effective action to be satisfied.

3.3 Spectral decomposition proof

As already mentioned, to perform the spectral decomposition proof we have to revise some hypotheses of Theorem 1. We will consider the group G to be global continuous and uniform. The continuity of G allows to define Noether currents. Asking G to be global permits to evade gauge symmetries for which the conserved currents (defined through Noether's first theorem) are trivial (in the sense of the equivalence relation) and so, give trivial conserved charges [34,69]. As we will see, Goldstone modes rely on these conserved currents, and the starting point of the proof is a non-zero conserved charge. This is a technical explanation on why G cannot be local which adds to the already provided intuitive reasoning based on the Brout-Englert-Higgs mechanism. The last constraint on G, i.e. to be uniform, is imposed to avoid the case of spacetime symmetries (spacetime translations symmetry breaking will be ruled out by considering a homogeneous vacuum) and to ease the technicalities of the computations.

Now that we are ensured to have properly defined conserved currents (continuous global G), we have to guarantee to be able to associate to them conserved charges. This is done by specifying what we mean by locality. We will ask the interactions to be local enough such that

$$\int_{\partial V} dS_i \, j^i(x) = 0 \ , \tag{3.3.1}$$

where V is the spatial volume of the system. Hence,

$$\frac{dQ}{dt} = \int_V d^{d-1}x \, \partial_0 j^0 = -\int_V d^{d-1}x \, \partial_i j^i = -\int_{\partial V} dS_i \, j^i(x) = 0 \ . \tag{3.3.2}$$

A more precise statement is made in [43].

As cited above, we are not considering the spontaneous breaking of spacetime symmetries. It implies that our vacuum $|0\rangle$ is homogeneous (i.e. an eigenstate of P_μ). Also, since we do not include gravity, we chose $|0\rangle$ to be the zero of energy: $P_\mu |0\rangle = 0$.

Finally, the main hypothesis of Goldstone's theorem is that we have spontaneous symmetry breaking. Let Q be a generator of G such that Q is spontaneously broken. By definition, it exists a field Φ giving

$$\langle 0| \, [Q, \Phi(x)] \, |0\rangle \neq 0 \ . \tag{3.3.3}$$

To prove Goldstone's theorem under the aforementioned hypotheses, we study the spectral decomposition of (3.3.3) by injecting a closure relation where the basis vectors $|n_{\vec{k}}\rangle$ are eigenvectors of P_μ [9].

$$\langle 0| \, [Q, \Phi(x)] \, |0\rangle = \int d^{d-1}x' \, \langle 0| \, [j^0(x'), \Phi(x)] \, |0\rangle \tag{3.3.4}$$

$$= \int d^{d-1}x' \sum_n \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \Big(\langle 0| \, j^0(x') \, |n_{-\vec{k}}\rangle \langle n_{-\vec{k}}| \, \Phi(x) \, |0\rangle - \langle 0| \, \Phi(x) \, |n_{\vec{k}}\rangle \langle n_{\vec{k}}| \, j^0(x') \, |0\rangle \Big) \ . \tag{3.3.5}$$

As we did in (2.3.5), thanks to Q that generates a uniform symmetry, we translate the conserved current to the origin:

$$\langle 0| [Q, \Phi(x)] |0\rangle = \int d^{d-1}x' \sum_n \int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{ik_\mu x'^\mu} \big(\langle 0| j^0(0) |n_{-\vec{k}}\rangle \langle n_{-\vec{k}}| \Phi(x) |0\rangle - \langle 0| \Phi(x) |n_{\vec{k}}\rangle \langle n_{\vec{k}}| j^0(0) |0\rangle\big) \tag{3.3.6}$$

$$= \sum_n \int d^{d-1}k \, e^{iE_n(\vec{k})t} \varphi(\vec{k}) \big(\langle 0| j^0(0) |n_{-\vec{k}}\rangle \langle n_{-\vec{k}}| \Phi(x) |0\rangle - \langle 0| \Phi(x) |n_{\vec{k}}\rangle \langle n_{\vec{k}}| j^0(0) |0\rangle\big) \ , \tag{3.3.7}$$

where,

$$\int \frac{d^{d-1}x'}{(2\pi)^{d-1}} e^{-i\vec{k}\vec{x}} = \varphi(\vec{k}) \xrightarrow[V\to+\infty]{} \delta^{d-1}(\vec{k}) \ . \tag{3.3.8}$$

From (3.3.8), we have that only the modes in the zero-momentum limit intervene in the integral of (3.3.7). Furthermore, since we have $dQ/dt = 0$, it means that the only time dependence in (3.3.7) is coming from $\Phi(x)$. Thus, the exponential should not intervene. Therefore, only the modes with

$$E_n(\vec{k}) \xrightarrow[\vec{k}\to\vec{0}]{} 0 \ , \tag{3.3.9}$$

i.e. the massless modes, should contribute to the sum over n. Finally, with the hypothesis (3.3.3), the final result should be non-zero. Thus, we are ensured there exists at least one particle[8] $|n_{\vec{k}}\rangle$ which is massless and which is such that $\langle 0| \Phi(x) |n_{\vec{k}}\rangle \langle n_{\vec{k}}| j^0(0) |0\rangle \neq 0$. These are the NG modes, we learned that, besides being massless, they are created by the action of the broken symmetry on the vacuum (here represented by $j^0(0)|0\rangle$) in a way which still needs to be clarified/formalised (cf. the coset construction, Subsection 4.2.1).

The reason we did not write $\varphi(\vec{k})$ directly as a Dirac delta is to emphasise that the evaluation of (3.3.7) at $\vec{k} = 0$ should be understood as a limit (coming from the infinite volume limit). Hence, we should not consider isolated momentum eigenstates with zero eigenvalue (i.e. spurious states). So, the NG modes are indeed properly defined particles. A more detailed discussion on the spurious states can be found in the literature (e.g. [43, 52, 53, 70]). From this discussion, we learn that the hypothesis on locality is not only there to guarantee charge conservation but also to avoid spurious states which would invalidate our conclusion on (3.3.7). The final outcome on locality is that the theory should have a well behaved range of interactions (at most finite range or exponentially decreasing with distance). If it is not the case, it should be checked case by case if the Noether charges are time independent [9]. Notice that field theories with non-local interaction terms which cannot be written as a single spacetime integration could then be allowed [54].

To avoid computational heaviness, we looked at a homogeneous vacuum. But this hypothesis excludes a large area of applications in condensed matter. The spectral decomposition proof can be generalised to the case where the fundamental theory possesses

[8] The term "particle" is used in a generic way. Since these degrees of freedom are not the fundamental ones, we usually speak of quasi-particles or collective excitations.

continuous spatial translation symmetries in some directions and discrete ones in the remaining spatial directions. The spectral proof can also be extended to the situation where we have the spontaneous breaking of continuous spatial translation symmetries to discrete symmetries (i.e. we can include the formation of lattices) [41].

3.4 Toy model: spontaneous symmetry breaking of $U(1)$

Until now, we remained abstract in the development of what Goldstone's theorem is. The aim of this subsection is to illustrate the different results we obtained so far with a concrete example. To do so, let us consider the following toy model of a complex scalar field in $d \geqslant 2$ dimensions

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + M^2 \phi^* \phi - \lambda(\phi^* \phi)^2 \quad \text{with } M^2 > 0 \text{ and } \lambda > 0 \ . \tag{3.4.1}$$

The potential term $V(|\phi|) = -M^2\phi^*\phi + \lambda(\phi^*\phi)^2$ is the Mexican hat potential of Figure 1.

We can observe that the theory (3.4.1) is invariant under the $U(1)$ symmetry

$$\phi(x) \to e^{i\alpha}\phi(x) \ . \tag{3.4.2}$$

To find a stable background, we look for a particular solution of the EOM which minimises the energy. We ask this solution to be a non-zero constant ϕ_0 to minimise to kinetic energy[9]. Concerning the potential energy, we impose

$$\left.\frac{dV(|\phi|)}{d|\phi|}\right|_{\phi_0} = 0 \ \Leftrightarrow \ |\phi_0| = \sqrt{\frac{M^2}{2\lambda}} \equiv v \ . \tag{3.4.3}$$

From the energy minimisation, the phase remains unspecified (it corresponds to the $U(1)$ circle at the bottom of the Mexican hat), therefore, we will arbitrarily choose it to be zero. Notice that $\phi_0 = 0$ would also extremise the energy but it would correspond to a maximum and thus, to an unstable background. So, our particular solution is $\phi_0(x) = v$ (it is straightforward to check that it is indeed a solution of the EOM). It breaks spontaneously the $U(1)$ symmetry because it transforms non-trivially under it

$$v \to v\, e^{i\alpha} \ . \tag{3.4.4}$$

We have that all the hypotheses of Goldstone's theorem are satisfied. Hence, we expect to find at least one massless mode in the perturbation theory. To explicitly verify it, we parametrise the fluctuations as

$$\phi(x) = (v + \sigma(x))\, e^{i\theta(x)} \ . \tag{3.4.5}$$

The perturbation theory up to third order is

$$\mathcal{L} = \partial_\mu\sigma\partial^\mu\sigma + v^2\partial_\mu\theta\partial^\mu\theta - 2M^2\sigma^2 - 2\sqrt{2\lambda M^2}\,\sigma^3 + 2\,v\,\sigma\,\partial_\mu\theta\partial^\mu\theta + \mathcal{O}\left(\epsilon^4\right) \ . \tag{3.4.6}$$

[9] $E_{\text{kin}} = |\partial_t\phi|^2 + |\partial_i\phi|^2 \geqslant 0$.

where $\epsilon \sim \theta \sim \sigma$. We observe that $\theta(x)$ is a massless mode. Is it the predicted NG mode or is it a matter of luck ? We have that $\theta(x)$ parametrises a perturbation of the vacuum v in the direction of the action of $U(1)$

$$v \to v\, e^{i\theta(x)} \ , \tag{3.4.7}$$

where the arrow corresponds to a spacetime modulated $U(1)$ action. We see that $\theta(x)$ is by definition an NG mode – a broken symmetry originated massless mode – and so, the prediction of Goldstone's theorem is indeed satisfied for our toy model.

If we pay attention to the interaction term of the NG mode $\theta(x)$, it involves derivatives. If we go in Fourier space, we understand that this interaction term will go to zero at low energy. We recover the idea that NG modes are weakly coupled in the IR.

We could have guessed the shape of the interaction terms for $\theta(x)$ based on the $U(1)$ symmetry. Indeed, the fundamental theory (3.4.1) is invariant under $U(1)$ and so should be the perturbation theory (3.4.6). The transformation rule of $\theta(x)$ is given by

$$(v + \sigma(x))\, e^{i\theta(x)} \to (v + \sigma(x))\, e^{i\theta(x)} e^{i\alpha} \ , \tag{3.4.8}$$

so,

$$\theta(x) \to \theta(x) + \alpha \ . \tag{3.4.9}$$

We have that $\theta(x)$ transforms as a shift and $\sigma(x)$ is invariant. Therefore, the perturbation theory (3.4.6) is $U(1)$ invariant if on each $\theta(x)$ there is a derivative acting on it. This explains why $\theta(x)$ is massless and why its interactions go to zero at low energy. This can be verified empirically by coding a computer symbolic program (e.g. on Mathematica) for this toy model and run the code to any desired (finite) higher orders. Each term involving the field $\theta(x)$ will have at least one derivative in it.

We can now have a sense of why we say that the symmetries which are spontaneously broken are "hidden". It is because they are non-linearly realised in the perturbation theory (cf. (3.4.9)) which makes it not always convenient to see the symmetry invariance. In addition, NG modes are transforming non-homogeneously (cf. (3.4.9), even if we evaluate $\theta(x)$ at zero, it still transforms) which explains the systematic derivative operators acting on them.

We should emphasise that the intuition we acquired from Sections 3.1 and 3.4 is valid for the spontaneous breaking of internal symmetries. When the breaking of spacetime symmetries is involved, there are additional conceptual and technical difficulties which might spoil some of the intuitive results we derived so far (but the masslessness aspect of NG modes claimed by Goldstone's theorem remain robustly true).

A comment about the boundary conditions

In order to extract the EOM from the fundamental action, we apply the variational principle. To do so, we need to integrate by part which leads to a boundary term in the variation of the action. In standard QFT textbooks, we consider the fields to be zero at infinity which erases this boundary term. In our case, considering a particular non-zero constant solution $\phi_0(x) = v$ does not satisfy the zero-at-infinity assumption. However, the boundary part of the variational principle is of the form $\delta\phi^* \partial\phi$ which is zero for a constant field. Hence, $\phi_0(x) = v$ is a tolerated classical solution.

On a more general level, a possible consistent way to impose boundary conditions is the following. We regularised the (spacetime) volume by putting the system in a large finite-size (spacetime) box and we impose periodic boundary conditions, similarly to the Born–von Karman boundary conditions in condensed matter (it can be justified by an isotropic influence of the external world on our system for example). Because these boundary conditions are periodic, the boundary term from the variational principle will be equal to zero. We can then retrieve our initial model by sending the (spacetime) volume back to infinity. In particular, $\phi_0(x) = v$ is satisfying periodic boundary conditions.

Since the perturbations around the classical background $\phi_0(x) = v$ are the fields we are going to quantise, we require these perturbations to have a "gentle enough" decrease in the asymptotic regions, and eventually to vanish at infinity. This in order to allow for a Fourier transformation, which is necessary for the standard canonical quantisation procedure. Let us notice that the behaviour of the perturbation fields at infinity is consistent with the periodic boundary conditions (and correspond to the zero-at-infinity assumption of standard QFT textbooks).

3.5 Paradigm of the computations

The toy model we derived in the previous section illustrates well how the computations are done in practice, namely what is the philosophy of (almost) each computation we are going to perform in this thesis. First, we learned that to have SSB we need theories with interactions[10]. Therefore, the EOM are non-linear differential equations. Fortunately for us, we only need to find one particular stable non-trivial solution (if any exists). This is most of the time done by considering an Ansatz (motivated by some physical reasons like the energy minimisation) and by tuning this Ansatz in order to solve the EOM. Second, we are interested in the particle spectrum of the theory quantised around the classical particular solution we found (or we could even just be interested by solely the classical perturbative theory, e.g. for classical statistical field theory physics). This spectrum corresponds to the spectrum of the free theory obtained after renormalisation. More specifically, we want the masses and the dispersion relations. Hence, our computations will be at quadratic order in the perturbation theory. This makes the masses apparent and the EOM are linear differential equations which can be solved in Fourier space and so, they display the dispersion relations. If the perturbation theory is still too complicated, we can go at lower energy and integrate out the hierarchically most massive modes because only the hierarchically small massive modes are our primary interest (NG modes and as we will see, the related hierarchically small massive symmetry originated modes). The computations of Part II are done at the classical level. As a consequence, the results on the gaps and the dispersion relations might be spoiled by the renormalisation procedure once we quantise the theory. The argument is that our results are symmetry protected and so, they should hold as well in the quantum theory[11]. Naturally, a formal proof

[10]There are of course counter-examples such as the free real scalar relativistic theory $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi$ which is shift invariant and where the trivial solution breaks the shift symmetry. But most of the physical relevant SSB need interactions to occur.

[11]If an anomaly occurs – i.e. an explicit symmetry breaking – or if the renormalised theory does not show SSB anymore (for example, the parameters have changed such that we lose the Mexican hat shape for the potential) then we are outside the range of our hypotheses (we are discussing SSB patterns) and

should be given but as it will be seen, the models we are studying are involved and their quantisation constitutes several projects by their own. From the effective point of view, we can also imagine that there are physical cut-offs in our theory (e.g. lattice spacings) and so, our description is an EFT where the quantisation is naturally regularised (and so normalised) by these cut-offs – this could be a way out in case where our field theories are non-renormalisable. Concerning Part III, the quantisation is at the center of attention because we are precisely probing a peculiar change of behaviour between the classical theory and the quantum theory.

3.6 Directions of research

Goldstone's theorem provides an interesting observation. But as often in science, an interesting observation leads to new questionings. Here are listed some of the main open questions brought by Goldstone's theorem:

1. Goldstone's theorem predicts the existence of gapless modes when a global continuous SSB occurs but does not provide a precise statement on how many there will be in the perturbation theory. We thus need a counting rule for such modes, said otherwise, we need to know which NG candidates are dependent and independent from each other. To do such counting, we probably also need a classification of the NG modes.

2. The fundamental hypothesis of Goldstone's theorem is that we have an SSB. Therefore, it could be meaningful to probe what are the conditions to have a spontaneous symmetry breaking in a given theory. For example, Coleman's theorem [16] states that, at quantum level, for relativistic theories in two-dimensional spacetime, there are no SSB that could lead to NG modes.

3. If we know the number of NG modes and their statistics (for internal symmetries, these are bosons) it could be interesting to have their dispersion relations. This would, for example, allow us to compute thermodynamic observables. A generic study of the possible shape of dispersion relations, for instance [71], could then be of interest.

4. The NG modes are systematically present in the IR since they are massless. But they are not the only type of light particles. So, in the perspective of building effective field theories, we should understand how NG particles interact with other non-symmetry originated particles. An example could be the Cooper pairs where it is the interaction between the electrons and the phonons (NG modes coming from the discrete breaking of spatial translations) which display the effective attractive interaction between the electrons once we integrated out the phonons.

Partial answers have already been provided to this list of questions but, each of these points remains an active topic of research. This introduction to Goldstone physics plays the role of an extended state of the art, the questionings mentioned above will be further commented and clarified. Parts II, III and IV will humbly provide partial answers and

it is not inconsistent if our results do not hold in such situations.

some directions of investigation to a-few of the open questions stated in this part of the dissertation. It should be mentioned that there are as well some more fundamental extensions of the Goldstone theorem which are investigated in the litterature. For example, how is Goldstone's theorem modified on curved spacetimes [72–74] ? Does Goldstone theorem generalise for Higher-form symmetries [75] ? Can we study NG modes in thermal non-equilibrium systems [76] ? Nevertheless, we will not address these broader considerations in the rest of this thesis.

Chapter 4

Spontaneous breaking of internal symmetries

Since the statement of Goldstone's theorem in the early sixties, major progress has been achieved in the understanding of the NG modes and their physics. Most of these developments took place in the case of the spontaneous symmetry breaking of internal symmetries. The breaking of these symmetries is conceptually and technically less involved compared to the breaking of the non-uniform symmetries (especially the spacetime symmetries).

In this chapter, we specifically focus on the main results associated to the breaking of internal symmetries – with some comments on the case of uniform symmetries, the next chapter will then treat the situation with the breaking of spacetime symmetries (and by extension, with the SSB of non-uniform symmetries also).

The two first sections of the current chapter will be about the classifications and the counting rules for NG modes, the next sections will be dedicated to generalities on Goldstone physics.

4.1 Counting rule and classification based on dispersion relations

We already exposed the fact that Goldstone's theorem does not provide a precise statement on the number of NG modes we could expect from a given symmetry breaking pattern. In the original proofs of the theorem [4], it was already clear that for relativistic theories displaying solely spontaneous symmetry breaking of internal symmetries, the number of NG modes is equal to the number of broken generators. A pedagogical proof of this statement, at classical level, can be found in [24]. However, several examples in non-relativistic theories showed that the number of NG modes might be reduced compared to the number of broken generators (by definition, the number of NG modes is bounded from above by the number of broken generators). A textbook example could be ferromagnetism [25], where the breaking of two of the three $SU(2)$ generators due to the spins alignment leads to one magnon (the gapless spin-wave), see Appendix A. Therefore, a counting rule, and an associated classification, for NG modes is necessary. It remains an open question for a totally generic breaking pattern (including both uniform and non-uniform symmetries[1]), however, some progress and some strong results have been obtained through the past decades.

Nielsen and Chadha proposed a first classification relying on the dispersion relations of the NG modes [70]. This classification led to a counting rule. One of the original hypotheses of Nielsen and Chadha was that the considered fundamental theory should

[1] A more general way to say internal symmetries and spacetime symmetries.

have continuous spatial translations symmetry. This has been relaxed in [41] such that discrete spatial translations are tolerated.

The guideline of the proof of Nielsen and Chadha's counting rule is the spectral decomposition of (2.3.6), in the same spirit as the argument we displayed for Goldstone's theorem at Section 3.3. We will not present the proof because it will neither provide additional intuitive knowledge nor new technical computational tools.

Theorem 2 (Nielsen and Chadha's theorem)**.** *Let us consider a fundamental (field) theory such that the locality of the interactions implies that, at quantum level, if $A(x)$ and $B(0)$ are any two local operators, then*

$$|\,\boldsymbol{x}\,| \to \infty : |\,\langle 0|\,[A(\boldsymbol{x},t), B(0)]\,|0\rangle\,| \to e^{-\tau|\boldsymbol{x}|},\ \tau > 0\ . \tag{4.1.1}$$

If

- *n_{BG} generators of the uniform symmetries of our fundamental theory are spontaneously broken (while the non-uniform ones remain untouched),*
- *the notion of gap is well defined,*
- *the dispersion relations of the NG modes are isotropic and can be written as a polynomial expansion at low momentum,*

then we classify as type I the NG modes with an energy which is proportional to an odd power of the momentum at long wavelengths, n_I is the number of such modes. The ones with an even power are called type II and n_{II} is their number. The amount of NG particles satisfies the inequality

$$n_I + 2n_{II} \geqslant n_{BG}\ . \tag{4.1.2}$$

We can notice that asking the dispersion relation to be isotropic is an implicit hypothesis of rotational symmetry of the fundamental theory. Relaxing this hypothesis is discussed in [41]. The locality condition (4.1.1) ensures an analytic behaviour of the Fourier transform and so, indirectly of the dispersion relations. Specifying that the latter should have a polynomial expansion at long wavelengths could be considered as tautological – this tautology is not present in the original statement of Nielsen and Chadha, it is due to the reformulation made by the author of this dissertation.

As a consistency check, we have that for relativistic theories, the massless dispersion relations are always linear (with a model independent velocity equal to the speed of light) and so, the NG modes are always type I ($n_{\text{II}} = 0$). The counting rule 4.1.2 informs us that $n_{\text{I}} \geqslant n_{\text{BG}}$. Complemented with $n_{\text{I}} + n_{\text{II}} \leqslant n_{\text{BG}}$ by definition of the NG modes, we recover the known result $n_{\text{I}} = n_{\text{BG}}$ for relativistic theories with SSB of internal symmetries.

The Nielsen and Chadha counting rule is an inequality, a stronger counting rule with an equality has been obtained later on by Brauner, Murayama and Watanabe. The derivation of this stronger counting rule is the goal of the next section. However, despite being weaker, the classification based on dispersion relations is still used in the literature since in some cases it is more practical.

4.2 Counting rule through the coset construction

To derive the counting rule involving an equality, we will do the hypothesis that G is internal and compact (in addition to be global and continuous)[2]. We are going to establish, by using the coset construction, the most generic shape of the dominant terms in the IR of an effective field theory describing NG modes resulting from a given breaking pattern $G \to H$. Then, we will count the number of canonical independent degrees of freedom contained in this generic theory which will provide us a counting rule for the NG modes and a classification based on the broken generators. The obtained counting rule was conjectured and partially proved by Brauner and Watanabe [54] and proved by Murayama and Watanabe [77,78] in the early decade of 2010. Their work is based on several progress in the counting and the classification of NG modes, a (non-exhaustive) list of relevant papers could be [9, 41, 57–59, 70, 79–84].

4.2.1 The coset construction

From a general perspective, the coset construction is the classification of the non-linear realisations of a given continuous symmetry group[3] G which reduce to linear representations when considering a continuous subgroup $\tilde{H}$ of G. Then, Lagrangians, which consist of an expansion in the fields and their derivatives (this defines locality), are built such that they are invariant under these specific realisations. We thus understand that the name "coset construction" comes from the quotient space $G/\tilde{H}$, the part of G which is non-linearly realised, and from the construction method to get invariant Lagrangians.

The coset construction was first established in the sixties in the domain of elementary particles physics. Indeed, effective theories were built in order to describe light mesons. Such theories were displaying non-linear realisations of respectively the chiral group $SU(2) \times SU(2)$ and the chiral group $SU(3) \times SU(3)$ (see for example the non-exhaustive list [85–88]). It is Coleman, Wess and Zumino who, in the end of the sixties, established a classification of all the non-linear realisations respecting the criteria defined in the preceding paragraph for a generic connected compact semi-simple internal symmetry group G [55]. Just afterwards, with the additional help of Callan, they set up a method which permits to build invariant local Lagrangians and to gauge the symmetry [56]. The coset construction is sometimes referred to CCWZ construction, the initials of the previously cited authors.

Intuitively, if we take back our $U(1)$ Mexican hat example – see Section 3.4, we have that the Lagrangian we obtained after the spontaneous symmetry breaking reproduces non-linearly the $U(1)$ symmetry through the shift of the phase field which is nothing else than the NG mode. The obtained perturbation theory could then be likened to a particular case of the coset construction. This intuition and so, the interest of the coset construction for Goldstone physics, was formally noticed in [89]. They showed that

[2]Of course the fundamental theory can have additional non-internal symmetries which are not spontaneously broken, but they should have particular commutation relations with the broken internal symmetries. These requirements will be discussed later on, however, since spacetime symmetries commutes with internal G, they are tolerated (the non-trivial discussion is for the non-uniform symmetries which are not spacetime symmetries).

[3]More precisely, it concerns only the realisation of the elements of the group which are connected to the identity.

effective field theories describing NG modes where corresponding to CCWZ invariant Lagrangians. From a coset construction they were able to recover all the properties of NG fields, i.e. massless modes (or with a light mass when a small explicit symmetry breaking occurs) and fields which are weakly coupled at small energy. Furthermore, by gauging the symmetries, they retrieve the Brout-Englert-Higgs mechanism. The non-linear transformations of NG fields are also discussed in [57].

From now on, we will consider the coset construction only in the framework of Goldstone physics. Since the transformation rules for the NG fields are settled by the transformation rules of the fundamental fields, we do not need the mathematical machinery of classifying the different equivalent transformation laws. We therefore can relax some of the hypotheses of the papers [55, 56]. This introduction to the coset construction heavily relies on the review [18], another relevant review is [19].

4.2.1.1 Hypotheses on the symmetry group

Let G be an internal continuous compact group which is faithfully linearly realised in the fundamental theory where the dimension of the realisation[4] is finite. In such a case, we can always chose a basis of the Lie algebra so that [24]

$$\mathrm{Tr}\,(G_\alpha G_\beta) = \delta_{\alpha\beta} \ , \tag{4.2.1}$$

$$G_\alpha^\dagger = G_\alpha \ , \tag{4.2.2}$$

where $\{G_\alpha\}$ is the realisation of the generators of G. We will do a misnomer by denoting G_α as a generator of G. The choice made such that (4.2.1) and (4.2.2) are satisfied implies that the structure constants of the algebra are fully anti-symmetric:

$$[G_\alpha, G_\beta] = i f_{\alpha\beta}{}^{\gamma}\, G_\gamma \ , \tag{4.2.3}$$

where $f_{\alpha\beta}{}^{\gamma}$ is anti-symmetric in its 3 indices.

4.2.1.2 Comment on the algebra

We will consider that G is spontaneously broken to a continuous subgroup H. Let us call X_a the broken generators of G, and T_A the unbroken ones. Since H is a subgroup, we have

$$[T_A, T_B] = i f_{AB}{}^{C}\, T_C \quad \Leftrightarrow \quad \forall a,\ f_{AB}{}^{a} = 0 \ . \tag{4.2.4}$$

By using the full anti-symmetry of the structure constants and (4.2.4), we have

$$\forall a,\ f_{Aa}{}^{B} = 0 \quad \Leftrightarrow \quad [T_A, X_a] = i f_{Aa}{}^{b}\, X_b \ . \tag{4.2.5}$$

We observe that $\{X_a\}$ is a representation of H (more precisely, the space generated by $\{iX_a\}$ is a representation of the Lie algebra of H). Let us emphasise that this last observation is always true for compact groups. This is one of the main reasons why we take G as being compact in our hypotheses.

[4] We use the more general terminology "realisation" instead of "representation" since, as we will see, the action of G will be non-linear for the specific parametrisation we will choose for the fields.

We can notice that this classification between broken and unbroken generators is not unique since one can always redefine a broken generator as $X_a \to X_a + {c_a}^A T_A$, where the ${c_a}^A$ are arbitrary coefficients. On the condition that the newly defined generators still form a representation of H, it will not alter our generic discussion on the spontaneous breaking of internal symmetries. In practice, these possible redefinitions can be used to simplify technical computations.

4.2.1.3 The coset construction for NG modes

A field of the fundamental theory – called a fundamental field – ϕ can be parametrised as [18]

$$\phi(x) = U(\pi(x))\,\chi(x) \ , \tag{4.2.6}$$

with

$$U(\pi(x)) \equiv e^{i\pi^a(x)X_a} \ , \tag{4.2.7}$$

where $\pi^a(x)$ and $\chi(x)$ are general functions[5]. If we particularise to $\chi(x) = v$ with v being the constant VEV (no spacetime SSB), we have that

$$e^{i\pi^a(x)X_a}v \tag{4.2.8}$$

corresponds to a spacetime modulated fluctuation around the VEV in a spontaneously broken direction of G. It is, by definition, a NG mode. The NG modes are therefore naturally parametrised by $\pi^a(x)$, i.e. the coordinates of a mapping between spacetime and the connected patch to the identity[6] of G/H. Henceforth, we refer to π^a as NG candidates (we still have to determine which ones are independent, i.e. which one are NG modes) and we consider them as small perturbation fields. The Equation (4.2.7) is called the coset parametrisation.

To get the transformation rules of π^a, we use the realisation of G on the fundamental field which leads us to

$$g\,U(\pi(x))\,\chi(x) = U(\tilde{\pi}(x))\,\tilde{\chi}(x) \ , \tag{4.2.9}$$

where the tildes refer to the transformed fields and $g \in G$ is a shortcut to denote the realisation of g. We will keep using this misuse as long as it leads to no ambiguity. Since $gU(\pi(x)) \in G$ by associativity of the product and because[7]

$$\forall \alpha^a,\ \exists f^a(\alpha), g^A(\alpha) \quad | \quad e^{i\alpha^\beta G_\beta} = e^{if^a(\alpha)X_a}\, e^{ig^A(\alpha)T_A} \ , \tag{4.2.11}$$

[5]It is proven in [18] that we can indeed always write (4.2.6). An argument to be convinced is that $\chi(x)$ is a totally generic function and that it is reasonable to consider $e^{i\pi^a(x)X_a}$ to be invertible since it is a group element.

[6]This is mathematically quickly said, it should need a more formal description. But the gist of what this mapping is geometrically is enough for our purpose.

[7]We use the Baker–Campbell–Hausdorff formula where the lie algebra elements are considered as small since NG candidates are perturbations around the VEV, thus

$$e^y e^x = e^{x+y+\frac{1}{2}[y,x]} \ , \tag{4.2.10}$$

where the approximation $[x,[x,y]] \approx 0$ is considered. Then, we conclude with the commutation relation (4.2.5).

we can write

$$gU(\pi(x)) = U(\tilde{\pi}(x))\, e^{iu^A(\pi,g)T_A} \ , \tag{4.2.12}$$

where u^A is some function of π and g. The transformation rules $\pi \to \tilde{\pi}$, $\chi \to \tilde{\chi}$ therefore satisfy

$$gU(\pi(x)) = U(\tilde{\pi}(x))\, e^{iu^A(\pi,g)T_A} \ , \tag{4.2.13}$$

$$\tilde{\chi}(x) = e^{iu^A(\pi,g)T_A}\chi(x) \ . \tag{4.2.14}$$

It can be checked that it is indeed a good realisation of G (i.e. the product of the realisation is the realisation of the product). In general, these transformation rules are non-linear through the dependency of u^A in π. The transformation law for π^a is complicated, however, the one for χ is technically easier. This because it is a (non-linear) covariant transformation under G built on a covariant realisation of H.

If we particularise the transformation laws (4.2.13), (4.2.14) for $g = h \in H$ and that we use (4.2.5), we obtain

$$hU(\pi(x)) = U(\tilde{\pi}(x))\, h \Leftrightarrow \tilde{\pi}^a(x)X_a = h\, \pi^a(x)X_a\, h^{-1} \ , \tag{4.2.15}$$

$$\chi(x) = h\, \tilde{\chi}(x) \ . \tag{4.2.16}$$

We observe that our fields transform linearly under H while generically, according to (4.2.13) and (4.2.14), they transform non-linearly under G. Thus, Nambu-Goldstone modes and their effective field theories could indeed be well described by the coset construction formalism where $\tilde{H}$ corresponds to H (concerning internal symmetries at least). We remind that $\tilde{H}$ is defined as the continuous subgroup of G which is linearly realised.

It could be instructive to develop a bit further the transformation rule of π^a when g is generated by broken generators. Infinitesimally, the left-hand side of (4.2.13) is

$$gU(\pi(x)) = e^{i\omega^a X_a} e^{i\pi^b(x)X_b} \tag{4.2.17}$$

$$= (1 + i\omega^a X_a + \mathcal{O}(\omega^2))(1 + i\pi^b(x)X_b + \mathcal{O}(\pi^2)) \tag{4.2.18}$$

$$= 1 + i(\pi^a(x) + \omega^a)X_a + \mathcal{O}(\epsilon^2) \ , \tag{4.2.19}$$

where $\pi \sim \omega \sim \epsilon$. It should be equal to the right-hand side of (4.2.13)

$$U(\tilde{\pi}(x))\, e^{iu^A(\pi,g)T_A} = (1 + i\tilde{\pi}^a(x)X_a + \mathcal{O}(\tilde{\pi}^2))(1 + iu^A(\pi,g)T_A + \mathcal{O}(u^2)) \tag{4.2.20}$$

$$= 1 + iu^A(\pi,g)T_A + i\tilde{\pi}^a(x)X_a + \mathcal{O}(\epsilon^2) \ , \tag{4.2.21}$$

where $\tilde{\pi} \sim u^A \sim \epsilon$. By comparing (4.2.19) to (4.2.21), we get

$$\tilde{\pi}^a(x) = \pi^a(x) + \omega^a + \mathcal{O}(\epsilon^2) \ , \quad u^A(\pi,g) = \mathcal{O}(\epsilon^2) \ . \tag{4.2.22}$$

The transformation of the NG modes is inhomogeneous. This is a signature that in the EFTs describing NG modes, there cannot be any mass terms for π^a and the interacting terms involving NG modes should contain derivatives (i.e. weakly coupled in the IR). Moreover, it provides an argument that the spontaneous breaking of gauge symmetries (at classical level) cannot lead to NG modes. In fact, ω^a being a broken symmetry parameter, if this symmetry is gauged, ω^a is then an arbitrary function that should be fixed through a gauge choice. From (4.2.22), we understand that the associated NG candidate can be suppressed by a suitable gauge fixing. The coset construction allows us to recover all the characteristic features predicted by Goldstone's theorem.

We can even go further and recover the idea that the NG modes emanating from the breaking of internal symmetries are bosons. Let us consider that the fundamental field is in a linear representation of the Lorentz group (even if Lorentz's group is not necessarily included in the symmetry group of the fundamental theory), then[8]

$$e^{i\omega^{\mu\nu}L_{\mu\nu}}U(\pi(x)) = e^{i\omega^{\mu\nu}L_{\mu\nu}}e^{i\pi^b(x)X_b} \tag{4.2.24}$$

$$= e^{i\omega^{\mu\nu}(i(x_\mu\partial_\nu - x_\nu\partial_\mu)+S_{\mu\nu})}e^{i\pi^b(x)X_b} \tag{4.2.25}$$

$$= e^{i(\pi^b - \omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\pi^b)X_b}e^{i\omega^{\mu\nu}L_{\mu\nu}} \tag{4.2.26}$$

$$\equiv e^{i\tilde{\pi}^b(x)X_b}e^{iu^{\mu\nu}(\pi,\omega)L_{\mu\nu}} \;, \tag{4.2.27}$$

where, thanks to $[S_{\mu\nu}, X_a] = 0$ because X_a is internal, π^a transforms as a spinless field. Furthermore, π^a fits in the standard classification of fields based on the representation of the Poincaré group because, in addition to be spinless, it transforms canonically under translations:

$$e^{ia^\mu P_\mu}U(\pi(x)) = e^{-a^\mu\partial_\mu}e^{i\pi^b(x)X_b} \tag{4.2.28}$$

$$= e^{i(\pi^b(x) - a^\mu\partial_\mu\pi^b(x))X_b}e^{-a^\mu\partial_\mu} \tag{4.2.29}$$

$$\equiv e^{i\tilde{\pi}^b(x)X_b}e^{iu^\mu(\pi,a)P_\mu} \;. \tag{4.2.30}$$

We conclude that $\pi^a(x)$ is a real scalar field which displays all the expected features of an NG mode and is therefore a consistent NG candidate.

Building an invariant Lagrangian directly using the transformation law of π^a under G is an involved process. We need an object which transforms covariantly with respect to G and will therefore, constitute our building block. This object is obtained through the Maurer-Cartan 1-form: $dx^\mu U(\pi)^{-1}\partial_\mu U(\pi)$. It is a 1-form which takes its values in the Lie algebra of G. We can thus write:

$$U(\pi)^{-1}\partial_\mu U(\pi) = -i\,\mathcal{A}_\mu^A(\pi)\,T_A + i\,e_\mu^a(\pi)\,X_a \;. \tag{4.2.31}$$

With $\gamma = e^{iu^A(\pi,g)T_A}$ and

$$U(\tilde{\pi}(x)) = gU(\pi(x))\,\gamma^{-1} \;, \tag{4.2.32}$$

coming from (4.2.13), we have

$$U(\tilde{\pi})^{-1}\partial_\mu U(\tilde{\pi}) = \gamma\,U(\pi)^{-1}\partial_\mu U(\pi)\gamma^{-1} + \gamma\,\partial_\mu\gamma^{-1} \;. \tag{4.2.33}$$

Hence, by recalling (4.2.4) and (4.2.5), the transformation rules are given by:

$$e_\mu^a(\tilde{\pi})\,X_a = \gamma\left(e_\mu^a(\pi)\,X_a\right)\gamma^{-1} \;, \tag{4.2.34}$$

$$-\,i\,\mathcal{A}_\mu^A(\tilde{\pi})\,T_A = \gamma\left(-i\,\mathcal{A}_\mu^A(\pi)\,T_A\right)\gamma^{-1} + \gamma\,\partial_\mu\gamma^{-1} \;. \tag{4.2.35}$$

[8] We use the Baker–Campbell–Hausdorff formula with the same approximation as in (4.2.10)

$$e^y e^x = e^{x+y+\frac{1}{2}[y,x]} = e^{x+[y,x]}e^y \;. \tag{4.2.23}$$

We thus have that $e_\mu(\pi) \equiv e^a_\mu(\pi) X_a$ transforms in a covariant way under G from a realisation of H. Therefore, $e_\mu(\pi)$ is the building block we were looking for. Indeed, any function of $e_\mu(\pi)$ which is covariantly invariant under H will automatically be invariant under G. However, the transformation laws are spacetime dependent through their dependency in the fields $\pi^a(x)$[9]. So, $\partial_\nu e_\mu(\pi)$ does not transform covariantly. We need to define a good differential operator. If we look at (4.2.35), we observe that $\mathcal{A}^A_\mu(\pi)$ transforms as a gauge field and can then be used to define a covariant derivative for $e_\mu(\pi)$:

$$(D_\mu e_\nu)^a \equiv \partial_\mu e^a_\nu + f_{Bc}{}^a \, \mathcal{A}^B_\mu \, e^c_\nu \ . \tag{4.2.36}$$

The covariant behaviour of (4.2.36) can be verified. With an infinitesimal expansion of $\gamma = e^{iu^A(\pi,g)T_A}$ in (4.2.34) and (4.2.35), we find

$$\delta e^a_\mu \equiv e^a_\mu(\tilde{\pi}) - e^a_\mu(\pi) \approx f_{bA}{}^a \, e^b_\mu \, u^A \ , \tag{4.2.37}$$
$$\delta A^A_\mu \equiv A^A_\mu(\tilde{\pi}) - A^A_\mu(\pi) \approx \partial_\mu u^A - f_{BC}{}^A u^B A^C_\mu \ . \tag{4.2.38}$$

By injecting these transformation law in (4.2.36), we have

$$\delta \left(D_\mu e_\nu\right)^a \approx f_{bA}{}^a \, \left(D_\mu e_\nu\right)^b \, u^A \ . \tag{4.2.39}$$

It is the same transformation law as e^a_μ. We can then be convinced that (4.2.36) is indeed a covariant derivative.

Before commenting on the construction of an invariant Lagrangian, we can try to have a sense on the way $e_\mu(\pi)$ and $\mathcal{A}^A_\mu(\pi)$ depend on π^a. We have that

$$U(\pi)^{-1}\partial_\mu U(\pi) = e^{-i\pi^a X_a}\partial_\mu \, e^{i\pi^b X_b} \tag{4.2.40}$$
$$= i\,\partial_\mu \pi^b \left(e^{-i\pi^a X_a} X_b \, e^{i\pi^c X_c}\right) \ . \tag{4.2.41}$$

We can notice the global $\partial_\mu \pi^b$ factor, and by comparing the obtained expression with (4.2.31), we can conclude that $e_\mu(\pi)$ and $\mathcal{A}^A_\mu(\pi)$ will systematically contain a derivative of π^a. In particular, by developing (4.2.41) to quadratic order, we have

$$e^a_\mu(\pi) = \partial_\mu \pi^a - \frac{1}{2}\partial_\mu \pi^b \pi^c f_{bc}{}^a + \mathcal{O}(\pi^3) \ , \tag{4.2.42}$$
$$\mathcal{A}^A_\mu(\pi) = \frac{1}{2}\partial_\mu \pi^a \pi^b f_{ab}{}^A + \mathcal{O}(\pi^3) \ . \tag{4.2.43}$$

We can build a G invariant Lagrangian by using e_μ and $D_\mu e_\nu$ by asking this Lagrangian to be H covariantly invariant:

$$\mathcal{L}(e_\mu, D_\mu e_\nu, \ldots) \text{ such that } \mathcal{L}(h\, e_\mu\, h^{-1}, h\, D_\mu e_\nu\, h^{-1}, \ldots) = \mathcal{L}(e_\mu, D_\mu e_\nu, \ldots) \ , \tag{4.2.44}$$

where the ellipses denote higher covariant derivatives. Let us mention that we could also add other fields than the NG modes through $\chi(x)$ which transforms covariantly (4.2.14) (the derivative operator would then be given by $D_\mu \chi(x)$).

[9] Let us emphasise/remind that we are looking to a global realisation of G, G is not gauged.

4.2.1.4 Effective field theories for NG modes

Said crudely, an effective theory is the most general theory respecting some given symmetry constrains which can be written as an expansion in energy.

So, to establish the effective theory for NG modes associated to a given symmetry breaking pattern, we need to show that the most general G invariant Lagrangian for π^a built with $e_\mu(\pi)$ is equivalent to the most general G invariant Lagrangian directly constructed with π^a. This is proven in [18], we will not repeat the proof and accept the statement[10].

Furthermore, since a generic construction based on $e_\mu(\pi)$ is indeed totally general, we have that each term of the Lagrangian contains a derivative of π^a (cf. the paragraph below (4.2.41)). The expansion in energy is therefore rather natural. We are thus able to write a generic EFT and to guess the first dominant terms. In fact, we know that the EFT will systematically have a derivative in each term, the dominant terms will be the ones with the minimum number of derivatives.

For the relativistic case, since we need to contract the Lorentz indices, the minimum number of derivatives we can have is two. Hence[11],

$$\mathcal{L}(\pi) = \frac{1}{2} g_{ab}(\pi)\partial_\mu \pi^a \partial^\mu \pi^b + \mathcal{O}(\partial^4) , \tag{4.2.45}$$

where $g_{ab}(\pi)$ is a symmetric matrix. To ensure the kinetic energy to be positive, we take $g_{ab}(\pi)$ to be positive definite for all π^a. The additional constrains we have to impose on $g_{ab}(\pi)$ in order for $\mathcal{L}(\pi)$ to be G invariant will give us a geometric interpretation.

Let us see the G transformation

$$\tilde{\pi}^a = \pi^a + \xi^a(\pi) , \tag{4.2.46}$$

as a diffeomorphism on G/H, where $\xi^a(\pi)$ generically depends on π^a. The latter statement can be seen by considering the higher terms in (4.2.22). We get

$$\begin{aligned} \mathcal{L}(\tilde{\pi}) &= \frac{1}{2} g_{ab}(\pi + \xi)\, \partial_\mu(\pi^a + \xi^a)\, \partial^\mu(\pi^b + \xi^b) + \mathcal{O}(\partial^4) && (4.2.47)\\ &= \mathcal{L}(\pi) + \frac{1}{2}\left(\xi^a\, \partial_a g_{bc} + g_{ac}\, \partial_b \xi^a + g_{ba}\, \partial_c \xi^a\right)\, \partial_\mu \pi^b \partial^\mu \pi^c + \mathcal{O}(\xi^2) , && (4.2.48) \end{aligned}$$

where we Taylor expanded $g_{ab}(\pi + \xi)$. Imposing $\mathcal{L}(\tilde{\pi}) = \mathcal{L}(\pi)$ to have the G invariance is equivalent to ask

$$\xi^a\, \partial_a g_{bc} + g_{ac}\, \partial_b \xi^a + g_{ba}\, \partial_c \xi^a = 0 . \tag{4.2.49}$$

[10] This in accordance with the paradigm of this introduction to Goldstone physics: we develop the technical proofs only when we acquire new intuitive knowledge on NG modes from it or when it permits to master a new computational tool (e.g. here the coset construction is illustrated and it permits us to understand better the different features of NG modes).

[11] As it can be observed, in the subdominant terms, we included possible higher time-derivative terms. Going higher than the second derivative in time leads to the Ostrogradsky instabilty [90–93]. It is argued in the cited papers that this instability can be avoided in the perturbation theory. In particular, in the search of possible IR modifications of General Relativity, ghost condensation models have been proposed with higher time-derivative terms [94]. From the moment we have either a one-time derivative or a two-time derivative as dominant term, these higher time-derivative term will not influence our discussion. We will assume to be in this situation.

By seeing π^a as coordinates on G/H and its transformation under G as a diffeomorphism, we can interpret $g_{ab}(\pi)$ as a metric with the isometry group G. Indeed, (4.2.49) is the Lie derivative with respect to $\xi^a(\pi)$, thus the constraint of being G invariant is given by

$$\mathcal{L}_\xi g = 0 \ . \tag{4.2.50}$$

This provides an interesting geometric picture because it shows that finding the dominant term of the most general relativistic G invariant EFT is equivalent to looking for the most generic positive definite G invariant metric on G/H.

The geometrical interpretation of the coset construction has been established in [95], this in the goal to make a link with general relativity. Volkov enriched this geometrical approach in [96].

A bit more of details: *It is shown in [18] that when Span$\{iX_a\}$ forms a completely reducible representation of H (we have already seen that it is a representation, cf.* (4.2.5)*), the set of positive definite G invariant metrics on G/H is parametrised by n positive parameters, where n is the number of irreducible representations of H in Span$\{iX_a\}$. In particular, when Span$\{iX_a\}$ is an irreducible representation of H, the metric is unique up to a positive normalisation factor.*

For the non-relativistic case, we are going to keep the spatial rotation symmetry[12]. This leads us to

$$\mathcal{L}(\pi) = c_a(\pi)\partial_t\pi^a + \frac{1}{2}g_{ab}(\pi)\partial_t\pi^a\partial_t\pi^b - \frac{1}{2}\bar{g}_{ab}(\pi)\partial_i\pi^a\partial_i\pi^b + \mathcal{O}(\partial_t^3, \partial_t\partial_i^2, \partial_i^4) \ , \tag{4.2.51}$$

where $g_{ab}(\pi)$ and $\bar{g}_{ab}(\pi)$ are generic positive definite G invariant metrics on G/H, they are taken in that way to ensure the positivity of the kinetic energy, even when we go at higher energy and that ∂_t^2 becomes dominant compared to ∂_t. Notice that $g_{ab}(\pi)$ and $\bar{g}_{ab}(\pi)$ are proportional to each other when $\{X_a\}$ is an irreducible representation of H. To have an invariant theory, we need the function $c_a(\pi)$ to be a generic covector field on G/H which transforms under the G diffeomorphism (4.2.46) as a global (π-)derivative

$$\mathcal{L}_\xi c_a = \partial_a\Omega_\xi(\pi) \ , \tag{4.2.52}$$

where $\Omega_\xi(\pi)$ is an unconstrained function on G/H. Indeed,

$$c_a(\tilde{\pi})\partial_t\tilde{\pi}^a = (c_a(\pi) + \partial_b c_a\xi^b)(\partial_t\pi^a + \partial_b\xi^a\partial_t\pi^b) + \mathcal{O}(\xi^2) \tag{4.2.53}$$

$$= c_a(\pi)\partial_t\pi^a + (\xi^b\partial_b c_a + c_b\partial_a\xi^b)\partial_t\pi^a + \mathcal{O}(\xi^2) \tag{4.2.54}$$

$$= c_a(\pi)\partial_t\pi^a + \partial_t\Omega_\xi + \mathcal{O}(\xi^2) \ , \tag{4.2.55}$$

where we Taylor expended till the first order and where we used the definition of the Lie derivative acting on a covector. We notice that under the constraint (4.2.52), the Lagrangian transforms up to a global derivative. Our goal is to have an invariant theory

[12] This is not too restrictive for condensed matter because we are looking at phenomena occurring on large spatial scales compared to the lattice spacing. Hence, in the same way the discrete translation symmetries are smoothened, the discrete rotation symmetries appear continuous in such regime.

rather than an invariant Lagrangian. So, this transformation behaviour is tolerated for our EFT.

Let us mention that, in the context of effective theories for NG modes, terms transforming up to a global derivative are called Wess-Zumino-Witten terms [19]. To be totally generic, the classification of such terms should be added to our discussion. This classification is outside the scope of this dissertation[13]. However, a discussion of such terms for our particular case can be found in [78] which ensures that (4.2.51) is indeed totally general. Some possible directions to look at to get acquainted with these Wess-Zumino-Witten terms are [97–102].

4.2.1.5 Recap on the hypotheses made for the EFT

We established that (4.2.51) corresponds to the most general dominant terms of an EFT describing NG modes in the IR – let us notice that the limit of low energy is consistent with the massless aspect of NG modes. Some of the hypotheses were explicitly stated and others have been implicitly considered. We thus here provide a recap under which hypotheses (4.2.51) has been obtained.

- The spontaneous symmetry breaking pattern $G \to H$ is such that G is a continuous global compact internal group with a faithful finite dimensional linear representation on the fundamental theory. And H is either a continuous subgroup of G or the trivial subgroup containing only the identity. The constrains on G might look extensively restrictive but they are satisfied in many physical cases. Indeed, $U(1)$, $SO(3)$, $SU(N)$ are for example common groups encountered in physics where they are realised through faithful matrix representations. Hence, they do satisfy the requirements on G. Let us remind that we took G to be compact mainly to ensure that $\{X_a\}$ is a representation of H. A specificity that we notably used when displaying the transformation rules. It suggests that we might relax the hypothesis on the compactness of G on the condition that (4.2.5) is still satisfied.

- We kept some bonds with the fundamental theory through the parametrisation (4.2.6) and (4.2.7). This implied that we had to consider G to be linearly realised on the fundamental theory. As we will see in Section 5.2, we are able to build the same EFTs purely based on group theory, more specifically, purely on the Lie algebra structure. Therefore, we can anticipate that G should be realised on the fundamental theory but not necessarily linearly. Furthermore, since the only group theory result we used was (4.2.5) ((4.2.4) is automatically verified), we can anticipate that G can be generalised to any continuous internal symmetry group respecting (4.2.5) – it was ensured by considering G being compact. Beside the symmetry group G, the fundamental theory can have any other non-internal unbroken symmetries on the condition that (4.2.5) remains true. In particular, we can consider any unbroken spacetime symmetries since they commute with the internal broken symmetries. The EFT construction remains exactly the same, we only have to add the constraint to be

[13]For completeness, let us at least state that the classification of the Wess-Zumino-Witten terms can be reduced to finding the $d+1$ de Rham cohomology group $H^{d+1}(G/H)$. From mathematics, it seems that for general G/H, $H^{d+1}(G/H)$ is not available. But when G/H is itself a Lie group, the de Rham cohomology group is well known. Furthermore, the construction of $H^{d+1}(G/H)$ is simpler when G is compact which shows once again why the coset construction is systematic for compact symmetry groups [97].

invariant under these additional non-internal unbroken symmetries. We implicitly did it when we were imposing Poincaré invariance on (4.2.45) for relativistic theories, or when we asked for spatial rotation symmetries in (4.2.51).

- We considered that NG modes are the only massless modes (or light modes) in the spectrum. So to speak, we went to enough low energy such that all the massive modes have been integrated out. Of course, the discussion remains to be generalised to the case of additional massless perturbations and/or very light modes.

- In addition to continuous rotation symmetries, we made the hypothesis of continuous spacetime translation symmetries. In particular, the notion of mass is well defined. Let us mention that lattice physics could be discussed through our EFT in the continuum limit which is consistent with the IR limit.

- We made the physically realistic assumption that the theory possesses at least one term with either one time-derivative or two time-derivatives (i.e. higher time-derivative terms, if any, are subdominant in the IR). The choice for the metrics g and $\bar{g}$ to be positive definite is motivated by the stability of the theory (positivity of the energy).

- The locality of the theory has been taken such that only fields at the same spacetime position interact (modulo the infinitesimal difference due to the finite number of derivatives).

- The dimension of spacetime d has been chosen to be strictly greater than two. Indeed, we took in consideration spatial rotation symmetries, spatial rotations require at least two spatial directions. If we would have studied the case $d = 2$, additional terms as $\partial_t\pi\partial_x\pi$, etc. would have been tolerated. It is discussed in [78]. Furthermore, the case $d = 2$ is singular (mainly for the relativistic case) due to Coleman's theorem (see Subsection 4.5.1) and requires specific attention.

- There might be possible additional hypotheses that we overlooked due to some lack in the mathematical rigour of the development. Since we do not claim to have an axiomatic approach of physics, we will consider these hypotheses to be encompassed in the term "physical theory".

Let us mention that, when writing a general EFT, we have to keep in mind that this EFT should be consistent with a possible UV completion (as discussed in Section 2). This could for example impose some constraints on the parameters of the theory, to avoid superluminal speeds for example.

4.2.2 Counting rule and classification based on the broken generators

All the NG candidates in (4.2.51) are not necessarily dynamically independent. In fact, canonical conjugation between π^a fields can appear if the Lagrangian is of the form

$$\mathcal{L} \sim \pi^1\partial_t\pi^2 - \pi^2\partial_t\pi^1 + \dots \,. \tag{4.2.56}$$

By using the definition of the canonical momentum

$$P_a \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \pi^a)} , \tag{4.2.57}$$

we find that the canonically conjugated pairs are $(\pi^1, -\pi^2)$ and (π^2, π^1). Hence, we have that π^1 and π^2 are canonically conjugated and both of them form one degree of freedom[14]. From this schematic reasoning, we understand that the term $c_a(\pi)\partial_t\pi^a$ in (4.2.51) is crucial for our counting rule.

To count the number of independent NG modes, we have to label which modes are conjugated to which other modes. To do so, we need a classification. A natural classification could be based on the broken generators since, as we have seen in the coset construction, it is them which generate the NG modes. Hence, we need to establish a link between the Q_a's (the conserved charges which after quantisation correspond to the broken generators) and the $c_a(\pi)$'s. This is done thanks to Noether currents.

The guidelines of this subsection will be to compute the Noether currents. It will provide us a non-explicit relation between the conserved charges and the $c_a(\pi)$ coefficients. By introducing a new quantity and with the preceding relation in mind, we will be able to establish a direct contact between the Q_a's and the $c_a(\pi)$'s. It will then remain to express $c_a(\pi)\partial_t\pi^a$ in terms of the Q_a's in our Lagrangian. So, the canonical structure of the theory will be given in terms of the Q_a's. We will thus end up with a classification and a counting rule based on the broken generators. This subsection leans on [54, 77, 78].

Let us start by re-expressing a bit differently the transformation law of π^a.

$$\pi^a(x) \xrightarrow{G} \pi^a(x) + \xi^a(\pi) = \pi^a(x) + \omega^\alpha h^a_\alpha(\pi) , \tag{4.2.58}$$

such that

$$h^a_\alpha(\pi) = \begin{cases} h^a_b(\pi) & \text{if } \alpha = b \\ \pi^c h^a_{Ac} & \text{if } \alpha = A \end{cases} , \tag{4.2.59}$$

where h^a_{Ac} is constant, because π^a transforms linearly under H. The action of the generators of G is then given by the operators

$$G_\alpha = h^a_\alpha(\pi)\partial_a . \tag{4.2.60}$$

We have seen that the Lagrangian (4.2.51) transforms up to a global derivative. It permits to define K^0_α in such manner that $\delta\mathcal{L}(\pi) \equiv \omega^\alpha \partial_t K^0_\alpha$. In such case, the (zero component of) the Noether currents are given by (cf. (2.1.7))

$$j^0_\alpha(\pi) = \frac{\partial \mathcal{L}}{\partial(\partial_t \pi^a)} h^a_\alpha(\pi) - K^0_\alpha(\pi) . \tag{4.2.61}$$

Let us find out the expression of K^0_α.

$$\begin{aligned}
\delta\mathcal{L}(\pi) &= \partial_t \Omega_\xi = (\mathcal{L}_\xi c_a)\, \partial_t\pi^a && (4.2.62)\\
&= \left(\omega^\alpha h^b_\alpha(\pi)\partial_b c_a(\pi) + c_b(\pi)\omega^\alpha \partial_a h^b_\alpha(\pi)\right) \partial_t\pi^a && (4.2.63)\\
&= \omega^\alpha h^b_\alpha(\pi)\partial_b c_a(\pi)\partial_t\pi^a + c_b(\pi)\omega^\alpha\partial_t h^b_\alpha(\pi) && (4.2.64)\\
&= \partial_t \left(c_b(\pi)\omega^\alpha h^b_\alpha(\pi)\right) + \omega^\alpha h^b_\alpha(\pi)\left(\partial_b c_a(\pi) - \partial_a c_b(\pi)\right)\partial_t\pi^a , && (4.2.65)
\end{aligned}$$

[14] We define the degrees of freedom as the quantities for which we have fixed their instantaneous speed and their values at initial time to unequivocally fix the dynamics.

where to go from (4.2.64) to (4.2.65) we used an integration by part. We know that the Lagrangian transforms up to a global derivative. So, it must exist a function $r_\alpha(\pi)$ such that

$$\partial_a r_\alpha(\pi) = h^b_\alpha(\pi)\left(\partial_b c_a(\pi) - \partial_a c_b(\pi)\right) \ . \tag{4.2.66}$$

Thus

$$\delta\mathcal{L}(\pi) = \omega^\alpha \partial_t \left(c_b\, h^b_\alpha + r_\alpha\right) \ , \tag{4.2.67}$$

which gives

$$K^0_\alpha = c_b(\pi)\, h^b_\alpha(\pi) + r_\alpha(\pi) \ . \tag{4.2.68}$$

We are now able to compute the (zero component of) Noether currents:

$$j^0_\alpha(\pi) = \frac{\partial\mathcal{L}}{\partial(\partial_t\pi^a)} h^a_\alpha(\pi) - K^0_\alpha(\pi) \tag{4.2.69}$$

$$= \frac{1}{2} g_{ab}(\pi) h^a_\alpha(\pi)\partial_t\pi^b - r_\alpha(\pi) \ . \tag{4.2.70}$$

The vacuum expectation value of j^0_α is given by its classical vacuum value. The argument is that a renormalisation factor $j^0_\alpha \to Z j^0_\alpha$ would spoil the commutation algebra $[j^0_\alpha, j^0_\beta] = i f_{\alpha\beta}{}^\gamma j^0_\gamma$, [78][15]. Evaluating (4.2.70) on the vacuum corresponds to consider the fields π^a as vanishing fields. This can be understood from (4.2.6) and (4.2.8). So,

$$\langle 0|\, j^0_\alpha(x)\, |0\rangle = j^0_\alpha(x)\big|_{\pi=0} = -\, r_\alpha|_{\pi=0} \ . \tag{4.2.71}$$

Through equations (4.2.66) and (4.2.71), we have a non-explicit relation between the broken charge densities and the c_a's. To make this connection more explicit, let us study the quantity

$$\rho_{\alpha\beta} \equiv \lim_{V\to+\infty} \frac{-i}{V} \langle 0|\, [Q_\alpha, Q_\beta]\, |0\rangle \ , \tag{4.2.72}$$

where V is the spatial volume of our system. We can develop this expression by using the definition of a conserved charge, by translating the conserved current at the origin, by taking into account the internal aspect of Q_α and by considering a homogeneous vacuum

$$\rho_{\alpha\beta} = \lim_{V\to+\infty} \frac{-i}{V} \int_V d^{d-1}x\, \langle 0|\, [Q_\alpha, j^0_\beta(x)]\, |0\rangle \tag{4.2.73}$$

$$= -i\, \langle 0|\, [Q_\alpha, j^0_\beta(0)]\, |0\rangle \ . \tag{4.2.74}$$

With the help of (2.1.14), (4.2.60) and (4.2.71),

$$\rho_{\alpha\beta} = -\, \delta_\alpha j^0_\beta(0)\big|_{\pi=0} = -\, h^a_\alpha(\pi)\partial_a j^0_\beta(0)\big|_{\pi=0} = h^a_\alpha(\pi)\partial_a r_\beta(\pi)\big|_{\pi=0} \ . \tag{4.2.75}$$

Let us notice that if we particularise at $\beta = A$, we have $\partial_a r_A|_{\pi=0} = 0$ because $h^b_A(\pi = 0) = 0$ (cf. (4.2.66), (4.2.59)). Thus,

$$\rho_{\alpha A} = 0 = -\rho_{A\alpha} \ . \tag{4.2.76}$$

[15] An alternative approach based on the Ward-Takahashi identities recovers the similar result that a non-zero VEV of non-abelian charge densities induces a one-time derivative term in the effective Lagrangian [58].

This is consistent with $Q_A |0\rangle \propto |0\rangle$, since Q_A is unbroken, which implies $\rho_{\alpha A} = 0 = \rho_{A\alpha}$. We will therefore focus on ρ_{ab}. With (4.2.66),

$$\rho_{ab} = h_a^c(0) h_b^d(0) \left(\partial_d c_c - \partial_c c_d\right)\big|_{\pi=0} \tag{4.2.77}$$

$$\Leftrightarrow \partial_{[d}c_{c]}\big|_{\pi=0} = \frac{1}{2}\rho_{ab} \left(h^{-1}(0)\right)_d^a \left(h^{-1}(0)\right)_c^b , \tag{4.2.78}$$

where the brackets $[\ldots]$ on the indices correspond to an anti-symmetrisation of these indices. The intuition that we can indeed invert $h_b^d(0)$ is that π^a transforms inhomogeneously under the action of broken generators and so, $h_b^d(0) \neq 0$. More formally, $\{X_a\}$ forms a basis of the tangential space of G/H at $\pi = 0$ and these generators are faithfully realised by $\{h_b^a(\pi)\partial_a\}$. Hence, the latter expression, evaluated at $\pi = 0$, is as well a basis of the tangential space of the coset space at the identity. Thus, $h_b^a(0)$ is a full ranked matrix.

We are now able to express the canonical structure of our theory in terms of ρ_{ab}. From our schematic reasoning (4.2.56), we can limit ourselves to the quadratic part of the Lagrangian to probe the canonical conjugated fields. Therefore, let us expand the first term of the Lagrangian (4.2.51) till the quadratic order in π:

$$c_a(\pi)\partial_t \pi^a = (c_a(0) + \partial_b c_a \,|_{\pi=0}\, \pi^b)\partial_t \pi^a \tag{4.2.79}$$

$$= (c_a(0) + \partial_{[b} c_{a]} \,|_{\pi=0}\, \pi^b + \partial_{\{b} c_{a\}} \,|_{\pi=0}\, \pi^b)\partial_t \pi^a , \tag{4.2.80}$$

where the braces $\{\ldots\}$ on the indices denote a symmetrisation of these indices. To continue the development, we can notice that

$$\partial_t(\partial_{\{b} c_{a\}} \,|_{\pi=0}\, \pi^b \pi^a) = \partial_{\{b} c_{a\}} \,|_{\pi=0}\, \partial_t \pi^b \pi^a + \partial_{\{b} c_{a\}} \pi^b \partial_t \pi^a \tag{4.2.81}$$

$$= 2\partial_{\{b} c_{a\}} \,|_{\pi=0}\, \pi^b \partial_t \pi^a . \tag{4.2.82}$$

Using the last equality in (4.2.80)

$$c_a(\pi)\partial_t \pi^a = \partial_{[b} c_{a]} \,|_{\pi=0}\, \pi^b \partial_t \pi^a + \partial_t(c_a(0)\partial_t \pi^a + \frac{1}{2}\partial_{\{b} c_{a\}} \pi^b \pi^a) . \tag{4.2.83}$$

We can drop the term with the total derivative because it will lead to a surface term in the expression of the action and due to the fact that we neglect the boundary effects, it will not influence the evolution of the system. The first term of (4.2.83) can be re-expressed with (4.2.78)

$$c_a(\pi)\partial_t \pi^a = \frac{1}{2}\rho_{ab} \left(h^{-1}(0)\right)_c^a \left(h^{-1}(0)\right)_d^b \pi^c \partial_t \pi^d \tag{4.2.84}$$

$$= \frac{1}{2}\rho_{ab} \tilde{\pi}^a \partial_t \tilde{\pi}^b , \tag{4.2.85}$$

where we did a field redefinition, i.e. a change of coordinate on G/H induced by the full ranked matrix $h_b^a(0)$. Let us do the misnomer ρ as being the matrix ρ_{ab} instead of $\rho_{\alpha\beta}$. Since ρ is a real and an anti-symmetric matrix, there exists an orthogonal change of basis such that (let us suppose that we work with this new basis since the beginning)

$$\rho = \begin{pmatrix} M_1 & & & & & \\ & \ddots & & & & \\ & & M_m & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \text{ with } M_i = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix} , \tag{4.2.86}$$

where $\lambda_i \neq 0$ for $i = 1, \ldots, m$. We emphasise that

$$\text{rank}(\rho) = \sum_{i=1}^{m} \text{rank}(M_i) = 2m \Leftrightarrow m = \tfrac{1}{2}\text{rank}(\rho) . \tag{4.2.87}$$

From (4.2.85) and (4.2.86), we have

$$c_a(\pi)\partial_t \pi^a = \sum_{i=1}^{m} \frac{1}{2}\lambda_i(\tilde{\pi}^{2i}\partial_t\tilde{\pi}^{2i-1} - \tilde{\pi}^{2i-1}\partial_t\tilde{\pi}^{2i}) . \tag{4.2.88}$$

By comparison with our schematic reasoning (4.2.56), we have that the $\tilde{\pi}^{2i}$ field is canonically conjugated with the $\tilde{\pi}^{2i-1}$ field. Hence, they do form one single degree of freedom instead of two. The associated independent NG mode is called a type B NG mode. Since i is running from 1 to m, we have that the number of type B NG modes, n_B, is given by

$$n_\text{B} = \tfrac{1}{2}\text{rank}(\rho) . \tag{4.2.89}$$

Concerning the π^a fields lying in the null part of ρ, cf. (4.2.86), they do not intervene in the single time-derivative term of the Lagrangian and are therefore canonically independent from the other fields. Each of these π^a represents one degree of freedom. We denote these NG modes as type A NG modes.

Conceptually, a type B NG mode is generated by two broken generators[16] while a type A NG mode is produced by one broken generator. However, in practice, this classification might not be robust with respect to an arbitrary choice of basis in the algebra. This is discussed in [54].

It is interesting to notice that by looking at (4.2.51) and to its Fourier transform, type B NG mode will systematically have a quadratic dispersion relation ($\omega \sim q^2$) while type A will in general have a linear dispersion relation ($\omega \sim q$). In fact, $\bar{g}_{ab}(\pi)$ might be semi-definite positive in some cases (we, here, extend a bit our hypotheses) and thus zero for some directions. In such situation, the dispersion relations are dictated by the $\mathcal{O}(\partial_i^4)$ term and so, type A NG mode would have a quadratic dispersion relation ($\omega \sim q^2$). Hence, type B NG modes are always type II NG modes while type A NG modes are often type I NG modes but in particular situations, it can be type II. This explain the inequality of Nielsen and Chadha's counting rule (4.1.2): $n_\text{I} + 2n_\text{II} = n_\text{BG}$ is when type A are indeed

[16]Qualitatively speaking, we are going to say that two broken generators Q_i and Q_j are conjugated if $\langle 0| [Q_i, Q_j] |0\rangle \neq 0$. In such case, we consider that Q_i and Q_j generate one type B NG mode. But it remains to show that, in the chosen basis, the generators are either conjugated by pairs or are independent. Otherwise the classification is spoiled.

type I but $n_{\rm I} + 2n_{\rm II} \geqslant n_{\rm BG}$ is when type A are type II and that they are erroneously counted twice by Nielsen and Chadha's counting rule.

With all these developments, we have recovered (with some shortcuts) the theorem established by Brauner, Murayama and Watanabe in [54, 77, 78]. This theorem can be stated as

Theorem 3 (Brauner-Murayama-Watanabe's theorem)**.** *Let us consider a physical field theory living in* $2+1$ *or above Minkowski spacetime which is invariant under translations and rotations (at least at long distances) and where no terms contain fields at two separated spacetime points (it could eventually be relaxed to an exponentially decrease of the interactions with distance). If the fundamental theory, in addition to spacetime symmetries which are not spontaneously broken, has a faithfully realised global continuous internal compact symmetry group* G *generated by* $\{Q_\alpha\}$ *such that it is either completely spontaneously broken or partially spontaneously broken to a continuous subgroup* H*, this without any anomalies and explicit symmetry breaking being involved, then, considering that the associated NG modes are the only massless modes, the number of NG bosons* n_{NG} *is related to the number of broken symmetry generators* n_{BG} *by the equality*

$$n_{NG} = n_{BG} - \frac{1}{2} rank(\rho) \ , \tag{4.2.90}$$

with

$$\rho_{ab} \equiv \lim_{V\to\infty} \frac{-i}{V} \langle 0| \left[Q_a, Q_b\right] |0\rangle \ , \tag{4.2.91}$$

where V *is the spatial volume of our system in spacetime and* $\{Q_a\}$ *are the broken generators.*

Let us mention that the hypothesis on the locality of the fundamental theory should ensure the effective field theory to be itself local, as we required in our preceding developments.

We conclude this discussion on the Brauner-Murayama-Watanabe's counting rule with several observations and remarks.

First, on the condition that the symmetry algebra satisfies ("gen." stands for "generator")

$$[\text{broken gen.}\,,\ \text{unbroken gen.}] = \sum \text{broken gen.}\ , \tag{4.2.92}$$

the theorem holds if there are additional unbroken internal symmetries and unbroken non-uniform symmetries. Still under the condition that (4.2.92) is verified, the counting rule remains true for G being non-compact. Let us stress that, for the moment, we do not tolerate the spontaneous breaking of non-uniform symmetries (in particular, spacetime symmetries). It will be discussed in Chapter 5.

Second, Brauner-Murayama-Watanabe's counting rule is not totally model independent since ρ is the VEV of the commutators of the broken generators of G. So, there is a dependency on the vacuum of the fundamental theory. Based on a thorough analysis of the topology/geometry of the coset space G/H and of the presymplectic structures which can live on G/H, Murayama and Watanabe completely classified the possible combinations of numbers of type A and type B NG modes for a given breaking pattern $G \to H$ [78]. Hence, it is partial information on the number of NG modes which rely only on the symmetries and so, is totally model independent.

Third, the counting rule can be sensible to the central extension of the Lie algebra at the quantum level since ρ depends on the commutators of the generators acting on the Hilbert space rather than on the phase space (the projective Hilbert space). A pedagogical example can be found in [9], it is the free non-relativistic complex scalar field

$$\mathcal{L} = i\phi^* \partial_t \phi - \frac{1}{2m} \partial_i \phi^* \partial_i \phi \; . \tag{4.2.93}$$

This theory is invariant under $U(1) \cong SO(2)$ and under the complex shift $\phi \to \phi + z$ where z is a complex constant, hence described by two real numbers. The symmetry group is then the internal continuous $ISO(2)$ group (I stands for inhomogeneous and corresponds to the shifts). Its realisation on ϕ is generated by three generators satisfying the algebra

$$[Q_3, Q_1] = -iQ_2 \; , [Q_3, Q_2] = iQ_1 \; , [Q_1, Q_2] = 0 \; , \tag{4.2.94}$$

where Q_1 and Q_2 respectively generates the real and imaginary shift while Q_3 generates $U(1)$. One possible vacuum of (4.2.93) is the trivial field $\phi = 0$, which means that the spontaneous breaking pattern is $ISO(2) \to SO(2)$. Despite $ISO(2)$ being non-compact, the condition (4.2.92) is verified and we can apply the counting rule. At classical level, the counting rule predicts two NG modes and indeed, the perturbation theory of (4.2.93) around $\phi = 0$ is nothing else than (4.2.93) (where ϕ is now the small perturbation) and we do indeed have two massless real modes. However, when we canonically quantise the theory, the Noether charges are such that a central charge appears

$$[Q_1, Q_2] = 2iV \; , \tag{4.2.95}$$

where V is the volume of the theory. Hence at quantum level, the counting rule predicts one NG mode. And it is verified by an explicit computation that the quantum spectrum has one massless mode. We refer to [9] for the details of the computations. Let us quote that for a relativistic free field, such central charge would have not appeared and we would have consistently had $n_{\text{NG}} = n_{\text{BG}}$. Let us mention that the central extensions of a Lie algebra $\mathfrak{g}$ are classified by the second Chevalley-Elenberg cohomology group $H^2(\mathfrak{g})$. This group is trivial for semi-simple finite-dimensional algebras, hence, such algebras do not have central extensions [103, 104].

Fourth, we have seen that it is the one-time derivative term which canonically combined the NG modes. Since the relativistic EFT (4.2.45) does not possess this term, we can safely conclude that all the NG modes are independent and so, their number corresponds to the number of broken generators. This is in fact displayed by (4.2.90). For relativistic theories, no charged operator under Lorentz group can acquire a VEV otherwise Lorentz symmetry would be broken. Considering no central extension, we have for relativistic theories

$$\rho \sim \langle 0| \left[Q_a, Q_b\right] |0\rangle = i f_{ab}{}^{c} \langle 0| Q_c |0\rangle \sim \langle 0| j^0_c |0\rangle = 0 \; , \tag{4.2.96}$$

because j^0_c is a component of the conserved current which is a Lorentz-vector [10]. Thus, for relativistic theories $\rho = 0$ and $n_{\text{NG}} = n_{\text{BG}}$ as expected. Let us emphasise that $n_{\text{NG}} = n_{\text{BG}}$ would not necessarily be true if we consider spontaneous breaking of spacetime symmetries in a relativistic fundamental theory (see Chapter 5).

Fifth, if we force the effective theory to depend only on the independent NG modes, the obtained EFT will be complicated. Indeed, if we try to integrate out one of the two canonically conjugated fields π^a of a type B NG mode, it will lead to non-local interaction terms in $\mathcal{L}_{\text{eff}}$ [78]. Furthermore, it will spoil the classification since a type B would then be described by one π^a which might be interpreted as a type A. We thus understand that the considered locality is necessary for the classification to make sense.

Sixth, if we go to higher enough energy, the two-time derivative term become dominant compared to the single time-derivative term. The canonical structure would then rather be determined by $g_{ab}(\pi)\partial_t\pi^a\partial_t\pi^b$ instead of $c_a(\pi)\partial_t\pi^a$. Since we consider g_{ab} as being a positive definite metric, and so a non-degenerate metric, the two π^a fields associated to a type B NG mode will be canonically independent. This means that each massless type B NG mode has a massive partner called an almost NG mode. The mass of an almost NG mode scales with the Lorentz breaking parameter c_a [105]. It is interesting to notice that its mass is not necessarily hierarchically smaller than other massive modes in the theory. Furthermore, the scaling with the Lorentz breaking parameter is consistent with the relativistic case: when $c_a = 0$, all the NG modes are type A and do not have partners. So, the almost NG mode should become a type A NG mode when we send c_a to zero (assuming a continuous limit), i.e. a massless mode. The case where g_{ab} is semi-positive definite, thus degenerate, is discussed in [105] where a counting rule is provided to give the number of almost NG mode we could expect. In the same paper, it is discussed the possibility to relax the hypothesis on spatial rotation symmetry in Theorem 3. Notice that, in our derivation, it might spoil the computation of the conserved current (4.2.69). It is as well commented that, because of the term $c_a\partial_t\pi^a$, time-reversal symmetry breaking might be a signature of type B NG modes. However, we should be cautious because as shown in [78], there is no systematic implication.

Seventh, the derivation of Theorem 3 has been (mainly) done at classical level. Going to higher orders, i.e. including the interactions in the EFT, and quantising the theory would lead to a renormalisation of the bare parameters of the EFT. The argument that the counting rule (4.2.90) still holds is that it mostly relies on the presence or not of the c_a coefficients and not on their specific values (the counting rule is not fine tuned). The way the parameters are arranged in the EFT is constraint by the symmetries. Therefore, we expect the canonical structure of the EFT to be symmetry protected and so is the counting rule. Naturally, this should be confirmed by formal computations. The historic papers on which Theorem 3 is based already proved that $\langle 0| [Q_a, Q_b] |0\rangle \neq 0 \Rightarrow n_{\text{NG}} < n_{\text{BG}}$ and this, with concrete computations both at classical and quantum level [9, 41, 57–59, 70, 79–84]. Brauner and Watanabe conjectured the counting rule (4.2.90) in [54] and partially proved it by a spectral decomposition in analogy to the proof of Nielsen and Chadha's counting rule and of the proof in Section 3.3. These quantum computations provide evidences of the quantum validity of Theorem 3. Futhermore, the stability of the classification of type A and type B NG mode under renormalisation in the presence of interactions has been studied in [106].

Eight, it should be mentioned that the counting rule (4.2.90) has been independently obtained in [107] by Mori projection operator method. Furthermore, in this article they extend the discussion to the finite temperature case. During the mid-2010 decade, [108] re-derived the counting rule (4.2.90) by the Bogoliubov theory and discussed how we could take into consideration other gapless modes than the NG modes in the analysis and discussed also how we could deal with spontaneously broken spacetime symmetry breaking.

Finally, while Brauner and Watanabe conjectured the counting rule in [54], they provided a partial proof which requires the symmetry to be uniform instead of compact and internal. Furthermore, the proof of the counting rule relies heavily on the shape of the EFT (4.2.51) and on the fact that the single time-derivative is due to a non-zero VEV of a charge density. Leutwyler recovered these two ingredients thanks to an EFT building method based on the Ward-Takahashi Identities, where he used similar hypotheses than the ones we imposed for the coset construction except that he does not require G to be compact [57, 58]. In addition, several examples we can find in the already cited literature (e.g. the acoustic phonon analysis in [77]) also point towards the idea that the counting rule could be extended to uniform symmetries – mainly because the shape of (4.2.51) can correspond to EFTs not necessarily coming from compact groups. Moreover, lattice systems (and the associated breaking of spatial translations and spatial rotations) could be encompassed in the discussion since we are in the IR which is consistent with a spatial continuum limit. A commonly accepted conjecture is then that Theorem 3 holds for G being a uniform continuous symmetry group which can include the spatial rotation and without the necessity to satisfy (4.2.92).

These abstract notions and the usefulness of the coset construction as well as of the Theorem 3 are illustrated in Appendix A with a concrete physical example: ferromagnestism.

4.3 Pseudo Goldstone modes

In the context of spontaneous symmetry breaking, there might be similar modes to the NG modes which are present in the theory. They also are symmetry-originated and have a mass (partially) settled by the symmetries. To understand their origin, we can take back the intuitive picture we have for the NG modes. We spontaneously break the symmetries of the system by looking for a non-trivial solution which minimises the energy. Taking this solution to be homogeneous (in this section, we are mostly interested by the breaking of internal symmetries), the kinetic part of the theory does not intervene in the discussion which simplifies the reasoning. Then, the flat directions of the potential around the background correspond to opportunities to build massless on-shell fluctuations. The broken symmetries parametrise some of these flat directions and are thus, the origin of the NG modes. However, there can be additional flat directions:

- The potential part of the action might have a bigger continuous symmetry group than the action as a whole. If these additional symmetries are spontaneously broken by the background, it leads to additional flat directions.
- The equations for the potential minimisation might see some emergent symmetries which with the SSB mechanism could correspond to additional flat directions.
- There might be a fine tuning among the Lagrangian parameters which makes that for a specific vacuum, the potential at this particular point has flat directions.

The additional symmetries coming from the potential are called approximate symmetries (since they are not exact symmetries of the full action). We understand that if we fluctuate the system around the vacuum along the flat directions associated to the approximate symmetries, we are getting massless modes [109, 110]. These modes are called quasi NG modes since at classical level, they are massless but their mass is not symmetry protected when we quantise the theory (or when we follow the RG flow). The quasi NG modes represent a limitation of Brauner-Murayama-Watanabe's counting rule because it has been established considering the NG modes as being the only massless modes in the theory. A first counting rule was derived in [110] but the generalisation of Brauner-Murayama-Watanabe's counting rule is discussed in [108, 111].

There is another family of modes which are closely related to the Nambu-Goldstone mechanism. They appear when a symmetry is explicitly broken by a small parameter m in the Lagrangian compared to the VEV. If the VEV would have spontaneously broken the symmetry then, we would have had an NG mode. If we consider the theory to have a continuous behaviour in the limit $m \to 0$, we can expect that the would-be NG mode has a small mass which goes to zero in the zero limit of m. This intuition was notably used by Nambu and Jona-Lasinio to describe light mesons in [2], one of the articles which led to the conjecture of Goldstone's theorem. It is Gell-Mann, Oakes and Renner who first established, still in the context of quantum chromodynamics (QCD), that the square of the mass of the would-be NG mode scales linearly with m [112]. This relation bears their name and is abbreviate as the GMOR relation. This result has been derived in several ways in the literature, mostly in the context of QCD, by using the Ward-Takahashi identities (e.g. [113, 114]) or by following the effective theory approach (e.g. [18, 115]). In a more general context than QCD, a derivation of the GMOR relation has been done for generic relativistic effective field theories in [57]. A proof of the GMOR law, at the level of the fundamental theory, for abelian internal symmetries in relativistic theories has been established in [22]. It can be worth to mention that the GMOR relation has been recovered in non-relativistic toy models [116] as well as in holographic models [21] displaying a homogeneous symmetry breaking of spatial translation.

We close this subsection with a clarification on the nomenclature. We call approximated symmetries, transformations which either leave the potential part of the action invariant but not the action as a whole or which are symmetries explicitly broken by a small parameter. The modes which have for origin the first case of approximate symmetries are called quasi NG modes. When the symmetry is explicitly broken by a small parameter and that the associated mode follows the GMOR relation, we call such mode a pseudo NG mode. Finally, the term massive NG mode can refer either to the massive partner of a type B NG mode (i.e. an almost NG mode) either to a massive symmetry-originated mode obtained in the context of the introduction of a chemical potential (cf. next section).

4.4 Goldstone physics at finite density

As we have seen, spontaneous symmetry breaking occurs in several areas of physics. It could then be interesting to see how Goldstone's theorem is implemented in these different domains. With the quantum field theory formalism we employ, the implementation of the theorem in particle physics is rather straightforward. We could now look how does it apply for many-body systems at equilibrium. A first step to it is to consider QFTs still at zero temperature but with a chemical potential. This is the aim of this subsection.

For a statistical system we can associate a chemical potential to each of its conserved charges – the microscopic dynamics is as usual dictated by either a Hamiltonian or a Lagrangian which might possess some symmetries and therefore, some conserved quantities are defined. Switching on the chemical potential μ_Q associated to the charge Q means that we consider that the external world acts on our statistical system such that it can vary the "conserved" charge Q. We are thus working in the grand canonical ensemble where μ_Q scales the energy cost when we vary Q. Considering our statistical system to be at equilibrium, its thermal state is dictated by the grand canonical partition function

$$Z = \text{Tr}\left[e^{-\beta\left(H-\mu_Q Q\right)}\right] . \tag{4.4.1}$$

The trace in (4.4.1) represents a summation/integration on the phase space. Hence, in the zero temperature limit, i.e. $\beta \to +\infty$, we can do the saddle-point approximation. This means that the thermal state is given by the minimisation of $H - \mu_Q Q$ and by the small fluctuations around the minimum.

From this brief recap of statistical physics, we emphasise that the microscopic dynamics is given by H and that the thermal state of the system is settled by $\tilde{H} \equiv H - \mu_Q Q$. So, switching on a chemical potential means that the background, which will spontaneously break the symmetries of H, minimises $\tilde{H}$ instead of H. Thus, compared to the zero chemical potential case, the vacuum might change and also the symmetry breaking pattern $G \to H_G$ (the unbroken subgroup is now written with a subindex to not be mistaken with the Hamiltonian). Furthermore, it is the fluctuations around $\tilde{H}$ which are physically important, thus, the gap (the mass) will be defined with respect to $\tilde{H}$ instead of H.

Let us call $|\mu\rangle$ the microscopic state which minimises $\tilde{H}$. It is therefore an eigenvector of the latter operator and since we do not consider gravity, we can redefine the energy scale such that the eigenvalue is zero:

$$\tilde{H}\,|\mu\rangle = (H - \mu_Q Q)\,|\mu\rangle = 0 . \tag{4.4.2}$$

The current literature on Goldstone physics at finite density deals in three ways with the chemical potential case at zero temperature:

1. We can interpret $\tilde{H}$ as generating an evolution of the system in a new time direction. Thermally speaking, it is the fluctuations evolving along this new time direction which interest us. Hence, we can effectively consider that the dynamics and the mass is given by $\tilde{H}$. Furthermore, from (4.4.2), $|\mu\rangle$ does not break spontaneously time-translation with respect to the new definition of time. We see that we recover exactly the setup of Goldstone's theorem where the theory is changed from H to $\tilde{H}$ and that the considered symmetry group should be the one of $\tilde{H}$. This idea is recovered in [79,80][17]. It is an efficient way to proceed to "get back on our feet" and extract, without additional costs, information on the low energy thermal excitations.

[17] These papers are written in the Lagrangian field theory approach. In such case, a chemical potential μ can be effectively described by gauging the symmetry to which it is related and by fixing the gauge field to be $A^\nu = \mu_Q\,\delta^{\nu 0}$ [117]. Schematically, the Lagrangian is thus of the form:

$$L = D_\mu\phi^* D^\mu\phi + \ldots = (\partial_0 + i\mu_Q)\phi^*(\partial_0 - i\mu_Q)\phi - \partial_i\phi^*\partial_i\phi + \ldots . \tag{4.4.3}$$

Of course, since we disregard some of the physical aspects, we lose some results, as it will be confirmed later.

2. We can tackle the problem with the standard spectral decomposition approach as we did in Section 3.3, where the time evolution of the microscopic states (i.e. the kets) is driven by H and where the gap is computed with respect to $\tilde{H}$. It is the results of this method that we present here below. It has the advantage of keeping track of the physical origin of μ_Q.

3. When Q is spontaneously broken by $|\mu\rangle$, the vacuum $|\mu\rangle$ evolves non-trivially in H-time as it can be noticed from (4.4.2). Therefore, time translation symmetry is spontaneously broken as well as maybe other spacetime symmetries (such as boosts for example). An approach based on the study of spacetime symmetry breaking can thus be used. The references [13,46] explicitly deal with such problematic. The EFTs built on a generalisation of the coset construction for spacetime symmetries permit to extract additional results compared to the ones we get with the standard spectral decomposition approach. These aspects will be discussed in Section 5.3.

Nicolis, Piazza [11] and Brauner, Murayama, Watanabe [12] showed, by a spectral decomposition of the order parameter, that the number of gapless NG modes is given by the Brauner-Murayama-Watanabe's counting rule where the considered broken generators are the broken symmetry generators of $\tilde{H}$, i.e. the broken symmetry generators of H which commute with Q. They also showed that the remaining broken symmetry generators of H lead to gapped modes where the gap is entirely fixed by μ_Q and by group theory. This can be summarised in the following theorem (NPBMW stands for the initials of the authors of [11, 12]).

Theorem 4 (NPBMW theorem)**.** *Let us consider a system satisfying the hypotheses of Theorem 3 (i.e. Brauner-Murayama-Watanabe's theorem's hypotheses). We switch on a chemical potential, μ_Q, for a particular symmetry generated by Q. The thermal state of the system is driven by the free energy and the notion of gap is defined according to it. The free energy has a symmetry group $\tilde{G}$ such that $\tilde{G} \subseteq G$. The number of massless NG bosons n_{NG} is related to the number of broken symmetry generators n_{BG} of $\tilde{G}$ by the equality*

$$n_{NG} = n_{BG} - \frac{1}{2} rank(\tilde{\rho}) \ , \tag{4.4.4}$$

with

$$\tilde{\rho}_{a,b} \equiv \lim_{V \to +\infty} \frac{-i}{V} \langle\mu| [\tilde{Q}_a, \tilde{Q}_b] |\mu\rangle \ , \tag{4.4.5}$$

where V is the volume of our system in spacetime, $|\mu\rangle$ is the vacuum and $\{\tilde{Q}_a\}$ are the broken generators of $\tilde{G}$.
The spectrum possesses some gapped modes as well, where their gap is entirely fixed by group theory and by μ_Q. The number of the massive NG modes n_{mNG} is given by

$$n_{mNG} = \frac{1}{2} \left[rank(\rho) - rank(\tilde{\rho}) \right] \ , \tag{4.4.6}$$

where ρ is defined in a similar fashion than $\tilde{\rho}$ but using the broken generators of G, $\{Q_a\}$, instead of the ones of $\tilde{G}$ ($\{\tilde{Q}_a\} \subseteq \{Q_a\}$). Under an appropriate choice of basis for the Lie *algebra of G, the massive NG modes are generated by pairs of broken generators $\{Q_{\pm\sigma}\}$ ($\notin \{\tilde{Q}_a\}$) and their gaps are $\mu_Q\, q_\sigma$ where $[Q, Q_{\pm\sigma}] = \pm q_\sigma Q_{\pm\sigma}$.*

Let us notice that the generators being at the origin of the gap-fixed modes are the ones which are explicitly broken by the parameter μ_Q in $\tilde{H}$. However, the gaps do not follow the GMOR relation. There are several points which permit to evade the GMOR relation: here, we are studying a gap with respect to $\tilde{H}$ while the dynamics is still driven by H for which the considered generators are symmetry generators. Moreover, in Section 5.3, we will see that, under the condition Q being spontaneously broken, all the physics can be solely derived from H. Hence, there is no notion of explicit symmetry breaking (only spontaneous breaking are present).

From an effective theory approach, Brauner, Murayama and Watanabe [12] have seen that there are additional massive NG modes for which the mass goes to zero when μ_Q is sent to zero but, this mass is different form the one of the gapped modes predicated by Theorem 4. Assuming a continuous behaviour of the theory with the limit $\mu_Q \to 0$ and that the new vacuum in this limit still displays the same breaking pattern $G \to H_G$, the number of such additional massive NG modes can be obtained by counting the number of NG modes at zero chemical potential and by substracting the number of NG modes and massive NG modes of Theorem 4 at finite chemical potential. To understand the nature of these additional modes, their dispersion relations etc. a deeper analysis should be done. This is the subject of [13], and Section 5.3, based on seeing the introduction of a chemical potential as breaking the time translation symmetry. We can, however, already guess candidates for such additional massive NG modes. If we consider a relativistic microscopic theory at finite density (Lorentz symmetry is thus generally broken), the partners of the massive NG modes of Theorem 4 can also be massive[18]. When we send μ_Q to zero, we recover our relativistic theory and so, all NG modes are type A, including our initial massive NG modes of Theorem 4. Therefore, their massive partners should also be massless, it makes them part of the additional massive NG modes we have when the chemical potential is switched on.

4.5 No spontaneous symmetry breaking at low dimensions

Spontaneous symmetry breaking is the fundamental hypothesis of Goldstone's theorem. It is therefore consistent to ask whenever a spontaneous symmetry breaking is possible. In this section we will enunciate some theorems which state that at lower spacetime dimensions, some spontaneous symmetry breaking patterns are impossible. In accordance with this dissertation, we will mainly discuss the zero temperature case, then, a comment will be made for the finite temperature case.

4.5.1 Coleman's theorem

From Section 2.4, at quantum level, the quantum superposition of the possible classical vacua is avoided thanks to the large volume of spacetime: the energy required for the

[18] A massive NG mode of Theorem 4 corresponds here to a combination of two generators. We can therefore consider a partner which is given by the orthogonal combination of the two same generators.

system to switch from one classical vacuum to another is proportional to the system volume (even if the potential directions are flat, there is still kinetic energy involved during the switching). Naively said, the lower the dimension of the spacetime is, the lower is the volume of the system. Hence, we could guess that at sufficiently low dimensions, the quantum fluctuations will be large enough to give rise to a symmetric quantum vacuum. For example, if we take the $U(1)$-circle of the Mexican hat of Figure 1, the specific classically selected point of the circle playing the role of the classical vacuum will be forgotten by the system due to the large quantum fluctuations around such point. Indeed, the fluctuations go all over the circle giving then a zero average state. The VEV being the order parameter, we lose the spontaneous symmetry breaking at quantum level.

This idea has been formally stated by Coleman [16] under the theorem:

Theorem 5 (Coleman's theorem). *For relativistic physical field theories in two-dimensional spacetime, at the quantum level, there cannot be any spontaneous breaking of continuous internal global symmetries.*

The proof of Coleman [16] is rather mathematical, hence, we will sketch the proof of [118] which is more physical and closer to the intuition we proposed earlier. It is a proof by contradiction, spontaneous symmetry breaking implies massless modes at quantum level. In two-dimensional spacetime, such massless modes induce an IR divergence which makes vanish the VEV (the order parameter) and so, we lose spontaneous symmetry breaking. The consistent picture is thus that we never have spontaneous symmetry breaking in such context.

To see this, we start with a relativistic fundamental field theory and we spontaneously break some of its continuous internal symmetries. To get the asymptotic spectrum of the QFT obtained by quantisation around the chosen classical vacuum, we perturb our fundamental theory around our background and we consider the free theory of the perturbations[19]. The NG modes being independent from each other – the free part of their action is of the form of (4.2.45), so there are no canonical conjugated pairs –, we do not lose much generality by considering the specific abelian case of the $U(1)$ symmetry spontaneously broken. The unique associated NG mode is denoted by $\theta(x)$. A relativistic massless free theory is a conformal field theory (CFT). We can obtain the shape of the two-point correlator of $\theta(x)$ through the Ward-Takahashi identities [119]. The symmetry under the scaling $x^\mu \to \lambda\, x^\mu$ imposes

$$\langle \theta(\lambda x)\theta(\lambda y)\rangle = \lambda^{-2\Delta_\theta} \langle \theta(x)\theta(y)\rangle \ , \tag{4.5.1}$$

where Δ_θ is the scaling dimension of $\theta(x)$. From the invariance under Lorentz and under translations, with $\theta(x)$ being a scalar field and since the only scaling object at disposal in our CFT is x^μ, we have

$$\langle \theta(x)\theta(0)\rangle \propto \frac{1}{|x|^{2\Delta_\theta}} \ . \tag{4.5.2}$$

We can already see that there is a change of behaviour at large distance ($|x| \to +\infty$), i.e. in the IR, following Δ_θ being positive or negative. In particular, we will have an IR

[19] Another way to argue why we focus on the free theory of the perturbations is to anticipate the fact that we will get an IR divergence. Hence, it is the low energy regim of the theory which interests us. We know that, at low enough energy, the relativistic Goldstone modes are described by (4.2.45) and are thus free.

divergence when

$$\Delta_\theta \leqslant 0 \ . \tag{4.5.3}$$

Let us mention that the case $\Delta_\theta = 0$ should be treated with more care, but as we will see, it leads to an IR divergence as well.

We still have to determine Δ_θ. When we mentioned that the only scaling object at disposal was x^μ, we did a small shortcut because actually, there is a scale in the theory which is the VEV v coming from the $U(1)$ fundamental theory. But this v, dynamically speaking, can be reabsorbed by a redefinition of the field $\theta(x)$. Looking back to (3.4.6), the free theory of a $U(1)$ NG mode is

$$S_{\text{free}}[\theta] = \int d^d x\, v^2\, \partial_\mu\theta\partial^\mu\theta \ , \tag{4.5.4}$$

where d is the spacetime dimension. By absorbing v into the canonical dimensionless $U(1)$ phase θ, we get a dynamical field θ with a scaling dimension[20]

$$\Delta_\theta = \Delta_v = \Delta_\phi \ , \tag{4.5.5}$$

where ϕ is the $U(1)$ fundamental field. The canonical dimension of a relativistic scalar field is $\frac{d-2}{2}$, under the relativistic scaling[21] $x^\mu \to \lambda\, x^\mu$, the scaling dimension is equal to the canonical dimension. Hence,

$$\Delta_\theta = \frac{d-2}{2} \ . \tag{4.5.6}$$

By injecting (4.5.6) in (4.5.2), and by adding a bit more of details on the coefficients necessary to understand the case $\Delta_\theta = 0$, we obtain

$$\langle\theta(x)\theta(0)\rangle \propto \frac{\Gamma\left(\frac{d}{2}-1\right)}{|x|^{d-2}} \underset{d=2}{\propto} -\ln\left(\frac{|x|}{|x_0|}\right) \ . \tag{4.5.7}$$

The IR divergence region given by (4.5.3) corresponds to $d \leqslant 2$. Since, in this dissertation, we consider $d \geqslant 2$, it leaves the case $d = 2$. Mentioning the coefficient $\Gamma\left(\frac{d}{2}-1\right)$ was important to understand why the case $d = 2$ is singular[22]. We allowed ourselves to not derive the origin of this Γ as well as other details of the following computations, the reason is that we do explicitly similar computations in Part III of this thesis.

We observe that we have a radical change of behaviour when the spacetime dimension d is equal to two. With a proper regularisation, we get a logarithmic behaviour for $d = 2$ where x_0 is the regulator. We then have both a UV divergence ($|x| \to 0$) and an IR

[20] A brief recap on the nomenclature: we redefine a field in a field theory such that one of the terms in the Lagrangian has a coefficient equal to one. All the terms which acquire a coefficient one thanks to this redefinition are called the "kinetic part" of the theory. Usually, we chose the lowest time-derivative order term to have a coefficient one. The canonical dimension of the field is then settled by its kinetic term(s). The scaling dimension of the field is the parameter which tells us how the field transforms under a given dilatation of the spacetime coordinates. Classically, the scaling dimension of a field is equal to its canonical dimension. At quantum level, from the RG flow, there might be some (anomalous) corrections which makes differ the scaling dimension from the canonical dimension. Here, we are looking to a Gaussian RG fixed point, no such corrections will appear.

[21] Relativistic because time and space scale the same way.

[22] The Γ function is divergent when evaluated at zero.

divergence ($|x| \to +\infty$). The UV divergence can be dealt with the usual procedures of renormalisation. However, the IR divergence will remain[23]. It is worth to emphasise that this IR divergence is due to the Goldstonic nature of $\theta(x)$, i.e. to its masslessness. To observe it, we go to Fourier space, from (4.5.4) and after integrating ω

$$\langle\theta(x)\theta(0)\rangle \sim \int d^{d-1}k \, \frac{e^{-i(\omega(k)t-k_i x^i)}}{\omega(k)} \underset{IR}{\sim} \int d|\vec{k}| \, \frac{|\vec{k}|^{d-2}}{|\vec{k}|} \, , \tag{4.5.8}$$

where the dispersion relation of θ is $\omega(k) = |\vec{k}|$, we used the spherical coordinates and focused on the IR part of the integral ($|\vec{k}| \approx 0$). The lack of a mass in the denominator and the polynomial order of the dispersion relation (which by the way is linked to Δ_θ) are the reasons why the integral diverges when $d = 2$ and when $|\vec{k}| \to 0$. Let us notice that the regulator $|x_0|$ in (4.5.7) can be seen as a mass regulator $1/|x_0|$. To get rid of this IR divergence, we could get rid of the massless mode, i.e. of the Goldstone mode. This is what naturally does the theory. In fact, we will now show that the VEV of the fundamental $U(1)$ theory is set to zero due to this IR divergence.

Considering $\theta(x)$ as a free field, we have[24] [35]

$$\langle\theta(x)\theta(0)\rangle = \left\langle\theta^+(x)\theta^-(0)\right\rangle = \left\langle\left[\theta^+(x), \theta^-(0)\right]\right\rangle \, , \tag{4.5.9}$$

with $\theta \equiv \theta^+ + \theta^-$ where θ^+ is associated to the positive energy modes and is proportional to an annihilation operator, θ^- is associated to the negatives energy modes and is proportional to a creation operator. We now evaluate the one-point function of the fundamental $U(1)$ complex field ϕ:

$$\begin{aligned}\langle\phi(x)\rangle &= \left\langle(v+\sigma(x))e^{i\theta(x)}\right\rangle \\ &= v\left\langle e^{i\theta(x)}\right\rangle = v\left\langle e^{i\theta^-(x)}e^{i\theta^+(x)}e^{1/2\left[\theta^-(x),\theta^+(x)\right]}\right\rangle = v\, e^{-1/2\left\langle\left[\theta^+(x),\theta^-(x)\right]\right\rangle} \\ &= v\, e^{-1/2(\langle\theta(0)\theta(0)\rangle)} \\ &= 0 \quad \text{for d=2} \, ,\end{aligned} \tag{4.5.10}$$

where $\sigma(x)$ is the small massive norm perturbation and v is the classical chosen vacuum. To go through the different equalities, we are at the free level, hence the relativistic one-point correlator of σ is zero. The fields σ and θ are not interacting, thus, their mixed correlator splits into the shape $\langle\sigma\rangle\left\langle e^{i\theta(x)}\right\rangle$. Thanks to $\langle\sigma\rangle = 0$, the mixed correlator vanishes. Then, we used the Baker–Campbell–Hausdorff formula (up to the quadratic order). By using (4.5.7) and translational symmetry, we notice that in two-dimensional spacetime, the VEV vanishes which makes inconsistent the initial hypothesis that $U(1)$ symmetry is spontaneously broken at quantum level. This concludes the sketch of the proof by contradiction[25].

[23] Naively said, the bare parameters are already used to deal with the UV divergence. Furthermore, as it will be displayed, the IR divergence is due to the massless modes only. An efficient way to renormalise the IR divergence will be to simply get rid of these massless modes. These massless modes being the NG modes, we can remove them by suppressing the SSB.

[24] This is the non time-ordered 2-point correlator. However, we will evaluate it at the two same space-time points. Hence, we will be able to use the time order propagator (4.5.7) to evaluate it.

[25] The physical interpretation we gave to Coleman's theorem was that it is due to large quantum

It should be emphasised that it is the massless nature of the NG modes which leads to the logarithmic behaviour (4.5.7) – cf. (4.5.8). If they were massive, we would not have had a CFT and the two-point correlator would then be an exponential decrease with an argument weighted by the mass of the considered particle. This explains why Coleman's theorem does not exclude the spontaneous breaking of discrete symmetries in two dimensions. This is because such symmetries do not lead to Goldstone modes, i.e. to massless particles "free" in the IR. The same comment holds for small explicit breaking of symmetries. In this case, the NG candidates will be pseudo-Goldstone modes with masses hierarchically smaller than the rest of the massive spectrum. These masses can anyway play the role of IR regulators and so they permit to avoid the IR divergence. Hence, explicit symmetry breaking (even a small one) is tolerated by Coleman's theorem.

A wondering we could have on our sketch of the proof could be that we worked all the time with the free theory of the perturbations. Intuitively, switching on the interactions will lead to another UV renormalisation but, this new renormalisation will not alter the massless nature of the NG modes since it is symmetry protected. Hence, the IR divergence will remain and the consistent way to fully renormalise the theory will be to set to zero the VEV.

Coleman's theorem can fail for specific relativistic theories. Theories with a large number N of constituents are known to have ordered phases in the $N \to \infty$ limit [26, 27]. It can be seen that the large quantum fluctuations are actually suppressed by a $1/N$ power [28]

$$\langle \phi(x)\phi(0) \rangle \underset{|x|\to\infty}{\propto} \frac{1}{|x|^{1/N}} \underset{N\to\infty}{\longrightarrow} cst \ , \tag{4.5.12}$$

where ϕ is the fundamental field and its two-point correlator probes the ordered structure of the vacuum[26]. This is precisely the case for theories which have a holographic dual. It was shown in [121] that indeed AdS_3 holography allows for spontaneous symmetry breaking in its dual two-dimensional QFT (we will come back on what holography is in Part II – for now, holography is a tool allowing us to probe large N field theories).

Let us notice that large N theories could a priori be seen as QFT curiosities rather than proper physical theories. So, the failure of Coleman's theorem is a rather academic

fluctuations that the VEV is set to zero. It is in fact what we have here. A quantum measurement is statistical. Therefore, a measurement is given by its expectation value and by its variance. The variance is what is called the quantum fluctuation. In our case, the variance of a measurement on θ is

$$(\Delta\theta(0))^2 = \left\langle \theta(0)^2 \right\rangle - \langle \theta(0) \rangle^2 = \langle \theta(0)\theta(0) \rangle \ , \tag{4.5.11}$$

where $\langle \theta(0) \rangle = 0$ since θ is driven by a relativistic free theory. Notice that this result is true at all spacetime position thanks to the symmetry of translations. Thus, the divergence of the 2-point correlator is indeed a divergence of the quantum fluctuations.

[26]Large scale ordered structure is, under some circumstances, indeed synonym to SSB. The cluster decomposition tells us that for relativistic theories, at large distance (space-like distance) we have [120]

$$\langle \phi(x)\phi(0) \rangle \underset{|x|\to\infty}{\sim} \langle \phi(x) \rangle \langle \phi(0) \rangle \ , \tag{4.5.13}$$

where x is a spacelike vector. A non-zero result reflects the fact that two measurements at non-causal distance can be correlated but, these measurements cannot influence themselves (cf. the splitting into one-point correlators). So, if we have large scale ordering, i.e. large scale correlation, it means that the VEV is non-zero. Large distance ordering is indeed a sign of SSB.

discussion. However, in the framework of holographic dualities, such models could describe sensible gravitational physics.

4.5.2 Non-relativistic extension of Coleman's theorem

The non-relativistic scenario has been commented by Griffin, Grosvenor, Horava and Yan [17]. The main difference is that now the free theory of NG modes can lead to a non-trivial canonical conjugation among NG modes as it has been seen for the type B NG modes by studying (4.2.51). Furthermore, the free theory is not anymore constrained by Lorentz symmetry, thus, the order of the spatial derivatives can be higher than the usual second order. Having this in mind, [17] based their argumentation on the intuition that no spontaneous symmetry breaking can occurs when the scaling dimension of the NG modes is smaller or equal than zero, in analogy to the relativistic case (4.5.3). The sketch of the reasoning goes as follow, if we idealise the free theory of the NG modes, we respectively have for the type A NG modes and for the type B NG modes[27]

$$S_{\text{Free Type A}}[\pi] = \int dt d^{d-1}x \left(\partial_t \pi \partial_t \pi - c\, \partial_i^n \pi\, \partial_i^n \pi\right) \ , \tag{4.5.14}$$

$$S_{\text{Free Type B}}[\pi^1, \pi^2] = \int dt d^{d-1}x \left(\pi^1 \partial_t \pi^2 - \pi^2 \partial_t \pi^1 - c_j\, \partial_i^{\frac{n}{2}} \pi^j\, \partial_i^{\frac{n}{2}} \pi^j\right) \ , \tag{4.5.15}$$

where we have already reabsorbed the VEV factor into the definition of the π fields which are the NG fields (this means that the π fields have the same canonical dimensions than the fundamental fields, and by extension, the same scaling dimensions). For simplicity, we considered the two canonicaly conjugated type B NG fields to have the same order in the spatial derivatives. In (4.5.14) and (4.5.15), we can observe that we have a non-relativistic scaling

$$x^i \to \lambda\, x^i \ , \tag{4.5.16}$$

$$t \to \lambda^n\, t \ , \tag{4.5.17}$$

$$\pi \to \lambda^{-\Delta_\pi}\, \pi \ , \tag{4.5.18}$$

where, respectively for type A NG modes and for type B NG modes, we have

$$\text{Type A:} \qquad \Delta_\pi = \frac{d-1-n}{2} \ , \tag{4.5.19}$$

$$\text{Type B:} \qquad \Delta_\pi = \frac{d-1}{2} \ . \tag{4.5.20}$$

Moreover, we still have the spacetime translation symmetries and the spatial rotation symmetries. Thus, the 2-point correlator at zero time but at two saptial different points has the shape

$$\langle \pi(0,x)\pi(0,0)\rangle \propto \frac{1}{|x|^{2\Delta_\pi}} \ . \tag{4.5.21}$$

We recover the idea that we have an IR divergence ($|x| \to +\infty$) when $\Delta_\pi \leqslant 0$ (again, the case $\Delta_\pi = 0$ should be dealt with more care). This suggests that, thanks to (4.5.19), when

[27] Since we are taking for granted the classification type A and type B NG modes, it means that we implicitly make the assumption to be in the range of Theorem 3 and its extended conjecture.

we solely have type A NG modes, we do not have SSB of continuous internal symmetries at quantum level when

$$\text{Solely type A, no SSB when:} \qquad d \leqslant n+1 \; , \tag{4.5.22}$$

where $2\,n$ is the number of spatial derivatives in the free theory. Thanks to (4.5.20), when we solely have type B NG modes, we do not have SSB of continuous internal symmetries at quantum level when

$$\text{Solely type B, no SSB when:} \qquad d \leqslant 1 \; , \tag{4.5.23}$$

which never occurs since $d \geqslant 2$.

We can notice that for type A NG modes, we consistently recover the relativistic case when $n = 1$ (the true relativistic case is when the coefficient $c = 1$ also). Concerning the type B NG modes, they do not prevent the spontaneous symmetry breaking at quantum level in any dimensions. The intuition could be that a type B NG mode has a gapped partner which also corresponds to a fluctuation in one of the broken symmetry directions. Because this mode is gapped, it does not lead to IR divergences (cf. (4.5.8) for example) and so, the associated symmetry direction can be spontaneously broken. Of course, it is a non-trivial mechanism which leads to the idea that the symmetry direction associated to the gapless type B NG mode can then as well be broken. Let us mention that type B NG modes can only occur in non-relativistic systems, hence, there is no conflict with Coleman's theorem.

The result for the type A NG modes can also be obtained through the Fourier space where the dispersion relation is $\omega \sim |\vec{k}|^n$:

$$\langle \pi(x)\pi(0)\rangle \sim \int d^{d-1}k \, \frac{e^{-i(\omega(k)t - k_i x^i)}}{\omega(k)} \underset{IR}{\sim} \int d|\vec{k}| \, \frac{|\vec{k}|^{d-2}}{|\vec{k}|^n} \; . \tag{4.5.24}$$

The last integral diverges in the region $|\vec{k}| \to 0$ when $d-2-n \leqslant -1$, i.e. when $d \leqslant n+1$, consistently with (4.5.22). If we reproduce the same reasoning with type B NG modes, we will not recover (4.5.23). This is because our computations do not take into account the canonical conjugation structure of the type B NG modes, therefore, this case should be discussed and analysed more carefully.

We close this discussion on the non-relativistic extension of Coleman's theorem by stressing the fact that the presented results come from an argumentation in [17] and no formal rigorous proofs or computations have been done in this paper. Murayama and Watanabe presented a computation in [78] where they confirm that SSB leading to only non-relativistic type A NG modes with linear dispersion relations do indeed follow Coleman's theorem (no SSB at quantum level when $d = 2$). The idea remains the same than the Fourier space approach except that they consider a generalisation of (4.2.51) (at two dimensions there are additional symmetry-allowed terms). Furthermore, they also emphasise the fact that there is nothing that prevent SSB at quantum level if only type B NG modes are generated. The non-relativistic scenario of Coleman's theorem is as well discussed in [5].

Some of the possible open questions that remain or some points to clarify, at zero temperature, could be:

1. What is the critical dimension (i.e. the dimension below which we cannot have SSB at quantum level) when the theory possesses both type A and type B NG modes ?
2. More explicit computations to consolidate the two statements of [17] are needed.
3. In the strict large N limit, is the spontaneous symmetry breaking recovered at quantum level like in Coleman's case ?
4. How Coleman's theorem extends to the breaking of spacetime symmetries ? The gist of why it is more complicated is that the dispersion relations of the NG modes in this case are more involved. Indeed, we might not have isotropic dispersion relations anymore (we might lose rotational symmetries) and so, possibly have non-analytic dispersion relations. The Fourier space analysis of the IR divergences is thus less straight forward.

In Part III of this dissertation we provide an explicit computation to assert the statement of [17] that no spontaneous symmetry breaking that lead to only type A NG modes can occur at quantum level when $d \leqslant n + 1$. This computation is actually a bit more general since there, n is taken to be real positive greater than one rather than to be a natural number greater than one. These kinds of theories are called Lifshitz theories and will be presented in Part III. In the same part of this thesis, we will also show that it is possible to evade the statement on the critical dimension if we consider strict large N theories. This will be seen through a holographic computation.

4.5.3 Mermin-Wagner-Hohenberg theorem

A bit prior to Coleman work, a similar discussion has been done at finite temperature where the thermal fluctuations play a similar role as the quantum fluctuations on the parameter order. It is the Mermin-Wagner-Hohenberg theorem [122, 123]. It states that at finite temperature, no continuous spontaneous symmetry breaking can occur for $d \leqslant 3$ where d is the spacetime dimension. Let us notice that the critical value for d is stricter than the one for the zero temperature case (Coleman's theorem) which is consistent with the idea that now, both quantum fluctuations and thermal fluctuations add up in order to vanish the order parameter.

In this thesis, we are not discussing thermal field theory, hence, we will not expand on how Goldstone physics fits with Mermin-Wagner-Hohenberg theorem. A discussion on the possible NG modes we can have in a thermal theory, at low spacetime dimension, is done (for example) in [5, 78, 124].

Chapter 5

Spontaneous breaking of spacetime symmetries

Until now, we concentrate our study of NG modes (counting rules and classifications) on the breaking of continuous internal (compact) symmetry groups. We also discussed and argued that it is reasonable to think that we can extend the already obtained results to the breaking of continuous uniform symmetries. But concerning the breaking of continuous non-uniform symmetries, and more specifically of continuous spacetime symmetries, the analysis is much more involved. A generic counting rule for such symmetry breaking patterns is still unknown and represents a current active research topic. Because this field is not yet well established, we will remain perfunctory in our discussion. The latter will focus on the spacetime symmetry case, the extrapolation to non-uniform symmetries in general will be commented.

5.1 Spacetime symmetry specificities

We can have a feeling of the reasons why spacetime symmetry breaking is a complex problem. First of all, many of the spacetime symmetries are non-compact – e.g. dilatation, translations, boosts,... It means that the useful group theory properties we used so far are not anymore systematically verified. We did not expand much on it, but we have noticed that the coset construction can be related to differential geometry. The geometric study of coset spaces G/H where G is non-compact is more involved. More broadly, taking G totally general makes more difficult a generic classification of the NG modes, of all the possible terms in a symmetric-invariant EFT,...

Furthermore, breaking spacetime symmetries usually means to work with a spacetime dependent background where before it was purely constant. Thus, the functional aspect of QFT is emphasised. Indeed, to find a stable vacuum, we minimise the energy. When we look for a constant solution, the energy can be seen as a function defined on a set of numbers (i.e. a real or a complex space). But when we tolerate for spacetime modulated vacua, we are forced to consider the energy as it is, i.e. a functional.

Another difficulty is that the effective Lagrangians are less constrained and thus, are more complicated. If we look at (4.2.45) and (4.2.51), we were able to write them in a compact form thanks to respectively Lorentz symmetry and rotation symmetry. The dispersion relations for the NG modes are therefore more involved with less constrained EFTs and can even be non-analytic. We understand that the classification based on dispersion relations might have some flaws.

Finally, Derrick theorem [24] suggests that we would need to have higher derivative terms in the fundamental theory to have stable and physical solitonic solutions. Of course, this should be qualified but it displays the tendency that even toy models are difficult theories. See for instance [45, 116, 125], some of the papers which are the core of Part II.

We just gave a gist of the technical difficulties associated to spacetime spontaneous symmetry breaking, but does it affect the counting rules we already know ? The answer is yes. In fact, even for relativistic fundamental theories, the number of NG modes can be reduced compared to the number of broken generators. Indeed, [96, 126] studied the spontaneous breaking of the conformal group down to the Poincaré group. They found that (for some cases) only one massless mode was present in the spectrum. It appears that it is the NG mode associated to the breaking of dilatation while the breaking of the special conformal transformations is not providing additional NG modes[1]. A simpler example is the spontaneous breaking of translation and rotation symmetries to a discrete subset by a crystal lattice. With an explicit computation of the oscillation modes of the lattice, it can be noticed that the number of NG modes are linked to the breaking of translations and that the rotations do not provide additional massless excitations. We can intuitively understand this last result as explained in the paragraph below.

By looking in Figure 5.1, we have an infinite straight rope disposed in a plane Oxy. By choosing this specific position/configuration, we spontaneously break translation symmetry in the $x-$direction and the rotation symmetry of the plane. We can observe that a global rotation acting on the rope is equivalent to a modulated action of the x-translation where the modulation is linear with y. If we extrapolate this information, we have that a local action of rotation on our rope can always be reproduced by a local action of x-translation. By definition, an NG mode is a spacetime modulation of the background in the direction of one of the spontaneously broken symmetries. Hence, the NG mode generated by the broken rotation is equivalent to the NG mode generated by the x-translation. So, unlike to the internal case, even before discussing dynamical conjugation between NG modes, we can already have locking between some broken directions; and thus, a reduction of the number of independent NG modes. This instinctive reasoning has been formalised and talked through in [127–129]. We will come back on it once we have generalised the coset construction for the breaking of spacetime symmetries (we could even say generalised for a generic breaking pattern – modulo some assumptions).

5.2 The coset construction for spacetime symmetries

The coset construction has been extrapolated to particular cases of spontaneous spacetime symmetries. In fact, we already mentioned that [96, 126] studied the spontaneous breaking of the conformal group down to the Poincaré group. Volkov in [96] displayed a coset construction for an unspecified group G spontaneously broken to a subgroup H containing the Poincaré group. This particular extension to spacetime symmetries is explained in the lecture notes of Ogievetsky [130]. In this framework, it was noticed that, for some specific symmetry breaking patterns, it is possible to build invariant Lagrangians without

[1] A possible handwaving argument to understand this result is that there is a conjecture which states that a Poincaré invariant theory invariant under scaling symmetry is a CFT (meaning, it is as well invariant under the action of the special conformal transformations). Therefore, starting with a relativistic EFT, we only need to add one field to it in order to make it scale invariant. For the same prize, we make it invariant under the full conformal group. Hence, the EFT of a CFT where dilatation and special conformal transformations have been spontaneously broken needs only one NG field (the one coming from dilatation breaking) to be invariant under the conformal group [127].

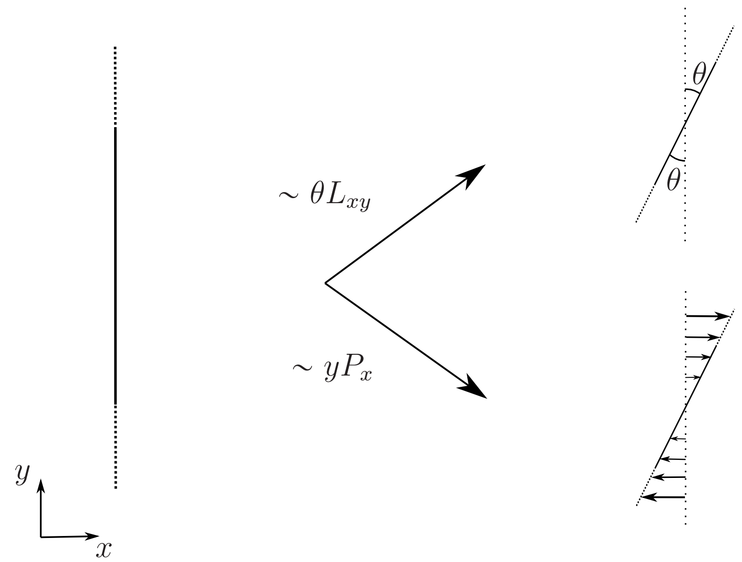

Figure 5.1: In this cartoon, we illustrate how a modulated translation on a line can reproduce a global rotation of this line. This is the schematic reasoning providing that some NG modes associated to broken spacetime symmetries can be locked together.

requiring all the NG mode candidates. Hence, some of them might be non-physical or can be massive (e.g., the single dilaton associated to the breaking of dilatation symmetry and of special conformal transformation symmetries). The conditions when this situation occurs have been investigated by Ivanov and Ogievetsky in [131]. Based on the more detailed review [19], we provide in this section a description of the prescription provided by [96, 130, 131]. The hypotheses of validity of this prescription are not yet well established. Thus, the following results should be taken with care and any application of the prescription to an extended case should carefully be checked.

5.2.1 Effective field theories purely based on group theory

If we try to directly extend the reasoning we made at Subsection 4.2.1 to the case where G possesses some spacetime symmetries, the process and the computations become relatively involved[2]. This is due to the fact that we use the field realisation of G acting on the fundamental field(s). For internal symmetries, there is not much a difference between the Lie algebra and its field representation. This is why we were doing the misnomer of calling $\{G_\alpha\}$ the generators instead of the realisation of the generators. However, for spacetime symmetries, it is not the case. This because they act as well on the spacetime coordinates, their field realisation involves derivatives. For example, the field realisation of a Lorentz generator is $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}$. Since the NG fields are the coefficients of the realisation of the broken generators, due to their coordinates dependence, the usual commutation relation coming from the algebra is not satisfied anymore [58]:

$$[G_\alpha, \pi^a(x) X_a] = [G_\alpha, \pi^a(x)]\, X_a + \pi^a(x)\, [G_\alpha, X_a] \neq \pi^a(x)\, [G_\alpha, X_a] \ , \tag{5.2.1}$$

[2]For this discussion on the coset construction for the spacetime symmetry case, we use the same nomenclature and notations as in Subsection 4.2.1. Few details might change but they will explicitly be stated.

because the derivatives in the realisation of G_α will act on $\pi^a(x)$. Therefore, all the computations involving the commutation of two group elements, e.g. to compute (4.2.13), are more involved compared to the internal case where the usual commutation laws of the algebra are sufficient ($[G_\alpha, \pi^a(x) X_a] = \pi^a(x)\,[G_\alpha, X_a]$).

To avoid this technical problem, we need to go back to a more abstract picture and forget (for a while) the fundamental theory. We are going to build tools which will permit to write EFTs purely based on group theory considerations. So, the prescription proposed by Volkov et al. is the pure EFT picture while the reasoning proposed at Subsection 4.2.1 was still a bit relying on the fundamental theory (through the way the symmetry group is realised on the theory and through the way the fundamental fields are parametrised in terms of the NG fields and other perturbation fields). In some sense, we are recovering the original idea of the coset construction: the classifications of the non-linear realisations of $G/\tilde{H}$ where $\tilde{H}$ is a linearly realised subgroup of G. We will not do a mathematical discussion and we do not claim any mathematical rigour. We will follow the guidelines given in the review [19] and of course use some other relevant papers which will be cited at the appropriated time.

Because we use a totally abstract reasoning, we have to build everything from scratch. In particular, we have to build the NG candidate fields and to define consistent transformation laws for them (and for eventually additional matter fields and/or gauge fields). We know that we have at disposal a generic group G and we make the assumption that it contains the Poincaré group and that the latter is not spontaneously broken. We thus have an algebra containing, among others, broken generators $\{X_a\}$ and spacetime translation generators $\{P_\mu\}$. To define NG candidate fields, in the subspace generated by $\{X_a,\, P_\mu\}$, we consider a submanifold parametrised by $\pi^a(x)$, where x^μ are the algebra coefficients of P_μ and π^a are the algebra coefficients of X_a. The goal is now to build/to define transformation laws under G such that x^μ can be interpreted as Minkowski spacetime coordinates and $\pi^a(x)$ as usual QFT fields (i.e. with usual transformation laws under the Poincaré group for both x^μ and $\pi^a(x)$). These transformation laws for $\pi^a(x)$ should as well be non-linear for G/H, where H is the unbroken subgroup (notice that $G/H \subseteq G/\tilde{H} \Rightarrow \tilde{H} \subseteq H$). This in order for the $\pi^a(x)$ to parametrise NG candidates.

Let us emphasis that, since we are at the Lie algebra level, we do have that

$$[G_\alpha, \pi^a(x) X_a] = \pi^a(x)\,[G_\alpha, X_a]\ . \tag{5.2.2}$$

From this observation, the coset construction for the spacetime symmetry case is vastly similar to what we did in Subsection 4.2.1. A last important remark is that G_α and P_μ are both labelised by Greek indices. Since G_α will not be used in the explicit computations, it will always be clear that, in the rest of this section, a Greek index refers to a spacetime translation generator.

5.2.2 Hypotheses on the symmetry group

We consider a continuous global symmetry group G which can include spacetime transformations as well as non-uniform symmetries in general. This group G is spontaneously broken to a continuous subgroup H which contains at least the Poincaré group. We denote X_a the broken generators, P_μ the unbroken translation generators and T_A the remaining unbroken generators – so, T_A contains the Lorentz generators and some other unbroken generators (notice the difference with Subsection 4.2.1, where T_A was denoting all the unbroken generators). The subspace generated by T_A is, as it will be confirmed a posteriori, $\tilde{H}$ and, on the contrary to the internal case, it is different from the unbroken subgroup H. We ask for the following relations

$$[X_a, T_A] = i f_{aA}{}^{b}\, X_b \ , \tag{5.2.3}$$
$$[P_\mu, T_A] = i f_{\mu A}{}^{\nu}\, P_\nu \ . \tag{5.2.4}$$

Moreover, we ask $\tilde{H}$ to be a subgroup, thus,

$$[T_A, T_B] = i f_{AB}{}^{C}\, T_C \ . \tag{5.2.5}$$

Notice that it is not anymore systematic because $\tilde{H} \neq H$. As already commented at Subsubsection 4.2.1.2, this classification between broken and unbroken generators is not unique. However, a redefinition of the generators should still satisfy the commutation relations requested.

5.2.3 The coset parametrisation and the transformation laws

With a similar approach to the internal case, we define the transformation laws and we construct the covariant building blocks in the perspective to write invariant EFTs under G for the NG candidates.

The coset parametrisation is given by

$$U(x, \pi(x)) = e^{ix^\mu P_\mu} e^{i\pi^a(x) X_a} \ . \tag{5.2.6}$$

A possible intuition on why there is an additional $e^{ix^\mu P_\mu}$ factor compared to the internal case is that the coset parametrisation is built with the objects transforming non-linearly under the symmetry group[3]. Now that the translations are explicitly listed in G, and because the coordinates x^μ transform as a shift (so, non-linearly) under the action of the translations, we intuitively understand that we have to introduce the $e^{ix^\mu P_\mu}$ factor in the coset parametrisation[4].

Let us notice that we could write the coset parametrisation in another way, in a product of exponentials with each of them containing one broken generator for example. This can be useful in practice to simplify some computations.

The Lie algebra elements will be considered as small[5] and so, each expansion intervening in the following will be carried out till the order two included. It means for example that the Baker–Campbell–Hausdorff formula implies

$$e^y e^x = e^{x+y+\frac{1}{2}[y,x]} = e^{x+[y,x]} e^y \ , \tag{5.2.7}$$
$$e^y\, x\, e^{-y} = x + [y, x] \ , \tag{5.2.8}$$

[3] Let us recall that $\tilde{H}$ in the coset construction $G/\tilde{H}$ is the subgroup which is linearly realised. The coset parametrisation parametrises $G/\tilde{H}$.

[4] In the language of cosets, x^μ parametrise the coset space (Poincaré)/(Lorentz) which is another way to argue that it should appear in the coset parametrisation [127].

[5] We consider group elements close to the identity which for NG candidates is consistent since they do represent perturbations around a VEV.

where the approximation $[x,[x,y]] \approx 0$ is considered.

We are now ready to compute the transformation laws. In particular, we will verify that our coset parametrisation indeed makes sense: x^μ are Minkowski spacetime coordinates and π^a are NG candidate fields.

Like for the pure internal case, we define the transformation rules under G as

$$gU(x,\pi) \equiv U(\tilde{x},\tilde{\pi})e^{iu^A(g,\pi(x))T_A} = e^{i\tilde{x}^\mu P_\mu}e^{i\tilde{\pi}^a(\tilde{x})X_a}e^{iu^A(g,x,\pi(x))T_A} \ , \tag{5.2.9}$$

where $g \in G$.

If we look to the transformations under the action of an element generated by P_μ, we get:

$$e^{ia^\nu P_\nu}e^{ix^\mu P_\mu}e^{i\pi^a(x)X_a} = e^{i(x^\mu+a^\mu)P_\mu}e^{i\pi^a(x)X_a} \ , \tag{5.2.10}$$

because $[x^\mu P_\mu, a^\nu P_\nu] = 0$. By comparison with (5.2.9), we obtain

$$\tilde{x}^\mu = x^\mu + a^\mu \ , \tag{5.2.11}$$
$$\tilde{\pi}^a(\tilde{x}) = \pi^a(x) \ , \tag{5.2.12}$$
$$u^A(g,x,\pi(x)) = 0 \ . \tag{5.2.13}$$

If the action of P_μ is seen as a translation, these transformation laws are consistent with our interpretation of x^μ being spacetime coordinates and π^a being fields defined on Minkowski spacetime. To confirm it, we look how these objects transform under the Lorentz group:

$$e^{i\omega^{\pi\kappa}L_{\pi\kappa}}e^{ix^\mu P_\mu}e^{i\pi^a(x)X_a} = e^{i(\Lambda x)^\mu P_\mu}e^{i\omega^{\pi\kappa}L_{\pi\kappa}}e^{i\pi^a(x)X_a} \tag{5.2.14}$$
$$= e^{i(\Lambda x)^\mu P_\mu}e^{i\left(\pi^a(x)-\omega^{\pi\kappa}\pi^b f_{(\pi\kappa)b}{}^a\right)X_a}e^{i\omega^{\pi\kappa}L_{\pi\kappa}} \ , \tag{5.2.15}$$

where, at the first line we used the product structure of the Poincaré group, namely,

$$(\Lambda,0)(I,x) = (\Lambda,\Lambda x) = (I,\Lambda x)(\Lambda,0) \ , \qquad \Lambda \equiv e^{i\omega^{\pi\kappa}L_{\pi\kappa}} \ . \tag{5.2.16}$$

We reached the second line (5.2.15) thanks to (5.2.7) and (5.2.3) ($L_{\mu\nu} \in \{T_A\}$). It gives us

$$\tilde{x}^\mu = \Lambda x^\mu \ , \tag{5.2.17}$$
$$\tilde{\pi}^a(\tilde{x}) = \pi^a(x) - \pi^b(x)\omega^{\pi\kappa}f_{(\pi\kappa)b}{}^a \ , \tag{5.2.18}$$
$$u^{\pi\kappa}(g,x,\pi(x)) = \omega^{\pi\kappa} \ , \tag{5.2.19}$$

where the brackets in the subindices of the structure constant is because one Lorentz generator is labelised in such a way, $L_{\pi\kappa}$. We then have that the x^μ transform as Minkowski coordinates under Lorentz and that the π^a fields have a priori non exotic transformation laws under Lorentz since the latter are based on the usual commutation relations we have in the spacetime algebras (such as the conformal algebra). In particular, if X^a are internal symmetry generators, the commutation relations with Lorentz generators are trivial and the π^a are spinless field, consistently with our results from Subsection 4.2.1.

Now that we have verified the consistency of our parametrisation from the point of view of QFT, we can compute the transformation laws under respectively the unbroken subgroup (we already have done it for P_μ, it remains under $\tilde{H}$) and the broken subset of G. In particular, we still have to check that the π^a transform linearly under $\tilde{H}$ and non-linearly under $G/\tilde{H}$.

Under the action of $e^{im^A T_A} \in \tilde{H}$, we have

$$\tilde{x}^\mu = x^\mu - x^\nu m^A f_{A\nu}{}^\mu \,, \tag{5.2.20}$$
$$\tilde{\pi}^a(\tilde{x}) = \pi^a(x) - \pi^b(x) m^A f_{Ab}{}^a \,, \tag{5.2.21}$$
$$u^A(g, x, \pi(x)) = m^A \,. \tag{5.2.22}$$

If T_A are internal generators ($f_{A\nu}{}^{\cdots} = 0$, where the ellipses indicate any indices), we consistently have $\tilde{x}^\mu = x^\mu$ since such symmetries are expected to not act on spacetime coordinates. In any case, we have that the π^a fields transform linearly under $\tilde{H}$.

Under the action of $e^{in^b X_b} \in G/H$, we have

$$\tilde{x}^\mu = x^\mu - x^\nu n^a f_{a\nu}{}^\mu - \frac{1}{2} n^b \pi^a(x) f_{ba}{}^\mu \,, \tag{5.2.23}$$
$$\tilde{\pi}^a(\tilde{x}) = \pi^a(x) + n^a - \frac{1}{2}\pi^c(x) n^b f_{bc}{}^a - n^b x^\mu f_{b\mu}{}^a \,, \tag{5.2.24}$$
$$u^A(g, x, \pi(x)) = -n^a x^\mu f_{a\mu}{}^A - \frac{1}{2} n^b \pi^a(x) f_{ba}{}^A \,. \tag{5.2.25}$$

We observe that the π^a fields transform non-linearly under G/H as they should in order to be NG candidates. If we particularise to the action of internal symmetries ($f_{b\mu}{}^{\cdots} = 0 = f_{ba}{}^\mu$) and restrict ourselves to the first order, we recover (4.2.22).

All these results have been obtained by using the definition (5.2.9), by considering the approximated version of the Baker–Campbell–Hausdorff formula (5.2.7) and by applying the commutation relations of Subsection 5.2.2 in order to shift the g element from the left to the right and stop when we reached the correct shape of the right-hand side of (5.2.9).

5.2.4 The Maurer-Cartan 1-form

The transformation laws of our fields are non-linear and involved. Therefore, we need to construct objects which transform covariantly, the latter will be the building blocks of the EFTs. As we have already seen in the pure internal case, this is achieved by considering the Mauer-Cartan 1-form

$$\begin{aligned} &dx^\mu \, U(x,\pi)^{-1} \partial_\mu U(x,\pi) = \\ &dx^\mu \left(-i\, \mathcal{A}_\mu{}^A(x,\pi)\, T_A + i\, r_\mu{}^\alpha(x,\pi)\, e_\alpha{}^a(x,\pi)\, X_a + i\, r_\mu{}^\alpha(x,\pi)\, P_\alpha \right) \,. \end{aligned} \tag{5.2.26}$$

This equality holds because the Mauer-Cartan 1-form takes its values in the Lie algebra of G and we have an additional term compared to the internal case due to the additional P_μ in the coset parametrisation (5.2.6). Moreover, we expressed the coefficients of the X_a generator as a product of r and e, it is a convenient choice for later. It assumes that r is invertible (e is r^{-1} times the coefficient of X_a) and we will keep this hypothesis in the rest of the development.

By computing $d\tilde{x}^\mu \, U(\tilde{x},\tilde{\pi})^{-1} \tilde{\partial}_\mu U(\tilde{x},\tilde{\pi})$, we obtain the transformation properties of e,

r and $\mathcal{A}$. With $\gamma = e^{iu^A(g,x,\pi)T_A}$,

$$d\tilde{x}^\mu\, U(\tilde{x},\tilde{\pi})^{-1}\tilde{\partial}_\mu U(\tilde{x},\tilde{\pi}) = \frac{\partial\tilde{x}^\mu}{\partial x^\nu}dx^\nu\gamma\, U(x,\pi)^{-1}\frac{\partial x^\kappa}{\partial\tilde{x}^\mu}\partial_\kappa\left(U(x,\pi)\gamma^{-1}\right) \tag{5.2.27}$$
$$= dx^\mu\left(\gamma U(x,\pi)^{-1}\partial_\mu U(x,\pi)\gamma^{-1} + \gamma\partial_\mu\gamma^{-1}\right) \tag{5.2.28}$$
$$\begin{aligned} &= dx^\mu\left(-i\,{\mathcal{A}_\mu}^A(x,\pi)\,\gamma T_A\gamma^{-1}\right.\\ &\qquad +i\,{r_\mu}^\alpha(x,\pi)\,{e_\alpha}^a(x,\pi)\,\gamma X_a\gamma^{-1}\\ &\qquad \left.+i\,{r_\mu}^\alpha(x,\pi)\,\gamma P_\alpha\gamma^{-1} + \gamma\partial_\mu\gamma^{-1}\right)\ . \end{aligned} \tag{5.2.29}$$

Thanks to the commutation relations of Subsection 5.2.2, we can write

$$\gamma P_\alpha\gamma^{-1} = {h_\alpha}^\beta(g,x,\pi)P_\beta\ , \tag{5.2.30}$$
$$\gamma X_a\gamma^{-1} = {h_a}^b(g,x,\pi)X_b\ , \tag{5.2.31}$$

where $h(g,x,\pi)$ is a representation of $\tilde{H}$, with the different indices (Greek letters, lowercase Latin letters and capital Latin letters) corresponding to different representations of the same element of $\tilde{H}$. This is coming from the fact that $\gamma \in \tilde{H}$. Thus, we see that the group G is non-linearly realised through the covariant representation of $\tilde{H}$, namely $h(g,x,\pi)$, where the non-linearity is hidden in the non-trivial dependency of h on g, x, π. By injecting these results into (5.2.29) and by remembering that $d\tilde{x}^\mu = \frac{\partial\tilde{x}^\mu}{\partial x^\nu}dx^\nu$ with $\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\frac{\partial x^\beta}{\partial\tilde{x}^\mu} = \delta^\beta_\nu$, we can extract the transformation laws of the Mauer-Cartan 1-form coefficients

$${r_\mu}^\alpha(\tilde{x},\tilde{\pi}) = \frac{\partial x^\nu}{\partial\tilde{x}^\mu}{r_\nu}^\beta(x,\pi){h_\beta}^\alpha(g,x,\pi)\ , \tag{5.2.32}$$
$${e_\alpha}^a(\tilde{x},\tilde{\pi}) = {\left(\left(h^{-1}(g,x,\pi)\right)\right)_\alpha}^\nu\,{e_\nu}^b(x,\pi){h_b}^a(g,x,\pi)\ , \tag{5.2.33}$$
$$d\tilde{x}^\mu{\mathcal{A}_\mu}^A(\tilde{x},\tilde{\pi})T_A = dx^\mu\left({\mathcal{A}_\mu}^A(x,\pi)\gamma T_A\gamma^{-1} + i\gamma\partial_\mu\gamma^{-1}\right)\ , \tag{5.2.34}$$

where ${\mathcal{A}_\mu}^A T_A$ is dealt with in a different fashion compared to e and r, this because we will use it to define a covariant derivative and so, we will need T_A to be in a specific representation of $\tilde{H}$. By analogy to gauge theories, and by the similarity of the laws of transformation between ${\mathcal{A}_\mu}^A$ and a gauge field, we call ${\mathcal{A}_\mu}^A$ the coset connection.

By looking at (5.2.33), we notice that we have an object which depends on the π^a fields and which transforms covariantly under the action G despite that the π^a fields transform non-linearly. This object is the building block we will use in order to write invariant Lagrangians under G. However, the spacetime coordinates are as well affected by the action of G since it includes spacetime symmetries. Therefore, the measure d^dx of integration in the action of the EFT is not invariant. To have an invariant theory from an invariant Lagrangian, we need an invariant measure. This is achieved thanks to (5.2.32), where r has similar transformation properties than the vielbein in general relativity under local Lorentz transformations [19,132]. We then call ${r_\mu}^\alpha$ the coset vielbein. The measure $d^dx\det(r)$ is invariant under G. Indeed, by using matrix products, we have

$$d^d\tilde{x}\det(r(\tilde{x},\tilde{\pi})) = d^dx\det\left(\frac{\partial\tilde{x}}{\partial x}\right)\det\left(\frac{\partial x}{\partial\tilde{x}}h(g,x,\pi)\,r(x,\pi)\right) \tag{5.2.35}$$
$$= d^dx\det\left(\frac{\partial\tilde{x}}{\partial x}\right)\det\left(\frac{\partial\tilde{x}}{\partial x}\right)^{-1}\det\left(h(g,x,\pi)\right)\det\left(r(x,\pi)\right)\ , \tag{5.2.36}$$

where $\det\left(h(g,x,\pi)\right) = 1$ since ${h_\beta}^\alpha$ is defined by

$$dx\, r\, \gamma\, P\, \gamma^{-1} = dx\, r\, h\, P \tag{5.2.37}$$
$$\Rightarrow \det\left(dx\, r\, P\right) = \det\left(dx\, r\, P\right)\det\left(h\right) \tag{5.2.38}$$
$$\Leftrightarrow \det\left(h\right) = 1\ ,\ \text{assuming}\ \left(dx\, r\, P\right) \neq 0\ . \tag{5.2.39}$$

Hence, indeed

$$d^d\tilde{x}\,\det(r(\tilde{x},\tilde{\pi})) = d^dx\,\det\left(r(x,\pi)\right)\ . \tag{5.2.40}$$

At present, we know how to construct an invariant EFT but, in order to allow us for more generality, we should define a covariant differential operator acting on $e(x,\pi)$. In fact, the usual ∂_μ does not give a covariant transformation law both because the covariant matrix h has an explicit and implicit dependence in x (we already had the implicit dependence in the pure internal case) and because x^μ is itself transforming under G.

Let us define r^{-1} such that

$$r(x,\pi)_\lambda{}^\kappa \left(r^{-1}(x,\pi)\right)_\kappa{}^\beta = \delta_\lambda{}^\beta\ . \tag{5.2.41}$$

Its transformation rule is obtained through

$$r(\tilde{x},\tilde{\pi})_\lambda{}^\kappa \left(r^{-1}(\tilde{x},\tilde{\pi})\right)_\kappa{}^\beta = \delta_\lambda{}^\beta \tag{5.2.42}$$
$$\Leftrightarrow \frac{\partial x^\nu}{\partial \tilde{x}^\lambda} r_\nu{}^\beta(x,\pi) h_\beta{}^\kappa(g,x,\pi) \left(r^{-1}(\tilde{x},\tilde{\pi})\right)_\kappa{}^\beta = \delta_\lambda{}^\beta\ . \tag{5.2.43}$$

So,

$$\left(r^{-1}(\tilde{x},\tilde{\pi})\right)_\kappa{}^\beta = \left((h^{-1}(g,x,\pi)\right)_\kappa{}^\lambda \left(r^{-1}(x,\pi)\right)_\lambda{}^\nu \frac{\partial \tilde{x}^\beta}{\partial x^\nu}\ . \tag{5.2.44}$$

From all this, we define a covariant derivative which acts on a field in a given covariant realisation of G through a linear representation of $\tilde{H}$ (like e),

$$D_\alpha \equiv \left(r^{-1}(x,\pi)\right)_\alpha{}^\mu \left(\partial_\mu - i\mathcal{A}_\mu{}^A(x,\pi) T_A\right)\ , \tag{5.2.45}$$

where T_A is the realisation of the generator T_A in the representation of $\tilde{H}$ in which the field acted on is. Of course, ∂_μ should be seen as a diagonal matrix (we are looking at finite dimensional linear realisation of $\tilde{H}$) where each non-zero entry is ∂_μ.

To see the covariance of D_α, let us consider a matter field ψ^i where i is the index of a finite dimensional linear realisation of $\tilde{H}$, such that ψ^i transforms under G as

$$\tilde{\psi}^i(\tilde{x}) = h^i{}_j(g,x,\pi)\psi^j(x)\ . \tag{5.2.46}$$

We have

$$\tilde{D}_\alpha\tilde{\psi}^l(\tilde{x}) = \left(r^{-1}(\tilde{x},\tilde{\pi})\right)_\alpha{}^\mu \left(\delta^l{}_j\,\tilde{\partial}_\mu - i\mathcal{A}_\mu{}^A(\tilde{x},\tilde{\pi})\,(T_A)^l{}_j\right)\tilde{\psi}^j(\tilde{x}) \tag{5.2.47}$$
$$= \left((h^{-1}(g,x,\pi)\right)_\alpha{}^\nu h^l{}_m(g,x,\pi) \tag{5.2.48}$$
$$\left(r^{-1}(x,\pi)\right)_\nu{}^\beta \left(\delta^m{}_i\,\partial_\beta - i\mathcal{A}_\beta{}^A(x,\pi)\,(T_A)^m{}_i\right)\psi^i(x)$$
$$= \left((h^{-1}(g,x,\pi)\right)_\alpha{}^\nu h^l{}_m(g,x,\pi) D_\nu\psi^m(x)\ , \tag{5.2.49}$$

where we developed thanks to (5.2.32), (5.2.34) and (5.2.46). The computation is straightforward, the only trick was at a moment to think about $h\partial h^{-1} = \partial(hh^{-1}) - \partial h h^{-1} = -\partial h h^{-1}$. The reasoning is the same when applied on ${e_\nu}^a$, but a bit more technical. We have thus shown that D_α is indeed a covariant derivative. Moreover, we have as well see how we can enrich the theory with additional matter fields with transformation laws of the form (5.2.46).

At the moment, we have covariant objects: ${e_\nu}^a$, D_α and possibly additional fields ψ^i. To have invariant terms in our effective Lagrangian, we need to be able to contract the indices. Hence, we have to define how to lower and to raise indices. Since we are working in Minkowski spacetime, we have at disposal the Minkowski metric $\eta_{\mu\nu}$. A possible way to define lowering and raising indices is through the matrices $g^{\alpha\beta}$ and g^{ab} defined as

$$g^{\alpha\beta} \equiv \eta^{\mu\nu} {r_\mu}^\alpha(x,\pi) {r_\nu}^\beta(x,\pi) \ , \tag{5.2.50}$$

$$g^{ab} \equiv \eta^{\mu\nu} {e_\mu}^a(x,\pi) {e_\nu}^b(x,\pi) \ , \tag{5.2.51}$$

where $g_{\alpha\beta}$ and g_{ab} are respectively numerically defined as $g_{\alpha\beta} = (g^{-1})^{\alpha\beta}$ and as $g_{ab} = (g^{-1})^{ab}$. As a consistency check, it can be computed that

$$D^\alpha \equiv g^{\alpha\beta} D_\beta \ , \tag{5.2.52}$$

transforms as

$$\tilde{D}^\alpha = D^\rho {h_\rho}^\alpha \ , \tag{5.2.53}$$

where it should be recalled that $\eta^{\mu\nu}$ transforms as a tensor under spacetime symmetries (i.e. diffeomorphisms if we focus solely on the action on the spacetime coordinates). Therefore,

$$\tilde{D}^\alpha \tilde{D}_\alpha = D^\rho {h_\rho}^\alpha \left(h^{-1}\right)_\alpha^{\ \nu} D_\nu = D^\rho D_\rho \ . \tag{5.2.54}$$

Let us mention that to be rigorous we should have made the computation by applying D_α on a field, for simplicity, if we take this field to be non-transforming (a scalar under G), we recover the computation we just did.

5.2.5 Effective field theories for a given symmetry breaking pattern

We are now able to write effective field theories for any given continuous global symmetry breaking pattern $G \to H$ respecting the assumptions, i.e. G and H should contain the Poincaré group and the algebra commutation relations of Subsection 5.2.2 should be verified. Such EFTs will have the form

$$S = \int d^d x \det\left(r(x,\pi)\right) \mathcal{L}\left({e_\alpha}^a(x,\pi), D_\alpha\left({e_\mu}^a(x,\pi)\right), \ldots\right) \ , \tag{5.2.55}$$

where $\mathcal{L}$ is required to be invariant under linear realisations of $\tilde{H}$, which as we have seen is enough to be invariant under G. The ellipses in $\mathcal{L}$ denote higher covariant derivatives.

Finding the most general effective field theory for a given symmetry breaking pattern is highly involved. First, we have to prove that the most general EFT we can build with $e, De, \ldots$ is equivalent to the most general EFT we can build directly with x and

π. This is proven for the pure internal case [18] but as far as the author knows, not for the generic case. Second, once we have the most general invariant Lagrangian, we still need to write all the possible (inequivalent) Wess-Zumino-Witten terms which again, as far as the author knows, there is no generic classification of such terms for spacetime symmetries/non-uniform symmetries. This could be seen as one of the technical reasons why a counting rule for the NG modes associated to a generic spontaneous symmetry breaking pattern $G \to H$ is still an open question.

It can be mentioned that the symmetries can be gauged, indeed, we have at disposal a coset connection $\mathcal{A}_\mu{}^A(x,\pi)$. In this dissertation, gauge symmetries are not our primary interest, we are therefore not commenting further this aspect.

The requirement to preserve Poincaré group unbroken (or even to ask that it is a symmetry in the first place) can be restrictive in the study of spacetime symmetry breaking since most of the physical systems showing a non-trivial condensate define a preferential frame (the rest frame) breaking then the boosts. On the conceptual level, asking Poincaré to be included in G can be relevant because high energy physics should be relativistic. Therefore, any fundamental non-relativistic theories can itself be seen as an EFT of a UV relativistic theory [32]. In practice, it is of course highly restrictive, especially for condensed matter applications. However, we can see the coset construction we just derived as a "recipe". In the sense that even if we do not satisfy all the assumptions of Subsection 5.2.2, we can apply, and modify according to our case, the procedure and carefully check step by step the process[6]. There are many examples in the literature where the EFTs derived from the coset construction despite being non-relativistic, or breaking translations and/or boosts, provide successful results, e.g. [32, 102, 128, 129, 134–136]. And of course, the coset construction we derived for non-relativistic theories in the pure internal case is one of these successful examples.

From the discussion we had in Section 5.1, and in particular form the example of Figure 5.1, it appeared that, in the case of spacetime symmetry breaking, the reduction of NG candidates can happen even prior any discussions on the dynamics. A reduction of NG candidates can be seen solely based on their definitions and their transformation properties. However, in our presentation of the coset construction, we did not see such events. We arrived at the point to write invariant EFTs which means that beyond that step, the reduction of NG candidates will be dynamical. The reason is that we skipped a step, we did not compute explicitly the functions $e_\alpha{}^a(x,\pi)$, $r_\mu{}^\alpha(x,\pi)$ and $\mathcal{A}_\mu{}^A(x,\pi)$. We are going to do it in the following subsection. As we will see, it will lead to a rich discussion on the potential removal of some NG candidates, where the key word is "inverse Higgs constraints".

[6]An example how the recipe can be extended is when some of the translation symmetries are spontaneously broken by extended objects such as membranes. In this case, the broken P_μ are included in the set $\{X_a\}$ and the unbroken translation generators, labelised as $\bar{P}_\mu$, are dealt with in the regular way in the coset parametrisation:

$$U(x,\pi(x)) = e^{ix^\mu \bar{P}_\mu} e^{i\pi^a(x) X_a} \,. \tag{5.2.56}$$

As we can notice, the NG candidate fields π^a are now function of only the unbroken coordinates. This can be understood from Figure 5.1, where the infinite rope breaking the x-direction tolerates a spatial modulation only following the unbroken direction, i.e. the fluctuations are spatially dependent of only the y-direction (considering we are in $2+1$-dimensional spacetime). More examples of the breaking of translations by extended objects can be found in [127, 133] and in the references therein.

5.2.6 Inverse Higgs constraints

Let us compute the functions ${e_\alpha}^a(x,\pi)$, ${r_\mu}^\alpha(x,\pi)$ and ${\mathcal{A}_\mu}^A(x,\pi)$ from their definitions (5.2.26). To do so, we use the commutation relations of Subsection 5.2.2, the Baker-Campbell-Hausdorff formula till second order and the relation [19]

$$e^{-ia^\alpha G_\alpha}\partial_\mu e^{ia^\alpha G_\alpha} = i\partial_\mu a^\alpha \left(G_\alpha + \frac{i}{2}a^\beta \left[G_\alpha, G_\beta\right]\right) + \mathcal{O}\left(a^3\right) \ . \tag{5.2.57}$$

At second order, we obtain

$${r_\mu}^\alpha(x,\pi) \approx {\delta_\mu}^\alpha - \pi^a {f_{\mu a}}^\alpha - \frac{1}{2}\partial_\mu \pi^a \pi^b {f_{ab}}^\alpha \ , \tag{5.2.58}$$

$${e_\alpha}^a(x,\pi) \approx \partial_\alpha \pi^a - \pi^b {f_{\alpha b}}^a - \frac{1}{2}\partial_\alpha \pi^b \pi^c {f_{bc}}^a + \pi^b {f_{\alpha b}}^\lambda \partial_\lambda \pi^a - \pi^c \pi^b {f_{\alpha c}}^\lambda {f_{\lambda b}}^a \ , \tag{5.2.59}$$

$${\mathcal{A}_\mu}^A(x,\pi) \approx \pi^a {f_{\mu a}}^A + \frac{1}{2}\partial_\mu \pi^a \pi^b {f_{ab}}^A \ . \tag{5.2.60}$$

We can notice that terms with no derivatives are present. Hence, contrary to the pure internal case – that incidentally can be recovered by considering ${f_{\mu a}}^{\cdots} = 0$, the effective Lagrangians may contain no derivative terms and so, some NG candidates can be massive (the discussion on NG candidates being systematically weakly coupled at low energy can as well be affected). In the literature, there are several examples of such a reduction of the number of massless NG candidates (e.g. [96, 126]). We are evading the intuitive picture we presented at Subsection 3.1, this intuitive reasoning was considering internal symmetries for simplicity, where now it is confirmed that the spacetime case/non-uniform case is much more elaborate.

To study the possibility to have massive NG candidates, we can restrict ourselves to the quadratic part of the effective theories. Since these theories are built on the contraction of indices of the covariant objects e, r, D, their quadratic part is of the form

$$S_{\text{quad}} \sim \int d^d x \det\left(r\right) \left({e_\alpha}^a {e^\alpha}_a + + D_\alpha \left({e_\mu}^a\right) D^\alpha \left({e^\mu}_a\right)\right) \ . \tag{5.2.61}$$

To be indeed quadratic, it means that e should be of order one and r and D should be of order zero. Hence, at quadratic order for the EFTs, we have:

$${r_\mu}^\alpha(x,\pi) \approx {\delta_\mu}^\nu \ , \tag{5.2.62}$$

$${e_\alpha}^a(x,\pi) \approx \partial_\alpha \pi^a - \pi^b {f_{\alpha b}}^a \ , \tag{5.2.63}$$

$${\mathcal{A}_\mu}^A(x,\pi) \approx 0 \ . \tag{5.2.64}$$

In the goal to have EFTs describing only massless NG candidates and so, to be able to establish a counting rule for the NG modes, we could eliminate the potentially massive NG candidates π^b in (5.2.63) by imposing by hand $e(x,\pi) = 0$. This constraint is consistent with the symmetries because e transforms covariantly and thus, the constraint is symmetric invariant. However, by doing so, we are trivialising the theory (5.2.61)! This is of course a cavalier manner to get rid of massive NG candidates. Ivanov and Ogievetsky [131] refined this idea and emphasised under which conditions it is possible to impose similar constraints without trivialising the EFTs.

The additional hypothesis required for the following discussion is that the broken generators X_a are in a complete reducible representation of $\tilde{H}$, the same for the translations P_μ. We will denote with brackets the different irreducible multiplets $X_a^{(i)}$ and $P_\mu^{(l)}$ (where the indices a and μ now labelise the different generators inside respectively the multiplet (i) and the multiplet (l))

$$[X_a^{(i)}, T_A] = i f_{aA}{}^b X_b^{(i)} \ , \qquad [P_\mu^{(l)}, T_A] = i f_{\mu A}{}^\nu P_\nu^{(l)} \ . \tag{5.2.65}$$

From this hypothesis, we are ensured that the $h(g, x, \pi)$ element of $\tilde{H}$ representing the action of $g \in G$ defined at (5.2.30) and (5.2.31) does not mix the different multiplets. Therefore, from (5.2.33), $e_\alpha{}^a(x, \pi)$ is itself a complete reducible representation of $\tilde{H}$ and by extension of G (since it transforms under G through a covariant realisation of $\tilde{H}$). The different multiplets are denoted by $e_\alpha{}^a(x, \pi)|_{(l)}{}^{(i)}$. This notation $|_{(l)}{}^{(i)}$, when necessary, we will use it to indicate the multiplet nature of the indices.

From the point of view of the transformation laws, each multiplet $e_\alpha{}^a(x, \pi)|_{(l)}{}^{(i)}$ are independent from each other. This means that, we can use only few of them to build an invariant effective Lagrangian. In particular, for a given multiplet, setting $e_\alpha{}^a(x, \pi)|_{(l)}{}^{(i)} = 0$ will not trivialise the theory and it will not spoil the symmetries.

If

$$[P_\mu^{(l)}, X_b^{(j)}] \supset i f_{\mu b}{}^a X_a^{(i)} \ , \ f_{\mu b}{}^a|_{(l)(j)}{}^{(i)} \neq 0 \ , \tag{5.2.66}$$

it means that, from (5.2.63),

$$e_\alpha{}^a(x, \pi)|_{(l)}{}^{(i)} = \partial_\alpha \pi^a|_{(l)}{}^{(i)} - \pi^b|^{(j)} f_{\alpha b}{}^a|_{(l)(j)}{}^{(i)} + \dots \ . \tag{5.2.67}$$

We understand that if we impose (5.2.67) to be zero, the obtained equation will be solvable and we can express the $\pi^b|^{(j)}$ NG candidates in terms of $\partial_\alpha \pi^a|_{(l)}{}^{(i)}$. As a consequence, we are able to eliminate some potentially massive NG candidates without trivialising the effective theory we are establishing and without spoiling the invariance of the latter.

The constraint

$$e_\alpha{}^a(x, \pi)|_{(l)}{}^{(i)} = 0 \ , \tag{5.2.68}$$

under the hypothesis (5.2.65) and the condition (5.2.66), bears the name of inverse Higgs constraint (IHC) [131]. The name comes from the case when we gauge the symmetries. In such a situation there are additional gauge fields and sometimes, we can impose a constraint like (5.2.68) to eliminate some of these gauge fields in terms of the NG candidates. Namely, it is sort of the Brout-Englert-Higgs mechanism in reverse.

As always, a good consistency check is that if we retrieve the pure internal case from this more general picture. In fact, by considering internal symmetries, the commutator $[P_\mu, X_a]$ is always zero, hence the condition (5.2.66) is never satisfied. No inverse IHC can be imposed and we recover the idea that there is no apparent reduction of the number of massless NG candidates without a careful study of their dynamics.

Speaking of the condition (5.2.66), when some translation symmetries are broken, it is only the unbroken translation generators which have to be considered in (5.2.66) – consistently with Footnote 6.

After imposing the inverse Higgs constraint, the coset construction is equivalent to the one we would have done with G', the reduced symmetry group where we subtracted the broken generators we got rid of. However, a trace of G would remain in the relative numerical values of the coefficients of the Lagrangian.

5.2.7 Does imposing inverse Higgs constraints lead to a loss of generality ?

Having the possibility to impose inverse Higgs constraints indicate that we do not need all the NG candidates in order to build an invariant effective Lagrangian under the action of G. If we want to build the most general effective Lagrangian for massless NG candidates, for example to establish a generic counting rule for the NG modes based on a non-specific breaking pattern $G \to H$, we have to ensure that imposing by hand an IHC does not lead to a loss of generality. The argument is that the fact to be able to impose an IHC stipulates that the NG candidates we could suppress are potentially massive. Hence, even if we do not impose the IHC, we can always go at a low enough energy and integrate out these massive modes. Several examples support this argumentation, see for e.g. [133] and the references therein. The issue is that it is a reasoning and not a formal proof. Indeed, (5.2.63) seems to lead to the idea that the π^b will be massive. But nothing prevents the effective Lagrangian to have an unusual form with a non-trivial interplay among its terms such that π^b will end-up massless or with no derivative at all, even at higher orders, and so to be an auxiliary field with no dynamics or even that π^b can be entirely absorbed by a field redefinition. The conservative approach would then be to impose no IHCs at all, however, it significantly increases the technical difficulty since we would be dealing with all the NG candidates[7]. We thus need to be smart and try to see which IHCs we are sure we can impose without losing generality. This statement holds also if we want to keep the massive modes, which IHC can be used to eliminate spurious fields without getting rid of physical massive modes ? Here are some suggestions:

1. If we have access to the fundamental theory or at least to the VEV and the realisation of G on the fundamental fields, we can solve the equation

$$i\pi^a(x)X_a\phi(x) = 0 \;, \tag{5.2.69}$$

for π^a, where $\phi(x)$ is the VEV (with a dependence on x^μ since we break spontaneously spacetime symmetries) and the X_a are the realisation of the broken generators on the fundamental fields. If (5.2.69) has non-trivial solutions, it means that some of the NG candidates generates the same fluctuations around the VEV than a combination of other NG candidates. It is then straightforward that we can eliminate the dependent NG candidates[8]. This argument has been provided for

[7] And even with such a conservative approach, we would not be sure to encompass all the possible physical cases. Indeed, we built one possible non-linear realisation of G but, there could be other non-equivalent non-linear realisations which could parametrise the NG candidates [13]. From a more mathematical point of view, that is what the coset construction is about: classifying the non-linear realisations. In particular, the ambiguity on the IHCs could be a signature of this mathematical comment. Let us notice that we did not have such a problem on the different inequivalent non-linear realisations of G in the pure internal case. This because we started with the representation of G on our fundamental field theory, hence, its realisation on the fluctuations, specifically on the NG candidates, was fixed/given.

[8] This is the formalisation of the idea illustrated in Figure 5.1. Especially, for the example of the breaking of the x-translation and the θ-rotation, in $2+1$-dimensional spacetime (x, y, t), we have that the VEV has the dependency $\phi(x)$. Then,

$$\left(i\pi P_x + i\theta J_{xy}\right)\phi(x) = 0 \Leftrightarrow \pi = y\,\theta \;, \tag{5.2.70}$$

where $P_x = i\partial_x$ and $J_{xy} = i(x\partial_y - y\partial_x)$. We can in fact eliminate one of the two fields as an NG candidate.

classical relativistic cases in [127] and has been generalised at quantum level and for non-relativistic theories in [128, 137]. It can be connected to the IHCs in the following way

$$(5.2.69) \Rightarrow \bar{P}_\mu i\pi^a(x)X_a\phi(x) = 0 \Leftrightarrow \left(-\partial_\mu\pi^a - \pi^c f_{\mu c}{}^a\right) X_a\phi(x) = 0 \ , \qquad (5.2.71)$$

where $\bar{P}_\mu$ is the realisation of an unbroken translation generator ($\bar{P}_\mu\phi(x) = 0$). We notice that the factor inside the brackets on the right-hand side of (5.2.71) is an IHC (5.2.67) under the condition $[\bar{P}_\mu, X_c] \supset X_a$. The converse of (5.2.71) is not ensured to be true but, it means that an IHC might be a signature of a non-trivial solution of (5.2.69). Hence, if we show it to be true, we can safely impose the IHC to zero[9].

2. In [133], they provide several examples where the potentially massive NG candidates are in fact auxiliary fields which lead to algebraic equations of motion and can thus be eliminated from the dynamics. In such case, the IHCs are equivalent to the EOM. We could then try to see if for the particular physical example at study, the effective Lagrangian would lead to only massive terms with no derivatives at all for the potentially massive fields.

3. We have previously commented that the coset parametrisation is not unique (see Subsection 5.2.3). It appears that the IHCs are sensible to the choice of parametrisation [133]. Following how we describe the coset parametrisation, the equations (5.2.63) might be easier to solve. Sometimes, for a specific parametrisation of (5.2.6) it is possible that both π^a and π^b have derivatives in (5.2.67), which makes the IHC non algebraically solvable. This can be a clue on the relevance or not of the IHC.

4. If we have access to the fundamental fields, we know how many scalar fields are necessary to describe them. This provides an upper bound on the number of independent fluctuations and so, on the number of independent NG candidates. It is another indication on whether or not we should impose IHCs.

5. It is always easier when we know the answer in advance. If, by another method, we already have the effective field theory we are trying to build from the coset construction, it can indicate how to deal with the IHCs. From it, we can obtain intuition on general rules under which an IHC is physically relevant or not.

[9]In [129], they showed that if (5.2.69) has a non-trivial solution allowing to express one class of NG candidates − the class 1 − in terms of another class of NG candidates − the class 2, the Noether currents associated to the respective symmetries of the NG candidates, J^μ_{1a} and J^μ_{2a} where a are the parameters of the symmetries, are as well linked together. We have that it exists two functions f^b_a and N^μ_a satisfying

$$J^\mu_{2a}\partial_\mu f^b_a = \partial_\mu N^\mu_a \ , \qquad (5.2.72)$$

such that

$$J^\mu_{1a} = f^b_a J^\mu_{2a} - N^\mu_a \ . \qquad (5.2.73)$$

In particular it implies $\partial_\mu J^\mu_{1a} = 0 \Leftrightarrow \partial_\mu J^\mu_{2a} = 0$. The relation (5.2.73) can be useful in QFT. The spectrum of the quantum theory is so that the pole structure of the correlators permits to saturate the Ward-Takahashi identities. Hence, to satisfy the Ward-Takahashi identities associated to J^μ_{1a} and J^μ_{2a}, we would need only one NG mode instead of two.

A conceptual question which is remaining is what is the interpretation of the modes that we eliminate through IHCs ? When the IHCs eliminates non-physical modes, it is suggested that these modes are gauge redundancies of the theory [13, 133]. The simplest situation to see it, is to consider the case of (5.2.69). Let us consider that we have two NG candidates π^1 and π^2. These two modes parametrise a fluctuation around the fundamental VEV $\phi(x) + \delta\phi(\pi^1, \pi^2)$. Injecting this parametrisation in the fundamental theory, we can extract the perturbative action

$$S[\phi + \delta\phi] \equiv S_{\text{pert}}[\pi^1, \pi^2] \ . \tag{5.2.74}$$

If we have a non-trivial solution for (5.2.69) of the form

$$\pi^2 = f(x)\pi^1 \ , \tag{5.2.75}$$

then $S_{\text{pert}}[\pi^1, \pi^2]$ is invariant under the gauge transformation

$$\pi^1(x) \to \pi^1(x) + \epsilon(x) \ , \qquad \pi^2(x) \to \pi^2(x) + \epsilon(x)f(x)\pi^1(x) \ , \tag{5.2.76}$$

because $\delta\phi[\pi^1, \pi^2]$ is invariant thanks to (5.2.75) being a solution of (5.2.69). We therefore can fix the gauge $\epsilon(x)$ to eliminate $\pi^1(x)$. Discarding π^1 through an IHC is then a way to get rid of a gauge redundancy.

When the IHCs eliminate physical modes, it appears that these are always massive modes. Hence, it looks that imposing an IHC never eliminates a massless NG candidate that we should have kept. It discards either a spurious field either a massive NG candidate. This idea has been argued and tested on several examples in [129].

5.2.8 Commonly accepted conjecture on the inverse Higgs constraints

In [129], Brauner and Watanabe proposed a fairly complete picture which is nowadays commonly accepted. By requiring solely the non-breaking of translation symmetries (thus, non-relativistic systems are included in the following statement), it is argued that imposing an inverse Higgs constraint leads to the elimination either of a spurious field (gauge redundant field, auxiliary field) or of a physical massive field. Therefore, in the optics to study only massless NG candidates, we can impose all the inverse Higgs constraints. The most general effective theory built by imposing all the IHCs might differ from the most general effective theory obtained by imposing none of the IHCs (or only some of them) and by integrating out the massive fields. However, physically speaking, these two theories are equivalent since we are going to use either experiments or the fundamental theory to tune the coefficients of these two effective fields theories. Thus, they will provide the same physical predictions. The only ambiguity remaining from the IHCs is when we want to study the massive NG candidates as well. In this case, imposing or not an IHC might spoil the analysis since, solely based on group theory, we do not have (yet) concrete arguments to know in advance if this IHC should/can indeed be imposed. The only way out is through experimental measurement or via an access to the fundamental theory.

By using this conjecture, we can have a good intuition on how the counting rules for NG modes will operate for a generic symmetry breaking pattern $G \to H$, where the

Poincaré group is both in G and in H. In this case, each of the broken generator provides an NG candidate. We reduce this number with the number of independent IHCs and this should provide the number of NG modes since we have seen that a relativistic EFT does not lead to canonical conjugation among NG candidates. For non-relativistic theories, it is of course much more involved because we still have to build the most general EFT (with all the possible Wess-Zumino-Witten terms), somehow tune this EFT with the fundamental theory (we have already seen in the pure internal case that it was not possible to have a counting rule for non-relativistic theories solely based on group theory) and study the canonical conjugation structure of this EFT.

5.2.9 Closing words on the coset construction for spacetime symmetries

In this section, we have presented the coset construction for symmetry breaking patterns involving spacetime symmetries and non-uniform symmetries in general. The development has been established based on constraining hypotheses. The commutation relations of Subsection 5.2.2 appear to be primordial for the most important result which is that we successfully build objects which transform non-linearly under G but have covariant representation of $\tilde{H}$ – the key formulas are (5.2.30) and (5.2.31). It allows us to circumvent the difficulty to write invariant theory under non-linear realisations of G. However, concerning the assumption to consider Poincaré symmetry to be present and unbroken, it seems reasonable that we can relax this hypothesis. In fact, we mainly used it to show that the NG candidates we build at Subsection 5.2.1 are well defined fields in the language of QFT. This is a pure formal aspect which in practice, could be evaded by a proper discussion on the nature of the fields and how they connect with the spacetime they are defined on. Because we are mainly dealing with continuous scalar fields (the NG candidates), keeping only the continuous translation spacetime symmetries at the fundamental level (i.e. in G but not necessarily in H) seems a fair relaxation of the hypotheses. Several coset construction computations have provided good results in non-relativistic frameworks [128, 129, 134, 135].

The coset construction displays the interesting feature that it might be possible to get rid of spurious fields and of massive fields without doing any explicit dynamical computations. These possibilities are encoded in the inverse Higgs constraints. The conjecture of Brauner and Watanabe on when to impose these IHCs takes place in the relaxed hypothesis framework we commented just above. But it additionally requires that spacetime translations are not spontaneously broken. Following this conjecture, we can safely impose all the possible IHCs if we want to solely focus on the massless NG candidates, however, some ambiguities remain on which IHCs to impose if we want to include the massive NG candidates as well.

We illustrate the coset construction for spacetime symmetries and the IHCs by a concrete example in appendix B in Section B.2. The standard relativistic superfluid EFT [138] is derived. However, we recommend reading the next section on the chemical case before to have a glance at it. This because the superfluid condensate is usually made possible via the introduction of a chemical potential.

Some possible outlooks could be to test how strong the conjecture of Brauner and Watanabe is and to which extend it remains true. This can be done through toy models.

In Part II of this thesis, toy models displaying spacetime symmetry breaking patterns are studied. Some are in the exact assumption framework of the conjecture and other are slightly outside (for example, some are breaking spatial translations). By comparing the number of massless NG modes, we obtain by explicit computation of the spectrum to the number predicted by the conjecture (i.e. by imposing all the IHCs and by discussing the possible dynamical reduction of the remaining NG candidates) we will be able to do a consistency check of the conjecture and of its extension to spacetime translation breaking. Another possible interesting discussion might be to focus on the breaking of non-uniform symmetries[10]. These are less technical symmetries because they do not act on spacetime coordinates but they nevertheless remain subtle (we do not have yet a well proven counting rule for them). They are non-internal and so, the condition (5.2.66) to have IHCs can apply on them. The non-uniform symmetries seem to be a good in between case to acquire some knowledge and some intuition on the concern we have concerning the IHCs and the coset construction in general. The motivation is also practical because as it is explained in Appendix D in Part II, they play a major role (via the polynomial shift symmetries) in fracton physics – the physics of modes with reduced mobility where their restricted dynamics is symmetry originated.

5.3 A brief come back on Goldstone physics at finite density

We commented in Section 4.4 that introducing a chemical potential μ_Q for a conserved charge Q whose associated internal symmetry is spontaneously broken is equivalent to consider a background solution of the theory $|\mu\rangle$ which spontaneously breaks time translation following

$$(H - \mu_Q Q)\,|\mu\rangle = 0 \ . \tag{5.3.1}$$

As we can observe, if the internal symmetry generated by Q is spontaneously broken, time translation generated by H is also spontaneously broken, but, the diagonal direction $(H - \mu_Q Q)$ is not[11]. We can then study the low energy spectrum of the theory from the perspective of the coset construction for spacetime symmetry breaking. And see if we recover similar results to Theorem 4.

Such an analysis has been done in [13, 46] in the relativistic case. In the paper [46], they generically study the spontaneous breaking of a uniform symmetry concomitantly with time translation (Lorentz boosts are as well broken) so that the diagonal direction

[10] In our discussion of the coset construction, we might have had this dichotomic view of internal symmetries VS. spacetime symmetries, but nothing in our development prevents G to include non-uniform symmetries. Notice that there are non-uniform symmetries which are neither internal neither spacetime symmetries. For example the linear shift $\phi(x) \to \phi(x) + \alpha_i x^i$.

[11] In the language of fields, we can illustrate this situation with the example where the fundamental field is a complex scalar field $\phi(x)$ with a $U(1)$-symmetric dynamics. If the VEV is given by $v\, e^{i\mu_Q t}$ where v is a constant, we have that $U(1)$ and time translation symmetries are spontaneously broken, however the transformation

$$t \to t + a \ , \qquad \phi \to e^{-ia\,\mu_Q}\phi \ , \tag{5.3.2}$$

still leaves the background invariant. Hence, the diagonal direction $H - \mu_Q Q$ in the symmetry Lie algebra is unbroken.

of these two symmetries is not spontaneously broken. They call such symmetry breaking pattern the spontaneous symmetry probing (SSP). They confirmed the physical validity of SSP by proving that it can indeed be realised, the only requirement is that the diagonal direction should be fixed in time, namely, the coefficient μ_Q in (5.3.1) has to be time independent. They display the non-exotic nature of SSP by showing that from any SSB of a uniform symmetry with a finite density ($j^0 \neq 0$, where j^μ is the corresponding Noether current), we can always build a VEV leading to a SSP (the converse is not true). The iconic example being the non-zero chemical potential case! Strong of this analysis, in [13] a counting rule emerged. We will label the theorem according to the initials of the authors: Nicolis, Penco, Piazza and Rosen.

Theorem 6 (NPPR theorem)**.** *We consider a Poincaré invariant fundamental theory satisfying Goldstone's theorem hypotheses with an internal continuous compact symmetry group G. We switch on a chemical potential μ_Q associated to the internal symmetry generated by Q. According to this, the background state of the theory acquires a VEV leading to the spontaneous breaking, among other internal generators, of Q. It corresponds to a spontaneous symmetry probing. We assume that the spontaneous breaking of time translation and of Lorentz boosts is only due to μ_Q (i.e. no underlying medium) – in particular, the symmetries of spatial tanslations and of rotations are preserved. Finally, we consider μ_Q much smaller than the coupling scale of the fundamental theory and that the limit $\mu_Q \to 0$ is smooth and without any phase transition, meaning that we can compare our result directly with the $\mu_Q = 0$ case which displays the same symmetry breaking pattern for the internal symmetry group G (the Poincaré group is now unbroken). Then, the observed hierarchically small massive modes of the spectrum are classified in four categories*

1. *Linear gapless: modes with low momentum dispersion relation $\omega \propto k$,*
2. *Quadratic gapless: modes with low momentum dispersion relation $\omega \propto \frac{k^2}{\mu_Q}$,*
3. *Fixed gap: gapped modes with a low momentum gap $\omega \propto \mu_Q$, completely determined by the symmetry breaking pattern (the proportional factor is linked to the structure constants),*
4. *Unfixed gap: gapped modes with a low momentum gap generically of order μ_Q, but dependent on the free parameters,*

where categories 1 *and* 2 *are respectively type I and type II NG modes, and where categories* 3 *and* 4 *are massive NG modes. The number of these modes follows the counting rule*

$$n_2 \leqslant n_4 \leqslant n_2 + n_3 \ , \tag{5.3.3}$$

with n_i, the number of modes of category i.

This counting rule has been derived with the most general effective theory obtained via the coset construction method for spacetime symmetry breaking where the unbroken time translation generator $\bar{P}_0$ has been taken to be $H - \mu_Q Q$. Let us mention that we are simplifying some technicalities, we refer to [13] for the full discussion and for the supplementary – non physically constraining – assumptions.

Since in addition to the breaking of internal symmetries, the breaking of spacetime symmetries is as well considered (namely the Lorentz boosts, and homogeneously time translation), we should obtain equal or larger number of NG modes and of massive NG modes compared to Theorem 4. This because, more NG candidates are at disposal. Type A NG modes can be either type I or type II, but most of the time it is type I. Concerning type B NG modes, these are always type II. The general EFT is involved and it does not permit to formally affirm that the type I NG modes are in this scenario type A. However, it is discussed and argued that it should be the case. Thus, if we accept this statement, the computation in the paper [13] recover the same number of NG modes (i.e. symmetry-originated massless modes) compared to Theorem 4. From now on, we consider category 1 to be type A NG modes and category 2 to be type B NG modes.

The new results of [13] lie in the numbering of the massive NG modes. The counting rule (5.3.3) is an inequality on the number of massive NG modes. This can be explained by the conjecture of Brauner and Watanabe on the IHCs (cf. previous section). Indeed, we do not have any ambiguity on n_1 and n_2, the number of massless NG modes, but due to the questioning on whether or not an IHC can be applied, the number of massive modes is not unequivocally determined. From the computations, n_3 is unambiguously known and it corresponds to the massive modes of Theorem 4. Therefore, all the uncertainties focus on n_4. The counting rule (5.3.3) informs us that the category 4 modes are the potential partners of category 2 and category 3 modes. In fact, by referring to Theorem 4, these latter modes are created by the conjugation of two broken generators which leaves room for an associated symmetry-originated mode. Actually, the partners of type B NG modes (i.e. category 2) are systematically category 4. Indeed, thanks to the assumption of the smooth limit $\mu_Q \to 0$, we recover the same breaking pattern of G but in a pure relativistic case. Thus, all the previously type B become type A, and their partners have no choice than becoming type A as well (each broken generator gives one independent NG mode). Thus, the partner of type B is either category 3 or 4 (a mass going to zero with the zero limit of μ_Q) – this is consistent with the generic feature of almost NG modes: their masses scale with the Lorentz breaking parameter which here is represented by μ_Q. But from Theorem 4, category 3 modes come from generators which do not commutes with Q while the type B NG modes are created by generators commuting with Q. Hence, the only possibility for type B partners is to be category 4. This explains the lower bound of (5.3.3). The upper bound, is when all the partners of category 3 mode are category 4. This is not systematic because these partners can be spurious modes. Indeed, IHCs can be applied on them since they are associated to broken generators X_a such that $[Q, X_a] \neq 0$, which leads to $[\bar{P}_0, X_a] \neq 0$ – where $\bar{P}_0$ is the unbroken time translation used in the coset parametrisation. The possibility to be eliminated by an IHC can be a signature of being a spurious field (IHCs either eliminate physical massive modes or they eliminate non-physical modes). Let us notice that no IHCs can be applied on the partners of type B modes because $[Q, X_a] = 0 = [H, X_a]$, so $[\bar{P}_0, X_a] = 0 = [P_i, X_a]$, none of the unbroken translations give a non-zero commutator which is part of the condition (5.2.66) to have an IHC. Each type B partner is a physical massive field and as already discussed, is a category 4. This explains once again the lower bound of (5.3.3).

A final comment on Theorem 6 could be on the assumption to take μ_Q smaller than the coupling scale of the theory. This is in order to make sure that the massive modes of categories 3 and 4 are visible in the EFT, i.e. they are not integrated out. This is also to introduce a hierarchy with the rest of the gapped mode of the theory and to focus on Goldstone physics, the physics coming from the symmetry breaking pattern.

Appendix B has for purpose to illustrate the coset construction for spacetime symmetries. It is done via an example in superfluidity. Since a chemical potential is often necessary to initiate the condensation, this example can also illustrate Theorem 6. However, the chemical potential being not the central focus, the illustration is a relatively simple case of Theorem 6 and so, does not display the interesting modes in this case, namely the massive NG modes. In Part II of the thesis, we analyse relativistic and non-relativistic toy models at finite density showing breaking patterns with spacetime symmetries. This evades a little bit the assumptions of Theorem 6 and so, it permits to see if the counting rule holds anyway or if at least we recover some qualitative features such as the unfixed gapped modes.

Chapter 6

State of the art

Goldstone physics takes place in the framework of the spontaneous symmetry breaking mechanism. The latter is when the state of the system displays a lower number of symmetries compared to the dynamics, i.e. compared to the equations of motion or to any objects from which the EOM are derived. The two main assets of Goldstone physics are the universality of the results we get and the exactness of these results. The reason for that is the almost purely symmetry based approach we have to describe effectively the phenomena at study. Symmetries being a generic feature of physics equations, Goldstone physics is recovered in many areas, ranging from particle physics to condensed matter physics passing by astrophysics. In the same way that symmetries lead to exact conserved quantities, the spontaneous breaking of the former leads to exact knowledge on the hierarchically small masses of the spectrum. This is encoded in the theorem of Goldstone.

Goldstone's theorem states that when a physical system, in Minkowski spacetime, has some continuous global symmetries which are spontaneously broken, its spectrum contains at least one gapless mode. These gapless modes which have a symmetry origin are called Nambu-Goldstone modes (NG modes). This theorem is very general because the mathematical requirements on the symmetry group as well as the physical requirements on the theory (locality, stability and UV completion – let us emphasise that we do not need to be relativistic) are loose. Furthermore, the theorem is valid both at classical and at quantum level with or without temperature and/or chemical potentials. However, it heavily relies on the possibility to have spontaneous symmetry breaking (SSB). Hence, the analysis under which conditions an SSB is possible have to be investigated. The same holds for a counting rule and a classification of the NG modes; indeed, Goldstone's theorem does not give a precise statement on the number of such modes. Moreover, we mentioned that Goldstone physics appears in many domains of physics; it means that we have to look into what is the interplay between the NG modes and these domains.

We have a strong classification and so, a powerful counting rule, when the system, at zero temperature, possesses unbroken spacetime symmetries (it must include rotations and translations) and a compact internal continuous symmetry group G which is partially or completely spontaneously broken. We can relax the hypothesis on the unbroken symmetries by tolerating to add any kinds of unbroken symmetries on the condition that the commutation relations between broken generators X_a and unbroken generators T_A remain systematically in the set of the broken generators (or is zero)

$$[X_a, T_A] \in \text{Span}\{X_b\} \ . \tag{6.0.1}$$

Let us notice that it is trivially true when T_A are spacetime generators confronted to the internal broken generators. It is as well conjectured that the classification and the counting rule stand even if G is a continuous group of uniform symmetries. The classification goes as follow: each broken generator X_a provides an NG candidate. The NG candidates might be matched in order to give one massless mode, a true NG mode, and a massive partner, the almost NG mode. This happens when

$$\langle 0| [X_a, X_b] |0\rangle \neq 0 \ , \tag{6.0.2}$$

where $|0\rangle$ is the vacuum of the fundamental theory, i.e. the theory prior symmetry breaking and so, prior the perturbation study around the vacuum. The NG mode created by a pair of broken generators satisfying (6.0.2) is called a type B NG mode. Otherwise, if it is created by a single broken generator, it is a type A NG mode. The total number of NG modes is then given by the sum of the numbers of type A and type B NG modes. It exists another classification which from an abstract point of view provides a weaker counting rule (because it is based on an inequality) but turns out to be sometimes more convenient in practice. An NG mode having a monomial dispersion relation at low-momentum with an odd-power in spatial momentum is called a type I NG mode, otherwise it is called a type II NG mode. It has been proven that $n_{\rm I} + 2n_{\rm II} \geqslant n_{\rm BG}$, where n_i are respectively the number of type I NG modes, type II NG modes and the number of broken generators.

From the preceding discussion, we do understand that the case of the breaking of uniform symmetries is well established. Nevertheless, there are of course some open questions remaining concerning these kinds of symmetry breaking patterns. A first example could be the search of the critical dimension of spacetime under which no spontaneous breaking of uniform symmetries can occur at quantum level. For relativistic theories, this critical dimension has been found and proved to be two – this is Coleman's theorem (nevertheless, large N QFTs in the strict infinite N limit evade this statement). For non-relativistic theories, there are no results which are known for a totally generic case. But it is conjectured that if the breaking pattern leads to solely type A NG modes, the critical dimension is $n+1$, where n is half of the number of spatial derivatives in the kinetic term of the effective theory describing the NG modes – i.e. when the scaling dimension of the non-trivial vacuum of the fundamental theory is zero. Concerning the situation where there are only type B NG modes, it appears and it has been computed that there is no critical dimension, SSB can occur at any dimensions of spacetime. The situation where both type A and type B NG modes are present remains unclear concerning the existence of a critical dimension of spacetime.

A second example of unsolved issues is the uncertainty on the number of symmetry originated massive modes when we switch on a chemical potential. The counting rule based on type A/type B NG modes has been generalised at finite density in order to count the massless modes (the NG modes) but also to count the massive modes with a gap entirely fixed by group theory and by a linear dependence on the chemical potential. However, there are other massive symmetry originated modes which are also scaling with the chemical potential but this dependency is model dependent. For relativistic fundamental theories, the number of such modes is bounded from below and from above. In any case, relativistic or non-relativistic, we do not have yet a strict counting rule for these symmetry originated modes with a model dependent gap.

The extension of all the aspects we mentioned so far to a totally generic symmetry breaking pattern, i.e. where the breaking of non-uniform symmetries is tolerated, constitutes the main current problematic of Goldstone physics. In the literature, the focus is mainly done on the breaking of spacetime symmetries but it should be mentioned that the discussion concerns non-uniform symmetries in general. We do not have, if it exists, a counting rule for a generic breaking pattern. Nevertheless, we have a powerful effective field theory (EFT) building tool which relies only on group theory: the coset construction. It relies on the assumption that the commutation of broken generators with unbroken generators (modulo the unbroken translation symmetries) always remains in the set of broken generators. Furthermore, it requires that spacetime translations are present and unbroken – there are some tricks to anyway describe the breaking of translations, e.g. a homogeneous breaking where the breaking of translation is compensated by an additional internal symmetry breaking. Schematically, the requirements on how the commutation relations should be are

$$[X_a, T_A] \in \text{Span}\{X_b\} \ , \tag{6.0.3}$$

$$[P_\mu, T_A] \in \text{Span}\{P_\nu\} \ , \tag{6.0.4}$$

where P_μ are the unbroken translation generators and where $P_\mu \notin \{T_A\}$. In the internal symmetric case, the reduction of the number of NG candidates is done purely at the dynamical level. Indeed, it is the canonical conjugation structure of the most general EFT which has led to the condition (6.0.2) which eventually causes a reduction of the number of NG modes compared to the number of broken generators. It appears from the coset construction, and under the associated assumptions, that for non-uniform symmetries there can be a reduction of the NG candidates prior any dynamical considerations. Or more precisely before any dynamical computations, because of course, this reduction of NG candidates is justified and recovered by the dynamics in the EOM. This apparent non-dynamical reduction can be directly seen from the transformation laws of the NG candidates and from their intrinsic definitions. It has been formalised under the notion of inverse Higgs constraint (IHC). In order to eliminate the NG candidate associated to the broken generator X_b, an IHC can be imposed if

$$[P_\mu, X_b] \supset X_a \ , \tag{6.0.5}$$

where P_μ is an unbroken translation generator (there are additional technical requirements behind the condition (6.0.5); in this summary we focus only on the gist of what Goldstone physics is). It is conjectured that imposing all the inverse Higgs constraints does not lead to a loss of generality on the numbering of the (massless) NG modes. Therefore, indication on their number for a given symmetry breaking pattern can be obtained through the number of independent IHCs we can impose and on the canonical conjugation structure of the EFT containing solely the massless modes. However, imposing IHCs might introduce a loss of generality on the massive symmetry originated modes. So, establishing the criteria on which we can impose an IHC without any loss of physics constitutes a remaining research task. In particular, this ambiguity on the number of massive symmetry originated modes due to the uncertainty revolving around the IHCs is at the origin of the problematic mentioned in the preceding paragraph on Goldstone physics at finite density.

The study of the breaking of non-uniform symmetries, and in particular of spacetime symmetries, is conceptually and technically more involved than the pure internal case. Tackling the different open questions directly with a generic approach might be too ambitious. The general strategy for this thesis and for the future research projects is the following.

As a first step, toy models will be studied. These toy models should be complex enough to encode physical subtleties but simple enough to allow for (partial) analytic computations. We mean by physical subtleties that the model should display breaking patterns or physical situations which are in the range of the assumptions of the different conjectures, this in order to check their consistency. It could be interesting to look at cases which are slightly outside the assumptions of the known theorems and of the conjectures to probe how far these known results could be extended. We want as much as possible to remain at the level of analytic computation. The reason is that it allows to keep track of how the free parameters of the theory interconnect to finally give the masses, the dispersion relations, the perturbation interactions and most importantly, it permits to discriminate the final spectrum. Indeed, the parametrisation of the perturbations which diagonalises the perturbation kinetic matrix is not necessarily the parametrisation which displays explicitly the NG candidates. Hence, it is not trivial to make the connection between the final dispersion relations and the NG candidates (which ones have been suppressed, which ones become massive and which ones are true NG modes ?). It is only possible if we keep track of all the steps of the computations, where usually in numerical simulations we are blind with respect to these subtleties. Moreover, if we are analytic, it is easier to study some parameter limits of our theory, for example when we switch on a chemical potential and we want to study the mass hierarchy when we send to zero this chemical potential. The fundamental idea behind the analysis of toy models is to acquire intuition on our final goal: find generic results.

The second step is to see how strong the intuition we acquired by studying toy models is. To do so, we have to reproduce the specific studied breaking patterns in exotic field theories. Holography is the right tool for that. Holography is a strongly VS. weakly coupled duality between a quantum field theory and a gravitational theory living in a one higher dimensional spacetime. When the quantum field theory is a strongly coupled large N theory with N going "strictly" to infinity, its gravity dual is purely classical and weakly coupled. Therefore, with a classical perturbative computation on the gravity side of the duality, we can obtain (information on) the quantum correlators of the strongly coupled large N QFT! Thus, if we succeed to encode the desired breaking pattern in the framework of holography, we will be able to test our acquired intuition at quantum level and for complicated theories.

The final step would then be a generic study based on the coset construction where our intuition from the two preceding steps can serve as guidelines, for example in the case of the criteria under which we have to impose inverse Higgs constraints. Spacetime symmetries are technically complicated because they act both on the internal space of fields but also on the manifold coordinates on which the fields are defined on. An in-between case could be the non-uniform symmetries which are not spacetime symmetries, e.g. the spatial polynomial shift symmetries ($\phi(x) \to \phi(x) + \beta_i x^i$). They have the same conceptual subtleties than spacetime symmetries, in particular they can as well be involved in IHCs since they do not commute with translations. But they are technically less elaborated since they do not act on spacetime coordinates. Another reason to focus on non-uniform non-spacetime symmetries is that they can be connected to the notion of subsystem symmetries, the latter play a major role in fracton physics – see Part II Appendix D. As we said earlier, because Goldstone physics is universal, it is important not only to study NG modes from a pure abstract viewpoint but also to see how it fits in concrete physical systems and how they interact with other fields. Especially, fracton physics is recent (it started in the mid-decade of 2010 while Goldstone physics started in the early decade of 1960) and already provided promising results. Therefore, establishing a connection with this area of physics can help us in both ways: consolidate and improve the early results of fracton physics but also learn new things about Nambu-Goldstone modes.

The three steps we mentioned above should of course not be seen as strictly sequential: there might be interconnections and overlap projects between these three approaches. And as often in research, new elements can make us deviate from the original path. These are just guidelines. In this thesis, the two first approaches are used. We close this state of the art with the specific open problematic we partially address in this thesis.

- In Part II, we build toy models which display spacetime symmetry breaking, first the breaking of dilatation alone and then the concomitant breaking of dilatation and spatial translation. The case with a chemical potential is also dealt with. These models slightly evade the hypotheses of the known results for spacetime symmetries and for Goldstone physics at finite density. Thus, it permits a non-trivial discussion on the counting rules and on the dispersion relations but also on new tools to discriminate in practice which dispersion relation is associated to which NG candidate(s). Connection with fractonic physics will be made and commented. Finally, these toy models have the right properties to play the role of Landau-Ginzburg's models effectively describing superfluid condensates. Let us mention that the breaking of translation is homogeneous; the inhomogeneous breaking will be briefly discussed in Appendix C in Part II.

- In Part III, with an explicit computation in QFT, we prove the conjecture on the critical dimension of spacetime under which there is no spontaneous symmetry breaking which would lead to solely type A NG modes. Let us notice that since we use the classification type A and type B NG modes, it means that we are assuming that only uniform symmetries are broken. Then, from a holographic computation, with a counter-example, we show that strict large N limit QFTs are not subject to this critical dimension. Nothing generic prevents an SSB at lower dimensional spacetime for strict large N QFTs.

- In Part IV, we provide outlooks based on the general guidelines mentioned above. Especially, we propose a holographic model which would be able to reproduce the toy model features of Part II and we discuss on the feasibility of this future research project. The thesis is eventually concluded with a discussion summarising the highlights of this dissertation.

Appendix A

Concrete example: ferromagnetism

The aim of this chapter is to illustrate how the results and the technology we introduced in Chapter 4 can be implemented on a specific observable physical phenomenon. We do not intend to do precise phenomenological predictions but rather to show how an analysis of the symmetries alone can already provide the behaviour of the observations, and how the coset construction can lead to crude quantitative predictions. The two main references for this part of the script are [18, 25].

The example we are going to look at is ferromagnetism: below a critical temperature T_c (usually between 10^2 K and 10^3 K) certain materials acquire a spontaneous magnetisation. This magnetisation come from the magnetic moment the elementary constituents (atoms, molecules, ions, electrons,...) of the considered material can have due to their spin and their orbital momentum. At high temperature, the thermal agitation randomly orients the different magnetic momenta and so, by average, there is no global magnetisation. By decreasing the temperature and depending how the elementary constituents interact, a magnetic ordering can appear, the global alignment of the individual magnetic momenta can generate a spontaneous global magnetic field.

We are going to study ferromagnet at low temperature ($T \ll T_c$), low enough such that we can do the approximation to study the microscopic theory at zero temperature in order to establish the fundamental state and the excitation spectrum. Afterwards, to get the thermodynamic quantities, we will apply the statistics on our microscopic spectrum.

Since our purpose is mainly illustrative, we can limit ourselves to a coarse model. As a first approximation, we can reasonably consider the electrons to be localised on their corresponding atoms. These atoms will be taken as identical and we will assume that they are placed at the sites of a 3 (spatial)-dimensional Bravais lattice. Each of them should possess a non-zero total angular momentum $\vec{S}$. It is standard practice to call this angular moment "spin" in reference to the original Heisenberg Hamiltonian, cf. later. We will already start from an effective theory where the Coulomb interactions combined with the Pauli exclusion provide effective interactions among momenta described by the Heisenberg Hamiltonian. We will consider that our system can indeed be effectively described by the Heisenberg Hamiltonian:

$$\hat{H} = -\sum_{\mathbf{RR'}} J_{\mathbf{RR'}} \, \hat{\mathbf{S}}_{\mathbf{R}} \hat{\mathbf{S}}_{\mathbf{R'}} \, , \tag{A.0.1}$$

where bold letters correspond to 3-dimensional vectors, $\mathbf{R}$ labelises the Bravais lattice sites and $J_{\mathbf{RR'}}$ depends on $\mathbf{R}$ and $\mathbf{R'}$ only by the difference $\mathbf{L} = \mathbf{R} - \mathbf{R'}$. We will suppose $J_{\mathbf{RR'}}$ to decrease fast enough with $\mathbf{L}$ such that our requirements on locality are satisfied. Furthermore, in the case of ferromagnetic materials: $J_{\mathbf{RR'}} > 0 \; \forall \, \mathbf{R}, \mathbf{R'}$.

To minimise the energy, we should maximise the scalar product in (A.0.1). To do so, all the spins should be aligned. The direction of the global alignment is not fixed by the energy minimisation principle, let us arbitrarily chose that all the spins align in the

$x-$direction.

A.1 Analysis based on solely the symmetries

The interactions between spins depend on the relative orientation of the spins (A.0.1). Hence, a global rotation of the spins will not alter the dynamics. We thus have a global $SU(2)$ symmetry. From the coset construction and the counting rule of Section 4.2, we learned that it is mainly the algebra which matters. Since $su(2) \cong so(3)$, for a better visualisation, we will consider the dynamics to have a global internal $SO(3)$ symmetry. It is indeed an internal symmetry since we rotate the spins around their attach points – we do not rotate the crystal (the space).

The vacuum has been established and chosen such that all the spins are aligned along the $x-$direction. We thus have a spontaneous symmetry breaking of $SO(3)$ to $SO(2)$ since the vacuum is invariant under a rotation along the Ox axis but does transform under any other kind of rotations. We thus have that the generators $\hat{S}_y$ and $\hat{S}_z$ are spontaneously broken.

Another important symmetry breaking for the coset construction is the spontaneous symmetry breaking of time reversal symmetry. Indeed, we can visualise the spin of each atom due to orbital rotation and to intrinsic rotation (the actual spin) of their constituents. By inverting the flow of time, the direction of rotation will change and so, the spins will flip. The dynamics is invariant under this flip since only the relative orientation between the spins matter. However, our vacuum will transform from an alignment along x to an alignment along $-x$.

From the breaking pattern $SO(3) \to SO(2)$, and from the hypotheses of our model, we can apply both Goldstone's theorem and the Brauner-Murayama-Watanabe's counting rule. We have that

$$\langle 0| [\hat{S}_j, \hat{S}_j] |0\rangle = 0 \quad , j = y, z \; , \tag{A.1.1}$$

$$\langle 0| [\hat{S}_y, \hat{S}_z] |0\rangle = i \langle 0| \hat{S}_x |0\rangle = i\, n\, V S \; , \tag{A.1.2}$$

where n is the density of atoms, V is the volume of our lattice and S is the norm of the spin of one atom. So,

$$\rho = \lim_{V\to\infty} \frac{-i}{V} \begin{pmatrix} 0 & i\, n\, V S \\ -i\, n\, V S & 0 \end{pmatrix} = \begin{pmatrix} 0 & n\, V S \\ -\, n\, V S & 0 \end{pmatrix} \Rightarrow \text{rank}(\rho) = 2 \; . \tag{A.1.3}$$

The low energy excitation spectrum will contain one NG mode and it will be of type B. Indeed,

$$n_{\text{NG}} = n_{\text{BG}} - \frac{1}{2}\text{rank}(\rho) = 2 - 1 = 1 \; , \tag{A.1.4}$$

$$n_B = \frac{1}{2}\text{rank}(\rho) = 1 \; , \tag{A.1.5}$$

$$n_A = n_{\text{NG}} - n_B = 0 \; . \tag{A.1.6}$$

Since the action of $SO(3)$ does not modify the Lorentz representation of the spins, the perturbations around the vacuum in the broken directions of $SO(3)$ will be scalars. The type B NG mode will thus be a boson and because it is a type B, it will have a quadratic dispersion relation. In condensed matter literature, this excitation is either called a magnon or a spin wave (since it is a fluctuation in the spin orientation which propagates through the system).

We have all the necessary tools to get the behaviour of certain thermodynamic quantities. We work in the natural units $c = \hbar = k_B = 1$, where k_B is the Boltzmann constant. Let us focus on the heat capacity per unit of volume:

$$c(T) \equiv \frac{d\epsilon}{dT} , \tag{A.1.7}$$

where ϵ is the energy per unit of volume.

The magnetic contribution to $c(T)$ is computed from

$$\epsilon_m = \epsilon_0 + \sum_{\mathbf{q}} \omega_{\mathbf{q}} \langle n_{\mathbf{q}} \rangle_T , \tag{A.1.8}$$

where ϵ_m is the magnetic energy density, ϵ_0 is the vacuum energy density and $\langle n_{\mathbf{q}} \rangle_T$ is the average number density of magnetic NG modes of wave vector $\mathbf{q}$ at temperature T. Because we are working at finite volume (ferromagnet materials are of finite size), the values for the wave vector are discretised due to the Born–von Karman boundary conditions

$$q_i = 2\pi \frac{k_i}{L_i} \text{ with } k_i \in \mathbb{Z},\ i = 1, \ldots, d-1 , \tag{A.1.9}$$

where L_i is the length of the system in the i-direction.

Since the excitation modes are bosons, $\langle n_{\mathbf{q}} \rangle_T$ is given by the Bose-Einstein statistics:

$$\langle n_{\mathbf{q}} \rangle_T = \frac{1}{V(e^{\frac{\omega_{\mathbf{q}}}{T}} - 1)} . \tag{A.1.10}$$

By going to the large volume limit (the volume is large compared to the lattice spacing), we can switch the sum for an integral in (A.1.8) and use (A.1.9) to determine the density of states in the integration measure. We roughly obtain

$$\epsilon_m = \epsilon_0 + \frac{1}{(2\pi)^3} \int d^3q \frac{\omega_{\mathbf{q}}}{e^{\frac{\omega_{\mathbf{q}}}{T}} - 1} \sim \epsilon_0 + \int_0^{+\infty} dq \frac{q^4}{e^{\frac{q^2}{T}} - 1} \sim \epsilon_0 + T^{5/2} \int_0^{+\infty} ds \frac{s^{3/2}}{e^s - 1} , \tag{A.1.11}$$

where we used the quadratic shape of the dispersion relation, we went to spherical coordinates and we made a change of variable. The most right-hand side integral is given by the Riemann zeta function

$$\int_0^{+\infty} ds \frac{s^{3/2}}{e^s - 1} = \zeta\left(\frac{5}{2}\right) \Gamma\left(\frac{5}{2}\right) , \tag{A.1.12}$$

which is non-zero and finite.

Finally, we have that the magnetic contribution $c_m(T)$ to the specific heat evolves with temperature as

$$c_m(T) \equiv \frac{d\epsilon_m}{dT} \sim T^{3/2} . \tag{A.1.13}$$

This thermal behaviour at low temperature is the standard textbook law obtained through more "usual" condensed matter computations [139]. To compare it with experiment, from the hypotheses of our model, we have to consider the specific heat as well coming from the phonons, i.e. the crystal oscillations. These modes are the NG modes coming from the spontaneous symmetry breaking of continuous spatial translation symmetries to their discrete subset due to the crystal lattice itself. Since translations commute with themselves and with internal symmetries, phonons are type A NG modes (as we have discussed, it is reasonable to consider the classification of Theorem 3 to hold for uniform symmetries) and they are bosons since translations do not modify the Lorenz representation of the field they act on. Because the interactions between elementary constituents of the lattice (atoms or molecule) can be simulate with harmonic interactions among closest neighbours[1], the interactions are closely localised in space. Hence, we can suppose that the EFT will then contain lower order spatial derivative terms, added to spatial rotation symmetry, the EFT will have a second order spatial derivative term as dominant term. In such a case, type A NG modes are type I NG modes. Thus, the phonons have linear relation dispersions. In (A.1.11), phonons will contribute to the energy density following T^4. So, the specific heat (A.1.13) goes as T^3. Let us mention that there are no additional NG modes due to the breaking of spatial rotations, this is explained in Chapter 5.

Solely based on the symmetries involved in our model describing ferromagnets, we displayed that the specific heat receives two contributions[2] such that at low temperature, it has the following behaviour

$$c(T) = a\,T^{3/2} + b\,T^3 \ , \tag{A.1.14}$$

where the a and b coefficients quantify respectively the magnetic contribution and the crystal contribution. If we conduct a measurement on a ferromagnetic material with close enough properties to our models and that we plot $c(T)\,T^{-3/2}$ in function of $T^{3/2}$, we should get a straight line permitting to determine a and b. This can be observed for Yttrium Iron Garnet from 1.5 to 4.2 K [140]. The plot is given in Figure A.1.

A.2 Coset construction for ferromagnetism

In order to have more quantitative results for our analysis of ferromagnetism, we can build the effective field theory describing magnons through the coset construction. Indeed, the breaking pattern $SO(3) \to SO(2)$ does satisfy the criteria with which we introduced the coset construction.

The coset space is

$$G/H = SO(3)/SO(2) \cong S^2 \ . \tag{A.2.1}$$

The 2-sphere can be parametrised by the azimuthal angle $\varphi \in [0, 2\pi[$ and by the angular angle $\theta \in [0, \pi]$. Thus, our candidates NG modes π are

$$\pi(x) = (\theta(x), \varphi(x)) \ . \tag{A.2.2}$$

[1] Each of the elementary constituent is at an equilibrium position, i.e. at the bottom of a potential well. The potential function expanded at lower order around the equilibrium position is then well approximated by a parabolic function.

[2] We made the assumption that the electrons are localised, hence, no additional electronic contribution to the specific heat is considered.

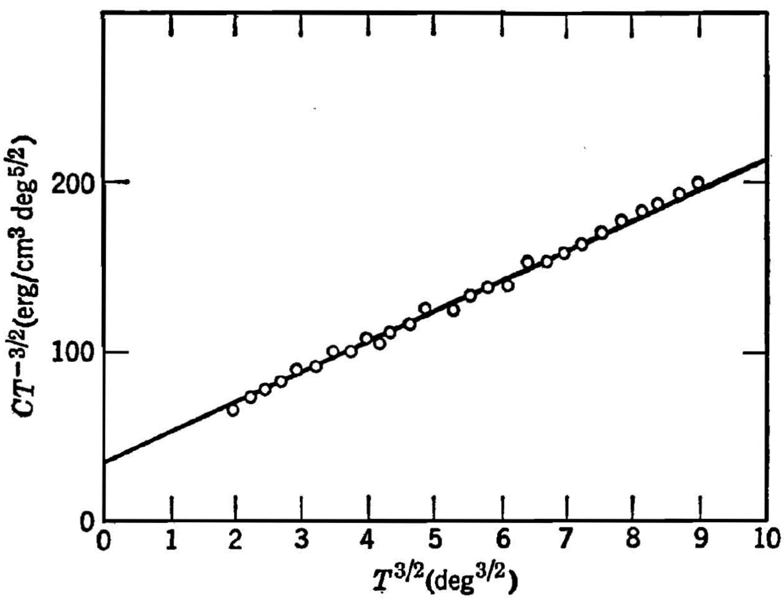

Figure A.1: Heat capacity of yttrium iron garnet at low temperature [139, 140].

Time reversal symmetry is broken, thus the EFT should contain a single time-derivative term and this term will be dominant with respect to $\mathcal{O}\left(\partial_t^2\right)$ since we consider to be (deep enough) in the IR. Furthermore, the IR region is consistent with the continuum limit, and we will thus make the approximation that we have continuous spatial translation symmetries and spatial rotation symmetries. Hence, the shape of the effective Lagrangian is

$$\mathcal{L}(\pi) = c_a(\pi)\partial_t\pi^a - \frac{1}{2}\bar{g}_{ab}(\pi)\partial_i\pi^a\partial_i\pi^b + \mathcal{O}(\partial_t^2, \partial_t\partial_i^2, \partial_i^4) \ . \tag{A.2.3}$$

The coefficient $\bar{g}_{ab}(\pi)$ should be a generic metric on S^2 invariant under $SO(3)$. Since the broken generators form an irreducible representation[3] of $SO(2)$ and from the emphasised comment in Subsubsection 4.2.1.4, $\bar{g}_{ab}(\pi)$ is given up to a global factor, a natural particular metric is the canonical metric of the 2-sphere, thus our generic metric is:

$$\bar{g}_{ab}(\pi) = u_1 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} \ , \tag{A.2.4}$$

where u_1 is a general constant. Another way to recover (A.2.4) as the most generic metric is to solve the system given by the vanishing Lie derivatives of the metric with respect to the $SO(3)$ generators[4].

[3]The real vector space generated by $\{iS_y, iS_z\}$ is an irreducible representation of $so(2)$ – a real algebra – where the action of $so(2)$ is defined by $[\cdot, iS_x]$ [18]. Indeed, if we try to find $a, b, c \in \mathbb{R}$ such that $[aiS_y + biS_z, iS_x] = c(aiS_y + biS_z)$, we have the constraints $a = bc$ and $c^2 = -1$. The constraint on c cannot be satisfied because $c \in \mathbb{R}$.

[4]The $SO(3)$ generators realising rotations on the 2-sphere are

$$\xi^a_{(1)} = \delta^a_\phi \ , \ \xi^a_{(2)} = -\left(cos(\phi)\delta^a_\theta - cot(\theta)sin(\phi)\delta^a_\phi\right) \ , \ \xi^a_{(3)} = sin(\phi)\delta^a_\theta + cot(\theta)cos(\phi)\delta^a_\phi \ . \tag{A.2.5}$$

The coefficient $c_a(\pi)$ is a generic covector which should transform as

$$\mathcal{L}_\xi c_a = \partial_a \Omega_\xi(\pi) \ , \tag{A.2.6}$$

where $\Omega_\xi(\pi)$ is an unconstrained function on S^2 and where ξ is any of the Killing vectors corresponding to the generators of $SO(3)$. It means that the anti-symmetric tensor

$$F_{ab} \equiv \partial_a c_b - \partial_b c_a \ , \tag{A.2.7}$$

is invariant under the isometry generated by ξ. Furthermore, since F_{ab} is defined on a two-dimensional manifold (the 2-sphere), it is defined by one scalar function $u_0'(\pi)$:

$$F_{ab} = u_0' \sqrt{\mathrm{Det}(\bar{g})}\, \epsilon_{ab} \ . \tag{A.2.8}$$

The invariance of F_{ab} reduces to

$$\mathcal{L}_\xi u_0' = \xi^a \partial_a u_0' = 0 \ , \tag{A.2.9}$$

for all the rotation Killing vectors ξ. This condition implies that $u_0'(\pi)$ is in fact independent of π. Combining this observation with (A.2.4), (A.2.7) and (A.2.8) we obtain an equation for $c_a(\pi)$

$$\partial_\theta c_\varphi - \partial_\varphi c_\theta = u_0'\, u_1 \sin(\theta) \ . \tag{A.2.10}$$

Because the transformation (A.2.6) of $c_a(\pi)$ is a symmetry of the theory and it is driven by an arbitrary function $\Omega_\xi(\pi)$, we can use it to eliminate one of the components of $c_a(\pi)$. Let us set to zero $c_\theta(\pi)$. Thus, a generic solution for (A.2.10) is

$$c(\pi) = (0, u_0 \cos(\theta)) \ \text{ with } u_0 \equiv -u_0' u_1 \ . \tag{A.2.11}$$

We can explicitly check that $\mathcal{L}_\xi c_a = \partial_a \Omega_\xi(\pi)$ is satisfied for each Killing vector (A.2.5) when the solution (A.2.11) is considered.[5]

Our effective Lagrangian is now of the form

$$\mathcal{L}\,(\pi) = u_0 \cos(\theta) \partial_t \varphi - \frac{u_1}{2} \left(\partial_i \theta \partial_i \theta + \sin^2(\theta) \partial_j \varphi \partial_j \varphi \right) + \mathcal{O}(\partial_t^2, \partial_t \partial_i^2, \partial_i^4) \ . \tag{A.2.12}$$

To connect our NG fields, $\theta(x)$ and $\varphi(x)$, to a physical interpretation, we express the 2-sphere with Cartesian coordinates

$$\begin{cases} s_x = \sin(\theta)\cos(\varphi) \ , \\ s_y = \sin(\theta)\sin(\varphi) \ , \\ s_z = \cos(\theta) \ , \end{cases} \tag{A.2.13}$$

were the Cartesian coordinates correspond to the "spin field" (this field provides the spin we have in our material at the spacetime position x). The fundamental state is when all

[5] $\Omega_{\xi_{(1)}} = cst.$, $\Omega_{\xi_{(2)}} = u_0 \frac{\sin(\varphi)}{\sin(\theta)} + cst.$, $\Omega_{\xi_{(3)}} = u_0 \frac{\cos(\varphi)}{\sin(\theta)} + cst.$.

the spins are aligned in the $x-$direction, so, our spin field should be constant over space and time, and it should point towards the $x-$direction. This means that

$$\begin{cases} \theta_0(x) = \dfrac{\pi}{2} \ , \\ \varphi_0(x) = 0 \ . \end{cases} \tag{A.2.14}$$

NG modes correspond to small fluctuations around the vacuum in the broken directions, hence, to properly describe NG modes we have to infinitesimally fluctuate around (A.2.14)

$$\begin{cases} \theta(x) = \dfrac{\pi}{2} + \theta'(x) \ , \\ \varphi(x) = \varphi'(x) \ . \end{cases} \tag{A.2.15}$$

Till quadratic order, we have

$$\mathcal{L}(\pi) = \frac{u_0}{2}\varphi'\partial_t\theta' - \frac{u_0}{2}\theta'\partial_t\varphi' - \frac{u_1}{2}\left(\partial_i\theta'\partial_i\theta' + \partial_j\varphi'\partial_j\varphi'\right) + \mathcal{O}(\partial_t^2, \partial_t\partial_i^2, \partial_i^4) \ . \tag{A.2.16}$$

From Fourier transform we can notice that we will get a unique dispersion relation

$$\omega = \frac{u_1}{u_0}q^2 \ . \tag{A.2.17}$$

Therefore, we recover the fact that we have a single particle corresponding to a type B NG mode[6].

The two unknown coefficients u_0 and u_1 can be obtained experimentally. For example, by inelastic neutron scattering through the medium: if we know how much energy and momentum have the neutrons before and after the propagation, we can deduce how much energy and momentum were provided to the system in order to excite the modes of the solid. We thus get a curve of ω in terms of q. Of course, we have to discriminate the phonon excitations and the magnon excitations. The magnons are sensible to temperature, hence, by doing the experiment at various temperature we can discriminate the two kinds of excitations and extract the dispersion relation of the magnons. Once u_0 and u_1 are determined, we can explicitly compute (A.1.11) and obtained a value for (A.1.13) or at least, an order of magnitude.

[6]Let us stress that we developed the EFT solely for the spins, the phonons were not considered in this coset construction. In fact, including the non-homogeneous breaking of translations through a lattice structure in the coset construction is out of the scope of both coset constructions presented respectively in Chapter 4 and in Chapter 5. A starting point could be [134].

Appendix B

Field-theoretical approach of superfluidity

In this appendix, we are going to illustrate how certain types of superfluid can be phenomenologically described by field theories in the spirit of Landau-Ginzburg's models[1]. The motivation is twofold. First, the effective description of superfluidity is one of the less involved examples of the coset construction for spacetime symmetry breaking. Second, in Part II of this thesis, some of the studied toy models are good candidates to phenomenologically describe some superfluid, in this appendix we explain why it is the case.

Let us stress that we are doing here a schematic discussion to give a gist on how certain superfluid can be described by field theories, we do not claim to be exhaustive nor to be rigorously precise.

A superfluid is a fluid whose main feature is that it can flow through a thin capillary with zero resistance [142, 143]. For such kind of flow, the superfluid has zero viscosity. However, if a cylinder is placed in a superfluid bath and rotated, there is a momentum transfer from the rotating cylinder to the superfluid, hence, the viscosity is not zero for this specific kind of motion [144]. We understand that a superfluid does not exactly behave as a perfect fluid (an analogy can be made with the difference between a superconductor and a perfect conductor). This specificity of superfluidity can be captured by the two-fluid model where the superfluid is seen as a mixture of a perfect fluid (zero viscosity) of density ρ_s and a normal fluid (non-zero viscosity) of density ρ_n. The phase transition is at the thermodynamical point where $\rho_s = \rho_n$ [8]. In this brief illustration of superfluid physics, we are going to only consider superfluid phase occurring at low temperature. Furthermore, consistently with the general hypotheses of this thesis, the zero temperature

[1] Landau-Ginzburg's theory is a generic approach to study phase transitions in statistical field theories at equilibrium. The phase transitions are classified through their symmetry breaking patterns. Hence, a field order parameter $m(\vec{x})$ is defined and the free energy is re-written in terms of $m(\vec{x})$: $F[m(\vec{x})]$. The latter is called the Landau-Ginzburg free energy. The partition function is now expressed as [141]:

$$Z_\beta[m(\vec{x})] = \int \mathcal{D}m(\vec{x})\, e^{-\beta F[m(\vec{x})]} \,. \tag{B.0.1}$$

We observe that in the zero temperature limit, through the saddlepoint approximation, the thermodynamical state of the system is given by the stationary point of $F[m(\vec{x})]$, i.e. the solution of the equations of motion derived from $F[m(\vec{x})]$. The thermodynamical conditions can modify the shape of $F[m(\vec{x})]$. So, under some conditions, the solution will be trivial – $m(\vec{x}) = 0$ – meaning that there is no SSB, this is one possible phase. Under other external conditions, the solution will be non-trivial – $m(\vec{x}) \neq 0$ – meaning that an SSB occured, which corresponds to another thermodynamical phase. Hence, at zero temperature, the shape of $F[m(\vec{x})]$ dictates the phase we are in, and $m(\vec{x})$ inform us of the behaviour of the system in the associated phase. Let us mention that not all the phase transitions are captured by Landau-Ginzburg's classification, e.g. topological phase transitions do not fit in this classification.

limit will be assumed. In such limit, the perfect fluid component of the mixture prevails. Thus, the phase transition point will not be reachable, only the superfluid phase itself will be described and this, with a single fluid model.

B.1 Bose-Einstein condensate superfluids

The low temperature superfluids correspond to bosonic particles which have undergo a Bose-Einstein condensation and where, given some criteria which will be detailed later, the condensate displays a superfluid behaviour [8].

A Bose-Einstein condensate (BEC) of a gas of weakly interacting bosons can phenomenologically be described by a classical field. Indeed, at high temperature, the thermal agitation is such that the individual bosons are in a quantum superposition of several momenta. Their individual probability wave functions (a superposition of plane waves labelled by the momenta) are therefore compact wave packets with definite individual positions. When the temperature is reduced, the bosons start to reach quantum states with lower momenta. Thus, the individual wave functions start to spread and individual positions start to be less clearly defined. When the phase transition occurs, all the bosons are in the same zero momentum state, i.e. they form the BEC. The individual probability wave functions fully overlap each other which leads to a one single collective wave function – physically, it can be seen as a density wave. This collective wave function is then phenomenologically described by a field theory [145, 146]. Since we started with weakly interacting bosons, this field theory is weakly coupled. A perturbative study can thus be made. From the Feynman diagrammatic description of QFT, we know that the expansion in terms of the coupling constants can be resumed into an expansion in number of loops where each loop is weighted by an $\hbar$ factor. The corrections due to the loops are small, therefore, the classical limit $\hbar \to 0$ gives a good picture of the behaviour of the collective wave function. Hence, the BEC field behaves like a classical wave. This assertion has been proven experimentally where two BECs have been collided and it provided a classical interference pattern[2] [147].

As mentioned earlier, the collective wave function is interpreted as a density wave and since it comes from the quantum formalism, this collective wave is a complex field. Hence, the field theory ruling the density wave has the $U(1)$ symmetry where the associated Noether charge corresponds to the density of particles. This conclude the fact that we can describe BEC phase transition through Landau-Ginzburg's theory: at low temperature, we have a classical field theory with a symmetry where the order parameter is the field itself. When the symmetry is spontaneously broken, it means that the system is in the BEC phase (non-trivial collective wave). Otherwise, the BEC phase transition has not occurred (trivial collective wave). This is the spirit of Landau-Ginzburg's models.

To obtain the different possible flows of the BEC field, we study the different particular solutions of the equations of motion derived from the classical Landau-Ginzburg field theory phenomenologically describing the BEC. Being in the zero temperature limit, this corresponds indeed to the thermodynamical state. One way to do it is to seek for a static

[2]Quantum waves (i.e. probability waves), due to the collapse of the quantum states after the measurement, form an interference pattern only after several repetitions of the experiment. Here, classical interference pattern means that the pattern has been obtained in a single shot, showing that the two considered waves behave indeed as classical waves.

solution (easier to find) and then obtain a moving solution by applying a boost on our static solution (usually, the theory has either Lorentz symmetry or Galilei symmetry). This concept is illustrated in Chapter 7 in Part II of this thesis.

To concretely define "what is moving" and so to define what we mean by the velocity of the fluid, we use the $U(1)$ internal symmetry of the Landau-Ginzburg field theory. The associated Noether charge corresponds to the quantity which is travelling. In the case of $U(1)$, it is matter transportation. The velocity of the fluid, v^μ, is then defined thanks to the Noether current schematically as $j^\mu \propto v^\mu$ [8].

In order for our BEC to be a superfluid, it should satisfy some criteria. First, it should be able to flow and so to be a fluid even at very low temperature. In our above discussion, we implicitly made the assumption that the gas of bosons does not solidify when lowering the temperature. The usual example of a bosonic material which, at atmospheric pressure, does not solidify even at extreme low temperature is ^{4}He. The phonon vibrations are larger than the lattice spacing causing the lattice structure to vanish and so, making it impossible for ^{4}He to solidify [144]. The second condition to be a superfluid is that when the fluid encounter an obstacle, its velocity remains constant. There are two possibilities to satisfy this last requirement:

1. The flow is topologically protected. For example, the classical solution of the considered Landau-Ginzburg model can have a vortex structure [148, 149]. From the $U(1)$ SSB, the coset space[3] is $G/H \cong U(1) \cong S^1$. In two-dimensional space, we can then classify topological solitons through the first homotopy group $\pi_1(S^1) \cong \mathbb{Z}$, where the equivalence classes of the topological solitons are labelised by the winding number of the circle at spatial infinity on the internal circle of the coset space [24]. Hence, a flow with a given winding number cannot be continuously deformed to a flow with another winding number, said otherwise, small perturbations in the system cannot change the winding number. So, the velocity – which is related to the winding number – is stable with respect to these small perturbations.

2. The excitation modes of the fluids are such that they cannot be activated by an interaction with an obstacle when the fluid has a velocity below a non-zero critical velocity. Thus, there cannot be energy and momentum exchange between the fluid and the external world. The velocity remains therefore constant. To evaluate this critical velocity, Landau proposed an idealised situation: we consider a one spatial dimensional flow of speed v going around a finite size obstacle of mass M at rest [144, 150]. By boosting the system, we can choose our frame such that the fluid is at rest and that the obstacle is moving at speed v (it is a handwaving argument, we do not keep track of the unphysical sign). In order to slow down, the obstacle should exchange momentum and energy with the fluid, by conservation of these two quantities we have

$$Mv = Mv' + p \,, \tag{B.1.1}$$

$$\frac{1}{2}Mv^2 = \frac{1}{2}Mv'^2 + E \,, \tag{B.1.2}$$

where v' is the final speed of the obstacle, p is the exchanged momentum and E is

[3]Here it represents all the possible values of the non-trivial homogenous vacua connected by the $U(1)$ symmetry.

the exchanged energy. We get

$$v' = v - \frac{p}{M} \ , \tag{B.1.3}$$

$$\frac{1}{2}Mv^2 = \frac{1}{2}\left(Mv^2 - 2vp + \frac{p^2}{M}\right) + E \ . \tag{B.1.4}$$

By considering the mass of the macroscopic obstacle to be large compared to p, E and v, we can neglect $\frac{p^2}{M}$ and obtain

$$E = v\, p \ . \tag{B.1.5}$$

Coming back to the beginning of our reasoning, v is the velocity of the fluid, E is the exchanged energy meaning that it corresponds to the energy of the excited modes of the fluid. These modes are obtained by a perturbative fluctuation around the background solution of the Landau-Ginzburg model and the dispersion relations of these perturbations $\omega(p)$, after quantisation, correspond to the energy of the excited modes. In the language of dispersion relations, there is a momentum and energy exchange between the obstacle and the fluid when the following equality is satisfied:

$$\omega(p) = v\, p \ . \tag{B.1.6}$$

The critical velocity is the minimal velocity for which (B.1.6) can be satisfied. Graphically, in the (ω, p)-plane, it corresponds to the minimal slope of the straight line starting from the origin and intersecting the dispersion curve $\omega(p)$. A cartoon of a standard dispersion relation for a Helium superfluid is represented in Figure B.1 [8]. These kinds of curves are obtained experimentally[4]. We can observe that the critical velocity is indeed non-zero for a superfluid. Hence, when v is smaller than v_{critical}, there is no energy and no momentum exchange which are possible because there are no excitation modes which can "absorb" them. Thus, the fluid keeps its speed constant. Let us notice that for a strictly linear dispersion relation, the critical velocity corresponds to the slope of the dispersion relation. For a quadratic dispersion relation, the critical velocity is zero since the p-axis is tangential at the origin to the quadratic dispersion relation.

A typical example of BEC superfluid satisfying Landau's criteria has been worked out by Bogoliubov. He derived the Landau-Ginzburg model and the associated (low energy) dispersion relations of a gas of weakly repulsive bosons with a positive chemical potential [151]. The positive chemical potential means that it is energetically favourable for the system to increase the particle density despite the repulsive interaction. The presence of the chemical potential allows the BEC to form. Other kinds of superfluid have been obtained through the introduction of a chemical potential [117, 152]. It is therefore not surprising to study Landau-Ginzburg toy models at finite density in the perspective of superfluid description.

[4]Usually, only an analytic effective study can be made and so, analytically we only have access to the low energy part of the curve (the linear part in the cartoon of Figure B.1).

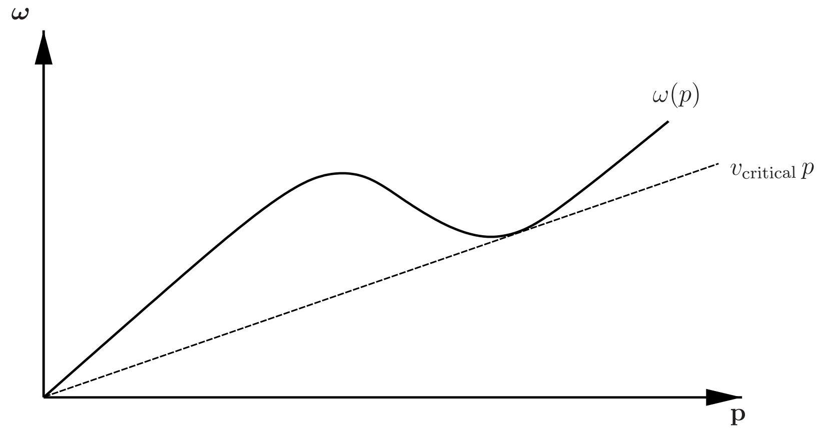

Figure B.1: Typical dispersion relation $\omega(p)$ of a superfluid, cartoon modified from [8].

Summary of BEC superfluids

As a summary of the above discussion, we explain why some of the toy models of Part II are good candidates to be Landau-Ginzburg like models describing low temperature Bose-Einstein condensate superfluids. The studied toy models are theories of classical fields with a $U(1)$ symmetry, therefore, matter transportation is defined through the $U(1)$ conserved charge and through the $U(1)$ Noether current, the fields could then represent BEC density waves. Furthermore, we are working at zero temperature, so the classical solutions are the thermodynamical states (in the Landau-Ginzburg spirit). Since the $U(1)$ symmetry is spontaneously broken, it means that the BEC condensate has formed and we are currently describing the BEC phase. Finally, the dispersion relations are of "linear" type[5] and so, the critical velocities seem to be non-zero – at least from what we can observe at low energy. The fact that a chemical potential is switched on is, as we have seen, not inconsistent to what we can find in the literature specifically dedicated to BEC superfluids. Finally, having a boost symmetry can ease the process of finding time-dependent (moving) solutions.

B.2 Effective theory for superfluids

In the language of Goldstone physics, the Landau-Ginzburg model describing a low temperature BEC superfluid is seen as the fundamental theory. In the BEC superfluid phase, there is a symmetry breaking pattern. We can use the tools provided by Goldstone physics, namely the coset construction, to obtain information on the effective theory. From it, we can extract the low energy excitation modes and, with a similar development we made in Appendix A, derive some thermodynamical properties.

Here, we are going to illustrate the coset construction for spacetime symmetries by computing the effective theory of a zero temperature relativistic superfluid in a 3 + 1-dimensional spacetime with a chemical potential μ_Q associated to a $U(1)$ symmetry. The spontaneous breaking of the $U(1)$ symmetry leads to the formation of a condensate, indi-

[5]We put brackets because the dispersion relations are not always analytic but $\omega \sim \sqrt{q_x^2 + q_y^2}$ can roughly be interpreted as being linear.

cating we are in the superfluid phase. This condensate defines a privileged frame, the one where the condensate is at rest. Hence, the Lorentz boosts are also spontaneously broken. As we have seen in Section 5.3, this condensate with a chemical potential corresponds to a spontaneous symmetry probing, which means that the diagonal direction of $U(1)$ symmetry and of time translation symmetry is left unbroken. The symmetry generators of the system are Q, P_μ, K_i and J_i corresponding respectively to $U(1)$, translations, boosts and rotations. The symmetry breaking pattern is as follow

$$\begin{aligned}
\text{Unbroken} &= \begin{cases} \bar{P}_i \equiv P_i & \text{spatial translations} \\ \bar{P}_0 \equiv P_0 - \mu_Q Q & \text{time translation} \\ J_i & \text{rotations} \end{cases} , \\
\text{Broken} &= \begin{cases} Q & U(1) \text{ symmetry} \\ K_i & \text{Lorentz boosts} \end{cases} .
\end{aligned} \tag{B.2.1}$$

The commutation relations are obtained from the Poincaré algebra and by using the fact that $U(1)$ is an internal symmetry. Hence,

$$[Q, \ldots] = 0 \ , \tag{B.2.2}$$
$$[\bar{P}_\mu, \bar{P}_\nu] = 0 \ , \tag{B.2.3}$$
$$[J_m, J_n] = \epsilon_{mn}{}^k J_k \ , \tag{B.2.4}$$
$$[J_m, \bar{P}_n] = i\epsilon_{mn}{}^k \bar{P}_k \ , \tag{B.2.5}$$
$$[J_i, \bar{P}_0] = 0 \ , \tag{B.2.6}$$
$$[J_m, K_n] = i\epsilon_{mn}{}^k K_k \ , \tag{B.2.7}$$
$$[K_i, \bar{P}_i] = i\eta_{ik} \left(\bar{P}_0 + \mu_Q Q\right) \ , \tag{B.2.8}$$
$$[K_i, \bar{P}_0] = -i\bar{P}_i \ , \tag{B.2.9}$$
$$[K_m, K_n] = -i\epsilon_{mn}{}^k J_k \ , \tag{B.2.10}$$

where η is the matrix of mostly minus Minkowski's metric and ϵ is the Levi-Civita symbol. We can observe that we respect the algebra criteria of Subsection 5.2.2. Indeed, we schematically have $[T, X] \sim X$, $[T, \bar{P}] \sim \bar{P}$ and the $\tilde{H}$ subalgebra is generated by the rotation generators $[J, J] \sim J$ (and as always, the unbroken generators form a subalgebra H: $[T, T] \sim T$). Moreover, $\{\bar{P}_i\}$ and $\{\bar{P}_0\}$ are two distinct multiplets of $\tilde{H}$, the same holds for $\{K_i\}$ and $\{Q\}$ respectively.

The coset parametrisation is chosen to be parametrised as

$$U(x, \pi, \lambda) = e^{ix^\mu \bar{P}_\mu} e^{i\pi(x)Q} e^{i\lambda^l(x)K_l} \ . \tag{B.2.11}$$

We can then compute the Maurer-Cartan 1-form

$$U^{-1}\partial_\mu U = i\Lambda_\mu{}^\alpha P_\alpha + i\left(-\mu_Q \delta^0_\mu + \partial_\mu \pi + \mu_Q \Lambda_\mu{}^0\right) Q + e^{-i\eta^l\lambda_l}\partial_\mu e^{i\eta^l\lambda_l} \ , \tag{B.2.12}$$

where the last term is not developed because we will not need it. Because, as we will see, the boost modes will be entirely expressed in terms of the $U(1)$ mode and from (B.2.10), it will also construct the coset connection that we are not going to use since we will remain at lower order in derivatives (and/or because we will anyway have a general enough expression). To get (B.2.12), we used the $M_{\tau\mu}$ labelling of the Lorentz generators:

$$\left(M_{\tau\mu}\right)_{\beta}{}^{\alpha} = -\delta_{\mu}{}^{\alpha}\eta_{\tau\mu} + \delta_{\tau}{}^{\alpha}\eta_{\mu\beta} \ , \tag{B.2.13}$$

$$\left(K_{l}\right)_{\mu}{}^{\alpha} = \left(M_{0l}\right)_{\mu}{}^{\alpha} = -\delta_{l}{}^{\alpha}\eta_{0\mu} + \delta_{0}{}^{\alpha}\eta_{l\mu} \ , \tag{B.2.14}$$

$$\Lambda_{\mu}{}^{\alpha} = \left(e^{\lambda^{l}K_{l}}\right)_{\mu}{}^{\alpha} \ . \tag{B.2.15}$$

We introduce the field $\psi = \pi - \mu_Q t$, and we get that the Maurer-Cartan coefficient associated to Q is

$$e_{\mu}(x,\psi,\lambda) = \left(\Lambda^{-1}\right)_{\mu}{}^{\alpha}\left(\partial_{\alpha}\psi + \mu_Q\Lambda_{\alpha}{}^{0}\right) = \left(\Lambda^{-1}\right)_{\mu}{}^{\alpha}\partial_{\alpha}\psi + \mu_Q\,\delta_{\mu}{}^{0} \ . \tag{B.2.16}$$

Because $\{\bar{P}_k\}$ and also $\{Q\}$ are multiplets of $\tilde{H}$, we could impose $e_k(x,\psi,\lambda) = 0$ without spoiling the symmetries and without trivialising the theory. This would correspond to IHCs to eliminate the boost modes since we have (B.2.8). In our physical situation, this is a physically consistent constraint. We can show it with the argument (5.2.69). The fundamental field can be considered as a complex scalar field condensing into a VEV of the form $ve^{i\mu_Q t}$ where v is a constant. The realisation of the broken generators on this background is

$$K_i = i\left(t\,\partial_i - x^i\partial_t\right) \ , \tag{B.2.17}$$

$$Q = \frac{1}{\mu_Q}\left(P_0 - \bar{P}_0\right) = \frac{i}{\mu_Q}\partial_t \ , \tag{B.2.18}$$

where $\bar{P}_0 = 0$ on the VEV because it is unbroken. We have that

$$\left(i\pi Q + i\lambda^{l}K_{l}\right)ve^{i\mu_Q t} = 0 \tag{B.2.19}$$

$$\Leftrightarrow \pi = x^{l}\lambda^{l} \ . \tag{B.2.20}$$

We conclude that, for each spacetime modulated boost K_l on the VEV, there is a spacetime modulation of the action of $U(1)$ on the same VEV that reproduces the same fluctuation. Hence, the boost NG candidates are all equivalent to the $U(1)$ NG candidate. We can therefore safely impose the IHC

$$e_k(x,\psi,\lambda) = 0 \ \Leftrightarrow\ \lambda^k = -\frac{\partial_k\psi}{\partial_0\psi} \ . \tag{B.2.21}$$

The building block of our EFT can then be computed. In (B.2.16), by expanding Λ^{-1} based on (B.2.15) till the second order and by using (B.2.21), we have

$$e_0(x,\psi,\lambda) = \left(\Lambda^{-1}\right)_{0}{}^{\alpha}\partial_{\alpha}\psi + \mu_Q \tag{B.2.22}$$

$$= \partial_0\psi - \frac{1}{2}\frac{\partial_l\psi\partial_l\psi}{\partial_0\psi} + \mu_Q \tag{B.2.23}$$

$$= \sqrt{\partial_0\psi\partial_0\psi - \partial_l\psi\partial_l\psi} + \mu_Q \tag{B.2.24}$$

$$= \sqrt{\partial_{\mu}\psi\partial^{\mu}\psi} + \mu_Q \ , \tag{B.2.25}$$

where to reach the third line, we Taylor developed the square root since (B.2.21) indicates $\partial_0\psi$ to be much larger than $\partial_k\psi$ considering λ^k to be a small perturbation.

Because μ_Q is a constant, we can get rid of it. Also, the coset vielbein is a Lorentz matrix, which mean that its determinant is one. Thus, d^4x is an invariant measure. A general EFT at leading order is then

$$S = \int d^4x \, P\left(\sqrt{\partial_\mu\psi\partial^\mu\psi}\right) \,, \tag{B.2.26}$$

where P is a generic polynomial function. This result is of course far from being an original computation, the coset approach for superfluids has already been widely studied in the literature, e.g. [76, 134]. The result (B.2.26) has been first derived through other methods [138] and thus, it corresponds to a consistency check of the coset construction for spacetime symmetries. The generic P function has to be determined either by the fundamental theory or by experiments [153].

Let us notice that the counting rule of Theorem 6 is verified with $n_1 = 1$ (the EFT is relativistic, so the NG mode is type A and type I) and $n_2 = n_3 = n_4 = 0$.

Part II

Spontaneous symmetry breaking of dilatation and spatial translations

Preamble Part II

We have seen in Part I that the spontaneous symmetry breaking mechanism for spacetime symmetries is both conceptually and technically involved. In this part of the dissertation, we build and analyse toy models which respectively display a spontaneous symmetry breaking of dilatation and a concomitant spontaneous breaking of dilatation and spatial translation symmetries. The goal is to put ourselves close to the edge, and even a bit beyond, of the validity range of the already established conjectures and theorems on spacetime symmetry breaking, see Part I. This in order to acquire intuition on how generic these currently known results are. Another aim for these explicit field theory computations is to develop new tools to efficiently study a given symmetry originated spectrum, i.e. from the dispersion relations, to be able to discriminate which NG candidates led to which dispersion relations. The Ward-Takahashi identities will prove to be useful for that.

This field theory toy model approach is mainly inspired from [116]. It should be mentioned that it already exists holographic models which study the spontaneous breaking of dilatation [31, 154, 155] and translations [21, 47]. However, the bottom-up approach of the holographic duality is blind with respect to the exact shape of the field theory it describes through the gravitational theory[6]. In holography, we have a control on the parameters which tune the spontaneous and the explicit symmetry breaking as well as other physical quantities as the chemical potential for example but, we do not have access to the free parameters of the QFT. Hence, we cannot see how a given term in the action influence the final spectrum. Since we do not have the full story, part of the behaviour of the dispersion relations cannot be explained. As we will see in Chapter 8, it is the emergent symmetries in the EFT of the toy model at study which permits to explain some of the specific features of the dispersion relations. It would have been difficult to observe this phenomenon from a pure holographic toy model. So, holography offers a powerful tool to study particular exotic examples of some symmetry breaking patterns but we have to know in advance what we should expect from these holographic analyses such that we understand which correlators to compute (often, we want to recover the Ward-Takahashi identities) and how to interpret the obtained dispersion relations. Field theory toy models are therefore necessary.

The motivation to focus specifically on dilatation symmetry and spatial translation symmetry is that these two symmetries play a major role in condensed matter. In the context of phase transitions, the physics around critical points in the phase diagram is scale invariant, moving away from these points generates, or is induced by, a breaking of dilatation. The study of the breaking of spatial translation has the aim to offer a fundamental description to lattices formation and to the associated phonons (the lattice oscillations). Indeed, the effective field theories of phonons are usually built on an already given lattice structure[7]. It could be fulfilling to provide possible models which can phenomenologically

[6]We have briefly introduced what holography is in Part I Chapter 6, we will come back in more details on this duality in Part III.

[7]In typical condensed matter circumstances, there is a large hierarchy in energy between the physics of a crystal formation/melting and its low-energy excitations. These latter determine the thermodynamic and linear response properties, which can be usually described by low-energy effective theories without

describe phonons by seeing them as emerging from the spontaneous symmetry breaking of spatial translations in a continuous fundamental theory. However, because it is a first step towards toy model building, in this thesis we are going to concentrate on homogeneous symmetry breaking[8] of spatial translation because these symmetry breaking patterns are technically easier to handle. It does not permit to describe lattices since the observables (e.g. energy density) remain homogeneous despite the breaking of translation. More concretely, with a homogeneous symmetry breaking of translation, in the wavevector space, there is no unit cells which are defined. Hence, we are not describing modes evolving on a lattice with discrete translation symmetries (e.g. the Bloch's waves [25]). A brief comment on the non-homogeneous breaking case is made in Appendix C.

In practice these toy models will be build following some guidelines allowing us to write the simplest models as possible with the required features. In usual field theories, the scaling symmetry is not present because the parameters of the theory have non-zero canonical dimensions which introduce specific scales into the system. Hence, a way to make such theories scale invariant (at least at classical level) is to promote these parameters to be compensator fields and to add kinetic terms for the latter to make them dynamical. To break spontaneously the scale symmetry, we need a non-trivial background, since the canonical dimensions (and so, the scaling dimensions at classical level) of the fields are non-zero (we will pay attention to be in the right dimensions of spacetime). To do so, we can start with the standard $U(1)$ Mexican hat theory (cf. Part I Section 3.4) and promote scaling parameters to be compensator fields. The additional $U(1)$ symmetry will as well allow us to switch on a chemical potential to probe situations closer to standard laboratory conditions. The non-trivial background will spontaneously break dilatation and $U(1)$ symmetry, which offers the opportunity to study the interplay between these two symmetries. Being relativistic, the zero chemical potential case is in the exact framework of the conjecture on inverse Higgs constraints (Subsection 5.2.8 in Part I). When the chemical potential is switched on, we slightly evade the hypotheses of the conjecture since time translation will be homogeneously broken. Furthermore, by breaking the dilatation, we moderately run away also from the assumptions of Theorem 6.

To break homogeneously one spatial translation symmetry, we will proceed as in [116]. The idea comes from Q-lattice holography [47] where the original instigator can be traced back to be Coleman for his Q-ball solutions [156]. The guidelines are the following, we need an additional uniform symmetry breaking in order to compensate the translation symmetry breaking and make it to be homogeneous. This additional symmetry will be $U(1)$ (or internal shifts according to the situation) and we will start with a Mexican hat potential to break the latter spontaneously. According to Derrick's theorem [24], we need higher spatial derivative terms to allow for a space modulated solution. Thus, the usual Mexican hat potential will now be built with gradient terms, i.e. the quadratic part of the potential will contain two spatial derivatives and the quartic term will contain four spatial derivatives. Contrary to the "pure field" Mexican hat potential, the gradient one has many possibilities for the terms it involves (the different possible ways to combine four derivatives). We will keep it as simple as possible by considering terms with as many fields as there are derivatives. We will also consider only one quartic term. The results will

considering the dynamical origin of the lattice.

[8] See Subsection 2.5.2 in Part I.

then be generalised by adding complementary allowed quartic terms. The $U(1)$ symmetry will also be used to switch on a chemical potential. Let us observe that the considered symmetry breaking pattern evade us a little bit from the assumptions of the conjecture on IHCs in Subsection 5.2.8 (because we break spatial translation) and we clearly evade the hypotheses of Theorem 6 since we are dealing with a non-relativistic model and we are breaking dilatation symmetry as well as a spatial translation symmetry.

All these toy models are tuned in order to respect the required symmetries but also to allow for non-trivial background in order to spontaneously break these symmetries. It means that the parameters have to be related to each other in specific fashions (for example, in the standard Mexican hat potential, the mass parameter should have an opposite sign compared to the coupling constant). But this is not fine-tuning since these parameters can anyway have a range of continuous possible values. Moreover, these models are tuned as well because we look for the simplest cases in order to enable us to do an analytic analysis. Hence, all the possible terms are not considered. Nevertheless, the sensitivity of our results will be tested by the introduction of additional possible terms (and/or by discussing the influence of the dimension of spacetime and by looking for other solitonic backgrounds).

Finally, we will remain at the classical level. This can be physically relevant in the perspective of Ginzburg- Landau-like models. In particular, our theories have the required characteristics to describe superfluid phases (cf. Appendix B in Part I). Let us mention that we expect our fundamental toy models to be non-renormalisable[9] and therefore, they should be seen as effective field theories.

Let us emphasise that chapters 7, 8 and appendices E, F, G are a minor editing of the papers [157] and [125], which are published works by the author and his collaborators. Appendix C corresponds to carefully selected parts of the publication [45] written by Daniele Musso and by the author of this thesis.

[9]Introducing a compensator field, asking for dilatation symmetry and tuning the coefficients to allow for spontaneous symmetry breaking reduce the (relative) number of available independent bare parameters of the theory (compared to the number of different fields). We then might not have enough bare parameters to absorb the divergences. Furthermore, anomalies might appear and evade us from the (classical) fixed point. By power counting, derivatives tend to reduce the scaling dimension of the coefficients associated to higher derivative terms. So, outside the fixed point, we might have coefficients with negative scaling dimension which would make the theory (superficially) non-renormalisable. Of course, all these ideas should be formally verified by an explicit quantum analysis of the models.

Chapter 7

Gapped dilatons in scale invariant superfluids

In this chapter, slightly edited from the paper [157] published by the author of this dissertation and by his collaborators, we study a paradigmatic model in field theory where a global $U(1)$ and scale symmetries are jointly and spontaneously broken. At zero density the model has a non-compact flat direction, which at finite density needs to be slightly lifted. The resulting low-energy spectrum is composed by a standard gapless $U(1)$ Nambu-Goldstone mode and a light dilaton whose gap is determined by the chemical potential and corrected by the couplings. Even though $U(1)$ and scale symmetries commute, there is a mixing between the $U(1)$ Nambu-Goldstone and the dilaton that is crucial to recover the expected dynamics of a conformal fluid and leads to a phonon propagating at the speed of sound. The results rely solely on an accurate study of the Ward-Takahashi identities and are checked against standard fluctuation computations. We extend our results to a boosted superfluid and comment the relevance of our findings to condensed matter applications.

7.1 The context

Scale invariance plays a special role in many-body and high-energy physics. It underlies the emergence of universality in many instances, such as critical phenomena, Landau-Fermi liquids or cold atoms at unitarity, to name a few. Scale transformations are a symmetry either at very low or very high energies compared to the intrinsic scales. In most cases they represent only an approximate symmetry valid in a restricted regime, requiring typically a certain degree of fine-tuning in the interactions, the thermodynamic variables, the external parameters, or the support of additional symmetries. When a scale invariant system is considered at non-zero particle number or at finite charge density, scale symmetry is spontaneously broken; such breaking is directly relevant to characterise the dynamics of the system mentioned above but it can also be useful to extract properties of large charge operators of a CFT via the state-operator correspondence [158–161].

From Part I of this thesis, we know through Goldstone's theorem that whenever global continuous symmetries are spontaneously broken, one expects to encounter gapless excitations in the form of Nambu-Goldstone (NG) modes. When the breaking involves spacetime symmetries, the counting of modes is involved and not yet well established, yet the presence of a Nambu-Goldstone mode associated to scale invariance, commonly known as dilaton, is still a possibility.

In the pure internal breaking case, we have seen that in the presence of a non-zero charge density μ for a conserved charge Q, gapped modes emerge when the effective

Hamiltonian[1] $\widetilde{H} = H - \mu Q$ does not commute with the broken generators. The gap is fixed by group theory considerations and is proportional to the chemical potential. In general, there can also be additional modes whose gap, although proportional to μ, is not protected by symmetry.

The breaking of scale invariance at finite density presents some similarities with the story above due to the fact that the generator of dilatations D does not commute with the Hamiltonian $[D, H] = iH$.[2] Accordingly, the commutator with the effective Hamiltonian is

$$[\widetilde{H}, D] = -iH \ . \tag{7.1.1}$$

For simplicity let us assume that Q is the generator of an Abelian $U(1)$ symmetry. If this symmetry is spontaneously broken, the ground state is not an eigenstate of Q. However, it must be by definition an eigenstate of $\widetilde{H}$, so time-translations generated by H are spontaneously broken too. In fact, time translations and the $U(1)$ symmetry are broken to a diagonal subgroup and there is just a single NG mode associated to both generators.

Equation (7.1.1) implies that there is a mixing between the $U(1)$ NG and the dilaton, then – even though Q commutes with $\widetilde{H}$ – the state produced by the corresponding charge density J^0 applied to the vacuum at some initial time is not an eigenstate of time evolution. If the dilaton were not dynamical, or if it were integrated out, the mixing implied by (7.1.1) would be manifested in the form of an inverse Higgs constraint. Indeed, $\widetilde{H}$ is interpreted as an unbroken time translation generator, and H which is locked with Q corresponds to a broken generator associated to the $U(1)$ NG candidate. We therefore are in the condition of possibly imposing an IHC which would express the dilaton in terms of the $U(1)$ mode.

Although interesting, one might wonder whether it is sensible to discuss the physics of a dilaton in the first place, since the energy density is in general non-zero at non-zero charge density[3]. In that case, a scale transformation would change the vacuum energy density (as determined by the temporal component of the energy-momentum tensor $T^{\mu\nu}$) by an amount proportional to itself

$$\delta \left\langle T^{00} \right\rangle \sim \left\langle -i[D, T^{00}] \right\rangle = (d+1) \left\langle T^{00} \right\rangle , \tag{7.1.2}$$

where $d+1$ is the number of spacetime dimensions. Both here and henceforth, we assume a relativistic theory, thus there cannot be a NG mode associated to the spontaneous breaking of scale invariance unless $\langle T^{00} \rangle = 0$.[4] This is quite restrictive. Since a gapless mode requires a degeneracy of ground states, the theory needs to have a moduli space of

[1] We are proceeding in analogy to [11, 12, 135, 162].

[2] Nevertheless, it is a conserved charge because $\partial_t D = H$, so its total time-derivative in the Heisenberg picture vanishes.

[3] Generically, we have massless modes when the potential has flat directions around the considered background. If we consider constant solutions, this requirement is recovered at the level of the energy: we should have flat directions for the energy. Since energy has scaling dimensions, the only possibility to have a flat direction in the direction of dilatation is that the energy is zero for a non-trivial constant (scaling) background. Notice that these flat directions associated to dilatation are non-compact. Hence, from the energetic viewpoint, the requirement to be able to speak of a dilaton is to have a zero energy at a constant solution and non-compact flat directions around it.

[4] Note also that the combination of Lorentz invariance (which fixes the expectation value of the energy-momentum tensor to $\langle T^{\mu\nu} \rangle = \Lambda \eta^{\mu\nu}$) and the Ward-Takahashi identity for scale invariance, $\left\langle T^{\mu}_{\ \mu} \right\rangle = 0$, fixes $\left\langle T^{00} \right\rangle = 0$.

vacua in addition to scale invariance: these are flat directions in the potential, supposing we refer to a field theory with a Lagrangian.[5]

Maybe contrary to expectations, the situation at finite density is similar despite the fact that the energy density is non-vanishing. If the ground state is homogeneous and isotropic, the expectation value of the components of the energy-momentum tensor correspond to constant energy density and pressure

$$\left\langle T^{00}\right\rangle = \varepsilon, \qquad \left\langle T^{ij}\right\rangle = p\,\delta^{ij} \; . \tag{7.1.3}$$

Scale invariance implies that the expectation value of the trace of the energy-momentum tensor will vanish $\left\langle T^{\mu}{}_{\mu}\right\rangle = 0$, which fixes the equation of state $\varepsilon = dp$, where d is the number of spatial dimensions. In addition, we have the usual relation between thermodynamic potentials at zero temperature, $\varepsilon + p = \mu\rho$, where $\rho = \left\langle J^0\right\rangle$ is the $U(1)$ charge density. Combining the two, the energy density of the scale invariant theory is $\varepsilon = d/(d+1)\mu\rho$. At finite density the relevant quantity is not the energy density, but the free energy (density) given by the effective Hamiltonian $T^{00} - \mu J^0$. A scale transformation changes the expectation of the effective energy density as follows:

$$\delta\left\langle T^{00} - \mu J^0\right\rangle \sim \left\langle -i[D, T^{00}]\right\rangle - \mu\left\langle -i[D, J^0]\right\rangle = (d+1)\varepsilon - d\mu\rho = 0 \; . \tag{7.1.4}$$

Then, under quite general assumptions, scale transformations do not shift the free energy of a finite density state in a scale invariant theory and it is legitimate to discuss the physics of a dilaton mode, at least at zero temperature[6].

The observation above does not directly imply the existence of a gapless (or gapped) mode. In the absence of a general argument that would allow us to fix the properties of a dilaton mode, we study a concrete model of spontaneous breaking of scale invariance at non-zero density. We restrict the analysis to a relativistic theory in $3+1$ dimensions, and keep the analysis classical. Such simple model is informative because it can be interpreted as an effective action à la Ginzburg-Landau for the order parameter.

The principal highlights of the present study are two. On one side, the characterisation of the dilaton dispersion relation and particularly its gap. This concerns mainly the effects of the chemical potential and its role in defining the effective low-energy spectrum. On the other side, we propose and check a method based uniquely on the study of Ward-Takahashi identities, that in our setup just correspond to classical conservation equations.

The current chapter is structured as follows. Section 7.2 introduces the model at zero density, where we emphasise the need for flat directions in the potential. This condition is relaxed in Section 7.3, where we study the model at non-zero density. In Section 7.4 the analysis is extended to allow for non-zero superfluid velocity. Each section has a subsection dedicated to the analysis of the Ward-Takahashi identities, together with a check of the latter method against standard Lagrangian computations for the fluctuations. We conclude the chapter in Section 7.5 with further comments on the results, their interpretation, their applications and possible extensions.

[5] A related discussion about fine-tuning the cosmological constant to zero in order to have a flat dilatonic direction is contained in [163–166].

[6] Let us notice that from the point of view of the free energy, we are looking at a constant background since it is an eigenstate of $\tilde{H}$.

7.2 The Model

Consider the standard Goldstone model for a global $U(1)$ symmetry in 4 spacetime dimensions

$$S = \int d^4x \, \left[\partial_\mu \psi \partial^\mu \psi^* - \lambda(|\psi|^2 - v^2)^2\right] \, , \tag{7.2.1}$$

where ψ is a scalar complex field charged under the $U(1)$ symmetry, which acts as $\psi \to e^{i\alpha}\psi$, while λ and v represent – respectively – a dimensionless and a dimensionful coupling. Given the presence of a dimensionful coupling, the model (7.2.1) does not enjoy scale invariance. We can nonetheless make it scale invariant if we replace v with a dynamical real scalar field ξ acting as a compensator:

$$S = \int d^4x \, \left[\partial_\mu \psi \partial^\mu \psi^* + \frac{1}{2}\partial_\mu \xi \partial^\mu \xi - \lambda(|\psi|^2 - \xi^2)^2\right] \, . \tag{7.2.2}$$

The equations of motion are given by

$$\partial^2 \psi + 2\lambda(|\psi|^2 - \xi^2)\psi = 0 \, , \qquad \partial^2 \xi - 4\lambda(|\psi|^2 - \xi^2)\xi = 0 \, , \tag{7.2.3}$$

and the generic stationary solution is

$$\xi = v \, , \qquad |\psi|^2 = v^2 \, . \tag{7.2.4}$$

The space of solutions (7.2.4) has two moduli, ξ itself and the phase of ψ. Consider the fluctuations around (7.2.4), parameterised as follows

$$\begin{aligned} \psi &= e^{i\frac{\vartheta}{\sqrt{2}v}}\left(v\, e^{\frac{\tau}{\sqrt{3}v}} + \frac{\rho}{\sqrt{6}}\right) \simeq v + \frac{\tau}{\sqrt{3}} + \frac{\rho}{\sqrt{6}} + i\frac{\vartheta}{\sqrt{2}} \, , \\ \xi &= v\, e^{\frac{\tau}{\sqrt{3}v}} - 2\frac{\rho}{\sqrt{6}} \simeq v + \frac{\tau}{\sqrt{3}} - 2\frac{\rho}{\sqrt{6}} \, , \end{aligned} \tag{7.2.5}$$

where τ, ρ and θ are real. The quadratic action for the fluctuations is given by

$$S_{\text{quad}} = \int d^4x \, \left[\frac{1}{2}\partial_\mu \tau \partial^\mu \tau + \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{1}{2}\partial_\mu \vartheta \partial^\mu \vartheta - 6\lambda v^2 \rho^2\right] \, . \tag{7.2.6}$$

We thus see that ρ gets a mass $12\lambda v^2$ while τ and ϑ are massless. We identify the latter two with the Goldstone bosons for broken scale invariance, the dilaton, and for broken $U(1)$ symmetry, the $U(1)$ NG. The dispersion relations are trivially relativistic, since Lorentz symmetry is preserved.

In order to study the low-energy modes about (7.2.4), one can alternatively rely entirely on symmetry considerations and, specifically, on the Ward-Takahashi identities. As we will show in the next subsection, such symmetry-aware approach permits to obtain the equations of motion for the low-energy modes in a direct way, which is usually more transparent than the standard Lagrangian study of the fluctuations.

7.2.1 Ward-Takahashi identities and low-energy modes

Model (7.2.2) features a conserved $U(1)$ current given by

$$J_\mu = i(\partial_\mu \psi^* \psi - \psi^* \partial_\mu \psi) \ , \qquad \partial^\mu J_\mu = 0 \ , \tag{7.2.7}$$

while the improved energy-momentum tensor is

$$T_{\mu\nu} = 2\partial_{(\mu}\psi^* \partial_{\nu)}\psi + \partial_\mu \xi \partial_\nu \xi - \eta_{\mu\nu}\mathcal{L} + \frac{1}{3}(\eta_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu)\left(\frac{1}{2}\xi^2 + |\psi|^2\right) \ . \tag{7.2.8}$$

This expression satisfies on-shell the following Ward-Takahashi identities [7]

$$T_{[\mu\nu]} = 0 \ , \qquad \partial^\mu T_{\mu\nu} = 0 \ , \qquad T^\mu_{\ \mu} = 0 \ . \tag{7.2.9}$$

We expand around the vacuum (7.2.4) by considering the fluctuation parametrisation (7.2.5). Up to linear order in the fields, the $U(1)$ current is given by

$$J_\mu \simeq \sqrt{2} v \partial_\mu \vartheta \ , \tag{7.2.10}$$

so that its conservation equation gives the equation of motion for the $U(1)$ NG mode

$$0 = \partial^\mu J_\mu \simeq \sqrt{2} v \partial^2 \vartheta \ . \tag{7.2.11}$$

The energy-momentum tensor expanded to linear order is

$$T_{\mu\nu} \simeq \frac{v}{\sqrt{3}}(\eta_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu)\tau \ , \tag{7.2.12}$$

and the trace Ward-Takahashi identity yields the equation of motion for the dilaton

$$0 = T^\mu_{\ \mu} \simeq \sqrt{3} v \partial^2 \tau \ . \tag{7.2.13}$$

From (7.2.11) and (7.2.13) we can observe that we recover the two massless modes of (7.2.6). The Ward-Takahashi computation, however, descends directly from symmetry arguments, being therefore more convenient (and easier) to apply, especially when dealing with models more complicated than (7.2.2). In particular, this approach allows to identify immediately and without ambiguities the nature of each Goldstone boson, simply by associating every (gapless) mode to the Ward-Takahashi identity that yields its equation of motion.

It is important to stress that the model (7.2.2) is fine-tuned. Indeed, (classical) scale invariance dictates that the potential should contain only quartic terms in the scalars, but the fact that the potential is a perfect square constitutes a fine-tuning, specifically considered to the purpose of having a flat direction. The latter is of course a necessary condition for the presence of a low-energy dilaton mode.

The simple argument is as follows. In such a relativistic set-up, scale invariance implies the absence of any reference scale in the (effective) Lagrangian. If scale invariance is to be broken spontaneously by a vacuum expectation value (VEV), then the latter must

[7] The trace Ward-Takahashi identity requires the improvement introduced in (7.2.8).

be arbitrary. Hence this VEV parameterises a non-compact flat direction. Moreover, the absence of any reference scale means that the flat direction must also correspond to a vanishing vacuum energy. The particle which corresponds to moving along this flat direction is the dilaton. We conclude that any effective theory that aims at describing spontaneous scale symmetry breaking (among others), must allow for a non-compact flat direction in its potential.

For instance, if we added a generic term preserving scale invariance but breaking the exchange symmetry between $|\psi|$ and ξ, namely (without loss of generality)

$$V = \lambda(|\psi|^2 - \xi^2)^2 + \lambda'(|\psi|^2)^2 \ , \tag{7.2.14}$$

the equations extremizing the potential would become

$$\lambda\psi(|\psi|^2 - \xi^2) = -\lambda'|\psi|^2\psi \ , \tag{7.2.15}$$
$$\lambda\xi(|\psi|^2 - \xi^2) = 0 \ . \tag{7.2.16}$$

Considering $\lambda' > 0$ for V to be bounded from below, the only solution is $\xi = 0 = \psi$, i.e. the flat direction is completely lifted, even though scale invariance is respected.

7.3 Spontaneous symmetry breaking at finite density

In this section we depart from the Lorentz-invariant set-up discussed above, by introducing a non-zero chemical potential μ for the charge associated to the global $U(1)$ symmetry. As we will see, we will still be able to identify the dilaton and the $U(1)$ NG, though their dispersion relations will be modified in an interesting way.

We start with a scale-invariant theory defined by the action

$$S = \int d^4x \left[\partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - \lambda(|\psi|^2 - \xi^2)^2 - \lambda'(|\psi|^2)^2\right] \ , \tag{7.3.1}$$

whose potential corresponds to the extension already introduced in (7.2.14). We are going to switch on a chemical potential μ for the $U(1)$ symmetry. As it will be seen later on, this chemical potential introduces a quadratic term going as $-\mu^2$ in the potential which tends to give a run away behaviour. This is the reason why we introduced the additional λ' term into (7.3.1), to stabilise the theory. This will be further commented. As discussed before, at finite chemical potential, the ground state is no longer determined by the Hamiltonian H but by the effective Hamiltonian $\widetilde{H} = H - \mu Q$, where Q is the $U(1)$ charge operator. As we will discuss, this modifies the effective potential of the theory and allows the fields to acquire a non-zero value. Notably, one can recover the zero chemical potential symmetry breaking case described by (7.2.2) by means of an appropriate limit for both μ and λ'. The main result of the present section is to show that the dilatonic mode acquires a gap, which depends on μ and λ'.

A nonzero chemical potential can be implemented by extracting a time-dependent phase from the complex field

$$\psi = e^{i\mu t}\phi \ , \qquad \psi^* = e^{-i\mu t}\phi^* \ . \tag{7.3.2}$$

The equations of motion then read

$$\partial^2\phi + 2i\mu\,\partial_0\phi - \mu^2\phi + \partial_{\phi *}V(|\phi|,\xi) = 0\ , \qquad \partial^2\xi + \partial_\xi V(|\phi|,\xi) = 0\ , \tag{7.3.3}$$

where $V(|\phi|,\xi) \equiv V(|\psi|,\xi)$ is given by (7.2.14). Note that these equations can equivalently be obtained introducing (7.3.2) in (7.3.1), identifying a new effective potential $V_\phi(|\phi|,\xi) = V(|\phi|,\xi) - \mu^2|\phi|^2$ and taking the variation with respect to ϕ^*, ξ. Although V_ϕ is not the true potential (indeed, the energy density is $E \sim V(|\phi|,\xi) + \mu^2|\phi|^2$), the extrema of V_ϕ correspond to solutions of the equations of motion of the original action (7.3.1). We will show in the following that V_ϕ determines the ground state for the effective Hamiltonian $\widetilde{H}$.

7.3.1 Effective Hamiltonian and ground state

In order to determine the effective Hamiltonian and the associated ground state we need to find expressions for the $U(1)$ charge Q and Hamiltonian. We will use the usual definitions in terms of the temporal components of the energy-momentum tensor $T_{\mu\nu}$ and $U(1)$ current J_μ

$$H = \int d^3x\, T_{00}\ , \qquad Q = \int d^3x\, J_0\ . \tag{7.3.4}$$

Then, the effective Hamiltonian at finite chemical potential is determined by the temporal component of an effective energy-momentum tensor $t_{\mu\nu}$

$$\widetilde{H} = \int d^3x\,(T_{00} - \mu J_0) \equiv \int d^3x\, t_{00}. \tag{7.3.5}$$

The $U(1)$ current can be written as follows

$$J_0 = 2\mu|\phi|^2 + j_0\ , \qquad J_i = j_i\ , \tag{7.3.6}$$

where

$$j_\mu = i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi)\ . \tag{7.3.7}$$

Similarly, for the energy-momentum tensor[8]

$$T_{00} = \mu J_0 + t_{00}, \tag{7.3.8}$$

$$T_{0i} = T_{i0} = \mu J_i + t_{0i} = \mu j_i + t_{0i}, \tag{7.3.9}$$

$$T_{ij} = t_{ij} + \delta_{ij}\left(\mu J_0 - 2\mu^2|\phi|^2\right) = t_{ij} + \delta_{ij}\mu j_0, \tag{7.3.10}$$

where

$$t_{\mu\nu} = 2\partial_{(\mu}\phi^*\partial_{\nu)}\phi + \partial_\mu\xi\partial_\nu\xi - \eta_{\mu\nu}\mathcal{L}_\phi + \frac{1}{3}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\left(\frac{1}{2}\xi^2 + |\phi|^2\right), \tag{7.3.11}$$

and

$$\mathcal{L}_\phi = \partial_\mu\phi^*\partial^\mu\phi + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - \lambda(|\phi|^2 - \xi^2)^2 - \lambda'(|\phi|^2)^2 + \mu^2|\phi|^2. \tag{7.3.12}$$

[8] The notations are such that the capital letters (T_{00} etc.) refer to the dynamics of (ψ, ξ) (and by extension, of ϕ) dictated by (7.3.1). The low case letters refer instead to the dynamics given by (7.3.12) which is not the Lagrangian for $(\phi,\ \xi)$ but it shares the same potential.

Notice that from (7.3.9) we have that $T_{0i} = T_{i0}$ implying that the Ward-Takahashi identities for boost transformations are satisfied, so that the full Lorentz symmetry is still preserved in the presence of a non-vanishing chemical potential.

The effective potential for $\mathcal{L}_\phi$ is the one we had identified previously in the equations of motion (7.3.3)

$$V_\phi = \lambda(|\phi|^2 - \xi^2)^2 + \lambda'(|\phi|^2)^2 - \mu^2|\phi|^2, \tag{7.3.13}$$

Since t_{00} determines the effective Hamiltonian (7.3.5), we see that the ground state will correspond to the minimum of the effective potential. The effective potential has three extrema[9]

$$\xi = \phi = 0 \; ; \qquad \xi = 0, |\phi|^2 = v^2 = \frac{\mu^2}{2(\lambda + \lambda')} \; ; \qquad \xi^2 = |\phi|^2 = v^2 = \frac{\mu^2}{2\lambda'} \; . \tag{7.3.14}$$

Out of the three extrema (7.3.14), the first two are saddle points and only the last is a minimum, which is the true ground state of the system. Note that for the true minimum to exist, and for V_ϕ to be bounded from below, we need to have $\lambda' > 0$. In other words, we need to lift the flat direction that we had at $\mu = 0$ in order to have a minimum, and symmetry breaking, when $\mu \neq 0$.

We now proceed to investigate the low-energy spectrum around this (degenerate) minimum.

7.3.2 Nambu-Goldstone dynamics from Ward-Takahashi identities

We perturb the fields around the ground state $\xi^2 = |\phi|^2 = v^2 = \frac{\mu^2}{2\lambda'}$. We use the same parameterisation as in (7.2.5), though adapted to the field ϕ

$$\phi = e^{i\frac{\vartheta}{\sqrt{2}v}} \left(v e^{\frac{\tau}{\sqrt{3}v}} + \frac{1}{\sqrt{6}}\rho \right) \; , \qquad \xi = v e^{\frac{\tau}{\sqrt{3}v}} - \frac{2}{\sqrt{6}}\rho \; . \tag{7.3.15}$$

As before, the kinetic terms are diagonal and canonically normalised for ϑ, τ and ρ. We still identify ϑ as the fluctuation of the phase of the condensate and τ as a fluctuation of its magnitude, while ρ corresponds to an orthogonal direction of increasing potential energy. For $\mu = 0$, ϑ and τ are naturally associated to the $U(1)$ NG and dilaton, while ρ enters as a Higgs fluctuation. This simple picture is a bit complicated when $\mu \neq 0$, as the would-be Goldstones undergo some mixing and also a non-vanishing gap for one linear combination. We will study this effect in some approximation here and in more detail in the next section.

When the perturbation (7.3.15) is introduced in the effective potential (7.3.13) and expanded to quadratic order, one finds no term for ϑ and the following mass matrix for (τ, ρ)

$$M = \frac{4v^2}{3} \begin{pmatrix} 2\lambda' & \sqrt{2}\lambda' \\ \sqrt{2}\lambda' & \lambda' + 9\lambda \end{pmatrix} . \tag{7.3.16}$$

In principle both perturbations are massive and mixed, but in the limit $\lambda' \ll \lambda$ in which there is an almost flat direction in the original potential (7.2.14), the mixing becomes

[9] Note that these uniform and static solutions are extrema of the effective potential (7.3.13), but not of the energy (7.3.8).

very small and there is a large hierarchy between the mass of τ, $m_\tau^2 \sim \lambda' v^2 \sim \mu^2$, and the mass of ρ, $m_\rho^2 \sim \lambda v^2$. In the following we will assume that we are in this situation, in which case the Higgs fluctuation ρ can be set to zero in the low energy description to a good approximation[10].

The dynamical equations for the remaining fluctuations can be derived from the Ward-Takahashi identities. When evaluated on-shell the $U(1)$ current should be conserved and the trace of the energy momentum tensor should vanish

$$\partial_\mu J^\mu = 0 \ , \qquad T^\mu_{\ \mu} = 0. \tag{7.3.17}$$

This gives two equations, which is sufficient to determine the dynamics of ϑ and τ. The trace of the energy-momentum tensor, to linear order in the fluctuations, is

$$T^\mu_{\ \mu} \simeq \sqrt{3}v\left(\partial^2\tau + \frac{4}{3}\mu^2\tau - 2\sqrt{\frac{2}{3}}\mu\partial_0\vartheta\right) \ , \tag{7.3.18}$$

whereas the divergence of the current is

$$\partial^\mu J_\mu \simeq \sqrt{2}v\left(\partial^2\vartheta + 2\sqrt{\frac{2}{3}}\mu\partial_0\tau\right) \ . \tag{7.3.19}$$

This translates into the set of coupled equations

$$\begin{aligned} &\partial^2\tau + \frac{4}{3}\mu^2\tau - 2\sqrt{\frac{2}{3}}\mu\partial_0\vartheta \simeq 0 \ , \\ &\partial^2\vartheta + 2\sqrt{\frac{2}{3}}\mu\partial_0\tau \simeq 0 \ . \end{aligned} \tag{7.3.20}$$

As suggested by the general analysis in the introduction, the chemical potential introduces a mixing between the $U(1)$ NG and the dilaton. The equations can be diagonalised using expansions in Fourier modes

$$\tau(x^0,\mathbf{x}) = \int \frac{d\omega d^3q}{(2\pi)^4} e^{-i\omega x^0 + i\mathbf{q}\cdot\mathbf{x}}\,\widetilde{\tau}(\omega,\mathbf{q}) \ , \quad \vartheta(x^0,\mathbf{x}) = \int \frac{d\omega d^3q}{(2\pi)^4} e^{-i\omega x^0 + i\mathbf{q}\cdot\mathbf{x}}\,\widetilde{\vartheta}(\omega,\mathbf{q}) \ . \tag{7.3.21}$$

Expanding at low momentum $q^2/\mu^2 \ll 1$, the equations have solutions when the modes satisfy the dispersion relations

$$\omega^2 \simeq \frac{q^2}{3} \ , \qquad \omega^2 \simeq 4\mu^2 + \frac{5}{3}q^2. \tag{7.3.22}$$

Therefore, there is a gapless mode π and a gapped mode σ, which at low momentum correspond respectively to the combinations

$$\widetilde{\pi} \simeq \widetilde{\vartheta} - i\,\mathrm{sign}(\omega/q)\frac{q}{\sqrt{2}\mu}\widetilde{\tau} \ , \qquad \widetilde{\sigma} \simeq \widetilde{\tau} - i\sqrt{\frac{2}{3}}\,\mathrm{sign}(\omega/\mu)\left(1 + \frac{q^2}{24\mu^2}\right)\widetilde{\vartheta}. \tag{7.3.23}$$

[10]In the EOM, at low energy, the derivative terms will be negligible compared to the massive term (it is explicitly seen with the Fourier transformation). At zero order, only the mass term remain which algebraically set to zero the associated field on-shell.

A few comments are in order. In the first place, the dispersion relation of π in (7.3.22) is such that it moves at the speed of sound as fixed by conformal invariance[11] $c_s^2 = 1/3$, i.e. it can be identified as a conformal superfluid phonon, while σ is the gapped dilaton. This identification is consistent with an effective field theory approach, see e.g. [161]. Note that the mixing is necessary for this to happen, otherwise the phonon would move at the speed of light due to relativistic invariance of the rest of the terms. The second observation is that the gap of σ is fixed by the chemical potential $m_\sigma = 2\mu$, and independent of the couplings λ and λ' in this approximation. This is very reminiscent of the massive Goldstone bosons appearing when internal symmetries are spontaneously broken in the presence of a chemical potential. A last observation is that because of the mixing, it is no longer true that each Ward-Takahashi identity is tied to one specific mode. Indeed reexpressing τ and ϑ in terms of π and σ, one can easily see that both fields appear in both equations (7.3.20).

7.3.3 Exact dispersion relations

The results obtained from the Ward-Takahashi identities are easy to interpret physically but we had to introduce several approximations to derive them, in particular we used the hierarchy between the masses of the Higgs fluctuation and the dilaton to freeze out the first. In order to go beyond this approximation we need to include the Higgs mode in the analysis, whose dynamics is not captured by the Ward-Takahashi identities since it is not a symmetry originated mode. This can be more simply done using the effective Lagrangian.

Consider again the vacuum $\xi^2 = |\phi|^2 = v^2 = \frac{\mu^2}{2\lambda'}$ and the fluctuations (7.3.15) around it. The quadratic Lagrangian for the fluctuations is

$$\begin{aligned}\mathcal{L}_{\text{quad}} &= \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\partial_\mu\vartheta\partial^\mu\vartheta + \frac{1}{2}\partial_\mu\tau\partial^\mu\tau \\ &\quad + 2\sqrt{\frac{2}{3}}\,\mu\,\tau\partial_t\theta + \frac{2}{\sqrt{3}}\,\mu\,\rho\partial_t\theta - \frac{2}{3}\sqrt{2}\mu^2\tau\rho - \frac{2}{3}\mu^2\tau^2 - \mu^2\frac{9\lambda+\lambda'}{3\lambda'}\rho^2\ .\end{aligned} \tag{7.3.24}$$

By going to Fourier space, we get

$$\mathcal{L}_{\text{quad}} = \frac{1}{2}\, y^T(-\omega,-q)\cdot M(\omega,q)\cdot y(\omega,q)\ , \qquad y = (\vartheta,\rho,\tau)\ , \tag{7.3.25}$$

where

$$M = \begin{pmatrix} \omega^2 - q^2 & i\frac{2}{\sqrt{3}}\mu\omega & i\frac{2\sqrt{2}}{\sqrt{3}}\mu\omega \\ -i\frac{2}{\sqrt{3}}\mu\omega & \omega^2 - q^2 - \frac{2(9\lambda+\lambda')}{3\lambda'}\mu^2 & -\frac{2\sqrt{2}}{3}\mu^2 \\ -i\frac{2\sqrt{2}}{\sqrt{3}}\mu\omega & -\frac{2\sqrt{2}}{3}\mu^2 & \omega^2 - q^2 - \frac{4}{3}\mu^2 \end{pmatrix}\ . \tag{7.3.26}$$

[11] A conformal field theory has $T^\mu{}_\mu = 0$. By looking at a free theory, its stress energy tensor corresponds to the one of a perfect fluid. Hence, in $3+1$ dimension, we have $\epsilon = 3p$. The speed of first sound in hydrodynamics is given by $\sqrt{\partial\epsilon/\partial p} = \sqrt{1/3}$. By an explicit computation, it is shown that the slope of the dispersion relation of a $U(1)$ NG mode in a relativistic theory at zero temperature and finite density is $\sqrt{\partial\epsilon/\partial p}$. Since the relativistic massless $U(1)$ NG mode is driven by a free theory, we expect the conformal result $\sqrt{\partial\epsilon/\partial p} = \sqrt{1/3}$ [8]. The coset construction approach is to particularise (B.2.26) to the conformal case. It means to get the stress energy tensor of the EFT (by deriving the action with respect to the metric) and impose $T^\mu{}_\mu = 0$. It will fix up to a global constant the function P. Then, the first speed of sound can be extracted. In 4-dimensional spacetime, it provides $c_s^2 = 1/3$ [167, 168].

Studying the zeros of the determinant of M, one finds one massless mode, the $U(1)$ NG boson, and two gapped modes:

$$\begin{aligned} \omega_1^2|_{q=0} &= 0 \ , \\ \omega_{2,3}^2|_{q=0} &= \frac{3\mu^2}{\lambda'}\left(\lambda+\lambda'\pm\sqrt{\lambda^2-\frac{2}{3}\lambda\lambda'+\lambda'^2}\right) \ . \end{aligned} \tag{7.3.27}$$

Expanding for low momentum q and for $\lambda' \ll \lambda$, we get

$$\omega_1^2 \simeq \frac{1}{3}q^2 \ , \tag{7.3.28}$$

$$\omega_2^2 \simeq 6\mu^2\frac{\lambda}{\lambda'}\left(1+\frac{\lambda'}{3\lambda}\right)+\left(1+\frac{2\lambda'}{9\lambda}\right)q^2 \ , \tag{7.3.29}$$

$$\omega_3^2 \simeq 4\mu^2\left(1-\frac{\lambda'}{3\lambda}\right)+\left(\frac{5}{3}-\frac{2\lambda'}{9\lambda}\right)q^2 \ . \tag{7.3.30}$$

Comparing with the dispersion relations in (7.3.22), we observe that the speed of the phonon is not modified by corrections depending on λ', while the mass of the gapped dilaton is corrected, though mildly. Indeed, contrary to massive NG bosons associated to internal symmetries, the mass of the gapped dilaton is not protected by the symmetry.

For $\lambda' \ll \lambda$, ω_1^2 and ω_3^2 reduce to the dispersion relations obtained in (7.3.22) from the study of the Ward-Takahashi identities, and we have a hierarchy between the two massive modes. Furthermore, in the limit

$$\mu \to 0 \ , \qquad \lambda' \to 0 \qquad \text{with} \qquad \frac{\mu^2}{2\lambda'} \to v^2 \ , \tag{7.3.31}$$

we recover the masses (7.2.6) of the relativistic model (7.2.2)

$$\begin{aligned} \omega_1^2|_{q=0} &= 0 \ , \\ \omega_2^2|_{q=0} &= 12v^2\lambda \ , \\ \omega_3^2|_{q=0} &= 0 \ , \end{aligned} \tag{7.3.32}$$

and ω_3 describes the massless dilaton. This suggests a connection between the corrections to the mass of the gapped dilaton at finite chemical potential and the lack of a flat direction in the potential at zero chemical potential. Indeed, at zero chemical potential, the λ' term lifts the flat directions which means that the dilaton in (7.3.1) is already massive even before switching on the chemical potential. The masses of gapped NGs might be protected only if there are flat directions associated to them, of course this will always be the case for internal symmetries.

7.4 Boosted superfluid

Since the chemical potential breaks Lorentz invariance, it is interesting to study the effect on the NG modes when the superfluid is set on motion relative to the frame determined by the effective Hamiltonian induced by the chemical potential, that one can identify as the "laboratory" frame. We consider again (7.2.2) and introduce both a chemical potential and a superfluid velocity

$$\psi = e^{i\mu_0 u_\mu x^\mu}\phi\ , \qquad \psi^* = e^{-i\mu_0 u_\mu x^\mu}\phi^*. \tag{7.4.1}$$

Where $u_\mu = \gamma(1, -\vec{\beta})$, $\gamma = 1/\sqrt{1-|\beta|^2}$ is a time-like four-velocity $u_\mu u^\mu = +1$. The chemical potential is $\mu = \gamma\mu_0$, and the time direction in the laboratory frame is x^0. The background plane wave (7.4.1) is the same as (7.3.2) seen by a boosted observer, compared to the laboratory frame. Since (7.2.2) is Lorentz invariant, the dispersion relations for the gapless low-energy modes can be obtained by boosting those obtained from (7.3.1) (i.e. the case with just a chemical potential). For the sake of providing an explicit check, we repeat the exercise of computing them directly through the Ward-Takahashi identities and through the perturbative Lagrangian approach.

7.4.1 Effective Hamiltonian and ground state

We proceed in a similar fashion to the case of zero velocity. The Hamiltonian and the charge are still determined by the energy-momentum tensor and the current as in (7.3.4), and the effective Hamiltonian at nonzero chemical potential by (7.3.5). Because of the boost, the expressions for the current and the energy-momentum tensor are slightly modified.

$$\begin{aligned} J_\mu &= 2\mu_0|\phi|^2 u_\mu + j_\mu\ , \\ T_{\mu\nu} &= 2\mu_0^2 u_\mu u_\nu |\phi|^2 + \mu_0(u_\mu j_\nu + u_\nu j_\mu) - \eta_{\mu\nu}\mu_0 u^\alpha j_\alpha + t_{\mu\nu}(\mu_0)\ . \end{aligned} \tag{7.4.2}$$

Where j^μ and $t_{\mu\nu}$ take the same form as before (7.3.7) and (7.3.11), replacing μ by μ_0. Recalling that the chemical potential is $\mu = \mu_0 u_0 = \mu_0\gamma$, the effective Hamiltonian is

$$H - \mu Q = \int d^3x\ (T_{00} - \mu J_0) = \int d^3x\,(t_{00}(\mu_0) - \mu\vec{\beta}\cdot\vec{j}). \tag{7.4.3}$$

Since j_i vanishes for constant ϕ, the extrema of the effective potential are the same as before (7.3.14) replacing μ by the effective chemical potential in the rest frame of the fluid μ_0. The ground state is thus $\xi^2 = |\phi|^2 = v_0^2 = \frac{\mu_0^2}{2\lambda'}$.

7.4.2 Nambu-Goldstone dynamics from Ward-Takahashi identities

We can use the same parametrisation for perturbations of the ground state as in (7.3.15), replacing v by v_0. The same considerations about the mass hierarchy of τ and ρ apply, so in this analysis we will assume $\lambda' \ll \lambda$ and freeze ρ. The dynamics of the low-energy modes are determined by the conservation equations for the current and the energy-momentum tensor. For the boosted superfluid they take the form

$$\begin{aligned} \partial^\mu J_\mu &= 2\mu\left(\partial_0 + \vec{\beta}\cdot\vec{\nabla}\right)|\phi|^2 + \partial^\mu j_\mu, \\ T^\mu{}_\mu &= 2\mu_0^2|\phi|^2 - 2\mu_0 u^\mu j_\mu + t^\mu{}_\mu(\mu_0) = 2\mu_0^2|\phi|^2 - 2\mu(j_0 + \vec{\beta}\cdot\vec{j}) + t^\mu{}_\mu(\mu_0). \end{aligned} \tag{7.4.4}$$

Therefore, we should just replace the terms with a single time-derivative by the material derivative $\mu\partial_0 \to \mu D_0 = \mu(\partial_0 + \vec{\beta}\cdot\vec{\nabla})$ and otherwise change μ by the effective μ_0:

$$\begin{aligned} \partial^\mu J_\mu &\simeq \sqrt{2}v_0\left(\partial^2\vartheta + 2\sqrt{\frac{2}{3}}\mu D_0\tau\right), \\ T^\mu_{\ \mu} &\simeq \sqrt{3}v_0\left(\partial^2\tau + \frac{4}{3}\mu_0^2\tau - 2\sqrt{\frac{2}{3}}\mu D_0\vartheta\right). \end{aligned} \tag{7.4.5}$$

From this, we obtain the equations

$$\begin{aligned} &\partial^2\tau + \frac{4}{3}\mu_0^2\tau - 2\sqrt{\frac{2}{3}}\mu D_0\vartheta \simeq 0 , \\ &\partial^2\vartheta + 2\sqrt{\frac{2}{3}}\mu D_0\tau \simeq 0 . \end{aligned} \tag{7.4.6}$$

The dispersion relation for the gapless mode can be more easily found by noting that $\mu = \gamma\mu_0$ and using comoving coordinates. Taking $\vec{\beta}$ parallel to the x^3 direction, we introduce

$$x^0 = \gamma(x^0_\beta + \beta x^3_\beta) , \qquad x^3 = \gamma(x^3_\beta + \beta x^0_\beta) , \qquad x^1 = x^1_\beta , \qquad x^2 = x^2_\beta. \tag{7.4.7}$$

Then

$$\frac{\partial}{\partial x^0_\beta} = \gamma(\partial_0 + \beta\partial_3) , \qquad \frac{\partial}{\partial x^3_\beta} = \gamma(\partial_3 + \beta\partial_0) , \qquad \partial^2 = \partial^2_\beta . \tag{7.4.8}$$

The equations become

$$\begin{aligned} &\partial^2_\beta\tau + \frac{4}{3}\mu_0^2\tau - 2\sqrt{\frac{2}{3}}\mu_0\partial_{x^0_\beta}\vartheta \simeq 0, \\ &\partial^2_\beta\vartheta + 2\sqrt{\frac{2}{3}}\mu_0\partial_{x^0_\beta}\tau \simeq 0. \end{aligned} \tag{7.4.9}$$

These are the same as before (7.3.20), replacing μ by μ_0. We introduce an expansion of the modes in the rest frame in plane waves

$$\begin{aligned} \tau(x^0_\beta, \mathbf{x}_\beta) &= \int \frac{d\omega_\beta d^3q_\beta}{(2\pi)^4} e^{-i\omega_\beta x^0_\beta + i\mathbf{q}_\beta\cdot\mathbf{x}_\beta}\,\widetilde{\tau}(\omega_\beta, \mathbf{q}_\beta), \\ \vartheta(x^0_\beta, \mathbf{x}_\beta) &= \int \frac{d\omega_\beta d^3q_\beta}{(2\pi)^4} e^{-i\omega_\beta x^0_\beta + i\mathbf{q}_\beta\cdot\mathbf{x}_\beta}\,\widetilde{\vartheta}(\omega_\beta, \mathbf{q}_\beta). \end{aligned} \tag{7.4.10}$$

We recover the expected low momentum dispersion relations in the rest frame

$$\omega^2_\beta \simeq c_s^2 q^2_\beta , \qquad \omega^2_\beta \simeq 4\mu_0^2 + \frac{5}{3}q^2_\beta , \tag{7.4.11}$$

where $c_s^2 = 1/3$ is the speed of sound of the scale invariant theory. These expressions can be translated to frequency and momentum in the laboratory frame using that

$$\omega = \gamma(\omega_\beta + \beta q_{\beta\,3}) , \qquad q_3 = \gamma(q_{\beta\,3} + \beta\omega_\beta) , \qquad q_1 = q_{\beta\,1} , \qquad q_2 = q_{\beta\,1}. \tag{7.4.12}$$

Note that the dispersion relations (7.4.11) are valid for low momentum in the rest frame of the fluid $|q_\beta| \ll |\mu_0|$. For the gapless modes they can be matched with a low momentum expansion in the laboratory frame $|q| \ll |\mu|$, however for the gapped modes this is not possible, as for generic β, $q_3 \sim \omega_\beta \sim \mu$. Therefore, finding the dispersion relations of the gapped modes at low momentum in the laboratory frame requires solving (7.4.6) directly.

We classify the dispersion relations of the gapless modes taking as reference the direction of the superfluid velocity in the laboratory frame. The dispersion relations for the longitudinal modes are

$$\omega_\parallel = \pm\frac{c_s \pm \beta}{1 \pm \beta c_s} q_3 \ , \quad q_1 = q_2 = 0 \ , \tag{7.4.13}$$

while the dispersion relation for the transverse modes is

$$\omega_\perp^2 = c_s^2 \frac{q_1^2 + q_2^2}{\gamma^2(1-\beta^2 c_s^2)} \ , \qquad q_3 = 0 \ . \tag{7.4.14}$$

These expressions agree with the ones obtained by relativistic addition of velocities. Note that for $|\beta| > c_s$ both (positive frequency) longitudinal modes (7.4.13) propagate in the same direction as the superfluid velocity. This is the reason why we expressed linearly the dispersion relations.

For the gapped modes, the low momentum dispersion relations are

$$\begin{aligned} \omega_\parallel &= 2\sqrt{1-\beta^2 c_s^2}\mu - \frac{2}{3}\frac{\beta q_3}{1-\beta^2 c_s^2} + \frac{5+\beta^2 c_s^2}{12\gamma^2(1-\beta^2 c_s^2)^{5/2}}\frac{q_3^2}{\mu} \ , \quad q_1 = q_2 = 0, \\ \omega_\perp^2 &= 4(1-\beta^2 c_s^2)^2 \mu^2 + \frac{5-\beta^2}{3(1-\beta^2 c_s^2)}(q_1^2 + q_2^2) \ , \quad q_3 = 0 \ , \end{aligned} \tag{7.4.15}$$

where again the longitudinal dispersion relation is expressed linearly. The gap is reduced by the superfluid velocity, but in this approximation remains finite even in the limit $\beta \to 1$, where the condensate vanishes (i.e. at fixed μ). Also note that at leading order for momenta in the same direction of the flow, the frequency is reduced.

7.4.3 Exact dispersion relations

We now study the effects of including the Higgs fluctuation ρ, and the corrections for finite λ'/λ. We thus resort to expanding the full Lagrangian. According to (7.4.1), we switch on a chemical potential $\mu = \mu_0\gamma$ and a background wave vector $k_3 = \mu_0\gamma\beta$. The effective potential is now:

$$V = \lambda(|\phi|^2 - \xi^2)^2 + \lambda'|\phi|^4 - (\mu^2 - k_3^2)|\phi|^2 \ , \tag{7.4.16}$$

For stationary solutions, the situation is not very different from the case with just μ, in fact one just needs to replace μ^2 by $(\mu^2 - k_3^2) = \mu_0^2$ in (7.3.13). Therefore, we consider the solution

$$\xi^2 = |\phi|^2 \ , \qquad |\phi|^2 = \frac{\mu_0^2}{2\lambda'} \ . \tag{7.4.17}$$

The fluctuation around (7.4.17) are still given by (7.3.15) where however $v = v_0 = \frac{\mu_0}{\sqrt{2\lambda'}}$. Writing $k_3 = \beta\mu$, the quadratic Lagrangian for the fluctuations is

$$\begin{aligned}\mathcal{L}_{\text{quad}} &= \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\partial_\mu\vartheta\partial^\mu\vartheta + \frac{1}{2}\partial_\mu\tau\partial^\mu\tau \\ &+ 2\sqrt{\frac{2}{3}}\,\mu\,\tau(\partial_t + \beta\partial_3)\theta + \frac{2}{\sqrt{3}}\,\mu\,\rho(\partial_t + \beta\partial_3)\theta \\ &- \frac{2}{3}\sqrt{2}\,\mu_0^2\rho\,\tau - \frac{2}{3}\mu_0^2\tau^2 - \mu_0^2\frac{9\lambda + \lambda'}{3\lambda'}\rho^2 \,.\end{aligned} \tag{7.4.18}$$

In analogy to (7.3.25) and (7.3.26), by going to Fourier space we get the kinetic matrix:

$$\begin{pmatrix} \omega^2 - q^2 & i\frac{2}{\sqrt{3}}\mu\,(\omega - \beta q_3) & i\frac{2\sqrt{2}}{\sqrt{3}}\mu\,(\omega - \beta q_3) \\ -i\frac{2}{\sqrt{3}}\mu\,(\omega - \beta q_3) & \omega^2 - q^2 - \frac{2(9\lambda+\lambda')}{3\lambda'}\mu_0^2 & -\frac{2\sqrt{2}}{3}\mu_0^2 \\ -i\frac{2\sqrt{2}}{\sqrt{3}}\mu\,(\omega - \beta q_3) & -\frac{2\sqrt{2}}{3}\mu_0^2 & \omega^2 - q^2 - \frac{4}{3}\mu_0^2 \end{pmatrix} . \tag{7.4.19}$$

From the determinant of (7.4.19), one can find the exact dispersion relations. First of all, setting the momenta $q = 0$ one finds that there is a massless mode corresponding to the $U(1)$ NG boson and two gapped modes:

$$\begin{aligned}\omega_1^2|_{q=0} &= 0\ , \\ \omega_{2,3}^2|_{q=0} &= \frac{3}{\lambda'}\Bigg[\lambda\mu_0^2 + \lambda'\mu^2(1 - c_s^2\beta^2) \\ &\qquad \pm\sqrt{\lambda^2\mu_0^4 - \frac{2}{3}\lambda\mu_0^2\lambda'\mu^2(1 - c_s^2\beta^2) + \lambda'^2\mu^4(1 - c_s^2\beta^2)^2}\Bigg],\end{aligned} \tag{7.4.20}$$

where $c_s = 1/3$ as before.

Now, expanding at low frequencies and momenta, one can extract analytically the dispersion relation for the $U(1)$ NG mode:

$$\omega_1 = \frac{c_s}{1 - c_s^2\beta^2}\left(2c_s\beta q_3 \pm \sqrt{(1 - \beta^2)^2 q_3^2 + (1 - \beta^2)(1 - c_s^2\beta^2)(q_1^2 + q_2^2)}\right) . \tag{7.4.21}$$

Notice that the above expression is independent of the ratio λ'/λ. Indeed, one can check that in the longitudinal and transverse case, it correctly reproduces the expressions (7.4.13) and (7.4.14), respectively.

For the massive modes, one has to expand the frequencies around the respective gaps. To first order in momenta and in λ'/λ, the dispersion relations for the gapped dilaton are:

$$\begin{aligned}\omega_\parallel^{(\text{gapped})} &= 2\mu\sqrt{1 - c_s^2\beta^2}\left[1 - \frac{\lambda'}{6\lambda}\frac{1 - c_s^2\beta^2}{1 - \beta^2}\right] \\ &\quad - \frac{2\beta}{3(1 - c_s^2\beta^2)}\left[1 - \frac{\lambda'}{3\lambda}\frac{1 - c_s^2\beta^2}{1 - \beta^2}\right] q_3 + \ldots \\ \omega_\perp^{(\text{gapped})} &= 2\mu\sqrt{1 - c_s^2\beta^2}\left[1 - \frac{\lambda'}{6\lambda}\frac{1 - c_s^2\beta^2}{1 - \beta^2}\right] \\ &\quad + \frac{1}{12\sqrt{1 - c_s^2\beta^2}}\left[\frac{5 - \beta^2}{1 - c_s^2\beta^2} + \frac{\lambda'}{6\lambda}\right]\frac{q_1^2 + q_2^2}{\mu} + \ldots\end{aligned} \tag{7.4.22}$$

For $\lambda'/\lambda = 0$ they agree with the dispersion relations obtained from the Ward-Takahashi identities (7.4.15). Again, we observe that the gap receives corrections in λ'/λ, so it is not protected by the symmetry.

Finally, we present the dispersion relations of the Higgs mode, to the same order:

$$\begin{aligned}
\omega_{\parallel}^{(\text{heavy})} &= \sqrt{\frac{6\lambda}{\lambda'}}\sqrt{1-\beta^2}\,\mu\left[1+\frac{\lambda'}{6\lambda}\frac{1-c_s^2\beta^2}{1-\beta^2}\right]-\frac{2\lambda'}{9\lambda}\frac{\beta}{1-\beta^2}\,q_3+\dots \\
\omega_{\perp}^{(\text{heavy})} &= \sqrt{\frac{6\lambda}{\lambda'}}\sqrt{1-\beta^2}\,\mu\left[1+\frac{\lambda'}{6\lambda}\frac{1-c_s^2\beta^2}{1-\beta^2}\right]+\frac{1}{2\sqrt{1-\beta^2}}\sqrt{\frac{\lambda'}{6\lambda}}\,\frac{q_1^2+q_2^2}{\mu}+\dots
\end{aligned} \tag{7.4.23}$$

Note that the $\beta \to 1$ limit at fixed μ seems to be ill-defined. However, this is an artifact of the expansion. For instance, inspecting (7.4.20) and noticing that in this limit $\mu_0 \to 0$, we find that the gap of the dilaton actually goes to zero, while the gap of the Higgs mode stays finite, but scales with μ, which might be slightly non-intuitive (recall that in this limit there is no condensate).

7.5 Summary and discussion

The two main highlights of the present chapter are:

1. The analysis of the low-energy mode associated to spontaneously broken scale symmetry and the characterisation of how its zero-temperature gap depends on the finite density.

2. The description of a generic method based on Ward-Takahashi identities alone to study the low-energy modes of an effective field theory.

The analysis pursued in the present chapter indicates that the spontaneous breaking of the scale symmetry at zero temperature gives rise to a light dilatonic mode whose gap is directly proportional to the chemical potential. This generic expectation can be relevant for the low-energy content of zero-temperature systems where the chemical potential, too, is small with respect to the UV cut-off of the effective description (related to some other physical scale such as an external magnetic field [169]).

Ward-Takahashi identities in quantum field theory are known to be a key tool for the study of symmetries, either when these are preserved or broken, and even when the breaking is explicit [39, 109]. The present chapter stresses that Ward-Takahashi identities alone seem to provide a sufficient framework to study the dispersion relations of the low-energy modes of an effective field theory, providing an alternative – generally simpler – approach than the direct fluctuation analysis at the level of the Lagrangian. The method is generic, but we applied it to the specific study of scale symmetry breaking to the purpose of elucidating the characteristics of the resulting low-energy dynamics. It would be interesting to look for a gapped dilaton in a strongly coupled theory, by means of the holographic duality. This might be achieved by combining holographic models with a gapless dilaton in Poincaré invariant vacua [155, 170] (see also [171]) and models with type II and gapped NG modes [172, 173]. In Part IV, an explicit holographic will be discussed.

We first examined a relativistic field-theory model (7.2.2) in four spacetime dimensions where scale symmetry and a global $U(1)$ symmetry are concomitantly and spontaneously broken. The scale-invariant potential must have two flat directions which translate into two gapless NG modes, the dilaton and the $U(1)$ NG both relativistic and both propagating at the speed of light, (7.2.11) and (7.2.13). These results are consistent with the results of Part I. From the point of view of the inverse Higgs constraints, we indeed have two independent NG candidates since $[D, P_\mu] = iP_\mu$ and $[Q, P_\mu] = 0$. No IHC can be imposed, furthermore, the EFT for the NG modes is relativistic (we did not break Lorentz symmetry). Therefore, there is no canonical conjugation among the NG candidates. We could have indeed predicted the two massless modes. The counting rule of Nielsen and Chadha, even if we are out of the scope of the hypotheses, is holding as well: $n_{\mathrm{I}} = 2$, $n_{\mathrm{II}} = 0$, so $n_{\mathrm{I}} + 2n_{\mathrm{II}} \geqslant n_{\mathrm{BG}}$. Being relativistic, the classification of Theorem 3 systematically consider all the NG modes as being type A and despite being out of the range of the assumptions, it gives the right number of NG modes.

In order to realise the same symmetry-breaking pattern at finite density, the model must be stabilised by means of an extra scale-invariant term (7.3.1) which lifts the dilatonic flat direction without affecting the spontaneous nature of the breaking. The resulting low-energy modes are nonetheless altered: the $U(1)$ NG remains gapless but propagates at the conformal speed of sound, like a superfluid phonon; the dilaton acquires a gap of the order of the chemical potential μ whose value is however not protected by symmetry (7.3.22). The dilaton is light compared to other gapped modes only when the coefficient of the term that lifts the flat direction (7.3.1) is tuned to be very small, in that case we observe that the dilaton gap becomes independent of the couplings. Despite being slightly outside the range of the conjecture of Subsection 5.2.8 because time translation is broken, the counting rule based on IHCs provide the right number of massless modes. This time the IHC (7.1.1) is imposed which reduces the number of massless NG candidates to the number of one. From Goldstone's theorem we know there should be at least one NG mode. Hence, we could have predicted that only one massless mode would appear in the spectrum[12]. It is interesting to notice that related to Theorem 6, there is no category 2 and category 3 massive modes but there is anyway one category 4 massive mode. This is of course not in conflict with Theorem 6 since we are outside the range of the hypotheses (dilatation breaking and a phase transition occurring in the naive $\mu \to 0$ limit). But it could be worth to notice that the dilatation breaking offers a new mechanism to create a category 4 massive mode (and that in this situation, the IHC would eliminate a physical massive mode). If we observe the situation through the eyes of Theorem 4, we can obtain the correct result for the massless modes by considering the fact that dilatation is not a symmetry of the free energy (7.1.1), only the $U(1)$ symmetry is, which explain the single gapless mode. The number of gapped fixed mode is correctly guessed to be zero. And of course, the massive dilaton being an unfixed gapped mode could not have been derived from Theorem 4 this because even in the pure internal case, the theorem is blind with respect to this type of massive NG modes.

Our results at nonzero density belong to the line of research on gapped NG modes [11–13,105,174]. In this context, a natural future perspective is to embed the present analysis into a systematic Maurer-Cartan effective framework, thus assessing its universality and possible generalisations.

[12]The Lorentz boosts are as well broken when the chemical potential is switched on. As for the superfluid case in Appendix B in Part I, there is no additional massless NG modes because boosts and $U(1)$ are locked together through IHCs.

One interesting field of applications is provided by condensed matter. The presence of a wide critical region in the phase diagram is a characteristic shared by many – generally strongly correlated – systems, among which the cuprates. The critical phase is associated to interesting phenomena like bad and strange metallicity and non-Fermi liquid behaviour [175]. It is also often conjectured to lie at the basis of the mechanism for high-temperature superconductivity, see for instance [176].

The defining property of such critical region is the validity of simple scaling rules whose origin, however, can involve complicated and often elusive dynamics related to the presence of a quantum critical point [177–179] or, more generally, to the presence of a scaling sector [176, 180]. This is sometimes referred to as generic scale invariance [181] and can be assumed among the defining symmetries of an effective description.

Another paradigmatic example is provided by cold atoms at unitarity, where there is an emergent non-relativistic conformal symmetry [182], known as Schroedinger symmetry. Gapped NG modes are known to appear when the Hamiltonian is deformed by some of the symmetry generators of the Schroedinger algebra [183]. An extension of our analysis, along the lines of [162], to systems with Galilean rather than Lorentz invariance would be quite interesting.

As another remark, still related to condensed matter but in the context of standard metals, it is relevant to mention that the low-energy modes of our analysis would not destabilise a Landau-Fermi liquid coexisting with them. This can be appreciated by means of an extension of the results of [184] to dilatations, a symmetry which does not commute with either spatial or temporal translations: one can show that the linear interaction term between the fermionic quasiparticles and, respectively, the $U(1)$ NG and the dilaton are both vanishing.

The model adopted here allows for generalisations in which the $U(1)$ symmetry is coupled to translations and the symmetry-breaking preserves only a linear combination of the two [116]. This would realise a spatial version of the pattern described above when $\mu \neq 0$ and only a diagonal component of the product of internal $U(1)$ and time translations was preserved. Such breakings are referred to as homogeneous because they do not yield any spacetime modulation of the energy density,[13] they however provide acoustic phonon modes. It is an interesting open question to study whether and how these phonons would coexist with a dilatonic mode [125]. The relevance of the question is three-fold: it relates to the counting problem of NG modes for spacetime symmetries [10, 127]; it concerns condensed matter systems where a critical scaling and the breaking of translations are intertwined;[14] it provides insight regarding holographic models where scaling and translation symmetries are broken together [21, 186–188].[15] This concomitant breaking between translation and dilatation from the field theory perspective is the subject

[13] An example of inhomogeneous breaking of spatial translations in field theory was studied in [45]. Studying the cubic polynomial in ω^2 associated to (7.3.26), one can exclude the presence of complex solutions. Similarly, a numerical study of (7.4.19) showed no hints of finite-momentum instabilities. Thus, for the purpose of studying translation symmetry breaking, the models introduced in the main text need to be enriched and generalised.

[14] Such as in the region of the phase diagram of cuprates overlapping with the critical, strange metal phase and the so-called pseudo-gap phase [175, 185].

[15] Merging the last two points, there is a current in the literature addressing the critical breaking of translations relevant for the study of strongly-correlated electron systems (specifically strange and bad metals), see for instance [47, 189–195].

of the next chapter.

Chapter 8

Fractons in effective field theories for spontaneously broken translations

This chapter is minorly edited from the original work of the author and his collaborators published in the paper [125].

We study the concomitant breaking of spatial translations and dilatations in Ginzburg-Landau-like models, where the dynamics responsible for the symmetry breaking is described by an effective Mexican hat potential for spatial gradients. We show that there are fractonic modes with either subdimensional propagation or no propagation altogether, namely, immobility. A journalistic overview on fracton physics is provided in Appendix D. The class of effective field theories studied in this chapter encompasses instances of helical superfluids and meta-fluids, where fractons can be connected to an emergent symmetry under higher moment charges, leading in turns to the trivialisation of some elastic coefficients. The introduction of a finite charge density alters the mobility properties of fractons and leads to a competition between the chemical potential and the superfluid velocity in determining the gap of the dilaton. The mobility of fractons can also be altered at zero density upon considering additional higher-derivative terms.

8.1 The context

An interesting aspect of low-energy effective theories is that of emergent symmetries[1]. In the simplest setup of a complex scalar field with a Mexican hat potential, the $U(1)$ symmetry associated to phase rotations of the scalar is spontaneously broken and the low-energy effective theory is described by a massless Nambu-Goldstone boson. At sufficiently low energies, the effective action of the theory is that of a massless scalar field, which not only enjoys the original $U(1)$ symmetry in the form of a constant shift of the Nambu-Goldstone field, but it is also conformal invariant and has an infinite set of conserved higher-spin currents associated to coordinate-dependent shifts of the Nambu-Goldstone field. Neither the conformal nor the coordinate-dependent shifts are symmetries of the full theory, and they are broken when higher-derivative corrections to the low-energy action are considered. Nevertheless, they can leave an imprint in the properties of the low-energy effective theory.

Similar emergent symmetries at low energies appear in other contexts like low-energy excitations of a Fermi surface, independent spin and spatial rotation symmetries in non-relativistic theories, etc. Here we want to explore low-energy effective theories with

[1]Emergent symmetries are symmetries which are not present in the fundamental theory but while integrating out the higher energy modes, i.e. following the RG flow, the parameters of the theory change and this rearrangement might lead to additional symmetries in the EFT. These supplementary symmetries are the emergent symmetries.

emergent symmetries that lead to (gapless) fractonic modes. Fractons are excitations that are able to move only along a restricted set of spatial directions, or are even completely immobile [196, 197]. Gapless fractons appear in a variety of systems such as spin liquids [198–202], dipole-conserving lattice models [203–207] and quantum elasticity [208–219]. Hydrodynamics of fractons has been studied in [220–222]. Models with spontaneous breaking of symmetries have also been studied [223, 224]. At low energies, the models we are going to discuss have similarities to these last, but with the important difference that it is not necessary to impose any exact coordinate-dependent phase rotation or shift symmetry in order to obtain fractonic dispersion relations.[2]

A second aspect that we want to explore is the effect of spontaneous breaking of space-time symmetries in the counting of Nambu-Goldstone bosons. In the previous chapter, we observed that an interesting case is when time translation is broken by a finite chemical potential. If scale invariance is spontaneously broken together with a global symmetry, the dilaton will get a gap proportional to the chemical potential since the generator of dilatations does not commute with the Hamiltonian. Integrating out the gapped modes and keeping only the gapless modes would be equivalent to applying the inverse Higgs constraints.

If, instead of time translations, space translations are homogeneously broken, we expect to find some qualitative similarities. There will be unbroken generators of space translations of the form $\widetilde{P}_i = P_i - k_{ia}Q^a$, where P_i are the ordinary generators of space translations and Q^a are the generators of spontaneously broken global symmetries. The generator of dilatations D does not commute with the unbroken generators $[D, \widetilde{P}_i] = iP_i$, so this might produce a gap for the dilaton dependent on k_{ia}. However, due to the breaking of spatial symmetries, the dispersion relations of the modes can depend in a non-trivial way on the spatial momenta, so the intuition from the chemical potential does not entirely apply to this more complicated situation. A more straightforward intuition could also be that our models in this chapter are non-relativistic, there is no equivalence between time and space in such theories.

What we will do in this part of the dissertation is to examine these questions using a simple $2+1$-dimensional model which can be viewed as a generalisation of the ordinary Mexican hat model for spatial derivative terms of a complex scalar field. Scale invariance is ensured by introducing an additional real scalar acting as a compensating field. It turns out that there is a large space of possible ground states breaking translation invariance, and the effective theory depends crucially on the symmetry realisation of the ground state. We restrict to states leading to homogeneous effective theories. We find emergent symmetries leading to fractonic dispersion relations and a strong dependence on spatial momentum that affects both the dispersion relations and the composition of the modes. We also study generalisations to finite chemical potential and to $3+1$ dimensions for some cases (see Appendix G).

The chapter is organised as follows. In Section 8.2 we introduce the model and discuss its ground states and symmetries. In Section 8.3 we compute the dispersion relations for linearised fluctuations around the ground states and identify the associated Nambu-

[2]Let us emphasise that, as the cited references in this paragraph suggest, the term “fracton” is interpreted in extended ways compared to its original introduction in condensed matter. In this work, we use the term “fracton” in the already customary wider sense, encompassing both discrete and continuum theories.

Goldstone modes. In Section 8.4 we extend our results to finite density and in Section 8.5 we introduce a deformation that removes some of the degeneracy of the simpler model and study its effect on the dispersion relations. In Section 8.6 we try to compare our results with theorems determining the number of gapless Nambu-Goldstone bosons and finally we conclude in Section 8.7 with a discussion of possible physical systems where similar symmetry realisations and exotic Nambu-Goldstone modes might be found. We have collected several technical results and generalisations to 3 + 1 dimensions in the Appendices.

8.2 Translation-breaking Mexican hat model

We consider a 2 + 1-dimensional model with two scalar fields, one complex and one real, governed by the following Lagrangian density

$$\mathcal{L} = \partial_t\Phi^*\partial_t\Phi + A\partial_i\Phi^*\partial_i\Phi + \frac{1}{2}\partial_t\Xi\partial_t\Xi - \frac{1}{2}\partial_i\Xi\partial_i\Xi - B\frac{(\partial_i\Phi^*\partial_i\Phi)^2}{\Xi^6} - H\Xi^6 \ . \tag{8.2.1}$$

The "couplings" A, B and H are all real and positive. The real scalar field Ξ presents a standard kinetic term and plays the role of a "compensator field," introduced in order to ensure scale invariance. The detailed scaling dimensions of the couplings and of the fields (and of the expectation values that we will introduce below) are

$$[A] = [B] = [H] = 0 \ , \quad [\Phi] = [\rho] = \frac{1}{2} \ , \quad [\Xi] = [v] = \frac{1}{2} \ , \quad [k] = [\partial] = 1 \ , \tag{8.2.2}$$

where we considered natural units of energy.

The complex scalar field Φ presents instead a non-standard kinetic term. Specifically, given the positivity of A, the quadratic term with spatial gradients has the opposite sign with respect to the standard relativistic action. This is a key ingredient for triggering the breaking of translation symmetry through configurations with non-vanishing gradients. Intuitively, the "wrong" sign in the gradient term for Φ can be thought in analogy to the negative squared mass term of the standard Mexican hat potential. Thus, we say that (8.2.1) features a "gradient Mexican hat" for Φ [45, 116].

The condition $A > 0$ corresponds to a gradient instability. An example where such a feature can play an important role is in cosmological scalar models for dark energy, where it relates to experimentally testable properties of the cosmological equation of state [225, 226]. In the context of spatially-modulated phases, a similar Lorentz-invariant model has been studied in [227]. The term associated to the B coupling in (8.2.1) has been devised for the symmetry-breaking purposes of the present study. Its real-world realizability as an effective description of the broken phase is not obvious and needs further investigation.

The equations of motion are given by

$$\partial_t^2\Phi + A\,\partial_i^2\Phi - 2B\,\partial_i\left(\frac{\partial_i\Phi}{\Xi^6}\partial_j\Phi^*\partial_j\Phi\right) = 0 \ , \tag{8.2.3}$$

$$\partial_t^2\Xi - \partial_i^2\Xi - \frac{6}{\Xi}\left[B\frac{(\partial_i\Phi^*\partial_i\Phi)^2}{\Xi^6} - H\Xi^6\right] = 0 \ . \tag{8.2.4}$$

8.2.1 Ground states

There is a large class of possible ground states that break spontaneously translation invariance with different patterns, but it is strongly restricted if we demand that the effective action for perturbations around the ground state is homogeneous, leaving just two possible types (see Appendix E). Following the symmetry breaking pattern they exhibit we dub the first "helical superfluid" and the second "meta-fluid". We will discuss both, pointing out the similarities and differences between the two types of ground states.

- **Helical superfluid:**
  We consider the following static ansatz for the solutions

$$\Phi(t,x,y) = \rho\, e^{ikx} \ , \tag{8.2.5}$$
$$\Xi(t,x,y) = v \ , \tag{8.2.6}$$

  where the compensator field is spatially constant, while the complex field configuration corresponds to a plane-wave of amplitude ρ and wave-vector k. All the parameters in the ansatz, ρ, k and v, are assumed to be non-zero, and without loss of generality also real and positive.

  The equations of motion descending from (8.2.1), when considered upon the ansatz (8.2.5) and (8.2.6) reduce to

$$\rho^2 k^2 \left(2Bk^2\rho^2 - Av^6\right) = 0 \ , \tag{8.2.7}$$
$$Bk^4\rho^4 - Hv^{12} = 0 \ . \tag{8.2.8}$$

  We can rewrite (8.2.7) and (8.2.8) as follows:

$$A = 2B\xi \ , \tag{8.2.9}$$
$$H = B\xi^2 \ , \tag{8.2.10}$$

  where we have introduced the dimensionless combination

$$\xi = \frac{k^2\rho^2}{v^6} = \frac{A}{2B} = \sqrt{\frac{H}{B}} \ , \tag{8.2.11}$$

  which parameterises the space of non-trivial static solutions. Positivity (and reality) of ξ implies $AB > 0$ and $HB > 0$. This is indeed satisfied by our choice of taking A, B and H all positive. Consistency of all the relations in (8.2.11) requires the following relation on the Lagrangian coefficients

$$H = \frac{A^2}{4B} \ , \tag{8.2.12}$$

  necessary to have non-trivial solutions, i.e. $v \neq 0$, $k \neq 0$ and $\rho \neq 0$; notice that this amounts to a fine-tuning. The significance of the fine-tuning becomes apparent when looking at the energy density for a static configuration. For (8.2.12) it takes the form of a complete square

$$\varepsilon = B\Xi^{-6}\left(\partial_i\Phi^*\partial_i\Phi - \frac{A}{2B}\Xi^6\right)^2 = Bv^{-6}\left(k^2\rho^2 - \xi v^6\right)^2 . \tag{8.2.13}$$

When evaluated on (8.2.11), the energy density is zero, so these are minimal energy solutions.

It is easy to see that there are two directions of marginal stability; in fact, we are fixing only the combination ξ given in (8.2.11), but the ansatz (8.2.5) and (8.2.6) has three independent parameters. In other words, we have a two-dimensional space of ground states for this particular ansatz.

We will expand for small fluctuations around this ground state using the parameterisation

$$\Phi(t,x,y) = \rho\, e^{ikx}\left[1+\phi(t,x,y)\right] = \rho\, e^{ikx}\left[1+\sigma(t,x,y)+i\chi(t,x,y)\right] \ , \tag{8.2.14}$$

$$\Xi(t,x,y) = v\,\left[1+\tau(t,x,y)\right] \ . \tag{8.2.15}$$

- **Meta-fluid:**
  We still consider model (8.2.1), but with a different background ansatz, namely

$$\Phi = b\,(x+iy) \ , \tag{8.2.16}$$

$$\Xi = v \ , \tag{8.2.17}$$

where b and v are respectively a complex and a real constant. In principle there can be more complicated solutions of this type where one introduces two complex constants b_x and b_y such that $\Phi = b_x x + b_y y$. The main difference with the case we study is that (8.2.16) keeps a combination of spatial and phase rotations of the complex field unbroken, while the more general solution does not. Since we are mainly interested in the breaking of translation symmetry, we keep to the isotropic case in order to avoid further complications.

The equation of motion (8.2.3) for Φ is automatically solved by the ansatz, while that for Ξ, (8.2.4), eventually leads to

$$v^6 = \frac{4B}{A}|b|^2 \ , \tag{8.2.18}$$

where we have used the condition on the coefficients (8.2.12). This guarantees that the energy density of the configuration vanishes, so these are also minimal energy solutions of the same model. We will perform an expansion of small fluctuations around the background

$$\Phi = b(x+iy) + b\left[u_x(t,x,y) + iu_y(t,x,y)\right] , \tag{8.2.19}$$

$$\Xi = v + \tau(x,y,z) \ . \tag{8.2.20}$$

The fluctuations u_i can be interpreted as displacement fields in a solid, in the spirit of the effective actions proposed in [134,228,229]. The reason for adopting the name meta-fluid will become apparent from the study of the elastic response in Section 8.2.3, specifically, from the vanishing of the shear elastic coefficient.

Finally, it is worth mentioning that there is not really an unbroken phase, even for $A < 0$. Indeed, the compensator field Ξ appears in the denominator in the interaction term with coefficient B in (8.2.1), and hence the limit $v \to 0$ is not well-behaved. We henceforth always keep $v > 0$.

8.2.2 Symmetries and Ward-Takahashi identities

The action defined by the Lagrangian (8.2.1) presents the following symmetries

- $U(1)$ symmetry:
$$\Phi \to e^{i\alpha}\Phi \ , \qquad\qquad \Xi \to \Xi \ , \tag{8.2.21}$$

- Complex shift symmetry:
$$\Phi \to \Phi + a_R + i\, a_I \ , \qquad\qquad \Xi \to \Xi \ , \tag{8.2.22}$$

- Dilatation symmetry:
$$x^\mu \to e^{-\eta} x^\mu, \qquad \Phi \to e^{\eta/2}\Phi \ , \qquad\qquad \Xi \to e^{\eta/2}\Xi \ , \tag{8.2.23}$$

Note that the $U(1)$ and complex shift symmetries are not independent, we can always use a $U(1)$ transformation to rotate a complex shift into a real one. The set of independent symmetries we discuss will then be dilatations and either the $U(1)$ and real shift or the complex shifts.

In the helical state the $U(1)$ symmetry is broken together with translations along the x direction to a diagonal combination. Real shifts and dilatations are also broken. The symmetry breaking pattern is quite different in the meta-fluid. In this case, it is the complex shift symmetry the one broken with translations, in both x and y directions, to a diagonal combination. A $U(1)$ symmetry that combines the phase change of the complex field and spatial rotations survives, so this phase is rotationally invariant. As in the previous case, dilatation symmetry is also broken.

The naïve counting of Nambu-Goldstone bosons would give us three gapless modes in each case: the Nambu-Goldstone modes associated to $U(1)$, real shift and dilatations in the helical state and the Nambu-Goldstone modes associated to real and imaginary shifts and dilatations in the meta-fluid state. As we will see the naïve counting fails and a mode becomes gapped. We will return to the issue of this counting in Section 8.6.

In the meta-fluid state the identification of the fluctuations is more or less evident, u^i should be associated to spatial translations/complex shifts while τ should correspond to scale transformations. In the helical state χ is clearly related to $U(1)$ rotations/translations in the x direction, but the role of σ and τ is not so obvious. In order to help with the identification of the modes in the following we will consider the Ward-Takahashi identities associated to symmetries. A more detailed derivation of the identities can be found in Appendix F.

The Ward-Takahashi identities at linear order in the fluctuations return different combinations of the linear equations of motion that we will obtain from the Lagrangian in (8.3.3), (8.3.4) and (8.3.5). The extra information we get from the Ward-Takahashi identities is that, when considering the decoupling or high momentum limit (which we will implement by formally taking $k \to 0$, though of course we keep the premise that $k \neq 0$ for symmetry breaking to happen), one can establish a connection between the fluctuation fields χ, σ and τ and the $U(1)$, the real shift and the dilatation symmetries. Similarly, for the meta fluid one can identify the dispersion relations that correspond to each mode at high momentum. Note that in the perspective where (8.2.1) is already an effective theory, the dispersion relations at high momentum would in principle be modified by putative higher derivative terms not included in the Mexican hat model we are studying (scale invariance would be explicitly broken by such corrections). However, those would come suppressed by a mass scale that we assume to be much larger than any of the scales in the model, so it is still sensible to discuss a high momentum limit.

$U(1)$ symmetry

The $U(1)$ current corresponding to the Lagrangian $\partial_\mu\Phi^*\partial^\mu\Phi$ has the form

$$j_\mu = \frac{i}{2}\left(\Phi\partial_\mu\Phi^* - \Phi^*\partial_\mu\Phi\right) \ . \tag{8.2.24}$$

Thus, for the model (8.2.1) we have

$$J_0 = j_0 \ , \tag{8.2.25}$$

$$J_i = -\left(A - 2B\frac{\partial_j\Phi^*\partial_j\Phi}{\Xi^6}\right) j_i \ , \tag{8.2.26}$$

whose conservation is encoded in the continuity equation[3]

$$\partial^\mu J_\mu = 0 \ . \tag{8.2.27}$$

Expanding to linear order in the fluctuations of the helical superfluid we have

$$\partial_t^2\chi - 2A\partial_x\left[k(\sigma - 3\tau) + \partial_x\chi\right] = 0 \ , \tag{8.2.28}$$

In the $k \to 0$ limit one finds

$$\partial_t^2\chi - 2A\partial_x^2\chi \simeq 0 \ , \tag{8.2.29}$$

indicating that at large frequency and momentum compared to k, the perturbation χ maps to the Nambu-Goldstone boson of the $U(1)$ symmetry, with a dispersion relation

$$\omega^2 \simeq 2Aq_x^2, \qquad q_x \gg k. \tag{8.2.30}$$

This mode has an unusual dispersion relation, and we will refer to it as a 'lineon' since it moves on a line. We will discuss this in more detail when we introduce the connection to fractons.

Shift symmetry

The (complex) shift current corresponding to the Lagrangian $\partial_\mu\Phi^*\partial^\mu\Phi$ is given by

$$j_\mu^{(s)} = \partial_\mu\Phi \ , \tag{8.2.31}$$

[3] In our conventions $\partial^\mu = (\partial_t, -\partial_i)$.

where the s label stands for "shift". The current is linear in the field because the field variation is a constant. Using (8.2.31), the current for the model (8.2.1) can be expressed as follows:

$$J_0^{(s)} = j_0^{(s)} \ , \tag{8.2.32}$$

$$J_i^{(s)} = -\left(A - 2B\frac{\partial_j \Phi^* \partial_j \Phi}{\Xi^6}\right) j_i^{(s)} \ . \tag{8.2.33}$$

The associated continuity equation is

$$\partial^\mu J_\mu^{(s)} = 0 \ , \tag{8.2.34}$$

which, at linear level in the fluctuations of the helical superfluid, gives two linearly independent equations, (8.2.28) and

$$\partial_t^2 \sigma + 2Ak\left[k(\sigma - 3\tau) + \partial_x \chi\right] = 0 \ . \tag{8.2.35}$$

In the $k \to 0$ limit, we get

$$\partial_t^2 \sigma \simeq 0 \ . \tag{8.2.36}$$

Therefore, at large frequencies and momenta compared to k, the perturbation σ can be identified with the Nambu-Goldstone mode of (real) shifts. Again, the unusual dispersion relation $\omega^2 \simeq 0$ will be discussed later on.

For the meta-fluid it is convenient to study only the Ward-Takahashi identity of complex shifts. To linear order in the fluctuations the conservation of the complex shift current produces the equations

$$v\left(\partial_t^2 u_i - A\partial_i \partial_k u_k\right) + 6A\partial_i \tau = 0 \ . \tag{8.2.37}$$

At high momentum, τ is decoupled and the displacements u_i combine in two modes with dispersion relations

$$\omega^2 \simeq 0 \ , \qquad \omega^2 \simeq A(q_x^2 + q_y^2) \ , \tag{8.2.38}$$

where the trivial mode corresponds to the transverse component $\partial_k u_k = 0$ and the propagating mode to the longitudinal component.

Dilatation symmetry

The Lagrangian (8.2.1) being scale invariant ensures us that we can improve the energy-momentum tensor such that the dilatation conserved current takes the form

$$D^\mu = \mathcal{T}^\mu{}_\nu x^\nu - V^\mu \ , \tag{8.2.39}$$

where V^μ is called the virial current. Therefore, the conservation equation

$$\partial_\mu D^\mu = 0 \tag{8.2.40}$$

is equivalent to say that, on-shell, the trace of the improved energy-momentum tensor is zero up to a total divergence of the virial current

$$\mathcal{T}^\mu{}_\mu = \partial^\mu V_\mu \ . \tag{8.2.41}$$

The improved energy-momentum tensor contains the following terms

$$\mathcal{T}^{\mu}{}_{\nu} \equiv T^{\mu}_{\nu} + \left(\Box\delta^{\mu}_{\nu} - \partial^{\mu}\partial_{\nu}\right)\left(\frac{1}{4}|\Phi|^2 + \frac{1}{8}\Xi^2\right) + \frac{A+1}{2}\theta^{\mu}{}_{\nu} \tag{8.2.42}$$

where

$$T^{\mu}_{\nu} = \frac{\delta\mathcal{L}}{\delta\partial_{\mu}X^I}\partial_{\nu}X^I - \delta^{\mu}_{\nu}\mathcal{L} \ , \tag{8.2.43}$$

$$\theta^{i}{}_{j} \equiv \left(\partial_k^2\delta_{ij} - \partial_i\partial_j\right)|\Phi|^2 \ . \tag{8.2.44}$$

By injecting the equations of motion in the trace of (8.2.42), we have that the virial current is given by

$$V_0 = 0 \ , \tag{8.2.45}$$

$$V_i = \frac{B}{\Xi^6}\left(\partial_k\Phi^*\partial_k\Phi\right)\partial_i|\Phi|^2 \ . \tag{8.2.46}$$

We now have an explicit expression for (8.2.41), which at linear order in the fluctuations of the helical superfluid gives

$$v^2(\partial_i^2\tau - \partial_t^2\tau) = 2\rho^2\partial_t^2\sigma + 8k\rho^2 A\left(k(3\tau - \sigma) - \partial_x\chi\right). \tag{8.2.47}$$

In the $k \to 0$ limit (assuming v, ρ can be kept fixed), one obtains

$$\partial_i\partial^i\tau - \partial_t^2\tau = 0 \ , \tag{8.2.48}$$

where we have used (8.2.36). Then, for large values of frequency and momenta, τ can be identified with the Nambu-Goldstone boson for dilatations. In this case the dispersion relation is the usual one for a relativistic massless mode

$$\omega^2 \simeq q_x^2 + q_y^2 \ . \tag{8.2.49}$$

For the meta-fluid, the dilatation Ward-Takahashi identity produces the following equation

$$\left(\partial_i^2 - \partial_t^2\right)\tau = \frac{12A|b|^2}{v^2}\left(6\tau - v\partial_k u^k\right). \tag{8.2.50}$$

At high momentum, the displacement fields decouple and the dilaton has a relativistic dispersion relation (8.2.49) as in the helical superfluid.

8.2.3 Connection to fractons

The unusual dispersion relations we have found in (8.2.30) and (8.2.36) are not just a peculiarity of the decoupling limit but they are also observed at small frequency and momentum, as we will show in the next sections. A possible way to understand their origin is through emergent symmetries of linearised perturbations around the translation-breaking ground states. These symmetries involve coordinate-dependent shifts of the fields similar to those introduced in fracton models [196,197] and are linked to excitations that are immobile or restricted to move in a subdimensional space.

In order to identify the emergent symmetry more easily, we will proceed by studying the quadratic Lagrangian of the perturbations and integrating out the gapped mode. The resulting effective Lagrangian admits a derivative expansion where the symmetry becomes manifest.

Helical superfluid

The action to quadratic order in the fluctuations is

$$\mathcal{L} = \frac{v^2}{2}\partial_\mu \tau \partial^\mu \tau + \rho^2(\partial_t \chi)^2 + \rho^2(\partial_t \sigma)^2 - 2A\rho^2 \left[\partial_x \chi + k(\sigma - 3\tau)\right]^2 . \tag{8.2.51}$$

In this form, we already observe emergent coordinate-dependent shift symmetries, namely

$$\delta\chi = \alpha(y) + \beta(x,y), \quad \delta\sigma = -\frac{1}{k}\partial_x \beta(x,y) + 3\delta + 3\gamma_i x^i, \quad \delta\tau(x,y) = \delta + \gamma_i x^i. \tag{8.2.52}$$

The emergence of these symmetries may explain in part the fractonic behaviour observed from the analysis of the Ward-Takahashi identities. The dilaton τ has the symmetry of a massless field, the symmetry under the transformation β can be used to introduce an arbitrary dependence on both x and y in σ while the remaining transformation α allows an arbitrary dependence on y in χ. In this way, the identification of σ as a fracton and χ as a lineon appears naturally. Note that these are not symmetries of the full action, so it is expected that terms of higher order in the fluctuations will not be invariant under them, however this only affects indirectly the dispersion relations by radiative corrections.

We can diagonalise the mass terms by performing a rotation of the fields

$$\begin{pmatrix} v\tau \\ \sqrt{2}\,\rho\sigma \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \eta \\ \varphi \end{pmatrix} , \tag{8.2.53}$$

by an angle

$$\tan\theta = \frac{v}{3\sqrt{2}\,\rho} . \tag{8.2.54}$$

The action becomes

$$\begin{aligned} \mathcal{L} = {} & \rho^2(\partial_t \chi)^2 + \frac{1}{2}(\partial_t \varphi)^2 + \frac{1}{2}(\partial_t \eta)^2 - \frac{1}{2}[\partial_i(\cos\theta\eta + \sin\theta\varphi)]^2 \\ & - 2A\rho^2 \left[\partial_x \chi - \frac{m_\eta}{2\sqrt{A}\rho}\eta\right]^2 . \end{aligned} \tag{8.2.55}$$

The mass of η equals to

$$m_\eta^2 = \frac{36Ak^2\rho^2}{v^2}\left(1 + \frac{v^2}{18\rho^2}\right) = 2Ak^2\left(1 + 18\frac{\rho^2}{v^2}\right) . \tag{8.2.56}$$

We can now group terms linear in η inside a squared term (after integrating by parts) and subtract the appropriate η-independent terms

$$\begin{aligned} \mathcal{L} = {} & \frac{1}{2}(\partial_t \eta)^2 - \frac{1}{2}\cos\theta^2(\partial_i \eta)^2 - 2A\rho^2\left[\partial_x \chi + \frac{\sin\theta\cos\theta}{2m_\eta\sqrt{A}\rho}\partial_i^2 \varphi - \frac{m_\eta}{2\sqrt{A}\rho}\eta\right]^2 \\ & + \rho^2(\partial_t \chi)^2 + \frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2}\sin^2\theta(\partial_i \varphi)^2 \\ & + \frac{2\sqrt{A}\rho}{m_\eta}\sin\theta\cos\theta\partial_x \chi \partial_i^2 \varphi + \frac{\sin^2\theta\cos^2\theta}{2m_\eta^2}(\partial_i^2 \varphi)^2 . \end{aligned} \tag{8.2.57}$$

Next, we integrate out η expanding its solution in derivatives, starting at lowest order with

$$\eta \simeq \frac{2\sqrt{A}\rho}{m_\eta}\left(\partial_x\chi + \frac{\sin\theta\cos\theta}{2m_\eta\sqrt{A}\rho}\partial_i^2\varphi\right) . \tag{8.2.58}$$

Then, up to the fourth order in derivatives, we get

$$\begin{aligned}\mathcal{L} =&\rho^2(\partial_t\chi)^2 + \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}\sin^2\theta(\partial_i\varphi)^2 + \frac{2\sqrt{A}\rho}{m_\eta}\sin\theta\cos\theta\partial_x\chi\partial_i^2\varphi \\ &+ \frac{2A\rho^2}{m_\eta^2}\left[(\partial_t\partial_x\chi)^2 - \cos^2\theta(\partial_i\partial_x\chi)^2\right] + \frac{\sin^2\theta\cos^2\theta}{2m_\eta^2}(\partial_i^2\varphi)^2 .\end{aligned} \tag{8.2.59}$$

Both χ and φ are gapless and have constant shift symmetries so there are corresponding conserved charges. Furthermore, up to total derivatives in the Lagrangian (8.2.59), χ can be shifted by a term depending on the coordinates

$$\chi \to \chi + a_i x^i + c_{ij}x^i x^j + f(y) . \tag{8.2.60}$$

Symmetry under shifts by linear terms imply that the dipole moment of the charge is conserved, while shifts under quadratic terms imply the conservation of quadrupole and second radial moment. This is characteristic of models of fractons that are immobile. Although higher derivative terms might spoil the shift symmetries, this would only affect the dispersion relations at higher order.

To quadratic order in momentum, the dispersion relations of the gapless fluctuations are

$$\begin{aligned}&\omega_\chi^2 \simeq 0 , \\ &\omega_\varphi^2 \simeq \sin^2\theta\, q_i^2 = \frac{v^2}{18\rho^2 + v^2}q_i^2 .\end{aligned} \tag{8.2.61}$$

Meta-fluid

To linear order, the spatial derivatives of Φ are

$$\partial_i\Phi = b(\delta_i^x + i\delta_i^y) + b(\partial_i u_x + i\partial_i u_y) \qquad \Rightarrow \qquad \partial_i\Phi^*\partial_i\Phi = |b|^2\left(2 + 2\partial_i u_i + (\partial_i u_j)^2\right) . \tag{8.2.62}$$

Then, expanding the action to quadratic order in the fields, we find

$$\mathcal{L} = \frac{1}{2}\partial_\mu\tau\partial^\mu\tau + |b|^2\partial_t u_i\partial_t u_i - A|b|^2\left(\partial_i u_i - \frac{6}{v}\tau\right)^2 . \tag{8.2.63}$$

We can also write this action in the following way

$$\mathcal{L} = \frac{1}{2}\partial_\mu\tau\partial^\mu\tau - \frac{1}{2}m_\tau^2\tau^2 + |b|^2\left(\partial_t u_i\partial_t u_i - C^{ijkl}\partial_i u_j\partial_k u_l + \frac{12K}{v}\tau\partial_i u_i\right) . \tag{8.2.64}$$

The coefficients C^{ijkl} are the components of the elasticity tensor, that in this case only has a bulk component

$$C^{ijkl} = K\delta^{ij}\delta^{kl}, \qquad K = A , \tag{8.2.65}$$

with K the bulk modulus, which also enters in the coupling between the dilaton and the bulk strain. The mass of the dilaton is

$$m_\tau^2 = 72K\frac{|b|^2}{v^2} \ . \tag{8.2.66}$$

A large v limit would make the dilaton massless and decoupled from the elastic theory at low energies, this latter remaining otherwise unaffected. Roughly, if there is a big hierarchy between the spontaneous breaking of dilatations and that of translations, one does not expect the low-energy elastic theory to be sensitive to the dilaton physics.

Since the shear modulus vanishes, any deformation with $\tau = 0$, $\partial_i u_i = 0$ has zero energy. Then, the elastic part describes a fluid or a meta-fluid. Note that constant changes in volume can be compensated with a shift of the dilaton, so scale invariance is preserved in this sense. This implies that there is a zero mode associated to the dilatation symmetry and a massive mode which corresponds to the combination squared in (8.2.63).

We can separate the gapped and gapless modes by doing the shift

$$\tau \to \sigma + \frac{v}{6}\partial_i u_i \ . \tag{8.2.67}$$

Then

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}m_\tau^2\sigma^2 + \frac{v}{6}\partial_\mu\sigma\partial^\mu\partial_i u_i + |b|^2\partial_t u_i\partial_t u_i + \frac{v^2}{72}\partial_\mu\partial_i u_i\partial^\mu\partial_j u_j \ . \tag{8.2.68}$$

We can further complete the square

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}m_\tau^2\left(\sigma + \frac{v}{6m_\tau^2}\partial_\mu\partial^\mu\partial_i u_i\right)^2 + |b|^2\partial_t u_i\partial_t u_i \\ + \frac{v^2}{72}\partial_\mu\partial_i u_i\partial^\mu\partial_j u_j + \frac{v^2}{72m_\tau^2}\left(\partial_\mu\partial^\mu\partial_i u_i\right)^2 \ .\end{aligned} \tag{8.2.69}$$

Integrating out σ implies solving order by order in derivatives with the leading term

$$\sigma \simeq -\frac{v}{6m_\tau^2}\partial_\mu\partial^\mu\partial_i u_i \ . \tag{8.2.70}$$

To sixth order in derivatives in the action, we are left with

$$\mathcal{L} \simeq |b|^2\partial_t u_i\partial_t u_i + \frac{v^2}{72}\partial_\mu\partial_i u_i\partial^\mu\partial_j u_j + \frac{v^2}{72m_\tau^2}\left(\partial_\mu\partial^\mu\partial_i u_i\right)^2 \ . \tag{8.2.71}$$

In this form, we also observe that the shear strain has zero energy and that the action is symmetric under constant changes of the bulk strain. This implies that there are linear and quadratic shift symmetries

$$\delta u_i = a_i + b_{ij}x^j + c_{ijk}x^j x^k \ . \tag{8.2.72}$$

Then, we have that, not only the charges associated to the constant shifts, but also their dipole and second moments are conserved, this is characteristic of fractonic models. The larger symmetry associated to arbitrary shear and rotational strains corresponds to transverse transformations

$$\delta u_i = \epsilon_{ik}\partial_k\omega(\boldsymbol{x}) + \left(\partial_i\partial_j - \delta_{ij}\partial_k^2\right)V^j(\boldsymbol{x}) \ . \tag{8.2.73}$$

8.3 Dispersion relations

By a standard perturbation analysis of the model (8.2.1) around the respective backgrounds of the helical superfluid and the meta-fluid, we will compute the dispersion relations of the fluctuations. This will support and refine some of the results and interpretations which we already derived in the preceding sections.

8.3.1 Helical superfluid

As stated in Subsection 8.2.1, we perform a fluctuation of the model (8.2.1) around a plane-wave background where we consider the parameterisation

$$\Phi(t,x,y) = \rho\, e^{ikx}\left[1+\phi(t,x,y)\right] = \rho\, e^{ikx}\left[1+\sigma(t,x,y)+i\chi(t,x,y)\right] \; , \tag{8.3.1}$$

$$\Xi(t,x,y) = v\,\left[1+\tau(t,x,y)\right] \; . \tag{8.3.2}$$

The equations of motion at linear order for the fluctuations are[4]

$$2A(k+i\partial_x)\left[k(\sigma-3\tau)+\partial_x\chi\right]+\partial_t^2(\sigma-i\chi)=0 \; , \tag{8.3.3}$$

$$2A(k-i\partial_x)\left[k(\sigma-3\tau)+\partial_x\chi\right]+\partial_t^2(\sigma+i\chi)=0 \; , \tag{8.3.4}$$

$$12Ak\rho^2\left[k(\sigma-3\tau)+\partial_x\chi\right]-v^2\left(-\partial_x^2-\partial_y^2+\partial_t^2\right)\tau=0 \; . \tag{8.3.5}$$

Going to Fourier space, we obtain a homogeneous algebraic system determined by the kinetic matrix:

$$M=\rho^2\begin{pmatrix} \omega^2-2Ak^2 & -2iAkq_x & 6Ak^2 \\ 2iAkq_x & \omega^2-2Aq_x^2 & -6iAkq_x \\ 6Ak^2 & 6iAkq_x & \frac{1}{2}\left(\omega^2-q_x^2-q_y^2\right)\frac{v^2}{\rho^2}-18Ak^2 \end{pmatrix} , \tag{8.3.6}$$

where the first row corresponds to σ, the second one to χ and the third one to τ.

In order to have non-trivial solutions for the fluctuations, the determinant for the kinetic matrix should vanish,

$$\det(M)=\frac{\omega^2\rho^4}{2}\left\{v^2\left(\omega^2-q_x^2-q_y^2\right)\left[\omega^2-2A\left(k^2+q_x^2\right)\right]-36Ak^2\omega^2\rho^2\right\}=0 \, . \tag{8.3.7}$$

This leads to a set of conditions for the frequency and momenta that determine the dispersion relations. The fluctuation determinant (8.3.7) has a ω^2 factor, producing a gapless mode whose dispersion relation is trivial, i.e. identically zero, $\omega=0$. Apart from such a trivial mode, the spectrum features a gapless and a gapped mode

$$m_2^2 = 0 \; , \tag{8.3.8}$$

$$m_3^2 = 2Ak^2\left(1+18\frac{\rho^2}{v^2}\right) \; . \tag{8.3.9}$$

[4] We remind the reader that the parameters k, ρ and v are not independent, but related by (8.2.11). We will refrain from expressing one of the parameters in terms of the others, but instead aim at writing the various expressions in their simplest form, here and in the rest of the chapter.

Let us remark that the mass (8.3.9) agrees with the mass coming from the effective field theory analysis (8.2.56), $m_3 \equiv m_\eta$.

Proceeding to compute the dispersion relations, we obtain:

$$\omega_1^2 = 0 \ , \tag{8.3.10}$$

$$\omega_{2,3}^2 = \frac{1}{2}\left[2Aq_x^2 + q_x^2 + q_y^2 + m_\eta^2 \mp \sqrt{\Delta}\right] \ , \tag{8.3.11}$$

with

$$\Delta = \left\{2Aq_x^2 + q_x^2 + q_y^2 + m_\eta^2\right\}^2 - 8A\left(k^2 + q_x^2\right)\left(q_x^2 + q_y^2\right) \ . \tag{8.3.12}$$

First note that $\Delta^2 \leqslant (2Aq_x^2 + q_x^2 + q_y^2 + m_\eta^2)^2$ so that, if $\Delta \geqslant 0$, $\omega_{2,3}^2 \geqslant 0$ leads to a real dispersion relation. Indeed one can see this is the case for all momenta, by using the expression for m_η, which is given by (8.3.9), we see that $(2Aq_x^2 + q_x^2 + q_y^2 + m_\eta^2)^2 \geqslant (2A(k^2 + q_x^2) + q_x^2 + q_y^2)^2$. From this, it follows that $\Delta \geqslant \left(2A(k^2 + q_x^2) - (q_x^2 + q_y^2)\right)^2 \geqslant 0$.

The expansion at low momenta provides

$$\omega_1^2 = 0 \ , \tag{8.3.13}$$

$$\omega_2^2 = \frac{v^2}{18\rho^2 + v^2}\left(q_x^2 + q_y^2\right) + \mathcal{O}(q^4) \ , \tag{8.3.14}$$

$$\omega_3^2 = m_\eta^2 + 2Aq_x^2 + \frac{18\rho^2}{18\rho^2 + v^2}\left(q_x^2 + q_y^2\right) + \mathcal{O}\left(q^4\right) \ . \tag{8.3.15}$$

To recover the results obtained from the Ward-Takahashi identities of Section 8.2.2, and in particular the fractonic behaviour, we consider the large-momentum behaviour of (8.3.11). Of course, the exact trivial mode will remain trivial in any q limit. In order to take the large-momentum limit, we simply take $q_x, q_y \gg m_\eta$. Then, we find

$$\omega_2^2 \simeq \begin{cases} q_x^2 + q_y^2 & \text{if} \quad (2A-1)q_x^2 - q_y^2 > 0 \\ 2Aq_x^2 & \text{if} \quad (2A-1)q_x^2 - q_y^2 < 0 \end{cases} \ , \tag{8.3.16}$$

and

$$\omega_3^2 \simeq \begin{cases} 2Aq_x^2 & \text{if} \quad (2A-1)q_x^2 - q_y^2 > 0 \\ q_x^2 + q_y^2 & \text{if} \quad (2A-1)q_x^2 - q_y^2 < 0 \end{cases} \ . \tag{8.3.17}$$

Thus, ω_2 and ω_3 swap their roles depending on the sign of $(2A-1)q_x^2 - q_y^2$, which in general depends on the direction in the momentum plane. Note that if $A \leqslant 1/2$ this quantity is always negative, so in that case ω_2 and ω_3 do not change with direction.

8.3.2 Identification of the modes

In order to study the Nambu-Goldstone nature of (8.3.13), (8.3.14) and (8.3.15), we need to determine how they relate to a local spacetime modulation of the various symmetry-originated zero modes. The study of the Ward-Takahashi identities gave us already a glance into such associations at large momentum. At low momentum, instead, one can get useful information from the effective action, which we have already derived to establish the connection to fractons. Accordingly, we will identify χ with the $U(1)$ Nambu-Goldstone mode and τ and σ with the dilaton and shifton respectively.

1. Low momentum:
   Comparing (8.3.13), (8.3.14) and (8.3.15) with (8.2.61) and (8.2.56), we can make the following identifications

   i Trivial mode: ω_1, mostly χ.

   ii Gapless mode: ω_2, mixture of τ and σ. According to (8.2.53) and (8.2.54) if $v \gg \rho$ ($k/\rho^2 \gg 1$) it would be mostly τ and if $v \ll \rho$ ($k/\rho^2 \ll 1$) it would be mostly σ.

   iii Gapped mode: ω_3, mixture of τ and σ orthogonal to the gapless mode.

2. High momentum:
   Comparing (8.2.36), (8.2.30) and (8.2.48) with (8.3.16) and (8.3.17), we can identify

   i Trivial mode: ω_1 mostly σ.

   ii Lineon: ω_3 (for $(2A-1)q_x^2 - q_y^2 > 0$) or ω_2 (for $(2A-1)q_x^2 - q_y^2 < 0$), mostly χ.

   iii Relativistic mode: ω_2 (for $(2A-1)q_x^2 - q_y^2 > 0$) or ω_3 (for $(2A-1)q_x^2 - q_y^2 < 0$), mostly τ .

These identifications unveil a strong change in the nature of the modes as a function of momentum, this being a reflection of the mixing induced by the breaking of translation symmetry. Indeed, the breaking of translation requires higher derivative terms which lead to a non-trivial dependency in momentum in the kinetic matrix, and in particular in the non-diagonal terms. Furthermore, the breaking of translation following one spatial direction introduce anisotropy in the kinetic matrix. For $A > 1/2$, the transmutation does not only occur in the transition from low to large momentum but also depending on the direction in the momentum plane.

We would like now to pause a moment to comment on the relation with the non-scale invariant model of [116], where precisely the helical ground state was considered. The model is basically the same as the present one, where however the fluctuation τ is frozen. The spectrum is easily obtained from the determinant of the upper-left 2-by-2 submatrix of (8.3.6). It consists of a trivial fractonic mode, and a gapped lineon. Hence, we see that the compensator field enforcing scale invariance is a highly non-trivial addition to the model, yielding non-trivial mixing among the modes, and their identification.

8.3.3 Meta-fluid

In order to get the dispersion relations for the meta-fluid, we proceed in a similar fashion as for the helical superfluid. By referring to Section 8.2.1, we study the fluctuations of the model (8.2.1) around the background given in (8.2.16) and (8.2.17) with the following parametrisation:

$$\Phi(t,x,y) = b\Big[x + iy + u_x(t,x,y) + iu_y(t,x,y)\Big] , \tag{8.3.18}$$

$$\Xi(t,x,y) = v + \tau(t,x,y) . \tag{8.3.19}$$

At first order in the fluctuations, the equations of motion are given by

$$\partial_t^2 u_x - A\partial_x \left(\partial_x u_x - \frac{6}{v}\tau\right) = 0 \ , \tag{8.3.20}$$

$$\partial_t^2 u_y - A\partial_y \left(\partial_y u_y - \frac{6}{v}\tau\right) = 0 \ , \tag{8.3.21}$$

$$\partial_t^2 \tau - \partial_i^2 \tau + 12A\frac{|b|^2}{v}\left(\frac{6}{v}\tau - \partial_i u_i\right) = 0 \ . \tag{8.3.22}$$

The quadratic fluctuation matrix in Fourier space is

$$M = |b|^2 \begin{pmatrix} \omega^2 - Aq_x^2 & -Aq_xq_y & -6i\frac{A}{v}q_x \\ -Aq_xq_y & \omega^2 - Aq_y^2 & -6i\frac{A}{v}q_y \\ 6i\frac{A}{v}q_x & 6i\frac{A}{v}q_y & \frac{1}{2|b|^2}\left(\omega^2 - q_x^2 - q_y^2\right) - 36\frac{A}{v^2} \end{pmatrix} \ , \tag{8.3.23}$$

where the two first lines correspond to u_x and u_y while the last line is associated to τ. The determinant of this matrix is given by

$$\det(M) = \frac{|b|^4\omega^2}{2}\left[(\omega^2 - q^2)(\omega^2 - Aq^2) - 72\frac{A|b|^2}{v^2}\omega^2\right] \ , \tag{8.3.24}$$

where we have used $q^2 = q_x^2 + q_y^2$. Take notice that the above expression is completely isotropic. The dispersion relations, given by the roots of the determinant, are the following

$$\omega_1^2 = 0 \ , \tag{8.3.25}$$

$$\omega_{2,3}^2 = \frac{1}{2}\left\{(1+A)q^2 + m_\tau^2 \mp \sqrt{[(1+A)q^2 + m_\tau^2]^2 - 4Aq^4}\right\} \ , \tag{8.3.26}$$

where we have used (8.2.66). Again, one can see that the argument of the square root is always strictly positive, and $\omega_2^2 \geqslant 0$.

At low momenta, we find the following dispersion relations:

$$\omega_1^2 = 0 \ , \tag{8.3.27}$$

$$\omega_2^2 = \frac{A}{m_\tau^2}q^4 + \mathcal{O}\left(q^6\right) \ , \tag{8.3.28}$$

$$\omega_3^2 = m_\tau^2 + (1+A)q^2 + \mathcal{O}\left(q^4\right) \ . \tag{8.3.29}$$

We obtain a similar mass spectrum as for the helical superfluid: two massless modes, one being exactly trivial, and a gapped mode. However, we have some qualitative differences in the dispersion relations. Indeed, the non-trivial massless mode has a quadratic dispersion relation while in the helical background it has a linear behaviour. This can be traced back to the effective theory for gapless modes (8.2.71), where there are terms with two time-derivatives and four spatial derivatives but there are no terms with just two spatial derivatives.

An additional qualitative difference with the helical case is that all the dispersion relations are isotropic in the meta-fluid case. As it was mentioned in Section 8.2.2, the meta-fluid background preserves an effective rotation symmetry from the diagonal breaking of $U(1)$ and spatial rotations. This is actually due to the particular ansatz (8.2.16) for the solution. It is possible to choose a more general solution which will lead to anisotropies in the determinant of the kinetic matrix and hence in the spectrum. The other features of the latter would however be unchanged. Hence, we prefer to deal with the isotropic meta-fluid, for a better clarity of the resulting expressions. On the other hand, in the plane wave background spatial rotations are necessarily broken by the choice of a preferred direction in the solution.

At small momentum, the massive mode (8.3.29) is associated to the fluctuation τ, as it can be observed in the diagonalisation of (8.3.23) in the $q_i \to 0$ limit. We notice that the association to τ matches the effective study of Section 8.2.3 and that we recover the mass (8.2.66).

For large momenta, the non-trivial modes have the following dispersion relations

$$\omega_{2,3}^2 \simeq \frac{q^2}{2}\Big[1 + A \mp |1-A|\Big] , \tag{8.3.30}$$

so that when $A \leqslant 1$ we have that $\omega_2 \simeq \sqrt{A}|q|$ and $\omega_3 \simeq |q|$ (with $|q| = \sqrt{q^2}$), while when $A > 1$ we have the opposite, $\omega_2 \simeq |q|$ and $\omega_3 \simeq \sqrt{A}|q|$. By looking at the kinetic matrix in the large q limit, i.e. neglecting all non-leading terms in ω or q, we find that the mode with $\omega \simeq |q|$ is always aligned with τ, while the mode with $\omega \simeq \sqrt{A}|q|$ is aligned with the longitudinal combination of the u_i (the other being always the trivial immobile mode). Hence, when $A > 1$, the modes ω_2 and ω_3 switch nature when going from low to high momenta. On the other hand, note that for the meta-fluid, the trivial mode is always the transverse part of u_i, for all momenta.

Again, let us comment briefly on the possibility to have a meta-fluid ground state in the model of [116], where τ is frozen. In this case, we can easily see that there is no scale in the spectrum. Eventually, the spectrum consists of a trivial fractonic mode, and a gapless isotropic mode with linear dispersion relations. Hence, we notice that also in this case, the addition of the compensator field changes quite radically the spectrum, due to non-trivial mixing.

8.4 Finite density

As it has already been established for internal compact symmetries, working at finite density modifies the spectrum of Nambu-Goldstone modes associated to spontaneous symmetry breaking [11–13, 157]. We would like to probe how these results extend in our specific spacetime symmetry breaking pattern. To do so, we switch on a chemical potential μ for the $U(1)$ symmetry of the theory (8.2.1) in the framework of the helical superfluid background. We do not extend the analysis to the meta-fluid because in the presence of a chemical potential the effective action is no longer homogeneous.

The chemical potential introduces a new term in the effective potential $\sim -\mu^2\rho^2$, that makes it unbounded from below and would produce a run away behaviour. Something similar occurs in the model of the previous chapter for the simultaneous breaking of scale invariance and an internal symmetry. In that simpler case the issue was solved by introducing a small deformation of the model that lifts the space of minimal energy states at zero chemical potential and stabilises it at finite chemical potential. The results at zero density can be recovered by simultaneously sending the chemical potential and the deformation to zero. Following the same reasoning we introduce a new term with coupling λ^2 that preserves the $U(1)$ and dilatation symmetries

$$\mathcal{L} = \partial_t\Phi^*\partial_t\Phi + A\partial_i\Phi^*\partial_i\Phi + \frac{1}{2}\partial_t\Xi\partial_t\Xi - \frac{1}{2}\partial_i\Xi\partial_i\Xi - B\frac{(\partial_i\Phi^*\partial_i\Phi)^2}{\Xi^6} - H\Xi^6 - \lambda^2(\Phi^*\Phi)^3 \ . \tag{8.4.1}$$

The additional term breaks explicitly the shift symmetry, and would introduce an explicit dependence on the coordinates in the effective action of the meta-fluid.

The equations of motion are given by

$$\partial_t^2\Phi + A\,\partial_i^2\Phi - 2B\,\partial_i\left(\frac{\partial_i\Phi}{\Xi^6}\partial_j\Phi^*\partial_j\Phi\right) + 3\lambda^2\Phi^{*2}\Phi^3 = 0 \ , \tag{8.4.2}$$

$$\partial_t^2\Xi - \partial_i^2\Xi - \frac{6}{\Xi}\left[B\frac{(\partial_i\Phi^*\partial_i\Phi)^2}{\Xi^6} - H\Xi^6\right] = 0 \ . \tag{8.4.3}$$

To achieve a similar spontaneous symmetry breaking pattern as in Sections 8.2.1 and 8.2.2, we mimic the helical ansatz (8.2.5), (8.2.6) where the chemical potential is implemented by a time-dependent phase in the $U(1)$-direction. Written explicitly, it provides

$$\Phi(t,x,y) = \rho\, e^{i(\mu t + kx)} \ , \tag{8.4.4}$$

$$\Xi(t,x,y) = v \ , \tag{8.4.5}$$

where the parameters v, ρ, k and μ are all real and non vanishing, and assumed to be positive for simplicity. The equations of motion are

$$\rho^2\left(Ak^2 - \frac{2Bk^4\rho^2}{v^6} - 3\rho^4\lambda^2 + \mu^2\right) = 0 \ , \tag{8.4.6}$$

$$Bk^4\rho^4 - Hv^{12} = 0 \ . \tag{8.4.7}$$

We keep the same relation between the coefficients of the action

$$H = \frac{A^2}{4B} \ , \tag{8.4.8}$$

so that the relation (8.2.11) remains unchanged, but there is an additional condition

$$\mu^2 = 3\rho^4\lambda^2 \ . \tag{8.4.9}$$

Therefore ρ is fixed in terms of μ/λ. The zero density limit can be taken keeping ρ fixed if both μ and λ are taken to zero at the same rate.

The chemical potential μ is seen as an external parameter that fixes the ensemble. Therefore, ρ, k and v are parameters of the solution that should be solved in terms of A, B and μ. This can alternatively be achieved by minimizing the effective potential:[5]

$$V_{\text{eff}} = \frac{B}{v^6}\left(k^2\rho^2 - \frac{A}{2B}v^6\right)^2 + \lambda^2\rho^6 - \mu^2\rho^2 \ . \tag{8.4.10}$$

[5]The terminology "effective" comes from the fact that at finite density, it is customary to look for ground states which minimise the effective Hamiltonian $\tilde{H} = H - \mu Q$ where Q is the $U(1)$ conserved charge. This formulation is equivalent to searching for ground states of the Hamiltonian H, evolving in time along the $U(1)$-direction. Our ansatz (8.4.4), (8.4.5) is precisely doing so, and by considering μ as being an external parameter, ρ, k and v parametrise a static solution minimizing the effective Hamiltonian.

In the present case the ratio v/ρ is fixed by k/μ, more precisely $v^6/\rho^6 \sim k^2/\mu^2$. Then, if $k \gg \mu$ we expect the results to be quite similar to the $\mu = 0$ case with $v/\rho \gg 1$, in which case the gapless mode would be mostly τ. On the other hand, for $\mu \gg k$ they are expected to be closer to the case $v/\rho \ll 1$, where the mode with a gap proportional to k is mostly τ.

8.4.1 Dispersion relations

We are now ready to perform the fluctuations around our background. The fluctuations are parameterised as follows:

$$\Phi(t,x,y) = \rho\, e^{i(\mu t + kx)} \left[1 + \phi(t,x,y)\right] = \rho\, e^{i(\mu t + kx)} \left[1 + \sigma(t,x,y) + i\chi(t,x,y)\right] \,, \tag{8.4.11}$$

$$\Xi(t,x,y) = v\,\left[1 + \tau(t,x,y)\right] \,. \tag{8.4.12}$$

The linearised equations of motion are[6]

$$2A(k + i\partial_x)\left[k(\sigma - 3\tau) + \partial_x\chi\right] + \partial_t^2(\sigma - i\chi) + 2i\mu\partial_t(\sigma - i\chi) + 4\mu^2\sigma = 0 \,, \tag{8.4.13}$$

$$2A(k - i\partial_x)\left[k(\sigma - 3\tau) + \partial_x\chi\right] + \partial_t^2(\sigma + i\chi) - 2i\mu\partial_t(\sigma + i\chi) + 4\mu^2\sigma = 0 \,, \tag{8.4.14}$$

$$v^2\left(-\partial_x^2 - \partial_y^2 + \partial_t^2\right)\tau - 12Ak\rho^2\left[k(\sigma - 3\tau) + \partial_x\chi\right] = 0 \,. \tag{8.4.15}$$

Notice that the term $4\mu^2\sigma$ in (8.4.13) and (8.4.14) spoils the space-modulated shift symmetry we had in the case $\mu = 0 = \lambda$. We therefore do not expect a trivial mode in the spectrum.

In Fourier space, the kinetic matrix for the fluctuations is

$$M = \rho^2 \begin{pmatrix} \omega^2 - 2Ak^2 - 4\mu^2 & -2i(Akq_x + \omega\mu) & 6Ak^2 \\ 2i(Akq_x + \omega\mu) & \omega^2 - 2Aq_x^2 & -6iAkq_x \\ 6Ak^2 & 6iAkq_x & \frac{1}{2}\left(\omega^2 - q_x^2 - q_y^2\right)\frac{v^2}{\rho^2} - 18Ak^2 \end{pmatrix} , \tag{8.4.16}$$

where, as before, the first line corresponds to σ, the second one to χ and the third one to τ. Its determinant is given by

$$\begin{aligned} \det M = \frac{\rho^4}{2}\Big\{&\omega^2 v^2\left(\omega^2 - 8\mu^2\right)\left(\omega^2 - q_x^2 - q_y^2\right) \\ &- 2A\left[v^2\left(\omega^2 - q_x^2 - q_y^2\right)\left(\omega^2\left(k^2 + q_x^2\right) + 4k\omega q_x\mu - 4q_x^2\mu^2\right)\right. \\ &\left.+18k^2\rho^2\omega^2\left(\omega^2 - 8\mu^2\right)\right]\Big\} \,. \end{aligned} \tag{8.4.17}$$

Setting the momenta to zero, one gets

$$\det M = \frac{\rho^4\omega^2}{2}\left[\omega^2 v^2\left(\omega^2 - 2Ak^2 - 8\mu^2\right) - 36Ak^2\rho^2\left(\omega^2 - 8\mu^2\right)\right] , \tag{8.4.18}$$

whose zeros give the mass spectrum. One thus finds a gapless mode $m_1^2 = 0$ and two gapped modes, whose squared gaps are

$$m_{2,3}^2 = Ak^2\left(18\frac{\rho^2}{v^2} + 1\right) + 4\mu^2 \mp \sqrt{\left[Ak^2\left(18\frac{\rho^2}{v^2} + 1\right) + 4\mu^2\right]^2 - 288Ak^2\mu^2\frac{\rho^2}{v^2}} \,, \tag{8.4.19}$$

[6] We remind that here and in the following we keep using the parameters that make the expressions simplest. However, we must always recall that the relations (8.2.11) and (8.4.9) hold.

both real and positive. The reduction of the number of massless modes compared to the zero-density case is expected due to the explicit breaking of the shift symmetry by the pair μ and λ. Intuitively, such breaking leads to one less flat direction and hence, to one fewer gapless mode.

If we take $\mu \ll k$ while keeping ρ and v fixed, one gets

$$m_2^2 = \frac{144\rho^2}{18\rho^2 + v^2}\mu^2 + \mathcal{O}\left(\frac{\mu^4}{k^2}\right) , \tag{8.4.20}$$

$$m_3^2 = m_\eta^2 + \mathcal{O}(\mu^2) , \tag{8.4.21}$$

where we recall that m_η^2 as given in (8.3.9) is of $\mathcal{O}(k^2)$. Note that since as we already noticed, we have that $\mu/k \sim (\rho/v)^3$, the leading term in (8.4.20) goes to zero really as $m_2^2 \sim \mu^{8/3}k^{-2/3}$. In any case, the zero density limit returns the spectrum computed in Section 8.3.1 as expected.

In the opposite limit, $k \ll \mu$, we have

$$m_2^2 = 36Ak^2\frac{\rho^2}{v^2} + \mathcal{O}\left(\frac{k^4}{\mu^2}\right) , \tag{8.4.22}$$

$$m_3^2 = 8\mu^2 + \mathcal{O}(k^2) . \tag{8.4.23}$$

Again, note that taking into account the behaviour of ρ/v in the limit, we have that $m_2^2 \sim k^{4/3}\mu^{2/3}$, still very much suppressed with respect to $m_3^2 \sim \mu^2$. The upshot is thus that in both limits there is a large separation between the larger and smaller gap $m_3 \gg m_2$.

It is also possible to compute analytically the dispersion relation for ω_1, by taking the limit $\omega, q \ll \mu, k$ in (8.4.17). Actually, one can solve the resulting equation by further postulating $\omega \ll q$, so that we get:

$$\omega_1^2 = \frac{v^2}{36\rho^2 k^2}\, q_x^2\left(q_x^2 + q_y^2\right) + \mathcal{O}(q^5) . \tag{8.4.24}$$

Note that this expression is valid for momenta smaller than the chemical potential. Taking the zero density limit for any fixed momentum one recovers that ω_1 is the trivial mode. At large momentum, in any non-zero q_x direction, we can again solve for $\omega, k, \mu \ll q_x$ and get

$$\omega_1^2 = 4\mu^2 + \mathcal{O}\left(\frac{\mu^4}{q_x^2}\right) , \tag{8.4.25}$$

while in the pure transverse direction we have the exact dispersion relation:

$$\omega_1(0, q_y) = 0 , \tag{8.4.26}$$

as can be seen from the fact that in this case an ω^2 factorises again in (8.4.17). So, up to a correction that shifts the dispersion relation by a constant proportional to the chemical potential, ω_1 should be identified with the trivial mode.

For generic momenta, we compute the dispersion relations of the modes numerically and plot them in Figure 8.1. The asymptotic dispersion relations shown in (8.4.24), (8.4.25) and (8.4.26) match the blue curve in the numerical results of Figure 8.1. We further provide two three-dimensional plots of the low-momentum dispersion relations of ω_1 in Figure 8.2.

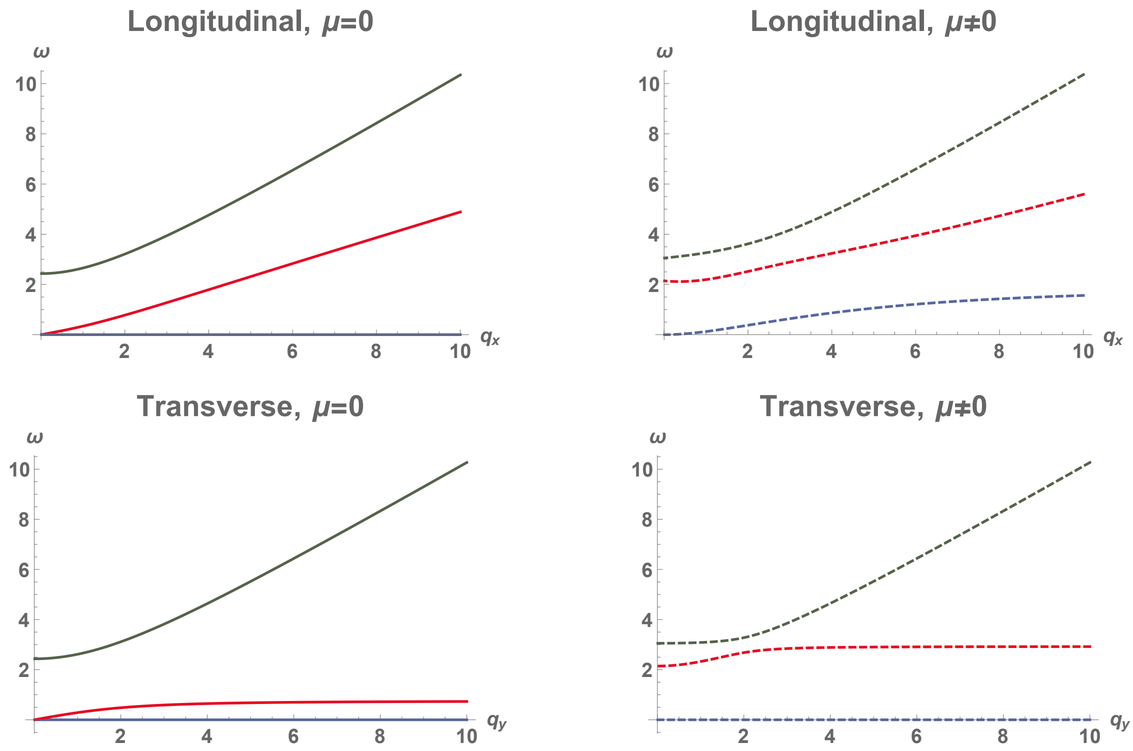

Figure 8.1: This figure displays the dispersion relations of the three modes ω_1 (blue), ω_2 (red) and ω_3 (green). They have been obtained by a numerical analysis of the roots of the determinants (8.3.7) and (8.4.17). The array of plots is such that each line corresponds respectively to the longitudinal direction ($q_y = 0$) and the transverse direction ($q_x = 0$). The columns refer to the case of zero and non-zero chemical potential – to make it more visual, the zero chemical plots are the solid curves while the non-zero chemical ones are dashed. All plots are done with $A = 0.125$, $B = 0.25$, $k = 1.5$; the left column is obtained with $\mu = 0 = \lambda$ while the right column is obtained with $\mu = 1$ and $\lambda = 0.5$. The VEV value ρ is fixed in the $\mu \neq 0$ case by the preceding cited parameters but it is not so in the zero chemical potential case. For practicality, we took the same value for ρ in both cases. Since $k > \mu$, it means that $v > \rho$. Hence, at low momentum, the green curve is mostly shiftonic while the red curve is mostly dilatonic.

8.4.2 Identification of the modes

The presence of a chemical potential does not fundamentally alter the equations of motion at very large momentum discussed in Section 8.3.1. Therefore, we expect that the high momentum identification of the modes remains unchanged at finite density, with the trivial mode ω_1 being mostly σ, the lineon ω_2 or ω_3 being mostly χ (depending on the value of A and the direction in the momentum plane) and the relativistic mode ω_3 or ω_2 being mostly τ. This is confirmed by Figure 8.1 where we have the same trends at large momentum for $\mu = 0$ and $\mu \neq 0$, the only difference being the non-zero plateau of the blue curve, which formally is produced by a subleading contribution at high momentum.

Similarly, in the regime $k \gg \mu$ we do not expect the identification at low momenta of the modes to be significantly altered. Therefore ω_1 would be mostly χ and, given that

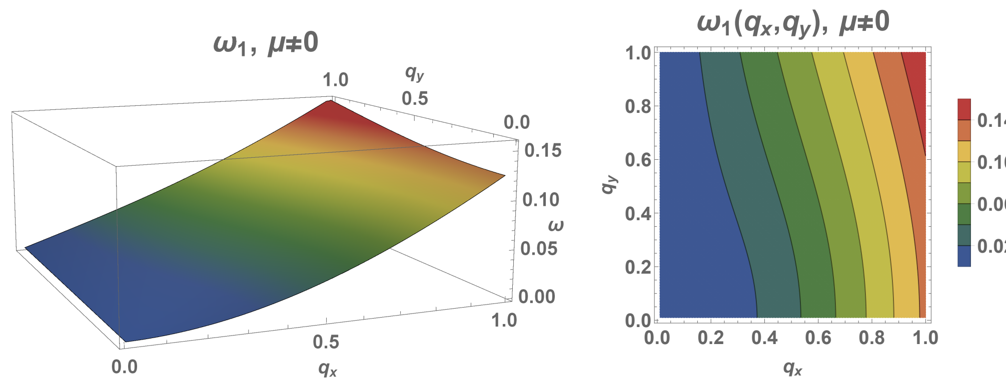

Figure 8.2: This figure displays the numerical ω_1 mode at low momentum, i.e. the $U(1)$ Nambu-Goldstone mode. Both plots represent the same graph and have been obtained with $A = 0.125$, $B = 0.25$, $k = 1.5$, $\mu = 1$ and $\lambda = 0.5$. On the left, a 3D plot is provided while on the right it is a contour plot.

$v/\rho \gg 1$ in this regime, the lower gapped mode ω_2 with gap $m_2 \sim (\mu^4/k)^{1/3}$ would be mostly τ and the higher gapped mode ω_3 with gap $m_3 \sim k$ would be mostly σ.

Finally, when $\mu \gg k$ the most relevant terms producing the mixing of different modes is changed. Taking the matrix (8.4.16) at zero momentum and $\mu \gg k$ leads to

$$M_{q=0,\,\mu\gg k} = \rho^2 \begin{pmatrix} \omega^2 - 4\mu^2 & -2i\omega\mu & O(k^2) \\ 2i\omega\mu & \omega^2 & 0 \\ O(k^2) & 0 & \frac{1}{2}\omega^2\frac{v^2}{\rho^2} + O(k^2) \end{pmatrix}. \tag{8.4.27}$$

In principle both χ and τ become gapless in this limit (they are eigenvectors of M with zero eigenvalue for $\omega = 0$). However, taking into account the $O(k^2)$ corrections we see that τ acquires a gap proportional to k while χ remains as the true gapless mode to leading order. Finally, the gapped mode with $\omega \simeq 2\sqrt{2}\mu$ is a linear combination $\sim \sigma - \frac{i}{\sqrt{2}}\chi$. Summarizing, the identification of the modes is

	$k \gg \mu \gg q$	$\mu \gg k \gg q$	$q \gg k, \mu;\ A < 1/2$
ω_1	χ	χ	σ
ω_2	τ	τ	χ
ω_3	σ	$\sigma - \frac{i}{\sqrt{2}}\chi$	τ

8.5 Removing the degeneracy

The Mexican hat model we have studied in the previous sections has a large emergent symmetry that results in the presence of trivial modes in the spectrum. We can remove partially the emergent symmetry and generate non-trivial dispersion relations for all the modes by introducing additional terms to the action, while at the same time keeping the same symmetry breaking pattern. At fourth order in spatial derivatives and fields, there are two possible extensions[7]

$$\Delta\mathcal{L} = G\,\Xi^{-6}\partial_i\Phi^*\partial_i\Phi^*\partial_j\Phi\partial_j\Phi + F\,\Xi^{-6}\Phi^*\Phi\partial_i\partial_j\Phi^*\partial_i\partial_j\Phi \ . \tag{8.5.1}$$

However, they do not produce qualitatively different results. For simplicity we will set $F = 0$ in the following. We will then study the extended model

$$\begin{aligned}\mathcal{L} =& \partial_t\Phi^*\partial_t\Phi + A\partial_i\Phi^*\partial_i\Phi + \frac{1}{2}\partial_t\Xi\partial_t\Xi - \frac{1}{2}\partial_i\Xi\partial_i\Xi \\ &+ \frac{1}{\Xi^6}\left[-B\left(\partial_i\Phi^*\partial_i\Phi\right)^2 + G\,\partial_i\Phi^*\partial_i\Phi^*\partial_j\Phi\partial_j\Phi\right] - H\Xi^6 \ ,\end{aligned} \tag{8.5.2}$$

where the G-term is the additional part.

In general, the new term will change the energy of the solutions. If we want to ensure that the helical superfluid background (8.2.5), (8.2.6) is a minimal energy solution we have to modify the relation between the coefficients to

$$H = \frac{A^2}{4(B-G)} \ . \tag{8.5.3}$$

With this choice the dimensionless combination

$$\xi = \frac{k^2\rho^2}{v^6} \ , \tag{8.5.4}$$

remains fixed as a function of the coefficients of the action through the relation

$$A = 2(B-G)\xi \ . \tag{8.5.5}$$

Equation (8.5.4) leaves therefore a moduli space with two flat directions since the static energy on-shell is identically zero.

Using the same basis of fluctuations for the helical superfluid (8.3.1)–(8.3.2), the new term introduces a contribution to the quadratic action of the form

$$\mathcal{L}_G = -4G\xi(\partial_y\sigma)^2. \tag{8.5.6}$$

This breaks partially the symmetry characterised by the transformation β in (8.2.52). With the new term, β is restricted to be a function at most linear in y, but yet arbitrary in x. So, the emergent symmetry with nonzero G is

$$\begin{aligned}&\delta\chi = \alpha(y) + \beta(x) + \epsilon(x)y, \quad \delta\sigma = -\frac{1}{k}\left[\beta'(x) + \epsilon'(x)y\right] + 3\delta + 3\gamma_i x^i, \\ &\delta\tau(x,y) = \delta + \gamma_i x^i.\end{aligned} \tag{8.5.7}$$

Following the same reasoning we did previously, we expect χ and σ to be both lineons, with χ moving along the x direction and σ along the y direction.

For the meta-fluid the ansatz (8.2.19) introduces a term in the action for fluctuations

$$\mathcal{L}_G \simeq \frac{2G(|b|^2)^2}{v^6}u_{ij}u_{ij} = \frac{4G(|b|^2)^2}{v^6}\partial_i u_j\partial_i u_j \ , \tag{8.5.8}$$

[7] We can also have additional higher derivative terms for the real scalar Ξ, but since the background value of Ξ is constant we are not interested in those.

where $u_{ij} = \partial_i u_j + \partial_j u_i - \delta_{ij}\partial_k u_k$ is the shear strain, and the second equality is obtained up to total derivatives. Note that, in contrast to the helical superfluid, we do not have to change the relation of H with the other coefficients since this term does not give a contribution to the energy density of the background. Furthermore, the meta-fluid is stable for $G < 0$, while the helical fluid is stable for $G > 0$, since this gives the right sign to the kinetic terms in (8.5.6) and (8.5.8). So, with the new term, only one of the two states would be realised depending on the values of the coefficients we choose to extend the model.

In the meta-fluid the new term introduces a shear modulus $\mathcal{G} = -4G|b|^2/v^6$, that removes most of the symmetries in (8.2.72) and (8.2.73), leaving just the symmetries for massless fields.

Integrating out the massive dilaton as before will remove the zero-momentum bulk modulus, but the higher derivative terms only affect the dispersion relation at higher order in momentum. Then, the effective low-energy theory is almost the same as ordinary elasticity, the dispersion relation for the fluctuations u_i is at lowest order in momentum

$$\omega^2 \simeq \mathcal{G}q^2 \ . \tag{8.5.9}$$

8.5.1 Ward-Takahashi identities

Helical superfluid

For the helical superfluid the $U(1)$ Ward-Takahashi identity (8.2.28) does not change when G is introduced, so the $U(1)$ Nambu-Goldstone mode has the same dispersion relation at high momentum. The real shift symmetry Ward-Takahashi identity becomes

$$\partial_t^2\sigma - 4\xi G\partial_y^2\sigma + 2Ak\left[k(\sigma - 3\tau) + \partial_x\chi\right] = 0 \ . \tag{8.5.10}$$

The dilatation Ward-Takahashi identity also acquires a new contribution

$$v^2(\partial_i^2\tau - \partial_t^2\tau) = 2\rho^2\left(\partial_t^2\sigma - 4\xi G\partial_y^2\sigma\right) + 8k\rho^2 A\left(k(3\tau - \sigma) - \partial_x\chi\right) \ . \tag{8.5.11}$$

Since ξ is a fixed quantity, in the high momentum limit $k \to 0$, the dispersion relation of the shifton is modified to

$$\omega_\sigma^2 \simeq 4\xi G q_y^2 = \frac{2AG}{B - G}q_y^2 \ . \tag{8.5.12}$$

On the other hand, the dilaton keeps a relativistic dispersion relation in this limit. This confirms our analysis of the emergent symmetries where we predicted that σ would behave as a lineon moving along the y direction.

Meta-fluid

When G is introduced, the dilatation Ward-Takahashi identity for the meta-fluid does not change, but there is a new term in the complex shift Ward-Takahashi identity

$$v(\partial_t^2 u_i - A\partial_i\partial_k u^k - \mathcal{G}\partial_k^2 u_i) + 6A\partial_i\tau = 0. \tag{8.5.13}$$

At high momentum this gives two modes with dispersion relations

$$\omega^2 \simeq \mathcal{G}q^2, \qquad \omega^2 \simeq (A + \mathcal{G})q^2, \tag{8.5.14}$$

with the first mode corresponding to the transverse and the second to the longitudinal components of u_i.

8.5.2 Dispersion relations for the helical superfluid

We obtain the following equations of motion at linear order for the fluctuations:

$$2A(k + i\partial_x)\left[k(\sigma - 3\tau) + \partial_x\chi\right] + \partial_t^2(\sigma - i\chi) - 4G\xi\partial_y^2\sigma = 0 \ , \tag{8.5.15}$$

$$2A(k - i\partial_x)\left[k(\sigma - 3\tau) + \partial_x\chi\right] + \partial_t^2(\sigma + i\chi) - 4G\xi\partial_y^2\sigma = 0 \ , \tag{8.5.16}$$

$$12Ak\rho^2\left[k(\sigma - 3\tau) + \partial_x\chi\right] - v^2\left(-\partial_x^2 - \partial_y^2 + \partial_t^2\right)\tau = 0 \ . \tag{8.5.17}$$

Sending the parameter G to zero (keeping the parameter k, ρ and v fixed) permits to recover the vacuum as well as the equations of motion of the $G = 0$ model (8.2.1). Hence, in this specific limit, we expect to recover smoothly the original spectrum.

Going to Fourier space, we obtain a homogeneous algebraic system for the equations of motion driven by the kinetic matrix:

$$M = \rho^2 \begin{pmatrix} \omega^2 - 2Ak^2 - 4G\xi q_y^2 & -2iAkq_x & 6Ak^2 \\ 2iAkq_x & \omega^2 - 2Aq_x^2 & -6iAkq_x \\ 6Ak^2 & 6iAkq_x & \frac{1}{2}\left(\omega^2 - q_x^2 - q_y^2\right)\frac{v^2}{\rho^2} - 18Ak^2 \end{pmatrix} \tag{8.5.18}$$

The determinant evaluates to

$$\det(M) = \det(M)_{G=0} - 2G\xi\rho^4 q_y^2\left[v^2(\omega^2 - q_x^2 - q_y^2)(\omega^2 - 2Aq_x^2) - 36Ak^2\rho^2\omega^2\right] \ , \tag{8.5.19}$$

where $\det(M)_{G=0}$ is given by (8.3.7).

If we specifically look for a trivial root of the determinant, we do not find one:

$$\det(M)|_{\omega=0} = -4AG\xi\rho^4 v^2 q_x^2 q_y^2\left(q_x^2 + q_y^2\right) \ . \tag{8.5.20}$$

This immediately tells us that there is no longer a trivial mode. This is consistent with the analysis we made based on the emergent shift symmetries.

The spectrum features two gapless modes and one gapped mode

$$m_1^2 = 0 \ , \tag{8.5.21}$$

$$m_2^2 = 0 \ , \tag{8.5.22}$$

$$m_3^2 = 2Ak^2\left(1 + 18\frac{\rho^2}{v^2}\right) \ . \tag{8.5.23}$$

Notice that the difference with the case $G = 0$ is hidden in the relation among the parameters, where the correction is given by a factor $(B - G)$ instead of simply B. So sending G to zero smoothly provides the masses of the $G = 0$ case.

Proceeding to compute the dispersion relations at small momenta, we obtain:

$$\omega_1^2 = \frac{4G\rho^2 v^2 (q_x^2 + q_y^2) q_x^2 q_y^2}{72k^2\rho^4 G q_y^2 + v^8 (q_x^2 + q_y^2)} + \mathcal{O}(q^6) \ , \tag{8.5.24}$$

$$\omega_2^2 = \frac{v^2}{18\rho^2 + v^2}\left[q_x^2 + \left(1 + \frac{72Gk^2\rho^4}{v^8}\right) q_y^2\right] + \mathcal{O}(q^4) \ , \tag{8.5.25}$$

$$\omega_3^2 = 2Ak^2\left(1 + 18\frac{\rho^2}{v^2}\right) + 2Aq_x^2 \tag{8.5.26}$$

$$+ \frac{18\rho^2}{18\rho^2 + v^2}\left[q_x^2 + \left(1 + \frac{4Gk^2\rho^2}{v^6}\right) q_y^2\right] + O\left(q^4\right) \ .$$

We recover smoothly the $G = 0$ case in the limit of zero G.

Note that the expression for the mode ω_1 has an unusual non-analytic dependence with momentum but overall it goes like $\omega_1 \sim q^2$, while the other gapless mode is linear $\omega_2 \sim q$. The dispersion relation of ω_1 is confirmed by the numerical study of Figure 8.3. In fact, the plots display the trivialisation of the dispersion relations for $q_x = 0$ and $q_y = 0$, and the non-trivial bump in the quadrant in between. Also, the analytic expression predicts the changes in slope we observed in the 3D plot. Indeed, according to (8.5.24), the starting slope of the dispersion relation at fixed $q_x > 0$ is larger than the starting slope at fixed $q_y > 0$. Hence, in order for the dispersion relations to join continuously, the fixed $q_x > 0$ dispersion relation should bend downwards.

A final comment is that at large momentum we observe that (8.5.18) diagonalises. In particular, the modes ω_1 and ω_2 are respectively transverse and longitudinal lineons with the dispersion relations

$$\omega_1 \sim 2\sqrt{G\xi}\, q_y \ , \quad \omega_2 \sim \sqrt{2A}\, q_x \quad \text{when } q_x, q_y \gg k \ . \tag{8.5.27}$$

This is in agreement with the analysis of Ward-Takahashi identities we did previously.

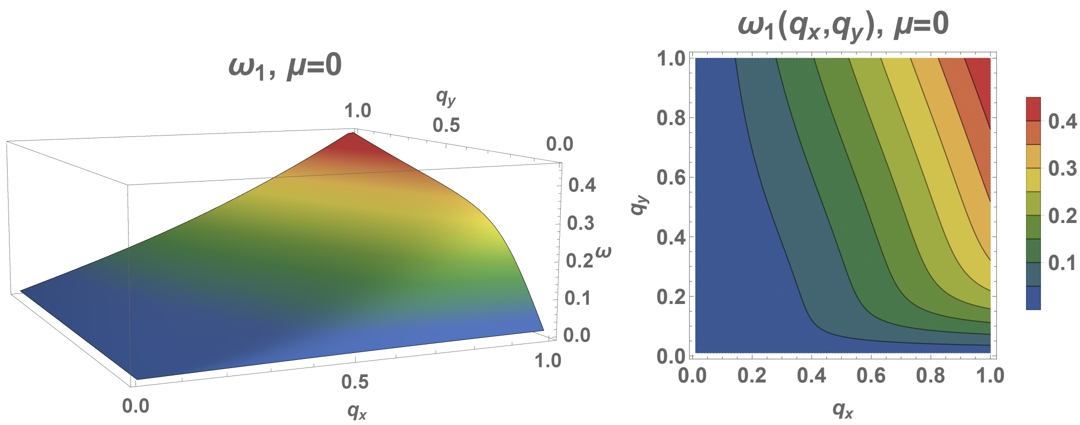

Figure 8.3: This figure displays the numerical ω_1 mode at low momentum, i.e. the $U(1)$ Nambu-Goldstone mode. Both plots represent the same graph and have been obtained with $A = 2$, $B = 1.5$, $G = 1$, $k = 1$, $\rho = 0.54$ and $\mu = 0 = \lambda$. On the left, a 3D plot is provided while on the right it is a contour plot.

Identification of the modes

Since the new term proportional to G adds a contribution to the kinetic matrix proportional to q_y^2, the separation between gapless and gapped modes at low momentum is the same as for $G = 0$. Indeed, the gap in (8.3.15) and the velocity in the x direction of the gapless mode (8.3.14) are the same with nonzero G (8.5.24). The nature of the high momentum modes is easily identified with the help of the Ward-Takahashi identities. So, the identification of the modes is essentially the same as in section 8.3.2, except ω_1 has a non-analytic behaviour at low momentum and becomes a lineon propagating in the y direction at high momentum.

8.5.3 Dispersion relations for the meta-fluid

Let us directly consider the kinetic matrix

$$M = |b|^2 \begin{pmatrix} \omega^2 - Aq_x^2 - \mathcal{G}q^2 & -Aq_xq_y & -6i\frac{A}{v}q_x \\ -Aq_xq_y & \omega^2 - Aq_y^2 - \mathcal{G}q^2 & -6i\frac{A}{v}q_y \\ 6i\frac{A}{v}q_x & 6i\frac{A}{v}q_y & \frac{1}{2|b|^2}\left(\omega^2 - q^2\right) - 36\frac{A}{v^2} \end{pmatrix} , \tag{8.5.28}$$

where we recall that $\mathcal{G} = -4G|b|^2/v^6 > 0$. We notice that only the first two diagonal terms are modified compared to the $G = 0$ case. Therefore, we expect that only two of the three dispersion relations will be more significantly affected by the correction, namely the lightest modes.

The determinant of the kinetic matrix reads as follows

$$\det(M) = \frac{|b|^4}{2}(\omega^2 - \mathcal{G}q^2)\Big[(\omega^2 - q^2)(\omega^2 - Aq^2 - \mathcal{G}q^2) - m_\tau^2(\omega^2 - \mathcal{G}q^2)\Big] . \tag{8.5.29}$$

It rightly reduces to (8.3.24) when $G = 0$. From this expression one can immediately see that what was formerly the immobile fracton, acquires isotropic and linear dispersion relations which are valid for any momenta, and are entirely controlled by G. One can further find the exact analytical expression for the other two modes, which will depend non-trivially both on G and m_τ. At low-momentum $\omega, q \ll m_\tau$, one can see that the condition $\det(M) = 0$ gets an additional factor of $(\omega^2 - \mathcal{G}q^2)$, giving the two gapless modes expected from the low-energy effective theory.

In more detail, at low momentum we have the expansions

$$\omega_1^2 = \mathcal{G}q^2 , \tag{8.5.30}$$

$$\omega_2^2 = \mathcal{G}q^2 + A(1 - \mathcal{G})\frac{q^4}{m_\tau^2} + \mathcal{O}\left(q^6\right) , \tag{8.5.31}$$

$$\omega_3^2 = m_\tau^2 + (1 + A)q^2 + \mathcal{O}\left(q^4\right) . \tag{8.5.32}$$

We recover the expected results from the effective analysis as well as the idea that two of the three modes are more substantially affected by G. Looking at (8.5.30), (8.5.31) and (8.5.32) we get back the dispersion relation of the original model when we send G to zero.

At large momentum we can drop the last term in (8.5.29), so that the determinant completely factorises and the modes will behave as

$$\omega_1 = \sqrt{\mathcal{G}}|q| , \qquad \omega_2 \simeq \sqrt{(A + \mathcal{G})}|q| , \qquad \omega_3 \simeq |q| . \tag{8.5.33}$$

When $G \to 0$, we recover the original large momentum behaviour.

According to our previous analysis, at low momentum we can identify ω_1 with the transverse component of the displacement u_i, while ω_2 corresponds to the longitudinal part. The gapped mode is mostly the dilaton. At high momentum, the Ward-Takahashi identities keep the same identification for the modes as for low momentum.

8.5.4 Extended model at finite density

We generalise the helical superfluid solutions to finite density, as already done in Section 8.4 for $G = 0$. In order to stabilise the ground states, we add a shift symmetry breaking λ-term in the Lagrangian,

$$\begin{aligned}\mathcal{L} =& \partial_t\Phi^*\partial_t\Phi + A\partial_i\Phi^*\partial_i\Phi + \frac{1}{2}\partial_t\Xi\partial_t\Xi - \frac{1}{2}\partial_i\Xi\partial_i\Xi \\ &+ \frac{1}{\Xi^6}\left[-B\left(\partial_i\Phi^*\partial_i\Phi\right)^2 + G\,\partial_i\Phi^*\partial_i\Phi^*\partial_j\Phi\partial_j\Phi\right] - H\Xi^6 - \lambda^2\left(\Phi^*\Phi\right)^3 \ .\end{aligned} \tag{8.5.34}$$

Given the condition (8.5.3), the plane wave ansatz (8.4.4)–(8.4.5) is a solution to the equations of motion minimizing the effective potential provided

$$v^6 = \frac{2(B-G)}{A}k^2\rho^2 \ , \qquad \rho^2 = \left|\frac{\mu}{\sqrt{3}\,\lambda}\right| \ . \tag{8.5.35}$$

Setting G to zero, we recover the background solution of the finite density $G = 0$ model.

The linearised equations of motion are

$$\begin{aligned}&2A(k+i\partial_x)\left[k(\sigma-3\tau)+\partial_x\chi\right] + \partial_t^2(\sigma - i\chi) && (8.5.36)\\ &\qquad -4G\xi\partial_y^2\sigma + 2i\mu\partial_t(\sigma - i\chi) + 4\mu^2\sigma = 0 \ , \\ &2A(k-i\partial_x)\left[k(\sigma-3\tau)+\partial_x\chi\right] + \partial_t^2(\sigma + i\chi) && (8.5.37)\\ &\qquad -4G\xi\partial_y^2\sigma - 2i\mu\partial_t(\sigma + i\chi) + 4\mu^2\sigma = 0 \ , \\ &v^2\left(-\partial_x^2 - \partial_y^2 + \partial_t^2\right)\tau - 12Ak\rho^2\left[k(\sigma-3\tau)+\partial_x\chi\right] = 0 \ . && (8.5.38)\end{aligned}$$

The kinetic matrix associated to the equations of motion is

$$M = \rho^2\begin{pmatrix} \omega^2 - 2Ak^2 - 4\mu^2 - 4G\xi q_y^2 & -2i(Akq_x + \omega\mu) & 6Ak^2 \\ 2i(Akq_x+\omega\mu) & \omega^2 - 2Aq_x^2 & -6iAkq_x \\ 6Ak^2 & 6iAkq_x & \frac{1}{2}\left(\omega^2 - q_x^2 - q_y^2\right)\frac{v^2}{\rho^2} - 18Ak^2 \end{pmatrix} . \tag{8.5.39}$$

Since G only contributes by terms proportional to the momentum, there are no qualitative differences in the gaps, it is enough to replace $B \to B - G$ in the expressions found in Section 8.4. The high momentum behaviour will once more be the same as for zero density. For low and intermediate momenta, we resort to numerics, our results are plotted in Figure 8.4 and in Figure 8.5. Comparing with Figure 8.3, we observe that the ω_1 mode is lifted at $q_y = 0$ when $\mu \neq 0$, as also happened at $G \neq 0$. On the other hand, comparing Figure 8.2 and Figure 8.5, the effect of G is to introduce a change in the slope of the dispersion relation in the q_x direction. The identification of the modes will be the same as that made at $G = 0$ in Section 8.4.

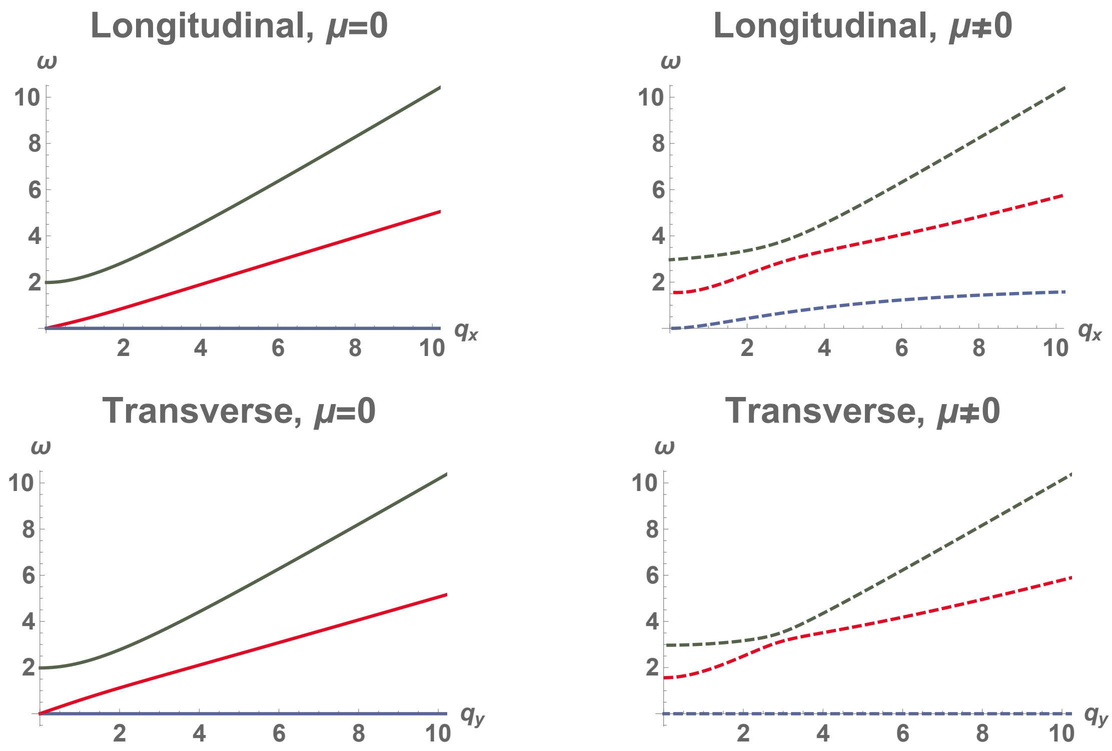

Figure 8.4: This figure displays the dispersion relations of the three modes (each mode has its own colour, as in Figure 8.1). The array of plots is such that each lines corresponds respectively to the longitudinal direction ($q_y = 0$) and the transverse direction ($q_x = 0$). The columns refer to the case of zero and non-zero chemical potential – to make it more visual, the zero chemical plots are the solid curves while the non-zero chemical ones are dashed. All plots are done with $A = 0.125$, $B = 0.5$, $G = 0.25$, $k = 1.5$; the left column is obtained with $\mu = 0 = \lambda$ while the right column is obtained with $\mu = 1$ and $\lambda = 1$. The VEV value ρ is fixed in the $\mu \neq 0$ case by the preceding cited parameters but it is not so in the zero chemical potential case. We took the same value for ρ in both cases for practical reasons. We have that $v > \rho$, hence, at low momentum, the green curve is mostly shiftonic while the red curve is mostly dilatonic. Notice that since $\mu < k$, the identification of the modes for the finite density case matches the one with zero chemical potential.

8.6 Counting the Nambu-Goldstone modes

A comparison of the specific results found above with the general knowledge on Nambu-Goldstone counting (see Part I) is interesting because non-trivial. To this purpose, we recapitulate in Table 8.1 the Nambu-Goldstone modes found explicitly from the study of the helical fluid and meta-fluid fluctuation Lagrangians, as well as their dispersion and analyticity properties. In particular, we stress that we found in general two gapless and a gapped mode.

For internal symmetries, the number of Nambu-Goldstone modes n_{NG} is generically bounded by the number of spontaneously broken symmetries n_{BG}. If there are no terms with single time-derivatives in the effective action, however, all the Nambu-Goldstone

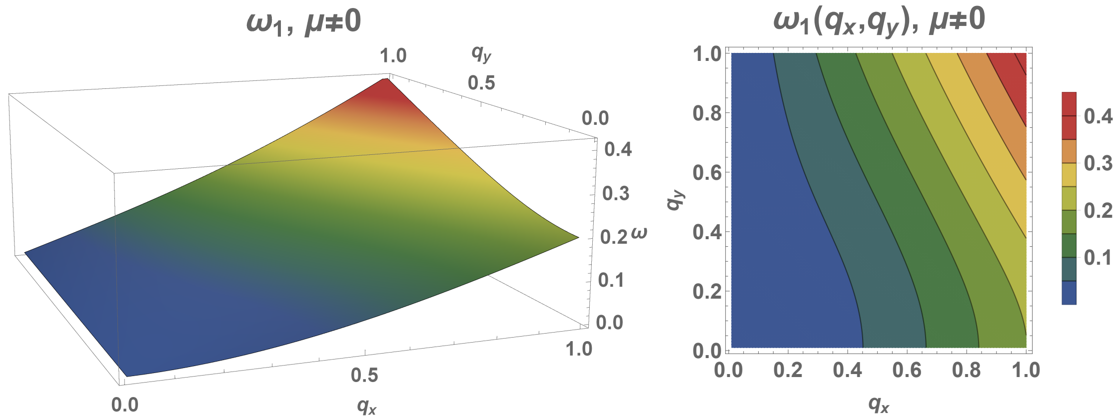

Figure 8.5: This figure displays the numerical ω_1 mode at low momentum, i.e. the $U(1)$ Nambu-Goldstone mode. Both plots represent the same graph and have been obtained with $A = 2$, $B = 1.5$, $G = 1$, $k = 1$, $\mu = 0.5$ and $\lambda = 1$. On the left, a 3D plot is provided while on the right it is a contour plot.

vacuum	ω_1	ω_2	ω_3
Helical ($G = 0$)	0	$q_\star$	gapped
Helical ($G \neq 0$)	$q_\star^2$	$q_\star$	gapped
Meta-fluid ($G = 0$)	0	q^2	gapped
Meta-fluid ($G \neq 0$)	$q_\star$	$q_\star$	gapped

Table 8.1: Dispersion and analyticity properties of the Nambu-Goldstone modes as found from the low-energy study of the fluctuation Lagrangian. The $\star$ subindex indicates non-analyticity.

modes are of type A in the classification of [77, 78] and we have $n_{\text{NG}} = n_{\text{BG}}$. A priori, that would be the case for the effective actions we found (we set $\mu = 0$ for the moment). However, we are not dealing with internal symmetries only, hence the notion of n_{BG} has to be qualified, as we will do shortly.

An alternative classification, perhaps more pertinent to our situation, is provided by counting theorems which split the total number of Nambu-Goldstone modes n_{NG} according to specific dispersion properties. Defining as type I/type II the modes with an odd/even dispersion relation, respectively, [70] established that

$$n_I + 2n_{II} \geqslant n_{\text{BG}} \ . \tag{8.6.1}$$

Turning to broken spacetime symmetries, there are no general counting rules (e.g. the counting rule (8.6.1) is out of the scope because it has been derived considering the breaking of uniform symmetries, where for us, dilatation is not uniform), yet it is known that the number of independent modes can be reduced [13, 96, 127–129, 131, 133]. In essence, if Q_a denotes the generators of broken symmetries, P_i denotes the unbroken translations and $\langle \Phi(\boldsymbol{x}) \rangle$ denotes the expectation value of the order parameter, then the following set of identities allows to reduce the number of independent fields

$$[P_i, Q_a] \langle \Phi(\boldsymbol{x}) \rangle = c_{iab} \, Q_b \langle \Phi(\boldsymbol{x}) \rangle \ , \tag{8.6.2}$$

where c_{iab} indicates the relevant structure constant of the symmetry algebra. This introduces a constraint such that would-be Nambu-Goldstone bosons appearing on each side of the identity are not independent in general.

Let us discuss how the explicit results found in the previous sections relate to the general counting rules. To the counting purposes, we have to consider the following symmetries:

- Spacetime: Translations P_1, P_2, rotations R and dilatations D;
- Internal: $U(1)$ transformations (phase rotations) Q, complex (i.e. real + imaginary) shifts S_R, S_I.

Let us report the symmetry content of the two kinds of ground states separately.

Helical superfluid: the unbroken symmetries are two translations

$$P_1 - kQ \ , \qquad P_2 \ , \tag{8.6.3}$$

which leaves in principle five broken symmetries $n_{\rm BG} = 5$. However, the commutation relation of unbroken translations with the broken generators results in additional conditions

$$\begin{aligned} &[P_1 - kQ, D] \propto P_1, \qquad [P_2, R] \propto P_1, \\ &[P_1 - kQ, S_R] \propto S_I, \qquad [P_1 - kQ, S_I] \propto S_R \ . \end{aligned} \tag{8.6.4}$$

This would imply that rotations, dilatations and broken translations are described by a single mode, and there would be a single mode associated to both real and imaginary shifts (indeed, we had already commented earlier on about this). Effectively we would be left with a number of independent broken symmetries $n_{\rm BG}^{\rm (in)} = 2$, where the up index stands for independent (we speak of independent broken generators in the perspective of independent massless NG modes).

Meta-fluid: the unbroken symmetries are two translations and a rotation

$$P_1 - S_R \ , \qquad P_2 - S_I \ , \qquad R - Q \ , \tag{8.6.5}$$

so there would be four broken symmetries in this case $n_{\rm BG} = 4$. The commutation relations of the unbroken translations with the broken symmetries would produce additional conditions

$$[P_1 - S_R, D] \propto P_1, \qquad [P_2 - S_I, D] \propto P_2. \tag{8.6.6}$$

Note that commutators with $R + Q$ result in unbroken translations. This would imply that broken translations and dilatations are described by a single mode. Effectively this reduces the number of independent broken symmetries to $n_{\rm BG}^{\rm (in)} = 2$.

We report the values of the countings in Table 8.2, which requires some discussion. For $G = 0$ in the helical superfluid, there is a trivial gapless mode and a type I mode. The trivial mode could be counted as type I or II. Still in the helical case, when $G \neq 0$, we have almost the same type of modes except that the trivial mode has a non-analytic dispersion relation, but we will still consider it as type II, since at low momenta $\omega \sim O(q^2)$. In

vacuum	n_I	n_{II}	n_0	n_{BG}	$n_{\text{BG}}^{(\text{in})}$	$n_A = n_{\text{NG}}$	$n_I + 2n_{II}$
Helical ($G = 0$)	1	0	1	5	2	2	2 or 3
Helical ($G \neq 0$)	1	1	0	5	2	2	3
Meta-fluid ($G = 0$)	0	1	1	4	2	2	3 or 4
Meta-fluid ($G \neq 0$)	2	0	0	4	2	2	2

Table 8.2: Summary of the countings for the two kinds of vacuum. With n_0 we denote the number of trivial modes.

so doing, the counting rule (8.6.1) is satisfied if we use $n_{\text{BG}}^{(\text{in})}$ as the number of broken symmetries.

Turning to the meta-fluid case, for $G = 0$ there is an analytic/type II mode and the trivial mode, which could be counted either as type I or II. On the other hand, if $G \neq 0$, the two gapless modes are type I/non-analytic. Also in this case we observe that the counting rule (8.6.1) is satisfied with $n_{\text{BG}}^{(\text{in})}$ as the number of broken symmetries.

Finally, switching on a chemical potential in the helical superfluid gaps one of the two massless modes. This could be seen as a consequence of introducing a mixing through a single time-derivative term in the effective Lagrangian, so effectively we would be left with a single type B Nambu-Goldstone mode in the classification of [77, 78], which turns out to be also type II if we just count the power of the momentum and ignore the breaking of rotational invariance. Note that in going to finite density we introduced an additional coupling that breaks the shift symmetry, so that in this case the remaining gapped mode is a pseudo-Goldstone mode and not a true massive Nambu-Goldstone. The single massless mode at finite density can also be recovered through the IHCs. Now that the shift symmetries are explicitly broken, the spontaneously broken generators are P_0, P_1, R, D and Q. The homogeneous breaking of time translation and spatial translation leaves one independent broken generator among P_0, P_1 and Q. Moreover, we can impose two IHCs:

$$[P_1 - kQ, D] \propto P_1, \qquad [P_2, R] \propto P_1 \ . \tag{8.6.7}$$

Thus, it remains one single independent broken generator to create one massless NG mode. A final comment is the following. A finite chemical potential can be implemented as a linear time dependence in the phase of the charged field. In so doing, however, time-translations would be broken by our choice of ensemble and not, strictly speaking, by a dynamical feature described at the level of the Lagrangian. In contrast, the space modulations that we consider in the helical case, although being formally similar in some aspects, are determined by the gradient Mexican hat potential. Some further comments on this are given in relation to ghost condensation in the next section.

8.7 Discussion

The concomitant breaking of dilatations and spatial translations constitutes the main focus of the present study, which adopts the framework of effective field theory. Together with dilatations and translations, an internal Abelian symmetry is broken, too. This symmetry serves two purposes, one physical and another technical. The former consists in modelling a conserved current, thus providing the possibility of considering finite density circumstances; the latter consists in the fact that the breaking of translations and a $U(1)$ symmetry to their diagonal subgroup allows for homogeneous symmetry breaking, in the spirit of Q-lattice models [47].

We have utilised a simple non-relativistic field theoretical setup which allows one to characterise the low-energy modes and derive their effective description in different regimes. The breaking of translations is dynamically induced by a gradient Mexican hat mechanism, namely the competition among a quadratic gradient term driving towards instability then stabilised by higher terms [116]. The gradient Mexican hat, when discretised, connects to lattice models with frustrated interactions [230].

By means of a neat particular example, we clarified the generic fact that the analysis of the modes revolves about three relevant bases: the basis given by the fluctuations of the fields appearing in the Lagrangian; the basis of the fluctuations which diagonalise the lowest-order dynamics at low energy; and the basis associated to the symmetries of the model (this latter is possibly incomplete). Whenever the connection among such bases is non-trivial, we have mixing phenomena. For instance, the Nambu-Goldstone mode associated to a specific symmetry can results from different combinations of the UV or the IR modes, as a function of momenta.

We showed the presence of two degenerate classes of vacua, one associated to a plane wave configuration and possessing a helical structure (i.e. a global phase rotation can be compensated by a suitable translation along the wave-vector of the plane wave), the other associated to complex field configurations which are linear in the coordinates. The latter class admits a specific subclass of isotropic solutions, where a global phase rotation of the background can be compensated by a suitable spatial rotation. We referred to the latter subclass as meta-fluids, because they show a trivial shear elastic response alongside isotropy.

An important feature of the model studied here is the presence of low-energy modes with reduced propagation properties. This fractonic behaviour can be associated to enhanced polynomial shift symmetries of the low-energy, linear effective theory and translates into the trivialisation of some elastic coefficients. More specifically, we have encountered both completely immobile modes and subdimensional modes, like lineons propagating only along one spatial direction. The immobile fractons can be thought intuitively as plastic deformations which cost zero energy, corresponding to an enlarged vacuum degeneracy, a property which can be compared to the diverging zero-temperature entropy of some fractonic lattice models in their continuum limit [231].

Despite describing an elastic effective field theory with fractonic excitations, the models studied in the present chapter differ from the setup where fracton-elasticity duality has been demonstrated [211, 212]. There, the two gapless modes of a symmetric gauge field represent the dual encoding of the transverse and longitudinal phonons. Our models lack a gauge field and encode the phonons as Nambu-Goldstone modes, dynamically generated by the interactions. Furthermore, the fractons described in the present chapter do not correspond to defects or non-perturbative configurations. They are the low-energy encoding of a trivial (or partially trivial) elastic response.

Immobile fractons cannot move when in isolation but can move due to interactions [211]. Analogously, we expect that the flat fractonic dispersion relations that we encountered are in general "bent" when considering higher non-linear terms in the effective theory, leading to non-trivial propagation.

To conclude, we briefly comment the relation of the present study with some applications in condensed matter and cosmology.

1. **LOFF (or FFLO) superconductors:** The Larkin-Ovchinnikov-Fulde-Ferrell state (LOFF state) [232, 233] is characterised by a spatially modulated order parameter and a gapless phonon associated to the breaking of translation invariance, and it might be realised in some unconventional superconductors [234, 235]. A similar colour superconducting state can also arise at high density in QCD [236, 237]. The Mexican hat model is similar to the Ginzburg-Landau functional used to describe superconducting states in the particle-hole symmetric case [238, 239]. It could be interesting to revisit the description of the LOFF state and other translation-breaking superconducting states to explore possible emergent symmetries and the fractonic nature of the associated modes.[8]

2. **Wigner crystals:** In the low electron density regime, Wigner crystals can be treated according to classical elastic theory [243, 244]. There, the chemical potential μ can possibly be sufficiently low as to allow for a μ-gapped dilaton to become relevant for the low-energy collective-mode description of the crystal response. Despite the non-vanishing μ, in a clean Wigner crystal, all the phonons (either longitudinal or transverse) are gapless. This matches with what we observed in the meta-fluid model, which however suggests that the standard elastic description for Wigner crystals could lack an extra (gapped) dilatonic degree of freedom.

3. **Charge density waves:** Optical conductivity measurements show a rich structure of peaks [245, 246]. In the presence of disorder, the general pattern is characterised by a low-frequency gapped mode corresponding to a pinned collective sliding mode of the density wave condensate. At the opposite end of the spectrum, there is a high-frequency mode associated to the single excitation through the density wave gap. The intermediate region features non-universal peaks corresponding to substrate modes, e.g. due to the impurities. It would be interesting to investigate whether the intermediate structure could conceal a gapped dilatonic peak.[9]

4. **Ghost condensates:** In the search of possible infrared modifications of General Relativity, a mechanism similar to the gradient Mexican hat has been proposed in the time-derivative sector, this is usually referred to as ghost condensation [225].[10] A relativistic generalisation of (8.2.1) is possible, yet it leads to trivial results. Specifically, one can consider the model

$$\mathcal{L} = A\partial_\mu\Phi^*\partial^\mu\Phi - B\frac{(\partial_\mu\Phi^*\partial^\mu\Phi)^2}{\Xi^6} - \frac{1}{2}\partial_\mu\Xi\partial^\mu\Xi - H\Xi^6 - \lambda^2(\Phi^*\Phi)^3 \ . \tag{8.7.1}$$

For $\lambda \neq 0$ the equations of motion imply $\Phi = 0$. Whereas, for $\lambda = 0$, the resulting low-energy effective theory features just a relativistic gapless mode, the other two

[8] A holographic discussion of FFLO phases has been commented in [240–242].

[9] An analogous question would be interesting also in relation to holographic realisations of charge density waves, see for instance [190, 191, 194, 195, 247].

[10] We refer to [248] for a discussion involving a dilatonic ghost. Ghost condensation is related to the spontaneous development of a harmonic time dependence, as such, is related to Floquet systems (see for a holographic discussion [249]).

degrees of freedom in (8.7.1) being associated to an emergent gauge redundancy at the quadratic level in the fluctuations.

Chapter 9

Key points of the field theory analysis

In this part of the thesis, we studied field theory toy models displaying the breaking of customary spacetime symmetries, namely the dilatation and the translation symmetries. The first observation we can make is that even the simplest cases are complicated! This is in agreement with the specificities of spacetime symmetries we mentioned in Section 5.1 in Part I. We considered symmetry breaking patterns which were slightly evading the hypotheses of validity of the different known results for NG modes presented in Part I. The outcome is that these general knowledge on NG modes seems to hold. Indeed, the analysis of the symmetry originated massless spectrum was systematically correctly guessed by imposing the IHCs and with a discussion on the canonical structure of the EFT we will have; for an EFT where the dominant term is a double time-derivative (e.g. relativistic EFTs), we do not expect canonical conjugation among the modes but when we introduce a chemical potential, we have to be more cautious since a single time-derivative term in the EFT might be introduced leading to a possible conjugation among the NG candidates. The classification of Nielsen and Chadha as well as the associated counting rule appeared to be appropriate in situations involving spacetime symmetry breaking, on the condition to be a bit loose concerning the analyticity of the dispersion relations (e.g. $\omega = \sqrt{q_x^2 + q_y^2}$ should be considered as linear despite it has no Taylor expansion).

Concerning massive NG modes coming from the introduction of a chemical potential, some subtleties where observed. The model in Chapter 7 suggests that dilatation breaking through a chemical potential leads to a creation mechanism of an unfixed gapped NG mode which is not predicted in Theorem 6 (this is not in conflict with the latter theorem since we are out of the scope of the assumptions). Working at finite density in the context of dilatation symmetry breaking imposed us to lift the flat directions[1]. Which means that at zero chemical potential (still with the flat directions lifted), no condensate would be present and the dilaton would be massive. This (partially) explains why the gap of the dilaton cannot be fixed solely by the chemical potential and by group theory, its gap already has a model dependent contribution coming from lifting the flat directions. The situation in the model of Chapter 8 is even more subtle because we have noticeably more symmetries at hand and only three real degrees of freedom (the three fluctuation modes). Lifting the flat directions in order to be able to switch on a chemical potential without destabilising the theory breaks explicitly the symmetries of internal shift. We already had a massive mode, so the fact to explicitly break symmetries would lead to the second

[1] In the pure internal case, the run away behaviour introduced by the term $-\mu^2\phi^2$ is already involved in the wrong sign of the mass of the Mexican hat potential. So, the theory is qualitatively speaking already stabilised with respect to such run away quadratic behaviour. In scale invariant theories, we do not have such a mass term. Hence, the run away behaviour due to the chemical potential is new and it destabilises the theory. This is corrected by lifting the flat directions.

observed massive mode (we have a non-trivial mixing among the NG candidates in the final spectrum, hence, the shiftonic part of the dispersion relation can provide it a mass). The remaining massless mode is explained by Goldstone's theorem which ensures at least one massless mode. Therefore, because the additional massive mode can be traced back to an explicit symmetry breaking (which also explains the dependence in both the chemical potential and the coupling parameter lifting the flat directions), it can be interpreted as a pseudo NG mode rather than a massive NG mode.

These toy models analysis permitted us as well to emphasise a possible link between two main currently active research topics in physics, namely Goldstone physics and fracton physics. The present-day paradigm for the study of fractonic modes in continuous model is to search for guidelines which would permit to build continuous field theories showing fractonic behaviour. The main direction of research to reach this goal is to directly start with models which enjoy strong subsystem symmetries or at least conservation of multipole moments. Our fundamental Goldstone model for translation symmetry breaking does not have such symmetries, however, it has the interesting feature that subsystem symmetries are emerging in the IR. And indeed, we recovered fractonic dispersion relations in our spectrum. The intuition is the following, in order to spontaneously break translation, we need higher derivative terms. It suggests that the effective theory for NG modes coming from the breaking of translations and of additional internal symmetries will have higher derivative terms. Usual NG modes already have shift like symmetries. Thanks to these higher derivative terms, these shift like symmetries might be promoted to polynomial shift symmetries[2]. It would then induce a conservation of multipole moments and so, a possible restriction on the modes' motion. If the derivatives of higher order are well arranged, we could even have the emergence of strong subsystem symmetries, for instance, an arbitrary space modulated shift symmetry. Therefore, all the intuitive requirements to have fractonic modes are met!

Beside the conceptual questionings on NG modes, some technical aspects have also been worked out. Usually, the parametrisation of the NG candidates is different from the way we parametrise the fluctuations around the background fields which is itself different from the parametrisation which diagonalises the kinetic matrix. The consequence is that the dispersion relations we obtain at the end are the ones associated to the diagonalising parametrisation. To discriminate which dispersion relations is connected to which NG candidates is highly involved. Even more that these connections depend on the norm of the spatial-momentum but also on its orientation. The breaking of translation symmetry has the tendency to display such intricate connections. Indeed, by breaking translation in one specific spatial direction, we establish anistropy. Moreover, the necessary higher derivative terms in the fundamental theory leads to a non-trivial dependence on the spatial-momentum in the kinetic matrix which leads to a non-trivial spatial-momentum dependent mixing while diagonalising the matrix. In the process of discriminating the NG candidates from the spectrum, we noticed that the Ward-Takahashi identities proved to be useful. Indeed, hasty said, it directly provides the dynamics of the associated NG candidates while the EOM provide the dynamics of all the fluctuations where the NG candidates are mixed between them and with the other degrees of freedom. In practice, it is of course not so black-and-white but it gives a basis on which to start to work on.

[2]The idea that the most relevant terms of the EFT describing NG modes enjoy polynomial spatial shift symmetries has already been suggested and studied in [106, 250, 251].

The reason the Ward-Takahashi identities focus on the dynamics of the associated NG candidates could be seen from the way the conserved current is defined in standard QFT textbooks. A usual method, sometimes known under the name "Noether's trick", is to promote the global continuous symmetry to be local, its parameter is now a spacetime function $\alpha(x)$. We then vary the action along the artificially-gauged transformation and we notice that we can re-write this variation under the form [5, 252]

$$\delta_{\alpha(x)} S = -\int d^d x \, \partial_\mu j^\mu(x) \, \alpha(x), \tag{9.0.1}$$

which defines the conserved current $j^\mu(x)$. We know that on-shell $\partial_\mu j^\mu = 0$ which ensures that on-shell $\delta_{\alpha(x)} S = 0$ for any spacetime modulated variation $\alpha(x)$ in the considered symmetry direction. The EOM are the equations which should be satisfied in order for the fields to be a stationary point of the action. Hence, the classical Ward-Takahashi identity $\partial_\mu j^\mu = 0$ is equivalent to the equation of motion for a fluctuation parametrised in the direction of the symmetry. It is the equation of motion of the NG candidate associated to the symmetry leading to the considered Ward-Takahashi identity. In practice the parametrisation of the fundamental fields is not necessarily the parametrisation which explicitly displays the NG candidates. This is why the Ward-Takahashi identities usually involve a mixing in the parametrisations and so, do not permit a clear discrimination among the modes. But by playing with some decoupling limits, we can extract some information.

The natural prospect is to look for more complex systems with the same symmetry breaking patterns. Holographic models are good candidates (specifically these are quantum computations with strong interactions), this will be briefly discussed in Part IV. On a larger time scale, a generic approach based on the coset construction could be implemented. In particular, concerning the link between fractons and NG modes, a coset construction similar to [253] could be considered where a polynomial shift symmetry is spontaneously broken. In addition, it will provide general knowledge on non-uniform symmetry breaking patterns.

By remaining in the realm of field theory toy models, we could focus on the relations between fractons and translation symmetry breaking by considering models displaying solely translation breaking rather than the concomitant breaking of dilatation and translation. A starting point will then be the model of [116]. Since the emergence of fractons from fundamental toy models displaying translation symmetry breaking does not require subsystem symmetries for the UV theory, we can try to build a toy model which reduces to the Shao-Seiberg's model in the IR after an SSB. It would provide some clues on the questions relative to the IR/UV mixing, and possibly solving it since we would have a scale, the VEV, at which the Shao-Seiberg's model would be cut-off. Another direction for toy model building could be to find higher derivative models with a usual kinetic term (the usual sign for the double spatial derivative term) enjoying translation SSB. It would help to make connections with known physical theories. From high energy physics perspective, it could be interesting to find relativistic UV completion theories of our non-relativistic toy models. And from experimental physics viewpoint, it would be worth to investigate if our toy models can play the role of phenomenological models for some physical processes.

Finally, the discussion should be extended to the non-homogeneous breaking of translation symmetries, in order to be able to describe the emergence of lattice structures. A glance at such kind of breaking pattern is provided in Appendix C.

Appendix C

A glance at non-homogeneous spatial translation symmetry breaking

In the present appendix, which entirely relies on the paper [45] written by Daniele Musso and by the author of this thesis, we present some of the peculiarities of a non-homogeneous breaking of spatial translation symmetry. It will provide the gist of the reason why such breaking is more involved compared to the homogeneous one. However, we will restrain our discussion to the breaking of continuous spatial translation symmetries to a discrete subgroup. The motivation is two-fold, the discussion will be simpler and it is physically relevant because the aim is to describe crystal structures, and their excitation modes, emerging from continuous fundamental theories.

The guidelines to build toy models showing such a symmetry breaking pattern remain the same as for the homogeneous breaking, namely we have to write a Mexican hat potential for the spatial gradients. Nevertheless, we do not need anymore an additional uniform symmetry to compensate the breaking of translation. Hence, the simplest model will then contain only one real scalar field in $1+1$-dimensional spacetime with no (apparent) symmetries beside the spacetime translation symmetry. This model will then be slightly changed to incorporate a shift symmetry as well, this in order to discuss the possible interplay between two symmetries which are spontaneously broken.

C.1 A real scalar model in $(1+1)$ dimensions

We consider a canonical kinetic term, in particular we avoid higher time-derivatives which would lead to Ostrogradsky instabilities [90,92]. We impose both spatial parity, $\partial_x \leftrightarrow -\partial_x$, and field-space parity, $\phi \leftrightarrow -\phi$. In an effective field theory spirit, we consider only terms up to the 4^{th} order in ϕ and up to the 8^{th} order in the spatial derivatives. Specifically, we take the model[1]

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{A}{2}\phi'^2 - \frac{m^2}{2}\phi^2 + \frac{B}{4}\phi'^4 + \frac{C}{2}\phi'^2\phi''^2 + D\phi''^4 \ , \tag{C.1.1}$$

where the dot indicates a time-derivative while the prime denotes a derivative along the only spatial direction x. We have considered a mass term so to break the rigid shifts $\phi \to \phi + c$ in the simplest possible way. The equation of motion descending from (C.1.1) is

$$\begin{aligned}\ddot{\phi} = \phi'' \left(A - 3B\phi'^2 + 4C\phi'''\phi' + 24D\phi'''^2\right) \\ + C\phi''^3 + C\phi'^2\phi'''' + 12D\phi''^2\phi'''' - m^2\phi \ .\end{aligned} \tag{C.1.2}$$

[1]This is not the most general Lagrangian we could write under our requirements. Nevertheless, the terms in (C.1.1) provide a simple setup able to capture the translation symmetry breaking mechanism that constitutes the focus of the present analysis.

We consider the static and harmonic ansatz

$$\phi(t,x) = \rho\cos(kx) \ , \tag{C.1.3}$$

characterised by a constant modulus ρ and a constant wave-vector k. It spontaneously breaks the translation symmetry along x to a discrete subgroup. In fact, the discrete symmetry transformations

$$x \to x + \frac{2\pi\, n}{k} \ , \tag{C.1.4}$$

where $n \in \mathbb{Z}$, remain unbroken. Plugging the ansatz (C.1.3) into the equation of motion (C.1.2), we get

$$\begin{aligned} 3k^4\rho^2\left[B + 2k^2(C - 6Dk^2)\right]\sin^2(kx) \\ - \left(Ak^2 + Ck^6\rho^2 - 12Dk^8\rho^2 + m^2\right) = 0 \ , \end{aligned} \tag{C.1.5}$$

which is satisfied for

$$A = -\frac{m^2}{k^2} - k^4\rho^2\left(C - 12Dk^2\right) \ , \tag{C.1.6}$$

$$B = -2k^2(C - 6Dk^2) \ . \tag{C.1.7}$$

Of course, these two equalities have to be reversed to rather expressed ρ and k in terms of the free parameters of the theory. To keep the mathematical developments brief, we will fix ρ and k and set accordingly A and B. This is not a fine tuning since we do not have apparent constraints on which values we chose for ρ and k.

We compute the diagonal components of the stress-energy tensor for a solution (C.1.3),

$$\begin{aligned} T_{tt} = \frac{1}{8}\Big\{&4\rho^2\left(m^2 - 4Dk^8\rho^2\right)\cos(2kx) \\ &+k^6\rho^4\left[-C + 12Dk^2 + \left(C - 4Dk^2\right)\cos(4kx)\right]\Big\} \ , \end{aligned} \tag{C.1.8}$$

and

$$T_{xx} = -\frac{1}{2}\rho^2\left(m^2 + 6Dk^8\rho^2\right) \ . \tag{C.1.9}$$

The model is invariant under translations, which translates into the Ward-Takahashi identity

$$\partial_\mu T^\mu_{\ \nu} = \partial_x T_{xx} = 0 \ . \tag{C.1.10}$$

Given the static character of the solution (C.1.3), in order to satisfy the 1-point Ward-Takahashi (C.1.10), the pressure T_{xx} needs to be x-independent. Nonetheless, the energy density T_{tt} is spatially modulated. This constitutes a signature of the non-homogeneous breaking of spatial translation symmetry.

We define the fluctuations

$$\phi(t,x) = \rho\cos(kx) + \varphi(t,x) \ . \tag{C.1.11}$$

The Lagrangian at linear order in the fluctuations is a total derivative, $\mathcal{L}_1 = \mathcal{B}'$, where

$$\mathcal{B} = -k^4\rho^3\cos(kx)\left(4Dk^2\cos(kx)^2 + C\sin(kx)^2\right)\varphi' - \frac{m^2\rho\sin(kx)\varphi}{k} \ . \tag{C.1.12}$$

The quadratic Lagrangian density is

$$\begin{aligned}\mathcal{L}_2 =& \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}m^2\varphi^2 \\ &+ \frac{2m^2 - 3k^6\rho^2(C-4Dk^2) + k^6\rho^2\cos(2kx)(7C-36Dk^2)}{4k^2}\varphi'^2 \\ &+ \frac{1}{2}k^2\rho^2\left[C\sin^2(kx) + 12Dk^2\cos^2(kx)\right]\varphi''^2 + Ck^3\rho^2\sin(2kx)\varphi'\varphi'' \ .\end{aligned} \tag{C.1.13}$$

The coefficients of the quadratic Lagrangian (C.1.13) are space-dependent. This is by definition the difference between homogeneous and non-homogeneous breaking of the spatial translation symmetry. Let us notice that the explicit dependence in x are through trigonometric functions such that the discrete transformations (C.1.4) are indeed still symmetries. This explicit space-dependency will be recovered in the EOM where the latter will be a linear differential equation with non-constant coefficients. Therefore, a single wave function in general will not be a solution anymore. We will thus need to redefine what we mean by modes. Moreover, energy is still well defined but not momentum since we lost the continuous spatial translation symmetry. Hence, a discussion will be needed to clarify what a mass is and what a dispersion relation is. Fortunately for us, the discrete translation symmetry will allow us to extend our already established definitions.

Despite the explicit space coordinate dependence, let us go to Fourier space and, as we could have expected, the various harmonic components of the fluctuation field are thereby mixed,

$$\begin{aligned}\tilde{\mathcal{L}}_2 =& a_0(k,\omega,q)\tilde{\varphi}(-\omega,-q)\tilde{\varphi}(\omega,q) + a_+(k,\omega,q)\tilde{\varphi}(-\omega,-2k-q)\tilde{\varphi}(\omega,q) \\ &+ a_-(k,\omega,q)\tilde{\varphi}(-\omega,2k-q)\tilde{\varphi}(\omega,q) \ ,\end{aligned} \tag{C.1.14}$$

where we only show the schematic form of the Fourier Lagrangian in a way that the important features are emphasised. What should be noticed is that the mixing occurs only among modes with fluctuation momenta that differs by $2k$, which is reminiscent from the discrete symmetry (C.1.4).

This peculiar mixing informs us that a solution will be a combination of plane waves where the wave vectors differ by an integer multiple of $2k$ and all these plane waves will have the same pulsation ω. We call the crystal wave vector, $q_{\text{cryst.}}$, the class of equivalent wave vectors[2] where the equivalence relation is that we identify all the wave vectors which differ by $2k\,n$, with $n \in \mathbb{Z}$: $q \sim q + 2k\,n$. Thus, a solution of the EOM can be labelised by one ω and one $q_{\text{cryst.}}$. Indeed, these two quantities are conserved through the evolution of the solution. The pulsation will be seen (quantumly speaking) as the energy and the crystal wave vector as a generalisation of the momentum. This is standard in condensed matter textbooks [25]. We can now extend what a mode is. A mode is a combination of solutions $(\omega, q_{\text{cryst.}})$ where ω is a periodic function of q of period $2k$. A massless mode is a mode for which it exists a particular crystal wave vector q_* which can be continuously approached such that

$$\lim_{q\to q_*} \omega(q) = 0 \ . \tag{C.1.15}$$

It is the generalisation of the definition we provided in Section 2.5 in Part I where the philosophy remains the same, a massless mode is a mode which can be static.

[2] In practice, $q_{\text{cryst.}}$ is the representative of the equivalence class and we take it to be in $[0, 2k[$.

For the analytic discussion on the stability of the fluctuations and on the resolution of the equation of motion, we refer to [45]. The goal of this appendix is to go straight to the point, for that reason we solely analyse a numerical resolution.

Depending on the specific values of the coefficients in the Lagrangian (C.1.1), one can find either stable or unstable modulated solutions. We consider a specific stable case, which is representative of a stability region within the coupling space, namely we take

$$m = 1 \, , \qquad C = -1 \, , \qquad D = -\frac{1}{10} \, , \tag{C.1.16}$$

We fix the coefficients A and B in the Lagrangian (C.1.1) according to (C.1.6) and (C.1.7) and requiring that the model admits a solution (C.1.3) with

$$k = 1 \, , \qquad \rho = 1 \, . \tag{C.1.17}$$

The numerics dispersion relations are provided in Figure C.1. The whole mode structure features two kinds of dispersion relations: an acoustic "bouncing" lower branch and optical upper branches which are concave and repeated for any multiple of $2k$. Usually in QFT we have one dispersion relation per real scalar fluctuation field. This is because the kinetic matrix can be diagonalised and that one single wave vector can be a solution. Therefore, each real scalar fluctuation has one dispersion relation. In our case, the situation is much more intricate, and due to the mixing among plan waves, several dispersion relations are possible despite having only one real scalar field. This is why we have this tower of modes in Figure C.1.

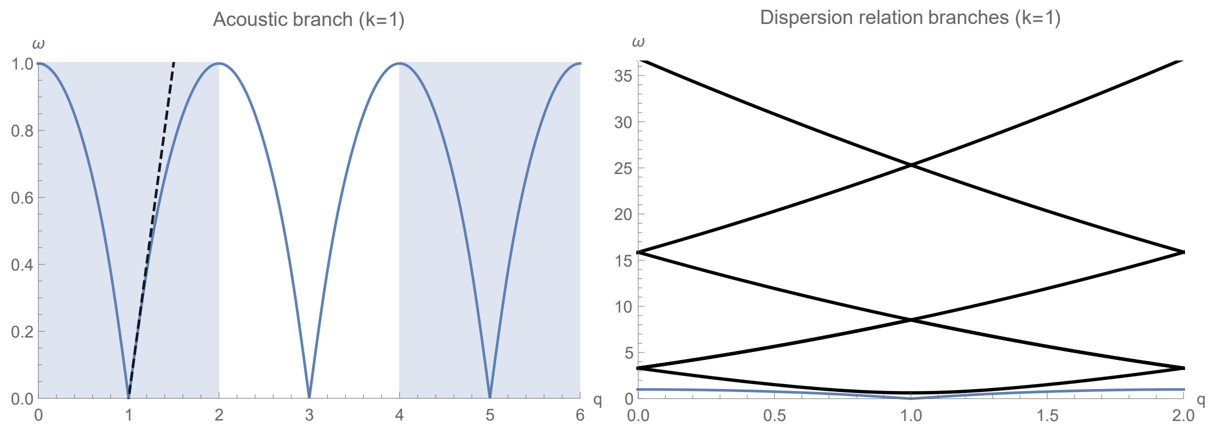

Figure C.1: Spectrum of the the linear fluctuations of model (C.1.1) around a solution (C.1.3) (the plots refer to the specific case (C.1.16) and (C.1.17)). Left: lower bouncing branch corresponding to acoustic phonons; the dashed line indicates the phonon propagation speed for $q \sim k = 1$. Right: The blue line is again the same acoustic phonon branch of the left panel, the black lines are instead the optical branches; these latter have polynomial concave shape (see Figure C.2, right panel) and there is a branch for any $2k$ multiple.

Formally, we already noticed that we have a Brillouin zone structure in the reciprocal space through the definition of $q_{\text{cryst.}}$. Thanks to Figure C.1, we can now concretely visualise the periodicity of $2k$ of $\omega(q)$ and how it permits to represent all the modes in only the first Brillouin zone (the right panel of Figure C.1). We therefore are getting closer to the description of a lattice structure. Our fluctuation is a continuous field and so, it could represent a lattice of periodicity given by (C.1.4) with an infinite number of elementary constituents (atoms, molecules ...) per unit cell. This interpretation is confirmed by recalling that the lowest phononic branch of a chain with any number of different atoms in the unit cell can be thought of as the mode of a chain with only one kind of atom; in fact, the lowest mode corresponds to the unit cell moving rigidly without deforming. Thus, we can compare our acoustic branch to the eigenfrequencies of the chain with only one kind of elementary constituent (such chain is modelled as a discrete chain of balls and springs) given by (see for instance [254])

$$\omega(q) = 2\sqrt{\frac{g}{\bar{m}}}\left|\sin\left(\frac{q}{2}\right)\right| \ , \tag{C.1.18}$$

where $\bar{m}$ is the mass of the balls and g is the second derivative of the potential between two neighbouring balls. The comparison is made in Figure C.2. To push further the comparison to discrete chains, if we look to a chain with several types of atoms, the internal oscillations within a unit cell correspond to optical modes. This might give a physical picture of the reason why we have a tower of optical modes in our continuous model.

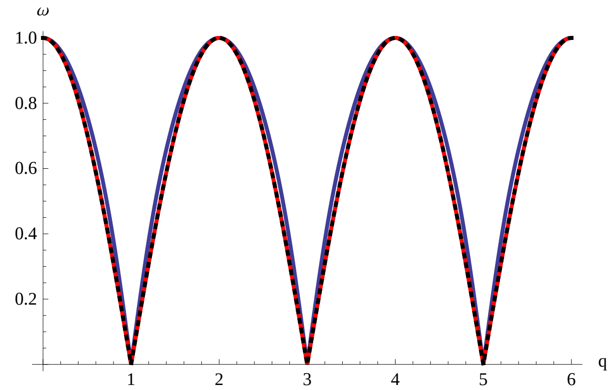

Figure C.2: Comparison of the phonon dispersion relation of model (C.1.1) with that of the one-dimensional chain: the black dashed line corresponds to (C.1.18) with $q \to \pi(q-1)$ and $\frac{g}{\bar{m}} = \frac{1}{4}$; the blue line is the phonon dispersion relation of Figure C.1; the red line is the phonon dispersion relation obtained with $k = \rho = m = 1$ and $C = -\frac{173}{50}$, $D = -2$, which approximates (C.1.18) to the .001 level.

Finally, we have broken one global continuous symmetry, according to Goldstone's theorem there should be a massless mode in the spectrum. In Figure (C.1) on the left panel, we do indeed observe a massless mode where the crystal wave vector corresponding to the static fluctuation is $q_* = 1 = k$ where we recall that for our numerical resolution $k = 1$. This value for the crystal wave vector can be understood. Our background (C.1.3) is static and under the infinitesimal action of translation symmetry parametrised by a, it provides a new static solution

$$\tilde{\phi}(t,x) = \rho\cos(kx) - \rho\, a\, k\, \sin(kx) \ . \tag{C.1.19}$$

We observe that the static fluctuation along the direction of the broken symmetry has a wave vector equal to k, this because our original background has itself a spatial modulation parametrised by k. Hence, our massless NG mode will be static when $q_* = k$. We name this NG mode as phonon due to its translation symmetry breaking origin. Let us mention that for the case of the breaking of internal symmetries, the background is constant, therefore, a linear transformation under the action of a symmetry provides a static fluctuation which is also constant. This is why the static part of NG modes takes place when their momentum is zero.

C.2 A shift-symmetric model

We modify (C.1.1) setting to zero the mass term and introducing a term with third derivatives,[3]

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{A}{2}\phi'^2 + \frac{B}{4}\phi'^4 + \frac{C}{2}\phi'^2\phi''^2 + D\phi''^4 + E\phi'^2\phi'''^2 \ . \tag{C.2.1}$$

The field ϕ appears in the Lagrangian only through its derivatives, so constant field shifts are a symmetry of (C.2.1). We consider again the ansatz (C.1.3), thus obtaining the following equation of motion

$$\begin{aligned} 3k^4\rho^2 \sin^2(kx)\left\{B + 2k^2\left[C + 2k^2(5E - 3D)\right]\right\} \\ - k^2\left\{A + k^4\rho^2\left[C + 12k^2(E - D)\right]\right\} = 0 \ , \end{aligned} \tag{C.2.2}$$

which is solved by

$$A = k^4\rho^2\left[12k^2(D - E) - C\right] \ , \tag{C.2.3}$$

$$B = -2k^2\left[C + 2k^2(5E - 3D)\right] \ . \tag{C.2.4}$$

The ansatz (C.1.3), when considered as a solution for model (C.2.1), breaks both translations and shift symmetry. We thus expect to have a massless mode both around $q_* = 0$ (the shifton[4]) and around $q_* = k$ (the phonon).

The quadratic Fourier Lagrangian is still of the form (C.1.14) but with other coefficients. Since the quadratic Lagrangian connects only wavevectors which differ by even multiples of k, the modes about $q = 0$ and those about $q = k$ can be studied separately and the shifton and phonon sectors "decouple".

Once again, we refer to [45] for the discussion on the analytic resolution. We restrict ourselves here to the numerical resolution. We consider the specific case[5].

$$C = 0 \ , \qquad D = -1 \ , \qquad E = -\frac{1}{10} \ , \tag{C.2.6}$$

[3] The new higher-derivative term appeared to be necessary to obtain stable backgrounds

[4] An infinitesimal transformation parametrised by a of the background along the shift symmetry direction is of the form

$$\tilde{\phi}(t,x) = \rho\cos(kx) + a \ . \tag{C.2.5}$$

The associated static fluctuation has no modulation along the spatial direction, which is the reason why we expect a massless mode around $q_* = 0$.

[5] If we just take $m = 0$ in (C.1.16), so considering $m = 0$, $C = -1$, $D = -1/10$ and $E = 0$, we find a case that features an imaginary propagation speed for the phonon.

and determine A and B using (C.2.3) and (C.2.4) upon requiring to have a solution for

$$k = 1 \ , \qquad \rho = 1 \ . \tag{C.2.7}$$

In Figure C.3 we see that the acoustic phonon branch has developed shiftonic dips. The overall periodicity is still $2k$, but we have light modes for any integer multiple of k.

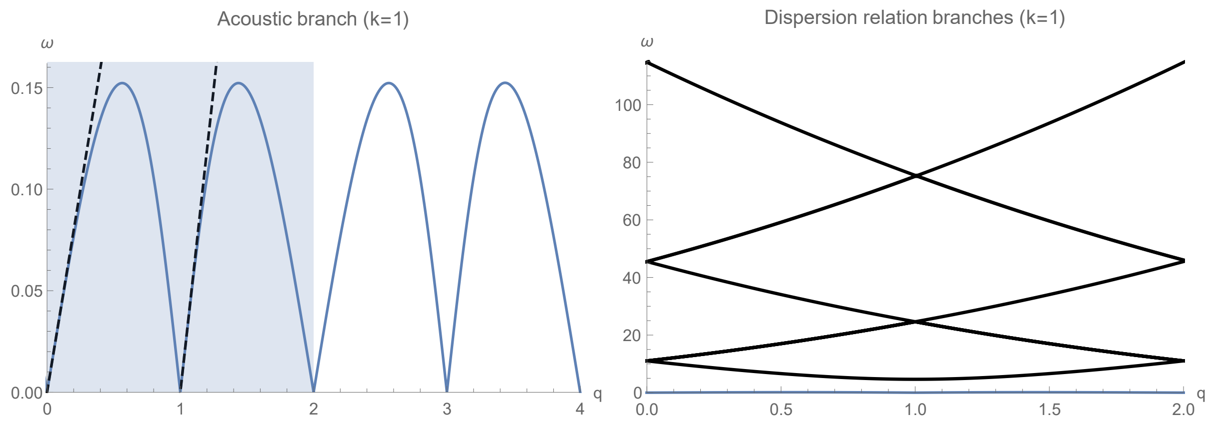

Figure C.3: Spectral structure of the shift-symmetric model (C.2.1) in the case (C.2.6) and (C.2.7). Left: acoustic branch featuring shifton ($q \sim 0{+}2k$) and phononic ($q \sim k{+}2k$) linear dispersion regions. Right: Tower of optical modes; the flat blue line coincides with the bouncing dispersion curve of the left plot.

Let us discuss the number of massless NG modes based on the general arguments presented in Part I of this dissertation. Strictly speaking, we have one massless mode which corresponds to the acoustic branch in Figure C.3 on the left panel. This can be explained by the argument (5.2.69). Indeed, if P_x and P_s are the generators of respectively the spatial translation transformation and the internal shift transformation, we have

$$\Big(\xi(t,x)P_x + \alpha(t,x)P_s\Big)\, \phi(x) = 0 \ , \tag{C.2.8}$$

$$\Leftrightarrow \Big(-\xi(t,x)\partial_x\, \rho\cos(kx) + \alpha(t,x)\Big) = 0 \ , \tag{C.2.9}$$

$$\Leftrightarrow \alpha(t,x) = -\xi(t,x)\, \rho\, k\, \sin(kx) \ . \tag{C.2.10}$$

Through their intrinsic definitions, the shifton and the phonon are equivalent. Only one massless NG modes is resulting. It is interesting to notice that the inverse Higgs constraints do not recover this result. The only unbroken translation symmetry is time translation (P_0) and we have

$$[P_0, P_x] = 0 \ , \qquad [P_0, P_s] = 0 \ . \tag{C.2.11}$$

None IHC can be imposed. We are out of the scope of the conjecture of Susection 5.2.8 in Part I and it appears that from the non-homogeneous breaking of translation, imposing all the IHCs is not enough to obtain all the independent massless NG mode prior any dynamical considerations.

Nevertheless, the discussion is subtler because if we go at enough low energy, the acoustic branch in Figure C.3 is truncated. Effectively, we then observe two distinct dispersion relations and so, two massless modes. We recover the same kind of discussion in Bose-Einstein condensate superfluid literature [8] where in Figure B.1, the linear part of the curve is seen as one excitation mode and where the local quadratic minimum is seen as another excitation mode. The thermal properties of superfluids at low temperature are computed with these two separated excitation spectra and it provides results fitting with experiments.

The goal of this appendix was not to present a self-contained and a self-explanatory discussion on the non-homogeneous breaking of translation symmetries but rather to display the difficulties associated to such breaking pattern. As we have seen, compared to standard QFTs, the definitions of modes and mass should be re-think, the dispersion relations are more involved and the discussion on the numbering of massless NG modes is subtle and might differ from the general statements of Part I of this thesis.

Appendix D

Journalistic overview of fracton physics

In this appendix, we do a journalistic overview of a specific species of excitation modes, namely the fractonic modes. It is a recent research area which started in the mid 2010 decade and which has ramifications in a wide variety of domains in physics, ranging from condensed matter lattice spin models to fundamental QFT passing by elasticity theory, higher rank gauge theories and many other areas. Since the study of fractonic modes is a new research topic with implications and promising results in many fields, it is a very dynamical subject with still a large number of open questions. The literature associated to fractonic modes is then vast and evolving on a daily basis. This is the reason why we emphasise the fact that we are going to do here only a journalistic introduction to the field. This introduction is mainly based on the three following reviews [196, 197, 255]. We refer the reader to the references therein for more details. However, concerning the necessary elements of fracton physics needed for this thesis, the specific papers will be explicitly cited and commented.

A fractonic mode is an excitation, or a perturbation, with reduced mobility. It means that it cannot propagate by its own in some spatial directions: either we need to activate some additional excitations (i.e. we need to pay an extra energy cost) to make our initial fractonic mode move either our fractonic mode is moving with no additional energy cost but, other excitations already existent in the system should move according to the motion of the fractonic mode. This definition of reduced mobility will be illustrated via some examples all along this appendix. When a mode is totally immobile, we call it a fracton[1]. When a mode can only move along a line, we call it a lineon and if it can only move along a plane, we call it a planeon.

There are several motivations to be interested in the study of fractonic modes. One could be their universality, the definition of reduced mobility is generic and so, it is not surprising to recover this concept in many areas of physics. Schematically, we could think to lattice spin models where a local excitation, by moving, tends to modify the spins orientation in its neighbourhood and so inducing additional local excitations along its trajectory. We could as well think of crystal structure where it is already known that crystal defects such as disclinations have constrained motion. From elasticity theory we can then make a connection with field theory and naively, recover the notion of fractonic modes in QFT. Another motivation for performing research on fractonic modes could be the fact that they can have explicit phenomenological signature. For example, their immobility tends to make them harder to thermalise and so, we have an observable increase of the thermalisation time. A third motivation is the potential usability of fractonic modes

[1] It should be mentioned that the term "fracton" has been previously used to refer to small-scale thermal vibrations of fractal structures [256].

in applied science, e.g. their reduced mobility in q-bit lattice systems could be exploited for quantum information storage ("hard drive" for quantum computers). Finally, the study of fractonic modes leads to conceptual questions. As we will see, it challenges some of the paradigms in QFT, in particular the effective field theory approach. This last motivation is maybe the most important for us since the main interest of Goldstone physics relies on the Wilsonian effective field description. We will illustrate the UV/IR mixing at the origin of this conflict between QFT for fractonic modes and standard QFT but, we will not expand on it. This because the discussion is still open in the literature and also (mainly) because the author is not familiar with a pure axiomatic approach of QFT.

D.1 Fractonic modes in solvable lattice models

The term "fractonic mode", in the sense of reduced mobility, appeared for the first time in the literature of lattice spin systems. Of course, prior to the introduction of this nomenclature in 2015, there were already existing (lattice) models displaying excitations with constrained motion, e.g. [257, 258]. It is in [259], that Vijay, Haah and Fu, noticed this recurrent kind of excitations with specific features and proposed the term "fracton". In their paper [260], they offer a first attempt of a generic description and classification of fractonic modes. Since discrete models are not encompassed in the framework of this thesis which focus on continuous field theories, we are going to restrict ourselves to an example of fractonic lattice model and illustrate through this example the philosophy of the generic approach initially proposed by Vijay, Haah and Fu. The continuous case will be discussed more thoroughly in the next section.

A standard example of a solvable lattice spin model displaying fractonic modes is the X-cube model. The geometry is a 3 spatial-dimensional cubic lattice with a q-bit (i.e. a spin $\frac{1}{2}$) at each edge, as represented in Figure D.1. The Hamiltonian is academically given by

$$H = -\sum_c B_c - \sum_v \left(A_{v_x} + A_{v_y} + A_{v_z}\right) \ , \tag{D.1.1}$$

where c runs over all the cubes and v runs over all the vertices of the cubic lattice. The B_c and A_{v_j} ($j = x, y, z$) operators are defined as

$$B_c = \prod_i \sigma^z_{c_i} \ , \qquad A_{v_j} = \prod_k \sigma^x_{v_{j,k}} \ , \tag{D.1.2}$$

where σ^l is a Pauli matrix ($l = x, y, z$) with the subscript indicating on which q-bit it acts on. We have that i runs over all the q-bits of the considered cube c (so, a product of 12 Pauli matrices) and where k runs over all the q-bits which are direct neighbours to the considered vertex v and which lie in the plane orthogonal to the j-direction[2] (so, a product of 4 Pauli matrices). For a schematic visualisation[3], cf. Figure D.1.

The X-cube model is solvable because the operators H, B_c and A_{v_j} mutually commute with each other $\forall c, v_j$. Indeed, $[B_c, A_{v_j}] = 0$ is direct if $v \notin c$ since they do not share any

[2] I.e. the normal vector of the considered plane is parallel to the j-direction.

[3] The cubic and crosslike geometries of the interactions motivate the name "X-cube" model.

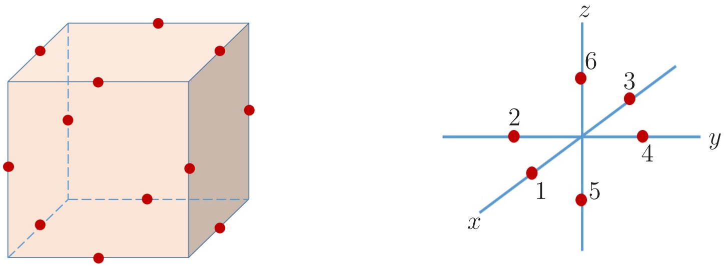

Figure D.1: On the left, an elementary cube of the cubic lattice is represented. At each red dots there is a q-bit. These red dots indicate as well where the 12 Pauli matrices σ^z of the operator B_c act on. On the right, it is the disposition of the closest neighbour q-bits seen by a vertex. It permits to visualise the definition of the A_{v_j} operators. For example, $A_{v_x} = \sigma^x_{v_x,2}\sigma^x_{v_x,6}\sigma^x_{v_x,4}\sigma^x_{v_x,5}$. These cartoons have been modified from [197].

q-bits. If $v \in c$, the vertex and the cube share 2 q-bits, so

$$B_c\, A_{v_j} = \prod_i \sigma^z_{c_i} \prod_k \sigma^x_{v_{j,k}} = (-1)^2 \prod_k \sigma^x_{v_{j,k}} \prod_i \sigma^z_{c_i} = A_{v_j}\, B_c \ , \tag{D.1.3}$$

where we used $\sigma^z\sigma^x = -\sigma^x\sigma^z$. Finally, $[B_c, B_{c'}] = 0 = [A_{v_j}, A_{v'_k}]$ because $[\sigma^l, \sigma^l] = 0$ with $l = x, y, z$. Therefore, schematically we have

$$[H, B] = [H, A] = [B, A] = [B, B] = [A, A] = 0 \ . \tag{D.1.4}$$

We can then simultaneously diagonalise them. Let us consider a ground state $|0\rangle$ where we normalise the ground state energy to zero, so, our chosen ground state is such that

$$H\,|0\rangle = 0\,|0\rangle \ , \tag{D.1.5}$$

$$B_c\,|0\rangle = b_c\,|0\rangle \ , \tag{D.1.6}$$

$$A_{v_j}\,|0\rangle = a_{v_j}\,|0\rangle \ , \tag{D.1.7}$$

where b_c and a_{v_j} are real numbers, since B_c and A_{v_j} are Hermitian ($(\sigma^l)^\dagger = \sigma^l$).

An important remark is that the ground state is largely degenerate. This is due to a large number of symmetries that the Hamiltonian has. Indeed, we can flip all the spins on a given plane of the cubic lattice, the obtained spin configuration will have the same energy than the state we started with. And this is true for any planes. Formally, flipping a spin is done by applying the σ^x operator on it: $\sigma^z\sigma^x = -\sigma^x\sigma^z$. It is direct that the operators A commute with the symmetry operator ($[\sigma^x, \sigma^x] = 0$). When we flip all the spin of a given plane, each cube sees either zero of its spin flipped or four of them. In any case, the operators B commute with the symmetry operator (commuting four σ^x operators on a B operator brings a factor $(-1)^4 = 1$). So, we indeed have that H commutes with the symmetry operator. These kinds of symmetries which act on a subregion of the lattice are called subsystem symmetries. In the language of QFT, a subsystem symmetry is a symmetry which acts on a subregion of the spatial part of spacetime. In the framework of our classification of symmetries established in Section 2.2 in Part I, subsystem symmetries are symmetries with generators that act non-trivially only on a spatial sub-manifold of the system [261]. They belong to the set of non-uniform symmetries.

It is not explicit in our development that the ground state $|0\rangle$ spontaneously breaks the planar subsystem symmetries but, from the literature, it does and it leads to a ground state degeneracy GSD on a $L_x \times L_y \times L_z$ cube of [196]

$$\log_2 \mathrm{GSD} = 2(L_x + L_y + L_z) - 3 \ . \tag{D.1.8}$$

The dependence on the size of the system can be understood by the fact that bigger the system is, more planes are available. And so, the number of planar subsystem symmetries is larger, which leads to an increase of the ground state degeneracy. A non-trivial computation leads to this exponential scaling with the size of the system, this is peculiar. Furthermore, this large GSD is not due to a fine tuning. Indeed, the subsystem symmetries are not affected by a tuning of the coefficients in H. Furthermore, we cannot lift the degeneracy with a local operator since a lot of planar subsystem symmetries will remain anyway. So, this exponential GSD scaling with the size of the lattice is robust.

Let us now focus on the core of this appendix: fractonic modes. We define, a priori, an excitation of only one cube, the cube c'. The system state $|1c'\rangle$ is such that

$$A_{v_j}|1c'\rangle = a_{v_j}|1c'\rangle \ , \tag{D.1.9}$$

$$B_c|1c'\rangle = \begin{cases} b_c|1c'\rangle & \text{if } c \neq c' \\ -b_{c'}|1c'\rangle & \text{if } c = c' \end{cases} \ , \tag{D.1.10}$$

$$\Rightarrow H|1c'\rangle = 2b_{c'}|1c'\rangle \ . \tag{D.1.11}$$

To create such state from $|0\rangle$, we have to act σ^x on one of the q-bit c'_i belonging to the cube c': $B_{c'}\sigma^x_{c'_i} = -\sigma^x_{c'_i}B_{c'}$. But by doing so, we excite as well the 3 other cubes which share with c' the q-bit c'_i. Hence, it is impossible to excite a single isolated cube from the vacuum[4]. However, we can get an isolated cube excitation by creating several cube excitations distant from each other. To do so, we have to apply a non-local operator, in this case, a membrane operator. Such operator is defined geometrically: we consider a rectangle R of finite size perpendicular to one of the Cartesian directions of the cubic lattice. To each edge it intersects, we apply σ^x on the q-bit associated to the vertex, cf. Figure D.2 for an illustration. So, if $R \bigcap c = 0$ then the cube c is not excited. If $R \bigcap c \neq 0$ and c is not at a corner of R, then there is an even number (either 2, either 4) of σ^x acting on c. Thus c is not excited. Finally, if $R \bigcap c \neq 0$ and c is at a corner of R, there is only one σ^x acting on c. As we have seen before, it means that this cube is excited. We end up with four isolated cube excitations, one at each corner of the membrane operator. To create these cube excitations, due to the discretisation of the energy spectrum of the quantum lattice model, it required a finite amount of energy (as it can be seen from (D.1.11)). Hence, we call such kind of excitations gapped modes.

One excited cube (obtained from the process described above) cannot propagate by its own. To individually propagate such excitation, we need to apply σ^x on one of the q-bit belonging to the cube. As we have seen, instead of just killing the cube excitation and create another one step further, it creates 3 excited cubes (see Figure D.2 for a cartoon of this reasoning). So, to move our initial excitation, we need to pay an extra cost in energy. This is what we mean by an immobile mode. One isolated excited cube is a fracton!

[4]Thus, the state $|1c'\rangle$ does not formally exist, it has been introduced to define the notion of an isolated excited cube.

Let us notice that starting from our four isolated fractons at the corner of our membrane operator, we can move them without an additional energy cost by applying the same membrane operator but bigger. A collective motion is thus tolerated, on the condition that it respects this "quadrupole" layout. As we will see with the continuous models, this analogy to the quadrupole moment conservation is not innocuous.

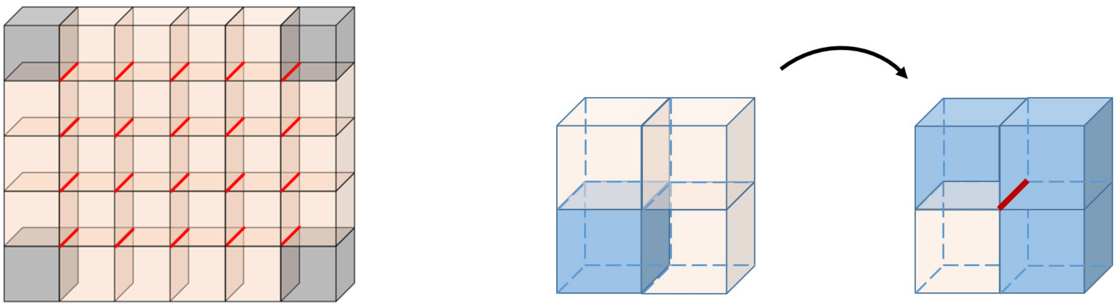

Figure D.2: On the left, we have a schematic representation of the action of a finite size membrane operator on the cubic lattice. The red edges are the ones on which a σ^x has been applied on. On the right, we illustrate the creation of two additional cubic excitations while trying to move a single isolated cubic excitation. These cartoons have been taken and modified respectively from [196] and [197].

The X-cube model possesses another fractonic mode. To see it, let us define the excitation of a vertex v' by the application of a σ^z on one of the closest neighbour q-bits to v'. It then excites two of the three operators $A_{v'_j}$. We cannot create one single isolated vertex excitation from $|0\rangle$. This due to the fact that exciting v' excites as well the vertex linked to v' through the edge containing the q-bit acted on with σ^z. However, by applying a non-local operator on $|0\rangle$, we can create two isolated vertex excitations. This time, the non-local operator is a straight line joining the two vertices v_1 and v_2 (see Figure D.3). On each q-bit on this line, we apply a σ^z. So, each vertex which is not at an extremity of the line (i.e. which is neither v_1 nor v_2) has two σ^z along a line acting on them, thus, none of the three A_{v_j} operators are excited. But the extremities, namely v_1 and v_2, have only one σ^z acting on them, so two of the three A_{v_j} operators are excited. Hence, the vertices at the extremities of the straight line operator are excited. One vertex excitation can move alone and without any additional energy cost only in the same direction than the line operator which has created it. To do so, we just need to apply σ^z on the next vertex in the line direction. If we make a perpendicular turn, then our initial excited vertex is still excited (but this time it is another pair of the three A_{v_j} operators which are switched on) as well as the vertex at the new extremity of the line operator. So, it required an additional excitation to make this turn. We conclude that a vertex excitation is a lineon!

Morally, the existence of fractonic modes in a lattice system can be explained by the strong conservation laws associated to the subsystem symmetries [261]. This intuitive reasoning is supported by the cartoon of Figure D.4. A symmetry charge associated to a planar subsystem symmetry, by charge conservation, can only move in the plane of this planar subsystem symmetry, it is a planeon. A similar charge at the intersection of 3 mutually orthogonal planes cannot move at all – it is a fracton, if the number of planes is reduced to 2, then the charge can only move along the intersection line of the 2 planes

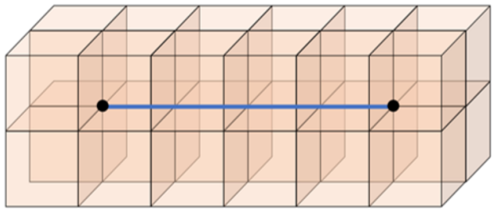

Figure D.3: This figure [196] shows the action of a line operator where on each blue edge, a σ^z operator has been applied. The two black dots correspond to the excited vertices.

– it is a lineon.

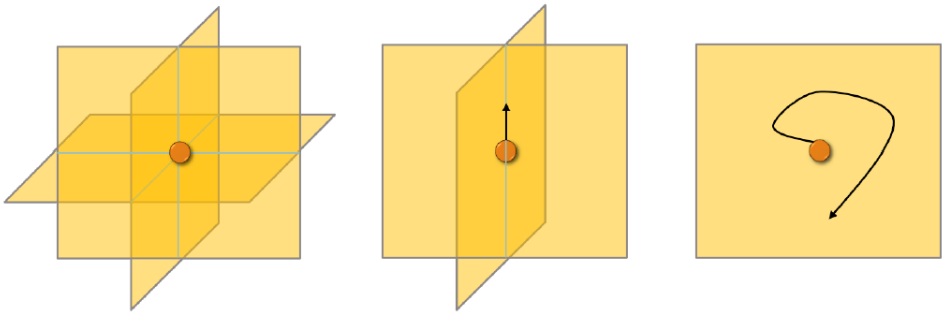

Figure D.4: From left to right, we have a charge fracton, a charge lineon and a charge planeon. This cartoon has been taken from [261].

In 2016, Vijay, Haah and Fu, proposed a first systematical approach to search for and to characterise fractonic modes [260]. Their idea is that a fractonic spin lattice model is dual to a generalised gauge lattice model obtained from the gauging of the subsystem symmetries that another spin lattice model has. The term "generalised" means that the gauge fields are at the center of the plaquettes (i.e. the faces of the cubes) rather than at the edges as usual. The duality is in the sense that the gauge model and the fractonic model share the same vacuum properties (e.g. the ground state degeneracy) and have the same excitation spectrum (e.g. the fractonic modes). The analysis of Vijay, Haah and Fu has been extended and improved by several other physicists. A starting point to look for the recent developments in this path to fracton classification could be [261]. In particular, we are going to use the general procedure for gauging subsystem symmetries proposed in [261] to display the dual gauge model of the X-cube model.

We consider a 3 spatial-dimensional cubic lattice model with a q-bit at each vertex v. The Hamiltonian is given by

$$H' = -\sum_v \tau_v^z \ , \tag{D.1.12}$$

where the τ_v operators are interpreted as the matter "fields". This model is relatively simple and H' is directly invariant under the local action of $\tau_{v'}^z$ on any vertices v' of the lattice. The X-cube model is one of the simplest examples of fractonic models, maybe "too simple", which might lead to some triviality when applying the procedure proposed by [260, 261]. So, just for the purpose of illustration, let us ignore that H' is already gauge invariant and let us apply the gauging procedure of [260, 261]. H' has the planar subsystem symmetries

$$U_{n,j} = \prod_{v_{n,j}} \tau^z_{v_{n,j}} \ , \tag{D.1.13}$$

where $j = x, y, z$ gives the perpendicular direction of the considered plane, $n \in \mathbb{N}$ labelises the possible planes perpendicular to the j-direction and v_{nj} are all the vertices inside the considered plane. To gauge these subsystem symmetries, we switch on gauge "fields" by introducing σ^l operators at the center of each plaquette of the cubic lattice, where $l = x, z$. The gauge symmetry we are going to implement is

$$G_{v'} = \tau^z_{v'} \, B_{v'} \ , \tag{D.1.14}$$

where

$$B_{v'} = \prod_i \sigma^z_{v'i} \ , \tag{D.1.15}$$

with i running over the 12 closest plaquette neighbours of v'. The gauge invariant model proposed by the procedure is

$$H'_g = -\sum_v \tau^z_v - \sum_{c,j} \prod_i \sigma^x_{c,i_j} \ , \tag{D.1.16}$$

where i runs over all the plaquettes of the cube c except the two plaquettes with their normal vector parallel to the j-direction ($j = x, y, z$). Pictorially, $\prod_i \sigma^x_{c,i_j}$ has a matchbox configuration, as represented in Figure D.5. H'_g indeed commutes with $G_{v'}$ $\forall v'$, it is direct for the first term in H'_g. Concerning the second term, the matchbox configuration of a given cube c shares either zero or two gauge fields with the closest gauge field neighbours of v'. Thus, the commutation leads to a factor of 1 or of $(-1)^2 = 1$.

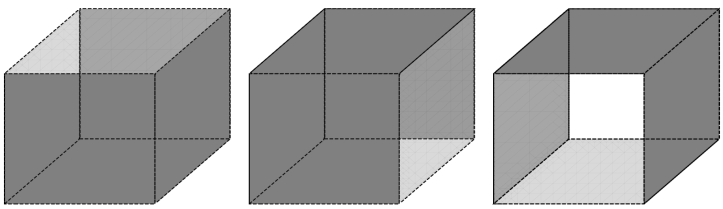

Figure D.5: This figure displays a pictorially visualisation of the operator $\prod_i \sigma^x_{c,i_j}$. For example, the rightest cartoon corresponds to $\prod_i \sigma^x_{c,i_x}$ where the grey plaquettes of the considered cube are the ones on which a σ^x is applied on. This figure has been taken from [262].

The extremely large degeneracy of states due to the gauge symmetry is unphysical. Hence, we have to fix the gauge. Let us impose the physical states of the system to be gauge invariant

$$G_{v'} \left|\psi\right\rangle = \left|\psi\right\rangle \ , \forall v' \qquad \Leftrightarrow \qquad \tau^z_{v'} \, B_{v'} \left|\psi\right\rangle = \left|\psi\right\rangle \ , \forall v' \ . \tag{D.1.17}$$

This can be satisfied with

$$B_{v'}\left|\psi\right\rangle = \tau^z_{v'}\left|\psi\right\rangle \ ,\forall v' \ , \tag{D.1.18}$$

because $\left(\tau^z_{v'}\right)^2 = I$, the identity. So, on our physical Hilbert space, our gauge model becomes ($\tau^z_v \leftrightarrow B_v$)

$$H'_{g,\text{fixed}} = -\sum_v B_v - \sum_{c,j}\prod_i \sigma^x_{c,i_j} \ . \tag{D.1.19}$$

We can redraw the cubic lattice model in order to place the gauge fields on the edges of the cubes rather than at the center of the plaquettes. With a non-trivial geometrical visualisation, we can convince ourselves that we recover the X-cube model:

$$H'_{g,\text{fixed}} = -\sum_c B_c - \sum_v \left(A_{v_x} + A_{v_y} + A_{v_z}\right) = H \ . \tag{D.1.20}$$

Beside the illustration of the duality proposed by Vijay, Haah and Fu, this computation offers an additional interpretation that as we will see, we recover in the continuous models. We can notice that (D.1.18) can be interpreted as a Gauss law if we see σ^z as the electric field. Indeed, if we develop the $B_{v'}$ operator, we have

$$\prod_i \sigma^z_{v'_i}\left|\psi\right\rangle = \tau^z_{v'}\left|\psi\right\rangle \ ,\forall v' \ . \tag{D.1.21}$$

This equality tells us that the "integration" of the electric field σ^z on a closed surface around the vertex v' gives the matter field at this vertex. This is similar to the standard Maxwell Gauss law

$$\oint_S d\vec{S}\,\vec{E} = Q \ , \tag{D.1.22}$$

where Q is the electric charge (more generically, the gauge charge) inside the volume surrounded by the closed surface S and $\vec{E}$ is the Maxwell electric field. Since B_v becomes B_c, and since we have seen that the excitation of a B_c is a fracton, the Gauss law (D.1.21) suggests that the fractons can be interpreted as the gauge charges of the gauge model dual to our fractonic model.

Through the example of the X-cube model we have displayed some of the main aspects of fracton physics in lattice models. Lattice models with fractonic modes are a still ongoing active research topic. Some of the directions of investigation are the influence of the geometry, of the topology and of the dimension of the manifold on which the lattice is drawn, the generalisation of the known toy models, the search of a generic classification of fractonic modes, the modelling of physical models allowing for experimental testing, the experimental measurements of fractonic modes and the engineering applications.

Key points of fractonic lattice models

What should be remembered about factonic lattice models is that such models display

1. A robust large ground state degeneracy which scales with the size of the system.

2. Excitations with reduced mobility, called fractonic modes. To create such excitations, due to the discretisation of the energy spectrum of quantum lattice models, the energy cost is finite. We therefore speak of gapped fractonic modes.

3. Subsystem symmetries. Morally, these subsystem symmetries imply the two first points of this list: point 1 because of spontaneous symmetry breaking, point 2 due to the conservation laws restraining the charges' motion. It is interesting to notice that gauging subsystem symmetries seems to play a role through the duality initially established in [260]. Especially, the fact that gauge charges appear to be the fractonic modes.

D.2 Fractonic continuous models

The paradigm of the Wilsonian renormalisation group flow is that low-energy physics is well captured by a (quantum) field theory description. According to this mindset, there exists an effective field theory for each lattice model, the latter being seen as the UV theory of the EFT. In particular, it should be true for fractonic lattice models and so, there should be an equivalent notion of fractonic modes in continuous field theory. However, the particular features of the fractonic lattice models that we recapitulated at the end of the previous section challenge our usual vision of field theory. Indeed, we should be able to find QFTs with a large number of degenerate vacua where this degeneracy scales with the dimension of the system. From standard QFT textbooks, we do not have explicit examples of such kind of field theories. In addition, we should also find a way to describe the robustness of this large degeneracy. The large number of vacua in spin lattice models means that there are many possible spin configurations giving the same zero energy. Among these spin configurations, some of them spatially change rapidly and can thus be interpreted as states with high momenta, despite the fact that they have a zero energy. We observe that we have a UV/IR mixing which comes into conflict with EFTs where physics at large distance is considered as equivalent to low energy. We can push a bit further the discussion by mentioning that a rapid spatial modulation of a state in a lattice could correspond to a non-continuous field configuration, which is usually not described by standard QFTs. Dealing with subsystem symmetries (and their gauging) is also a singular notion that we do not recover in the textbooks. Finally, fractonic modes in lattice models are gapped and so, a priori, they should not be captured by EFTs. On this last comment, as we will see, we will introduce a notion of gapless fractonic modes in the framework of continuous field theory[5].

Hence, the quantum field theories featuring fractonic behaviour are quite peculiar and might lead us to revisit the fundamental notions of QFT. Specifically for this thesis, this conflict with the standard EFT approach challenges some aspects of Goldstone physics, in particular the naive notion that NG modes can universally capture low-energy physics. That is, by definition, what makes an interesting and rich research topic!

As we have argued, continuous models for fractonic physics are involved. Thus, before trying to establish a generic description and classification of these models, the current strategy in the literature is to find toy models displaying the desired features and by doing so, to acquire some knowledge and intuition on the characteristics of fractonic continuous models. To achieve this goal, there is the top-down approach which consists of considering particular fractonic lattice models and to explicitly compute their continuous

[5]But it should be emphasised that in the literature (some) gapped fractonic modes are successfully described by field theories by considering them as defects of the field configuration [263].

limit. The other approach is the bottom-up one which, based on general concepts such as subsystem symmetries, tries to directly write field theories which have fractonic behaviour. For this journalistic overview, we are going to mainly focus on the bottom-up approach since it is the less "axiomatic" QFT perspective and so, this approach fits better with the guideline of this thesis: considering non-trivial toy models to display particular features of Goldstone physics and in this case, make a connection with fractonic modes.

D.2.1 Higher rank gauge theories

Historically, it is Pretko who, for the first time, noticed and labelised as it a fractonic behaviour in a field theory [200]. In the context of spin liquids, Pretko used spatial higher rank symmetric gauge theories, i.e. theories where the gauge field is of the form $A_{ij}(x)$, and observed that the gauge charges display a reduced mobility. The intuition is that gauge theories lead to gauge constraints, which are made explicit in the Hamiltonian formalism [264]. The Gauss law of Maxwell electromagnetism (D.1.22), where the electric field is the canonical conjugate momentum of the spatial components of the four-potential A_μ, is an example of such gauge constraints. If we switch on a source for the gauge field, these gauge constraints can be considered as conservation laws for the external sources. For higher rank gauge theories, we have more gauge constraints than compared to the usual 1-form gauge theories. Hence, the external sources are constrained by additional conservation laws which leads to a reduced mobility.

One of the less involved examples is the one of a gauge field transforming as $A_{ij} \to A_{ij} + \partial_i \partial_j \alpha$ for an arbitrary function of space α. The associated generalised Gauss law is

$$\partial_i \partial_j E^{ij} = \rho \ , \tag{D.2.1}$$

where E^{ij} is the generalisation of the electric field (canonical conjugate momentum of A_{ij}) and ρ is an external source, we can refer to it as well under the nomenclature of gauge density charge. This Gauss law implies some constraints on ρ:

$$\int dV \, \rho = \int dV \, \partial_i \partial_j E^{ij} = 0 \ , \tag{D.2.2}$$

$$\int dV \, x^k \rho = \int dV \, x^k \partial_i \partial_j E^{ij} = - \int dV \, \partial_j E^{kj} = 0 \ , \tag{D.2.3}$$

where the integration is on the entire space volume of Minkowski spacetime. We observe that the total net gauge charge should be zero and that the total dipole moment should as well be zero, and this, at any time. We thus have a global charge conservation and a total dipole moment conservation. It leads to a reduced mobility of the gauge charge. Indeed, in Figure D.6 on the left, we have an already existent positive pointlike charge (we suppose that somewhere else in space, there are other charges to have a global zero net charge). If we move only this pointlike charge, the dipole moment of the system will change. Thus, the motion is forbidden by the conservation rules, and so, in this system, a pointlike charge is a fracton. On the right side of Figure D.6, we have a dipole (once again, we assume there are other charges in space to satisfy (D.2.2) and (D.2.3)) and this dipole can translate without contradicting (D.2.3). We recover the idea that (some) collective motions of fractons are tolerated. We can also make a parallelism with the comment we made in the paragraph below (D.1.22). Finally, other transformation laws for the gauge field can be considered, leading to other generalised Gauss laws. In particular, the restriction in mobility is recovered with the conservation of other multipole moments.

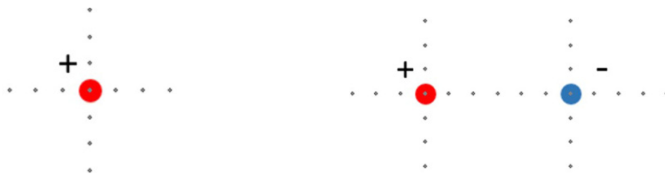

Figure D.6: On the left, an isolated pointlike charge is shown. On the right, it is an isolated dipole. These cartoons have been taken and modified from [200].

These spatial symmetric higher rank gauge theories are rather exotic compared to the usual 1-form gauge theories. There are still work to do to verify that such exotic gauge theories are well defined and that they do not lead to unphysical results. It should be mentioned that some checks and studies have already been performed, e.g. [201, 265]. Furthermore, it seems that spatial symmetric higher rank gauge theories are not only an academic curiosity but can have connection with real world physics. In fact, a clear connection between such theories and crystal elasticity has been made [211, 212].

D.2.2 Matter field theories

The preceding discussion suggests the idea that it should be possible to write matter field theories displaying reduced mobility, at least to separately describe the intrinsic dynamics of the gauge charge. Pretko successfully wrote such a model by looking for a theory which has for Noether conserved charges a $U(1)$ charge and the associated dipole [14]. Later on, Seiberg proposed a more general approach to find fractonic matter field theories [15]. We are now going to give a glimpse of the Seiberg strategy and from it, derive the Pretko model of [14] as well as the Shao-Seiberg model [263]. The obtained matter models can be gauged in order to make a connection with the preceding discussion on spatial symmetric higher rank gauge theories. But in the perspective of Goldstone physics, our primary interest does not lie in gauge theories (because gauge symmetries do not lead to NG modes). Hence, we will not further comment the gauging.

The intuition behind Seiberg approach is that dipole symmetry seems to be somehow connected with the notion of restricted mobility. A conserved dipole is a spatial vector Noether charge (as it can be seen from the matter description of the conservation law (D.2.3)). The first step in this intuitive reasoning is then to generalise this idea and to look for theories with (spatial) vector Noether charges, the associated symmetries are called (spatial) vector global symmetries. The conserved currents of this kind of symmetries have two symmetric indices, we could, for example, think of the stress-energy tensor $T^{\mu\nu}$ associated to the 4-momentum vector charge P^{μ}. But we can go further and somehow get rid of the Noether symmetry vision. To do so, the starting point is not the symmetries acting on the fields anymore but rather the assumption of the existence of a conserved current with two indices not necessarily symmetric (in particular, it can allow some connections with higher form symmetries). From the discussion on the spatial higher rank gauge symmetries, we know that time and space are not dealt with on the same footing. Based on this observation, we are going to ask our matter field theories to be non-relativistic. Thanks to that, we will be able to have richer conservation equations and so, richer conserved quantities. This will ensure the restriction on the mobility.

Seiberg point of view [15] is to assume that the fractonic matter toy model we are looking for contains a conserved current $J^{\mu\nu}$ satisfying

$$\partial_\mu J^{\mu i} = \partial_0 J^{0i} + \partial_j J^{ji} = 0 \ , \tag{D.2.4}$$
$$\partial_\mu J^{\mu 0} = \partial_0 J^{00} + \partial_i J^{i0} \neq 0 \ . \tag{D.2.5}$$

Thanks to (D.2.4), the following quantity

$$Q^j = \int dV\, J^{0j} \ , \tag{D.2.6}$$

where the integration is on the entire spatial volume of Minkowski spacetime, is conserved ($\partial_0 Q^j = 0$).

One way to realise (D.2.4) and (D.2.5) is to consider that the particular toy model we are looking for possesses a current (j^0, j^{ij}) such that

$$\partial_0 j^0 = \partial_i \partial_j j^{ij} \ . \tag{D.2.7}$$

Then, we define

$$J^{0j} = x^j j^0 \ , \tag{D.2.8}$$
$$J^{ij} = -x^j \partial_k j^{ki} + j^{ij} \ , \tag{D.2.9}$$

and by a direct computation, we see that (D.2.4) and (D.2.5) are satisfied.

D.2.3 Pretko's model

We now show that we can write a matter field theory with a dipole moment conservation and which displays a fractonic behaviour. From j^0, we can define the conserved charge

$$Q = \int dV\, j^0 \ , \tag{D.2.10}$$

thanks to (D.2.7). Then, by using (D.2.8), (D.2.6) is the dipole associated to Q:

$$Q^j = \int dV\, x^j\, j^0 \ , \tag{D.2.11}$$

where we already know that it is conserved.

In order to find a model with these conservation properties, we consider that Q is a $U(1)$ charge. It means that Q acts like a phase on the complex matter field ϕ. Indeed, hastily said, j^0 is the number density (as it will be verified a posteriori) $j^0 \sim \phi\partial_0\phi^* \sim \phi\, P_\phi$, where P_ϕ is the canonical conjugate momentum of ϕ. Hence,

$$\{\alpha Q,\, \phi\}_{\text{P.B.}} \sim \alpha\, \{\phi\, P_\phi,\, \phi\}_{\text{P.B.}} \sim \alpha\, \phi \ , \tag{D.2.12}$$

where we used the canonical conjugation through the Poisson brackets and α is a constant. From the expression (D.2.11) of Q^j, it means that it acts like a linear phase on the matter field. Therefore, the conserved charges Q and Q^j generates the symmetry

$$\phi \to e^{i(\alpha+\beta_j x^j)}\phi \ , \tag{D.2.13}$$

where β_j is a constant spatial vector. A possible invariant theory is

$$\begin{aligned}\mathcal{L} =& \partial_0\phi\partial_0\phi^* - m^2|\phi|^2 - \frac{\lambda}{4}\left(|\phi|^2\right)^2 - c_1\partial_i|\phi|^2\partial_i|\phi|^2 \\ &- c_2|\phi\partial_i\partial_j\phi - \partial_i\phi\partial_j\phi|^2 - c_3\left[(\phi^*)^2\left(\phi\partial_i\partial_i\phi - \partial_i\phi\partial_i\phi\right) + c.c.\right] \ ,\end{aligned} \tag{D.2.14}$$

where *c.c.* stands for complex conjugate. This is the Pretko model from his paper [14], the explicit expression (D.2.14) is taken from the review [255]. We can indeed verify a posteriori that

$$j^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\phi = \phi\partial_0\phi^* \ . \tag{D.2.15}$$

Taking $m^2 > 0$ and $\lambda > 0$, we have that the vacuum is $\phi = 0$. A small perturbation $\delta\phi$ around it provides

$$\mathcal{L}_{\text{quad}} \approx \partial_0\delta\phi\,\partial_0\delta\phi^* - m^2|\delta\phi|^2 \ . \tag{D.2.16}$$

The dispersion relation is $\omega = m$, meaning that the on-shell field configuration can have any spatial dependence and that the time evolution is just given by a phase e^{imt}. To understand this peculiarity, let us do the field redefinition

$$\delta\phi = e^{imt}\varphi \ , \tag{D.2.17}$$

then

$$\mathcal{L}_{\text{quad}} \approx \partial_0\varphi\,\partial_0\varphi^* - im\left(\varphi^*\partial_0\varphi - \partial_0\varphi^*\varphi\right) \ . \tag{D.2.18}$$

We can observe that a strong subsystem symmetry has emerged. In fact, the Lagrangian transforms up to a global derivative under an arbitrary space modulated complex shift

$$\varphi \to \varphi + c(\vec{x}) \qquad \Rightarrow \qquad \delta\mathcal{L}_{\text{quad}} \approx -i\,m\,\partial_0(c^*\varphi - \varphi^* c) \ . \tag{D.2.19}$$

This arbitrary spatially modulated shift informs us that any space dependence of φ can be a solution, and it reflects on $\delta\phi$. This provides an explanation for the $\omega = m$ dispersion relation.

If m is set to zero, the dispersion relation is $\omega = 0$. We have a massless mode which, in the language of plane waves, does not propagate. Furthermore, the plane waves solution of the theory can have any momenta. First, the vacuum is then highly degenerate with an infinite number of possible momenta. Second, we have a UV/IR mixing because even at large momenta, we remain at $\omega = 0$. We see that Pretko's model recovers the main features of fractonic lattice models: reduced mobility (but the mode is gapless instead of being gapped), large vacuum degeneracy (but no notion of system size nor robustness), UV/IR mixing and all that is induced by a subsystem symmetry – the arbitrarily spatial modulated shift.

The intuition that dipole moment conservation leads to fractonic behaviour is not always correct. Indeed, if we switch the sign of m^2, now it is $m^2 < 0$, we obtain a

Mexican hat potential. This allows a non-trivial vacuum $\phi = v$. If we focus on the free theory of the NG field π associated to the SSB of $U(1)$, the perturbation parametrisation is as follow

$$\phi = v\, e^{i\pi} \ , \tag{D.2.20}$$

and the theory is [255]

$$\mathcal{L} \approx v^2\,(\partial_0\pi)^2 - c_2\,v^4\,(\partial_i\partial_j\pi)^2 - c_3\,v^4\,(\partial_i\partial_i\pi)^2 \ . \tag{D.2.21}$$

The subsystem symmetry is reduced compared to the unbroken case since now it is an affine shift symmetry:

$$\pi \to \pi + \alpha + \beta_j\,x^j \ . \tag{D.2.22}$$

Actually, it is the usual non-linear realisation of the spontaneously broken symmetries acting on the NG fields. Hence, we can even say that compared to the fundamental theory (D.2.14), there is no emergent new subsystem symmetries. The dispersion relation is of the form $\omega^2 \sim k^4$ which does not display a reduced mobility of the associated plan waves. The subsystem symmetry is not constraining enough. This result of superfluid fractonic models is studied in [223, 224].

With the approach proposed by Seiberg, we have recovered Pretko's model. An alternative method is the "Noether vision" which consists into directly impose some symmetries on the theory, i.e. we start with explicit transformation laws acting on the fields, and make such symmetries strong enough to reduce the possible motion of the modes. This was the original approach of Pretko [14]. This construction of fractonic toy models by explicit symmetries relies on the intuition that multipole moments should be conserved (but, as we have seen there are counter-examples). Therefore, Gromov classified the algebras (called the multipole algebras) leading to such conserved quantities [266]. Among these algebras, there are the ones generating polynomial shift symmetries. It should be emphasised that the discussion on polynomial shift symmetries was first introduced, for a non-oriented fractonic purpose, in [17] where they recover Pretko's like models. The simplest polynomial shift symmetry is the affine shift symmetry which leads to a dipole conservation. As we have already seen the transformation law is

$$\phi \to \phi + \alpha + \beta_j\,x^j \ . \tag{D.2.23}$$

In order for the Lagrangian to be invariant, it should be a functional of higher spatial derivative terms so, it depends on $\partial_0\phi$ and $\partial_i\partial_j\phi$ for the simplest case. To compute the associated Noether currents, we have to use the Noether theorem for Lagrangian with higher derivatives [46]:

$$j^0_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \equiv \rho \ , \qquad j^i_\alpha = -\partial_j\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_j\phi)} \equiv \partial_j j^{ij}_\alpha \ , \tag{D.2.24}$$

$$j^{i0}_\beta = \rho\,x^i \ , \qquad j^{ij}_\beta = x^i j^i_\alpha - j^{ij}_\alpha \ . \tag{D.2.25}$$

Which indeed leads to the conservation of the shift charge and the associate dipole moment

$$Q = \int dV\,\rho \ , \qquad Q^i = \int dV\,x^i\,\rho \ . \tag{D.2.26}$$

Let us notice that from $\partial_\mu j^\mu_\alpha = 0$, we have

$$\partial_0 \, j^0_\alpha = \partial_i \partial_j \, j^{ij}_\alpha \; . \tag{D.2.27}$$

Hence, like the $U(1)$ affine symmetry, the affine shift symmetry is encompassed in the Seiberg prescription (D.2.7).

D.2.4 Shao-Seiberg's model

Seiberg method should be seen as a guideline to help us to be creative on writing fractonic continuous models. With the same start than for Pretko's model, we will derive another important fractonic model in the literature: the Shao-Seiberg model [263].

We want a fractonic toy model in $2+1$ Minkoswki spacetime. To satisfy (D.2.4) and (D.2.5), we again assume that our theory has a current (j^0, j^{ij}) verifying (D.2.7). In addition, we demand j^{ij} to be symmetric with $j^{xx} = 0 = j^{yy}$. Out of j^0, we build the usual conserved charge

$$Q = \int dxdy \, j^0 \; , \tag{D.2.28}$$

which generates a $U(1)$ symmetry on a complex scalar field ϕ. But this time, we are going to focus on the phase of ϕ and so, we write an action for a real scalar field θ which transform as a shift under a finite transformation generates by Q: $\theta \to \theta + \alpha$. Furthermore, we arbitrarily add the requirement that

$$Q_c \equiv \int dxdy \, c(x,y) \, j^0 \; , \tag{D.2.29}$$

is a conserved quantity. Of course, it is motivated by the fact that we want to constrain as much as possible the dynamics (without making it trivial) in order to get fractonic modes. For Q_c to be indeed conserved, we need

$$\partial_0 Q_c = \int dxdy \, c(x,y) \, \partial_0 j^0 = \int dxdy \, c(x,y) \, 2\partial_x \partial_y j^{xy} \tag{D.2.30}$$

$$= \int dxdy \, \partial_x \partial_y c(x,y) \, 2 j^{xy} = 0 \; , \tag{D.2.31}$$

$$\Leftarrow \partial_x \partial_y c(x,y) = 0 \Leftrightarrow c(x,y) = c_x(x) + c_y(y) \; , \tag{D.2.32}$$

where c_i is an arbitrary function of x^i. We have that Q_c induces the subsystem symmetry

$$\theta \to \theta + c_x(x) + c_y(y) \; . \tag{D.2.33}$$

A possible theory invariant under (D.2.33) is

$$\mathcal{L} = (\partial_0 \theta)^2 - \mu (\partial_x \partial_y \theta)^2 \; . \tag{D.2.34}$$

This is the Shao-Seiberg model [263]. It has the gapless dispersion relation

$$\omega^2 = \mu \, k_x^2 k_y^2 \; . \tag{D.2.35}$$

If we put either k_x or k_y to zero, we can make similar comments than the ones we made for the model of Pretko: for any value of the non-zero momentum, ω remains at zero. Thus, we have a ground state degeneracy, a UV/IR mixing and non-propagating plane waves. The arbitrary spatial modulation we can have along one direction when we set to zero the momentum in the other direction can be traced back to the subsystem symmetry (D.2.33). Finally, the field theory (D.2.34) is as well perculiar because it tolerates discontinuous field configurations. For example, θ can depends only on x and be discontinuous, the derivative along y will erase θ (if we permute the derivatives, we will first get Dirac deltas at the discontinuity points and ∂_y will make them disappear) and so, the theory is blind with respect to these discontinuities. From the mode point of view, once we set k_y to zero, we can send k_x to infinity and still keep a finite energy ($\omega = 0$).

Another counter-example that dipole moment conservation does not systematically lead to fractonic modes can be obtained by relaxing the requirement that the theory should be invariant under (D.2.33). Let us ask it to be invariant under (D.2.33) but with c_i as affine functions in x^i. In this case, Q_c is a dipole and a possible extension of (D.2.34) could be

$$\mathcal{L} = (\partial_0\theta)^2 - \mu(\partial_x\partial_y\theta)^2 - \mu_x(\partial_x^2\theta)^2 - \mu_y(\partial_y^2\theta)^2 \ , \tag{D.2.36}$$

where now the dispersion relation is of the form (without the coefficients) $\omega^2 \sim k_x^2 k_y^2 + k_x^4 + k_y^4$. This dispersion relation does not display fractonic features.

A brief additional comment should be made on the Shao-Seiberg model. This model is motivated by lattice physics and has been originally introduced from a top-down perspective, i.e. to find the field theory which corresponds to the continuous limit of a given fractonic lattice model. Shao-Seiberg model is then the effective theory of a fractonic lattice model. Once again, we can observe in (D.2.34) a pecularity for this EFT: it has higher spatial derivative terms as relevant term. It displays once again the UV/IR mixing.

In this overview of fracton physics, we focused solely on the bottom-up approach. For the top-down perspective, the reader can refer to [263, 267] and the associated companion papers. It permits to establish more rigorously the link between the fractonic excitations in lattice models and the features of the fractonic field theories, e.g. how to describe gapped fractonic modes while here we discussed gapless fractonic modes.

D.2.5 Key points of fractonic continuous models

We have seen that the QFT description of the fractonic excitations we have in some lattice models is involved and challenges the fundamentals of QFT. We do not know yet what are the necessary conditions and what are the sufficient conditions to have a field theory displaying fractonic behaviour. The current status of research is to find fractonic continuous toy models to acquire intuition. To find such models, it is not straightforward. We rely either on a top-down approach (continuous limit of a given fractonic lattice model) or on a bottom-up approach (model building based on general principles). In both cases, strong subsystem symmetries appear to be needed. To build a model with such symmetries (and so, we hope, with fractonic features), we either impose them by hand (that is what we did for the Shao-Seiberg model) or we start with a theory with multipole moment conservations and subsystem symmetries might emerge from it (as it happened for the Pretko model). One of the highlights of Part II of this thesis is that there might be another way to build toy models with emergent subsystem symmetries. In order to spontaneously break translations, we need higher derivative terms. It suggests that the effective theory for NG modes coming from the breaking of translations and of additional internal symmetries will have higher derivative terms. Usual NG modes already have shift like symmetries. Thanks to these higher derivative terms, these shift like symmetries might be promoted to polynomial shift symmetries[6]. It would then induce a conservation of multipole moments and so, a possible restriction on the modes' motion. If the derivatives of higher order are well arranged, we could even have the emergence of strong subsystem symmetries, for instance, an arbitrary space modulated shift symmetry. Therefore, all the intuitive requirements to have fractonic modes are met!

We close this appendix with a final comment about semantic. In Part II of this thesis, when we mention the term "fractonic modes", we refer to gapless modes with dispersion relations such that plane waves with (specific) non-zero momentum do not propagate. It is therefore not the gapped excitations we have in fractonic lattice models, but the philosophy is the same: a reduced mobility.

[6] The idea that the most relevant terms of the EFT describing NG modes enjoy polynomial spatial shift symmetries has already been suggested and studied in [106, 250, 251].

Appendix E

Homogeneous vacua

In this appendix we motivate the choice of the two different symmetry breaking vacua that we have discussed in the main text, namely the helical superfluid and the meta-fluid.

Solutions to the equations of motion (8.2.3) and (8.2.4) which minimise the energy necessarily imply a constant Ξ and a static Φ. The space-dependence of Φ is further constrained to satisfy[1]

$$\partial_i \Phi^* \partial_i \Phi = \frac{A}{2B} v^6 \equiv c^2 \ . \tag{E.0.1}$$

In principle, to explore the space of time-independent solutions, one must consider the most general Φ satisfying (E.0.1). Since we are considering field theories with 2 spatial dimensions, the field Φ represents a map from the real plane to the complex plane. The condition (E.0.1) restricts to maps whose complex gradient has constant modulus. The (functional) space of solutions is clearly very large.

However, we will add one physically motivated constraint, which is to require that the effective theory of the fluctuations around the vacuum solution be completely homogeneous. In other words, we require the effective Lagrangian of the fluctuating fields not to have any explicit space dependent function.

Suppose $\Phi_0(x_i)$ is a solution of (E.0.1). We expand the field around such solution as

$$\Phi(t, x_i) = \Phi_0(x_i) + f(x_i)\varphi(t, x_i) \ , \tag{E.0.2}$$

where φ is the fluctuating field, and $f(x_i)$ is a complex function, depending only on space coordinates, that takes into account the freedom in the definition of the fluctuating field. It will be fixed in order to have a homogeneous effective Lagrangian.

Let us first consider the term with the time-derivatives:

$$\partial_t \Phi^* \partial_t \Phi = |f|^2 \partial_t \varphi^* \partial_t \varphi \ . \tag{E.0.3}$$

Homogeneity is achieved requiring $|f|^2$ to be spacetime independent. Hence, f can be considered to have only a space-dependent phase.

Consider now the expansion of the expression squaring to the 'gradient Mexican hat,' to linear order in the fluctuations:

$$\begin{aligned} \partial_i \Phi^* \partial_i \Phi - c^2 &= \partial_i \Phi_0^* \partial_i \Phi_0 + \partial_i \Phi_0^* (\partial_i f \varphi + f \partial_i \varphi) + \partial_i \Phi_0 (\partial_i f^* \varphi^* + f^* \partial_i \varphi^*) - c^2 \\ &= \partial_i \Phi_0^* \partial_i f \varphi + \partial_i \Phi_0 \partial_i f^* \varphi^* + \partial_i \Phi_0^* f \partial_i \varphi + \partial_i \Phi_0 f^* \partial_i \varphi^* \ . \end{aligned} \tag{E.0.4}$$

The quadratic Lagrangian involves the square of the above expression, and will be homogeneous if and only if each coefficient of the four terms above is itself space-independent (or

[1] An analogous equation is described in [268] in relation to superfluids with constant superfluid velocity.

zero). Taking into account that they come in complex pairs, we have the two conditions relating f and Φ_0:

$$f\partial_i\Phi_0^* = ia_i \ , \qquad \partial_i f\partial_i\Phi_0^* = b \ , \tag{E.0.5}$$

where a_i and b are generic complex space-independent constants.

Let us now implement the fact that f must have all its space-dependence in a real phase:

$$f(x_i) = f_0 e^{i\theta(x_i)} \ . \tag{E.0.6}$$

From the first of (E.0.5) we get

$$\partial_i\Phi_0^* = i\frac{a_i}{f_0}e^{-i\theta} \ . \tag{E.0.7}$$

From the fact that $\partial_i\partial_j\Phi_0^* = \partial_j\partial_i\Phi_0^*$ we get that

$$a_i\partial_j\theta = a_j\partial_i\theta \ . \tag{E.0.8}$$

The second of (E.0.5) gives now

$$a_i\partial_i\theta = -b \ . \tag{E.0.9}$$

These last two sets of equations imply that $\partial_i\theta$ are both constant (and must be real for consistency).

If at least one of the constants is not zero (i.e. $b \neq 0$), then we can write $\theta = k_i x_i$, and we have $f \propto \Phi_0 = \rho\, e^{ik_i x_i}$, i.e. the helical solution (rotated towards a generic direction).

If on the other hand both constants are zero (i.e. $b = 0$), then θ is a constant that can be reabsorbed in f_0, the constant value of f. Then Φ_0 is linear, $\Phi_0 = b_i x_i$, i.e. we have the meta-fluid solution (generalised to the non-isotropic case).

Appendix F

Ward-Takahashi identities

The model in $2+1$ dimensions has a Lagrangian density

$$\mathcal{L} = \mathcal{L}(\partial_0 X^I, \partial_i X^I) \ , \qquad X^I = \{\, \Phi, \Phi^*, \Xi\} \ , \tag{F.0.1}$$

where

$$\begin{aligned} \mathcal{L} = &|\partial_0\Phi|^2 + \frac{1}{2}(\partial_0\Xi)^2 + A|\partial_k\Phi|^2 - \frac{1}{2}(\partial_k\Xi)^2 - H\Xi^6 \\ &- B\Xi^{-6}|\partial_k\Phi|^4 + G\Xi^{-6}|\partial_k\Phi^*\partial_k\Phi^*|^2. \end{aligned} \tag{F.0.2}$$

The Noether energy-momentum tensor is

$$T^\mu_{\ \nu} = \frac{\delta\mathcal{L}}{\delta\partial_\mu X^I}\partial_\nu X^I - \delta^\mu_\nu\mathcal{L}. \tag{F.0.3}$$

One can check that it is conserved on-shell $\partial_\mu T^\mu_{\ \nu} = 0$. The spatial components are symmetric

$$\begin{aligned} T^{(0)}_{ij} &= 2A\partial_{(i}\Phi^*\partial_{j)}\Phi - \partial_i\Xi\partial_j\Xi - \delta_{ij}\mathcal{L}, \\ T^B_{ij} &= -4B\Xi^{-6}(\partial_k\Phi^*\partial_k\Phi)\partial_{(i}\Phi^*\partial_{j)}\Phi, \\ T^G_{ij} &= 2G\Xi^{-6}\left[(\partial_k\Phi^*\partial_k\Phi^*)\partial_i\Phi\partial_j\Phi + c.c.\right], \end{aligned} \tag{F.0.4}$$

so the complete stress tensor is

$$T_{ij} = T^{(0)}_{ij} + T^B_{ij} + T^G_{ij}. \tag{F.0.5}$$

The T_{00} component is

$$T_{00} = 2\partial_0\Phi^*\partial_0\Phi + (\partial_0\Xi)^2 - \mathcal{L}. \tag{F.0.6}$$

Then, the trace is

$$T^\mu_{\ \mu} = T_{00} + \delta^{ij}T_{ij}. \tag{F.0.7}$$

The traces are

$$\begin{aligned} \delta^{ij}T^{(0)}_{ij} &= 2A\partial_k\Phi^*\partial_k\Phi - \partial_k\Xi\partial_k\Xi - 2\mathcal{L}, \\ \delta^{ij}T^B_{ij} &= -4B\Xi^{-6}(\partial_k\Phi^*\partial_k\Phi)^2, \\ \delta^{ij}T^G_{ij} &= 4G\Xi^{-6}|\partial_k\Phi^*\partial_k\Phi^*|^2, \end{aligned} \tag{F.0.8}$$

All together

$$\begin{aligned} T^\mu_{\ \mu} = &-|\partial_0\Phi|^2 - \frac{1}{2}(\partial_0\Xi)^2 - A|\partial_k\Phi|^2 + \frac{1}{2}(\partial_k\Xi)^2 + 3H\Xi^6 \\ &- B\Xi^{-6}|\partial_k\Phi|^4 + G\Xi^{-6}|\partial_k\Phi^*\partial_k\Phi^*|^2. \end{aligned} \tag{F.0.9}$$

Using the equation of motion for Ξ, we can write this as

$$\begin{aligned} T^\mu_{\ \mu} = &-|\partial_0\Phi|^2 - A|\partial_k\Phi|^2 - \frac{1}{4}\left(\partial_0^2 - \partial_k^2\right)\Xi^2 \\ &+ 2B\Xi^{-6}|\partial_k\Phi|^4 - 2G\Xi^{-6}|\partial_k\Phi^*\partial_k\Phi^*|^2. \end{aligned} \tag{F.0.10}$$

Using now the equations of motion for Φ, Φ^*

$$\begin{aligned} T^\mu_{\ \mu} &= -\frac{1}{2}(\partial_0^2 + A\partial_k^2)|\Phi|^2 - \frac{1}{4}\left(\partial_0^2 - \partial_k^2\right)\Xi^2 \\ &+ \partial_k\left(B\Xi^{-6}|\partial_m\Phi|^2\partial_k|\Phi|^2\right) - \frac{1}{2}\partial_k\left[G\Xi^{-6}(\partial_m\Phi^*\partial_m\Phi^*)\partial_k\Phi^2 + c.c.\right]. \end{aligned} \tag{F.0.11}$$

We can partially improve the energy momentum tensor

$$\mathcal{T}^\mu_\nu = T^\mu_{\ \nu} + \frac{1}{4}\left(\Box\delta^\mu_\nu - \partial^\mu\partial_\nu\right)\left(|\Phi|^2 + \frac{\Xi^2}{2}\right) + \theta^\mu_{\ \nu}. \tag{F.0.12}$$

Where $\partial^\mu = \eta^{\mu\alpha}\partial_\alpha$, $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ and the non-zero components of $\theta^\mu_{\ \nu}$ are

$$\theta^i_{\ j} = \frac{1}{2}(A+1)\left(\partial_k^2\delta^i_{\ j} - \partial_i\partial_j\right)|\Phi|^2. \tag{F.0.13}$$

Then, the trace is

$$\mathcal{T}^\mu_\mu = \partial^\mu V_\mu, \tag{F.0.14}$$

where $V^0 = 0$ and

$$V_i = B\Xi^{-6}|\partial_k\Phi|^2\partial_i|\Phi|^2 - \frac{1}{2}\left[G\Xi^{-6}(\partial_k\Phi^*\partial_k\Phi^*)\partial_i\Phi^2 + c.c.\right]. \tag{F.0.15}$$

There is a conserved current associated to scale transformations

$$D^\mu = \mathcal{T}^\mu_\alpha x^\alpha - V^\mu, \quad \partial_\mu D^\mu = \mathcal{T}^\mu_\mu - \partial^\mu V_\mu = 0. \tag{F.0.16}$$

Then, (F.0.14) is the Ward-Takahashi identity associated to dilatations.

F.1 Conserved current

The current is

$$J^\mu = \frac{i}{2}\left[\Phi\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi} - c.c.\right]. \tag{F.1.1}$$

The ordinary current is

$$j_\mu = \frac{i}{2}\left(\Phi\partial_\mu\Phi^* - \Phi^*\partial_\mu\Phi\right). \tag{F.1.2}$$

In this model, the components of the conserved current are

$$\begin{aligned} J_0 &= j_0, \\ J_i &= -\left(A - 2B\Xi^{-6}|\partial_k\Phi|^2\right)j_i - iG\Xi^{-6}\left[(\partial_k\Phi^*)^2\Phi\partial_i\Phi - (\partial_k\Phi)^2\Phi^*\partial_i\Phi^*\right]. \end{aligned} \tag{F.1.3}$$

And the current conservation equation is

$$\partial^\mu J_\mu = 0. \tag{F.1.4}$$

F.2 Shift symmetries

The Lagrangian has additional shift symmetry (we consider here only real shifts, to avoid overcounting)

$$\Phi \to \Phi + \alpha, \qquad \Phi^* \to \Phi^* + \alpha. \tag{F.2.1}$$

The Noether currents associated to this symmetry is

$$J_s^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} + \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi^*}. \tag{F.2.2}$$

If the action only depends on derivatives of Φ, then the current is conserved, since it is a combination of the equations of motion for Φ, Φ^*

$$\partial^\mu J_{s\,\mu} = 0. \tag{F.2.3}$$

Note that adding this equation makes the system of equations from the Ward-Takahashi identities equal to the system of equations from the Lagrangian, we have to solve for all the modes.

The components are

$$\begin{aligned} J_{s\,0} &= \partial_0 \Phi + \partial_0 \Phi^*, \\ J_{s\,i} &= -A\left(\partial_i \Phi + \partial_i \Phi^*\right) + 2B\Xi^{-6} |\partial_k \Phi|^2 \left(\partial_i \Phi + \partial_i \Phi^*\right) \\ &\quad - 2G\Xi^{-6}\left(\partial_k \Phi^* \partial_k \Phi^* \partial_i \Phi + \partial_k \Phi \partial_k \Phi \partial_i \Phi^*\right). \end{aligned} \tag{F.2.4}$$

F.3 Adding a chemical potential

We introduce a chemical potential

$$\Phi = e^{i\mu t}\phi, \qquad \Phi^* = e^{-i\mu t}\phi^*. \tag{F.3.1}$$

Then, the charge density becomes

$$J_0(\Phi) = j_0(\Phi) = 4\mu|\phi|^2 + j_0(\phi) \equiv J_0(\phi). \tag{F.3.2}$$

The time-time component of the energy-momentum tensor changes to

$$T_{00}(\Phi) = 2\mu^2|\phi|^2 + 2i\mu(\phi\partial_0\phi^* - \phi^*\partial_0\phi) + T_{00}(\phi) = \mu J_0(\phi) + t_{00}(\phi), \tag{F.3.3}$$

where

$$t_{00}(\phi) = T_{00}(\phi) - 2\mu^2|\phi|^2. \tag{F.3.4}$$

The effective Lagrangian is

$$\mathcal{L}_\phi = \mathcal{L} + 2\mu^2|\phi|^2. \tag{F.3.5}$$

The change in the trace is

$$T^\mu{}_\mu(\Phi) = T^\mu{}_\mu(\phi) - \mu^2|\phi|^2 - i\mu(\phi\partial_0\phi^* - \phi^*\partial_0\phi). \tag{F.3.6}$$

Appendix G

Generalisations to 3+1 dimensions

In this appendix, we briefly outline generalisations of Chapter 8 to 3 + 1-dimensional systems, to show that the essential features of both the helical superfluid and the meta-fluid are unchanged. The only difference is that we have to use different models to generalise the helical superfluid and the meta-fluid, respectively. We will keep the analysis of both models to a minimum, since it turns out that they are very similar to their 2 + 1-dimensional cousins.

G.1 3+1-dimensional helical superfluid

In order to generalise the helical superfluid, we keep the field content to be a complex scalar Φ and a real scalar Ξ. Only the scaling dimensions of the scalars changes, and hence the compensating powers of Ξ.

We thus start with the following Lagrangian, where we have already implemented a condition like (8.2.12):

$$\mathcal{L} = \partial_t\Phi^*\partial_t\Phi + \frac{1}{2}\partial_t\Xi\partial_t\Xi - \frac{1}{2}\partial_i\Xi\partial_i\Xi - \frac{B}{\Xi^4}\left(\partial_i\Phi^*\partial_i\Phi - \frac{A}{2B}\Xi^4\right)^2 . \qquad \text{(G.1.1)}$$

The equations of motion are solved for

$$\Phi = \rho\, e^{ikx} , \qquad \Xi = v , \qquad \text{with} \qquad \frac{k^2\rho^2}{v^4} = \frac{A}{2B} . \qquad \text{(G.1.2)}$$

The expansion is exactly as before

$$\Phi = \rho\, e^{ikx}(1+\sigma+i\chi) , \qquad \Xi = v(1+\tau) , \qquad \text{(G.1.3)}$$

so that the effective quadratic Lagrangian for the fluctuations about the helical vacuum is

$$\mathcal{L} = \frac{v^2}{2}\partial_\mu\tau\partial^\mu\tau + \rho^2(\partial_t\chi)^2 + \rho^2(\partial_t\sigma)^2 - 2A\rho^2\left[\partial_x\chi + k(\sigma-2\tau)\right]^2 , \qquad \text{(G.1.4)}$$

which is exactly similar to (8.2.51) except for a numerical coefficient. The spectrum will then be exactly the same: there is an immobile fracton, a gapless mode which has linear and isotropic dispersion relations at low momentum, but becomes a lineon at high momentum (propagating along x, now one out of three directions), and a gapped mode which has relativistic dispersion relations at high momentum (this is the spectrum for $A \leqslant 1/2$; if $A > 1/2$ then as before the lineon and the relativistic mode switch roles according to the direction of propagation). The gap is given by

$$m^2 = 2Ak^2\left(1 + 8\frac{\rho^2}{v^2}\right) . \qquad \text{(G.1.5)}$$

G.2 3+1-dimensional meta-fluid

In order to generalise the meta-fluid, the model has to contain as many (real) scalar fields as space directions, plus the compensator scalar field. Hence, we start with 3 real scalar fields Φ_i, to which we add Ξ. The Lagrangian is now

$$\mathcal{L} = \frac{1}{2}\partial_t\Phi_i\partial_t\Phi_i + \frac{1}{2}\partial_t\Xi\partial_t\Xi - \frac{1}{2}\partial_i\Xi\partial_i\Xi - \frac{B}{\Xi^4}\left(\partial_i\Phi_j\partial_i\Phi_j - \frac{A}{2B}\Xi^4\right)^2 . \tag{G.2.1}$$

The solution to the equations of motion is

$$\Phi_i = b\ x_i\ , \qquad \Xi = v\ , \qquad \text{with} \qquad \frac{3b^2}{v^4} = \frac{A}{2B} . \tag{G.2.2}$$

We take the fluctuations to be

$$\Phi_i = b(x_i + u_i)\ , \qquad \Xi = v + \tau\ , \tag{G.2.3}$$

and the quadratic Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\tau\partial^\mu\tau + \frac{1}{2}b^2\partial_t u_i\partial_t u_i - \frac{2}{3}Ab^2\left(\partial_i u_i - \frac{6}{v}\tau\right)^2 . \tag{G.2.4}$$

Again, this is very similar to (8.2.63), up to some numerical coefficients. However, now it involves 4 modes instead of three. But we can immediately see that the only modes that will have non-trivial dispersion relations are the mixtures of τ and the longitudinal component of u_i. Then, both transverse modes of u_i will be immobile fractons.

As for the non-trivial modes, one is gapped with gap given by

$$m^2 = 48A\frac{b^2}{v^2}\ , \tag{G.2.5}$$

and relativistic dispersion relation at high momentum, while the other is gapless with quadratic dispersion relation at low momentum

$$\omega^2 = \frac{4}{3}\frac{A}{m^2}q^4 + \mathcal{O}(q^6)\ , \tag{G.2.6}$$

and linear dispersion relation given by

$$\omega^2 \simeq \frac{4}{3}Aq^2\ , \tag{G.2.7}$$

at high momentum. If $A > 3/4$, the high momentum behaviour is switched between the two modes.

To summarise, we see that the generalisation to 3+1 dimensions yields physics very similar to the $2+1$-dimensional case that we have analysed in detail, so that we expect that the latter transposes to $3+1$ dimensions straightforwardly.

Part III

Are there Goldstone bosons in $d \leqslant z + 1$?

Preamble Part III

This part of the thesis is in the framework of the extension of Coleman's theorem for non-relativistic theories [16, 106, 122, 123]. More specifically, we are going to study spontaneous symmetry breaking patterns which lead to solely type A NG modes. Since we use the classification type A/type B, it is implicit that we set ourselves in the range of the hypotheses of Theorem 3. At sufficiently low energy, type A NG modes have a non-relativistic scaling symmetry where space scaling dimension is canonically 1 and time scaling dimension is n – see (4.5.14). The scaling dimension of the NG modes is then $\Delta = (d-1-n)/2$. By a scaling argument, we have seen in Part I Subsection 4.5.2 that it has been conjectured that no spontaneous symmetry breaking can happen when $\Delta \leqslant 0$, i.e. when $d \leqslant n+1$ [106]. Interestingly, in contrast to relativistic QFTs where only two dimensions are singled out as a particular case, in non-relativistic QFTs there is a whole range of dimensions in which spontaneous symmetry breaking is in principle forbidden. We will make an explicit QFT computation to confirm this conjecture. Furthermore, to give a physical motivation, we are going to perform this computation for Lifshitz theories invariant under time reversal – this particularisation to Lifshitz theories will not induce a loss of generality. As we will explain in the next paragraph, Lifshitz theories are directly connected with phenomenology in condensed matter and offer the expected scaling symmetry for the NG modes. The requirement for being time reversal invariant is to ensure to have a double time-derivative in the low-energy theory for the NG modes and so, to indeed describe type A NG modes.

Lifshitz theories are field theories invariant under spacetime translations, spatial rotations and which have the Lifshitz scaling symmetry

$$t \to \lambda^z t \ , \quad x^i \to \lambda x^i \ , \tag{9.2.8}$$

where $z \in \mathbb{R}$ is called the dynamical critical exponent [269]. These theories permit to portray physics around quantum critical points [270]. From phase transition physics, we know that at a critical point, the physics is scale invariant because the correlation length is undefined. In the language of field theory, the Landau-Ginzburg model has a scale symmetry such that the correlation length of the two-point correlators diverges. A quantum critical point is a critical point at zero temperature where phase transitions are driven by quantum effects. Around quantum critical points in the coupling space, the correlation length ξ evolves as

$$\xi^{-1} \sim |g - g_c|^\gamma \ , \tag{9.2.9}$$

where g is a coupling of the Landau-Ginzburg model and g_c is its critical value, γ is the associated critical exponent. Moreover, it is observed that the gaps Δ go to zero (otherwise there is no scaling symmetries) following

$$\Delta \sim \xi^{-z} \ . \tag{9.2.10}$$

From the latter equation, the scaling dimensions are

$$[x^i] = [\xi] \stackrel{!}{=} 1 \ , \quad [t] = [\Delta^{-1}] = z\,[\xi] = z \ . \tag{9.2.11}$$

Thus, we see that the scaling symmetry at a quantum critical point is the Lifshitz scaling symmetry (9.2.8). Let us emphasise that the parameter z is a measured parameter and so, it can take any real values. We refer the reader to [271–273] for concrete physical examples of Lifshitz systems and in particular to [270, 274] for an example where z is not an integer. The Lifshitz EFT is a theory for NG modes with a more general scaling symmetry than the one considered in Subsection 4.5.2 where only integer value were considered for the exponent n (notice that we recover the relativistic case when $z = 1$). It means that we generalise the conjecture by stating that no spontaneous symmetry breaking leading to solely type A NG modes can occur when $d \leqslant z + 1$.

As for the relativistic case [26–28, 121], we can probe if this statement holds true or not for strictly large N theories by performing a holographic computation for Lifshitz QFTs (see [269, 275–281]). An introduction to bottom-up holography is provided in Appendix H. Based on the relativistic experience, we guess it will not be true anymore. Therefore, we are looking for a counter-example rather than a generic argument. This allows us to try with a minimalist holographic model and to choose the parameters of the theory to ease as much as possible the computation. We will anyway remain generic in the discussion for spacetime dimension and for the critical exponent by only requiring $d \leqslant z + 1$ instead of completely fixing d and z in the window $d \leqslant z + 1$. Beside the motivation of Goldstone physics, we expect the holographic computation to be as well interesting from the point of view of holography in itself. Indeed, our arguments rely on symmetric arguments and so, consistency checks of the holographic conjecture for non-relativistic systems can be performed (e.g. recovering the Ward-Takahashi identities). Furthermore, the holographic renormalisation procedure heavily depends on the dimension of spacetime and on the asymptotic boundary expansion of the gravity-side fields. These expansions encode the scaling dimension of the QFT operators. Since we are precisely working with unusual scaling dimensions (negative ones instead of positive ones), we expect some subtleties in the holographic renormalisation procedure. The scaling dimension of an internal conserved current is $d - 1$. It can be seen from the variation of the action under the considered symmetry

$$\delta_\alpha S \sim \int d^d x \, \alpha \, \partial_\mu j^\mu \ . \tag{9.2.12}$$

The holographic dictionary tells us that a conserved current is sourced by a gravity-side gauge field A_μ. With

$$S_{\text{QFT}} \supset \int d^d x \, j^\mu A_\mu \ , \tag{9.2.13}$$

we see that the canonical dimension of A_μ is z. In the holographic language, in the temporal vector sector, we have that the source has a canonical dimension z and that the response has a canonical dimension $d - 1$. In the limit case $d = z + 1$, the source and the response are then at the same order in the asymptotic expansion. We then lose the intuitive idea that the leading coefficient is the source and that the subleading one is the response. We might be obliged to make well considered choices to be able to describe spontaneous symmetry breaking (or at least to extract the desired correlators to build the Ward-Takahashi identities), in addition to the technical difficulty of the apparition of logarithmic terms due to the mixing of the source and the response[1].

[1]For the critical relativistic case, $z = 1$ and $d = 2$, with the same scaling argument, we can predict

This part of the dissertation is organised as follows. In Chapter 10 we consider a QFT invariant under the Lifshitz group, time-reversal and a global $U(1)$ symmetry. We discuss under which conditions a one-point function vacuum expectation value survives quantum corrections. We find the condition to be $d > z + 1$, in agreement with a naive dimensional argument. We then proceed in Chapter 11 to analyse an equivalent holographic set-up. With the usual artillery of holographic renormalisation, we establish which counterterms need to be selected in order to obtain the correct gauge invariance of the generating functional, and hence reproduce the usual Ward-Takahashi identities for the conserved current. Such counterterms impose alternative quantisation for the temporal component of the bulk vector, i.e. the leading term in the near boundary expansion is identified with the VEV rather than the source. Finally, in Chapter 12 we comment on a few open questions.

Besides appendices H, I, J, K, this part of the thesis is a slight editing of the paper [125] of the author of this dissertation and his collaborators.

that both the temporal and the spatial vector sector will be at the Breitenlohner-Freedman bound. This prediction is indeed verified in [121], a paper of the author and his collaborators.

Chapter 10

Quantum corrections to the symmetry breaking VEV

We present in this chapter a generalisation of the argument by Coleman [16] for quantum field theories with Lifshitz scaling. As a reminder, Coleman's theorem states that for a relativistic theory in two-dimensional spacetime, at the quantum level, there cannot be any spontaneous breaking of symmetries that would lead to Goldstone bosons. The idea behind this argument is that for this specific spacetime dimension, massless scalars are ill-defined and so is the "would-be" Goldstone boson associated to the symmetry breaking. Physically, the interpretation is that quantum fluctuations are large enough to erase any notion of order, leading to the impossibility of having spontaneously broken symmetries.

The different Lifshitz theories being studied are identified by the number of spacetime dimensions d and the value of the dynamical critical exponent z. The argument is built with respect to a general action of the Lifshitz type invariant under a global continuous symmetry group. For simplicity, we consider the theory of a complex scalar ψ that is charged under a $U(1)$ global symmetry, invariant under time-reversal[1] and that possesses a potential V depending only on the modulus of ψ. To trigger the spontaneous symmetry breaking at the classical level, we suppose that V is minimal around a vacuum expectation value v for $|\psi|$, and there it takes the value zero for simplicity. The action is then given by

$$S\left[\psi\right] \quad = \quad \int \mathrm{d}t\mathrm{d}^{d-1}x \, \left(\partial_t\psi\partial_t\psi^* - (-1)^z\xi^2\psi\nabla^{2z}\psi^* - V(\psi\psi^*)\right) \ , \tag{10.0.1}$$

where z is the dynamical critical exponent, we take $z \geqslant 1$ for causality considerations, ξ is a positive real number without dimensions and $d \geqslant 2$ (to discuss Lifshitz scaling we need at least one spatial direction and one time direction). In Section I.1 of Appendix I we present low-energy Lifshitz theories and detail why the choice $z \geqslant 1$ is made. We note that ψ has dimension

$$[\psi] = \frac{d-1-z}{2} \ . \tag{10.0.2}$$

Doing a perturbation around the classical VEV, the physical field can be written as

$$\psi(x) \equiv \left(v + \sigma\left(x\right)\right) e^{i\theta(x)} \ , \tag{10.0.3}$$

where σ and θ are small fluctuations. The phase-field θ corresponds to the longitudinal direction of the action of $U(1)$ on the physical field, hence, it corresponds to the Goldstone boson if spontaneous symmetry breaking is allowed. Since we perform an analysis till the

[1] Namely, we want to study type A NG modes, time reversal symmetry ensures that the EFT will be type A like and since type A NG modes are canonically independent, there is not loss of generality to consider the $U(1)$ case.

quadratic order (small perturbations), the dynamics of θ is dictated by the free effective action

$$S[\theta] = \int \mathrm{d}t\mathrm{d}^{d-1}x\; v^2 \left(\partial_t\theta\partial_t\theta - (-1)^z\xi^2\theta\nabla^{2z}\theta - \xi^2\lambda^{2z}\theta^2\right) \; . \tag{10.0.4}$$

A mass term for θ with parameter λ is added by hand in order to confront the cases of spontaneous and explicit symmetry breaking. This parameter can also be viewed as an infrared regulator. Let us notice that we in fact recover the EFT of a type A NG mode with the desired scaling symmetry.

All we need for our argument is the two-point function of θ

$$\langle\theta(t,\vec{x})\theta(0)\rangle|_\lambda \;=\; \frac{\pi}{(2\pi)^d\xi v^2}\int \mathrm{d}^{d-1}p\, \frac{e^{i\vec{p}\cdot\vec{x}-i\xi\sqrt{p^{2z}+\lambda^{2z}}t}}{\sqrt{p^{2z}+\lambda^{2z}}} \; . \tag{10.0.5}$$

where $p \equiv \| \vec{p} \,\|$. On purely dimensional grounds, as displayed in Subsection 4.5.1 , the behaviour at large (spatial) separation of the propagator for θ is dictated, in the massless limit, by the dimension of ψ, (10.0.2). We thus expect the correlations to vanish at large separations only for positive dimensions, i.e. for $d > z+1$. Conversely, for $d \leqslant z+1$, we expect large long range correlations that can potentially spoil any vacuum expectation value.

We are now going to show that indeed, after renormalisation, the VEV is preserved in the former case, and is set to zero in the latter. We will follow an argument similar to the one given in [118] for the relativistic case, which is essentially equivalent to computing the one-loop correction to the ψ-tadpole[2].

First of all, if θ is approximated by a free field, we can write $\theta \equiv \theta^+ + \theta^-$ where θ^+ is associated to the positive energy modes and is proportional to an annihilation operator, θ^- is associated to the negatives energy modes and is proportional to a creation operator. If we consider the two-point function of θ, we find

$$\langle\theta(x)\theta(0)\rangle = \left\langle\theta^+(x)\theta^-(0)\right\rangle = \left\langle\left[\theta^+(x),\theta^-(0)\right]\right\rangle \; . \tag{10.0.6}$$

We now evaluate the one-point function of ψ using its decomposition in terms of the fluctuations σ and θ

$$\begin{aligned}\langle\psi(x)\rangle &= v\left\langle e^{i\theta(x)}\right\rangle = v\left\langle e^{i\theta^-(x)}e^{i\theta^+(x)}e^{1/2\left[\theta^-(x),\theta^+(x)\right]}\right\rangle = v\, e^{-1/2\left\langle\left[\theta^+(x),\theta^-(x)\right]\right\rangle} \\ &= v\, e^{-1/2\langle\theta(0)\theta(0)\rangle} \; ,\end{aligned} \tag{10.0.7}$$

where we used, besides the previous arguments, also the fact that σ is a massive perturbation around v. We thus see that in order to certify whether the VEV is maintained at the quantum level, we need the two-point function for θ at vanishing distance in time and space.

Obviously, such a limit $(t,\vec{x}) \to 0$ can lead to a UV divergence, naively giving an ill-defined one-point function above. However, it is known how to deal with such divergence through renormalisation. In order to disentangle potential IR divergences, we use the theory regulated by the small mass λ, guided by the expectation that explicit symmetry

[2] As it will be seen later, we will compute the propagator of θ between the two same spacetime points, which indeed corresponds to a loop.

breaking is always viable (switching on a mass for the NG modes is equivalent to see them as pseudo-Goldstone mode, which is a signature of an explicit symmetry breaking) and a non-zero value for the order parameter should be found in that case. In consequence, the limit $\lambda \to 0$ alone must have something to tell us about the possibility of spontaneous symmetry breaking.

We now use (10.0.5) evaluated at coinciding points to find the needed expression. Analogously, this computation can be seen as the evaluation of the one-loop correction to the tadpole. Following standard manipulations (see e.g. [282] or [283] for a similar context), we have

$$\begin{aligned}\langle\theta(0)\theta(0)\rangle|_\lambda &= \frac{\pi}{(2\pi)^d\xi v^2}\int \mathrm{d}^{d-1}p\,\frac{1}{\sqrt{p^{2z}+\lambda^{2z}}}\\ &= \frac{\Gamma\left((d-1)/2z\right)\,\Gamma\left((z+1-d)/2z\right)}{(4\pi)^{d/2}\Gamma((d-1)/2)\,z\,\xi v^2}\lambda^{d-1-z}\ .\end{aligned} \tag{10.0.8}$$

Cf. Appendix K Section K.1 for the details of the computation. We immediately note that the IR behaviour will be dictated by the power of λ, while the UV behaviour depends on the argument of the Gamma function $\Gamma\left((z+1-d)/2z\right)$ because the latter is independent of λ and it depends on d and z which set the scaling dimension of the NG field, i.e. the shape of the two-point correlator (cf. (4.5.21)). The IR behaviour will give a vanishing result for $d > z+1$, and a diverging one for $d < z+1$. At the same time, the Gamma function is always regular for $d < z+1$, while it can have singularities for $d \geqslant z+1$ (more specifically, it diverges for $d = z+1+2nz$, with n a positive or null integer). The limiting case is obviously $d = z+1$, actually the only one where we need to disentangle UV and IR divergences.

For $d = z+1$, let us treat this case with dimensional regularisation. Setting $d \to z+1-2z\epsilon$ gives first

$$\langle\theta(0)\theta(0)\rangle|_\lambda^\epsilon = \frac{\Gamma\left(1/2\right)}{(4\pi)^{(z+1)/2}\Gamma(z/2)\,z\,\xi v^2}\left(\epsilon^{-1}+\text{const.}-2z\ln\lambda+O(\epsilon)\right)\ . \tag{10.0.9}$$

We obtain a UV-regular expression keeping only the finite λ-dependent piece (and introducing for dimensional reasons the renormalisation scale μ):

$$\begin{aligned}\langle\theta(0)\theta(0)\rangle|_\lambda^{\mathcal{R}_{\mathrm{UV}}} &\equiv \lim_{\epsilon\to 0^+}\left(\langle\theta(0)\theta(0)\rangle|_\lambda^\epsilon - \langle\theta(0)\theta(0)\rangle|_\mu^\epsilon\right)\\ &= -\frac{\Gamma\left(1/2\right)}{(4\pi)^{(z+1)/2}\Gamma(z/2)\,\xi v^2}\ln\left(\lambda/\mu\right)^2\ .\end{aligned} \tag{10.0.10}$$

This expression is free from UV divergence thanks to renormalisation, but still has an IR divergence when $\lambda \to 0$. We can thus conclude that the massless same-point correlator diverges to $+\infty$ when $d = z+1$.

For $d < z+1$, we see from (10.0.8) that in the limit $\lambda \to 0$ the expression also diverges to $+\infty$ (recall we assume $d \geqslant 2$), without any need to regularise and renormalise in the UV.

For $d > z+1$, we would need to regularise and renormalise in certain cases as discussed above. However, we see in (10.0.8) that the result is multiplied by a positive power of λ, which will always win in the $\lambda \to 0$ limit against any term involving $\ln\lambda$. We thus conclude that the correlator in this case always vanishes in the massless limit.

Now, going back to the expression (10.0.7), inserting the UV-renormalised two-point function, we observe that the VEV is preserved when $d > z + 1$ while it is set to zero when $d \leqslant z + 1$. We summarise the results in the table below.

Condition for d and z	$\lim_{\lambda \to 0} \langle \theta(0)\theta(0) \rangle\|_{\lambda}^{\mathcal{R}_{\text{UV}}}$	$\langle \psi(x) \rangle^{\mathcal{R}_{\text{UV}}}$	Spontaneous symmetry breaking
$d > z + 1$	0	v	yes
$d \leqslant z + 1$	$+\infty$	0	no

We have thus generalised the Coleman theorem on the possibility of having spontaneous symmetry breaking to Lifshitz theories with time reversal invariance. The argument is essentally based on contradiction. By considering a generic $U(1)$ theory presenting Lifshitz scaling symmetry, the hypothesis that $U(1)$ is spontaneously broken leads to the presence of a massless field, the would-be Goldstone boson. We then observed that for $d \leqslant z + 1$ the latter is not well defined, leading to large quantum fluctuations that set the VEV to zero. Hence no spontaneous symmetry breaking can occur in those dimensions.

We now turn to discuss the same kind of theory, but with a large N number of constituents. We employ holography to study it, and enquire whether the large N limit can restore an ordered vacuum.

Chapter 11

Holographic renormalisation and symmetry breaking in $d \leqslant z+1$

By experience from the relativistic case [26–28,121], we expect that spontaneous symmetry breaking might occurs in $d \leqslant z+1$ when we consider strict large N theories. Therefore, we look for a counter-example of the statement made in the preceding chapter for such exotic QFTs. Thus, we consider a theory with the exact same symmetry properties, but from a holographic perspective. This is tantamount to say that the QFT under consideration, besides being in the large N limit, is also generically strongly coupled. Since we look for one counter-example, we can try with a minimal holographic model and tune the parameters such as the mass in order to make the holographic renormalisation easier. We will remain quadratic in the fluctuations. Indeed, our goal is to build the two-point Ward-Takahashi identity to probe for spontaneous symmetry breaking (2.1.16). Therefore, a quadratic renormalised holographic action is sufficient. Furthermore, we will not consider correlators involving the stress-energy tensor (we are dealing with internal symmetries, not spacetime ones). The metric can thus be considered as fixed. Because the Ward-Takahashi identities rely only on the relative values between correlators, we do not need an explicit expression for the latter. Hence, only a boundary analysis is sufficient and it can be a posteriori verified that the backreaction we would have had if we considered a dynamical metric is not modifying our final results. We will use a set-up in all similar to the one considered in [281] (a paper by the author and his collaborators), though we will implement time-reversal symmetry to be consistent with the discussion in the previous section.

On the bulk, gravity side of the holographic correspondence, we thus introduce a complex scalar ϕ charged under a $U(1)$ gauge symmetry. The charge is set to unity and the corresponding gauge field is A. To reproduce a QFT invariant under Lifshitz scaling, this matter content has to live on a curved spacetime in $d+1$ dimensions dominated by the presence of a background massive vector field B [276]. If it is defined as[1]

$$B \equiv \frac{\beta}{r^z}\mathrm{d}t \qquad \text{with} \qquad \beta \equiv \sqrt{\frac{2(z-1)}{z}}, \;\; z \geqslant 1 \;, \tag{11.0.1}$$

then the background metric reads (with a radius set to unity)

$$\mathrm{d}s^2 \equiv g_{mn}\mathrm{d}x^m\mathrm{d}x^n = \frac{\mathrm{d}r^2}{r^2} - \frac{\mathrm{d}t^2}{r^{2z}} + \frac{\mathrm{d}x_j^2}{r^2} \;, \tag{11.0.2}$$

with j running from 1 to $d-1$, and is isometric under a Lifshitz scaling and the rotations

[1] Besides the obvious requirement of keeping β real, the condition $z \geqslant 1$ has strong physical motivations, both in QFT and in holography [284]. Essentially $z < 1$ would lead to causality violations, see Section I.1, and might lead to a violation of the strong energy condition, see Section K.2.

of space coordinates. The part of this metric that is orthogonal to B is given by

$$\gamma_{mn} \equiv g_{mn} + \beta^{-2} B_m B_n \qquad \text{so that} \qquad \gamma_{mn} B^n = 0. \tag{11.0.3}$$

A general action invariant under the Lifshitz symmetry group and time-reversal for a scalar ϕ and a massless vector A is then given by

$$\begin{aligned} S\left[A_m, \phi\right] &= \int \mathrm{d}^{d+1}x \sqrt{-g} \left\{ -\frac{1}{4}\gamma^{mn}\left(\gamma^{pq} - \frac{2\kappa}{\beta^2}B^p B^q\right) F_{mp}F_{nq} \right. \\ &\left. - \left(\gamma^{mn} - \frac{1}{c^2\beta^2}B^m B^n\right)\left(D_m\phi\right)^* D_n\phi - m^2\phi^*\phi \right\}. \end{aligned} \tag{11.0.4}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$ and $D_m = \partial_m - iA_m$ as usual. Here, B as well as the metric g are meant as non-dynamical fields. Similarly, we will neglect backreaction of the scalar on them (in Section K.4, it is argued that this approximation has no influence on final results). This theory has three free parameters : κ, c^2 and m^2. Let us mention that for $\kappa = 1$ we recover the Maxwell kinetic term for the vector field and when $c^2 = 1$, we retrieve the Klein-Gordon kinetic term for the scalar field.

We list here the equations of motion that are obtained from the action above

$$\begin{aligned} \partial_m\left(\frac{\sqrt{-g}}{2}\left(\gamma^{mn}\left(\gamma^{pq} - \frac{2\kappa}{\beta^2}B^pB^q\right) - \gamma^{pn}\left(\gamma^{mq} - \frac{2\kappa}{\beta^2}B^mB^q\right)\right)F_{nq}\right) & \\ -i\sqrt{-g}\left(\gamma^{pq} - \frac{1}{c^2\beta^2}B^pB^q\right)\left(\phi^* D_q\phi - \phi\left(D_q\phi\right)^*\right) &= 0 \quad (11.0.5) \\ D_m\left(\sqrt{-g}\left(\gamma^{mn} - \frac{1}{c^2\beta^2}B^mB^n\right)D_n\phi\right) - \sqrt{-g}m^2\phi &= 0 \quad (11.0.6) \end{aligned}$$

Of course, when taking the variation of the action with respect to the dynamical degrees of freedom, one has to pay attention to the boundary terms that will play a prominent role in the holographic renormalisation.

Since the radial mode of the vector A does not source any operator on the QFT side of the correspondence, we can partially fix the gauge freedom by putting it to zero (i.e. we work in the radial, or holographic, gauge)

$$A_r = 0\ . \tag{11.0.7}$$

The spatial modes can be split into transverse and longitudinal modes

$$A_i \equiv T_i + \partial_i L \qquad \text{with the condition} \qquad \partial_i T_i = 0\ . \tag{11.0.8}$$

Finally, we consider the real and imaginary parts of the scalar separately

$$\phi \equiv \frac{\rho + i\pi}{\sqrt{2}}\ . \tag{11.0.9}$$

The gauge transformations in their infinitesimal form for the newly introduced fields read

$$\delta_\alpha\rho = -\alpha\pi\ , \qquad \delta_\alpha\pi = +\alpha\rho\ , \qquad \delta_\alpha A_t = \partial_t\alpha\ , \qquad \delta_\alpha L = \alpha\ , \tag{11.0.10}$$

where α is now a function of t and $\vec{x}$ only (to preserve the holographic gauge). All other quantities are gauge invariants.

We now want to switch on a background for the scalar to enforce the symmetry breaking in the QFT. So, we introduce ϕ_B that only depends on the r coordinate (we do not want to spontaneously break spacetime symmetries of the boundary QFT), and shift

$$\rho \quad \to \quad \phi_B + \rho \ . \tag{11.0.11}$$

Moreover, we prescribe that all the degrees of freedom that we described in the previous section are small fluctuations on top of this background. Assuming the gauge parameter is similarly small, gauge transformations now read

$$\begin{aligned} &\delta_\alpha \rho = -\alpha\pi \approx 0 \ , \qquad && \delta_\alpha \pi = +\alpha(\phi_B + \rho) \approx \alpha\phi_B \ , \\ &\delta_\alpha A_t = \partial_t \alpha \ , && \delta_\alpha L = \alpha \ . \end{aligned} \tag{11.0.12}$$

First, we find the equation for the background from (11.0.6) :

$$r\partial_r\left(r\partial_r\phi_B\right) - (d+z-1)r\partial_r\phi_B - m^2\phi_B \quad = \quad 0 \ . \tag{11.0.13}$$

The gauge fixing we performed in (11.0.7) gives us, taking $p = r$ in (11.0.5), the constraint

$$-\kappa r^{2z}\partial_r\partial_t A_t + r^2\partial_j^2\partial_r L - \phi_B\partial_r\pi + \pi\partial_r\phi_B \quad = \quad 0 \ . \tag{11.0.14}$$

Taking $p = t$ and $p = j$ in (11.0.5) gives the equations for the temporal and spatial modes of the vector. We also apply the projectors $\left(\delta_{ij}\partial_k^2 - \delta_{ik}\partial_k\partial_j\right)/\partial_k^2$ and ∂_j/∂_k^2 on the $p = j$ equation to separate equations for the transverse and longitudinal modes. The real and imaginary parts of (11.0.6) give rise to equations for the real and imaginary parts of the scalar respectively. All in all, the equations of motion for the dynamical degrees of freedom are

$$\begin{aligned}
r\partial_r\left(r\partial_r A_t\right) - (d-z-1)r\partial_r A_t + r^2\partial_j^2\left(A_t - \partial_t L\right) + \frac{1}{\kappa c^2}\left(\phi_B\partial_t\pi - \phi_B^2 A_t\right) &\quad = \quad 0 \ , && (11.0.15)\\
r\partial_r\left(r\partial_r T_i\right) - (d+z-3)r\partial_r T_i - \kappa r^{2z}\partial_t^2 T_i + r^2\partial_j^2 T_i - \phi_B^2 T_i &\quad = \quad 0 \ , && (11.0.16)\\
r\partial_r\left(r\partial_r L\right) - (d+z-3)r\partial_r L - \kappa r^{2z}\partial_t\left(\partial_t L - A_t\right) + \phi_B\left(\pi - \phi_B L\right) &\quad = \quad 0 \ , && (11.0.17)\\
r\partial_r\left(r\partial_r\rho\right) - (d+z-1)r\partial_r\rho - \frac{r^{2z}}{c^2}\partial_t^2\rho + r^2\partial_j^2\rho - m^2\rho &\quad = \quad 0 \ , && (11.0.18)\\
r\partial_r\left(r\partial_r\pi\right) - (d+z-1)r\partial_r\pi - \frac{r^{2z}}{c^2}\left(\partial_t^2\pi - \phi_B\partial_t A_t\right) & && \\
+r^2\partial_j^2\left(\pi - \phi_B L\right) - m^2\pi &\quad = \quad 0 \ . && (11.0.19)
\end{aligned}$$

Since we considered small fluctuations for the dynamical degrees of freedom, those equations are linear in the fields. The degrees of freedom ρ and T_i are both decoupled from the others.

We now turn to the asymptotic expansions of the fields near the boundary. Starting from the background for the scalar field, the exact solution is

$$\phi_B \quad = \quad w\, r^{\frac{\tilde{d}}{2}-\nu} + v\, r^{\frac{\tilde{d}}{2}+\nu} \ , \tag{11.0.20}$$

where w and v are real numbers and we have defined

$$\tilde{d} \equiv d + z - 1 \qquad \text{and} \qquad \nu \equiv \sqrt{\frac{\tilde{d}^2}{4} + m^2}\ . \tag{11.0.21}$$

This notation is convenient to draw a comparison with usual AdS holography. For simplicity we will take $0 < \nu < 1$.[2] For the fluctuations, the radial behaviour captured in the equations of motion imposes the following expansions. Leaving aside the spatial index i for the mode T_i, we get

$$\rho \overset{r\to 0}{\sim} \rho_0\, r^{\tilde{d}/2-\nu} + \tilde{\rho}_0\, r^{\tilde{d}/2+\nu} + \cdots \tag{11.0.22}$$
$$\pi \overset{r\to 0}{\sim} \pi_0\, r^{\tilde{d}/2-\nu} + \tilde{\pi}_0\, r^{\tilde{d}/2+\nu} + \cdots \tag{11.0.23}$$
$$A_t \overset{r\to 0}{\sim} \tilde{a}_0\, r^{-(2z-\tilde{d})} + \cdots + a_0 + \cdots \tag{11.0.24}$$
$$T \overset{r\to 0}{\sim} t_0 + \cdots + \tilde{t}_0\, r^{\tilde{d}-2} + \cdots \tag{11.0.25}$$
$$L \overset{r\to 0}{\sim} l_0 + \cdots + \tilde{l}_0\, r^{\tilde{d}-2} + \cdots \tag{11.0.26}$$

where all coefficients are fields with a (t, x_i) dependence. We have anticipated here the special case where $d \leqslant z+1$ (i.e. $2z \geqslant \tilde{d}$), the opposite case was treated in [281]. Dots between leading and subleading orders mean that one can find some more terms by adding powers of r two by two, if $\tilde{d}-2 > 2$ and/or $2z - \tilde{d} > 2$. Logarithms should also be taken into account starting from the order r^0 in the expansion of A_t if $\tilde{d} - 2z$ is even and from $r^{\tilde{d}-2}$ in the expansions of T_i if $\tilde{d} - 2$ is even. Finally, because of the presence of the background ϕ_B, and the particular shape of the Lifshitz metric, further powers in the expansions above appear. However, it can be checked that they are all subdominant with respect to the ones shown above (provided all our previous assumptions, that is $\nu < 1$, $z \geqslant 1$ and $d \geqslant 2$).

Coefficients crowned with a tilde symbol are leading or subleading modes that we do not want to play the role of sources in QFT. For the scalars, it is just a matter of choice (in this case, it identifies w as an explicit symmetry breaking parameter and v as a VEV), while for the gauge field, it is important because only vector modes without tilde symbol transform non-trivially under the gauge group and can actually play the role of sources for a conserved current in QFT.

Indeed, we can determine the gauge transformations for the coefficients. Since ρ does not transform at linear order under the gauged $U(1)$, we have non-trivial rules for coefficients of π only

$$\delta_\alpha \pi_0 = \alpha w\ , \qquad \delta_\alpha \tilde{\pi}_0 = \alpha v\ . \tag{11.0.27}$$

For the gauge vector, only two coefficients transform under the gauge transformation

$$\delta_\alpha a_0 = \partial_t \alpha\ , \qquad \delta_\alpha l_0 = \alpha\ . \tag{11.0.28}$$

It is important for the following to note that $\tilde{a}_0$ is a gauge invariant quantity and therefore cannot be the source for the temporal part of a conserved current.

[2] Taking $\nu \geqslant 1$ makes the procedure of renormalisation more involved. Note also that for $\nu > \tilde{d}/2$, we would need to set $w = 0$ in order for the background not to spoil the asymptotic Lifshitz scaling. See Section K.3 for additional details.

Note that in the limiting, relativistic, case where $d = 2$ and $z = 1$, i.e. $\tilde{d} = 2$, all leading and subleading terms of the vector modes have the same order in r respectively :

$$\tilde{d} = 2 \quad : \quad A_t \overset{r\to 0}{\sim} \tilde{a}_0 \ln r + a_0 + \cdots \tag{11.0.29}$$

$$T \overset{r\to 0}{\sim} \tilde{t}_0 \ln r + t_0 + \cdots \tag{11.0.30}$$

$$L \overset{r\to 0}{\sim} \tilde{l}_0 \ln r + l_0 + \cdots \tag{11.0.31}$$

This case was already discussed in an author and his collaborators' paper [121] – (see also [285, 286]) so we will keep $\tilde{d} > 2$ from now on.

We can now apply the procedure of holographic renormalisation [287, 288]. Applying the equations of motion in the expression (11.0.4), we find an action on the boundary. To regularise divergences, we evaluate it on a slice $r = \epsilon$ close to $r = 0$. This procedure defines the regularised action :

$$S_{reg} \equiv \int\limits_{r=\epsilon} \mathrm{d}^d x \, \frac{r^{-\tilde{d}}}{2} \Big\{ r^2 T_i \, r\partial_r T_i - r^2 L \, r\partial_r \partial_j^2 L - \kappa r^{2z} A_t \, r\partial_r A_t + \phi_B \, r\partial_r \phi_B + 2\rho \, r\partial_r \phi_B + \rho \, r\partial_r \rho + \pi \, r\partial_r \pi \Big\} . \tag{11.0.32}$$

We need to add some counterterms to get rid of the divergences and to see clearly which coefficient of each expansion seen before is a source for the action. To do it properly, we look at the variation that has to vanish to satisfy the variational principle (note that δS_{reg} is not the variation of S_{reg} given above, but the regularised variation of the action (11.0.4))

$$\delta S_{reg} = \int\limits_{r=\epsilon} \mathrm{d}^d x \; r^{-\tilde{d}} \Big\{ r^2 \delta T_i \, r\partial_r T_i - r^2 \delta L \, \partial_j^2 \, r\partial_r L - \kappa \, r^{2z} \delta A_t \, r\partial_r A_t + \delta\rho \, r\partial_r \left(\phi_b + \rho\right) + \delta\pi \, r\partial_r \pi \Big\} . \tag{11.0.33}$$

We will now renormalise this expression for the different sectors separately. We anticipate that the sector that will contain all the subtleties is the one of the temporal and longitudinal components of the vector. We start by treating the other sectors.

For the scalar sector, the procedure goes exactly as in [281]. We add the counterterm

$$S^{\phi}_{ct} \equiv \left(\tilde{d}/2 - \nu\right) \int\limits_{r=\epsilon} \mathrm{d}^d x \; r^{-\tilde{d}} \left\{ \phi^* \phi - \frac{\phi_B^2}{2} \right\} . \tag{11.0.34}$$

Using it to define the renormalised action for the scalar part, we find (neglecting terms of zeroth order in the fluctuations, which do not concern us here)

$$\begin{aligned} S^{\phi}_{ren} &\equiv \lim_{\epsilon \to 0} \left(S^{\phi}_{reg} - S^{\phi}_{ct} \right) \\ &= \nu \int \mathrm{d}^d x \; \{ 2 v \rho_0 + \rho_0 \, \tilde{\rho}_0 + \pi_0 \, \tilde{\pi}_0 \} . \end{aligned} \tag{11.0.35}$$

Then the overall variation reads

$$\begin{aligned} \delta S^{\phi}_{ren} &= \lim_{\epsilon \to 0} \left(\delta S^{\phi}_{reg} - \delta S^{\phi}_{ct} \right) \\ &= 2\nu \int \mathrm{d}^d x \; \{ \delta\rho_0 \left(\tilde{\rho}_0 + v \right) + \delta\pi_0 \, \tilde{\pi}_0 \} , \end{aligned} \tag{11.0.36}$$

showing explicitly that our counterterm selects ρ_0 and π_0 to play the role of the sources, i.e. their variations have to vanish on the boundary $r = 0$ to satisfy the variational principle.

For the transverse sector renormalisation, it is again exactly as in [281], to which we refer for the details. Suffice here to state the only relevant piece in the renormalised action

$$S^T_{ren} = \int \mathrm{d}^d x \, \left(\tilde{d}/2 - 1\right) (t_0)_i \, (\tilde{t}_0)_i \; , \tag{11.0.37}$$

up to possible local terms when $\tilde{d}$ is even and strictly bigger than 4. Considering the variation, we find that t_0 is identified with the source, as expected.

We finally consider the renormalisation of the temporal and longitudinal sectors. We will treat the case $d = z+1$ (i.e. $\tilde{d} = 2z$) in detail and see how the result is generalised to any d and z satisfying $d < z+1$.

When $d = z+1$, the expansions for the temporal and longitudinal modes until subleading order reads[3]

$$A_t \overset{r\to 0}{\sim} \tilde{a}_0 \ln r + a_0 + \cdots \tag{11.0.38}$$
$$L \overset{r\to 0}{\sim} l_0 + \tilde{l}_0 \, r^{2(z-1)} + \cdots \tag{11.0.39}$$

It leads to

$$S^{t/L}_{reg} = \int\limits_{r=\epsilon} \mathrm{d}^d x \, \left\{ -(z-1) l_0 \, \partial_j^2 \, \tilde{l}_0 - \frac{\kappa}{2} \, \tilde{a}_0 \, \tilde{a}_0 \, \ln r - \frac{\kappa}{2} \, a_0 \, \tilde{a}_0 + \cdots \right\} \tag{11.0.40}$$

and, for the variation

$$\delta S^{t/L}_{reg} = \int\limits_{r=\epsilon} \mathrm{d}^d x \, \left\{ -2(z-1) \delta l_0 \, \partial_j^2 \, \tilde{l}_0 - \kappa \, \delta\tilde{a}_0 \, \tilde{a}_0 \, \ln r - \kappa \, \delta a_0 \, \tilde{a}_0 + \cdots \right\} . \tag{11.0.41}$$

We see that the only divergence is logarithmic and takes place for the temporal component of the vector.

As in [121], we explore now two ways of renormalising this sector. Adding a mass-like counterterm

$$\tilde{S}^{t/L}_{ct} \equiv -\kappa \int\limits_{r=\epsilon} \mathrm{d}^d x \, \frac{\left(A_t - \partial_t L\right)^2}{2 \ln r} \tag{11.0.42}$$

gives the following renormalised expression for the variation

$$\delta \tilde{S}^{t/L}_{ren} = \int\limits_{r=\epsilon} \mathrm{d}^d x \, \left\{ -2(z-1) \delta l_0 \, \partial_i^2 \, \tilde{l}_0 - \kappa \, \tilde{a}_0 \, \partial_t \delta l_0 + \kappa \, \delta\tilde{a}_0 \, (a_0 - \partial_t l_0) \right\} . \tag{11.0.43}$$

which exhibits $\tilde{a}_0$ and l_0 in the role of the sources. This choice, which we can call ordinary quantisation, is not good since $\tilde{a}_0$ does not transform under the residual gauge transformation. Hence, it cannot reproduce a source for $\mathcal{J}_t$ on the QFT side of the correspondence if $\mathcal{J}_\mu$ is a conserved current.

[3]Note that in general, the equation of motion (11.0.17) leads to a simplification for the expansion of L, setting to zero all the possible coefficients between l_0 and $\tilde{l}_0$, and without logarithms for any $\tilde{d}$.

Inspired again by [121], we propose the following counterterm[4]

$$S^t_{ct} \;\equiv\; -\kappa \int_{r=\epsilon} \mathrm{d}^d x \; \frac{\ln r}{2} \left(r\partial_r A_t\right)^2 \; . \tag{11.0.44}$$

We note that it can be obtained by adding a term of the Legendre transform kind[5] to (11.0.42):

$$\lim_{\epsilon\to 0} S^t_{ct} \;=\; \lim_{\epsilon\to 0} \left(-\kappa \int_{r=\epsilon} \mathrm{d}^d x \left\{ (A_t - \partial_t L)\, r\partial_r A_t \right\} - \tilde{S}^{t/L}_{ct} \right) . \tag{11.0.45}$$

We find

$$\begin{aligned} S^{t/L}_{ren} \;&\equiv\; \lim_{\epsilon\to 0} \left(S^{t/L}_{reg} - S^t_{ct}\right) \\ &=\; \int \mathrm{d}^d x \left\{ -(z-1) l_0\, \partial_j^2\, \tilde{l}_0 - \frac{\kappa}{2}\, a_0\, \tilde{a}_0 \right\} , \end{aligned} \tag{11.0.46}$$

and the expression for the variation

$$\delta S^{t/L}_{ren} \;=\; \int_{r=\epsilon} \mathrm{d}^d x \left\{ -2(z-1)\delta l_0\, \partial_j^2\, \tilde{l}_0 - \kappa\, \delta a_0\, \tilde{a}_0 \right\} , \tag{11.0.47}$$

which is consistent with a_0 having the correct gauge transformation for being the source of the temporal component of a conserved current. Since the source is the subleading term in the expansion, we see that we have to choose “alternative quantisation” [291] for the bulk field A_t, and just for it.

Using the constraint (11.0.14), $\tilde{l}_0$ can be expressed in terms of other coefficients

$$-\kappa\partial_t \tilde{a}_0 + (2z-2)\partial_j^2 \tilde{l}_0 - 2\nu w \tilde{\pi}_0 + 2\nu v \pi_0 \;=\; 0 \; . \tag{11.0.48}$$

Plugging it inside our renormalised action, we find

$$\begin{aligned} S^{t/L}_{ren} \;&=\; \int \mathrm{d}^d x \left\{ -\frac{\kappa}{2} l_0\, \partial_t \tilde{a}_0 - \nu\, l_0 \left(w\tilde{\pi}_0 - v\pi_0\right) - \frac{\kappa}{2}\, a_0\, \tilde{a}_0 \right\} \\ &=\; \int \mathrm{d}^d x \left\{ -\frac{\kappa}{2} \left(a_0 - \partial_t l_0\right) \tilde{a}_0 - \nu\, l_0 \left(w\tilde{\pi}_0 - v\pi_0\right) \right\} . \end{aligned} \tag{11.0.49}$$

Generalizing now to the case $d < z+1$, the near boundary expansions of the bulk fields remain the same except for the temporal sector, where it is given by (11.0.24)

$$A_t \;\overset{r\to 0}{\sim}\; \tilde{a}_0\, r^{-(2z-\tilde{d})} + \cdots + a_0 + \cdots \tag{11.0.50}$$

with possibly also a $\ln r$ term if $z - \tilde{d}/2$ is a positive integer.

As in the case $d = z+1$, the longitudinal sector will not bring any divergence. Hence, we focus on the variation of the temporal part, whose relevant terms are

$$\delta S^t_{reg} \;=\; \kappa(2z-\tilde{d}) \int_{r=\epsilon} \mathrm{d}^d x \left\{ \delta\tilde{a}_0\, \tilde{a}_0 r^{-(2z-\tilde{d})} + \cdots + \delta a_0\, \tilde{a}_0 + \cdots \right\} . \tag{11.0.51}$$

[4]See also [289, 290] for similar counterterms, in different set-ups.

[5]Where the variables are $(A_t - \partial_t L)$ and $r\partial_r A_t$ which are gauge invariant under the radial gauge choice.

We directly go to alternative quantisation to see if an adapted version of the counterterm (11.0.44) remains a good choice. The numerical coefficient is fixed to cancel the hardest divergence of the regularised action. We propose

$$S^t_{ct} \equiv \frac{\kappa}{2} \int\limits_{r=\epsilon} \mathrm{d}^d x \, r^{2z-\tilde{d}} \frac{(r\partial_r A_t)^2}{2z-\tilde{d}} \,. \tag{11.0.52}$$

Note that this term carries the correct power of r to be covariantly defined with respect to the metric near the boundary. If $2z - \tilde{d} > 2$, we will also need to introduce further counterterms of the same kind as the one above to compensate all subleading divergences

$$S^t_{ct(k)} \equiv -\frac{\kappa}{2} \int\limits_{r=\epsilon} \mathrm{d}^d x \, r^{2z-\tilde{d}+2k} \frac{(r\partial_r A_t)\, \partial_j^{2k} \,(r\partial_r A_t)}{c_k} \,, \tag{11.0.53}$$

with k positive integers and c_k numerical coefficients that are straightforward to determine. None of the counterterms will affect the finite term proportional to $\tilde{a}_0$ in S^t_{reg}. As a consequence, a_0 remains a source of the renormalised action, as (11.0.51) is pointing. Putting temporal and longitudinal pieces together, we find

$$S^{t/L}_{ren} = \int \mathrm{d}^d x \, \left\{ \kappa(z - \tilde{d}/2)\, a_0 \, \tilde{a}_0 - (\tilde{d}/2 - 1)\, l_0 \partial_j^2 \tilde{l}_0 \right\} \,, \tag{11.0.54}$$

and for the variation

$$\delta S^{t/L}_{ren} = \int \mathrm{d}^d x \, \left\{ \kappa(2z - \tilde{d})\, \delta a_0 \, \tilde{a}_0 - (\tilde{d} - 2)\, \delta l_0 \partial_j^2 \tilde{l}_0 \right\} \,. \tag{11.0.55}$$

To get rid of $\tilde{l}_0$, we use again the constraint (11.0.14). Its first order now gives

$$\kappa(2z - \tilde{d})\partial_t \tilde{a}_0 + (\tilde{d} - 2)\partial_j^2 \tilde{l}_0 - 2\nu w \tilde{\pi}_0 + 2\nu v \pi_0 = 0 \,. \tag{11.0.56}$$

Then, we find

$$S^{t/L}_{ren}[a_0, l_0, \pi_0] = \int \mathrm{d}^d x \, \left\{ \kappa(z - \tilde{d}/2)\,(a_0 - \partial_t l_0)\, \tilde{a}_0 - \nu \, l_0 \,(w\tilde{\pi}_0 - v\pi_0) \right\} \tag{11.0.57}$$

up to possible local terms if $z - \tilde{d}/2$ is a positive integer. The latter would come from (11.0.53), these finite terms would be of the form $a_0 a_0$. These will not intervene in the Ward-Takahashi identities we will compute.

We can summarise our results for all $d \leqslant z + 1$ into the expression

$$S^{t/L}_{ren} = \int \mathrm{d}^d x \, \left\{ \frac{\bar{\kappa}}{2}\,(a_0 - \partial_t l_0)\, \tilde{a}_0 - \nu \, l_0 \,(w\tilde{\pi}_0 - v\pi_0) \right\} \,, \tag{11.0.58}$$

with

$$\bar{\kappa} \equiv \begin{cases} -\kappa & \text{if } d = z+1 \,, \\ \kappa(2z - \tilde{d}) & \text{if } d < z+1 \,. \end{cases} \tag{11.0.59}$$

The sum of the renormalised actions for every sector gives the complete gauge invariant effective action that can be used to define the partition function of the QFT

$$S_{ren} \equiv S^T_{ren} + S^{t/L}_{ren} + S^{\phi}_{ren} \,. \tag{11.0.60}$$

Thus, we find

$$\begin{aligned} S_{ren}\left[t_0, a_0, l_0, \rho_0, \pi_0\right] &= \frac{1}{2}\int \mathrm{d}^d x \, \Big\{(\tilde{d}-2)(t_0)_i(\tilde{t}_0)_i + \bar{\kappa}\left(a_0 - \partial_t l_0\right)\tilde{a}_0 \\ &\quad +2\nu\Big(\rho_0\tilde{\rho}_0 + \left(\pi_0 - w l_0\right)\left(\tilde{\pi}_0 - v l_0\right) \\ &\qquad + v\left(2\rho_0 + 2\pi_0 l_0 - w l_0 l_0\right)\Big)\Big\} \; . \end{aligned} \tag{11.0.61}$$

The equations of motion for the fluctuations relate, through the deep bulk (IR) boundary conditions, the gauge invariant combinations of the tilded coefficients to the gauge invariant combinations of the sources by non-local operators

$$\tilde{a}_0 = \mathfrak{F}_a\left(a_0 - \partial_t l_0\right) + \mathfrak{F}_\pi\left(\pi_0 - w l_0\right) \; , \tag{11.0.62}$$

$$\tilde{\rho}_0 = \mathfrak{G}_\rho\, \rho_0 \; , \tag{11.0.63}$$

$$\tilde{\pi}_0 - v l_0 = \mathfrak{H}_a\left(a_0 - \partial_t l_0\right) + \mathfrak{H}_\pi\left(\pi_0 - w l_0\right) \; , \tag{11.0.64}$$

$$(\tilde{t}_0)_i = \mathfrak{I}_t\,(t_0)_i \; , \tag{11.0.65}$$

where all these operators are non-polynomial functions of the derivatives ∂_t and ∂_i^2, and we have taken into account that the transverse and ρ sectors are decoupled.

We can thus finally write the renormalised action taking this into account

$$\begin{aligned} S_{ren}\left[t_0, a_0, l_0, \rho_0, \pi_0\right] &= \frac{1}{2}\int \mathrm{d}^d x \, \Big\{(\tilde{d}-2)(t_0)_i\,\mathfrak{I}_t\,(t_0)_i \\ &\quad +\bar{\kappa}\left(a_0 - \partial_t l_0\right)\Big(\mathfrak{F}_a\left(a_0 - \partial_t l_0\right) + \mathfrak{F}_\pi\left(\pi_0 - w l_0\right)\Big) \\ &\quad +2\nu\,\rho_0\mathfrak{G}_\rho\,\rho_0 + 2\nu\left(\pi_0 - w l_0\right)\Big(\mathfrak{H}_a\left(a_0 - \partial_t l_0\right) + \mathfrak{H}_\pi\left(\pi_0 - w l_0\right)\Big) \\ &\quad +2\nu v\Big(2\rho_0 + 2\pi_0 l_0 - w l_0 l_0\Big)\Big\} \; . \end{aligned} \tag{11.0.66}$$

Now, considering

$$S_{QFT} \supset \int \mathrm{d}^d x \, \Big\{(t_0)_i\mathcal{J}_i^T - l_0\partial_i\mathcal{J}_i - a_0\mathcal{J}_t + \rho_0\mathrm{Re}\mathcal{O} + \pi_0\mathrm{Im}\mathcal{O}\Big\} \; , \tag{11.0.67}$$

and the holographic correspondence, we can write for example

$$\langle\mathrm{Re}\mathcal{O}(x)\rangle = \frac{\delta i S_{ren}}{\delta i\rho_0(x)} \; , \tag{11.0.68}$$

or

$$\langle\mathrm{Im}\mathcal{O}(x)\partial_i\mathcal{J}_i(y)\rangle = \frac{\delta^2 i S_{ren}}{\delta i\pi_0(x)\delta(-i l_0(y))} \; . \tag{11.0.69}$$

In this way, we find

$$\langle\mathrm{Re}\mathcal{O}(x)\rangle = 2v\nu \; , \tag{11.0.70}$$

$$\langle\mathrm{Im}\mathcal{O}(x)\mathrm{Im}\mathcal{O}(0)\rangle = -i2\nu\mathfrak{H}_\pi\delta^d(x) \; , \tag{11.0.71}$$

$$-\langle\mathrm{Im}\mathcal{O}(x)\partial_t\mathcal{J}_t(0)\rangle + \langle\mathrm{Im}\mathcal{O}(x)\partial_i\mathcal{J}_i(0)\rangle = \left(-i2w\nu\mathfrak{H}_\pi + 2iv\nu\right)\delta^d(x) \; . \tag{11.0.72}$$

Technically speaking these are rather the connected correlators. In Section K.5, we argue that they are equal to the "full" propagators. These obtained relations can be reexpressed as

$$- \langle \mathrm{Im}\mathcal{O}(x)\partial_t \mathcal{J}_t(0)\rangle + \langle \mathrm{Im}\mathcal{O}(x)\partial_i \mathcal{J}_i(0)\rangle = w \langle \mathrm{Im}\mathcal{O}(x)\mathrm{Im}\mathcal{O}(0)\rangle + i \langle \mathrm{Re}\mathcal{O}\rangle \delta^d(x) \ , \tag{11.0.73}$$

which are the Ward-Takahashi identities for a current associated to a symmetry which is broken both spontaneously (by v) and explicitly (by w) – see Appendix J for the construction of the Ward-Takahashi identities when $U(1)$ is explicitly broken.

In the purely spontaneous case, the Ward-Takahashi identities imply the presence of a gapless mode, i.e. a Goldstone boson (see Section I.2). What our holographic analysis has shown is that the procedure of holographic renormalisation is still consistent with the presence of a non-zero VEV v. This then indicates that spontaneous symmetry breaking is indeed possible in holographically realised Lifshitz theories in $d \leqslant z+1$.

Chapter 12

Discussion and outlooks

In this part of the thesis, we have analysed the possibility to have spontaneous symmetry breaking in theories with Lifshitz scaling, depending on the dimensionality of spacetime. First, we considered the issue from the purely field theoretic perspective, and found the expected result: when the mass dimension of a scalar is zero or negative, i.e. when $d \leqslant z + 1$, large quantum fluctuations in the massless case erase any possibility of having an order, i.e. a VEV. We then proceeded to consider the same situation in a holographic set-up, suitable for a large N theory. We found that there is no consistency problem in having a non-zero VEV,[1] and hence a propagating massless scalar. This is consistent with the expectation that order can be restored in the $N \to \infty$ limit.

With respect to the previous analysis of the relativistic case in [121], we have seen that also in the present case we have to resort to alternative quantisation for the vector. However, and this is a novel feature, only the temporal component of the vector has to be treated in this way. Actually, it is the expected gauge symmetry of the renormalised action that ultimately dictates to us this asymmetric treatment of the temporal and spatial components of the bulk vector.[2]

We now comment on some issues that we did not address in the present part of the thesis, but that could be worth investigating.

- Having shown in this part that the holographic approach, being pertinent to the $N \to \infty$ limit, allows for spontaneous symmetry breaking, one can ask whether $1/N$ corrections can spoil this result and set the VEV to zero when $d \leqslant z + 1$. This amounts to computing corrections at leading order in the bulk interactions. Eventually, one is led to perform a one-loop integral in all similar to the one performed in section 10. This approach was followed in [292] for the case of $d = 3$ and finite temperature, finding that indeed large fluctuations erase the bulk scalar profile dual to the VEV. We expect a similar result also in the cases considered in the present part of the dissertation.

- Further, we can ask what happens when temperature is turned on. At high temperature, all the scales become negligeable and we might wonder if it is consistent to speak of Lifshitz scaling symmetry where time is dealt in a different way with respect to space. However, above a critical temperature, the thermal fluctuations do not tolerate an SSB in any cases. The question of SSB at lower dimension for Lifshitz theories is then consistent only if we look at finite temperature below a

[1] For instance, in principle a legitimate alternative result could have been to find that it was impossible to cancel all divergencies for $v \neq 0$.

[2] Note that a precondition to have a situation opposite to the one that we described, i.e. alternative quantisation only for the spatial components of the vector and ordinary quantisation for the temporal component, is to have $2z < \tilde{d} < 2$, i.e. $z < 1$. We can thus conclude that this possibility does not arise in physically sensible set-ups [284].

given scale, which could be the VEV. On the QFT side, a general argument like in [118] from thermal field theory (see for instance [33]) gives for a massless mode at finite temperature $T = 1/\beta$

$$\langle\theta(0,\vec{x})\theta(0)\rangle_T \propto \int \mathrm{d}^{d-1}p\, \frac{e^{i\vec{p}\cdot\vec{x}}}{p^z}\left(1+\frac{2}{e^{\beta p^z}-1}\right) \sim 2T\int \mathrm{d}^{d-1}p\, \frac{e^{i\vec{p}\cdot\vec{x}}}{p^{2z}} + \dots\ , \qquad (12.0.1)$$

where in the last step we have isolated the most IR divergent term. From the latter, we observe that at $T > 0$, such integral is generically IR divergent when $d \leqslant 2z+1$, hence increasing the critical dimension below which spontaneous breaking of continuous symmetries is prevented. Note that for $z = 1$, we recover the Mermin-Wagner-Hohenberg theorem [122, 123].[3] In holography, one should study scalar profiles in Lifshitz black hole spacetimes (see e.g. [293–297]). In the latter set-up, since our analysis was purely a boundary analysis, one does not expect any variation with respect to our results if the spacetime metric is asymptotic to the pure Lifshitz one. Bulk $1/N$ corrections should on the other hand be sensitive to the presence of the black hole horizon.

- On the conceptual level, it could be worth to understand how from some UV relativistic theories we could retrieve Lifshitz EFTs. This is partially discussed in [281].
- It would be interesting to explore possible realistic systems which display Lifshitz scaling (see [270] and references therein), in the $d \leqslant z + 1$ regime, to verify that indeed the spontaneous breaking of continuous symmetries does not take place. That would apply to systems in two spatial dimensions with $z \geqslant 2$, or in three spatial dimensions with $z \geqslant 3$. Finding such systems could open the way to an experimental verification of the phenomenon discussed in this part of the thesis.
- Finally, as usual in Goldstone physics, we can wonder how the known results for internal symmetries extend to spacetime symmetries. In this case, how Coleman's theorem should be adapted to encompass the spontaneous breaking of global continuous spacetime symmetries. Some subtleties we might expect could be that when we spontaneously break spatial translation symmetries, it can be seen as an effective reduction of spacetime dimension. We can think of an infinite membrane for example, where the fluctuations live on the membrane and so are effectively defined on a lower dimensional spacetime than the original spacetime into which the membrane is embedded. A similar, but less straightforward, reasoning can be made for the breaking of continuous translation symmetry towards crystal structures[4]. Thus, the Coleman critical dimension of spacetime will it be with respect to the original spacetime or to the effectively "reduced" spacetime ? Another specificity of spacetime symmetry breaking that might arise is that the NG modes could be fermionic. A Dirac free fermion has a single time-derivative in its Lagrangian. So, it

[3] Without time-reversal invariance, a similar argument would suggest that the critical dimension is now $d \leqslant z + 1$. These theories do not seem to respect the Mermin-Wagner-Hohenberg theorem, in the same way as they do not with the Coleman theorem at zero temperature.

[4] Let us emphasise that it is known that ^{4}He does not solidify at zero temperature and at atmospheric pressure due to large phonon vibrations compared to the lattice spacing [144]. This is a similar reasoning to what we did for the internal symmetries.

might lead to an ambiguity on the classification type A VS. type B NG modes (considering that we can extend this classification to non-uniform symmetries). From the experience we acquired from the internal symmetry case, it seems that what matters is if the massless NG mode has an associated massive partner or not. The intuition would then be that following the fermionic NG mode being independent or canonically conjugated to another NG candidate, it will lead to no spontaneous symmetry breaking allowed or to SSB allowed. If we are in the first situation, the fact that we have now a single time-derivative will influence the dispersion relation and then, probably the value of the critical dimension of spacetime.

Appendix H

Bottom-up approach of holography

In this appendix we present what holography is. More specifically, we focus on its bottom-up approach, which is the perspective used in this thesis. Holography is the conjecture that some physical systems can both be described by a quantum field theory in d dimension spacetime and by a gravitational theory in $d+1$ spacetime[1] [298]. The main interesting point of holography is that it is a strongly VS. weakly coupled duality, when the QFT is strongly coupled, the gravitational theory is not and vice versa. It is not yet well established under which conditions a QFT can be dual to a gravitational theory. However, it seems that to be a large N theory is one of the criteria. In order to have a feeling on why such kind of dualities should hold and also, to intuitively understand why the large N aspect plays a role, we start by briefly commenting the genesis of holography. Then, we will state general principles on which a holographic duality can be built on and how to extract information from it.

The historic duality is between a CFT and a gravitational theory living on anti-de Sitter spacetime (AdS), it has been derived by Maldacena in the end of the nineties [29]. In his paper, Maldacena considered a type IIB string theory with N $D3$-branes and studied it in two different regimes. In the perturbative low-energy regime, only the massless modes of the theory are surviving and the open strings ending on the N $D3$-branes lead to a $SU(N)$ $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) in 4-dimensional flat spacetime. In the classical limit, at low energy, the considered string theory reduces to a supergravity living on an $\mathrm{AdS}_5 \times S^5$ background geometry, where AdS_5 is AdS in five dimensions and S^5 is the five-sphere. These two radically different theories (SYM and supergravity) come from the same fundamental theory in two different regimes. If the fundamental theory behaves continuously going from one regime to the other, then the field theory is describing the same physics as the gravitational theory. This smooth transition is not proven and this is why the holographic duality is a conjecture. Nevertheless, many non-trivial checks of the duality have been performed by computing physical quantities on both sides [299]. To understand why holography provides us a tool to study strongly coupled large N field theories, we have to look at the coupling parameters of the involved theories. The SYM has its coupling g_{YM} which is related to the string coupling following $g_{\mathrm{YM}}^2 \sim g_{\mathrm{s}}$. If we take N to be large, from large N QFTs, we know that the effective coupling is the t'Hooft coupling [300] $\lambda \equiv N g_{\mathrm{YM}}^2 \sim N g_{\mathrm{s}}$. Concerning supergravity on $\mathrm{AdS}_5 \times S^5$, the radius L of AdS is given by $L^4 \sim g_{\mathrm{s}} N l_{\mathrm{s}}^4$ where l_{s} is the string length. The supergravity theory has been obtained in the classical limit, which corresponds to consider $g_{\mathrm{s}} \to 0$ and $L \gg l_{\mathrm{s}}$. This is achieved only if $g_{\mathrm{s}} N \gg 1$ and $N \to +\infty$. If the holographic conjecture holds true, the supergravity theory describes the same physics as a (strictly) large N SYM theory with a large t'Hooft coupling $\lambda \sim N g_{\mathrm{s}} \gg 1$. We can therefore use a classical gravitational

[1] In optical physics, a hologram represents a $3D$ image on a $2D$ object. In our case, it is the same principle: information of a $d+1$-dimensional space is encoded into a d-dimensional space (and vice versa). This explains the name "holography".

theory to obtain information on a strongly coupled large N quantum field theory! Let us notice that if N is not going strictly to infinity, some quantum corrections might to be considered in the supergravity theory.

It is believed that the holographic principle is more general than the original example, that there exists several QFTs which have a gravitational dual [270]. The historic conjectured duality as well as its extension has been tested in several physical cases. For example, a holographic computation permitted to recover some physical properties of a quark-gluon plasma (theoretical prediction: [301], experimental confirmation: [302]). These possible extended holographic dualities are studied following two approaches: the top-down approach and the bottom-up one. The top-down perspective is to start form one side of the duality and to explicitly build the other side with string theory oriented tools. It is an involved process but it has the advantage to not take (too much) for granted the conjecture and that we explicitly know the QFTs and the gravitational theories we are working with. The top-down construction is in the same fashion as the seminal paper [29]. These string theory considerations are not necessary for this dissertation and we will not comment them further. The interested reader can look at [303–309] for more information. In the bottom-up approach, we take for granted the holographic conjecture. Based on general symmetric principles, we build an explicit gravitational theory which is dual to a QFT from which we only know the symmetric properties and some operators content. The disadvantage of the bottom-up approach is that we do not know the explicit expression of the QFT we are describing, but, the advantage is that it is a generic method and it is usually simpler compared to a top-down approach. Since the holographic conjecture has been blindly accepted, a posteriori, some consistency checks are done. For example, we know that no matter how intricate the QFT is, the symmetry based results such as the Ward-Takahashi identities should be satisfied. The bottom-up perspective of holography is the method we are now going to present.

It should be mentioned that holography has a large impact on nowadays theoretical physics. Currently (early 2020 decade), the seminal paper of holography [29], published end of the nineties, has an order of magnitude of 17.000 citations while the seminal paper of Goldstone physics [4], published in the early sixties, has an order of magnitude of 2.000 citations[2]. By comparison, we understand that we are only doing a superficial presentation of what holography is. The bottom-up approach will be schematically introduced in a recipe fashion. This recipe oriented presentation is directly illustrated by the computations of Chapter 11. For additional technical details and for a pedagogical illustration of the "recipe", we refer the reader to the practical handbook [310].

Bottom-up holography consists in building an explicit gravitational theory dual to an unspecified QFT where only some of the symmetric features of the QFT are known. The guidelines for the construction of the gravitational model are based on symmetries and bare the name of "holographic dictionary". In order for the holographic conjecture to be true, the symmetries from both sides of the duality should match. Indeed, in the historic example we have that SYM is a conformal field theory in 4-dimensional flat spacetime, it has thus the $SO(2,4)$ symmetry and being a $\mathcal{N}=4$ supersymmetric theory, it includes a $SU(4)$ R-symmetry. The $SO(2,4)$ symmetry group matches with the isometry group of AdS_5 and $SU(4)$ matches with the $SO(6)$ isometry group of S^5. The intuition acquired from the already established dualities, led to the following statement for the holographic

[2]These citation numbers comes from Inspire HEP and the data was accessed in April 2022.

dictionary.

Conjecture 1 (Holographic dictionary)**.** *A strictly large N strongly coupled field theory with a spacetime symmetry group G_{st} and with an internal continuous symmetry group G_{int} living on a d-dimensional flat spacetime is dual to a classical gravitational theory with a gauge symmetry group G_{int} living on a $d+1$-dimensional curved spacetime where the geometry has the isometry group G_{st}. The correlators of the QFT are obtained by identifying the generating functionals of the two theories, where the sources of the QFT correspond to the asymptotic expansion of the field content of the gravitational theory.*

For some parts of the preceding statement, we already have a feeling why it is so (cf. large N, strong coupling, classical gravity, spacetime symmetries VS. isometries and $d = 4$ SYM is linked to a $d + 1 = 5$-dimensional AdS_5 geometry). Let us comment on the rest of this assertion.

The additional spacetime dimension of the gravitational theory, the radial direction of coordinate called r, is considered as a geometrisation of the energy scale of the QFT. A common interpretation is to consider the QFT living on a hyperplane $r = cst.$ of the gravitational theory. We can then have an intuition on how the radial coordinate scales with the energy scale of the QFT. If we have a physical event taking place in the QFT which is lying at $r_{\rm QFT}$, this event is seen gravitationally redshifted by an observer in the gravitational theory at $r_{\rm O}$ following

$$\frac{\nu_{\rm O}}{\nu_{\rm QFT}} = \sqrt{\frac{g_{00}(r_{\rm QFT})}{g_{00}(r_{\rm O})}} \ , \tag{H.0.1}$$

where g_{00} is the pure temporal component of the gravitational metric and ν is the frequency and so, a representation of the energy. In the case of AdS, in the Poincaré patch with the radius of AdS set to one, we have

$$ds^2 = \frac{1}{r^2}\left(dr^2 - dt^2 + dx^i dx^i\right) \ , \qquad r \in [0, +\infty[\ . \tag{H.0.2}$$

The ratio (H.0.1) tends to infinity when we send $r_{\rm QFT}$ towards the asymptotic boundary $r_{\rm QFT} \to 0$. It is customary to consider the QFT living at the asymptotic boundary of the gravitational theory, this in order to probe all the energy range of the QFT.

Saying that both side of the duality describe the same physics is concretely realised by identifying the generating functionals. Because the gravitational theory is classical, its generating functional reduces to the gravitational action evaluated on-shell thanks to the saddle point approximation ($\hbar \to 0$). Since the QFT is interpreted as living at the asymptotic boundary, the sources of the QFT are identified with the asymptotic expansion of the fields of the gravitational theory. The holographic prescription is [30]

$$W_{\rm QFT}[\varphi_0] = i S_{\rm grav}^{\text{on-shell}}[\varphi_0] \ , \tag{H.0.3}$$

where φ_0 is the asymptotic expansion of the field φ living in the gravitational theory sourcing an operator $\mathcal{O}_\varphi$ of the QFT. $W_{\rm QFT}$ is the generating functional of the connected correlators of the QFT and $S_{\rm grav}$ is the classical action of the gravitational theory.

If the QFT has an internal continuous symmetry group $G_{\rm int}$, it means that conserved currents are defined for this theory. We therefore should be able to compute correlators involving conserved currents and so, we need sources for them. The source of a conserved current is the gauge field associated to the symmetry leading to the considered conserved current[3]. In the context of holography, the sources of the QFT are the field content of the gravitational theory. Hence, the gravitational theory should contain gauge fields which intuitively explains why G_{int} is gauged in the gravitational side of the duality. Let us notice that the source of the stress energy tensor $T_{\mu\nu}$ is the gravitational metric $g_{\mu\nu}$.

To summarise, the holographic dictionary tells us that the field content of the gravitational theory are the sources of the QFT operators for which we want to compute the correlators. Namely, a scalar operator is sourced by a scalar field and a conserved current is sourced by a gauge field. Once we have the field content, we have to build a gravitational action by paying attention that the on-shell metric has the isometry group of the spacetime symmetries of the QFT. Since the QFT is any way non-specified, it is often enough to consider a minimal gravitational theory. When we have the gravitational action, we use (H.0.3) to obtain the QFT correlators. The simplest example of an AdS/CFT holographic model is a real scalar ϕ in AdS

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\left(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2\right) \ , \tag{H.0.5}$$

where g_{mn} is given by (H.0.2). In this example, for simplicity, the geometry is considered as being non-dynamical, a thorough discussion is made in Chapter 11 and in Section K.4 concerning this simplification. We will keep developing our generic recipe and illustrate some steps through this simplest holographic model.

Thus, we have to evaluate the gravitational action on-shell. The variational principle gives a bulk term and a boundary term. As usual, the bulk term provides the equations of motion and the boundary conditions are chosen such that the boundary term of the variational principle vanishes. As we will see later on, the boundary conditions at the asymptotic boundary ($r = 0$ in Poincaré AdS) are subtle and crucial because they influence the sources. Assuming we have the right boundary conditions, if the EOM are satisfied then the variational principle is also satisfied. The action on-shell can be simplified by integrating by part some terms to make appear the EOM inside the action. We can reduce the on-shell action to a pure asymptotic boundary term. For our example (H.0.5), we have

$$S_{\text{on-shell}} = \frac{1}{2}\int_{r=\epsilon} d^dx\, r^{-d}\,\phi\, r\,\partial_r\phi \ , \tag{H.0.6}$$

where we have anticipated a divergence by regularising $r = \epsilon$ with $\epsilon \to 0$. We consider minimal gravitational theories, therefore, if we want to compute correlators of one and two points, quadratic theories are enough (in our case, it will rather be the theory of the fluctuations which will be quadratic). From now on, we set ourselves in such a case.

[3]Let us illustrate it with the $U(1)$ example. We consider a $U(1)$ invariant theory $S[\phi] = \int \mathcal{L}$ with the standard nomenclature. If we promote $U(1)$ to be gauged, then S contains as well a gauge field A_μ. The variation of the theory under a gauge transformation is zero, if we expand, we have

$$\delta S[\phi, A_\mu] = \frac{\delta S}{\delta A_\mu}\delta A_\mu + \frac{\delta S}{\delta\phi}\delta\phi = \int \frac{\delta\mathcal{L}}{\delta A_\mu}\partial_\mu\alpha - \int j^\mu\partial_\mu\alpha = \int -\alpha\left(\partial_\mu\frac{\delta\mathcal{L}}{\delta A_\mu} - \partial_\mu j^\mu\right) = 0 \ . \tag{H.0.4}$$

As we can see, $\frac{\delta\mathcal{L}}{\delta A_\mu} = j^\mu$. Let us comment that the conserved current here is the one defined with an opposite sign compared to (2.1.7).

Hence, because the on-shell action is a pure asymptotic boundary action, the shape of the correlators is entirely provided by a boundary analysis of our holographic model. Of course, the explicit values of the QFT correlators depend as well of the deep bulk ($r \to +\infty$ in Poincaré AdS) because the coefficients of the asymptotic expansion of the fields are obtained through a resolution of the bulk EOM, in particular the deep bulk boundary conditions will "propagate" and will affect quantitatively the asymptotic boundary analysis. Nevertheless, in Chapter 11, we are only interested in recovering the Ward-Takahashi identities. Namely, we are interested in the relative values of the correlators between them to see if the Ward-Takahashi identities are saturated. Because we do not need the explicit values of the correlators, we will restrain ourselves to a pure asymptotic boundary study.

The on-shell asymptotic expansion of the fields is of the form [30, 291]

$$\phi \overset{r\to 0}{\sim} r^{\Delta}(\phi_0 + r^2\phi_1 + \ldots) + r^{\tilde{\Delta}}(\tilde{\phi}_0 + r^2\tilde{\phi}_1 + \ldots) \quad , \quad \Delta \leqslant \tilde{\Delta} \ , \tag{H.0.7}$$

where the coefficients are function of the boundary coordinates x^μ, and Δ and $\tilde{\Delta}$ depend on the dimension of spacetime, on the geometry and on the values of the free parameters of the theory. We present here an expansion for a scalar field but the idea remains the same for a vector field. The particular shape of this expansion is because generally the EOM are second order in the radial coordinate and with respect to this coordinate, they have a Euler's equation shape (this is coming from the radial dependency of the metric). For our example (H.0.5), the EOM is

$$r^{d+1}\partial_r\left(r^{-d+1}\partial_r\phi\right) + r^2\partial_\mu\partial^\mu\phi - m^2\phi = 0 \ . \tag{H.0.8}$$

The EOM are solved order by order in the limit $r \to 0$. Because the EOM are of second order in radial derivatives, we have two independent coefficients: ϕ_0 and $\tilde{\phi}_0$. The higher order coefficients are expressed respectively in terms of ϕ_0 and $\tilde{\phi}_0$ through the EOM. Let us mention that if $\tilde{\Delta}$ differs from Δ by an even integer, the two independent expansions overlap at some orders. It then leads to the introduction of logarithmic terms at the overlapping orders. Among ϕ_0 and $\tilde{\phi}_0$, one is playing the role of the source, the other one is called the response. To know which one plays which role, we have that the source is the coefficient which should be fixed by the boundary conditions at the asymptotic boundary in order to satisfy the variational principle. I.e. the variation of the source coefficient is zero and it permits to make vanish the boundary term coming from the variational principle. We understand that if we add carefully selected boundary terms to our gravitational theory, we can select which coefficient will be the source. We speak of ordinary quantisation when it is the dominant coefficient ϕ_0 which is the source and of alternative quantisation when it is the subdominant coefficient $\tilde{\phi}_0$ which is the source. We do not always have the possibility to choose between the two coefficients, sometimes there are no boundary terms permitting to change the quantisation. This has been generically studied for respectively a scalar and a vector in AdS. Since our goal is not to do a generic study of holography, we will not comment on this. We will just check by hand if in our considered cases (namely Lifshitz holography), we can find alternative boundary terms or not. The interested reader can look in the literature with the key word "unitarity bound" [291, 311–314]. When both independent expansion in (H.0.7) starts at the same orders, i.e. when $\tilde{\Delta} = \Delta$, an entire expansion will be weighted with a logarithmic factor[4]. We call such situation the Breitenlohner-Freedman bound (BF bound) [315]. For our use of holography, logarithmic terms do not bring additional conceptual difficulties but rather technical difficulties. The only subtlety can be that when we reach the BF bound by modifying the spacetime dimension or the parameter values of the theory, what was before the subleading coefficient ($r^{\tilde{\Delta}} \ll r^{\Delta}$ with $\Delta \leqslant \tilde{\Delta}$) will sort of become the leading coefficient ($\ln(r)\, r^{\tilde{\Delta}} \gg r^{\Delta}$ with $\Delta = \tilde{\Delta}$). Thus, if we want to keep sourcing the same QFT operator, we will need to perform a change of the quantisation (e.g. going from the ordinary quantisation to the alternative quantisation). This scenario in fact appears in Chapter 11 and this change of quantisation is crucial to be able to extract the Ward-Takahashi identities!

An important remark for our specific holographic considerations in Chapter 11 is that the canonical/scaling dimension of the sourced QFT operators are encoded in the asymptotic expansion of the gravity-side fields. Indeed, if for example ϕ_0 is the source, then

$$S_{\text{QFT}} \supset \int d^d x\, \phi_0\, \mathcal{O}_\phi \ , \tag{H.0.10}$$

where $\mathcal{O}_\phi$ is the QFT operator sourced by ϕ_0. The canonical dimensional analysis gives

$$[\mathcal{O}_\phi] = d - [\phi_0] \ . \tag{H.0.11}$$

In a gravitational theory of the form (H.0.5), the scalar field ϕ is non-dimensional. Thus $[\phi_0] = \Delta$, hence,

$$[\mathcal{O}_\phi] = d - \Delta \ . \tag{H.0.12}$$

The order at which the field expansion (H.0.7) starts is associated to the scaling dimension of the QFT operator.

We have commented that the on-shell action reduces to a pure asymptotic boundary action (e.g. (H.0.6)). Since we are on-shell, we have to inject the boundary asymptotic expansions of the field content which are of the form (H.0.7). Because the asymptotic boundary is at $r \to 0$, and that the field expansions might introduce polynomial terms in r with negative power (and also logarithmic terms), some infinities might appear in our on-shell gravitational action. We thus have to renormalise our theory, this procedure is known in the literature under the name of "holographic renormalisation" [287,316–319]. It consists into adding boundary counterterms to suppress the infinities without drastically modify our bulk theory. Then, we demand the counterterms to be local (as much as possible – e.g. with logarithmic divergences, a concession must be made) and invariant under all the symmetries induced on the boundary by the bulk action (at least on-shell – to not spoil the symmetries of the QFT correlators). It should be mentioned that the

[4]In our real scalar field in AdS example, we have

$$\Delta = d - \tilde{\Delta} \ , \qquad \text{with} \qquad \tilde{\Delta} = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2} \ . \tag{H.0.9}$$

We can observe that the solution becomes tachyonic when $m^2 \leqslant -\frac{d^2}{4}$. Let us notice that a negative mass for a free theory is energetically tolerated since AdS curvature compensates with a positive contribution to the energy. In the limit case $m^2 = -\frac{d^2}{4}$, we have $\tilde{\Delta} = \Delta$.

counterterms, being purely boundary terms, do not affect the bulk part of variational principle. Thus, they do not affect the EOM and so, the asymptotic expansion of the fields is unchanged. Hence, the nature of the infinities coming from putting the bulk action on-shell are not modified and we can indeed tune our boundary-counterterms to suppress them[5]. A particularity of logarithmic divergences is that it requests boundary counterterms weighted by logarithms. This means that we can also add finite boundary terms which are the same as the boundary counterterms but without the logarithms. Hence, the holographic renormalisation is not unique when there are logarithms in the expansion (H.0.7). The obtained results from the holographic model are then said to be scheme dependent.

The boundary counterterms do not alter the bulk part of the variational principle but they do intervene in the boundary part. Therefore, they can influence which coefficient between ϕ_0 and $\tilde{\phi}_0$ should be fixed by the asymptotic boundary condition in order for the variational principle to be satisfied. In other words, the holographic renormalisation allows us to select which coefficients of the asymptotic expansion of the fields will be the sources (as previously commented, sometimes only one choice is possible).

Once we have the renormalised action evaluated on-shell, it depends only on the sources and the responses. The final step is to express the responses in term of the sources in order to be able to implement the holographic prescription (H.0.3). By definition, QFT correlators are non-local functions since they are evaluated at separate spacetime positions. The derivation of our gravitational action with respect to the sources gives the correlators. This means that the responses are necessarily non-local functions of the sources. To get these dependencies, we need to solve the bulk EOM and to impose boundary conditions in the deep bulk ($r \to +\infty$ in Poincaré AdS). The latter should not alter the symmetries of the gravitational theory, for example, if we have a gauge symmetry, we know that the deep bulk boundary conditions will express a gauge invariant combination of responses in terms of a gauge combination of sources. Hence, without explicitly solve the bulk EOM, we can already guess how the responses will be expressed in terms of the sources up to unknown non-local functions. With this information, it is possible to verify if the Ward-Takahashi identities are saturated or not because these identities depend only on the relative values of the correlators and so, the dependence on the unknown non-local functions drops out.

Strategy to describe symmetry breaking in holography

In order to probe for spontaneous symmetry breaking in a QFT, we have to evaluate the VEV of an operator transforming non-trivially under the considered symmetry. Since in this part of the thesis we are interested in global internal symmetries, a (group multiplet) scalar operator is enough. The gravitational dual theory will then contain a (group multiplet) scalar field. To implement the QFT global internal symmetry, we need to gauge it in the gravitational part of the duality. Hence, we have the minimal field content of our holographic model: a scalar field, a gauge field and a metric. Since the QFT is any way unspecified, we build a minimal holographic model.

[5]A good practice is usually to find the counterterms which permit to delete the stronger infinities. The latter are mainly coming from a $\phi_0\phi_0$ product ("maximally dominant terms"). Then the weaker divergences come typically from a $\phi_0\phi_n$ product. We can then use the same counterterm as before but replace one ϕ_0 in terms of ϕ_n thanks to the EOM. For a concrete example, see the practical handbook [310].

Then, we look for a background, i.e. a particular solution of the EOM, which reproduce the symmetry breaking pattern we want to study. It means that we look for a metric with the right isometry group and for a scalar field profile which (classically) spontaneously breaks the gauge symmetry following the desired breaking pattern (the gravitational background can sort of be seen as the vacuum of the QFT). If the asymptotic behaviour of the field background is of the same order as the source (computed later with the fluctuations) then, switching on a background is similar to switching on permanently a source. We are then explicitly deforming the QFT theory with an operator transforming under the symmetry. We are describing an explicit symmetry breaking. If the background field is of the same order than the response, we are describing a spontaneous symmetry breaking of the symmetry. These assertions are formally verified by a computation in our specific case in Chapter 11. To obtain the dynamics, i.e. the correlators, we perform a fluctuation around the background and we follow the "recipe" presented before. Following how the Ward-Takahashi identities are satisfied, we will know if indeed we succeed to build a holographic model displaying the right symmetries and if we succeed to reproduce the symmetry breaking pattern we wanted. We refer the reader to Appendix J to see what are the general shapes of the Ward-Takahashi identities when $U(1)$ symmetry is spontaneously broken and when it is explicitly broken. Based on an analysis of the poles of the correlators intervening in the Ward-Takahashi identities, we are able to predict the presence of NG modes or not and as well some qualitative features on their dispersion relations. To know entirely the dispersion relations, a resolution of the EOM in the bulk is necessary as well as imposing boundary conditions in the deep bulk; this in order to determine the non-local functions of the correlators which are undetermined by the asymptotic boundary analysis.

Why should we do holography in the context of Goldstone physics ?

Holography is a conjecture, it means that its validity relies on multiple consistency checks. This is even more crucial when we look for holographic models which moves away from the standard AdS/CFT duality. In Goldstone physics, we are dealing with symmetry based results, hence, no matter how intricate the considered QFT is, we expect the results to hold true. In particular, the Ward-Takahashi identities should be saturated. Furthermore, Goldstone physics is taking place in non-relativistic theories, we are thus out of the scope of CFTs. Therefore, Goldstone physics provide well know examples in order to perform consistency checks of holographic dualities away from AdS/CFT. It permits to better understand the holographic dictionary, the holographic renormalisation procedure and the holographic conjecture in general.

The other way around, holography provides a tool to study exotic strongly coupled field theories at quantum level. It thus allows us to probe how the general results of Goldstone physics behave for such intricate QFTs.

To summarise these argumentations, we can mention the paper written by the author and his collaborators [173] where type B Goldstone bosons have been retrieved from a non-relativistic holographic model purely based on a boundary analysis. The Ward-Takahashi identities have been saturated, confirming the consistency of the holographic model. From the latter, the existence of NG modes and the shape of their dispersion relations have been extracted (cf. Section I.2 to see how it could be done). This strengthens the idea that Theorem 3 remains true even for exotic QFTs.

Appendix I

Ward-Takahashi identities in Lifshitz invariant field theories

In this appendix we collect some results concerning Ward-Takahashi identities and Goldstone bosons in Lifshitz field theories [281]. We start by deriving mixed correlators between currents and order parameters in low-energy effective field theories of Goldstone bosons. We then discuss how the qualitative features of these correlators can be extracted from the Ward-Takahashi identities.

I.1 Low-energy theories for Goldstone bosons

Consider the low-energy effective action for a Goldstone boson in a field theory which enjoys Lifshitz scaling $t \to \lambda^z t$, $x_i \to \lambda x_i$, and which is invariant under time reflections [17, 106, 250, 320]:

$$S = \int dt d^{d-1}x \; \frac{1}{2}\left(\partial_t\phi\partial_t\phi - (-1)^z \xi \phi \nabla^{2z}\phi\right) \;, \tag{I.1.1}$$

where $\nabla^2 = \partial_i\partial_i$, and the sign in front of the second term is chosen such that the dispersion relation reads

$$\omega^2 = \xi k^{2z} \;, \tag{I.1.2}$$

so that we can set ξ real and positive to avoid tachyonic behaviour.

The relativistic case is $z = 1$, $\xi = 1$. It can be more reassuring to think of z as an integer, but z is a measured parameter, so it can take any finite real values. To avoid any causality issues, we are going to restrain ourselves to the case $z \geqslant 1$. Indeed, from (I.1.2) we see that phase velocity and group velocity scale as k^{z-1}. When $z \geqslant 1$, at low momentum the velocities tend to zero. At large momentum the velocities tend to infinity but the energy ω tends as well to infinity. We know that our Lifshitz theory being non-relativistic must be UV completed by a relativistic theory. Hence, there is a UV cut-off and our Lifshitz theory remains consistent only if we look at momenta below the cut-off. Therefore, we do not see the infinite velocities. However, when $z < 1$, the velocities tend to infinity at low momenta and low energy. These supraluminal velocities indicate that our Lifshitz theory cannot be seen as the IR theory of a fundamental relativistic theory. The case $z < 1$ seems to be physically pathological.

Assuming ξ does not scale, the scaling dimensions are the following:

$$[\partial_t] = z \;, \quad [\partial_i] = 1 \;, \quad [\phi] = \frac{d-1-z}{2} \;. \tag{I.1.3}$$

The propagator for ϕ that one can extract from (I.1.1) is the following, in Fourier space:

$$\langle \phi(\omega, q)\phi(-\omega, -q)\rangle = \frac{i}{\omega^2 - \xi k^{2z}} \;. \tag{I.1.4}$$

It can be checked that it has the correct scaling dimension[1].

The action (I.1.1) has a shift symmetry $\phi \to \phi + v\alpha$, with v the VEV and α the parameter of the transformation. This is indeed expected for a Goldstone boson.

In order to find the current that generates this symmetry (which is broken by the VEV v) we promote α to a spacetime dependent function, and define

$$\delta S = \int dt d^{d-1}x \left(\partial_t \alpha J_t - \partial_i \alpha J_i\right) \ . \tag{I.1.5}$$

We then obtain

$$J_t = v\partial_t\phi \ , \qquad J^i = (-1)^{z-1}\xi v \partial_i \nabla^{2z-2}\phi \ . \tag{I.1.6}$$

They are linear, as it befits currents of a broken symmetry (at the lowest order). The conservation law is

$$\partial_t J_t - \partial_i J_i = v(\partial_t^2 + (-1)^z \xi \nabla^{2z})\phi = 0 \tag{I.1.7}$$

using the EOM. Note that it reads exactly as in the relativistic case, however the dimensions of the currents are now different:

$$[J_t] = d - 1 \ , \qquad [J_i] = d + z - 2 \ . \tag{I.1.8}$$

We can now check how the conservation law appears in two-point functions, i.e. in the Ward-Takahashi identities. Recall that here the operator breaking the symmetry is ϕ itself, with $\langle \delta_\alpha \phi \rangle = v$.

Using (I.1.4), we have

$$\langle J_t \phi \rangle = -iv\omega \langle \phi\phi \rangle = \frac{v\omega}{\omega^2 - \xi k^{2z}} \ , \tag{I.1.9}$$

$$\langle J_i \phi \rangle = i\xi v k_i k^{2z-2} \langle \phi\phi \rangle = -\frac{\xi v k_i k^{2z-2}}{\omega^2 - \xi k^{2z}} \ , \tag{I.1.10}$$

so that

$$i\omega \langle J_t \phi \rangle + ik_i \langle J_i \phi \rangle = \frac{iv\omega^2}{\omega^2 - \xi k^{2z}} - \frac{i\xi v k^{2z}}{\omega^2 - \xi k^{2z}} = iv \ . \tag{I.1.11}$$

This is the Ward-Takahashi identity

$$-\partial_t \langle J_t \phi \rangle + \partial_i \langle J_i \phi \rangle = i \langle \delta_\alpha \phi \rangle \ . \tag{I.1.12}$$

Forsaking T-invariance

We can briefly consider the case of a theory which has Lifshitz scaling but not time reversal symmetry. The low-energy action is then[2]

$$S = \int dt d^{d-1}x \ \left(i\phi^* \partial_t \phi - (-i)^z \zeta \phi^* \nabla^z \phi\right) \ . \tag{I.1.13}$$

[1] We are considering normalised correlator $\langle \phi(x)\phi(y) \rangle = \frac{\int \mathcal{D}\phi\, \phi(x)\phi(y)\, e^{iS[\phi]}}{\int \mathcal{D}\phi\, e^{iS[\phi]}}$ so, the canonical dimension of a n-point correlator is the canonical dimension of the product of the operators. Going to Fourier space includes an additional canonical dimension coming from the integration measure: $h(k) = \int d^d x h(x) e^{ikx}$. Thus, the Fourier transform of an n-point correlator is given by $n\,[\phi] - (z + (d-1))$.

[2] We need to write an action for a complex scalar, leading however to only one massless physical degree of freedom, the Goldstone boson, because of linearity in time-derivatives. Note that we assume boundary terms that make the action real.

Again, it is easier to consider z even, but it can be more general. Now the dimension of the Goldstone field is

$$[\phi] = \frac{d-1}{2} \ . \tag{I.1.14}$$

Note that it is always positive as long as $d > 1$. Its propagator is

$$\langle \phi\phi^* \rangle = \frac{i}{\omega - \zeta k^z} \ . \tag{I.1.15}$$

The currents read

$$J_t = -iv\phi \ , \qquad J_i = -(-i)^z \zeta v \partial_i \nabla^{z-2}\phi \ . \tag{I.1.16}$$

Their correlators are

$$\langle J_t\phi^* \rangle = \frac{v}{\omega - \zeta k^z} \ , \tag{I.1.17}$$

$$\langle J_i\phi^* \rangle = -\frac{\zeta v k_i k^{z-2}}{\omega - \zeta k^z} \ , \tag{I.1.18}$$

so that the Ward-Takahashi identities are realised again

$$i\omega\langle J_t\phi^* \rangle + ik_i\langle J_i\phi^* \rangle = \frac{iv\omega}{\omega - \zeta k^z} - \frac{i\zeta v k^z}{\omega - \zeta k^z} = iv \ . \tag{I.1.19}$$

I.2 From Ward-Takahashi identities to the Goldstone boson

Having seen how the Ward-Takahashi identities are realised in the prototypical example of the low-energy effective theory of the Goldstone bosons, we now reverse the logic and start from the Ward-Takahashi identities in order to find the Goldstone boson, i.e. a low-energy mode with gapless dispersion relation. We have

$$-\partial_t\langle J_t\mathcal{O} \rangle + \partial_i\langle J_i\mathcal{O} \rangle = i\langle \mathcal{O} \rangle \ , \tag{I.2.1}$$

for some operator which transforms under the symmetry generated by the currents, and which has a VEV that breaks the symmetry.

Using rotational symmetry, we parametrise the correlators in Fourier space as follows:

$$\langle J_t\mathcal{O} \rangle = f(\omega,k) \ , \qquad \langle J_i\mathcal{O} \rangle = k_i g(\omega,k) \ . \tag{I.2.2}$$

Note that $[f] = \Delta - z$ and $[g] = \Delta - 2$, where Δ is the dimension of the operator $\mathcal{O}$. The Ward-Takahashi identity then implies

$$\omega f + k^2 g = \langle \mathcal{O} \rangle \ . \tag{I.2.3}$$

Obviously, assuming $\langle \mathcal{O} \rangle$ finite and non-zero, when $\omega, k \to 0$, either f or g, or both, have to blow up, signalling the presence of a massless particle in the spectrum, the Goldstone boson.

Let us be more precise. Take first $k \to 0$ with $\omega \neq 0$. Then, assuming g finite in this limit, we have $f \to \frac{\langle \mathcal{O} \rangle}{\omega}$. similarly, when $\omega \to 0$ at $k \neq 0$, we have $g \to \frac{\langle \mathcal{O} \rangle}{k^2}$. We can then rewrite

$$f = \frac{\langle \mathcal{O} \rangle}{\omega} \tilde{f} \ , \tag{I.2.4}$$

where $\tilde{f}$ is a dimensionless function of ω and k, and by virtue of the Ward-Takahashi identity

$$g = \frac{\langle \mathcal{O} \rangle}{k^2}(1 - \tilde{f}) \ . \tag{I.2.5}$$

There are two trivial ways to satisfy the Ward-Takahashi identity, which is setting either $\tilde{f} = 1$ or $\tilde{f} = 0$. These two choices do not correspond to propagating degrees of freedom in the usual sense (i.e. they lead to degenerate dispersion relations $\omega = 0$ or $k^2 = 0$ respectively). We thus consider the only interesting case where $\tilde{f}$ is a non-trivial function. Requiring that the low-energy theory has Lifshitz scaling, then it must be a function of the ratio $x = \frac{k^z}{\omega}$. If we also impose time reversal symmetry, then it must be a function of x^2. The conditions on the $k \to 0$ and $\omega \to 0$ limits translate into

$$\tilde{f}(x = 0) = 1 \ , \qquad \tilde{f}(x = \infty) = 0 \ . \tag{I.2.6}$$

We can readily find simple functions that satisfy the above requirements and reproduce the correlators obtained previously. Without imposing time reversal symmetry, we can take

$$\tilde{f} = \frac{1}{1 - \zeta x} \tag{I.2.7}$$

so that

$$\langle J_t \mathcal{O} \rangle = \frac{\langle \mathcal{O} \rangle}{\omega} \frac{1}{1 - \zeta \frac{k^z}{\omega}} = \frac{\langle \mathcal{O} \rangle}{\omega - \zeta k^z} \ , \tag{I.2.8}$$

$$\langle J_i \mathcal{O} \rangle = k_i \frac{\langle \mathcal{O} \rangle}{k^2} \left(1 - \frac{1}{1 - \zeta \frac{q^z}{\omega}} \right) = -\frac{\zeta k_i k^{z-2} \langle \mathcal{O} \rangle}{\omega - \zeta k^z} \ . \tag{I.2.9}$$

These have the same form as (I.1.17)–(I.1.18).

Imposing now time reversal invariance, we can take

$$\tilde{f} = \frac{1}{1 - \xi x^2} \tag{I.2.10}$$

so that

$$\langle J_t \mathcal{O} \rangle = \frac{\langle \mathcal{O} \rangle}{\omega} \frac{1}{1 - \xi \frac{k^{2z}}{\omega^2}} = \frac{\omega \langle \mathcal{O} \rangle}{\omega^2 - \xi k^{2z}} \ , \tag{I.2.11}$$

$$\langle J_i \mathcal{O} \rangle = k_i \frac{\langle \mathcal{O} \rangle}{k^2} \left(1 - \frac{1}{1 - \xi \frac{k^{2z}}{\omega^2}} \right) = -\frac{\xi k_i k^{2z-2} \langle \mathcal{O} \rangle}{\omega^2 - \xi k^{2z}} \ . \tag{I.2.12}$$

These have the same form as (I.1.9)–(I.1.10).

Note that in both cases, more complicated functions can be taken. However, as soon as there is a denominator with a polynomial in x (or x^2), near its roots the function will be very close to the ones we have taken. It would be nice to understand better from general principles what possible analytic structures $\tilde{f}$ can have.

Appendix J

Explicit symmetry breaking and Ward-Takahashi identities

In this appendix, we illustrate how the Ward-Takahashi identities for a global internal $U(1)$ symmetry are modified when we consider its explicit breaking. We first comment the classical case and then derive the quantum version. This appendix is inspired by [310].

J.1 Classical Ward-Takahashi identities

We know that from a $U(1)$ invariant action

$$S_{\text{inv}}[\phi] = \int d^d x \, \mathcal{L}_{\text{inv}}(\phi, \partial\phi) \ , \tag{J.1.1}$$

where ϕ is a generic field transforming under $U(1)$, we can obtain a conserved current by[1]

$$j^\mu \equiv -\frac{1}{\alpha}\left(\frac{\delta\mathcal{L}_{\text{inv}}}{\delta(\partial_\mu\phi)}\delta_\alpha\phi\right) \ . \tag{J.1.2}$$

For simplicity, we considered $\mathcal{L}_{\text{inv}}(\phi, \partial\phi)$ to be invariant rather than invariant up to a global derivative. The classical Ward-Takahashi identity is

$$\partial_\mu j^\mu = 0 \ , \tag{J.1.3}$$

when ϕ satisfies the EOM, namely the Euler–Lagrange equation

$$\frac{\delta\mathcal{L}_{\text{inv}}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}_{\text{inv}}}{\delta(\partial_\mu\phi)}\right) = 0 \ . \tag{J.1.4}$$

Let us consider a complex operator $\mathcal{O}(\phi)$ transforming under $U(1)$ following the standard representation

$$\delta_\alpha\mathcal{O} = i\alpha\mathcal{O} \ , \quad \Rightarrow \quad \begin{cases}\delta_\alpha\text{Re}\mathcal{O} = -\alpha\text{Im}\mathcal{O} \\ \delta_\alpha\text{Im}\mathcal{O} = +\alpha\text{Re}\mathcal{O}\end{cases} . \tag{J.1.5}$$

We introduce this operator into the theory in order to break spontaneously the $U(1)$ symmetry. We have

$$S_{\text{tot}} = S_{\text{inv}} + S_{\text{w}} \ , \quad \text{with} \quad S_{\text{w}} = \frac{1}{2}w\int d^d x \,(\mathcal{O}(\phi) + \mathcal{O}^*(\phi)) = w\int d^d x \,\text{Re}\mathcal{O}(\phi). \tag{J.1.6}$$

[1] Let us notice the change of sign compared to (2.1.16) – a global sign does not alter the conservation property.

The Euler–Lagrange equation is now

$$\frac{\delta\mathcal{L}_{\text{inv}}}{\delta\phi} + w\frac{\delta\text{Re}\mathcal{O}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}_{\text{inv}}}{\delta(\partial_\mu\phi)}\delta\phi\right) = 0\ . \tag{J.1.7}$$

However, the invariant action S_{inv} is still invariant, so evaluated on-shell and under a $U(1)$ transformation, we obtain

$$\delta_\alpha S_{\text{inv}} = \int d^dx\, \delta_\alpha\mathcal{L}_{\text{inv}} \tag{J.1.8}$$

$$= \int d^dx \left[\frac{\delta\mathcal{L}_{\text{inv}}}{\delta\phi}\delta_\alpha\phi + \frac{\delta\mathcal{L}_{\text{inv}}}{\delta(\partial_\mu\phi)}\partial_\mu\delta_\alpha\phi\right] \tag{J.1.9}$$

$$= \int d^dx \left[\left(\frac{\delta\mathcal{L}_{\text{inv}}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}_{\text{inv}}}{\delta(\partial_\mu\phi)}\right)\right)\delta_\alpha\phi + \partial_\mu\left(\frac{\delta\mathcal{L}_{\text{inv}}}{\delta(\partial_\mu\phi)}\delta_\alpha\phi\right)\right] \tag{J.1.10}$$

$$= \int d^dx \left[-w\delta_\alpha\text{Re}\mathcal{O} - \alpha\partial_\mu j^\mu\right] \tag{J.1.11}$$

$$= \int d^dx \left[w\alpha\text{Im}\mathcal{O} - \alpha\partial_\mu j^\mu\right] \tag{J.1.12}$$

$$= 0\ . \tag{J.1.13}$$

We used (J.1.7) and (J.1.2). The last line is by definition of S_{inv} and so, for an explicit symmetry breaking of the form (J.1.6), the Ward-Takahashi identity takes the form

$$\partial_\mu j^\mu = w\,\text{Im}\mathcal{O}\ . \tag{J.1.14}$$

J.2 Quantum Ward-Takahashi identities

If we consider a generic function of the field $\mathcal{F}(\phi)$, the associated correlator is given by[2]

$$\langle\mathcal{F}(\phi(x))\rangle = \int\mathcal{D}\phi\,\mathcal{F}(\phi(x))\,e^{iS_{\text{tot}}[\phi]}\ . \tag{J.2.1}$$

The right-hand side of the equality is invariant with respect to a field redefinition. Let us assume that the functional integration measure is invariant under a $U(1)$ redefinition of the field $\mathcal{D}\phi' = \mathcal{D}\phi$. This a standard assumption when discussing symmetries at quantum level, it puts aside the treatment of the anomalies. If we take $\mathcal{F} = \mathbb{I}$, from an infinitesimal gauge $U(1)$ field redefinition, we have

$$0 = \delta_\alpha\int\mathcal{D}\phi\,e^{iS_{\text{tot}}[\phi]} \tag{J.2.2}$$

$$= \int\mathcal{D}\phi\,\delta_\alpha e^{iS_{\text{tot}}[\phi]} \tag{J.2.3}$$

$$= \int\mathcal{D}\phi\,i\delta_\alpha\left(S_{\text{inv}} + S_{\text{w}}\right)e^{iS_{\text{tot}}[\phi]} \tag{J.2.4}$$

$$= \int\mathcal{D}\phi\,i\int d^dx\left(-j^\mu\partial_\mu\alpha - \alpha w\text{Im}\mathcal{O}\right)e^{iS_{\text{tot}}[\phi]} \tag{J.2.5}$$

$$= i\alpha\int d^dx\left[\int\mathcal{D}\phi\,\partial_\mu j^\mu\,e^{iS_{\text{tot}}}[\phi] - \int\mathcal{D}\phi\,w\text{Im}\mathcal{O}\,e^{iS_{\text{tot}}}[\phi]\right]\ . \tag{J.2.6}$$

[2] The normalisation $\int\mathcal{D}\phi\,e^{iS_{\text{tot}}[\phi]} \stackrel{!}{=} 1$ is considered.

It gives us the relation

$$\langle \partial_\mu j^\mu \rangle = w \langle \mathrm{Im}\mathcal{O} \rangle \,, \tag{J.2.7}$$

which is the quantum version of (J.1.14).

If now we consider $\mathcal{F} = \mathrm{Im}\mathcal{O}$, the gauge $U(1)$ invariance of the path integral gives

$$\begin{aligned}
0 &= \delta_\alpha \int \mathcal{D}\phi \, \mathrm{Im}\mathcal{O} e^{iS_{\mathrm{tot}}[\phi]} && \text{(J.2.8)}\\
&= \int \mathcal{D}\phi \left[\delta_\alpha \mathrm{Im}\mathcal{O}\, e^{iS_{\mathrm{tot}}[\phi]} + \mathrm{Im}\mathcal{O}\, \delta_\alpha e^{iS_{\mathrm{tot}}[\phi]}\right] && \text{(J.2.9)}\\
&= \int \mathcal{D}\phi \left[\alpha\, \mathrm{Re}\mathcal{O}\, e^{iS_{\mathrm{tot}}[\phi]} + \mathrm{Im}\mathcal{O}\, i \int d^dx \left(\partial_\mu j^\mu \alpha - \alpha w \mathrm{Im}\mathcal{O}\right) e^{iS_{\mathrm{tot}}[\phi]}\right] && \text{(J.2.10)}\\
&= i\alpha \int d^dx \left[\int \mathcal{D}\phi(x') \right.\\
&\qquad \left.\left(-i\mathrm{Re}\mathcal{O}\delta^d(x-x') - w\mathrm{Im}\mathcal{O}(x)\mathrm{Im}\mathcal{O}(x') + \mathrm{Im}\mathcal{O}(x')\partial_\mu j^\mu(x)\right) e^{iS_{\mathrm{tot}}}[\phi(x')]\right] \,. && \text{(J.2.11)}
\end{aligned}$$

As a consequence, we have the Ward-Takahashi identity taking place when an explicit symmetry breaking parametrised by w occurs

$$\langle \mathrm{Im}\mathcal{O}(x)\partial_\mu j^\mu(0)\rangle = i \langle \mathrm{Re}\mathcal{O}\rangle \delta^d(x) + w \langle \mathrm{Im}\mathcal{O}(x)\mathrm{Im}\mathcal{O}(0)\rangle \,. \tag{J.2.12}$$

Considering the mostly plus signature for the Minkowski metric, we obtain the holographic identity of Chapter 11 (see equation (11.0.73)). Let us notice that by setting $w = 0$, and redefining $j^\mu \leftrightarrow -j^\mu$, we consistently recover the non-explicitly broken Ward-Takahashi identity (2.1.16) which is expressed with the mostly minus signature. Equation (2.1.16) already includes a record of the possible spontaneous symmetry breaking of $U(1)$ via the VEV $\langle \mathrm{Re}\mathcal{O}\rangle$.

Appendix K

Technicalities

We present here some technical aspects, computations and discussions, associated to Part III of the thesis.

K.1 Two-point function for a massive Lifshitz free scalar at null distance

We provide the details of the computation which led to (10.0.8). By path integral computation, from (10.0.4), we get

$$\langle\theta(t,\vec{x})\theta(0)\rangle|_{\lambda,\,t\geqslant 0} = \lim_{\epsilon\to 0^+}\int \frac{\mathrm{d}\omega \mathrm{d}^{d-1}p}{(2\pi)^d}\,\frac{i\,e^{-ip\cdot x}}{v^2\left(\omega^2-\xi^2p^{2z}-\xi^2\lambda^{2z}+i\epsilon\right)}\,. \tag{K.1.1}$$

where $p\equiv\|\vec{p}\|$. The residue theorem then leads to

$$\langle\theta(t,\vec{x})\theta(0)\rangle|_{\lambda,\,t\geqslant 0} = \frac{\pi}{(2\pi)^d\xi v^2}\int \mathrm{d}^{d-1}p\,\frac{e^{i\vec{p}\cdot\vec{x}}e^{-i\xi\sqrt{p^{2z}+\lambda^{2z}}t}}{\sqrt{p^{2z}+\lambda^{2z}}}\,. \tag{K.1.2}$$

We use (K.1.2) evaluated at null distance to find the expression below. Despite the fact that calculation is presented for $d>2$, it can be verified that the final result is also valid for $d=2$.

$$\begin{aligned}
\langle\theta(0)\theta(0)\rangle|_{\lambda} &= \frac{\pi}{(2\pi)^d\xi v^2}\int \mathrm{d}^{d-1}p\,\frac{1}{\sqrt{p^{2z}+\lambda^{2z}}}\\
&\overset{d>2}{=} \frac{\pi}{(2\pi)^d\xi v^2}\int_{\mathbb{S}^{d-2}}\mathrm{d}^{d-2}\Omega\int_0^\infty \mathrm{d}p\,\frac{p^{d-2}}{\sqrt{p^{2z}+\lambda^{2z}}}\\
&\overset{q(p)\equiv p/\lambda}{=} \frac{\pi\mathrm{Vol}(S^{d-2})}{(2\pi)^d\xi v^2}\lambda^{d-1-z}\int_0^\infty \mathrm{d}q\,\frac{q^{d-2}}{\sqrt{q^{2z}+1}}\,.
\end{aligned} \tag{K.1.3}$$

Using $\mathrm{Vol}(S^{d-2}) = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$ for $d > 2$ and the Gaussian integral,

$$\begin{aligned}
\langle\theta(0)\theta(0)\rangle|_\lambda &= \frac{2\pi^{(d+1)/2}}{(2\pi)^d\Gamma((d-1)/2)\,\xi v^2}\lambda^{d-1-z}\int_0^\infty \mathrm{d}q\,q^{d-2}\int_0^\infty \mathrm{d}a\,2\frac{e^{-(q^{2z}+1)a^2}}{\sqrt{\pi}} \\
&\overset{b(q)\equiv a^2q^{2z}}{=} \frac{1}{2^{d-2}\pi^{d/2}\Gamma((d-1)/2)\,\xi v^2}\lambda^{d-1-z} \\
&\quad \int_0^\infty \mathrm{d}a\int_0^\infty \mathrm{d}b\,b^{(d-1)/2z-1}e^{-b}\,\frac{e^{-a^2}}{2z\,a^{(d-1)/z}} \\
&= \frac{\Gamma\left((d-1)/2z\right)}{2^{d-1}\pi^{d/2}\Gamma((d-1)/2)\,z\,\xi v^2}\lambda^{d-1-z}\int_0^\infty \mathrm{d}a\,\frac{e^{-a^2}}{a^{(d-1)/z}} \\
&\overset{c(a)\equiv a^2}{=} \frac{\Gamma\left((d-1)/2z\right)}{2^{d}\pi^{d/2}\Gamma((d-1)/2)\,z\,\xi v^2}\lambda^{d-1-z}\int_0^\infty \mathrm{d}c\,c^{(z+1-d)/2z-1}e^{-c}\,. \qquad \text{(K.1.4)}
\end{aligned}$$

From the last integral, we recognise the expression of the Gamma function, which finally gives

$$\langle\theta(0)\theta(0)\rangle|_\lambda = \frac{\Gamma\left((d-1)/2z\right)\,\Gamma\left((z+1-d)/2z\right)}{(4\pi)^{d/2}\Gamma((d-1)/2)\,z\,\xi v^2}\lambda^{d-1-z}\,. \qquad \text{(K.1.5)}$$

K.2 Taking $z \geqslant 1$ satisfies the strong energy condition

We show that considering $z \geqslant 1$ ensures us to respect the strong energy condition of general relativity. From the Lifshitz metric

$$ds^2 = \frac{dr^2}{r^2} - \frac{dt^2}{r^{2z}} + \frac{dx_j^2}{r^2}\,, \qquad \text{(K.2.1)}$$

we extract the Ricci tensor [269]

$$\begin{aligned}
R_{rr} &= -\frac{1}{r^2}(z^2+d-2)\,, \qquad R_{tt} = \frac{1}{r^{2z}}z(z+d-2)\,, \\
R_{ij} &= -\frac{1}{r^2}(z+d-2)\delta_{ij}\,. \qquad \text{(K.2.2)}
\end{aligned}$$

Considering $d \geqslant 2$ and $z \geqslant 1$, for any timelike vector v^n, i.e.

$$g_{mn}v^mv^n = \frac{1}{r^2}v^rv^r - \frac{1}{r^{2z}}v^tv^t + \frac{1}{r^2}v^iv^i \leqslant 0\,, \qquad \text{(K.2.3)}$$

we have

$$\begin{aligned}
R_{mn}v^mv^n &= -\frac{1}{r^2}(z^2+d-2)v^rv^r + \frac{1}{r^{2z}}z(z+d-2)v^tv^t - \frac{1}{r^2}(z+d-2)v^iv^i \\
&\geqslant -\frac{1}{r^2}(z^2+d-2)v^rv^r + \frac{1}{r^2}z(z+d-2)(v^rv^r+v^iv^i) \\
&\quad - \frac{1}{r^2}(z+d-2)v^iv^i \\
&= \frac{(z-1)(d-2)}{r^2}v^rv^r + \frac{(z-1)(z+d-2)}{r^2}v^iv^i \\
&\geqslant 0\,, \qquad \text{(K.2.4)}
\end{aligned}$$

where to go from the first to second line, we used (K.2.3) and $z(z+d-2) \geqslant 0$. The last line is concluded directly thanks to $d \geqslant 2$ and $z \geqslant 1$.

The strong energy condition (K.2.4) is satisfied and it implies that the null energy condition is fulfilled as well. Pathologies arising from the violation of the null energy condition when $z < 1$ are discussed in [284].

K.3 Hypothesis on the mass of the scalar field

For the background of the scalar field and for its asymptotic boundary expansion, we restrain ourselves to the case $0 < \nu < 1$ with

$$\nu \equiv \sqrt{\frac{\tilde{d}^2}{4} + m^2} \ . \tag{K.3.1}$$

This is equivalent to constrain the mass m in terms of the considered spacetime dimension d and the considered dynamical scaling z. We clarify here why taking $0 < \nu < 1$ permits to ease the holographic computation. The discussion holds both for the real part and for the imaginary part of the scalar field. Based on (11.0.22) and (11.0.23), let us then generically write

$$\phi \overset{r \to 0}{\sim} r^{\Delta_-}\phi_0 + \ldots + r^{\Delta_+}\tilde{\phi}_0 + \ldots \ , \tag{K.3.2}$$

for the asymptotic expansion of the scalar field, with

$$\Delta_\pm = \frac{\tilde{d}}{2} \pm \nu \ . \tag{K.3.3}$$

Let us remind that the background field is given by

$$\phi_B = w\, r^{\Delta_-} + v\, r^{\Delta_+} \ . \tag{K.3.4}$$

Why $\nu > 0$?

The parameter ν is positive and if we take it to be zero, the order of the subleading coefficient and of the leading coefficient in (K.3.2) would be equal and we would need to introduce a logarithm in one of the two independent expansions. The same comment holds for the background (K.3.4). We would then be at the BF bound[1].

Why $\nu < 1$?

The case $\nu \geqslant 1$ leads to several possible issues and/or additional technicalities:

- If ν is an integer, at some order, the two independent expansions in (K.3.2) will overlap at some orders, the ones in the second ellipses of the sum. Logarithms would then be needed at these orders.

[1]Notice that the minimal value for m^2 is $m^2 = -\frac{\tilde{d}^2}{4}$ otherwise we would get an imaginary power in the field expansion. This is the same lower bound than for AdS where the spacetime dimension d has been replaced by the effective one $\tilde{d}$.

- In the QFT, in the ordinary quantisation, the source term is

$$\int d^d x \, \phi_0 \, \mathcal{O} \ , \tag{K.3.5}$$

and it is finite. We have that

$$[\mathcal{O}] = \tilde{d} - [\phi_0] = \tilde{d} - \Delta_- = \Delta_+ = [\tilde{\phi}_0] \ . \tag{K.3.6}$$

In the alternative quantisation, we would have had

$$[\mathcal{O}] = \tilde{d} - [\tilde{\phi}_0] = \tilde{d} - \Delta_+ = \Delta_- = [\phi_0] \ . \tag{K.3.7}$$

In both quantisation, the scaling dimension of the operator is the canonical dimension of the response. The fact that (K.3.5) is finite, suggests that the product $\phi_0\tilde{\phi}_0$ will be finite in the boundary regularised action. Thus, all orders below $r^{\Delta_-}r^{\Delta_+} = r^{\tilde{d}}$ will diverge and all orders above will vanish in the regularised boundary action. This can be a posteriori verified by looking at (11.0.33) – notice that each radial derivative in (11.0.33) is weighted by an r factor, hence, the orders are not mixed.

We conclude that all the terms which lead to a divergence are the ones between ϕ_0 and $\tilde{\phi}_0$ in the expansion (K.3.2). By taking $\nu < 1$, we suppress the first ellipses in (K.3.2) and so, we reduce drastically the number of divergences (only the strong divergence $\phi_0\phi_0$ is remaining). Let us mention that the term in the first ellipses of (K.3.2) can be locally expressed in terms of ϕ_0 thanks to the EOM. Thus, it is not conceptually complicated to suppress them once we know the counterterm for the strong divergence. It is just a matter of technical difficulties. Furthermore, they introduce local terms in the correlators ($\sim \phi_0\partial^k\phi_0$ rather than $\phi_0\tilde{\phi}_0 \sim \phi_0\phi_0/\partial^k$) which, physically speaking, is not relevant for us since we probe for the poles of the correlators to get information on the dispersion relations.

- When $\nu > \frac{\tilde{d}}{2}$, the background takes large values at the asymptotic boundary because of the term w which is now with negative power in the radial coordinates. It has two possible consequences. First, the background might start to influence the first ellipses in the expansions for the vector field (11.0.24), (11.0.25), (11.0.26) and so, introduce additional divergent terms. Second, the large value of the background might spoil the assumption that the backreaction of the metric is subleading in the expansion of the fields (cf. next section).

- In the ordinary quantisation, the canonical dimension of the QFT operator is growing with ν, see (K.3.6). Thus, bigger is ν, bigger is the possibility that the QFT contains parameters with negative canonical dimension. The QFT would then be non-renormalisable in the sense of power counting.

- Finally, by analogy to AdS holography, taking $0 < \nu < 1$ puts us in the window where both ordinary and alternative quantisation can be performed for the scalar field [321].

K.4 Backreaction of the metric

We give a schematic reasoning to argue that if we consider the metric as being dynamical in our holographic model, the backreaction of the metric on the field expansions will be subleading and so, it will not influence the qualitative results of Chapter 11. The goal is thus to establish the order at which starts the asymptotic expansion of the metric fluctuation.

In natural units, the Einstein equations are

$$G_{mn} = T_{mn} \ . \tag{K.4.1}$$

Let us call h_{mn} the fluctuation of the metric around the Lifshitz geometry (11.0.2). We then have

$$G_{mn}(h=0) + \delta G_{mn}(h) = T_{mn}(h=0,\phi_B) + \delta T_{mn}(h,\phi,A_m) \ . \tag{K.4.2}$$

If we absorb the massive vector field B_m into the definition of G_{mn}, then we have $G_{mn}(h = 0) = 0$ since (11.0.2) is a solution of Einstein gravity. The right-hand side of (K.4.2) is dominated by the background $T_{mn}(h = 0, \phi_B) \sim g_{mn} m^2 \phi_B^2$. From perturbative gravity, $\delta G_{mn}(h) \sim g_{mn}\partial^2 h$. We thus have

$$\partial^2 h \sim \phi_B^2 \ . \tag{K.4.3}$$

Our main interest is whether or not we can have a spontaneous symmetry breaking in our holographic model. Let us then consider the spontaneous symmetry case ($w = 0$) which is

$$\phi_B = v r^{\frac{\tilde{d}}{2}+\nu} \ . \tag{K.4.4}$$

So,

$$h \sim r^{\tilde{d}+2\nu+2} \ . \tag{K.4.5}$$

With $\tilde{d} \geqslant 2$ and $0 < \nu < 1$, we can directly verify that the backreaction of the metric fluctuation will be subleading in the field expansions (11.0.22), (11.0.23), (11.0.24), (11.0.25), (11.0.26), i.e. the corrections will appear in the second ellipses of the expansion. As we have seen in the previous section, these terms do not play any role in the holographic renormalisation and vanish in the renormalised action. Therefore, the correlators will keep the same qualitative shapes than the ones in fixed geometry. The Ward-Takahashi identities will still be verified and so, our example of a possible SSB when $d \leqslant z + 1$ in holography is still valid. Of course, the non-local unknown functions of the correlators will be affected by the dynamics of the metric because we need to solve the theory in the bulk to get them. The explicit symmetry breaking case requires more discussion but conceptually speaking, we already know that explicit symmetry breaking at low dimensions is tolerated.

K.5 Ward-Takahashi identities in terms of connected correlators

Strictly speaking, the holographic prescription provides us with the connected correlators. In this section, we show that the connected correlators we computed in Chapter 11 are equal to their equivalent "full" propagators. Connected propagators are here labelised with a c subindex.

$$\begin{aligned}
\langle \mathrm{Im}\mathcal{O}(x)\partial_\mu j^\mu(0)\rangle &= \langle \mathrm{Im}\mathcal{O}(x)\partial_\mu j^\mu(0)\rangle_c + \langle \mathrm{Im}\mathcal{O}(x)\rangle_c \langle \partial_\mu j^\mu(0)\rangle_c \\
&= \langle \mathrm{Im}\mathcal{O}(x)\partial_\mu j^\mu(0)\rangle_c \ , && \text{(K.5.1)} \\
\langle \mathrm{Im}\mathcal{O}(x)\mathrm{Im}\mathcal{O}(0)\rangle &= \langle \mathrm{Im}\mathcal{O}(x)\mathrm{Im}\mathcal{O}(0)\rangle_c + \langle \mathrm{Im}\mathcal{O}(x)\rangle_c \langle \mathrm{Im}\mathcal{O}(x)\rangle_c \\
&= \langle \mathrm{Im}\mathcal{O}(x)\mathrm{Im}\mathcal{O}(0)\rangle_c \ , && \text{(K.5.2)} \\
\langle \mathrm{Re}\mathcal{O}\rangle &= \langle \mathrm{Re}\mathcal{O}\rangle_c \ , && \text{(K.5.3)}
\end{aligned}$$

where we used $\langle \partial_\mu j^\mu(0)\rangle_c = 0$ (the one-point Ward-Takahashi identity) and the fact that the holographic computation gives $\langle \mathrm{Im}\mathcal{O}(x)\rangle_c = 0$.

Part IV

Outlooks and conclusion of the thesis

Preamble Part IV

To complete this dissertation, we detail a specific selected outlook. The purpose is to convince ourselves of the consistency of the methodology that we plan to apply in order to analyse a gapped dilaton in an intricate QFT. This future research project will be in collaboration with Daniel Areán, Jewel K. Ghosh, Daniele Musso and Ignacio Salazar Landea.

The closing words of the thesis will consist into a summary of the contributions we presented in this work to the open problems in Goldstone physics. A selection of concrete future research perspectives will as well be listed and commented.

Chapter 13

Towards a holographic gapped dilaton

In Part II Chapter 7, we observed that the concomitant spontaneous symmetry breaking of dilatation and $U(1)$ symmetries in a Lorentz invariant toy model at $U(1)$ finite density leads to, amongst others, a gapped dilaton. Its gap scales with the chemical potential μ but in a model dependent way. The reason why the gap scales with μ has been traced back to the inverse Higgs constraint that we can impose between the dilaton and the $U(1)$ NG mode (the latter is associated to both time translation breaking and $U(1)$ breaking). The model dependency of the gap has for origin that we needed to modify the initial fundamental theory to be able to switch on a chemical potential without facing instabilities. Namely, we had to lift the flat energy directions. Thus, even at zero chemical potential the dilaton would have been massive. At finite μ, the gap sort of keeps a memory of this fundamental mass which explains the model-dependency of the gap. We wish to probe how general these observations are. To do so, as already seen in Part III, holography is a powerful tool to check some results in highly non-trivial QFTs. Hence, in a future work with Daniel Areán, Jewel K. Ghosh, Daniele Musso and Ignacio Salazar Landea, we will build one (or several) holographic model(s) featuring the same symmetry breaking pattern as the field theory toy model of Chapter 7 and extract from it/them information on the dilaton at finite density. Holography is also a well suited formalism to introduce temperature in the discussion, this via setting a horizon in the deep bulk (QFT IR region) thanks to a black hole solution for example[1]. On a longer future timescale, the holographic study of the concomitant breaking of dilatation and translation of Chapter 8 could be performed through the holographic Q-lattice models. Finally, to study a dilaton, we need to break the dilatation symmetry of the QFT. It means that a non-trivial RG flow will be present. Due to the non-standard nature of the QFTs described by holography, it is already known that some exotic RG flows can be obtained and studied via a holographic approach [324–326]. Our project might then also provide some insight on this aspect.

Let us emphasise that the purpose of this chapter is to provide a concrete future perspective of the thesis and to show the feasibility of the proposed project. No specific original results will be presented here. Many papers have already display dilatation breaking in a holographic framework, see for example [154, 155, 327]. The first step of the project would then be to adapt the associated models to our case, i.e. at finite density, and to ensure ourselves that the symmetry breaking pattern can be tuned in order to be spontaneous rather than to be explicit. Afterwards, a Ward-Takahashi identities oriented analysis will be performed to study the symmetry originated modes. For now, we will

[1]A black hole temperature can be obtained from its horizon thanks to Hawking's formula for the temperature of a black hole. In the philosophy of the "recipe" presented in Appendix H (Part III), the holographic dictionary tells us that the temperature of a black hole in the gravitational theory corresponds to the temperature of the dual QFT [322, 323].

concentrate on the model of [327] and show that from it, at finite density, we can get a spontaneous symmetry breaking of dilatation and $U(1)$ symmetries starting with a UV conformal field theory. This should convince us that the proposed project is realistic.

13.1 Minimal holographic model

We want the same symmetry breaking pattern than in Chapter 7. We need a fundamental (i.e. a UV) theory with a $U(1)$ symmetry and it should be invariant under the Lorentz group and under dilatation. We want to switch on a chemical potential associated to $U(1)$ and to break spontaneously the dilatation symmetry and the $U(1)$ symmetry. Thanks to the holographic dictionary, we can get the minimal field content of the gravitational theory. We require a dynamical metric to compute correlators involving the stress-energy tensor (we are studying the breaking of spacetime symmetries), a scalar complex field to source a QFT operator charged under $U(1)$ and under dilatation (i.e. which has a non-zero scaling dimension) and a $U(1)$ gauge field to switch on a chemical potential and to source the $U(1)$ conserved current. In order for the fundamental QFT theory to be relativistic and invariant under dilatation, we will require our on-shell metric to be AdS in the asymptotic boundary region, the QFT will then be conformal in the UV[2].

A possible holographic candidate is then the holographic superconductor [328]:

$$S = \int d^{d+1}x\sqrt{-g}\left(R - \frac{1}{4}F_{AB}F^{AB} - \frac{1}{2}|\nabla\Phi - iqA\Phi|^2 - \mathcal{V}(\Phi,\Phi^*)\right) + 2\int d^dx\sqrt{-\gamma}K + S_{ct}\ , \tag{13.1.1}$$

where g is the metric, γ is the induced metric at the boundary and K is the extrinsic trace of the boundary, the associated term in the action is called the Gibbons–Hawking–York boundary term. The counterterms S_{ct} are required for holographic renormalisation[3]. We are assuming that the complex scalar field Φ is charged under the $U(1)$ real gauge field A_B and that the potential $\mathcal{V}$ is $U(1)$ invariant.

Let us mention that the Gibbons–Hawking–York boundary term is necessary in order for the variational principle to be satisfied. Indeed, the Ricci scalar R depends on the second derivative of the metric, schematically $\partial^2 g$. The boundary term of the variational principle coming from the integration by part will then contain terms involving ∂g. By fixing the metric at the boundary through boundary conditions, $\delta g = 0$, we are also fixing the tangential derivatives of the metric but not the perpendicular ones (the nomenclature "tangent" and "perpendicular" refers to the boundary manifold): $\partial_\perp \delta g \neq 0$. Thus, it will remain non-zero boundary terms, these terms are cancelled thanks to the introduction of the Gibbons–Hawking–York boundary term in the action. Because the discussion in this

[2]We did not expend much on it in the introduction to bottom-up holography, let us briefly comment it. The AdS metric in the Poincaré patch is given by $ds^2 = (1/r^2)(dr^2 + dx^\mu dx_\mu)$ where $r \in [0, +\infty[$ is the radial coordinate and x^μ are the boundary coordinates. We see that the metric is invariant under the scaling $r \to \lambda r$, $x^\mu \to \lambda x^\mu$ and that on a slice $r = cst.$, it is invariant under the Poincaré group acting on x^μ. The AdS metric has indeed the required isometries.

[3]Capital Latin letters will denote the bulk indices, small Greek letters will denote the boundary coordinates, and small Latin letters will refer to the boundary spatial coordinates.

dissertation will remain at the level of the bulk equation of motions, we will not further develop and comment the boundary terms.

The minimal model (13.1.1) is the starting point of our discussion but as we will see, we might need richer model to complete the gapped dilaton analysis (an intuition could be that in our field theory toy model of Chapter 7, we had to consider an additional neutral scalar compensator field to describe a dilatation invariant theory allowing a $U(1)$ SSB).

13.2 Suitable background

To show the consistency of the proposed project, we will display the existence of a background solution of the EOM of the theory (13.1.1) which induces the requested symmetry breaking pattern for the QFT. The bulk part of the variational principle applied on (13.1.1) gives the EOM

$$\begin{aligned} R_{AB} - \frac{1}{2}Rg_{AB} = \frac{1}{2}F_{AC}F_B{}^C - \frac{1}{8}g_{AB}F^2 + \frac{1}{2}\left(\nabla_A\Phi - iqA_A\Phi\right)^*\left(\nabla_B\Phi - iqA_B\Phi\right) \\ - \frac{1}{4}g_{AB}|\nabla\Phi - iqA\Phi|^2 - \frac{1}{2}g_{AB}\mathcal{V}(\Phi,\Phi^*)\ , \end{aligned} \tag{13.2.1}$$

$$\nabla^B F_{BC} - \frac{iq}{2}\left[\Phi^*(\nabla_C\Phi - iqA_C\Phi) - \Phi(\nabla_C\Phi - iqA_C\Phi)^*\right] = 0\ , \tag{13.2.2}$$

$$\frac{1}{2}D_A D^A\Phi - \frac{\partial\mathcal{V}(\Phi,\Phi^*)}{\partial\Phi^*} = 0\ , \tag{13.2.3}$$

where we have defined $D_B = \nabla_B - iqA_B$. The first equation is the Einstein equation, the second one is the Maxwell equation, and the third one is the Klein-Gordon equation.

The background we are looking for is non-trivial because we need a metric which asymptotically has a dilatation isometry but not in the bulk, this in order for our QFT to have a non-trivial RG flow induced by the breaking of its UV conformal symmetry group[4]. Moreover, the background should be non-zero for the scalar field to induce a spontaneous symmetry breaking of $U(1)$ and initiate the symmetry breaking of dilatation as well. Finally, the temporal component of the gauge field should also be non-zero to characterise a chemical potential (cf. later). If we add these difficulties to the fact that the EOM are a coupled system of non-linear differential equations, we understand that a natural approach is to set an ansatz and to solve the particularised EOM numerically. These are the guidelines we will follow.

Thanks to a shooting numerical method, we will screen different possible solutions until we find one which induces a spontaneous breaking of the symmetries rather than an explicit breaking. To do so, we have to choose a specific potential $\mathcal{V}(\Phi,\Phi^*)$ in (13.1.1). To find a suitable potential and to determine the number of parameters on which the shooting method will be performed, an analytic study of some general properties of the ansatz and of the symmetries of the EOM will be made. In particular, it will involve a boundary analysis both in the IR and in the UV.

[4]Let us remind that the radial coordinate in the bulk is a geometrisation of the energy scale of the dual QFT. Hence, moving along the radial coordinate is similar than moving along the RG flow. Therefore, the asymptotic boundary region is called the UV region and the deep bulk region is labelised as the IR region.

We can use the following ansatz for the different fields:

$$ds^2 = \frac{dz^2}{h(z)} + e^{2B(z)}\left(-h(z)dt^2 + dx_i dx^i\right), \; A = \chi(z)dt, \; \Phi = \phi(z) \; , \tag{13.2.4}$$

with $\phi \in \mathbb{R}$. For the choice of the coordinate system, we follow [327]. In the UV, we want to recover AdS which in the Poincaré patch and in the associated frame of coordinates is given by

$$ds^2 = \frac{1}{r^2}\left(dr^2 + dx^\mu dx_\mu\right) \; , \tag{13.2.5}$$

where $r \to 0$ is the UV and $r \to +\infty$ is the IR. Let us do the change of coordinates $r = e^{-z}$ (the radius of AdS is set to one). The AdS metric is now

$$ds^2 = dz^2 + e^{2z}\left(dx^\mu dx_\mu\right) \; , \tag{13.2.6}$$

with $z \in]-\infty, +\infty[$ such that $z \to +\infty$ is the UV and $z \to -\infty$ is the IR. As for the coordinate r, we will also call z the radial coordinate since it plays the same role. We see that the shape of the ansatz is consistent with the new coordinate system, in the UV we will require the function B to be linear and the function h to be constant. Moreover, will consider an RG flow between two fixed points. Hence, we will require that in the IR we recover as well AdS (not necessarily with the same radius than in the UV).

The choice for the shape of the vector background is justified by the fact that we want a non-zero $U(1)$ chemical potential. Let us recall that the gauge vector field is the source for the $U(1)$ conserved current. If we switch on a dominant term in the UV for the temporal component of the background vector, it corresponds to deform the QFT with a term[5]

$$\int d^d x \, A_t \, j^t = A_t \int d^d x \, \rho = A_t \int dt \, Q \sim \int dt \, \mu \, Q \; , \tag{13.2.7}$$

where we were able to bring outside the vector field of the integration because it is only modulated along the radial coordinate. It should be noticed that the temporal component of a current density is a charge density which by integration gives the charge. We then end up with a deformation of the theory which has the same form as the deformation of the free energy due to a chemical potential. This allows us to interpret the asymptotic dominant term of A_t to be the chemical potential [322].

Concerning the shape of the scalar background, this is entirely similar to what we did in Chapter 11. The only additional point of attention is that the response in the UV asymptotic expansion should have a non-zero scaling dimension in order for the VEV to transform under dilatation. And so, to induce an SSB of the latter symmetry in addition to the $U(1)$ symmetry.

For the ansatz (13.2.4), the Einstein tensors are:

$$G_{zz} = \frac{(d-1)\dot{B}\left(dh\dot{B} + \dot{h}\right)}{2h} \; , \tag{13.2.8}$$

$$G_{tt} = -\frac{d-1}{2} h e^{2B}\left[\dot{B}\dot{h} + h\left(d\dot{B}^2 + 2\ddot{B}\right)\right] \; , \tag{13.2.9}$$

$$G_{ij} = \frac{e^{2B}}{2}\left[\ddot{h} + (2d-1)\dot{B}\dot{h} + (d-1)h\left(d\dot{B}^2 + 2\ddot{B}\right)\right]\delta_{ij} \; . \tag{13.2.10}$$

[5]We learn this in Chapter 11 for the scalar profile and the explicit breaking of $U(1)$.

We also record the Ricci tensor

$$R = -\left[\ddot{h} + dh\left\{2\ddot{B} + (d+1)\dot{B}^2\right\} + (2d+1)\dot{B}\dot{h}\right] . \tag{13.2.11}$$

The Einstein tensors are related by the Bianchi identity:

$$\nabla^A G_{AB} = 0 . \tag{13.2.12}$$

Therefore, we can consider any two of them to be independent components.
The independent equations of motion are:

$$2(d-1)h\ddot{B} + h\dot{\phi}^2 + \frac{q^2\chi^2\phi^2}{he^{2B}} = 0 , \tag{13.2.13}$$

$$\ddot{h} + d\dot{B}\dot{h} = \frac{\dot{\chi}^2}{e^{2B}} + \frac{q^2\phi^2\chi^2}{he^{2B}} , \tag{13.2.14}$$

$$\ddot{\chi} + (d-2)\dot{B}\dot{\chi} - \frac{q^2\chi\phi^2}{h} = 0 , \tag{13.2.15}$$

$$\ddot{\phi} + \left(d\dot{B} + \frac{\dot{h}}{h}\right)\dot{\phi} + \frac{q^2\chi^2\phi}{h^2e^{2B}} - \frac{V'}{h} = 0 . \tag{13.2.16}$$

We have used the following notation

$$\dot{} \equiv \frac{d}{dz} , \quad ' \equiv \frac{d}{d\phi} ; \tag{13.2.17}$$

and we have defined a potential $V(\phi)$ such that

$$V'(\phi) = \left.\frac{\partial\mathcal{V}(\Phi, \Phi^*)}{\partial\Phi^*}\right|_{\Phi=\phi} . \tag{13.2.18}$$

It is to this potential $V(\phi)$ that we will refer from now on. The first equation of the EOM is Einstein equation for the component $G^z{}_z - G^t{}_t = T^z{}_z - T^t{}_t$. The second one is the equation $G^i{}_i - G^t{}_t = T^i{}_i - T^t{}_t$. In particular, these subtractions permitted to withdraw the dependency on $\mathcal{V}$ in the Einstein equations. The third equation of motion is the t component of the Maxwell equation, and finally Eq. (13.2.16) is the Klein-Gordon equation. We also have a constraint equation:

$$2(d-1)h\dot{B}\left(dh\dot{B} + \dot{h}\right) - h\left(h\dot{\phi}^2 - 2\mathcal{V}\right) - e^{-2B}\left(q^2\phi^2\chi^2 - h\dot{\chi}^2\right) = 0 . \tag{13.2.19}$$

13.2.1 Two monotonic functions

In this subsection, we will discuss monotonic properties of the functions χ and h. We start with the former. We can write Eq. (13.2.15) as:

$$\frac{d}{dz}\left\{e^{(d-2)B}\dot{\chi}\right\} = \frac{q^2\phi^2}{he^{(d-2)B}}\chi . \tag{13.2.20}$$

This can be integrated once to find an integro-differential equation:

$$\dot{\chi} = Ce^{-(d-2)B} + q^2e^{-(d-2)B}\int_{-\infty}^{z} d\rho \frac{\phi^2}{he^{(d-2)B}}\chi , \tag{13.2.21}$$

where C is the integration constant. Notice that this integration constant must be zero, otherwise χ will diverge at the IR. In fact, in the IR we want to recover AdS, which means that $B(z) \sim z$ and since $d > 2$, $exp(-(d-2)B)$ diverges when $z \to -\infty$. Therefore, we can write

$$\dot{\chi} = q^2 e^{-(d-2)B} \int_{-\infty}^{z} d\rho \frac{\phi^2}{he^{(d-2)B}} \chi \ . \tag{13.2.22}$$

If χ is positive at the IR, then the right-hand side is positive. That will make $\dot{\chi}$ to be positive. Consequently, that will increase χ. This process will continue, and χ will be a monotonically increasing function.

If χ is negative at the IR, the reverse will happen and χ will be a monotonically decreasing function. The other possibility is when $\chi = 0$ at the horizon. In this case, by the same argument we can conclude that $\chi = 0$ everywhere.

The other function having a monotonicity property is h. To show this, we write Eq. (13.2.14) as:

$$\frac{d}{dz}\left(e^{dB}\dot{h}\right) = \frac{\dot{\chi}^2}{e^{(2-d)B}} + \frac{q^2\phi^2\chi^2}{he^{(2-d)B}} \ . \tag{13.2.23}$$

Since the right-hand side is positive definite, $e^{dB}\dot{h}$ is a monotonically increasing function. At the IR, this takes zero value since $\dot{h}$ cannot diverge due to regularity of the spacetime. Since $e^{dB} \geqslant 0$, $\dot{h}$ is non-negative. Therefore h is a monotonically increasing function of z.

13.2.2 UV analysis

We now analyse the possible asymptotics of the different fields near the UV. The UV is reached when $z \to +\infty$. It is desired to recover the AdS$_{d+1}$ geometry near the UV. To obtain this, we demand:

$$B(z) \to \frac{z}{\ell_{UV}\sqrt{h_{UV}}}, \ h(z) \to h_{UV}, h'(z) \to 0 \ , \tag{13.2.24}$$

where ℓ_{UV} is the UV AdS length.

We assume that the UV is located at an extremum of the potential, and without a loss of generality, we can assume that the extremum is located at $\phi = 0$. Near $\phi = 0$, we can expand the potential as:

$$V(\phi) = -\frac{d(d-1)}{\ell_{UV}^2} + \frac{m_{UV}^2\phi^2}{2} + \cdots \ . \tag{13.2.25}$$

AdS is the solution of Einstein's equations without matter with a negative cosmological constant. Let us notice that the constant term of V plays the role of the cosmological constant and this is why the UV AdS radius is encoded in this term.

We can now solve Eqs. (13.2.15)-(13.2.16) to find:

$$\chi(z) = \mu + C_\chi e^{\frac{(2-d)z}{\ell_{UV}\sqrt{h_{UV}}}} + \cdots \ , \tag{13.2.26}$$

$$\phi(z) = \phi_- e^{-\frac{\Delta_- z}{\ell_{UV}\sqrt{h_{UV}}}} + \phi_+ e^{-\frac{\Delta_+ z}{\ell_{UV}\sqrt{h_{UV}}}} + \cdots \ , \tag{13.2.27}$$

where

$$\Delta_\pm = \frac{d \pm \sqrt{d^2 + 4m_{UV}^2\ell_{UV}^2}}{2} \ . \tag{13.2.28}$$

Considering the ordinary quantisation[6], in the dual field theory side, ϕ_- and ϕ_+ are interpreted as the source and the VEV of an operator $\mathcal{O}$ with scaling dimension Δ_+ dual to the bulk field ϕ.

Demanding the reality of the dimension implies that

$$m_{UV}^2 \geqslant -\frac{d^2}{4\ell_{UV}^2} \; , \tag{13.2.29}$$

which is the Breitenlohner-Freedman bound. Furthermore, we require $\Delta_+ \neq 0$ in order for the VEV to transform under dilatation and so, to be able to induce a spontaneous symmetry breaking of the scaling symmetry.

The goal is to fix the IR boundary conditions and to solve numerically the EOM to extract the parameters of interest: μ, ϕ_- and ϕ_+. The hope is to obtain a solution with $\phi_- = 0$ and $\phi_+ \neq 0$ that would represent a spontaneous breaking of the dilatation symmetry and the $U(1)$ symmetry. This kind of solution will be called a VEV flow. In the numerical analysis we will screen a range of possible boundary conditions, this is the shooting method. To know how many independent parameters we can tune to search for a VEV flow, we will study the scaling symmetries of the EOM. Then, the analytic IR analysis will provide us the shooting parameters.

13.2.3 Scaling symmetries

In this subsection, we will scrutinise the scaling symmetries of the EOM (they should not be confused with the QFT dilatation symmetry we have in the UV). In particular there are two types of scaling symmetry as mentioned in [327]. Here, a scaling transformation of a quantity is understood as

$$X \to \lambda^\alpha X \; , \tag{13.2.30}$$

where λ is the scaling parameter and α depends on which scaling symmetry we are focusing. The following tabular is giving the value of α for each quantity for both types of scaling symmetry:

type	dz	e^B	h	χ	ϕ
I	1	0	2	1	0
II	0	1	0	1	0

(13.2.31)

Now we will discuss their implications individually.

13.2.3.1 Type I scaling

If $\{B(z), h(z), \chi(z), \phi(z)\}$ is a set of solution of the equations of motion. Then, one can show that the following set $\left\{\tilde{B}(z), \tilde{h}(z), \tilde{\chi}(z), \tilde{\phi}(z)\right\}$ is also a set of solution where,

$$\tilde{B}(z) = B\left(\frac{z}{\lambda}\right) , \; \tilde{h}(z) = \lambda^2 \, h\left(\frac{z}{\lambda}\right) , \; \tilde{\chi}(z) = \lambda \, \chi\left(\frac{z}{\lambda}\right) , \; \tilde{\phi}(z) = \phi\left(\frac{z}{\lambda}\right) \; . \tag{13.2.32}$$

[6] In the UV we recover the AdS case and so, it is reasonable to re-use the boundary results of AdS holography.

This scaling symmetry does not change either the source or the VEV of the scalar operator. This can be seen as follows. Near the UV ($z \to +\infty$), we have the following expansions:

$$\tilde{\phi}(z) = \tilde{\phi}_- e^{\frac{-\Delta_- z}{\sqrt{\tilde{h}_{UV}}\ell_{UV}}} + \tilde{\phi}_+ e^{\frac{-\Delta_+ z}{\sqrt{\tilde{h}_{UV}}\ell_{UV}}} + \cdots \ , \tag{13.2.33}$$

$$\phi(z) = \phi_- e^{\frac{-\Delta_- z}{\sqrt{h_{UV}}\ell_{UV}}} + \phi_+ e^{\frac{-\Delta_+ z}{\sqrt{h_{UV}}\ell_{UV}}} + \cdots \ , \tag{13.2.34}$$

where $\tilde{h}_{UV}$ and h_{UV} are the UV values of $\tilde{h}(r)$ and $h(r)$ respectively. From Eq. (13.2.32), they are related by

$$\tilde{h}_{UV} = \lambda^2 h_{UV} \ . \tag{13.2.35}$$

Therefore, we can write Eq. (13.2.33) as

$$\tilde{\phi}(z) = \tilde{\phi}_- e^{\frac{-\Delta_- z}{\lambda\sqrt{h_{UV}}\ell_{UV}}} + \tilde{\phi}_+ e^{\frac{-\Delta_+ z}{\lambda\sqrt{h_{UV}}\ell_{UV}}} + \cdots \ . \tag{13.2.36}$$

Since $\tilde{\phi}(z) = \phi\left(z/\lambda\right)$, from Eq. (13.2.34) we have

$$\tilde{\phi}(z) = \phi_- e^{\frac{-\Delta_- z}{\lambda\sqrt{h_{UV}}\ell_{UV}}} + \phi_+ e^{\frac{-\Delta_+ z}{\lambda\sqrt{h_{UV}}\ell_{UV}}} + \cdots \ . \tag{13.2.37}$$

Comparing Eqs. (13.2.36) and (13.2.37), we can deduce that

$$\tilde{\phi}_- = \phi_-, \ \tilde{\phi}_+ = \phi_+ \ . \tag{13.2.38}$$

Thus, we can conclude that scaling changes neither the source nor the VEV.

13.2.3.2 Type II scaling

Now we will investigate the implication of the type II scaling symmetry:

$$\chi \to \lambda\chi \ , \tag{13.2.39}$$

$$e^B \to \lambda e^B, \ \Rightarrow B \to \ln\lambda + B \ . \tag{13.2.40}$$

From the transformation

$$B \to \ln\lambda + B \ , \tag{13.2.41}$$

we can observe that

$$\frac{dB}{dz} \to \frac{dB}{dz} \ . \tag{13.2.42}$$

It means that ℓ_{UV} is invariant under the type II scaling transformation. Therefore

$$\phi(z) = \phi_- e^{\frac{-\Delta_- z}{\sqrt{h_{UV}}\ell_{UV}}} + \phi_+ e^{\frac{-\Delta_+ z}{\sqrt{h_{UV}}\ell_{UV}}} + \cdots \tag{13.2.43}$$

is also invariant. Hence, both the source and the VEV do not change under the scaling.

13.2.4 IR analysis

We now proceed to the analysis of the deep IR region. Like [327] we demand that the IR geometry is again an AdS_{d+1} space. That means we demand that near $z \to -\infty$

$$B(z) \to \frac{z}{\ell_{IR}}, \ h(z) \to 1, \dot{h}(z) \to 0 \ , \tag{13.2.44}$$

where we set the IR value of the function h to one by using the type I scaling symmetry. It will not spoil our seek for a VEV flow since we have seen that neither ϕ_- nor ϕ_+ are affected by such a scaling transformation. Concerning ℓ_{IR}, it is the AdS length at the IR and it is related to the potential by:

$$V(\phi) = -\frac{d(d-1)}{\ell_{IR}^2} + \frac{m_{IR}^2(\phi - \phi_{IR})^2}{2} + \cdots \ . \tag{13.2.45}$$

In the UV, we have considered that the AdS geometry lies at an extremum of the potential. We reproduced this consideration for the IR, where now the extremum is at ϕ_{IR} and we have developed $V(\phi)$ around it.

Concerning the gauge field χ, in the leading order, Eq. (13.2.15) becomes

$$\ddot{\chi} + \frac{d-2}{\ell_{IR}}\dot{\chi} - q^2\phi_{IR}^2\chi = 0 \ . \tag{13.2.46}$$

This can be solved with the ansatz

$$\chi(z) = \chi_+ e^{\frac{\Delta_\chi z}{\ell_{IR}}} \ , \tag{13.2.47}$$

where Δ_χ satisfies the following relation:

$$\Delta_\chi(\Delta_\chi + d - 2) = q^2\phi_{IR}^2\ell_{IR}^2 \ . \tag{13.2.48}$$

More explicitly

$$\Delta_\chi = \frac{1}{2}\left[-(d-2) + \sqrt{(d-2)^2 + 4q^2\ell_{IR}^2\phi_{IR}^2}\right] \ , \tag{13.2.49}$$

where only one of the two solutions of (13.2.48) is considered because we know from Subsection 13.2.1 that χ is monotonic. Furthermore, we consider the chemical potential to be positive, to reach it we thus need an increasing monotonic χ with the condition $\chi_+ > 0$. The square root of Δ_χ is bigger than $d-2$ which explains why we took this root for Δ_χ, this in order to have an increasing χ along z.

By using the type II scaling symmetry, we can set χ_+ to one without spoiling the VEV flow analysis. From now on, we take $\chi_+ = 1$.

Using these information, we can integrate Eq. (13.2.14) to find:

$$h(z) = 1 + \frac{\left(\Delta_\chi^2 + \ell_{IR}^2 q^2\phi_{IR}^2\right)}{2(\Delta_\chi - 1)(d + 2\Delta_\chi - 2)} e^{\frac{2(\Delta_\chi - 1)z}{\ell_{IR}}} . \tag{13.2.50}$$

This requires $\Delta_\chi > 1$ for h to be increasing as seen in Subsection 13.2.1. With (13.2.48), this condition on Δ_χ gives

$$1 + q^2\ell_{IR}^2\phi_{IR}^2 \geqslant d \ . \tag{13.2.51}$$

The asymptotic behaviour of the scalar field in the IR is obtained by solving (13.2.16) in the $z \to -\infty$ limit and by imposing the solution to reach ϕ_{IR} in this limit. Thanks to $\Delta_\chi > 1$, the term involving the confrontation between χ and $exp(2B)$ vanishes. A solution is then

$$\phi(z) = \phi_{IR} + \phi_{IR}^{(-)} e^{-\frac{\Delta_{IR}^{(-)} z}{\ell_{IR}}} , \tag{13.2.52}$$

where

$$\Delta_{IR}^{(-)} = \frac{1}{2}\left(d - \sqrt{d^2 + 4m_{IR}^2 \ell_{IR}^2}\right) , \tag{13.2.53}$$

which only works for $m_{IR}^2 > 0$ such that the square root is bigger than d.

Thanks to the scaling symmetries and to the monotonic behaviours, we have entirely fixed the IR boundary conditions of h and χ. Moreover, the IR boundary condition of B is fixed by the potential as well as the value of ϕ_{IR}. We conclude that the only IR shooting parameter remaining for the numerical analysis is $\phi_{IR}^{(-)}$.

13.3 Numerical study

We now proceed to the numerical analysis. Following [327], we choose the potential $\mathcal{V}(\Phi, \Phi^*)$ to be such that[7]

$$V(\phi) = -\frac{d(d-1)}{\ell_{UV}^2} + \frac{m_{UV}^2}{2}\phi^2 + \frac{\lambda}{8}\phi^4. \tag{13.3.1}$$

We consider $m_{UV}^2 < 0$ and $\lambda > 0$ to have a Mexican hat potential shape. The extrema are located at:

$$\phi = 0, \ \pm\sqrt{\frac{-2m_{UV}^2}{\lambda}} . \tag{13.3.2}$$

Thus, we can observe that this choice of potential is consistent with the UV analysis we made. We take

$$\phi_{IR} = \sqrt{\frac{-2m_{UV}^2}{\lambda}} . \tag{13.3.3}$$

By expanding $V(\phi)$ around ϕ_{IR}, we extract

$$m_{IR}^2 = -2m_{UV}^2 , \tag{13.3.4}$$

$$\ell_{IR}^2 = \frac{2\lambda \ell_{UV}^2 d(d-1)}{2d\lambda(d-1) + \ell_{UV}^2 m_{UV}^2} . \tag{13.3.5}$$

We select the following parameters

$$\ell_{UV} = 1, \ m_{UV}^2 = -2, \ d = 3, \ q = 2, \lambda = 3 . \tag{13.3.6}$$

Then

$$m_{IR}^2 = 4 , \tag{13.3.7}$$

$$\ell_{IR}^2 = 0.9 . \tag{13.3.8}$$

[7] $\mathcal{V}(\Phi, \Phi^*) = -\frac{d(d-1)}{\ell_{UV}^2} + m_{UV}^2 \Phi\Phi^* + \frac{\lambda}{4}(\Phi\Phi^*)^2$.

We can verify that the requirements coming from our analytic analysis are satisfied: $m_{IR}^2 = 4 > 0$, $\Delta_\chi \approx 1.75 > 1$, $m_{UV}^2 = -2 \geqslant -(d^2)/(4\ell_{UV}^2) = -2.25$ and $\Delta_+ = 2 \neq 0$. The latter, being the scaling dimension of the QFT scalar operator $\mathcal{O}$, ensures that the VEV does transform under dilatation.

From the IR analysis, we know that there is only one remaining parameter to fix, namely $\phi_{IR}^{(-)}$, in order to completely settle the IR boundary conditions. In Figure 13.1, a numerical solution for $\phi_{IR}^{(-)} = 0.1$ is presented. As we can see, we indeed describe an RG flow connecting two fixed points since we have an AdS space both in the UV and in the IR. If we do the numerics with $\phi_{IR}^{(-)} = 1$, the scalar field changes directions before reaching to IR as is it can be seen from Figure 13.2. In the literature these types of flows are termed as bouncing flows [324, 325] and it opens a door for further analysis towards the study of exotic RG flows.

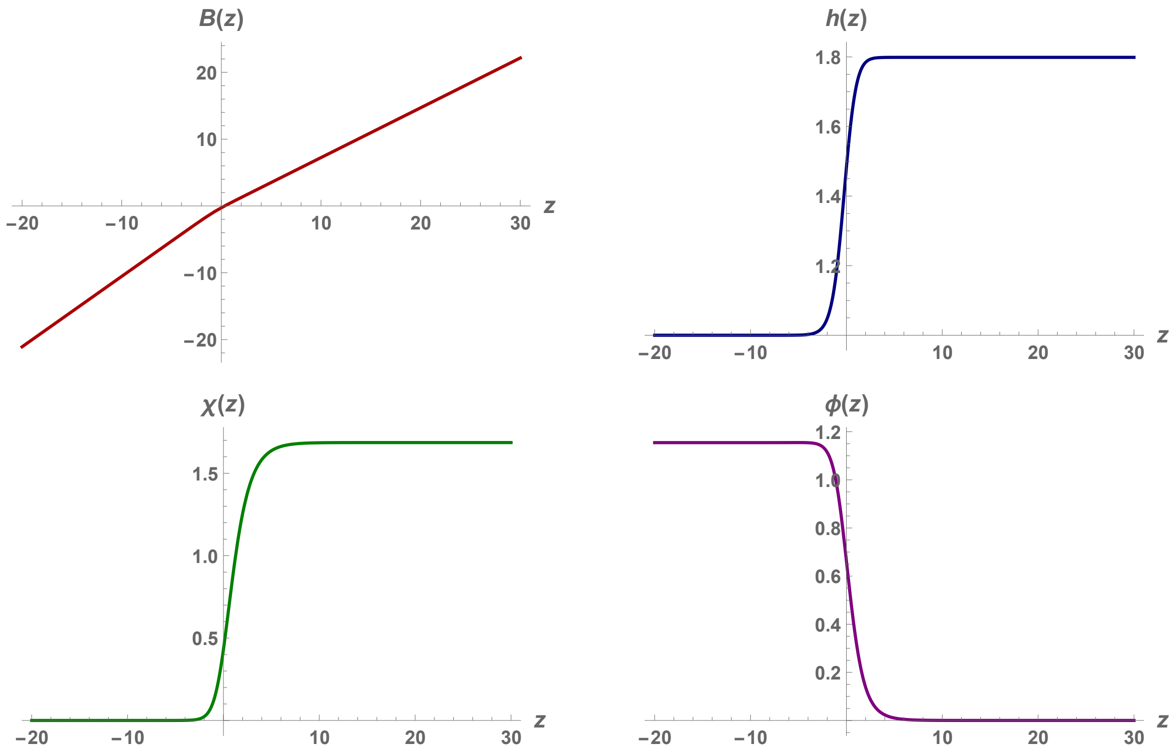

Figure 13.1: This figure displays the numerical solution for the four functions constituting our ansatz (13.2.4). The numerics have be done considering (13.3.6) and taking $\phi_{IR}^{(-)} = 0.1$. The linear behaviour of $B(z)$ in the IR and in the UV as well as the constant conduct of $h(z)$ in these same regions inform that we are indeed describing a geometric kink between two AdS spaces.

In any case, connecting two AdS spaces through a non-trivial geometric kink corresponds to a breaking of dilatation. It remains to find a solution which corresponds to a pure spontaneous breaking of dilatation. To do so, we will screen a range of values of $\phi_{IR}^{(-)}$ and plot ϕ_+ in function of ϕ_-. It will explicitly show if there exists a solution for which $\phi_- = 0$ and $\phi_+ \neq 0$.

The strategy to extract the source is as follows. We know that the leading behaviour

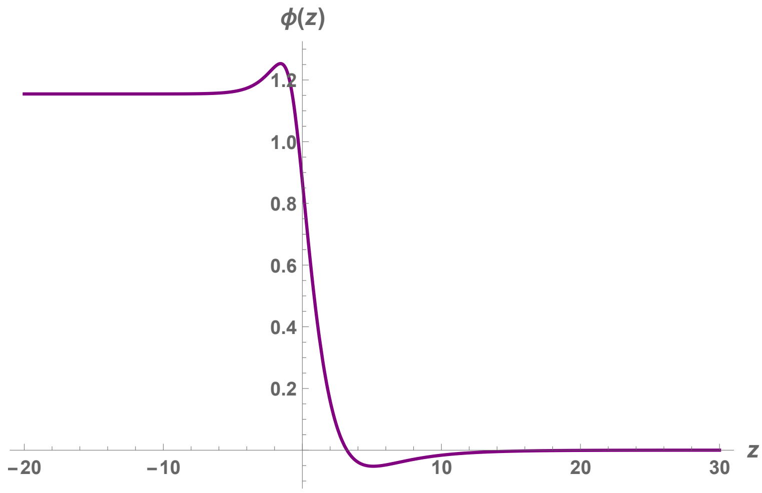

Figure 13.2: Here is presented the numerical solution for the scalar field when $\phi_{IR}^{(-)} = 1$. The other functions $B(z)$, $h(z)$ and $\chi(z)$ keep the same qualitative shape as in Figure 13.1.

of the scalar field is:

$$\phi(z) = \phi_- e^{-\frac{\Delta_- z}{\sqrt{h_{UV}}\ell}} + \phi_+ e^{-\frac{\Delta_+ z}{\sqrt{h_{UV}}\ell}} + \cdots \ . \tag{13.3.9}$$

By taking the logarithm, we find that the leading behaviour is:

$$\log|\phi(z)| = \log|\phi_-| - \frac{\Delta_-}{\sqrt{h_{UV}}\ell} z + \cdots \ , \tag{13.3.10}$$

where the terms in $\cdots$ are exponentially suppressed in the UV. Therefore, from a linear fit of $\log|\phi|$ near the UV, we can extract ϕ_-. To extract ϕ_+, we can construct the following function

$$S(z) = \dot{\phi}(z) + \frac{\Delta_-}{\sqrt{h_{UV}}\ell}\phi(z) \ . \tag{13.3.11}$$

From the leading behaviour of $\phi(r)$, we find that the leading behaviour of S is

$$S(z) = \frac{\Delta_- - \Delta_+}{\sqrt{h_{UV}}\ell}\phi_+ e^{-\frac{\Delta_+ z}{\sqrt{h_{UV}}\ell}} + \cdots \ . \tag{13.3.12}$$

By taking the logarithm of both sides, we can obtain the VEV.

In Figure 13.3, on the left panel, we can observe that it exists a pure VEV flow and from the right panel, that this pure VEV flow occurs at a finite chemical potential. We can thus be convinced of the consistency of our holographic study in the goal to describe a gapped dilaton at finite density.

Of course, this interpretation of spontaneous symmetry breaking should be checked through a computation of the correlators (at least from a pure asymptotic boundary perspective) and through the recovering of the Ward-Takahashi identities associated to the considered breaking patterns. The generic QFT Ward-Takahashi identities for the symmetries at hand will be obtain in a similar fashion than in [21], which is a generalisation

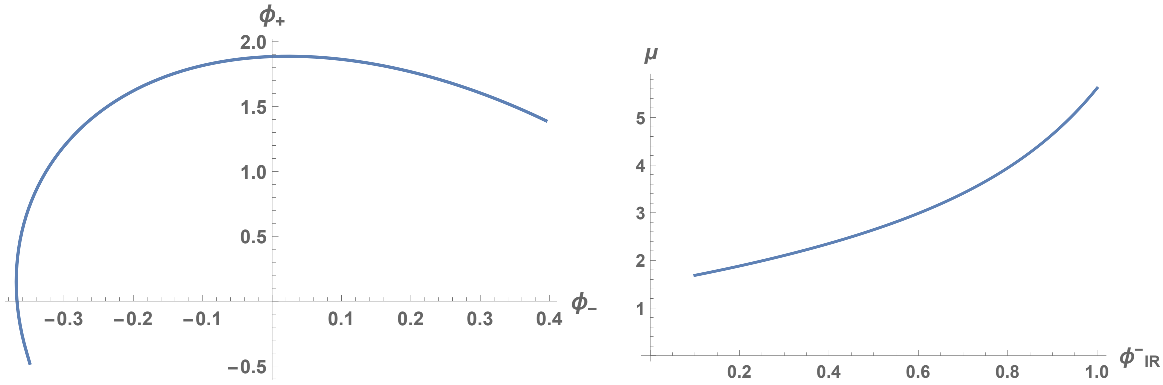

Figure 13.3: These two plots correspond to the numerics of a screening of the IR boundary parameter $\phi_{IR}^{(-)}$ ranging from 0.1 to 1 in increments of 0.01. On the left panel, the VEV ϕ_+ is plotted in function of the source ϕ_-. The curve is anti-clockwise oriented when increasing $\phi_{IR}^{(-)}$. The intersection with the vertical axis happens when $\phi_{IR}^{(-)} = 0.44 \pm 0.01$. On the right panel, the evolution of the chemical potential with ϕ_+ is shown. The chemical potential corresponds to the asymptotic UV value of $\chi(z)$. As we can observe, at the VEV flow $\phi_{IR}^{(-)} = 0.44 \pm 0.01$, the chemical potential has a non-zero finite value ($\mu \approx 2.5$).

of Appendix J Section J.2. Another possible check is to compute some thermodynamical quantity, for example, is the first law of thermodynamic satisfied ? Information on the dispersion relation of the expected gaped dilaton will be obtain from the correlators through either a pure asymptotic boundary computation for the fluctuations or a numerical bulk resolution for the fluctuations. To determine how the gap scales with the chemical potential, we should seek for other VEV flows at different μ. The truncated spiral shape of the plot on the left panel in Figure 13.3 suggests that other VEV flows are present. As a comparison, we reproduced the numerical analysis for the case $\mu = 0$. The monotonic nature of $\chi(z)$ we discussed in Subsection 13.2.1 indicates that it would correspond to entirely switch off the gauge field. We did so, and from a first numerical run, it seems that the VEV goes to zero with the source. We did not find a pure VEV flow. Hence, other holographic models might be needed to be able to describe both cases: the zero chemical potential situation and the finite density case. These other holographic models can be obtained for example by changing the scalar potential of the theory or by enriching the field content of our original model (following the philosophy of the toy model of Chapter 7, it could be the introduction of a neutral scalar field).

Let us emphasise that the main part of the discussion of the current chapter comes from [327]. The original features compared to [327] are the bouncing flow we might have for some values of $\phi_{IR}^{(-)}$, the strategy developed to seek for a VEV flow and the fact that we found a pure VEV flow.

Conclusion of the thesis

This dissertation has focused on some specific aspects of Goldstone physics. The latter is a broad subject due to its main asset: it is a universal approach of the infrared physics which relies on symmetries. Low-energy physics is mainly our daily life surroundings, which makes it observable by definition. Therefore, Goldstone Physics is a formal description of physics but it can almost straightforwardly provide material for phenomenology and for experiments. This explains the large scope of this area of science.

Goldstone physics is twofold. First, it is the study of the infrared spectrum from the perspective of symmetry-originated modes. When spontaneous symmetry breaking occurs, with additional not too restrictive conditions, the spectrum will contain massless particles (NG modes) and light particles (massive NG modes or pseudo NG modes following the considered mechanism). Theorems exist to provide us information on their number and their characteristics. An active research direction is to generalise and to enrich these theorems. We can emphasise that these predictions provide exact results (e.g. the masslessness of NG modes is not an approximation). It constitutes another major asset of Goldstone physics.

Second, Goldstone physics has the aim to build the most general shape of an effective field theory for a given symmetry breaking pattern. To do so, it uses several building methods, such as the coset construction. Consequently, it helps to improve these methods.

Putting everything together, knowing both the spectrum content and the shape of the theory at low energy, entirely fix the dynamics of the infrared physics (up to the theory parameters which should be obtained either from the ultraviolet theory or from experimental measurements). The approach is mainly based on symmetry concepts which are model independent. Hence, the outcomes are generic, which leads to universality.

As we mentioned, the theorems describing the number and the properties of the NG modes and their possible associated partners need to be extended. Indeed, most of the generic knowledge we have so far concerns the breaking of internal symmetries while for the breaking of spacetime symmetries – and non-uniform symmetries in general – we have some tools at disposal but no clearly established theorems. This problematic constitutes one direction of research.

Once a theorem is stated, it is done under some assumptions. A systematic hypothesis of Goldstone physics is that a spontaneous breaking of a global continuous symmetry group has occurred. Therefore, an upstream work is to be able to determine in advance if a given physical system can in fact display a spontaneous symmetry breaking. This is a mathematically involved problem because it requires to understand well the possible non-trivial solutions of non-linear equations of motion coming from interacting systems (cf. the quantum tunnelling effect and the singular limits). However, no-go statements have been made. For relativistic theories, no internal spontaneous symmetry breaking can occur when the dimension of spacetime is equal to two; this is Coleman's theorem. A non-relativistic version of the latter has been conjectured, where the critical dimension depends on the canonical dimension of the vacuum expectation value.

In order to make progress in the two above mentioned lines of research, the literature follows some guidelines. This thesis has been structured on the same paradigm. As a first step, toy models are studied. These are complex enough to display rich physics but simple enough to allow an almost complete analytical study. Intuition on general results is then acquired. The second step is to test this intuition on more intricate quantum field theories. Holography offers an adapted framework to do so because from a classical gravitational theory solved perturbatively, we can analyse the correlators of a strongly coupled large N quantum field theory. Finally, the remaining robust intuitive ideas are wording as conjectures. From generic effective field theory building tools such as the coset construction, we have the opportunity to attempt to prove these conjectures in a general fashion. Of course, some overlaps between the listed steps are seen in the literature; the sequential listing should not be considered as totally rigid.

Original contributions to the open problems

Our first original investigation (see Part II) has been to build field theories displaying respectively the spontaneous breaking of dilatation symmetry and the concomitant spontaneous breaking of dilatation symmetry and spatial translation symmetry. The physical motivation to focus on these two kinds of spacetime symmetries is condensed matter oriented. Indeed, dilatation symmetry refers to the scaling symmetry around critical points and translation breaking refers to crystal structures. It should be mentioned that we thoroughly studied the homogeneous breaking of translation and only had a glance at the non-homogeneous breaking. The latter is the one which indeed leads to lattices, the former is a simpler case which is more convenient for a first approach to translation breaking.

An observation was that minimal models describing spacetime symmetry breaking are already complicated, in particular in the case of the breaking of translation symmetry since higher derivative terms have to be introduced. A consequence was the intricate relation between the parametrisation of the fluctuations, the parametrisation which diagonalises the kinetic matrix – i.e. the one associated to the dispersion relations – and the parametrisation of the NG modes. The identification of the NG modes based on the dispersion relations in Fourier space was dependent on the norm of the momentum but also on its direction. The highlight was to use, amongst other things, the Ward-Takahashi identities to disentangle the issue of identification.

Let us notice that the brief example of non-homogeneous translation symmetry breaking we did showed that the dispersion relations spectrum is more subtle to handle and necessitates extra discussions on the modes labelling.

In this perturbative analysis, the inverse Higgs constraints were systematically providing the right number of massless NG modes (modulo a comment on the non-homogeneous translation symmetry breaking case). Hence, the tools coming from the coset construction formalism proved to be consistent for the considered breaking patterns even if they have been used slightly outside the scope of their original hypotheses.

Concerning the model involving the homogeneous breaking of translation symmetry, its key feature was the emergence of fractonic modes at low energy despite that the fundamental theory was not showing explicit signs of subsystem symmetries. The intuition on this peculiarity is the presence of higher derivative terms in the fundamental theory in order to break spontaneously spatial translation symmetries. The NG modes naturally display shift symmetries; once we go to low energy, the higher derivative terms rearrange themselves such that the shift symmetries are promoted to polynomial shift symmetries (and even to fully general spatially modulated shift symmetries). The apparition of these subsystem symmetries explains the reduced mobility observed in the dispersion relations of the spectrum. It offers a new perspective on how to build toy models for gapless fractonic modes.

Eventually, in order to be able to switch on a chemical potential without destabilising the dilatation invariant theories, we had to lift the energetic flat directions. Therefore, the dilaton was systematically getting a model dependent gap. In the theory displaying the concomitant breaking of translation and dilatation, lifting the flat directions was also explicitly breaking a shift symmetry. Therefore, the gapped dilaton has been interpreted as a pseudo NG modes. However, in the model of the individual breaking of dilatation, the dilaton had no other choice than being a massive NG mode. In fact, despite being model dependent, its gap was also scaling with the chemical potential. This type of massive NG modes is predicted by some of the already known theorems associated to NG modes at finite density. However, the latter, being mainly for internal symmetries, do not capture it in their counting rule. Therefore, this generating mechanism for massive NG modes remains to explore.

Our second original investigation (see Part III) was to confirm the conjectured critical dimension of spacetime under which a non-relativistic system cannot sustain, at quantum level, a continuous spontaneous symmetry breaking leading solely to type A NG modes. With an explicit computation in Lifshitz quantum field theory, without loss of generality, we attested that indeed this critical dimension is $z + 1$, where z is the dynamical critical exponent. Some relativistic large N field theories have shown that order can be restored in the $N \to +\infty$ limit. In fact, thanks to a holographic model, we observed that having a VEV in $d \leqslant z + 1$ does not lead to any pathological behaviour. A memory of the critical dimension was nevertheless encoded in the system since, during the holographic renormalisation, for $d \leqslant z + 1$, we had to perform an alternative quantisation for the temporal component of the vector gauge sector in order to be able to source conserved currents.

The take home message of the original contributions to the open problems in Goldstone physics made by the author and his collaborators and presented in this dissertation is the following:

- The Ward-Takahashi identities constitute an appropriated tool to perform the identification of the NG modes from the dispersion relation spectrum.

- Imposing the Inverse Higgs constraints when considering homogeneous breaking of translation symmetries appears to be physically consistent and it leads to the right counting rule for the massless symmetry-originated modes. This observation stands for spatial translation, as well as for time translation in the context of a non-zero chemical potential. Concerning an inhomogeneous breaking, the discussion is more subtle and requires further investigations.

- It seems there is a non-trivial interplay between non-compact flat directions and the ignition of a non-zero chemical potential. This particular relationship is leading to an unfixed gap for the massive NG mode associated to the spontaneous breaking of dilatation symmetry.

- The effective field theories for NG modes coming from the spontaneous breaking of translations are excellent candidates to display fractonic behaviours.
- From a generic $U(1)$ invariant Lifshitz theory with time-reversal symmetry, it has been confirmed that a non-relativistic system cannot sustain, at quantum level, a continuous spontaneous symmetry breaking leading solely to type A NG modes.

Outlooks

Outlooks can be obtained by generalising the hypotheses on which a given result has been established. For example, how is Goldstone's theorem modified for field theories on curved spacetimes ? Can we generalise this theorem for higher form symmetries ? These constitute important generic conceptual open questions. In the course of this thesis, we have provided several perspectives specific to our investigations. We will now recall the ones which, in a near future, are potentially the most realistic.

One of the main results of the thesis is the possible connection between fractonic modes and the spontaneous breaking of spatial translation symmetry. Continuous field theories for excitations with reduced mobility are peculiar because they notably present an IR/UV mixing. The emergence of fractonic modes in the low-energy limit of a fundamental theory which does not feature subsystem symmetries could potentially cure the IR/UV mixing thanks to a natural energetic cut-off coming from a symmetry breaking. Field theory models for the spontaneous breaking of spatial translation symmetry could be ideal UV candidates. Therefore, a future project could be to consider such systems and try to retrieve Shao-Seiberg's model in the IR. The toy models we considered so far in Part II have their spatial kinetic term with an opposite sign compared to the canonical case. Hence, in parallel to the already mentioned model building, we could seek for higher derivative term models with the usual kinetic term. This in order to facilitate the connection with phenomenology.

Another highlight of the dissertation was the peculiar gap for the dilaton when a chemical potential has been switched on. It is an example of a massive NG mode with a model depend gap. To probe the generality of this observation, a holographic computation can be done. The details have been provided in the preceding chapter. On a larger time scale, the concomitant breaking of dilatation and translation can also be holographically studied through Q-lattice models.

To know whether or not a spontaneous symmetry breaking can occur is one of the cornerstones of Goldstone physics. It could then be interesting to realise the thermal version of the QFT computation from Part III to determine the critical dimension of spacetime when non-relativistic theories are at finite temperature. Because the discussion has been made so far solely for internal symmetries, similar QFT oriented calculations – first at zero temperature – could be performed in the framework of the breaking of spacetime symmetries. Let us notice that the coset construction could provide the free effective field theory, i.e. the starting point of the computations. Nevertheless, we should pay attention that a similar mechanism than for the type B NG modes might happen: massive symmetry originated partners tend to allow the symmetry breaking to occur at any spacetime dimension. A careful analysis of the inverse Higgs constraints and of the canonical conjugation structure of the effective field theory will then be necessary to determine if massive symmetry originated modes are present.

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Abstract

This thesis lies in the framework of the spontaneous symmetry breaking mechanism. When such a mechanism occurs, Goldstone's theorem predicts the existence of massless modes, called Nambu-Goldstone modes (NG modes). The current knowledge on NG modes is classified following the types of symmetries involved in the considered breaking pattern. Spacetime symmetries are the ones for which most of the analysis remains to be done. From a perturbative approach, we separately and concomitantly study the breaking of dilatation symmetry and of spatial translation symmetry. It allows us to comment on the present-day conjectures concerning the counting of NG modes associated to breaking patterns involving spacetime symmetries. Moreover, we get closer to standard laboratory conditions by investigating the situation in presence of a chemical potential. The considered Landau-Ginzburg's like models constitute plausible effective field theories to describe superfluids. The higher derivative terms required to spontaneously break translations lead to emergent subsystem symmetries. A connection between NG modes and fractonic modes, i.e. excitations with reduced mobility, is then made.

Non-relativistic systems are less constrained by the symmetries compared to Lorentz invariant systems which make the former more general. Even for non-spacetime symmetries, some uncertainties on the physics of NG modes remain when dealing with non-relativistic models. One of them is the critical dimension of Minkowski spacetime under which no spontaneous symmetry breaking can occur. This dimension has been conjectured and we propose an explicit computation in order to attest this conjecture. However, through a holographic analysis, we discuss some way out for large N field theories.

All along the dissertation, concrete future research perspectives on the above-mentioned discussions are provided.

Key words: Spontaneous symmetry breaking, Goldstone physics, effective field theories, bottom-up holography.