Keplerian Black Holes and Gravitating Goldstones

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Constructing relativistic systems with hidden symmetry and the double copy of field theories

PhD thesis

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by

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‘En als hij danst
Danst op logaritmes
Walst in parabolen met een
Eindeloos ondeelbaar priemgetal
Door het raam van zijn heelal’
— Spinvis, uit *Hij Danst*

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Chapter 1

Introduction

Since the first unequivocal detection of gravitational waves in 2015 at the Laser Interferometer Gravitational-Wave Observatory (LIGO) in Livingston and Hanford [3], there has been a surge of interest in the accurate calculation of motion and radiation of binary compact objects, from which these waves originate. In popular terms, we have gained, next to our eyes, observing the spectrum of electromagnetic radiation from the heavens, now also ears, listening to the vibrations of spacetime itself, to learn about the universe. Interpreting what we hear as well as possible motivates further exploration of the current best theory of gravity we have: General Relativity (GR).

But there is a less optimistic reason too, that interest in a more thorough understanding of the predictions of GR, our best guess for the workings of the gravitational interaction, has seen an increase. The last decade and a half saw another large experiment, the Large Hadron Collider (LHC), starting runs and confirming some predictions, such as the Higgs boson discovery, but falling short of meeting other expectations (albeit less certain a priori), such as providing evidence for supersymmetry and string theory. That makes the 2010s a time in high energy physics where big hopes and dreams of unification of interactions - the weak, strong and electromagnetic forces as described in the standard model (SM) on one side and GR on the other - have gone unfulfilled and the most clear pathways to that unification have, in a sense, grown more obscure rather than being illuminated.

New opportunities in precision testing and perhaps a slight loss of momentum in finding better fundamental theories of nature motivate the community to understand more thoroughly those theories already there. An exciting example of a different approach of already existing theories is visible in the resurgence of the amplitude programme in recent decades,[1] essentially reordering calculations in quantum field theory (QFT) in a more insightful - and much quicker - way, and in the application of such techniques to classical, gravitational systems. Sure, it is exciting that formerly untrodden ground has been reached by some of these gravitational amplitude calculations in terms of post-Minkowskian contributions [27], but the calculational

[1] Actually, the modern amplitude programme has its origin [186, 209] arguably in supersymmetric theories and string theory, so it would be mistaken to present this as a consequence of susy not being found by LHC - but it is a good example of a different approach to existing theory.

precision needed for the current state of the art experiments can conceivably also be reached by more 'brute force' numerical approaches [111]. Arguably, the more interesting aspect of amplitude methods applied to gravitational problems, is that the new techniques *order* gravity in a different way.

In this thesis, the main question is whether there exist relativistic systems displaying classical dynamics and what physical realisations of those look like. This would yield possibly interesting toy models for other, more realistic relativistic systems, since they have the promise to be simpler than generic systems. And this simplicity would be very welcome: While the solar system through Kepler's ears rings as a harmonic symphony of (nearly) perfectly formed ellipses, the Einstein equation of a two-body system in General Relativity cannot be solved exactly even in the idealised situation where both bodies are point particles and there is no energy loss due to radiation.

An example of this quest for more simplicity in relativistic problems was given by Caron-Huot and Zahraee in 2019 [59], which looked at $N = 8$ supergravitational two-body problems. They show that the conservation of the same symmetry as the classical Kepler problem is related to the vanishing of a particular scattering amplitude in their relativistic theory [59]. By ordering their calculation in a gravitational theory in a particular way, they found an interesting connection between a classical observable and a scattering amplitude, holding more widely than just for the particular theory.

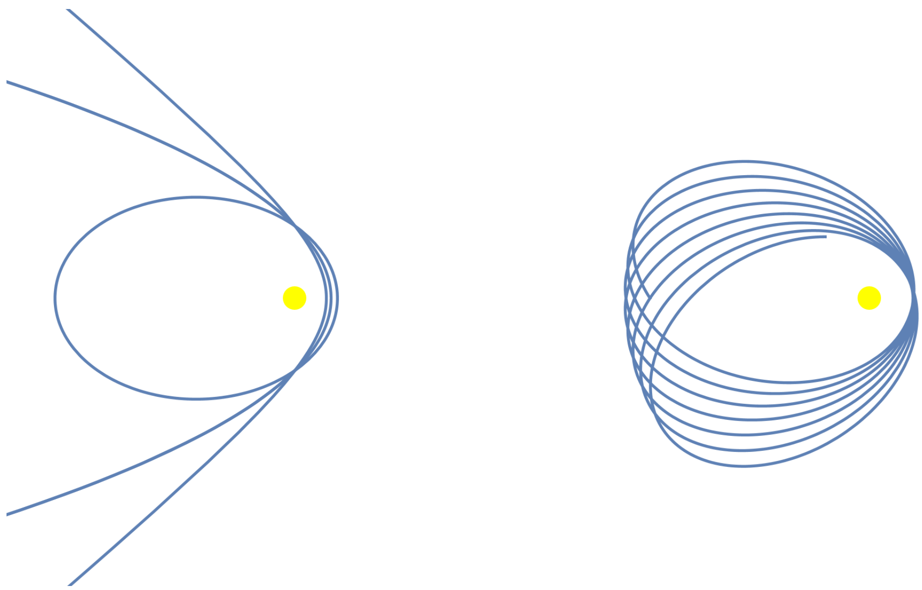
The supergravity theory mentioned above is not a viable candidate for an actual theory of nature, but it is interesting for different reasons. One way of understanding current best-guess theories better, is to describe not just what the way interactions work in nature *is*, but also in what ways nature *could* have worked consistently. The work in this thesis very much taps into this mode of thinking, asking questions to which we know the answer nature gives, but do not know the space of possible answers a consistent theory might give.

Philosophically, this thesis draws on two concepts with the promise of simplifying and ordering our understanding of physical theories. The first concept is that of symmetry, the second that of the double copy. We will introduce both concepts here, without technical details, in the context of their application in the following chapters.

1.1 Symmetries of the Kepler problem

The first original works in this thesis explore ways in which a symmetry known from the non-relativistic Kepler problem can or cannot be present in relativistic systems. The classical Kepler problem, in which a central body is orbited by a second, much smaller body, is well known for its large amount of symmetry and its simplicity. The orbits of the Kepler problem in Figure 1.1, ellipses, parabolas and hyperbolas respectively, are simple curves: conic sections. Focusing on bounded orbits, which are the ones relevant for observations of gravitational waves, we might consider another striking property: the ellipses close in on themselves, creating periodic motion. This periodicity of bounded orbits is a result of the symmetry the problem possesses, which is the maximal amount for any dynamical system in three dimensions.

The most obvious relativistic equivalents, in electromagnetism or GR, destroy part of this



(a) **(b)**

Figure 1.1: Keplerian orbits with the same angular momentum, for eccentricities $e = 0.7, 1, 1.3$ for the ellipse, parabola and hyperbola respectively. On the right, part of an orbit with (exaggerated) precession is given, as encountered in relativistic theories.

symmetry, resulting in differently shaped orbits as compared to those of Kepler shown in Figure 1.1. The question we ask in Chapters 3, 4 and 5 is whether there could exist relativistic theories, alternative to the ones nature seems to adhere to, that preserve the classical symmetry in the relativistic setting. The answer is a resounding yes, and we map out the possible shape of the Hamiltonians describing such systems.

1.2 The double copy: quantum and classical

The double copy relation between gauge and gravity describes how seemingly unrelated theories are in fact strongly linked together. As the topic of the double copy is much less well-known than the Kepler problem, we will delve into it a bit further.

Let us first consider why the double copy is interesting to begin with. The reasons are roughly twofold: First, as alluded to earlier, unification of all fundamental interactions has so far not been able to incorporate gravity. The double copy hints at a deep connection between gauge theories, describing all known particle forces, and gravity, which might open the door to a unified description of both. Second, and of a much more practical nature, the double copy relation allows one to construct gravity scattering amplitudes, which are notoriously difficult to compute, in terms of the simpler gauge theory ones. This can for example be applied fruitfully in the calculation of higher-order interactions in gravitational two-body systems [24].

The double copy is first and foremost a relation between scattering amplitudes of gauge theories and gravitational theories. Since its discovery, however, other versions of the same link have been shown. These differ both in the linked theories and in the kind of objects linked through double copy-like relations. For example, instead of gauge theories containing colour charges, one can relate theories with global symmetry containing flavour charges [75]. Also, double copy relations can be shown between equations of motion, fields and actions as well as scattering amplitudes [174, 72]. Additionally, several manifestations of *classical* double copies have been established [176, 195, 167, 123, 66], mapping classical (colour-)charged solutions through a simple prescription to gravity solutions.

Already in the 80s it was realised that duality relations between specific string amplitudes - those of open and closed strings - had important implications for the low-energy limits of those theories: gauge theory and gravity respectively [150]. At this point, the relation had a rather mystifying character: two vastly different theories in the non-string realm seemed to share a structure not visible in the usual descriptions in terms of actions.

A development that inspired a leap in understanding the connection between gravity and gauge theories was the finding of *colour-kinematics duality* [20]. For tree-level amplitudes, in which there are no loops in the Feynman diagrams describing them, the duality has been proven [26], but also at loop level progress has been made in the last decade [61, 25], see [30] for a review.

Through the lens of colour-kinematics duality, the scattering amplitude of a set of particles is given by three basic pieces of information: the colour factor c_i, which holds all information on the charges of the external particles related to the Lie algebra symmetry of the fields; the kinematic numerator n_i, accounting for all information on the momenta of the incoming and outgoing particles; and the denominator d_i, holding the denominators of the propagators for all internal particles being exchanged. The index i here enumerates the particular diagrams that contribute to the amplitude A_n with n external particles. Summing over these implicitly, we can write each n-point tree-level amplitude schematically as

$$A_n = \frac{n_i c_i}{d_i} \,. \tag{1.1}$$

But what are those particular diagrams we sum over? It turns out that it is sufficient to only consider *trivalent diagrams*, that is, diagrams with only 3-vertices. For four point, this is easy to see: take the numerator n_c of the contact diagram. Then $d_c = 1$, since there are no propagators in a contact diagram, and we can simply take $n'_c = n_c s$ and $d'_c = s$, with s the squared momentum of the s-channel exchanged particle, without changing the amplitude. Effectively, this rewrites the contact diagram as an exchange diagram. At higher point, similar generalised gauge transformations exist to remove all non-trivalent diagrams.

Since the colour factors are made up of structures obeying Lie algebras, there must be linear relations among the colour factors of different diagrams inherited from the Lie algebra, such as the Jacobi identity

$$c_i + c_j + c_k = 0 \,, \tag{1.2}$$

where the different diagrams i, j and k are related to each other by interchanging particles in a subdiagram in a particular way. For now, it is not important what the exact representations of these factors are, only that there are Jacobi-like identities that colour factors satisfy.

Colour-kinematics duality then is obeyed if these relations are similarly satisfied by the kinematic numerators, i.e. for the above relation

$$n_i + n_j + n_k = 0 \,. \tag{1.3}$$

Such relations are not generically satisfied (at least, at higher-than-four-point), yet, because of the generalised gauge transformations as the one above, at tree level it is always possible to change the numerators into a form in which the Jacobi-like relations are satisfied.

Once we have kinematic numerators and colour factors satisfying the same equations, or manifesting colour-kinematics duality, we can start interchanging the factors at will, without changing the fact that the end product remains a valid scattering amplitude. One option would be to take a Yang-Mills amplitude $A_{n,\mathrm{YM}} = \frac{n_i c_i}{d_i}$, and replace each colour factor c_i with its kinematic counterpart n_i. We would end up with a double copy of the kinematic part of the Yang-Mills amplitude:

$$A_{n,\mathrm{GR}} = \frac{n_i^2}{d_i} \,, \tag{1.4}$$

which turns out to be the graviton scattering amplitude.[2] This procedure is sometimes described as squaring gauge theory amplitudes to find gravity amplitudes, or called simply the double copy.

However, the double copy procedure can be applied to other theories than Yang-Mills, which describes a vector interaction, as well. For example, it can be applied to the Non-Linear Sigma Model (NLSM), which describes scalar fields having different, polarisation-free kinematical factors and flavour instead of colour. Moreover, the factors of all theories can be mixed and matched to yield consistent amplitudes. This creates a whole web of double copy-related theories, that has been extensively researched over the past decade or so [71, 30].

In chapter 6, we discuss a part of this web relating certain special scalar theories non-linearly realising symmetries. These non-linear symmetries imply the scalars are so-called Goldstones: scalar particles necessarily arising in the process of spontaneous symmetry breaking, i.e. the system settling into a ground state that is *non*-symmetric in a theory that *is* symmetric under the particular transformation.

Non-linear symmetries in theories lead to particular momentum structures in the resulting amplitudes, making sure they go to zero with a specific power of the momentum if we send any external momentum to zero: the so-called soft limit. The special scalar theories at hand have a larger soft limit than one might naively expect based on the number of derivatives per field (each of which generically contributes a momentum factor). In fact, simply specifying the soft limit and the number of derivatives per field for a scalar theory allows one to completely determine the form of interactions and hence the amplitudes, allowing only one free parameter [78]. This fact makes them an interesting testing ground for double copy investigations: they are defined by a single coupling constant, making them in a sense the scalar field theoretic analogue of Yang-Mills and General Relativity, which also have this property [71].

[2] For pure Yang-Mills, the double copy in principle also includes an anti-symmetric tensor and a scalar field, but these can often be projected out by making suitable assumptions on the contributing colour factors.

The mixing and matching of factors of the special scalar theories can lead to intriguing results. As we will see in Chapter 6, when taking two copies of a flavour factor, one finds, next to the usual NLSM amplitudes, also amplitudes stemming from gravitational interaction of the Goldstones.

Next to scattering amplitudes (on-shell), we also discuss the off-shell manifestation of the double copy for the special scalar theories. This makes the link between them on the level of the fields and non-linear symmetries, through a prescription systematically replacing flavour information with kinetic information, just like the scattering amplitude double copy. Finally, and in a very different context, the incarnation of the double copy in terms of classical solutions is applied in Chapter 5. Some solutions of the Einstein field equations of a specific linear form are also solutions of the Yang-Mills field equations [176]. These include (from the gravity perspective) the Schwarzschild and Kerr black holes, as well as some linear gravitational waves. The classical double copy, by exchanging constants in a suitable way, links these to charged solutions of electromagnetism (since non-linearities do not play a role, possible self-interactions of the bosons do not matter). As we will discuss, not just the backgrounds can be copied, dynamical systems of test particles on top of them can be, too.

1.3 Overview and notation

Another aspect of this thesis is that it aims to build a bridge between physics, in the form of field theory, relativity and gravity, and mathematics, in particular Hamiltonian mechanics and dynamical systems. This dual starting point leads to the chapters 2 and 3, which on one hand mean to explore both the basics of the relevant mathematics and physics, respectively, and on the other hand function to introduce particular material useful to our purposes in the subsequent chapters. Chapter 2 discusses symmetries and integrability in the context of classical mechanics, focusing specifically on the Kepler problem, while Chapter 3 treats relativistic generalisations of the two body problem, highlighting the one-center system and discussing the breaking or persistence of Keplerian symmetry in various cases.

The three chapters following all discuss our research contributions. In Chapter 4, we present a class of implicitly defined Hamiltonians, which we show to have an orbital equivalence to a Kepler Hamiltonian on each energy level. Furthermore, we show this form is realised by a two-body system in Einstein-Maxwell-dilaton theory with a specific scalar coupling constant, which both in the 1-center limit and in the 1PN limit possesses Keplerian symmetry. Subsequently, in Chapter 5 we show this theory is not alone, but it can be related through the classical double copy to a single - and zeroth copy system similarly preserving the symmetry in the 1-center system. We present a general class of Hamiltonians that we claim contains all relativistic Hamiltonians with Keplerian symmetries. The symmetry can be naturally interpreted from a 5-dimensional point of view, since there it becomes a property of the background spacetime, and classical and relativistic Hamiltonians are just a time-reparametrisation away from each other. Finally, Chapter 6 discusses another realisation of the double copy, both on- and off-shell, connecting scalar theories with non-linearly realised symmetries through flavour-kinematics duality.

Notational conventions. In what follows, unless differently specified, we will use the following notational conventions.

We will use (q, p) to denote canonical coordinates in $T^\star\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$. The radial momentum is denoted p_r, i.e. $p_r = \frac{(p \cdot q)}{r}$ where $r = r(q) = |q|$. We will use upper indices to denote vector components, therefore, V^i denotes the ith component of a vector V. Indices for relativistic, 4-dimensional objects are denoted by Greek letters, for 5-dimensional objects by capitalised Roman letters. Throughout, we will assume Einstein notation and omit explicit sums. For instance, 4-vectors are denoted by x^μ, the Lorentzian metric is denoted $g_{\mu\nu}$ and, therefore, the inner product of tangent vectors with respect to g is given by $g_{\mu\nu}\dot{x}^\mu\dot{y}^\nu$. For the metric we assume the signature $(- + ++)$.

For convenience, throughout the thesis we use units such that the speed of light, c, and the gravitational constant, G, are equal to 1.

Chapter 2

Symmetries, Hamiltonian mechanics and the Kepler problem

As we are interested in the symmetries underlying motion of celestial (and possibly other) bodies, it makes sense to describe the systems we study from the point of view most naturally catering to symmetric needs. This point of view is provided by the Hamiltonian formalism. In the Hamiltonian formalism, as we will see shortly, one of the central results on symmetries in physical systems, Noether's theorem, becomes almost tautological, as noted by [13]. Let us briefly consider how to see that.

Noether's theorem, in loose terms, states that for every continuous symmetry transformation, generated by e.g. Q, leaving the Hamiltonian H invariant, there exists a conserved quantity, and vice versa. An infinitesimal version of such a transformation of the Hamiltonian, with some small nonzero parameter ϵ is given by

$$\delta H = \epsilon \{H, Q\}\,, \tag{2.1}$$

which must vanish for the transformation to be a symmetry. The time derivative of the quantity Q meanwhile is given by

$$\dot{Q} = \{Q, H\}\,. \tag{2.2}$$

The definition of the curly (Poisson) brackets here is quite irrelevant to the point, only the fact that they are anti-symmetric, such that $\{H, Q\} = 0$ immediately implies $\{Q, H\} = 0$.

Moreover, symmetries and conserved quantities apparently hidden from the point of view of Lagrangian or Newtonian mechanics, become interpretable as properties of the phase spaces on which Hamiltonians are defined. The harmonic oscillator and the Kepler problem are the quintessential examples of this, though there are many more [54].

In this chapter, we will discuss some basics of the Hamiltonian formalism, symmetries and their consequences. We will not prove many of the treated statements, as this is done by other authors, better equipped to such goals. Here, we only provide an overview and explanation, for the main purpose of this thesis is to see how mathematical tools can be applied to physical, relativistic systems.

2.1 Hamiltonian formalism

In this section, we draw mostly from the succinct introduction into Hamiltonian systems in the first part of the review on hidden symmetries by Cariglia [54] and the discussion of Hamiltonian mechanics in [206], taking what we need and neglecting what is deemed unnecessary for our purposes.

In Hamiltonian mechanics, one studies a $2n$-dimensional manifold $\mathcal{P}$ with a symplectic 2-form $\omega = \omega_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, with Greek indices for now $\mu, \nu = 1, \dots, 2n$ and a smooth function $H : \mathcal{P} \to \mathbb{R}$ that is called the *Hamiltonian*. The coordinates x^μ here are local coordinates on the manifold $\mathcal{P}$.

Definition 1. *A* ***symplectic 2-form*** *is an anti-symmetric bilinear map on the tangent spaces to $\mathcal{P}$, satisfying the properties that it is closed, $d\omega = 0$, and non-degenerate, meaning* $\det(\omega_{\mu\nu}) \neq 0$*.*

We will study systems where $\mathcal{P} = T^*\mathcal{M}$, i.e. the phase space is the cotangent bundle of a configuration manifold $\mathcal{M}$ which has dimension n. The coordinates $x^\mu = (q^i, p_j)$, $i, j = 1, \dots n$ are generalised coordinates and momenta respectively (written without indices whenever no doubt can arise as to their meaning), and $q \in \mathcal{M}$ parametrises the configuration space. The 2-form can be written $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$. This is not a restriction, since locally there always exist coordinates, called Darboux coordinates, for which the above holds.

The Hamiltonian function induces dynamics on the phase space $\mathcal{P}$ according to

$$\dot{x}^\mu = \omega^{\mu\nu}\partial_\nu H \,, \tag{2.3}$$

where the dot denotes derivation with respect to the evolution parameter λ and $\omega^{\mu\nu}$ the inverse of the 2-form matrix, which exists because of non-degeneracy. In terms of local phase space coordinates this becomes the familiar equations of Hamilton:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \,. \tag{2.4}$$

In fact, we can define such a flow for any function f on phase space, creating a vector field associated to it having components

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} = \omega^{\mu\nu}\partial_\nu f =: X_f^\mu \,, \tag{2.5}$$

with σ being the particular evolution parameter of f. The complete vector field then is given by contracting the components above with basis $\frac{\partial}{\partial x^\mu}$, meaning it works as a differential operator on some other function g:

$$X_f(g) = \omega^{\mu\nu}\partial_\mu g \partial_\nu f \,, \tag{2.6}$$

which is anti-symmetric in the exchange of g and f, meaning $X_f(g) = -X_g(f)$.

Clearly, the matrix $\omega^{\mu\nu}$ plays an important role in comparing flows of different functions to each other. Let us therefore see what it amounts to in case we use Darboux coordinates, for which $\omega = \omega_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}p_i \wedge \mathrm{d}q^i$. Calculation of the inverse gives

$$\omega^{\mu\nu} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} , \tag{2.7}$$

where 0_n and I_n denote the null and identity square matrices of dimension n. Applying this to the vector field of f acting on g, we find a frequently reappearing pattern of partial derivatives, which we can summarise in a bracket:

$$X_f(g) = -\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} + \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} =: \{g, f\} . \tag{2.8}$$

In the last equality we define the Poisson bracket, which gives a Lie algebra structure to the smooth functions on phase space, as the brackets are anti-symmetric and satisfy Jacobi's identity. It is simple to show there is an anti-homomorphism of the Poisson algebra of functions on phase space to the Lie algebra of vector fields associated to these functions: For smooth functions f, g, h,

$$\begin{aligned} [X_f, X_g] (h) &= (X_f X_g - X_g X_f) (h) \\ &= \{\{h, g\}, f\} - \{\{h, f\}, g\} \\ &= -\{\{g, f\}, h\} \\ &= -X_{\{f,g\}}(h) , \end{aligned} \tag{2.9}$$

where Jacobi and anti-symmetry were used going to the third line and anti-symmetry going to the fourth.[1] This result allows us to talk almost without distinction about the Lie algebra of vector fields on a phase space and the Poisson algebra of smooth functions on the same phase space.

The equations of motion for the dynamical systems we will encounter are given by Hamilton's equations (2.4). These must of course coincide with the actual motion in a physical system we might want to describe. Let us therefore briefly examine whether that is indeed the case, on the way establishing the familiar link between Lagrangian and Hamiltonian. For future reference, let us define the former as follows.

Definition 2 ([201])**.** *Given a configuration space $\mathcal{M}$ the* ***Lagrangian*** *is a smooth function $L \equiv L(q, \dot{q}, \lambda) : T\mathcal{M} \times \mathbb{R} \to \mathbb{R}$.*

The Lagrangian, in this quite standard formulation, only depends on the position and its first derivative, such that the equations of motion are second order, like Newton's second law.[2] For mechanical systems, we often have a *natural Lagrangian* $L(q, \dot{q}, \lambda) = K(\dot{q}) - V(q)$, with K the kinetic and V the potential energy. Recall that the motion in classical (non-quantum) physical systems is given by those solutions $q(\lambda)$ extremising the action.

[1] Note that there is a typo in [54] concerning this.

[2] In the context of classical mechanics, in principle nothing stops us from building higher-order equations of motion, but in quantum physics this is distinctly undesirable, as higher-than-second-order equations of motion lead to Ostrogradsky's instability and the explosive production of particles, which is not in line with what we observe in the universe [222].

Definition 3. *The **action** S of a path $q(\lambda)$ is a functional*

$$S[q] := \int_{\lambda_1}^{\lambda_2} L(q(\lambda), \dot{q}(\lambda), \lambda)\ \mathrm{d}\lambda\,, \tag{2.10}$$

between fixed points $q(\lambda_1)$ and $q(\lambda_2)$.

This leads to the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}^i}\right) = \frac{\partial L(q,\dot{q})}{\partial q^i}\,. \tag{2.11}$$

Note that we will only consider Lagrangians not explicitly depending on time since this is sufficient for our purposes.

The relation between Lagrangian and Hamiltonian is given by the Legendre transform $LT : (q,\dot{q}) \to (q,p)$, mapping the tangent bundle $T\mathcal{M}$ to the cotangent bundle $T^*\mathcal{M}$. It is effectively casting the information stored in the Lagrangian into the Hamiltonian

$$H(q,p) = p\,\dot{q}(q,p) - L(q,\dot{q}(q,p))\,, \qquad p_i = \frac{\partial L(q,\dot{q})}{\partial \dot{q}^i}\,. \tag{2.12}$$

The Legendre transform hinges on the existence of the inversion of $p(q,\dot{q}) = \frac{\partial L(q,\dot{q})}{\partial \dot{q}^i}$ to find $\dot{q}(q,p)$. This means the Jacobian determinant of the transformation must be nonzero:

$$D(LT) = \begin{pmatrix} \frac{\partial q^i}{\partial q^j} & \frac{\partial q^i}{\partial \dot{q}^j} \\ \frac{\partial p_i}{\partial q^j} & \frac{\partial p_i}{\partial \dot{q}^j} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0_n \\ \frac{\partial^2 L(q,\dot{q})}{\partial \dot{q}^i \partial q^j} & \frac{\partial^2 L(q,\dot{q})}{\partial \dot{q}^i \partial \dot{q}^j} \end{pmatrix} \neq 0\,, \tag{2.13}$$

which reduces to the requirement that the matrix $\frac{\partial^2 L(q,\dot{q})}{\partial \dot{q}^i \partial \dot{q}^j}$ is non-degenerate, that is

$$\det\left(\frac{\partial^2 L(q,\dot{q})}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0\,. \tag{2.14}$$

A Lagrangian satisfying this is called a non-degenerate Lagrangian. In particular, for a natural Lagrangian $L_n(q,\dot{q}) = \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j - V(q)$ this implies the metric g must be non-degenerate.

Assuming the Lagrangian associated to a Hamiltonian is non-degenerate, we can use the form of the Hamiltonian (2.12) to rewrite the action as

$$S = \int_{\lambda_1}^{\lambda_2} \left(p\,\dot{q}(q,p) - H(q,p)\right)\ \mathrm{d}\lambda\,, \tag{2.15}$$

which then, when extremised, should yield the equations of motion. Importantly, the variation of the action should be done independently in both q and p, as these are in principle independent coordinates of phase space, though we have attached a physically meaningful connection to pairs (q^i, p_i) in our coordinate description. This is contrary to the situation in the Lagrangian formalism, where only q is varied as $\dot{q}$ variation follows from it [206]. This equal footing for coordinates and momenta in the Hamiltonian formalism is one of the reasons it is so convenient for describing symmetries, as we will see.

The variation of the action yields

$$\begin{aligned} \delta S &= \int_{\lambda_1}^{\lambda_2} \left(\delta p \ \dot{q} + p \ \delta \dot{q} - \frac{\partial H}{\partial q} \delta q - \frac{\partial H}{\partial p} \delta p \right) \ \mathrm{d}\lambda \\ &= \int_{\lambda_1}^{\lambda_2} \left(\left[\dot{q} - \frac{\partial H}{\partial p} \right] \delta p + \left[-\dot{p} - \frac{\partial H}{\partial q} \right] \delta q \right) \ \mathrm{d}\lambda + p \ \delta q|_{\lambda_1}^{\lambda_2} , \end{aligned} \tag{2.16}$$

where we use the assumption that $q(\lambda_1)$ and $q(\lambda_2)$ are fixed to drop the last term. For extremising paths, the relations in the square brackets must vanish, as δp and δq are nonzero. This implies the variation of the same action leading to the Euler-Lagrange equations for a physical system lead us to Hamilton's equations, showing they are equivalent.

2.2 Symmetries

Noether's theorem is one of the most important results in mathematical physics, tying an inextricable knot between global continuous symmetries on one hand and conserved quantities on the other. In the context of Hamiltonian mechanics, where we study the phase space of physical systems, the consequence of a symmetry is to effectively lower the dimension of phase space, implying calculation of trajectories becomes easier. Informally, if there is enough symmetry in a system to constrain the motion to a surface of half the dimension of the full phase space, we call the system integrable. In such cases the phase space acquires a very rigid structure, which can be seen as 'stacks' of these lower-dimensional surfaces. Examples of systems in which this situation occurs are well known: the isotropic harmonic oscillator, the rotation of a rigid body or planetary motion around one or two fixed centers. Some systems have symmetries that are not immediately apparent from their Hamiltonians or the configuration space. These are sometimes called 'hidden symmetries'.

In this section, we will explore all concepts above in more depth and greater detail. First, we discuss symmetries of Hamiltonian systems, then we will see how integrable systems are defined and that the Liouville-Arnold theorem shows there exist coordinates in which the motion is linear. Lastly, we discuss hidden symmetry and how to see which symmetry is hidden and which is not.

2.2.1 Noether's theorem

Symmetries are those transformations of the manifold that leave the equations of motion and hence the stationary paths invariant, implying they change the Lagrangian at most with a total derivative, resulting in a constant of integration in the action. Though there exist non-continuous, i.e. discrete symmetries, such as parity - or time reversal symmetry, we will only concern ourselves with continuous symmetries. In the Hamiltonian formalism, a continuous symmetry transformation is the flow of a vector field X_f related to a smooth function $f : \mathcal{P} \to \mathbb{R}$ on the phase space that leaves the Hamiltonian H invariant. In other words,

$$X_f(H) = \{H, f\} = 0 \, . \tag{2.17}$$

At the same time, we know the evolution of any phase space function (2.3) as generated by the Hamiltonian H will give us

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_{\text{flow } H} = \{f, H\}\,, \tag{2.18}$$

with λ the time parameter conjugate to the Hamiltonian. Assuming the infinitesimal symmetry X_f, the constancy in time of f is immediate and vice versa. This means that we have just shown the following theorem, known as Noether's theorem:

Theorem 1 (Infinitesimal Noether [206])**.** *Let $(\mathcal{P}, \omega, H)$ be a Hamiltonian system and X_f a vector field related to a smooth function $f : \mathcal{P} \to \mathbb{R}$. Then f is constant along the flow of H such that*

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_{\text{flow H}} = 0\,, \tag{2.19}$$

if and only if (2.17) *holds.*

To go beyond this infinitesimal statement to a global one, the exponential map $\{\Phi_f^t = \exp\{tX_f\}|\ t \in \mathbb{R}\}$ is used, establishing a one-parameter group of diffeomorphisms of the phase space which is a *symmetry* [88].

Conserved quantities are also called *first integrals* or *constants of motion*, though some authors differentiate the definitions based on the in- or exclusion of time dependency. We will always make it explicit if constants are time-dependent. As said, first integrals, and hence symmetries, effectively reduce the number of degrees of freedom of the phase space. This results from simply fixing one dimension of it, equating a function to a constant $f = c$. The $2n$-dimensional whole of phase space $\mathcal{P}$ then is constrained to the $(2n-1)$-dimensional level set $f^{-1}(c)$. This makes them an extremely useful tool in simplifying a dynamical system. Note however, that the level set can no longer be considered a phase space of a Hamiltonian system, as it is odd-dimensional. This will be addressed in Section 2.2.3 on symplectic reduction.

2.2.2 Integrable systems

What is striking about the above treatment of symmetries is, well, its symmetry. If one forgets for a moment that the flow parameter of the Hamiltonian is the physical evolution parameter, the statement that function f is a symmetry of the Hamiltonian H could just as well have been read as the statement that the Hamiltonian is a symmetry of the function f. From the point of view of mathematics then, it makes more sense to talk about the two functions f and H being in involution with each other, meaning that

$$\{f, H\} = -\{H, f\} = 0\,, \tag{2.20}$$

that is, they Poisson commute.

In a system with multiple symmetries, say m of them, one calls the collection of the conserved functions $F = (f_1, \ldots, f_m) : \mathcal{P} \to \mathbb{R}^m$ the *momentum* map. This is said to have *critical values* if the conserved quantities are not functionally independent, while all other cases are considered *regular values*. Independence here means $\mathrm{d}f_i$ are linearly independent.

A particular case of systems with symmetry is the class of integrable systems.

Definition 4 ([88])**.** *A Hamiltonian system* $(\mathcal{P}, \omega, H)$ *is an* ***integrable system*** *if it has* $n = \frac{1}{2}\dim\mathcal{P}$ *first integrals* $F = (f_1 = H, \ldots, f_n)$*, which are independent and satisfy*

$$\{f_i, f_j\} = 0\,, \tag{2.21}$$

for all i, j*.*

Locally, i.e. in a small enough neighbourhood of a point on the manifold, all Hamiltonian systems have this property [18]. Globally, however, integrability is far from trivial and its consequences are crucial.

The word integrability suggests that one can integrate the system, or solve its equations of motion. This is indeed the case, since one can systematically write down the integral expressions for the variables solving the equations, though in general there is no guarantee these integrals are computable themselves. However, it has been shown there is an additional structure to phase spaces of these systems, a much stronger statement than 'solvability' of a system alone.

Before getting to this structure, let us consider the notion of a Lagrangian submanifold.

Definition 5 ([201])**.** *A submanifold* $\mathcal{L} \subset \mathcal{P}$ *of the phase space with* $\dim\mathcal{L} = \frac{1}{2}\dim\mathcal{P}$ *is a* ***Lagrangian submanifold*** *if and only if the symplectic form vanishes on the tangent subbundle* $T\mathcal{L}$*.*[3]

A Lagrangian submanifold has the crucial property that all trajectories intersecting it are constrained to it [201].

A useful example of such a submanifold is the one defined by n independent constant functions $f_i = c_i$ in involution on it. The vector fields of these functions are n in number and independent, and as such a basis of the tangent space. Moreover, they all commute among themselves, implying $\omega = 0$ on this submanifold as

$$0 = \{f_j, f_i\} = X_{f_i}(f_j) = \omega^{\mu\nu}\partial_\mu f_j \partial_\nu f_i\,. \tag{2.22}$$

The phase spaces of integrable systems are then built up out of Lagrangian submanifolds displaying the above properties, as they have commuting, independent conserved functions almost everywhere in phase space.

This brings us to the structure of integrable systems. We will only quote the case for compact systems, though a similar statement can be made about non-compact systems.[4]

Theorem 2 (Arnol'd-Liouville [88])**.** *Let* $(\mathcal{P}, \omega, H)$ *be an integrable system of dimension* $2n$*, with momentum map* $F = (f_1 = H, \ldots, f_n)$*. Let* $c \in \mathbb{R}^n$ *be a regular value of* F*. The corresponding level* $F^{-1}(c)$ *is a Lagrangian submanifold of* $\mathcal{P}$*. Moreover, if the flows of the vector fields* $(X_{f_1}, \ldots, X_{f_n})$ *are complete and* $F^{-1}(c)$ *is compact and connected, then*

[3] Slightly more formally, the pullback by the inclusion map of the symplectic structure ω vanishes. The inclusion map is the map that sends an element in the first space to the same element in the second space, in our case $\iota : \mathcal{L} \hookrightarrow \mathcal{P}$. A pullback pulls a function on the target space back to the domain of the function one pulls back by. In our case, the pullback *by* the inclusion map *of* the symplectic structure is $\iota^*\omega : T\mathcal{L} \to \mathbb{R}$ and acts as the restriction of the form defined on the whole phase space to the submanifold.

[4] The role of tori will be played by cylinders in that case.

- *$F^{-1}(c)$ is diffeomorphic to $\mathbb{T}^n$ and*
- *there exist local Darboux coordinates $(\phi^1, \ldots, \phi^n, \psi_1, \ldots, \psi_n)$ such that ψ_i are integrals of motion called action coordinates and ϕ^i evolve linearly under the flow of the vector fields $(X_{f_1}, \ldots, X_{f_n})$ and are called angle coordinates.*

This means that the phase space is foliated by invariant tori, labelled by regular values of the momentum map $c \in \mathbb{R}^n$. On these tori, there are coordinates such that Hamilton's equations are solved by

$$\phi_i(\lambda) = \phi_i(0) + \Omega_i(c)\lambda\,, \qquad \psi_i(\lambda) = \psi_i(0)\,, \tag{2.23}$$

where Ω_i are constant frequencies. Taking an integrable system with a 4-dimensional phase space, a regular value of the momentum map would look like figure 2.1, with the two angles ϕ_1 and ϕ_2 denoted by the red and blue arrows. Generically, the path through phase space will then fill the torus, unless the frequencies Ω_1 and Ω_2 are rational, in which case the path will be periodic.

It is possible for a system of $2n$ dimensions to have more than n independent integrals of motion. Maximally n can be in involution, but every additional independent integral will further reduce the number of degrees of freedom in phase space and thus constrain the motion to a lower dimension. Considering this in terms of action-angle coordinates, this situation amounts to a combination of angle coordinates being conserved. Such a system is called superintegrable. The maximal number of independent integrals of motion is $2n - 1$, which constrains the motion in phase space to one dimension. Since the considered manifolds are compact, this implies *strictly periodic motion*.[5] Examples of maximally superintegrable systems are the isotropic harmonic oscillator in multiple dimensions, the Kepler problem (2D, 3D) and trivially every energy-conserving one-dimensional system.

Example 1 (The Kepler problem part 1)**.** *The Kepler problem, for which the Hamiltonian $H_{\text{Kep}} : T^*\mathbb{R}^3 \to \mathbb{R}$ is given by*

$$H_{\text{Kep}} = \frac{p^2}{2\mu} - \frac{k}{r}\,, \tag{2.24}$$

with $\mu, k \in \mathbb{R}^2$, $r := |q|$, clearly conserves energy $H_{\text{Kep}} = E$, as the Hamiltonian is independent from the time parameter itself. Hamilton's equations of motion then read

$$\begin{aligned} \dot{q} &= \frac{\partial H_{\text{Kep}}}{\partial p} = \frac{p}{\mu}\,, \\ \dot{p} &= -\frac{\partial H_{\text{Kep}}}{\partial q} = -k\frac{q}{r^3}\,. \end{aligned} \tag{2.25}$$

Angular momentum is also conserved, as the Hamiltonian depends only on rotation-invariant scalars $u_1 = q^2$ and $u_2 = p^2$. Explicitly, we have by Hamilton's equations

$$\dot{L} = \dot{q} \times p + q \times \dot{p} = 2\frac{\partial H_{\text{Kep}}}{\partial u_2} p \times p - 2q \times q\frac{\partial H_{\text{Kep}}}{\partial u_1} = 0\,, \tag{2.26}$$

[5] As opposed to *conditionally periodic motion*, which is the situation where multiple angle variables are independent, but the ratios of their frequencies Ω_i are rational.

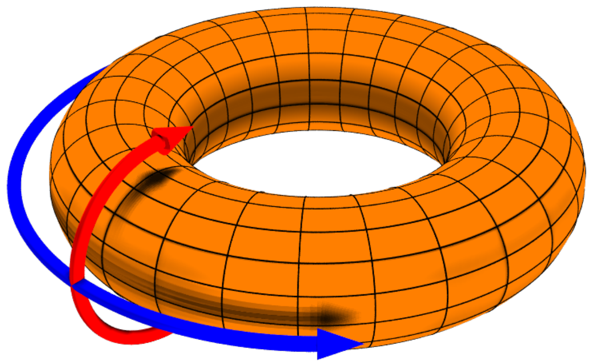


Figure 2.1: An integrable system of 4 dimensions, having two angle coordinates, of which the direction here is given by the two differently coloured arrows [107].

with the dot denoting derivation with respect to time λ*. Of course, the components* L^i *of the angular momentum vector are not in involution, since*

$$\{L^i, L^j\} = \epsilon^{ijk} L^k \,, \tag{2.27}$$

with ϵ^{ijk} *the Levi-Civita tensor, but the set* $(H, L^2 = L \cdot L, L_z)$ *is, as can be readily checked by computing the Poisson brackets among them. This means the Kepler problem, and in fact any central potential problem, is integrable and even superintegrable, since the three angular momentum components and energy all contribute independent first integrals.*

For a usual central potential problem, the energy and angular momentum form the full set of independent integrals of motion. The Kepler problem however is special, as it conserves another vector, the Laplace-Runge-Lenz (LRL) vector

$$A = p \times L - \mu k \frac{q}{r} \,. \tag{2.28}$$

As shown in Figure 2.2, the vector points along the ellipse for bounded orbits.

The LRL vector is constant because of the particular form of the potential, so using Hamilton's *equations (2.25) we have*

$$\begin{aligned}\dot{A} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[p \times L - \mu k \frac{q}{r}\right] ,\\ &= \dot{p} \times L + \frac{\mu k}{r^3}(\dot{q} \cdot q)q - \frac{\mu k}{r}\dot{q} ,\\ &= -\frac{k}{r^3} q \times L + \frac{k}{r^3}(p \cdot q)q - \frac{k}{r}p ,\\ &= \frac{k}{r^3}\left(q \times L - q \times L\right) ,\\ &= 0 ,\end{aligned} \tag{2.29}$$

where it was used that $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ for any three-vectors a, b, c.[6] *Of the three components of this vector, only one is an additional independent integral, as*

$$\begin{aligned} L \cdot A &= 0 ,\\ A^2 &= \mu^2 k^2 + 2\mu H L^2 ,\end{aligned} \tag{2.30}$$

where the last equality can be derived by using $L^2 = q^2 p^2 - (q \cdot p)^2$. This makes the total of independent first integrals for Kepler five, meaning it is maximally superintegrable and all bounded orbits are periodic. Note that the size of the LRL vector is related to the eccentricity e of the orbit, through $e^2 = A^2/(\mu^2 k^2)$.

Concretely, the integrability of the Kepler problem implies we can find the motion. The Lagrangian of the Kepler problem is given by

$$\mathcal{L} = \frac{\mu}{2}\dot{q}^2 + \frac{k}{r} , \tag{2.31}$$

which is a natural Lagrangian, such that the energy is only a sign flip away. It can be written in terms of the radial coordinate r and angular momentum L (taking one angle $\theta = \frac{\pi}{2}$ for simplicity) as

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r} , \tag{2.32}$$

reducing the number of variables in the equation to 1 so the system can be integrated. Now, it is useful to change variables to the so-called Binet variable, $u = \frac{1}{r}$, meaning $\dot{r} = -\frac{\mathrm{d}u}{\mathrm{d}\phi}\frac{L}{\mu}$. The above then becomes

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2 - c_1 u = c_0 , \qquad \textit{with } c_1 = \frac{2k\mu}{L^2},\ c_0 = \frac{2E\mu}{L^2} , \tag{2.33}$$

or slightly more suggestive

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + \left(u - \frac{c_1}{2}\right)^2 = c_0 + \frac{c_1^2}{4} . \tag{2.34}$$

[6] A generalisation to n dimensions of the LRL vector can be found for example in [134].

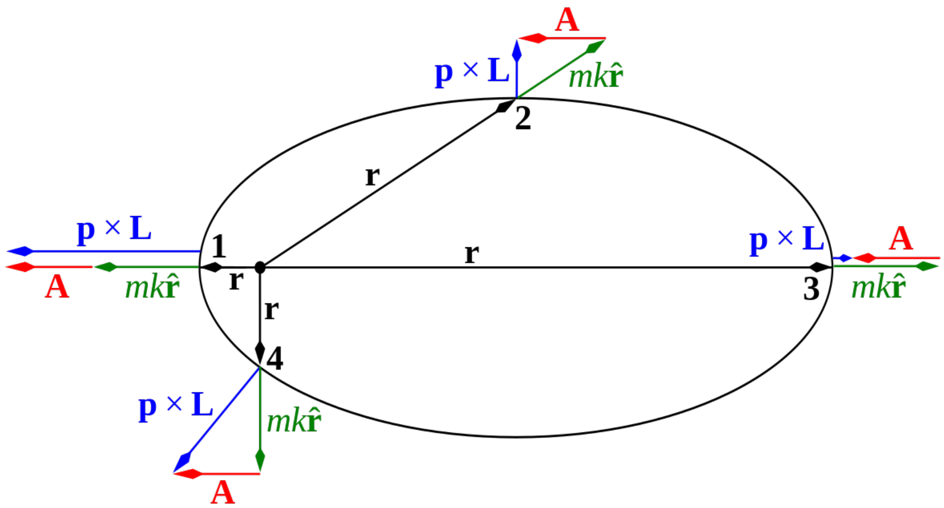


Figure 2.2: The Laplace-Runge-Lenz vector A, at 4 points along the elliptical orbit [105], denoting the mass with m instead of μ here.

One can check that this is solved by

$$u = \frac{1}{l}(1 + e\cos\phi)\,, \qquad \frac{1}{l} = \frac{c_1}{2}\,,\ e^2 = 1 + \frac{4c_0}{c_1^2}\,. \tag{2.35}$$

These of course give the famed conic sections for $r = 1/u$: ellipses for $e < 1$, parabolas for $e = 1$ and hyperbolas for $e > 1$. For the compact part of phase space, i.e. when $e < 1$, we indeed have strictly periodic motion because of the maximal superintegrability.

The flows of vector fields commuting with the Hamiltonian flow, the symmetries, together form a group, which lives on the manifold $\mathcal{P}$ and acts smoothly. In other words, the flows form a Lie group, of which the tangent space at the origin is the Lie algebra. Barring non-connectedness of the group, these are in one-to-one correspondence with each other. The Lie algebra is formed by the vector fields, whose generating functions form a Poisson algebra, a Lie algebra where the bracket is the Poisson bracket and hence satisfies the Leibniz rule. There is an anti-homomorphism between this Poisson algebra and the Lie algebra of the vector fields, as discussed earlier (2.9).

Example 2 (The Kepler problem part 2)**.** *To see what the total symmetry algebra of the Kepler problem amounts to, one needs to calculate the Poisson brackets among all symmetry generators. These read*

$$\{L^i, L^j\} = \epsilon^{ijk} L^k\,, \qquad \{L^i, A^j\} = \epsilon^{ijk} A^k\,, \qquad \{A^i, A^j\} = -2\mu H \epsilon^{ijk} L^k\,, \tag{2.36}$$

such that the sign (or vanishing) of the Hamiltonian H determines the nature of the algebra. On the negative energy set of the phase space $\Sigma_- = \{(q,p)|H < 0\}$, the LRL vector can be rescaled to $\bar{A} = -\frac{1}{\sqrt{-2\mu H}}\,A$ to yield the Poisson brackets

$$\{L^i, L^j\} = \epsilon^{ijk} L^k\,, \qquad \{L^i, \bar{A}^j\} = \epsilon^{ijk} \bar{A}^k\,, \qquad \{\bar{A}^i, \bar{A}^j\} = \epsilon^{ijk} L^k\,, \tag{2.37}$$

which define a Lie algebra isomorphic to so(4). If instead the energy is positive, the algebra becomes so(3,1), while in case the energy vanishes we get se(3); the algebra of isometries (rotations and translations) in three dimensions (also sometimes denoted iso(3)).

2.2.3 Symplectic reduction

Clearly, symmetries of Hamiltonian systems harbour a lot of power, reducing the dimensionality of the solution space and structuring solutions in simple ways. On top of this, a symmetry enables us to relate Hamiltonian systems to lower dimensional Hamiltonian systems. This is different from the reduced dimensionality implied directly by conservation of a function as discussed in Section 2.2.1, as conservation of a function $f = c$ implies the motion takes place on $f^{-1}(c)$, an odd-dimensional level set: definitely not a phase space. In symplectic reduction [88], one can use a symmetry group of dimension k to reduce the degrees of freedom of the $2n$-dimensional phase space of a Hamiltonian system $(\mathcal{P}, \omega, H)$ by $2k$ to yield another Hamiltonian system $(\mathcal{P}_{\text{red}}, \omega_{\text{red}}, H_{\text{red}})$, where $\dim \mathcal{P}_{\text{red}} = 2(n-k)$. How these phase spaces are related exactly, will be discussed in this section.

As a precursor borrowed from [88], let us consider a simple situation to illustrate how one symmetry can lead to the riddance of 2 degrees of freedom. Say the $2n$-dimensional phase space of our system $(\mathcal{P}, \omega, H)$ can be described in canonical coordinates (q, p), of which one momentum $p_n = c$ is conserved. The Hamilton equation for this momentum immediately gives

$$\dot{p}_n = -\frac{\partial H}{\partial q^n} = 0\,, \tag{2.38}$$

implying the Hamiltonian is independent of q^n. Setting the momentum to its constant value, we have

$$H(q^1, \dots, q^{n-1}; p_1, \dots, p_{n-1}, c) = H_{\text{red}}(q^1, \dots, q^{n-1}; p_1, \dots, p_{n-1})\,, \tag{2.39}$$

where H_{red} is defined on the same phase space, but taking only the first $n-1$ canonical coordinate pairs into account. The more general statement for k-dimensional symmetries then does not surprise us anymore, considering that locally one can always define coordinates such that the conserved functions are momenta and their conjugate coordinates are cyclic. Globally, however, it is not always possible to change phase space coordinates in this convenient way, and we need the Marsden-Weinstein theorem.

The Marsden-Weinstein theorem states that we can reduce phase spaces with a certain symmetry group to lower-dimensional phase spaces, effectively dividing out the symmetry.

This is done by considering a regular enough level set $F^{-1}(c) \subset \mathcal{P}$ of the phase space and identifying all points related to each other by symmetry transformations in the group G, creating a reduced phase space $\mathcal{P}_{\text{red}} = F^{-1}(c)/G$.

Theorem 3 (Marsden-Weinstein [172, 88])**.** *Let G be a k-dimensional Lie group leaving the symplectic structure on the manifold $(\mathcal{P}, \omega)$ invariant and $F : \mathcal{P} \to \mathbb{R}^k$ its momentum map. Let G act freely*[7] *on the set $F^{-1}(c)$ for some value $c \in \mathbb{R}^k$. Then $\mathcal{P}_{\text{red}} = F^{-1}(c)/G$*

[7] Acting freely means every group element sends every point of the manifold to another point, except the identity element.

is a $2(n-k)$-dimensional manifold. Moreover, if $\iota : F^{-1}(c) \hookrightarrow \mathcal{P}$ is the inclusion map (see footnote 3) and $\pi : F^{-1}(c) \to \mathcal{P}_{\text{red}}$ the projection to the reduced manifold $\mathcal{P}_{\text{red}}$, then $\iota^*\omega = \pi^*\omega_{\text{red}}$ defines a unique symplectic form ω_{red} on $\mathcal{P}_{\text{red}}$.

This means that the reduced phase space indeed has a symplectic structure. Since the fixing of the level set eliminates k degrees of freedom for a k-dimensional symmetry group and the quotient by the symmetry group does so too, the new dimension is indeed $2(n-k)$.

If two symmetry actions G and K commute, that is, all functions $g \in G$ are in involution with $k \in K$, the resulting reduced phase space $\mathcal{P}_{\text{red}}$ from the reduction with respect to G will itself possess the symmetry K and can thus be reduced further. This is the content of the second theorem of the Marsden-Weinstein paper [172]. For integrable systems, having n integrals mutually in involution, the process of reduction can continue until the manifold is two-dimensional and has only the Hamiltonian left as first integral.

Example 3. ***The Kepler problem part 0*** *An example of the power of reduction is the very construction of the Kepler problem itself. Consider the two body problem, consisting of masses m_1 and m_2 interacting through a gravitational potential $V(q_1, q_2) = -\frac{k}{|q_1-q_2|}$. Initially, denoting the collision set Δ, this system lives on a manifold $\mathcal{P} = T^*(\mathbb{R}^6 \backslash \Delta)$, which is 12-dimensional, and the Hamiltonian is given by*

$$H_{2b}(q_1, q_2; p_1, p_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{k}{|q_1 - q_2|}\,, \tag{2.40}$$

where q_i and p_i are the 3D position and momentum of the ith body. The total momentum $P = p_1 + p_2$ of this configuration is conserved, since

$$\{P, H\} = \left\{p_1 + p_2, -\frac{k}{|q_1 - q_2|}\right\} = +\frac{k}{|q_1 - q_2|^3}\{p_1, q_1\} - \frac{k}{|q_1 - q_2|^3}\{p_2, q_2\} = 0\,, \tag{2.41}$$

which holds by the assumption that the coordinates are canonical. Applying the vector field associated to the total momentum with parameters $b \in \mathbb{R}^3$ yields

$$\langle b, X_P \rangle = -\langle b, \{P, \cdot\} \rangle = b \cdot \frac{\partial \cdot}{\partial q_1} + b \cdot \frac{\partial \cdot}{\partial q_2}\,, \tag{2.42}$$

where $\langle B, C \rangle$ denotes the inner product. This flow of the momentum acts as a translation, or a $(\mathbb{R}, +)$ action for all three dimensions. Taking $P = 0 \in \mathbb{R}^3$ as regular value of the momentum map and reducing over the three commutative actions yields a 6-dimensional space according to Marsden-Weinstein, given by $\mathcal{P}_{\text{red}} = P^{-1}(0)/(\mathbb{R}^3)$. Since we want the new coordinates (q, p) to be canonical and the value of the total momentum suggests $p_1 = -p_2 \equiv p$, we see that in terms of the old brackets both $\{p, q\} = \{p_1, q\}$ and $\{p, q\} = \{-p_2, q\}$ must be true. This implies that we can take $q = q_1 - q_2$ as the new coordinates. Writing down the reduced Hamiltonian in these new phase space coordinates, we have the famous Kepler system as in equation (2.24).

2.2.4 Hidden Symmetry and Killing tensors

Symmetries often can be seen immediately from the Hamiltonian, for example by observing independence of certain coordinates, implying translational symmetry, or dependence only on the norm of a set of coordinates, meaning rotation in these coordinates leaves the Hamiltonian invariant. However, not all symmetries are spotted this easily. To make the notion of a hidden symmetry precise, we will introduce Killing vectors and tensors.

Recall that the phase space is the cotangent bundle $\mathcal{P} = T^*\mathcal{M}$ of a configuration manifold $\mathcal{M}$. Now, let the configuration manifold $\mathcal{M}$ have a metric g and inverse metric g^{-1}. A symmetry of the metric g is the flow of a vector field K on $\mathcal{M}$ such that

$$\nabla_{(i}K_{j)} = 0\,, \tag{2.43}$$

with ∇ the Levi-Civita connection for the metric and the brackets denoting symmetrisation over indices:

$$V_{(i}W_{j)} = \frac{1}{2}(V_iW_j + V_iW_j)\,. \tag{2.44}$$

The equations (2.43) are called the Killing equations and K a Killing vector.[8] Killing vectors generate symmetries of a metric, or isometries. These can be lifted to $\mathcal{P}$ to yield symmetries of the phase space, of which we know they imply the existence of conserved quantities. Explicitly, if a metric space possesses a Killing vector with components $K^i(q) = g^{ij}K_j(q)$, motion through the phase space conserves a quantity Q_K such that

$$Q_K = K^i p_i\,, \tag{2.45}$$

with p_i the momenta conjugate to the coordinates q^i. A conserved quantity for a Hamiltonian H of course satisfies

$$\{K, H\} = 0\,. \tag{2.46}$$

Indeed, expanding the above and organising it in terms of contractions with momenta through inverse metric g^{ij}, this becomes equivalent to the Killing equations.

In analogy to the Killing equations, the definition of a rank N Killing tensor $K_{i_1 \dots i_N}$ is given by [160]

$$\nabla_{(i_1}K_{i_2 \dots i_{N+1})} = 0\,. \tag{2.47}$$

One can show that a tensor satisfying the above guarantees the existence of a conserved quantity [215]

$$Q_{K_N} = K^{i_1 \dots i_N} p_{i_1} \cdots p_{i_N} \tag{2.48}$$

along geodesics.

Notice that the arrow of implication goes both ways: a conserved quantity Q_{K_N} that can be written as a tensor $K^{i_1 \dots i_N}$ contracted with N momenta also implies the existence of a Killing tensor. This type of conserved quantity, quadratic (or even higher order [118, 116, 212, 211]) in momenta, is tied to *hidden symmetries*. These are symmetries of the phase space that cannot be constructed as the lift of a symmetry of configuration space [54].

Killing vectors have a direct geometric interpretation as generators of isometry of the metric, but Killing tensors do not have such a clear interpretation directly related to structure of

[8] The equations can be seen to derive from the Lie derivative of the metric with respect to the vector field K, but here we choose the Killing equations as definition of Killing vectors, to draw the parallel with Killing tensors.

the metric [215]. Rather, they encode a symmetry of geodesic motion on the metric; they generate a symmetry of phase space transforming coordinates into momenta and vice versa. This can be seen as follows. If the projection from phase space to the configuration manifold is $\pi : \mathcal{P} = T^*\mathcal{M} \to \mathcal{M}$, the pushforward of the vector field $X_{Q_K}(\cdot) = \{\cdot, Q_K\}$ is given by $\pi_* X_{Q_K} = K$. In other words, the conserved phase space quantity generates a symmetry of the metric when restricted to configuration space. Instead considering the Killing tensor K_N, the similarly constructed vector field on $\mathcal{M}$ vanishes [56]

$$\pi_* X_{Q_{K_N}} = 0\,, \tag{2.49}$$

showing there is no intelligible action of the flow related to this phase space symmetry on the configuration manifold. As such, it is hidden.

There is another type of symmetry that deserves mention: conformal symmetry. Conformal Killing vectors are vectors $\hat{C}^i$ such that

$$\nabla_i \hat{C}_j = g_{ij}\lambda\,, \tag{2.50}$$

with λ some constant, and have the property of generating conserved quantities when the path is a null geodesic, that is, when $H = \frac{1}{2}g^{ij}p_i p_j = 0$. This implies that we can write the commutator for conformally conserved quantities $\hat{C}$ generated by conformal Killing vectors $\hat{C}^i$ as

$$\{\hat{C}, H\} = f(q,p)H\,, \qquad f > 0\,. \tag{2.51}$$

When f vanishes, $\hat{C}$ becomes an ordinary conserved quantity as it generates a symmetry that leaves the Hamiltonian invariant.

Similarly, one can construct conformal Killing tensors. We have a tensor $\hat{C}_{i_1 \dots i_n}$ such that

$$\nabla_{(i_1}\hat{C}_{i_2\dots i_{n+1})} = g_{(i_1 i_2}\lambda_{i_3\dots i_{n+1})} \tag{2.52}$$

guarantees the conservation for null geodesics of

$$\hat{C} = \hat{C}^{i_1\dots i_n} p_{i_1}\cdots p_{i_n}\,, \tag{2.53}$$

as the contraction of the former with momenta coincides with the Poisson bracket for conformally conserved quantities.

Example 4 (The hidden symmetry in Kepler)**.** *The LRL vector in the Kepler problem is conserved due to a hidden symmetry. Movement in the Kepler potential is of course not geodesic, though we can apply a lift to make it so, as we will see in Chapter 5, but the conserved quantities that are the components of the vector* (2.29) *clearly are quadratic in momenta. Moreover, applying the symmetry transformations related to the vector to the coordinates of phase space mixes coordinates q and momenta p. If X_{e^j} is the vector field of component j of the LRL vector, the transformation generated by it reads [178]*

$$X_{e^j}(q^i) = \frac{1}{\mu}\left(2q^j p^i - q^i p^j - (q\cdot p)\delta^{ij}\right)\,, \qquad X_{e^j}(p_i) = \frac{1}{\mu}\left(-p^2\delta_i^j + p^j p_i\right) + k\left(\frac{\delta_i^j}{r} - \frac{q^j q_i}{r^3}\right)\,, \tag{2.54}$$

where the indices are raised and lowered by δ^{ij} *or* δ_{ij}*. This mixes the coordinates and momenta, exactly what we would expect from a hidden symmetry.*

That this indeed leaves the Lagrangian invariant, can be seen by using $p_i = \mu v_i$*, and applying the above transformations with parameters* a_j*, giving*

$$\begin{aligned} L'_K - L_K &= a_j \left[\frac{1}{2} \mu X_{e^j}(v^2) - k X_{e^j}(\frac{1}{r}) \right] \\ &= a_j \left[k \left(\frac{v^j}{r} - \frac{(q \cdot v) q^j}{r^3} + \frac{k}{r^3} \left(q^j (q \cdot v) - q^2 v^j \right) \right) \right] = 0 \,. \end{aligned} \tag{2.55}$$

2.3 Canonical transformations

An advantage of symplectic geometry and a requirement of our description of physical systems is that they are independent of the choice of coordinates one uses. This means in particular that whenever we have a Hamiltonian system $(\mathcal{P}, \omega, H)$, there is a set of equivalent Hamiltonian systems where each can be written $(\mathcal{P}, \omega, H')$ with only the form of the Hamiltonian different. These systems are related through canonical transformations: those transformations of the generalised coordinates (q, p) such that the symplectic form ω and therefore Hamilton's equations are left invariant. In this section we introduce canonical transformations, their infinitesimal versions, and show how finite, continuous canonical transformations can be constructed.

If $\Phi : \mathcal{P} \to \mathcal{P}$ is a canonical transformation, we have

$$\Phi^* \omega = \omega \,. \tag{2.56}$$

If we denote the new coordinates (Q, P), we have for the Hamiltonians

$$H'(Q(q,p), P(q,p)) = H(q,p) \,. \tag{2.57}$$

Because we consider the momenta as coordinates of the phase space in their own right, these transformations contain more than just coordinate transformations of the configuration manifold, allowing independent redefinitions of momenta and even mixing between coordinates and momenta. The only demand on these transformations, that the symplectic 2-form is invariant, by (2.8) can also be formulated as the demand that the transformation is between sets of canonical coordinates:

$$\{Q^i(q,p), P_j(q,p)\} = \delta^i_j \,, \qquad \{Q^i(q,p), Q^j(q,p)\} = \{P_i(q,p), P_j(q,p)\} = 0 \,, \tag{2.58}$$

where the brackets are interpreted in the old coordinates (though, clearly, if the new coordinates satisfy the above, it works in brackets interpreted in new coordinates as well!).

In some cases, we will want to consider infinitesimally small changes of coordinates. The motivation for this will readily become apparent. These transformations are quite simple, with small ϵ they read [206]

$$Q^i = q^i + \epsilon A^i(q,p) \,, \qquad P_i = p_i + \epsilon B_i(q,p) \,, \tag{2.59}$$

such that the first of the above Poisson brackets gives to first order in ϵ

$$\frac{\partial A^i}{\partial q^j} + \frac{\partial B_j}{\partial p_i} = 0\,, \tag{2.60}$$

showing the functions A^i are not independent of B_j. In fact, making the Ansatz $A^i = \frac{\partial G}{\partial p_i}$ and $B_j = -\frac{\partial G}{\partial q^j}$ gives us just one function $G(q,p)$ to find. This function is called the generator of the infinitesimal canonical transformation.

Suppose now that we take ϵ as a flow parameter, such that we can continuously change (q,p) into $(q(\epsilon),p(\epsilon)) = \Phi_G^\epsilon(q,p)$ as a function of it, as ϵ ranges from zero to some (for now still small) value. Moreover, we interpret these coordinates as the same coordinates at a different point along a flow in phase space of which ϵ is the parameter:

$$q^i(\epsilon) = q^i(0) + \epsilon\frac{\partial G}{\partial p_i}\,, \qquad p_i(\epsilon) = p_i(0) - \epsilon\frac{\partial G}{\partial q^j}\,. \tag{2.61}$$

This perspective is often called active, because we let the position in phase space change as a function of the parameter [206]. The canonical transformation (2.59) in terms of the generating function $G(q,p)$ then gives

$$\frac{\mathrm{d}q^i}{\mathrm{d}\epsilon} = \frac{\partial G}{\partial p_i}\,, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}\epsilon} = -\frac{\partial G}{\partial q^i}\,, \tag{2.62}$$

which are nothing but Hamilton's equations for a Hamiltonian G, having evolution parameter ϵ.

In considering what a certain coordinate transformation does to a function on a particular point in phase space, we instead want to adopt a passive view, changing the labelling of the point [214]. This means the relevant coordinates for the evaluation of the function after flow by Φ_G^ϵ have originated from a point 'earlier' on the orbit. Specifically, we need the coordinates $\Phi_G^{-\epsilon}(q,p)$, as $\Phi_G^\epsilon\Phi_G^{-\epsilon} = \mathrm{Id}$. For the change of a general function $F(q,p;\epsilon)$ on phase space, transforming under a coordinate transformation Φ_G^ϵ we then get

$$\begin{aligned}\frac{\mathrm{d}F(q(-\epsilon),p(-\epsilon))}{\mathrm{d}\epsilon} &= \frac{\partial F}{\partial q^i}\frac{\mathrm{d}q^i(-\epsilon)}{\mathrm{d}\epsilon} + \frac{\partial F}{\partial p_i}\frac{\mathrm{d}p_i(-\epsilon)}{\mathrm{d}\epsilon}\\ &= -\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} + \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}\\ &= \{G,F\} = -\{F,G\}\,,\end{aligned} \tag{2.63}$$

which is only a minus sign different from the usual flow of a function $F(q,p)$ along a vector field X_G:

$$\frac{\mathrm{d}F(q,p)}{\mathrm{d}\epsilon} = X_G(F) = \{F,G\}\,, \tag{2.64}$$

showing how the function changes as the point on which it is evaluated flows from one to the other, as parametrised by ϵ. As all generators of continuous symmetries can be considered Hamiltonians with their own particular evolution parameter as discussed in Section 2.2.2, this then gives us another interpretation of symmetries: they are active canonical transformations that leave the Hamiltonian invariant, taking $(\mathcal{P}, \omega, H)$ to $(\mathcal{P}, \omega, H)$.

We can also define higher-order-in-ϵ transformations, essentially building an expansion of the non-infinitesimal or finite transformation. This is done through construction of a Taylor expansion around $\epsilon = 0$ [214]. To do this, it is useful to define the adjoint operator

$$\mathrm{ad}_G(\cdot) := \{G, \cdot\} \,. \tag{2.65}$$

We define $n > 0$ repeated iterations of the adjoint operator by

$$[\mathrm{ad}_G]^n := [\mathrm{ad}_G]^{n-1} \circ \mathrm{ad}_G, \qquad [\mathrm{ad}_G]^0 := \mathrm{Id}_{C^\infty(\mathcal{P})} \,. \tag{2.66}$$

Under the canonical transformation given by the flow of Hamiltonian G, a function $F(q, p; \epsilon)$ is then Taylor expanded around $\epsilon = 0$ as

$$F(q, p; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} [\mathrm{ad}_G]^n F(q, p; 0) \,. \tag{2.67}$$

In case we want to make an approximation to a certain order in ϵ, we simply truncate the sum at that term. This technique is applied in Section 4.2.2 to transform Kepler-like systems, to a certain order of approximation, to the classical Kepler problem.

2.4 Bertrand's theorem

As we have seen in Section 2.2.2, the Kepler problem is unusually symmetric. Now we know from the last section how different-yet-equivalent Hamiltonians can be related, it is natural to ask how unique, up to canonical transformations, the Kepler problem is. Bertrand's theorem answers part of this question.

Loosely speaking, Bertrand's theorem shows there are only two central potential systems that have the property that all bounded orbits close: the Kepler problem and the isotropic harmonic oscillator. These two systems are known for their large symmetry groups, making them maximally superintegrable, forcing their bounded orbits to close in on themselves. Bertrand's theorem in a sense then implies the reverse statement: there are no systems with exclusively closed bounded orbits that do *not* have the largest number of independent integrals of motion possible.

Theorem 4 (Bertrand [200])**.** *The only central potentials that result in closed orbits for all bound trajectories are the isotropic harmonic oscillator potential* $V_{\mathrm{HO}}(r) = kr^2$ *and the Kepler potential* $V_{\mathrm{Kep}} = -\frac{k}{r}$.

The reasoning in the proof is quite straightforward and can be found in for example [125]. As all attractive central potentials admit circular orbits, we can consider paths only slightly different from circular and demand that, still, the orbits close. At lowest order in the perturbation away from circularity, the equation of motion implies the potential is a power law $V(r) = kr^{\beta^2-2}$, with $\beta \in \mathbb{Q}$, and at higher order the only remaining possibilities are $\beta^2 = 1$, Kepler, or $\beta^2 = 4$, the harmonic oscillator, of which we know that the orbits close to all orders.

2.5 Kepler as free motion on a 3-sphere

The conservation of the LRL vector appears to be almost 'coincidental', certainly it is hidden in the sense of Section 2.2.4. For many years this puzzled mathematicians, until in 1935 Fock noticed that the Schrödinger equation for the hydrogen atom could be mapped with preservation of symmetry properties to an equation manifestly invariant under so(4) [109]. The hydrogen atom being the quantum mechanical equivalent of the Kepler problem, it was to be expected a similar mapping should exist in the classical case. Indeed, as shown for example by Moser in 1970 [177], the orbits of the Kepler problem can be mapped to geodesic motion on a 3-sphere, without changing symmetry properties. This implies that, while the so(4) can be regarded as hidden in the original phase space, there is another, perhaps more natural phase space in which we can observe bounded Kepler orbits, where the symmetry is evident immediately from the geometry of the configuration space. We will now briefly treat the construction as given by [133], highlighting important steps.

Working in the opposite direction of Guillemin and Sternberg in Section 7 of [133], we start with the Kepler Hamiltonian (in three dimensions)

$$H = \frac{p^2}{2} - \frac{1}{r}\,, \tag{2.68}$$

and note that on the energy surface $H = -\frac{1}{2}$ there is a function

$$J = r(H + \frac{1}{2})\,, \tag{2.69}$$

with a vector field

$$\mathrm{d}J = r\,\mathrm{d}H\,. \tag{2.70}$$

This *regularises* the Kepler problem, since the singularity in H at $r = 0$ is exactly cancelled making the function J defined everywhere in phase space. Note that we need this particular energy since the term in $\mathrm{d}J$ proportional to $\mathrm{d}q$ vanishes on this surface, though the reason for the addition of $\frac{1}{2}$ in J will only become clear in a moment. The factor of r multiplying the vector field has the effect, aside from the cancellation of the singularity, of reparemetrising time t to a new time s by

$$\mathrm{d}s = \frac{\mathrm{d}t}{r}\,, \tag{2.71}$$

with the new time ticking more slowly far away and faster closer to the center.

In terms of phase space variables our new Hamiltonian is given by

$$J = \frac{r}{2}(1 + p^2) - 1\,. \tag{2.72}$$

Let us now take the combination

$$K = \frac{1}{2}(J + 1)^2\,, \tag{2.73}$$

and see that the vector fields associated to these functions are the same: Since on the surface we are considering $J = 0$,

$$\mathrm{d}K = (J + 1)\,\mathrm{d}J = \mathrm{d}J\,. \tag{2.74}$$

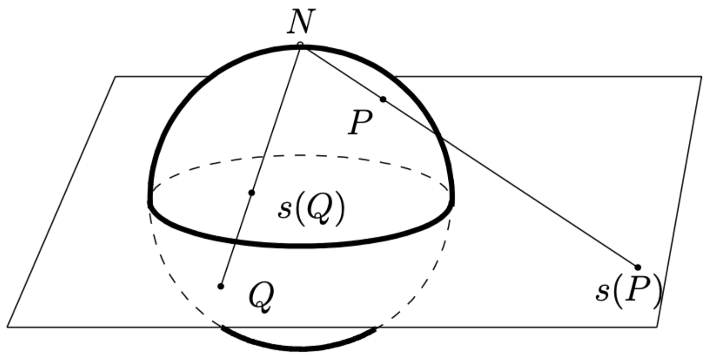


Figure 2.3: *The stereographic projection of points P and Q on a sphere to the plane [1].*

In terms of phase space variables we have

$$K = \frac{1}{8}(p^2+1)^2 r^2 \,. \tag{2.75}$$

This already looks a lot like free motion, with a momentum squaring as

$$\zeta^2 = \left(\frac{1}{2}(1+p^2)\right)^2 r^2. \tag{2.76}$$

In fact, it is exactly the momentum on a 3-sphere under the cotangent lift of an inverse stereographic projection, *provided we interchange the momentum and position.*

Let us unpack the specific transformation needed. This mapping takes us from $T^*\mathbb{R}^3$, the phase space of the regularised Kepler problem to $T^*S^3_N$, which is the phase space of a particle moving on a 3-sphere (without north pole), embedded in $\mathbb{R}^4$.

Firstly, recall that a stereographic projection takes each point P on a sphere to a point $s(P)$ on the plane through the equator, by drawing a line from the north pole through P and extending this to the plane, see Figure 2.3. Analysing the triangles made by this line and the vertical axis, one can show the coordinates w^i, with $i = 1, 2, 3$ of point $s(P)$ are related to the coordinates (y^0, y^i) of the point P on the 3-sphere by

$$w^i = \frac{y^i}{1-y^0} \,. \tag{2.77}$$

Moreover, the inverse reads

$$y^0 = \frac{w^2-1}{w^2+1} \,, \qquad y^i = \frac{2w^i}{w^2+1} \,, \tag{2.78}$$

where $w^2 = w^i w_i$ and indices are raised and lowered by δ^{ij}.

Secondly, let us consider what the phase space we are mapping to looks like. It can be parametrised by canonical coordinates $(y, \eta) \in T^*S^3_N$, which are those pairs taken from $\mathbb{R}^4 \times \mathbb{R}^4$ that satisfy

$$y^2 = 1, \qquad y \cdot \eta = 0, \tag{2.79}$$

reducing the number of dimensions of the phase space from 8 to 6 as should be the case to match the phase space dimensions of the Kepler problem.

For the cotangent lift of the inverse stereographic projection, meaning the transporting of the phase space on the plane to that on the 3-sphere, we need to use the fact that the tautological one-form must be conserved, that is

$$\eta_\mu \mathrm{d}y^\mu = \xi_i \mathrm{d}w^i \,, \tag{2.80}$$

where the sums are implicitly over $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$. This equation, upon substitution of the (inverse) stereographic projection (2.77) and (2.78) gives an expression for ξ in terms of the canonical coordinates of the 3-sphere phase space (y, η), which after use of the relations (2.79) reads

$$\xi_i = (1 - y^0)\eta_i + \eta_0 y_i \,, \tag{2.81}$$

and the inverse can be calculated by taking the inner product $\xi_i w^i$ and using some of the above relations to be

$$\eta_0 = \xi_i w^i \,, \qquad \eta_i = \frac{1}{2}(1 + w^2)\xi_i - \xi_j w^j w_i \,. \tag{2.82}$$

Having constructed the full cotangent lift, we can indeed conclude that the square of our new momentum η reads

$$\eta^2 = \frac{1}{4}\xi^2(1 + w^2)^2 \,, \tag{2.83}$$

meaning the interchange of the role of momentum and coordinates, a perfectly canonical transformation, indeed gives us the Hamiltonian K in (2.75).

The stereographic projection maps circles to circles, and so does the inverse. In our case, this means the circles in momentum space of the Kepler problem, as noted by [178], are mapped to circles on the 3-sphere. Moreover, as the motion is that of a free Hamiltonian on the 3-sphere, it will be geodesic motion, i.e. motion along the great circles. From this point of view, the so(4) symmetry is natural: the orbits, great circles on the sphere, are rotated into other orbits. In the language of hidden symmetries and Killing tensors one can say that the above construction linearises the quadratic-in-momenta constants of motion, which are the components of the LRL vector, manifesting the so(4) symmetry as an isometry.

2.6 Multi-center systems

A natural generalisation of the Kepler problem we have just discussed, is the multi-center system, in which there are more than one stationary objects orbited by another. Though it shares the number of phase space dimensions with the Kepler problem, it does not have the same integrability properties. This is obvious of course, as one of the symmetries of the Kepler problem is spherical symmetry, which is immediately broken by the inclusion of any other object.

A particular case is the two-center system, which is still integrable in classical mechanics. Any higher number of centers stops being integrable in general [42, 155, 156]. Physically, the two-center system is for example approximated by two stars orbited by a much smaller planet, where the stars are approximately stationary on the typical timescales of the movement of the planet. Considering the two-center problem in a rotating reference frame, one can use it to find the motion in the restricted three body problem, where two objects are much larger than a third.

In the following, we will consider the two-center problem and establish its integrability.

2.6.1 Separation of the two-center problem

The two-center problem is integrable, as it has exactly the same number of independent integrals of motion as dimensions of configuration space, 3, which are in involution. To have the opportunity to introduce some new concepts, instead of simply giving the independent integrals and verifying their existence, we will establish integrability in a slightly different way. Integrability can also be established through separation of the Hamilton-Jacobi equation, which is a sufficient, yet not necessary condition.

2.6.1.1 The Hamilton-Jacobi equation and its separation

The Hamilton-Jacobi equation provides an alternative perspective on classical dynamics, just like Lagrangian, Newtonian or Hamiltonian mechanics. In our discussion of it, we will draw heavily from [206].

Where in Lagrangian mechanics we look at the action and vary the path through configuration space, with fixed initial and final points $q_{\mathrm{in}}, q_{\mathrm{fin}}$, to derive the Hamilton-Jacobi equation we vary instead only the endpoint and assume the true path $q_{\mathrm{true}}(\lambda)$ is taken to begin with. We define this action evaluated on the true path

$$W(q_{\mathrm{in}}, q_{\mathrm{fin}}, \Lambda) = S[q_{\mathrm{true}}(\lambda)]\,, \tag{2.84}$$

such that W is a function of the initial point, which we can set to zero immediately, the final point q_{fin} and the time Λ it takes to get from the former to the latter. Using this function W, we can formulate $n+1$ first order partial differential equations, which together are equivalent to the n second order Lagrangian equations and the $2n$ first order Hamilton equations.

Recall that varying the action by varying the path gives

$$\begin{aligned}\delta S &= \int_0^\Lambda \mathrm{d}\lambda\ \frac{\partial L(q,\dot{q},\lambda)}{\partial q}\delta q + \frac{\partial L(q,\dot{q},\lambda)}{\partial \dot{q}}\delta\dot{q} \\ &= \int_0^\Lambda \mathrm{d}\lambda\ \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{q}}\right)\delta q + \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_0^\Lambda\,,\end{aligned} \tag{2.85}$$

where we integrated by parts to get the second line. In case of the Lagrangian formulation, the boundary term vanishes: $\delta q|_0 = \delta q|_\Lambda = 0$, giving the bracketed Euler-Lagrange equation in the integral. However, now we assume the EL equation is satisfied, but vary the final point, so we find

$$\frac{\partial W(q_{\mathrm{fin}},\Lambda)}{\partial q_{\mathrm{fin}}} = \frac{\partial L}{\partial \dot{q}} = p^{\mathrm{fin}}\,, \tag{2.86}$$

giving the first n partial differential equations. For the only other dependence, on Λ, it is not immediately clear how to derive a useful PDE. Let us therefore take the total derivative with respect to Λ, reading

$$\frac{\mathrm{d}W(q_{\text{fin}},\Lambda)}{\mathrm{d}\Lambda} = \frac{\partial W}{\partial \Lambda} + \frac{\partial W}{\partial q_{\text{fin}}}\dot{q}_{\text{fin}} = \frac{\partial W}{\partial \Lambda} + p^{\text{fin}}\dot{q}_{\text{fin}}\,. \tag{2.87}$$

But the total derivative of an action is also just the Lagrangian, or in the case of our action evaluated on the true path

$$\frac{\mathrm{d}W}{\mathrm{d}\Lambda} = L(q_{\text{true}}(\Lambda),\dot{q}_{\text{true}}(\Lambda),\Lambda) = L(q_{\text{fin}},\dot{q}_{\text{fin}},\Lambda)\,, \tag{2.88}$$

so we can express the partial derivative as

$$\begin{aligned}\frac{\partial W}{\partial \Lambda} &= L(q_{\text{fin}},\dot{q}_{\text{fin}},\Lambda) - p^{\text{fin}}\dot{q}_{\text{fin}}\\ &= -H(q_{\text{fin}},p^{\text{fin}},\Lambda)\,,\end{aligned} \tag{2.89}$$

which is our $n+1$th PDE. As we can freely choose any final position and total time, we can relabel $q_{\text{fin}} \to q$ and $\Lambda \to \lambda$. The Hamilton-Jacobi equation (HJE) then is the result of substituting the first n equations into the last, yielding

$$\frac{\partial W}{\partial \lambda} = -H(q,\frac{\partial W}{\partial q},\lambda)\,, \tag{2.90}$$

which in general is a non-linear first order equation. This need not be easy to solve, but if we can find a solution $W(q,\lambda)$, this tells us the evolution of the system through substitution.

A benefit of this approach, is that if (2.90) contains a separable coordinate, say q^1, there exists a constant function $\psi(q^1,\frac{\partial W}{\partial q^1})$. Therefore, we can use separability to find constants of motion and study integrability.

Definition 6 ([201])**.** *One calls q^1 a* ***separable coordinate*** *if the function W solving the Hamilton-Jacobi equation can be written as*

$$W(q,\lambda) = W_1(q^1) + \bar{W}(q^2,\dots,q^n;\lambda)\,, \tag{2.91}$$

and the Hamiltonian is of the form

$$H(q^1,q^2,\dots,q^n;p_1,p_2,\dots,p_n;\lambda) = H(\psi(q^1,p_1);q^2,\dots,q^n;p_2,\dots,p_n;\lambda)\,, \tag{2.92}$$

for some function $\psi(q^1,p_1)$.

The constancy of $\psi(q^1,\frac{\partial W_1}{\partial q^1})$ is clear: since we can independently vary q^1 and the equation (2.90) must still hold, the only option is that the function is constant, i.e. it is a constant of motion.

As the functions ψ_i of all separable coordinates q^i are in involution and independent, this implies that full separability, splitting the function W in n additive parts, guarantees integrability in the sense of Definition 4. This then gives a way to show integrability: find a generalised coordinate system in which the Hamilton-Jacobi equation separates completely.

Example 5 (Separating the two-center problem)**.** *The Hamiltonian of the classical two-center problem is given by*

$$H = \frac{p^2}{2m} - \frac{k_1}{|q - d_1|} - \frac{k_2}{|q - d_2|}, \tag{2.93}$$

with $d_1 = (d, 0, 0)$ *and* $d_2 = (-d, 0, 0)$ *the locations of the fixed masses, and* $k_i = M_i m$ *the product of the masses at one of the centers with the test mass and the coupling constant* G *which we set to one.*

Jacobi showed this Hamiltonian has a separable Hamilton-Jacobi equation in a special set of coordinates called prolate spheroidal coordinates, given by

$$q^1 = d \cosh \xi \cos \eta \,, \quad q^2 = d \sinh \xi \sin \eta \cos \phi \,, \quad q^3 = d \sinh \xi \sin \eta \sin \phi \,, \tag{2.94}$$

in which the Hamiltonian reads

$$H = \frac{1}{2m} \left(\frac{P_\phi^2}{d^2 \sinh^2 \xi \sin^2 \eta} + \frac{1}{d^2(\sinh^2 \xi + \sin^2 \eta)} \left[P_\xi^2 + P_\eta^2 \right] \right) - \frac{(k_1 + k_2) \cosh \xi + (k_2 - k_1) \cos \eta}{d(\cosh^2 \xi - \cos^2 \eta)} \,. \tag{2.95}$$

Substituting the Ansatz $S_0 = -Et + L\phi + S_\xi(\xi) + S_\eta(\eta)$ *gives a Hamilton-Jacobi equation*

$$(\partial_\xi S_\xi)^2 + (\partial_\eta S_\eta)^2 + L^2 \left(\frac{1}{\sinh^2 \xi} + \frac{1}{\sin^2 \eta} \right) - 2md \left[(k_1 + k_2) \cosh \xi + (k_2 - k_1) \cos \eta \right] = 2md^2 E(\sinh^2 \xi + \sin^2 \eta) \,. \tag{2.96}$$

This equation is separable into a part only dependent on ξ *and a part only dependent on* η*, such that*

$$(\partial_\xi S_\xi)^2 = -\frac{L^2}{\sinh^2 \xi} + 2md \left[m(M_1 + M_2) \cosh \xi + Ed \sinh^2 \xi \right] + C \tag{2.97}$$

$$(\partial_\eta S_\eta)^2 = -\frac{L^2}{\sin^2 \eta} + 2md \left[m(M_2 - M_1) \cos \eta + Ed \sin^2 \eta \right] - C \,. \tag{2.98}$$

This concludes the separation and thereby proof of integrability of the two-center problem.

Chapter 3

Relativistic two-body problems

In the weeks in November 1915 during which Einstein finished his general theory of relativity (GR), he tested it against a number called the anomalous precession of Mercury [208, 219]. Reproducing this number would go down in history as the first great success of his theory. In its calculation, Einstein used a couple of interesting concepts. First, he expanded his theory in what we would now call a post-Newtonian expansion, because exact calculation was utterly impossible. Then he took the limit in which the sun is infinitely heavier than Mercury. And finally, he calculated the rate of precession of the quasi-elliptical orbit of Mercury due to GR.

If one slightly distorts the elliptical motion in a Kepler problem, as Einstein's set-up did, the orbit becomes a quasi-ellipse, no longer exactly closing, but precessing ever so slightly every time the orbiter circles its host. The perihelion is the point of closest approach to the host body, and its shift the difference in the angle of two subsequent perihelion points, i.e. the amount the perihelion precesses in one circling of the host.[1]

This was historically the first test that General Relativity has stood: Einstein calculated the predicted precession of Mercury due to GR and compared it to the anomalous precession, known then for some 50 years [157]. The anomalous precession of Mercury is that part of the precession of its orbit not explained by the Newtonian pull of other planets. Other such tests were for example the bending of light and gravitational redshift.

The importance of these tests becomes clear when we consider that GR was not the only theory in contest at the time. Gunnar Nordström developed a rivalling theory in 1913, similarly aiming to extend Newtonian gravity in such a way as to make it compatible with special relativity. And he succeeded! The problem with his theory? It did not produce the anomalous precession of Mercury [220].[2]

In this chapter, we will discuss the relativistic two-body problem, developing tools on the way not unlike those Einstein used to get to know the predictions of his theory of gravity.

[1] Perihelion literally means 'near Sun'. Periapsis is the more general notion referring to the point in an ellipse or elliptical orbit, but we will stick to the word perihelion.

[2] Additionally, the first theory he developed did not satisfy the equivalence principle of inertial and gravitational mass, urging him to develop his second theory, that did so, according to him – though this is not entirely clear [220, 94]. It still did not give the observed anomalous precession however.

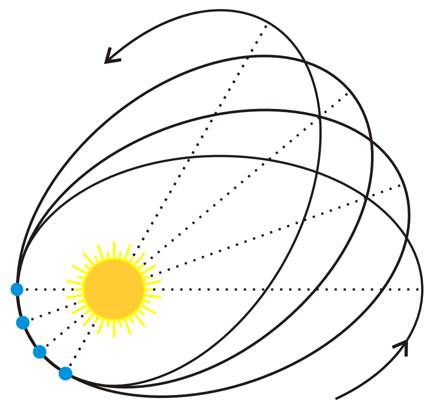


Figure 3.1: Schematic drawing of the perihelion precession of Mercury around the Sun [106].

Moreover, these tools will be tailored to look for relativistic problems displaying the same symmetries as the classical problem. The classical two-body problem, with $\frac{1}{r^2}$ force is simple in two ways that the relativistic version generically is not. Firstly, the reduction of the translations that we saw in the previous chapter renders it equivalent to a one-center problem, allowing us to focus on just one body in a simple potential. Relativistic physics has to contend with a finite speed of information, making it impossible to explicitly write down the Hamiltonian without additional assumptions in the first place, as we will see in this chapter. Secondly, the result of the reduction of the classical system, the Kepler problem, is maximally superintegrable, whereas the relativistic one-center problem is 'just' superintegrable. Both these aspects that are more complicated in relativistic physics as compared to Newtonian physics will be addressed, though our focus will be primarily on the latter.

Before focusing on relativistic one-center problems, we will briefly consider the relativistic two-body problem in full generality. The one-center system, seen as the limit of the full problem in which one mass is much larger than the other, is just one way of approximating it. Another way is an expansion in the coupling constant, such as the post-Minkowskian (PM) expansion for gravity in terms of Newton's constant G.[3] The most famous method of approximating the full problem, called the post-Newtonian (PN) expansion in the gravitational case, is taking a truncated expansion in inverse powers of the speed of light of the Lagrangian or Hamiltonian, or directly of an observable such as the scattering angle.

Both these approximation schemes have as added benefit that the first orders are free from dissipative effects, that is, the energy in the system remains conserved up to a certain order (2.5PN [39] and 3PM [48]). This is not the case for the first correction in the small mass ratio expansion, for which the dissipative term dominates the first correction to the test mass limit, its 'zeroth order' [192]. Historically, this seems to have lead to the situation where the conservative system is considered separately to higher order than the lowest dissipative one in the previous two, see e.g. [28, 111], but not in the small mass ratio expansion. Therefore, and because dissipation immediately destroys time-translation symmetry, we will only consider the post-Minkowskian and post-Newtonian approximations.

[3] In the context of GR, the test-mass limit and the first post-Minkowskian expansion of the two-body system in fact coincide [89].

Returning to our main purpose, recall that the maximal superintegrability of the Kepler problem implies that for Newton's description of revolving planets or Coulomb's description of the interaction of charged particles, the trajectories close and are truly periodic for negative energies. We will be looking for relativistic theories with additional dynamical symmetry, not necessarily describing naturally occurring systems. As the existence of hidden symmetry in the classical Kepler problem caused the closing of all bounded orbits, we will be looking for theories in which this special situation occurs as well. In essence, we will use the perihelion shift as a diagnostic tool for additional symmetry in relativistic Kepler-like problems.

As we have seen above in the discussion of GR and Nordström's theory, depending on what relativistic theory we consider, we find different corrections to physically observable quantities, such as the perihelion shift. In general however, given a one-center problem, we know there are at least 4 conserved quantities, as the Hamiltonian is conserved along with the 3 components of angular momentum. This makes the relativistic one-center system still superintegrable, though maximal superintegrability generically is lost.

Physical examples of relativistic one-center systems are those with charges or with masses. The former, which is the relativistic Coulomb problem, contains a fixed particle at the origin, with a certain charge and a smaller charge in its orbit. Alternatively, we can consider the relativistic gravitational problem, with one particle fixed at the origin (call it a black hole) in whose field another particle moves. This is just a particle in a Schwarzschild metric. Again, by the setup of the problem we are guaranteed conservation of angular momentum and therefore superintegrability. However, there is a difference between this problem and the previous, as expressing the perihelion precession in terms of appropriate constants yields a result in GR that is 6 times that in electrodynamics.

The above examples are two in a set of three theories that arguably are the most natural ones to consider, as they correspond respectively to spin-1 and spin-2, minimally coupled theories. The latter concept meaning they couple only to the lowest moment, that is, only to the charge or mass but not the dipole moments. The remaining minimally coupled theory is dilaton gravity, which is a spin-0 field coupled through a conformal factor in the metric. Interacting higher spin theories, so with spin-3 or higher force carriers, are known to be impossible due to gauge invariance [218]. We will consider one-center systems in all 3 theories and combinations of these, and compare their perihelion shifts in search for theories where the shift vanishes, as these might possess additional symmetry.

Let us, before commencing our theoretical discussion of the relativistic two-body problem, consider for a moment the possible value such efforts have in the context of current and future experimental endeavours.[4] From 2015 onward, the observatories LIGO, VIRGO and KAGRA have detected many instances of gravitational waves originating from binaries of neutron stars or black holes [82, 3]. These observations can be fit into a slightly different category than the previously mentioned 'classical' tests, like gravitational redshift and perihelion precession, as gravitational waves are an example of strong field effects. In this way, they test GR in a completely different energy regime. With more observing runs [2] and the space based telescope LISA upcoming, the dawn of the gravitational wave era provides strong motivation

[4]This paragraph is adapted from an introduction paragraph of [180].

for precision study of binary dynamics, particularly of the earlier stage of the merger [8, 7]. This earlier stage is typically approached with analytical tools [141].

Moreover, and in light of our focus on one-center systems, so-called extreme mass-ratio inspirals (EMRIs) are expected to provide a significant part of the observations for the space-based gravitational wave observatory LISA [31]. Also, numerical relativity analyses are computationally limited to comparable-mass inspirals, as the high computational burden excludes systems with more than hundreds of cycles (i.e. revolutions of the host) [216]. All this goes to say that an analytical approach to one-center-like relativistic systems is motivated by more than just theoretical interest.

In the following, we will briefly treat the relativistic two-body problem and what makes it complicated and discuss two important approximation schemes for such systems in Section 3.1. Subsequently, some relativistic incarnations of the Kepler problem are discussed in Section 3.2. To facilitate comparison of the perihelion shifts of the different theories, we will calculate them all in a similar way. Lastly, in Section 3.3 we will make a synthesis of sorts by presenting Einstein-Maxwell-dilaton theory, which combines the fields of the preceding systems and turns out to have remarkable properties in a particular tuning of coupling constants.

3.1 The general relativistic two body problem

The reduction of two bodies to one reduced mass in a central potential is straightforward in the classical case, but not in the relativistic case. This is a consequence of the relativity of time. A quick way to see this, is to consider the kinetic energy in the center of mass frame, which classically can be written as in Example 3 as the usual $\frac{p^2}{2\mu}$ in terms of the reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ and relative momentum $p = p_1 = -p_2$. In the relativistic case, we cannot parametrise both kinetic energies conveniently, so keeping it reparametrisable, we must write

$$K_{rel} = m_1 c^2 \sqrt{1 + \frac{p_1^2}{m_1^2 c^2}} + m_2 c^2 \sqrt{1 + \frac{p_2^2}{m_2^2 c^2}} \tag{3.1}$$

$$= (m_1 + m_2)c^2 + \frac{p^2}{2\mu} - \frac{1}{8}\frac{m_1^3 + m_2^3}{(m_1+m_2)^3}\frac{p^4}{c^2} + \dots\,, \tag{3.2}$$

where we expand the square roots in small $\frac{p^2}{c^2}$. Now we cannot rewrite this as the kinetic energy of just one body, but really need both kinetic terms separately. This might not seem like a big problem yet, but in introducing an interaction potential to the bodies, it quickly becomes intractable. Because of the finite speed of light, the potential terms for one body in principle depend on many past positions of the other body, generating an infinite amount of additional terms that one would need to calculate to find the explicit action or Hamiltonian for the reduced system.

Consider the two body problem in general relativity for example, in the limit where we take both bodies to be point particles and we disregard spin, such that the total action reads (see e.g. [91])

$$S = \int \mathrm{d}^4 x \sqrt{g} R - \sum_a m_a \int \mathrm{d}\tau_a \sqrt{-g_{\mu\nu}(x_a^\lambda)\dot{x}_a^\mu \dot{x}_a^\nu}\,, \tag{3.3}$$

with g the metric, R the curvature scalar, $a = 1, 2$ and τ_a is the world line parameter of body a. Note that we leave out some constants for legibility. While we may use reduction over translations in order to express the integrands in center-of-mass coordinates x^μ, we cannot easily rid ourselves of the two world line integrals without changing the physics. Moreover, if one were to try to establish a Hamiltonian for this system, in order to apply the machinery of the last chapter, the particle Lagrangians are required to be part of the same integrand. This entails writing for example a factor $\frac{\mathrm{d}\tau_1}{\mathrm{d}\tau_2}$ in the first of the two particle terms, parametrising it with τ_2. This function we do not know a priori, making it impossible to explicitly write down the Hamiltonian.

All systems with multiple world lines suffer from this, unless all but one of the world line integrals can be solved without resorting to the solution of the variational problem itself. That exceptional situation in our two-body example occurs for example when one body is infinitely heavier than the other: $m_2 \gg m_1$, implying the metric it feels is independent of the coordinates of the other body. The velocity of the heavy body will necessarily be constant (or zero in the center-of-mass frame) and the integral becomes the mass term only $m_2 \int \mathrm{d}\tau_2 \sqrt{-g_{\mu\nu}(x_2^\lambda)\dot{x}_2^\mu \dot{x}_2^\nu} = m_2$, because one can choose to parametrise the integral by the proper time of m_2. Another example would be the free case, in which neither particle is feeling the other, because both have negligible mass.

This multi-world-line problem is not limited to gravitational theories: it is simply the consequence of the fields being determined by the movement of particles and the velocity of information being finite [135]. A similar action to the above for electromagnetism was for example constructed in 1929 by [114].

The inability to write the relativistic two body problem as a central potential problem points to the more general fact that there is no closed form solution for the fields describing two moving comparable masses. What we do know, is that the phase space of a two-body problem can be reduced to one spanned only by the relative momentum and position (barring situations where translations are not a symmetry of the theory to begin with), making it 6-dimensional, and 3 angular momentum components are conserved along with energy in a conservative problem. This leads to a tragic situation in some sense: The conservative relativistic two-body problem with spherical symmetry would be superintegrable, if only we could write down its Hamiltonian.

The only known way to approach the problem is through approximations like the post-Newtonian and post-Minkowskian approximations for gravitational theories. These ideas essentially repackage separate, truncated expansions of the two integrals above in varying ways.

As a unifying picture to the problem we want to approach, scattering amplitudes and effective field theory (EFT) will be useful. Just as the classical, that is, non-quantum, components of scattering amplitudes can be determined from a classical Hamiltonian, one can also determine the classical Hamiltonian from scattering amplitudes [124, 74]. EFT adds the organising principle of scale separation to the field theoretic techniques, making a strong foundation for calculation and conceptual understanding.[5] This picture will be used, to the extent that we

[5] While the use of field theoretic techniques in GR is ongoing since the 1960s [32], the application of EFT

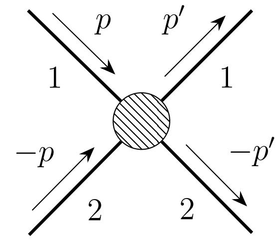


Figure 3.2: Time from left to right, scalars given by solid lines. The blob represents the scattering amplitude.

need it, to discuss the post-Minkowskian and post-Newtonian approximations, both of which have a clear interpretation in terms of amplitudes, as well as the fixing of redundant degrees of freedom in general two-body Hamiltonians.

3.1.1 Scattering amplitudes and EFT for classical binaries

Before starting our discussion of specific approximations and how Hamiltonians are derived in these, let us consider how scattering amplitudes and effective field theory can be used to constrain the possible shapes such Hamiltonians can take. Instead of reviewing all techniques in detail, we will only present some insights gained from these tools. For this Section, much use is made of [24], aimed primarily at the post-Minkowskian approximation. A review of these techniques in the context of the post-Newtonian approximation is e.g. [158].

From the point of view of quantum field theory, every interaction can be described by scattering of elementary particles, which are excitations of the fields. In the case of gravity, this field is the metric and the massless excitation is called the graviton. For two-body problems, there would additionally be 2 enormously heavy excitations of some scalar matter field. Through $2 \to 2$ scattering amplitudes, involving one or multiple gravitons in intermediate states, these then interact with each other, see Figure 3.2.

Because of the enormity of the 2 bodies compared to quantum scales, the 'quantum' part of the quantum field theory describing these excitations would be wholly redundant.[6] This is where effective field theory comes in, as there is a clear hierarchy between the size of the momentum $|p|$ of the bodies and the size of the momentum transfer $|p - p'|$ carried by internal gravitons, i.e.

$$|p| \gg |p - p'| \,. \tag{3.4}$$

Since individual scattering processes making up the total interaction of the binary are so small, we can average over their effects to retrieve a very reasonable approximation. From this averaged behaviour of the scattering of the 2 bodies, we can then learn which classical potential is resulting from the smaller interactions.

At first glance, it may seem strange to look at scattering amplitudes, with bodies in free states before and after the event, while being mostly interested in bound systems[7] (after all,

in the context of the two-body system, proposed in [124], is relatively new [112].

[6] In particular, particle production effects for the 2 classical bodies are not to be expected.

[7] The term 'bound' is borrowed from atomic physics, in which electrons are bound to a nucleus. In our

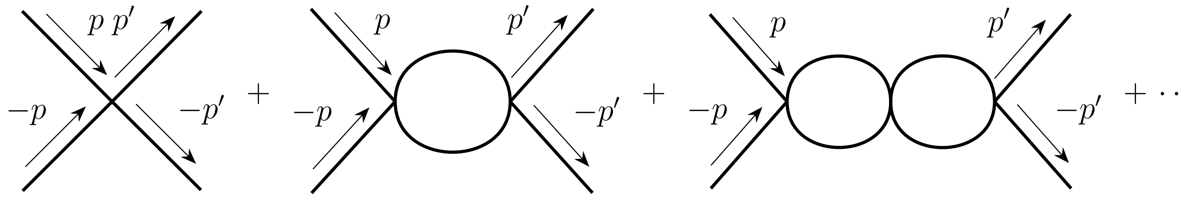
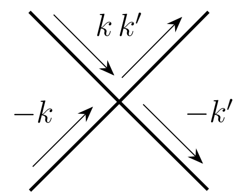


Figure 3.3: The scattering amplitude of the EFT is an expansion in terms of bubbles [24].

those are the systems we observe in LIGO, VIRGO and the like!). However, the underlying physics of both processes is the same, meaning studying scattering, we can find the potential that holds for bound systems as well.

Procedurally, one would first calculate the $2 \to 2$ scattering amplitudes in the EFT, built as a generic classical theory with a potential Ansatz $V(k, k')$, giving a vertex rule [74]



$$= -iV(k, k')\,, \tag{3.5}$$

where we use momentum labels k, k' to emphasise this rule holds for all momenta, not just the external ones. The scattering amplitudes for the EFT can be expanded purely in terms of bubbles, as in Figure 3.3, since in the EFT particle number is conserved [24]. Then, the free coefficients in the Ansatz can be fixed by calculating the amplitudes in the full theory and matching this to the EFT amplitudes [124, 24].

Now what does all this mean for the shape of the Hamiltonian, and more specifically, the potential? In principle, the most general Hamiltonian one could reasonably expect for the relativistic two-body problem, as mentioned above, will in reduced form only be dependent on relative momenta p and positions q. Moreover, because of spherical symmetry inherited from the Poincaré group, in the center-of-mass frame a general two-body Hamiltonian can be written solely in terms of the $SO(3)$ invariants p^2, $\sqrt{q^2} = r$ and $(p \cdot q)$. This would suggest a general form of the Hamiltonian [194][8]

$$H = \mu \sum_{(l,m,n)} \alpha_{l,m,n} \frac{(p^2)^l (p_r^2)^n}{r^m}\,, \tag{3.6}$$

writing the radial momentum $p_r = \frac{q \cdot p}{r}$.

However, calculating a Hamiltonian with the method outlined above based on the scattering amplitudes in a theory, we see this can be constrained further. Firstly, it is clear that a reasoning similar to that leading up to (3.6) must hold for scattering amplitudes: given $2 \to 2$ scattering as in Figure 3.2, the only dependence can be on invariant scalars p^2, p'^2 and $p \cdot p'$,

context it means systems with bounded orbits or objects kept together through some force.

[8]Non-integer powers of these scalars in the Hamiltonian are not possible in the context of conservative, non-spinning systems. This follows from the vertex rules and velocity corrections as we will see soon.

of the incoming momentum p and outgoing momentum p' (recall that we are in center-of-mass frame, hence $p_1 = -p_2 = p$). Moreover, by energy conservation, we know that the energy of one body, say m_1, is given by $E_1^2 = m_1^2 + p^2$ before and the same after, implying $p^2 = p'^2$, reducing the number of independent scalars to 2.

We can then take a basis in which $\frac{1}{2}(p^2 + p'^2) = p^2$ and $|p - p'|$ are the independent scalars building the amplitudes, since $\frac{1}{2}(p - p')^2 = p^2 - p \cdot p'$. The latter independent scalar, $|p - p'|$, is called the momentum transfer. Functions $f(|p - p'|)$ Fourier transform to functions $F(r)$ of the distance r between the objects, conjugate to the momentum transfer. This means there is a choice of field basis for scattering amplitudes, or equivalently, a choice of canonical coordinates, that allows us to build them solely from p^2 and $\frac{1}{r}$.[9] This implies that the coefficients in the potential, too, can be written in terms of p^2 and r. As we already know how we can write the kinetic part of the Hamiltonian as function of p^2, we have generally [194]

$$H = \mu \sum_{(l,m)} \alpha_{l,m} \frac{(p^2)^l}{r^m} \,. \tag{3.7}$$

This choice for canonical coordinates is called *isotropic gauge*.[10] This gauge choice still does not completely fix all unphysical degrees of freedom [93] in the Hamiltonian. One way to fix the residual degrees of freedom, is to require that the Hamiltonian, in the limit that the objects are infinitely far away from each other, reduces to a free Hamiltonian [74]. This can be written as

$$H_{\text{AG}}(r,p) = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} + \mu V(r,p)\,, \qquad V(r,p) = \sum_{l=0,m=1}^{\infty} \alpha_{l,m} \frac{(p^2)^l}{r^m}\,, \tag{3.8}$$

where $V(r,p)$ is the two-body potential and $\alpha_{l,m}$ are functions of the mass ratio ν. This is called *amplitude gauge* by [93], as will not surprise the reader, as this contains exactly the (Fourier transformed) potential Ansatz $V(r,p)$ one would take in order to map scattering amplitudes to a classical potential [74]. In so choosing the form of the Hamiltonian, we fix the $\alpha_{l,0}$ to be those as given by the expansion of the kinetic terms.

Another option, as argued by [93], is to write the Hamiltonian in a combined expansion in the Kepler Hamiltonian $H_{\text{Kep}} = \frac{p^2}{2} - \frac{1}{r}$ and $\frac{1}{r}$, and then fixing all terms in that expansion proportional to $\frac{1}{r}$ to zero, yielding

$$H_{\text{LRL}}(r, H_{\text{Kep}}) = \sum_{\substack{l=0,m=0\\ m\neq 1}}^{\infty} \bar{\alpha}_{l,m} \frac{H_{\text{Kep}}^l}{r^m}\,. \tag{3.9}$$

This is called *LRL gauge*, because it immediately shows conservation of the Newtonian LRL vector when all terms with powers of $\frac{1}{r}$ are absent, as H_{Kep} commutes with the LRL vector. And vice versa, in this gauge, all systems with additional symmetry are simply functions of the Kepler Hamiltonian [93].

[9] To add some more intuition, the external particles have just one independent momentum scalar: p^2. However, the exchanged particles are not on shell. This gives another independent momentum scalar, $|p - p'|$, being the magnitude of the exchange momentum, see e.g. [112].

[10] The word gauge is often used for a coordinate choice, as in the context of GR general coordinate transformations are recognised as a kind of gauge transformations: symmetries that arise solely because of our wish to write down something in terms of coordinates.

3.1.2 Post-Newtonian approximation

The post-Newtonian approximation is an expansion in terms of the inverse of the speed of light. The expansion terms can be corrections to the classical kinetic energy of the form $\frac{p^2}{m^2c^2}$. A relativistic free particle, for example, has Hamiltonian

$$H_{\text{fp}} = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} = mc^2\left(1+\frac{p^2}{2m^2c^2}-\frac{p^4}{8m^4c^4}+\mathcal{O}\Big(\frac{p^6}{m^6c^6}\Big)\right) . \tag{3.10}$$

The leading term in this expansion gives the rest-mass energy mc^2. But also other dimensionless parameters can be expanded in the inverse of the speed of light and are treated as similar in size, such as $\left(\frac{GM\omega}{c^3}\right)^{2/3}$ and $\frac{GM}{Rc^2}$, where Newton's constant G, a mass M, frequency ω and scale R are introduced. Here the equivalence $\omega^{2/3} \sim 1/R$ comes from Kepler's third law, while the equivalence $\frac{p^2}{m^2c^2} = \frac{v^2}{c^2} \sim \frac{GM}{Rc^2}$ is also known as the virial theorem at the Newtonian level. In physical terms (and in the context of conservative dynamics[11]), the expansion assumes the field to be weak and the speeds of the compact bodies to be much smaller than the speed of light.

In this approximation, observables like scattering angles or the perihelion precession can be calculated, or functions like the Hamiltonian or Lagrangian. The route that is historically taken (for example by Lorentz & Droste in 1917, translated version [161]) is to expand Einstein's field equations, solve them for a certain source term (two compact objects in the binary case) and analyse what compact-body Lagrangian or Hamiltonian would give rise to the equations of motion. The conservative two-body Lagrangian is known today to fourth post-Newtonian order, and has been calculated using EFT methods [111, 113].

To mathematically make sense of this approximation, we will give a formal definition of post-Newtonian expansion of functions on phase space, which will be useful in Chapter 4, taken from [180]. Denoting $\epsilon = \frac{1}{c^2}$ and ignoring other constants, we have the following.

Definition 7. *A Hamiltonian function $B(\epsilon; q, p)$ depending on a small parameter $\epsilon > 0$ is in **post-Newtonian expansion to Nth order** if it is of the form*

$$B(\epsilon; q, p) = \sum_{j=0}^{N} \epsilon^j B_j(q, p) + \mathcal{O}(\epsilon^{N+1}) , \tag{3.11}$$

for some regular enough Hamiltonian functions B_j.

This gives us an order counting system for terms occurring in a relativistic Hamiltonian. Such a Hamiltonian is given by

$$H_{\text{rel}} = \sum_{j=0}^{\infty} \epsilon^j \Lambda_j(\alpha), \qquad \Lambda_j(\alpha) = \sum_{\substack{(l,m,n)\in\mathbb{N}^3 \\ l+m+n=j}} \alpha_{l,m,n}\frac{(p^2)^l(p_r^2)^n}{r^m} , \tag{3.12}$$

written in general gauge here. Clearly, the PN orders of $\Lambda_j(\alpha)$ will be exactly j.

[11]For non-conservative systems, there is another expansion: the near-zone expansion, which again amounts to keeping track of powers of $\frac{1}{c^2}$ [39].

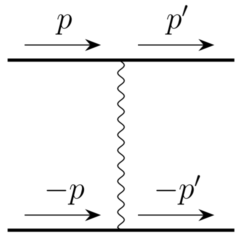


Figure 3.4: The tree-level diagram, with thick lines representing the compact objects and the wiggly line the exchanged graviton.

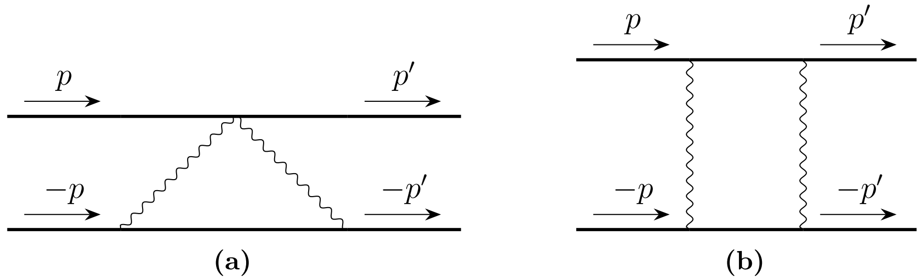


Figure 3.5: The triangle and box diagrams which contain the classical contributions at one-loop.

From an effective field theory perspective, the expansion in $1/c^2$ corresponds to an expansion in progressively more complicated Feynman diagrams of the full theory, as reviewed e.g in [112], where EFT is applied to the Binary problem in the post-Newtonian approximation. Since each scalar-scalar-graviton vertex contributes a factor G/c^2, and a cubic graviton vertex does so as well, the more loops in a diagram, the higher the order of the contribution. The simplest diagram of two classical scalars interacting through gravitons, is the one in Figure 3.4, a tree-level diagram. This is therefore the diagram that produces the $\frac{1}{r}$ term in the Kepler Hamiltonian.

However, since the vertices as well as the propagators are velocity-dependent, lower order diagrams keep contributing when calculating the higher order effective action or Hamiltonian. For example, while the Newtonian order is given by Figure 3.4, the 1PN order is given by Figure 3.5, as well as the first order velocity corrections to Figure 3.4. The diagrams in Figure 3.5 are given in a particular decomposition of the full amplitude, such that the one-loop with graviton 3-vertex can be ignored [24].

Written in the center of mass frame and normalising the relative momentum and radial coordinate conveniently, the Hamiltonian up to and including 1PN becomes [178, 180]

$$H_{(0+1)\mathrm{PN}} = \frac{1}{2}p^2 - \frac{\alpha}{r} + \frac{1}{c^2}\left\{\frac{1}{8}(3\nu - 1)p^4 - \frac{(3+2\nu)\alpha}{2r} + \frac{(\nu+1)\alpha^2}{2r^2}\right\}, \tag{3.13}$$

where $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$ denotes the so-called symmetric mass ratio and we have set $\mu = \frac{m_1 m_2}{(m_1+m_2)} = 1$. Here we see the two expansions, related through the virial theorem, taking place: one for the weak field which we have for large distances and small $1/r$ (shorthand for $\frac{GM}{Rc^2}$) and one for the small speed, corresponding to small p^2 (or $\frac{v^2}{c^2}$). The term in the middle is a mix of these two, stemming from the velocity corrections to the tree-level diagram 3.4.

For non-conservative systems, which the Binary Problem in General Relativity is due to the emission of gravitational waves, also terms proportional to odd powers of $\frac{1}{c}$ contribute. The first order at which corrections to the equations of motion due to radiation become important is $1/c^5$ or 2.5PN [39]. Current state-of-the-art calculations of the waveform of compact binaries has been carried out to 4.5PN [40].

3.1.3 Post-Minkowskian approximation

The post-Minkowskian approximation is an expansion in the gravitational constant G.[12] Clearly, such an expansion is also used in the previous case. However, in the post-Minkowskian approach there is no small speed approximation, so all results are valid to all orders in velocity. In the language of EFT and Feynman diagrams, the post-Minkowskian expansion is a loop expansion: each attached graviton leg carries a factor $\sqrt{G}$, and the gravitons are all considered internal, hence they need to be attached on both ends. The contributing diagrams are those with the correct number of loops, so Figure 3.4 at 1PM order (containing Newton and velocity corrections to all order in v^2/c^2) and Figure 3.5 at 2PM.

The first post-Minkowskian Hamiltonian is given by

$$H_{(0+1)\mathrm{PM}} = K_1 + K_2 + \frac{G}{r}\frac{1}{K_1 K_2}\left(m_1^2 m_2^2 - 2\left(K_1 K_2 + p^2\right)^2\right), \qquad K_i = \sqrt{m_i^2 + p^2}, \quad (3.14)$$

where we use a gauge in which the flat-space limit is the special-relativistic center-of-mass Hamiltonian. As we take a limit where the momentum goes to zero, we retrieve the rest mass term and the Kepler potential for a reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ orbiting a central mass $(m_1 + m_2)$.

An example of the use of the post-Minkowskian expansion is [29], where the third order conservative Hamiltonian is found using the methods described hand-wavingly here. The included type of terms in post-Newtonian and post-Minkowskian Hamiltonians for the binary problem follow a structure as in Figure 3.6.

From the above discussion, it may seem like the Nth post-Newtonian expansion is always contained within the $(N + 1)$th post-Minkowskian expansion, but this is not always true. An enlightening example is found in [89]. Here the scattering angle in the two body problem is written as an expansion of dimensionless rescalings of the angular momentum and non-relativistic energies, $j \equiv \frac{cJ}{G\mu M}$ and $\hat{E} \equiv \frac{E}{\mu c^2}$ respectively in their notation, with J the angular momentum. While every post-Minkowskian order $\sim G^n$ contains an infinite sum of orders of $\frac{1}{c^2}$, a post-Newtonian order $\sim (\frac{1}{c^2})^n$ similarly contains an infinite sum of orders of G. This is

[12] While the post-Newtonian expansion can be directly used in the context of other theories than GR, the post-Minkowskian is specific to GR. However, replacing G with the relevant coupling constant of another theory, one can obtain a similar expansion.

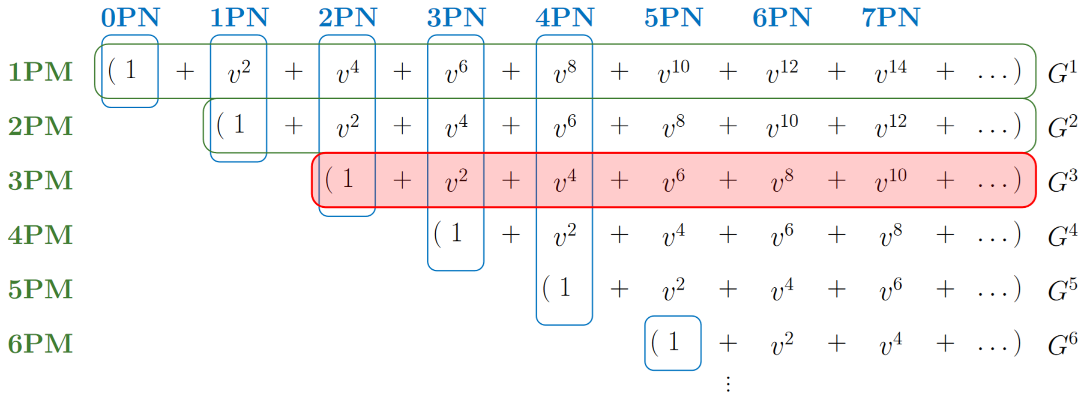


Figure 3.6: A schematic picture of the types of terms present in post-Newtonian and post-Minkowskian expansions, showing progress on the binary problem potential (recently extended to 4PM [27]). Taken from [24].

because we can write the scattering function in a combined expansion

$$\begin{aligned}\chi(j,\hat{E}) = &\frac{1}{j}\left(\chi_{1,-1}\hat{E}^{-(1/2)} + \chi_{1,0} + \chi_{1,1}\hat{E}^{1/2} + \dots\right)\\ &+ \frac{1}{j^2}\left(\chi_{2,-2}\hat{E}^{-1} + \chi_{2,-1}\hat{E}^{-(1/2)} + \chi_{2,0} + \dots\right)\\ &+ \frac{1}{j^3}\left(\chi_{3,-3}\hat{E}^{-(3/2)} + \chi_{3,-2}\hat{E}^{-1} + \chi_{3,-1}\hat{E}^{-(1/2)} + \dots\right)\,,\end{aligned} \tag{3.15}$$

with $\chi_{n,k}$ functions of the mass ratio. Here the first terms in each bracket contribute at Newtonian order, while the second terms each contribute at $\frac{1}{2}PN$ etc. Similarly, the first line gives the 1PM contributions, the second gives the 2PM terms. Because of the negative powers in $\hat{E}$, the number of terms contributing at NPN order indeed contain terms not present in the $(N+1)$PM order, e.g. the term $\sim \frac{\hat{E}^{-1}}{j^{(N+2)}}$. These do not necessarily vanish, see for example [34].

3.2 Relativistic one-center problems

Now we will direct our attention to relativistic incarnations of the Kepler problem, essentially the extreme mass-ratio limit of the two-body problem. To facilitate comparison with relativistic systems, one can glance back at Example 1, where we employed the Binet equation to find the shape of orbits in the classical Kepler case. We will now do the same for physically distinct relativistic corrections, showing how the classical ellipses are distorted in each case as parametrised by the strength of the interaction. This interaction strength is determined by considering the strength of the force experienced by a particle starting off stationary.

3.2.1 Maxwell

Since we want the particle Lagrangian of relativistic particles to be independent of the parametrisation, we generally write a square root for the kinetic term. However, sometimes it is useful for calculations to do without such a term. To still describe the physical Lagrangian, independent of parametrisation, we therefore need to manually put in this redundancy, by introducing an auxiliary variable h. For a relativistic particle in an electromagnetic field in flat space, the Lagrangian is then given by e.g. [205, 129]

$$\mathcal{L} = \frac{1}{2h}\dot{x}^\alpha \dot{x}^\beta \eta_{\alpha\beta} - \frac{hm^2}{2} - qA_\mu \dot{x}^\mu, \tag{3.16}$$

where the dot represents derivation with respect to the parameter λ, q its charge and $\eta_{\alpha\beta}$ the Minkowski metric. Indeed, the action built from this Lagrangian is invariant under a reparametrisation, combined with a transformation of the auxiliary variable

$$\lambda \to \tilde{\lambda}(\lambda)\,, \qquad h \to h \left(\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\lambda}\right)^{-1}\,, \tag{3.17}$$

with $\tilde{\lambda}(\lambda)$ a monotonic function of λ. Actions with this form for the Lagrangian are sometimes called a 'Polyakov-type' actions, and h is also called 'Einbein'. The familiar square-root form can be retrieved by solving the Euler-Lagrange equation for h and substituting it back in. However, when it is useful to, we will choose instead to parametrise affinely such that $h = \frac{1}{m}$ and we get

$$\mathcal{L} = \frac{m}{2}\dot{x}^\alpha \dot{x}^\beta \eta_{\alpha\beta} - \frac{m}{2} - qA_\mu \dot{x}^\mu\,. \tag{3.18}$$

Note that the Euler-Lagrange equation of h implies that

$$h^2 = -\frac{1}{m^2}\dot{x}^\alpha \dot{x}^\beta \eta_{\alpha\beta}\,, \tag{3.19}$$

such that the choice $h = \frac{1}{m}$ demands the constraint

$$-\eta_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 1\,, \tag{3.20}$$

which is the statement that the parameter λ is affine, in the case of a massive particle.

Assuming the static Coulomb potential $qA = (m\alpha/r, 0, 0, 0)$, we can calculate the force this particle will experience at rest to be

$$\ddot{r} = \frac{\alpha}{r^2}\hat{r}\,, \tag{3.21}$$

which is attractive for α negative, meaning the charges have opposite signs. Here we have made the choice to have a positive sign for $\dot{t}$, as we will always do unless explicitly mentioned.

The conjugate momenta to ϕ and t read

$$L = mr^2\dot{\phi}\,, \qquad E = -m\left(\dot{t} + \frac{\alpha}{r}\right)\,, \tag{3.22}$$

which implies we can write the velocity norm as

$$-\left(\frac{E}{m} + \frac{\alpha}{r}\right)^2 + \dot{r}^2 + \frac{L^2}{m^2 r^2} = -1\,. \tag{3.23}$$

Then changing to the Binet variable we find

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + c_2 u^2 - c_1 u = c_0 \,, \qquad \text{with } c_1 = \frac{2E\alpha m}{L^2},\ c_0 = \frac{E^2 - m^2}{L^2},\ c_2 = \left(1 - \frac{\alpha^2 m^2}{L^2}\right) . \tag{3.24}$$

Comparing this to the Binet equation for the Kepler problem, we see the coefficient of u^2 is no longer unity. This requires an additional constant in our equation, so that we get

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + \left(\sqrt{c_2}u - \frac{c_1}{2\sqrt{c_2}}\right) = c_0 + \left(\frac{c_1}{2\sqrt{c_2}}\right)^2 . \tag{3.25}$$

This is solved by

$$u = A(1 + e\cos\sqrt{c_2}\phi) \,, \tag{3.26}$$

which comes back to the same value when $\sqrt{c_2}\phi = 2\pi$, meaning the perihelion shift is

$$\Delta\phi = \frac{2\pi}{\sqrt{c_2}} - 2\pi \approx \pi\frac{\alpha^2 m^2}{L^2} \,, \tag{3.27}$$

with the approximation to first order, agreeing with [205].

3.2.2 Einstein

Another way to generalise the classical central potential problem is by considering the static, spherically symmetric solution of Einstein's field equations. This is fundamentally different from the electromagnetic case, because the force involved in the interaction is a different force, of a different spin. This can be seen from the number of indices, determining the way a field transforms under the Lorentz group, which make sure that after quantisation of the fields A_μ and $g_{\mu\nu}$ their operators give spin-1 and spin-2 excitations respectively, while a scalar, such as the dilaton we will see next, has spin-0.

The relevant solution is the Schwarzschild metric

$$g = -\left(1 - \frac{r_s}{r}\right)dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,, \tag{3.28}$$

with the Schwarzschild radius $r_s = \frac{2GM}{c^2}$. Taking a massive particle in such a background gives the Lagrangian

$$\mathcal{L} = \frac{m}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \frac{m}{2} \,, \tag{3.29}$$

where once again we have parametrised by proper time, meaning $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$. The static force is then

$$\ddot{r} = -\frac{r_s/2}{r^2}\hat{r} \,. \tag{3.30}$$

The angular momentum and energy are conserved here too, so that after a bit of rewriting we find the Binet equation, with a new coefficient

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2 - c_3 u^3 - c_1 u = c_0 \,, \qquad \text{with } c_1 = \frac{r_s m^2}{L^2},\ c_0 = \frac{E^2 - m^2}{L^2},\ c_3 = r_s \,. \tag{3.31}$$

As a comparison with the Kepler system shows (or an analysis tracking the occurrence of $1/c^2$ factors), the term with c_3 is a relativistic correction. Therefore, we can solve the equation perturbatively as demonstrated for instance in [207]. They find

$$\Delta\phi = \frac{3\pi}{2} c_1 c_3 = 6\pi \frac{(r_s/2)^2 m^2}{L^2}\,, \tag{3.32}$$

which is, under proper exchange of constants given by the attractive force, 6 times that of the electromagnetic case we have seen before.

3.2.3 Dilaton

A third force is the so-called dilaton. A scalar field that scales, or dilates, the kinetic term of a particle as a function of its position. It was an early contender for the relativistic theory of gravity [183, 102], but got rejected based on its predictions not matching reality, chief among which was the perihelion shift [94, 220].

For the one-center case of purely dilatonic gravity, we consider the relativistic Lagrangian for a free particle on a metric that is flat, except for a rescaling dependent on the dilaton field φ.[13] The particle Lagrangian reads

$$\mathcal{L} = \frac{m}{2} e^{2a\varphi} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{m}{2}\,, \tag{3.33}$$

where $\eta_{\mu\nu}$ is the flat metric with $(-\ +\ +\ +)$ on the diagonal and a is the coupling constant.

To see what the φ field is when we place a single, large point particle at the origin, we need to first note there is some ambiguity when talking about a dilaton field. The only way to couple matter to a scalar field is multiplying the kinetic term, because there are simply no indices available to combine with velocities as we have in the electromagnetic case and in the usual GR case. This means that the dilaton field functions as a conformal factor for the metric, and can equivalently be seen from a perspective incorporating it into the metric $\tilde{g}_{\mu\nu} = e^{2a\varphi}\eta_{\mu\nu}$. This is called the Jordan frame, and it changes the field action involved to one containing both gravity terms (due to the now non-flat metric) and dilaton terms, making it a kind of scalar-tensor theory.

In the frame we will consider, called Einstein frame, the field Lagrangian is given by

$$\mathcal{L}_{\text{field}} = -2(\partial\varphi)^2\,, \tag{3.34}$$

meaning we only have a dilaton field. Similar to before, this leads to a vacuum equation where we pick out the solution corresponding to a single charge at the origin, reading

$$\varphi = -\frac{aM}{r}\,, \tag{3.35}$$

with M the mass. Using this solution and the particle Lagrangian (3.33), we can derive the static force

$$\ddot{\vec{r}} = -\frac{a^2 M}{r^2}\hat{r}\,, \tag{3.36}$$

[13] It can be seen as the Goldstone boson of spontaneously broken conformal symmetry [132].

to first order in the PN expansion. Regardless of the sign of the dilaton coupling, this force is attractive.

Writing in spherical coordinates we spot the conserved angular momentum and energy

$$L = me^{2a\varphi}r^2\dot{\phi}\,, \qquad E = -me^{2a\varphi}\dot{t}\,. \tag{3.37}$$

Proper time parametrisation then gives

$$-\left(\frac{E}{m}\right)^2 e^{-2a\varphi} + \dot{r}^2 e^{2a\varphi} + \frac{L^2}{m^2r^2}e^{-2a\varphi} = -1\,, \tag{3.38}$$

yielding a binet equation, expanding the dilaton field term to first PN order:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + c_2u^2 - c_1u = c_0\,, \qquad \text{with } c_1 = \frac{2a^2Mm^2}{L^2}\,,\ c_0 = \frac{E^2-m^2}{L^2}\,,\ c_2 = \left(1 + \frac{2a^4M^2m^2}{L^2}\right). \tag{3.39}$$

The perihelion shift for a dilatonic force will therefore be

$$\Delta\phi = -2\pi\frac{a^4M^2m^2}{L^2}\,. \tag{3.40}$$

By making an appropriate exchange of constants (indicated by the static forces), we can compare this to the one of the electromagnetic force. As it turns out, the dilaton has a perihelion shift opposite to the one of the electromagnetic force and twice as large.

It is interesting to note however, that this result is not independent of the way we define the coupling of the scalar field to begin with, as discussed for example in [94]. Taking the field Lagrangian (3.34) as given, implying the vacuum solution is always as in (3.35), one can couple the scalar to matter in various ways parametrised by function $F(\varphi)$ in

$$\mathcal{L} = \frac{m}{2}\left(1 + F(\varphi)\right)\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{m}{2}\,. \tag{3.41}$$

Choosing $a = 1$ in the exponentially coupled case above (known as Nordströms first theory), this function reads

$$F(\varphi) = \varphi + \frac{b}{2}\varphi^2 + \dots\,, \tag{3.42}$$

with $b = 1$, whereas for Nordströms final theory it is simply $F(\varphi) = \varphi$, leading to $\Delta\phi = -\pi\frac{M^2m^2}{L^2}$: a measurably distinct prediction (in a hypothetical universe where such a scalar is present). In general we have to first order [94]

$$\Delta\phi = -\frac{1+b}{6}\Delta\phi_{\mathrm{GR}}\,, \tag{3.43}$$

with $\Delta\phi_{\mathrm{GR}}$ the perihelion shift as in GR. In particular, this implies we can cancel the first order perihelion shift, by taking $b = -1$. The higher order terms in $F(\varphi)$ similarly contribute to higher order perihelion shifts, however, such that one would need to specify the full function to find the all-order shift. As we will see in Chapter 5, it is indeed possible to find the full function of the field φ needed to close the ellipses to all orders.

Additionally, and interestingly in the light of the discussion surrounding Nordström's scalar theory in the years preceding GR, we can also exactly mimic the GR prediction, by taking $b = -7$. Still, other predictions, such as the deviation of light due to the presence of mass, remain different even with this choice, such that it would still be possible to distinguish a scalar theory from GR experimentally.

Let us conclude the discussion of scalar gravitational theories by explicating that the name dilaton is usually reserved for the exponentially coupled theory. While the dilaton is the Goldstone boson of conformal transformations [203], and both scalar theories mentioned here satisfy the weak equivalence principle stating that inertial and gravitational masses are the same for non-gravitationally bound bodies, only Nordström's final theory is argued to satisfy the strong equivalence principle, including self-gravitating bodies [94].

3.2.4 Maxwell-dilaton

As the perihelion shifts of the dilaton and Maxwell forces are opposite in direction of precession, it is natural to suppose there might be a balancing theory, combining both fields in such a way as to make the shift vanish. Let us therefore consider a naive combination of both theories, where the vector and scalar fields do not couple to each other. Constructing a one-center problem in such a theory amounts to placing a charge at the origin, coupling it conformally to a scalar field.

The calculation of the perihelion shift in the orbit of another charge, also conformally coupled to the scalar, is then to first order just a combination of the separate effects. This shows that, at least to first order in the post-Newtonian expansion, the perihelion shift can vanish for the correct size of the coupling constant a such that $\frac{m^2}{L^2}(\alpha^2 - 2a^4M^2) = 0$. However, this is only to first order. When we expand the calculation to 2PN, we no longer have the freedom to adjust the relative strength of the forces involved, and the perihelion precesses once more. The Binet equation, derived similarly to before, reads

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 - c_3u^3 + c_2u^2 - c_1u = c_0\,, \qquad \text{with } c_1 = \frac{2a^2Mm^2 + 2E\alpha m}{L^2},\ c_0 = \frac{E^2 - m^2}{L^2} \quad (3.44)$$
$$c_2 = \left(1 + \frac{(2a^4M^2 - \alpha^2)m^2}{L^2}\right),\ c_3 = \frac{4}{3}\frac{a^6M^3m^2}{L^2}\,.$$

Balancing the forces to make the first order perihelion shift vanish removes all c_2 contributions at higher order as well, but the contribution from c_3 (and higher orders) adds a term

$$\Delta\phi = 4\pi\frac{M^4m^4}{L^4}\left(a^8 + \frac{E\alpha a^6}{Mm}\right), \tag{3.45}$$

which does not vanish for the balancing value $a^2 = \sqrt{\frac{\alpha^2}{2M^2}}$. This shows that, at least in coupling the Maxwell and dilaton field in this naive way, there is no additional symmetry to orders past 1PN.

3.2.5 Einstein-dilaton

To address the probe limit (or test-mass limit) in a theory that combines a metric field with a dilaton, we cannot simply use known static background fields. This is because the known static, spherically symmetric solution, the Janis-Newman-Winicour (JNW) solution [143, 223, 213], is only defined for a subset of all dilaton couplings we would like to consider.[14] Moreover, it is not a black hole, but a naked singularity: the outer-most singularity in the coordinates of JNW is in fact a curvature singularity, implying it cannot be removed by a suitable coordinate redefinition in the way that the singularity on the boundary of the Schwarzschild black hole can. This is in line with the no-hair theorem, stating that black holes are completely described by their mass, angular momentum and possibly charge.

Since a sufficient solution for our purposes does not exist, our analysis will proceed perturbatively. Results obtained in this way in the present and many other scalar-tensor theories can be found in [92], and more recently the perihelion shift was presented too in [93], calculated through scattering amplitude methods.

To first post-Newtonian order, the tensor and scalar fields do not influence the form of their respective solutions. This means we can simply take the solutions as we saw them above. There is a caveat however, since the coordinate systems in which we gave the solutions are not the same as soon as we introduce a mass in the center. The Schwarzschild metric (3.28) was given in Schwarzschild coordinates, in which the coefficient of the spherical term in the metric is r^2, while the dilaton field was given in terms of an isotropic radius. To distinguish these from now on, we call the isotropic radius ρ. The Schwarzschild metric in isotropic coordinates has the form

$$g = -\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2}dt^2 + \left(1+\frac{M}{2\rho}\right)^4\left(d\rho^2+\rho^2 d\Omega^2\right)\,, \tag{3.46}$$

whereas the dilaton solution is still given by

$$\varphi = -\frac{aM}{\rho}\,. \tag{3.47}$$

A particle starting out static will then experience a force

$$\ddot{\rho} = -\frac{(1+a^2)M}{r^2}\hat{\rho}\,, \tag{3.48}$$

to first order, simply the addition of the effect of both forces.

The proper time parametrisation gives, restricting to the plane $\theta = \pi/2$,

$$-e^{2a\varphi}\left[\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2}\dot{t}^2 + \left(1+\frac{M}{2\rho}\right)^4\left(\dot{\rho}^2+\rho^2\dot{\phi}^2\right)\right] = -1\,, \tag{3.49}$$

[14] Furthermore, it lies on the wrong branch of solutions as viewed from the zero charge limit of the theory in the next subsection 3.3, see also Chapter 4.

from which the conserved quantities are calculated to be

$$L = e^{2a\varphi}\left(1+\frac{M}{2\rho}\right)^4 \rho^2\dot{\phi}\,, \qquad E = -e^{2a\varphi}\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2}\dot{t}\,, \tag{3.50}$$

such that the Binet equation becomes

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + c_2u^2 + c_1u = c_0\,, \qquad \text{with } c_0 = \frac{E^2-m^2}{L^2}\,,\ c_1 = \frac{2(1+a^2)m^2}{L^2}\,,$$
$$c_2 = 1 - \frac{2(3-a^2)(1+a^2)M^2m^2}{L^2}\,, \tag{3.51}$$

yielding a perihelion shift of

$$\Delta\phi = 2\pi\frac{M^2m^2}{L^2}(3-a^2)(1+a^2)\,. \tag{3.52}$$

As noted by [93], this first order perihelion shift vanishes for a specific tuning of the coupling $a = \sqrt{3}$, one not accessible in the JNW solution (a must be smaller than 1 in this case). The same perihelion can be found as the zero charge limit of the one for extremal Einstein-Maxwell-dilaton, see Section 4.3.1. However, beyond first order, once more no tuning of the parameters allows for exclusively closing orbits, as calculated by [93]. Moreover, the probe particle as well as the central mass are not properly 'matched' objects, in the sense of [145] that the internal structure of a point-like object is not trivial in scalar-tensor theory. Even for point-like objects, one typically needs an additional parameter β to describe the coupling to the dilaton field [92]. This requirement crucially vanishes for the extremal objects in Einstein-Maxwell-dilaton theory, owing to the fact that these objects do not experience self-gravitational effects.

3.2.6 Einstein-Maxwell

Whereas the above combinations of fields, due to the negative perihelion shift typically generated by the dilaton field, could be expected to yield zero precession in some specific instances, this is not a realistic expectation for the Einstein-Maxwell case. Both fields lead to positive perihelion shift, and indeed, combining them only exacerbates the non-closing of the orbits. For completeness, however, we briefly discuss the orbits in a one-center system in this theory here.

The relevant solution to the Einstein-Maxwell equations goes by the name of Reissner-Nordström, and it is a charged, non-rotating, stationary, spherically symmetric black hole. The perihelion shift, for charge-to-mass ratios Q and q for the center and the probe respectively, is given to first order by

$$\Delta\phi = \pi\frac{M^2m^2}{L^2}\left(6(1-Qq) + Q^2q^2 - Q^2\right)\,, \tag{3.53}$$

which agrees with [15] under the assumption that $q \ll Q$. The only way to make this zero without resorting to $q > 1$ or $Q > 1$, is to take both charges extremal, that is $q = Q = 1$.

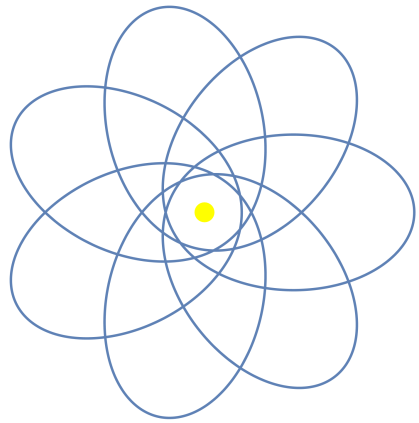


Figure 3.7: A conditionally periodic orbit: closing, yet without additional continuous symmetry.

Extremal charge is the maximal charge a black hole can have without becoming a naked singularity [147]. When both charges are extremal, the static force will vanish completely and bounded orbits are not possible.

3.2.7 Bertrand spacetimes

Another way to approach the question of which relativistic systems conserve classical symmetries, is to demand existence of the symmetries in the first place and backward-engineer the solutions capable of satisfying that demand. It is in this vein that Perlick constructed all Bertrand spacetimes, and pinpointed those with additional symmetry [189]. We will now give a brief overview of this work, highlight the case that is of interest to us, and mention a limitation.

Perlick constructs all Bertrand spacetimes, which is essentially the demand that all bounded time-like geodesics have vanishing perihelion shift.

Definition 8 ([189])**.** *A Lorentzian manifold* (M, g) *is a* ***Bertrand spacetime*** *if and only if*

1. *it is static and spherically symmetric*
2. *there is a circular trajectory through each point*
3. *any initial condition for the geodesic equation which is sufficiently close to that of a circular trajectory gives a periodic trajectory.*

As it turns out, this definition is enough to guarantee all Bertrand spacetimes have all bounded orbits closing, not just the (nearly) circular ones. This justifies the name referring to the classical Theorem 4.

However, there is a caveat: not all of the Bertrand spacetimes have strictly periodic orbits, which is required for additional symmetry. Strict periodicity is the situation where the body comes back to the same point after just one circling of the host, as opposed to a conditionally periodic orbit, where the perihelion shift is given by $2\pi/\beta$ with $\beta \neq 1$ and $\beta \neq 2$, such that the orbit forms a rosette closing after a number of revolutions around the center, as in Figure 3.7. The subset of spacetimes that does result in additional symmetry (in the form of Killing tensors, see 2.2.4), gives rise to Kepler- and harmonic oscillator-like ellipses.

Furthermore, we would like to specialise to those spaces which are asymptotically Minkowskian, meaning the geodesics far away from the origin, with $r \to \infty$, behave as in special relativity, since this is what we expect from a one-center-like system. Asymptotically Minkowskian Bertrand spacetimes are either of the form

$$g = -\frac{\mathrm{d}t^2}{G \mp r^2(1 + k^2r^2 \pm \sqrt{1 + 2k^2r^2})^{-1}} + \frac{2(1 + k^2r^2 \pm \sqrt{1 + 2k^2r^2})\mathrm{d}r^2}{1 + 2k^2r^2} + r^2\mathrm{d}\Omega^2\,, \quad (3.54)$$

where G and k are real constants with $Gk^2 > 1$ for the upper and $Gk^2 > -1$ for the lower signs, or

$$g = -\frac{\mathrm{d}t^2}{1 + kr^{-1}} + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \quad (3.55)$$

with $k > 0$ a real constant [189].[15] These particular gravitational fields then are isolated sources giving rise to bound systems with additional symmetry, the type of relativistic system we set out to find. However, they do not solve Einstein's equation in a vacuum, like the Schwarzschild solution, and moreover, they violate the so-called weak energy condition, meaning there are observers that will measure a negative energy density, which makes the sources quite exotic if not unphysical objects.

3.3 Extremal Einstein-Maxwell-dilaton systems

In the previous section, we have seen an array of different ways of relativising the classical one-center problem, and analysed whether these could give rise to exclusively closing bounded orbits. While it is possible to create situations in which the perihelion shift vanishes, it appears not so easy to do so to all orders (e.g. Maxwell-dilaton or Einstein-dilaton) or with objects obeying relations we expect from physical objects (Einstein-dilaton, Bertrand spacetimes).

In this section, we present a theory - Einstein-Maxwell-dilaton with dilaton coupling $a = \sqrt{3}$ - that does clear both bars raised above: the extremal-anti-extremal one-center system, obeying the expected field equations, has exclusively closing bounded orbits to all orders. Einstein-Maxwell-dilaton (EMD) theory is a generalisation of general relativity that is of interest due to its general nature of forces, comprising spin-0,1,2 background fields, as well as its ability to circumvent the 'no-hair theorem'. This states that a black hole cannot be described by properties other than its mass, charge, and angular momentum. EMD escapes this prohibition by introducing a non-trivial scalar field and charge.[16]

We will first discuss EMD in some generality, giving black hole solutions and the extremal backgrounds of interest. Then we proceed to ask the question what probe (or test) particles

[15] Note that the paper contains a mistake here: it excludes the lower signs in the first expression, while this is not necessary to achieve an asymptotically Minkowskian spacetime.

[16] This scalar field gives the black hole what is called *secondary* hair, as the scalar charge is completely determined in terms of mass and charge within a given theory, i.e. for a given value of the scalar coupling constant [80].

in such a theory would look like, and study integrability of the two-center system. As realised by [83], for a particular choice of coupling constants the two-center systems in EMD have surprising integrability properties, highly reminiscent of their classical equivalents. We will study this system for two kinds of test particles, massless and massive, and show how their Hamilton-Jacobi equations are separable - a sufficient condition for integrability, see Section 2.6.1. Implicitly, since one can adjust the two-center problem to make one of the centers irrelevant, we have then also shown the one-center problem in this EMD theory has the same integrability properties as the classical one. Subsequently, we take a look at the higher dimensional origin of EMD theory with our particular parameter tuning, hinting toward a deeper explanation which will be considered in later chapters.

Sections 3.3.1, 3.3.2 and 3.3.3 are adapted from [180], of which the remainder will be discussed in Chapter 4. The connection between the separation of the two-center system and the classical two-center system is made here for the first time.

3.3.1 Black holes with dilaton hair

The fields present in EMD theory are the metric $g_{\mu\nu}$, the four-potential A_μ and the dilaton field φ, exponentially coupled (through coupling constant a) to the electromagnetic field strength. Mathematically, these are respectively a pseudo-metric tensor, a covector and a function on the space-time manifold $\mathbb{R}^4$. A solution of the corresponding Einstein-Maxwell's equation is then given by the critical tensors of the action (see also e.g. [139, 84])

$$S[g_{\mu\nu}, A_\mu, \varphi] = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left(R - 2(\partial\varphi)^2 - e^{-2a\varphi}F^2\right)\,, \tag{3.56}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the pseudometric-tensor matrix, R is the Ricci scalar and $F = dA$ is Maxwell's field. Note the coupling between the dilaton and Maxwell fields, which makes this slightly different from the simple addition of the bulk Lagrangians we have seen so far. Different values of a correspond to different reductions of $N = 8$ supergravity [83], among which $a = \sqrt{3}$ will be the special value we will be interested in. In addition to diffeomorphism invariance and gauge symmetry, this action has a global symmetry that shifts the dilaton while rescaling the gauge vector.

Starting with the special case $a = 0$, the static and spherically symmetric solutions of this theory are given by the well-known Schwarzschild and Reissner-Nordström black holes, with possibly non-vanishing electric charge. In the electrically neutral case, the introduction of the dilaton does not introduce additional solutions; scalar-gravity is known to satisfy the no-hair theorem and hence cannot carry scalar charge [79]. In contrast, when introducing the dilaton (i.e. $a \neq 0$) in the charged case, the solution becomes more interesting and reads [119, 115]

$$\mathrm{d}s^2 = -\lambda^2\mathrm{d}t^2 + \lambda^{-2}\mathrm{d}r^2 + r^2\kappa^2\mathrm{d}\Omega^2\,, \qquad F_{tr} = \frac{e^{2a\varphi_0}Q}{r^2\kappa^2}\,, \qquad e^{2a\varphi} = e^{2a\varphi_0}\left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}}\,, \tag{3.57}$$

where

$$\kappa^2 = \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}}\,, \qquad \lambda^2 = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}}\,. \tag{3.58}$$

Note that one can set $\varphi_0 = 0$ by the shift symmetry of the dilaton, which we will subsequently do. This most general solution is parametrised by the locations of the inner and outer horizons $r_\pm$. These are related to the mass and charge of the object by

$$r_+ = M + \sqrt{M^2 + Q^2(a^2-1)}\,, \qquad r_- = \left(\frac{a^2+1}{a^2-1}\right)\left(-M + \sqrt{M^2+Q^2(a^2-1)}\right)\,, \tag{3.59}$$

Importantly, the 'horizons' labelled by the minus sign are singular for all $a > 0$ (i.e., the scalar curvature diverges at this point), whereas the ones labelled by the plus signs are not.

As mentioned above, this solution carries scalar charge, given by a simple integration over a spherical shell surrounding it [115]:

$$D = \lim_{\rho\to\infty} \frac{1}{4\pi} \oint \nabla^\mu \varphi \; \mathrm{d}^2\sigma_\mu = \frac{a}{a^2-1}\left(-M + \sqrt{M^2+Q^2(a^2-1)}\right)\,. \tag{3.60}$$

In order to have non-vanishing dilaton charge, one therefore needs both electric charge $Q \neq 0$ as well as non-vanishing scalar coupling $a \neq 0$. For a given theory and hence value of a, the mass and charge determine the dilaton charge, which is therefore not an independent parameter.

For completeness, we would like to mention that for the same set of charges (M, Q, D), a second solution exists, given by the above fields but with parameters

$$\tilde{r}_+ = M - \sqrt{M^2 + Q^2(a^2-1)}\,, \qquad \tilde{r}_- = \left(\frac{a^2+1}{a^2-1}\right)\left(-M - \sqrt{M^2+Q^2(a^2-1)}\right)\,, \tag{3.61}$$

where we have added a tilde to avoid confusion with the solutions that form our main interest. In the neutral case, these solutions are the Janis-Newman-Winicour solution for Einstein minimally coupled to a scalar field [166], as also referred to in the Einstein-dilaton discussion in Section 3.2.5. Note that they are in general different from Schwarzschild (when choosing $a \neq 0$); however, in this case the solution develops a naked singularity (that is, a non-removable singularity not cloaked by an event horizon). This is in agreement with the statement that scalar-gravity does not have any black hole solutions other than Schwarzschild. The introduction of the electric charge does not qualitatively change this singular property. For these reasons we will not consider this solution any further.

3.3.2 The extremal case

We now turn to the extremal case of the hairy black hole solutions (3.59). To this end, it is convenient to rewrite the relation (3.60) between the three charges as the quadratic relation

$$(D - aM)^2 = a^2(M^2 + D^2 - Q^2)\,. \tag{3.62}$$

The importance of the expression on the right-hand side lies in the extremality of the black hole. Imagine two such black holes; when this combination vanishes, attractive spin-0,2 forces between two such black holes (proportional to $M^2 + D^2$) would exactly cancel the repulsive spin-1 force (proportional to Q^2).

The dimensionless parameter

$$\chi^2 \equiv \frac{M^2 + D^2 - Q^2}{M^2}\,, \tag{3.63}$$

is therefore a measure of extremality, and interpolates between 0 and 1. When $\chi = 1$, this corresponds to a neutral black hole (i.e. the Schwarzschild solution). In contrast, the case $\chi = 0$ corresponds to an extremal black hole: in this case, the two sides of (3.62) vanish separately, and the black hole has extremal charges

$$D_{\mathrm{extr}} = aM\,, \qquad Q_{\mathrm{extr}} = \pm\sqrt{1+a^2}M\,, \tag{3.64}$$

that are both linearly proportional to the mass. For all values $a \neq 0$, the solutions will be singular in the extremal limit [184]. Moreover, the thermodynamics of such extremal objects are fundamentally different for $a \gtrless 1$ – in fact, it has been argued [139] that they resemble elementary particles more than black holes for $a > 1$. As this will be of no consequence for the dynamics, which is our concern here, we will still refer to these objects as black holes.

Due to the cancellation of forces between extremal black holes, one can also construct multi-center solutions. For the Einstein-Maxwell case, these are the Majumdar-Papapetrou solutions [170, 185], while the solutions with a non-minimally coupled dilaton field added in have been discussed in [84]. The fields in this case are given by

$$\begin{aligned} g &= -U^{-2/(1+a^2)}\mathrm{d}t^2 + U^{2/(1+a^2)}\mathrm{d}q\cdot\mathrm{d}q\,,\\ A &= \frac{1}{\sqrt{1+a^2}U}\mathrm{d}t\,,\\ e^{-\varphi} &= U^{a/(1+a^2)}\,, \end{aligned} \tag{3.65}$$

with

$$U(q) = 1 + (1+a^2)\sum_n \frac{M_n}{|q-q_n|}\,, \tag{3.66}$$

where the sum is over the extremal black holes with mass M_n and positions q_n of which there may be arbitrarily many. The no-force condition implies that all centers carry electric and dilaton charges (3.64) that are proportional to their masses. For a single charge, this solution corresponds to the extremal case of the general Einstein-Maxwell-dilaton metric. This can be seen by noting the two horizons of the EMD merge into $r_\pm = M(1+a^2)$ and switching to the isotropic radius $\rho = r(1 - \frac{r_\pm}{r})$.

3.3.3 Skeletonisation

Having discussed the relevant background solutions in EMD theory, we want to consider dynamical systems built by taking an extremal background and placing a test particle on it. To make our dynamical systems pertain to dilaton-charged black holes, simply taking a point particle with a mass and electric charge while keeping the universal dilaton coupling a nonzero does not suffice. For self-gravitating objects, even in the zero size limit, there will be a dependence on the background scalar field of the way the object couples to it. One can see this by considering the black hole presented in Section 3.3.1: both the dilaton charge and electric charge depend on the background scalar field, while the electric charge is conserved by a $U(1)$ symmetry.

In general, one can describe a particle in EMD by its conserved charge Q_p and a mass function $\mathfrak{m}(\varphi)$, absorbing the dependence on the dilaton field. It will show up in the Lagrangian describing the dynamics of the point particle, reading

$$L_{pp} = \mathfrak{m}(\varphi)\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - Q_p A_\mu \dot{x}^\mu . \tag{3.67}$$

We can compare the field generated by a particle in this parametrisation to the field we know belongs to a certain object, in order to find the mass function belonging to the zero size limit of the particular object. Taking a black hole as example, this leads to the matching condition [145, 152]

$$\frac{\mathrm{d}\mathfrak{m}(\varphi)}{\mathrm{d}\varphi} = \frac{a}{a^2-1}\left(-\mathfrak{m}(\varphi) + \sqrt{\mathfrak{m}(\varphi)^2 + Q_p^2 e^{2a\varphi}(a^2-1)}\right) . \tag{3.68}$$

For every value of the dilaton coupling a, the solution to this equation will depend on the charge and an integration constant, determined by the mass m and charge.

Note that the above ODE is fully analogous to the expression for the dilaton charge (3.60), with the identifications

$$(M, D, Q) \simeq (\mathfrak{m}(\varphi), \frac{d\mathfrak{m}(\varphi)}{d\varphi}, Q_p e^{a\varphi}) . \tag{3.69}$$

Indeed, one should think of the latter as the background-dependent charges, which go to their asymptotic values for $\varphi \to 0$. The mass function therefore determines more than only the masses. Its first derivative corresponds to the dilaton charge. Moreover, its second derivative is closely related to the extremality combination:

$$\frac{d^2}{d\varphi^2}\log\mathfrak{m}(\varphi) = \frac{a^2 Q_p^2 e^{2a\varphi}}{\mathfrak{m}^2(\varphi)} \frac{(a - \frac{\mathrm{d}}{\mathrm{d}\varphi})\mathfrak{m}(\varphi)}{a\mathfrak{m}(\varphi) + (a^2-1)\frac{\mathrm{d}\mathfrak{m}(\varphi)}{\mathrm{d}\varphi}} , \tag{3.70}$$

which is evaluated on the background to be

$$\beta := \frac{d^2}{d\varphi^2}\log\mathfrak{m}(\varphi)|_{\varphi=0} = \frac{a^2 Q^2}{m^2}\frac{\chi}{\chi + \frac{aD}{m}} , \tag{3.71}$$

clearly vanishing for extremal black holes.

In the extremal case, therefore, the coupling of the particle to the dilaton field is simply through an exponential $\mathfrak{m}(\varphi) = m e^{a\varphi}$. Looking at the Lagrangian (3.67), this shows the extremal particle couples like a particle without self-gravitation to the metric and dilaton field. In retrospect this is not surprising, since, if the extremal particle does not experience a net force from other extremal particles stationary with respect to it, why would it experience any force generated by itself? Equivalently, the extremal particle can be seen to couple to a metric given by

$$\tilde{g}_{\mu\nu} = e^{2a\varphi} g_{\mu\nu} . \tag{3.72}$$

If we make the transformation to the tilde metric, we switch from the Einstein frame to the Jordan frame, in which the bulk action takes the form [108, 152]

$$S_{\text{Jordan}} = \frac{1}{16\pi} \int \mathrm{d}^4\mathbf{x}\sqrt{-\tilde{g}}\; e^{-2a\varphi} \left(\tilde{R} + \left(6a^2 - 2\right)(\partial\varphi)^2 - F^2\right) . \tag{3.73}$$

In this frame, the extremal particle does not couple to the dilaton field at all, making its mass constant.

Remark 1. *In general, for non-extremal cases,* (3.68) *has no simple closed-form expression. An exception is the case* $a = 1$*, for which it is solved by* $\mathfrak{m}(\varphi)^2 = \mu^2 + \frac{1}{2}Q_p^2 e^{2\varphi}$*, where the integration constant* μ *is given by* $\mu^2 = m^2 - \frac{Q_p^2}{2}$*, showing it is a measure of deviation from extremality, since for* $a = 1$ *the particle is extremal when* $Q_p^2 = 2m^2$ *(setting the background field to zero).*

3.3.4 Separation of the two-center system

Now we know how to consistently introduce test particles in the relevant background, let us consider the integrability of the two-center system. The two extremal black holes have mass M_i, charge Q_i and dilaton charge $\Sigma_i = aM_i$ related by $Q^2 = M^2 + \Sigma^2 = M^2(1 + a^2)$. Placing an extremal test particle, with $Q_p^2 = m^2 + \sigma^2$ and $\sigma = am$, in this background, we have the Lagrangian as in equation (3.67), with $\mathfrak{m}(\varphi) = me^{a\varphi}$, leading to the equation of motion as in [83]:

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = \sqrt{1 + a^2}e^{-a\varphi}F^\alpha_\beta\dot{x}^\beta - a(\dot{x}^\alpha\dot{x}^\beta\nabla_\beta + \nabla^\alpha)\varphi\,. \tag{3.74}$$

Here we see the different forces playing out clearly, first the connection term next to the acceleration, and on the other side the electromagnetic force and dilaton force. The light-like case will have zero charge and dilaton charge, and will therefore only have the left-hand side.

Instead of solving the above equation of motion, however, we will show integrability through the separation of the Hamilton-Jacobi equations. As discussed in Section 2.6.1.1, separation of the HJE is sufficient to show integrability. Moreover, it gives a map between the classical mass and energy and those in the relativistic EMD theory.

3.3.4.1 Light-like case

Let us first consider the separation of the Hamilton-Jacobi equation for the case of null particles on our two-center background. The separated Hamilton-Jacobi equation found for the light-like case in the EMD theory with $a = \sqrt{3}$ and centers at $z = \pm 1$ was given by [83]:

$$(\partial_\xi S_\xi)^2 = -\frac{L^2}{\sinh^2\xi} + E'^2\left[4(M_1 + M_2)\cosh\xi + \sinh^2\xi\right] + D \tag{3.75}$$

$$(\partial_\eta S_\eta)^2 = -\frac{L^2}{\sin^2\eta} + E'^2\left[4(M_2 - M_1)\cos\eta + \sin^2\eta\right] - D. \tag{3.76}$$

Comparing this to Section 2.6.1.1, we see the light-like motion in EMD is equivalent to the classical motion for a mass and energy

$$m^2 = 2E'^2\,, \qquad E = \frac{|E'|}{2\sqrt{2}} = \frac{m}{4}. \tag{3.77}$$

In other words, the constant E' sets the mass as well as the energy of the two-center system, which is in a scattering state, since the classical energy E is necessarily positive. The quantity E' is the canonically conjugate momentum to the time coordinate. This means that, since the Hamiltonian is time independent, by reparametrising we can make it equal to any constant we like except zero.

Another way to see null geodesics on this EMD background are just scattering states in a classical two-center system, is by considering the Hamiltonian

$$\mathcal{H}_0 = g^{\mu\nu} p_\mu p_\nu = -U^{1/2} E'^2 + U^{-1/2} p^2 = 0\,, \tag{3.78}$$

summed over $i = 1, 2, 3$. A rescaled version

$$p^2 - U E'^2 = 0\,, \tag{3.79}$$

then after the choice $E' = 1$ comes down to

$$p^2 - 4\left(\frac{M_1}{|\mathbf{x} - \mathbf{x}_1|} + \frac{M_2}{|\mathbf{x} - \mathbf{x}_2|}\right) = 1, \tag{3.80}$$

where we indeed recognise the classical two-center problem. In the classical realm, there is of course no attraction between light and mass, so here the light-like case corresponds to a particle with a mass $m = \sqrt{2}$ and energy $E = \frac{1}{2\sqrt{2}}$. Turning on the mass of the extremal probe in the EMD system then takes us to energies below this, as we will see soon in (3.84).

3.3.4.2 Anti-extremal test particle

Having reviewed the slightly simpler case of null-geodesics in the EMD two-center system, we will now focus our attention on the motion of an anti-extremal, massive test particle in the same background. This case, too, is integrable, which we once more will be able to show through separation of the Hamilton-Jacobi equation. This will also show the two-center and therefore the one-center system in EMD are, in some sense, equivalent to their non-relativistic counterparts. The exact way in which this equivalence holds, will be discussed in Chapter 4.

As we will discuss in Section 3.3.5 in more detail, extremal test particles have been shown (by Gibbons and Wells [122]) to follow null geodesics in a five-dimensional metric, for which the Hamilton-Jacobi equation can be reduced to a four-dimensional one

$$\left(\frac{\partial S}{\partial x^\alpha} - m'\sqrt{1+a^2} A_\alpha\right)^2 = -m'^2 U^{-2a^2/(1+a^2)}\,, \tag{3.81}$$

where we introduced the mass m' of the particle and the index runs over time and three space dimensions. The case of $a = \sqrt{3}$ is integrable again, since the Hamilton-Jacobi equation is separable. This separation is the same as in the case of null-geodesics in the four dimensional metric (3.75), except for the two bold terms proportional to m':

$$(\partial_\xi S_\xi)^2 = -\frac{L^2}{\sinh^2\xi} + 4E'^2(M_1+M_2)\cosh\xi + \left(E'^2 + \mathbf{2m'E'}\right)\sinh^2\xi + D \tag{3.82}$$

$$(\partial_\eta S_\eta)^2 = -\frac{L^2}{\sin^2\eta} + 4E'^2(M_2-M_1)\cos\eta + \left(E'^2 + \mathbf{2m'E'}\right)\sin^2\eta - D. \tag{3.83}$$

The relation to the non-relativistic energy in section 2.6.1.1 is then shifted accordingly to

$$E = \frac{|E'| - 2m'}{2\sqrt{2}}\,, \tag{3.84}$$

where we have assumed the relativistic energy E' to be negative. The mass of the corresponding classical particle remains as in (3.77). In particular, for the values

$$|E'| < \frac{m'}{2} \tag{3.85}$$

the system corresponds to a classical two-center problem with negative energy E, thus having bounded orbits. All other energies E' give scattering orbits.

The one-center problem, our prime focus in this work, is contained in the system considered above. By simply sending one of the central masses M_1, M_2 to zero, as the dilaton and electric charge are proportional to the mass, we remove any effect from this center and obtain the one-center problem. As the same can be done in the non-relativistic case, this shows the EMD one-center is equivalent to its non-relativistic counterpart in the same sense as the two-center system. We can immediately conclude the bounded orbits of the EMD-one-center system are all elliptical.

3.3.5 Einstein-Maxwell-dilaton theory from Kaluza-Klein reduction

Having seen the special dynamics in some EMD systems, we will now consider the 5-dimensional origin of such systems. This will, in Chapter 5, help us understand why the dynamics is the way it is.

In this section, we will review Kaluza-Klein reduction of 5D pure gravity to 4D Einstein-Maxwell-dilaton theory. Subsequently, we will treat the reduction of geodesic motion on a Ricci-flat 5D space to 4 dimensions in this context, yielding the special systems discussed in the previous sections.

3.3.5.1 Reducing Einstein-Hilbert to Einstein-frame EMD

A 5-dimensional theory of pure gravity has as action the Einstein-Hilbert term, with appropriate gravitational constant $\hat{G}$, Ricci scalar $\hat{R}$ and determinant of the metric $\hat{g}$:

$$S_{5D} = \frac{1}{16\pi\hat{G}}\int\sqrt{-\hat{g}}\hat{R}\;\mathrm{d}^5x\,. \tag{3.86}$$

One of the 5 dimensions, denoted z, is taken compact and small, such that all modes in a Fourier expansion of the metric components will attain a very high mass, except for the massless zero mode. Now, we will assume that the metric is completely independent of the coordinate z, allowing a natural separation between metric components $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$ and $\hat{g}_{zz}$, where we have Greek indices $\mu = 0, 1, 2, 3$ and z is its own index. This already suggests the splitting of the 15 degrees of freedom of the symmetric metric tensor $\hat{g}_{AB}$ into 10 tensor, 4 vector and 1 scalar component as viewed from 4 dimensions. A convenient parametrisation of the Ansatz for the metric is[17]

$$\mathrm{d}\hat{s}^2 = e^{2\varphi/\sqrt{3}}\mathrm{d}s^2 + e^{-4\varphi/\sqrt{3}}\left(\mathrm{d}z + 2A_\mu \mathrm{d}x^\mu\right)^2 , \tag{3.87}$$

in terms of 4-dimensional fields φ, A, g (scalar, vector, tensor respectively) and writing $\mathrm{d}s^2$ for the 4-metric.

Working out the action in terms of these 4D fields (commonly done in Vielbein formalism [191]) and integrating over the compact dimension, one ends up with the action

$$S_{4D} = \frac{1}{16\pi G}\int \sqrt{-g}\left[R - 2(\partial\varphi)^2 - e^{-2\sqrt{3}\varphi}F^2\right]\ \mathrm{d}^4x\,, \tag{3.88}$$

where G is the usual gravitational constant, R the 4D curvature scalar and $F^2 = F_{\mu\nu}F^{\mu\nu}$ the square of the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Viewed from four dimensions, the additional compact spacial dimension thus gives rise to an Einstein-Maxwell-dilaton theory with dilaton coupling $a = \sqrt{3}$, and the diffeomorphism invariance, gauge symmetry and scale symmetry are all inherited from Ansatz-preserving coordinate transformations in 5 dimensions [191, 196].

Considering a massless particle in such a space as (3.87), we can write down the Polyakov-type Lagrangian [128]

$$\mathcal{L}_p = -\frac{1}{2h}\hat{g}_{AB}\dot{x}^A\dot{x}^B\,, \tag{3.89}$$

with the auxiliary variable h. After expressing this in 4D fields and Legendre transforming to get rid of $\dot{z}$ in favour of the conserved momentum in the compact dimension

$$P_z = -\frac{1}{h}e^{-4\varphi/\sqrt{3}}(\dot{z} + 2A_\mu\dot{x}^\mu)\,, \tag{3.90}$$

the above Lagrangian becomes

$$\mathcal{L}_p = \frac{1}{2}\left[\frac{1}{h}e^{2\varphi/\sqrt{3}}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - he^{-4\varphi/\sqrt{3}}P_z^2 - 4P_zA_\mu\dot{x}^\mu\right]. \tag{3.91}$$

Such a function, where one or more velocities in a Lagrangian are traded for a (conserved) momentum, is often called a Routhian. Finally, solving for h and plugging it back in (taking the positive branch to get the correct kinetic term), we have

$$\mathcal{L}_p = -\left[e^{\sqrt{3}\varphi}|P_z|\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + 2P_zA_\mu\dot{x}^\mu\right], \tag{3.92}$$

which is the action in EMD theory of an extremal particle with charge $q = 2P_z$, mass $m = |P_z|$ and dilaton charge $\sigma = \sqrt{3}|P_z|$.

[17] In principle the coefficients in the powers of the exponent are arbitrary, but we choose specific values to end up in Einstein-frame and match a conventional choice of coefficients in the final 4-dimensional action.

3.3.5.2 Geodesics on a 5D gravitational wave

Having established the link between 5-dimensional gravity and geodesics and 4-dimensional Einstein-Maxwell-dilaton, we will now look at an example of a Ricci-flat space with geodesic motion. The dynamics of the 5D geodesic is in 4D exactly that of an anti-extremal test particle on an extremal black hole background, as discussed for the two-center and one-center case in Section 3.3.4.2. In Chapter 5, we will discuss why this explains the classical-seeming dynamics of the EMD systems.

The particular, Ricci-flat, 5D metric we will discuss is given by

$$\begin{aligned} \mathrm{d}\hat{s}_5^2 &= U\left(\mathrm{d}z + \frac{\mathrm{d}t}{U}\right)^2 + U^{-1/2}\left(-U^{-1/2}\mathrm{d}t^2 + U^{1/2}\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}\right) \\ &= U\mathrm{d}z^2 + 2\mathrm{d}z\mathrm{d}t + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}\,, \end{aligned} \tag{3.93}$$

corresponding with 4-dimensional fields

$$e^{\varphi} = U^{-\sqrt{3}/4}, \qquad A = \mathrm{d}t/(2U), \qquad \mathrm{d}s_4^2 = -U^{-1/2}\mathrm{d}t^2 + U^{1/2}\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}\,, \tag{3.94}$$

with U a positive harmonic function of $\mathbf{x}$. Note that if U were to vanish, both t and z would be null-coordinates in the 5-dimensional metric, since the norms of $\mathrm{d}z$ and $\mathrm{d}t$ would vanish. Since U is nonzero, z no longer is a null direction (it is space-like because of the positive sign of U) and only t is a null-coordinate. However, in the 4-dimensional metric, coordinate t is timelike. Furthermore, viewed from 4 dimensions, the sign of the potential A only becomes meaningful when we fix the parametrisation $\dot{t}$ and introduce a charged particle (which the null geodesic in 5D provides).

This metric has at least two isometries because the harmonic function U is only dependent on the three spacial coordinates of spacetime. Its interpretation is that of a so-called pp-wave, which is short for plane-fronted wave with parallel rays. This type of spacetimes is defined by the presence of a covariantly constant null-vector, which in our coordinates is $\frac{\partial}{\partial t}$[144] and is larger than the class given above, since U might additionally depend on z. Moreover, note that the Ricci tensor satisfies

$$\hat{R}_{AB} = 0\,, \tag{3.95}$$

making the spacetime Ricci-flat.

As shown in [122], null-geodesics of this metric have equations of motion coinciding with the equations of motion of an extremal particle in an extremal (singular) background in Einstein-Maxwell-dilaton theory. That is, a particle with mass m, charge $|q| = 2m$ and dilaton charge $\sigma = \sqrt{3}m$, such that the static repulsive electric force of two of such particles would cancel the attractive force of gravity and dilaton. The background is an extremal black hole, though it becomes singular at the specific dilaton coupling considered.

For simplicity, we take the standard 5D geodesic action

$$\mathcal{L}_p = \frac{1}{2} g_{AB}\dot{x}^A\dot{x}^B\,, \tag{3.96}$$

with $h = -1$. This also means the time-parametrisation is now fixed by

$$\dot{t} = -\frac{1}{2\dot{z}}(U\dot{z}^2 + \dot{x}^2)\,, \tag{3.97}$$

implying that the sign of $\dot{t}$ also fixes the sign of $\dot{z}$. The conserved momentum conjugate to the coordinate z is given by $P_z = U\dot{z} + \dot{t}$. Additionally we use the translation symmetry of t, finding $P_t = \dot{z}$ is conserved and its sign tethered to the choice for $\dot{t}$. This can be used to eliminate $\dot{z}$ and $\dot{t}$, leading to the equations of motion

$$\ddot{x}^i = \tfrac{1}{2}\frac{\partial U}{\partial x^i}P_t^2\,, \qquad \ddot{t} = -P_t\left(\partial U \cdot \dot{x}\right)\,, \tag{3.98}$$

in 4 dimensions. This means there is a special value for the momentum where the force vanishes, that is $P_t = 0$. This is the case where $P_z = \dot{t}$, so that the extremal charge $q = 2P_z$ has the same sign as $\dot{t}$. In other words, if we fix parametrisation by having $\dot{t}$ positive, and agree upon taking the current sign of (3.93) to correspond to positive charge for the black hole, the sign for which this interesting situation occurs is positive too. Moreover, the null requirement immediately implies vanishing spacial momentum P_x. In 5 dimensions, this motion corresponds with a constant velocity in the null t-direction, while in 4D it is a stationary particle remaining at rest due to cancelling electromagnetic and dilaton/gravitational forces.

To learn the shape of the orbits in other cases, we use again the null requirement. We will be interested in the one-center system from now on, meaning $U = 1 + \frac{4}{|\mathbf{x}|}$, though the reduction works for any number of centers. As the t coordinate plays the role of time in 4 dimensions, the conjugate P_t will have the interpretation of energy. We have

$$g^{AB}P_AP_B = 2P_tP_z - UP_t^2 + P_x^2 = 0\,, \tag{3.99}$$

and we can reparametrise to find an implicit expression for the energy. Let us first assume positive P_z, yielding

$$-P_t + \frac{P_t^2}{2P_z} = \frac{P_x^2}{2P_z} - \frac{2P_t^2}{P_z|\mathbf{x}|}\,, \tag{3.100}$$

where we used $U = 1 + \frac{4}{|\mathbf{x}|}$. The (x, P_x) canonical coordinates are constrained to a surface on which the dynamics is that of a Kepler problem. Recall that our choice of parametrisation results in P_t being negative, which immediately shows the 'Keplerian energy' $-P_t + \frac{P_t^2}{2P_z}$ can only be positive, implying only unbounded orbits are possible for positive P_z. For negative P_z we consider

$$P_t + \frac{P_t^2}{2|P_z|} = \frac{P_x^2}{2|P_z|} - \frac{2P_t^2}{|P_z||x|}\,, \tag{3.101}$$

which is negative for $|P_t| < 2|P_z|$ and positive for $|P_t| > 2|P_z|$. This shows the negatively charged particle in four dimensions can access both bounded and unbounded orbits.

On first view, the Keplerian orbits of this relativistic system are astonishing. However, the 5-dimensional origin already gives a hint towards an explanation. One might suspect the Lorentzian 5D space itself to hold the explanation: Whereas an so(4) symmetry in 3d qualifies as 'enhanced', in a 5-dimensional Lorentzian space it might very well be an isometry. After all, 5D Minkowski has an so(4) subalgebra. We will see that this explanation in fact does not hold. However, as will be discussed in Chapter 5, the 5D space provides the most natural path to connect the system to the ordinary Kepler problem.

To recapitulate, we have seen there exists a specific combination of fields, with specific charges and couplings, for which the one-center problem has Kepler-like ellipses as bounded orbits. Moreover, the one-and two-center problems are related to their classical counterparts as can be learned from the separation of the Hamilton-Jacobi equation. Lastly, the specific combination of charges and couplings turns out to have its origin in a higher-dimensional theory. The exact nature of the connection to classical physics, and how this fits into a broader framework of relativistic systems displaying non-relativistic orbits, is the topic of the next chapters.

Chapter 4

Extremal Black Holes as Relativistic Systems with Kepler Dynamics

4.1 Introduction

As pointed out in Chapter 3, future experiments sensitive to a long inspiral phase require precise knowledge of the dynamics of binary compact objects. Though there is a good understanding of the PN expansion and other approximations in the context of these systems, both of the conservative and radiation part to high order, this stimulates the further development of analytical tools to limit the demand on computational resources.

This chapter, adapted slightly from [180], is therefore dedicated to the identification of specific relativistic systems for which the PN expansion results in a system of a much more manageable form. Often, systems with this kind of simplifications possess more symmetries and conservation laws, reducing the number of effective degrees of freedom. This is famously the case in the classical analogue of the relativistic binary systems; we will investigate to what extent the same holds for certain relativistic systems.

Recall from Section 2.2.3 that on the non-relativistic i.e. classical level, the two-body problem divides up nicely into the motion of a free particle (the total mass located at the center of mass) and the motion of a particle with reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ in the stationary potential generated by the total mass $M = m_1 + m_2$. The solutions to this problem then are the same ellipses as in the classical Kepler problem in celestial mechanics. The latter possesses, next to the expected spherical symmetry $SO(3)$ yielding the conservation of angular momentum, an additional symmetry which gives the conservation of the Laplace-Runge-Lenz (LRL) vector (2.28). Since this symmetry is not immediately obvious on the level of the Lagrangian it is often referred to as a hidden symmetry.

The three components of the angular momentum vector, the three components of the LRL vector and the total energy form seven conserved scalar quantities. As the length of the LRL vector is determined by angular momentum and energy, and the angular momentum vector is perpendicular to the LRL vector, only five of the scalar conserved quantities are independent. The joint levelsets of these five constants of motion in the six-dimensional phase space are hence one-dimensional. As a consequence, bounded orbits must be periodic and take the form of the famous elliptical orbits found in Kepler's model of the Solar System (while in General Relativity of course, the symmetry is broken and the perihelion – the point of closest approach – precesses).

For central force systems, there is a strong link between closing orbits and enhanced symmetry, in the form of Bertrand's theorem, discussed in Section 2.4. This states that the only two central forces whose bounded orbits are all closed curves are the Kepler potential and the isotropic harmonic oscillator [125], which are in fact related (see [10, 140] for a complex transformation relating the two in 2D, and [173] for a reduction of the 4D harmonic oscillator to the 3D Kepler system) and known for their large symmetry groups, $SO(4)$ and $SU(3)$ in 3 dimensions, respectively.

Since we know symmetries make problems more tractable and the non-relativistic problem possesses additional symmetry, it is natural to attempt to restore the non-relativistic symmetry in relativistic systems. While the closing of bounded orbits is not a sufficient condition for conservation of a LRL vector in a two-body problem, it is a necessary condition that is satisfied very rarely by relativistic theories. The closure of orbits can therefore be a useful tool for diagnosis of theories when looking for additional spacetime symmetries, as demonstrated by e.g. [60].

There has been previous work done on identifying relativistic systems that have exclusively closed bounded orbits. For example, Perlick has identified all spacetimes in General Relativity with that property, a sort of relativistic Bertrand theorem [189]. He considered all spherically symmetric spacetimes that have bounded timelike geodesics with a perihelion shift equal to $\frac{\pi}{\beta}$, with β rational. The cases $\beta = 1$ and $\beta = 2$, corresponding to relativistic versions of the Kepler problem and harmonic oscillator, are the only ones admitting an additional symmetry. However, Perlick's theorem is only taking into account gravity, without allowing other forces to be present. Additionally, there is the hydrogen-like system in $\mathcal{N} = 4$ super Yang-Mills theory, which has an additional conserved vector, coinciding with the classical LRL vector in the non-relativistic limit [57, 6]. Interesting follow-up results were derived in $\mathcal{N} = 8$ supergravity, where the two-body problem was shown to have a LRL vector to at least order 1PN and a vanishing perihelion shift to third order in the post-Minkowskian (PM) expansion [60, 188].[1] However, at 3PM there appears to be a hint that the quantum energy-level degeneracy linked to the LRL vector and present at 1PN might be lost. This suggests an interesting break in the bond between closed bounded orbits and hidden symmetry, which is present classically. Additionally, it was shown in [60] that the test-mass limit in $\mathcal{N} = 8$ supergravity has a zero perihelion shift to all orders in velocity.

Although relativistic corrections of the Kepler problem generically break the symmetry associated with the LRL vector, it follows from the above that specific systems manage to preserve it. These systems then, one might wonder, are perhaps not truly relativistic in some sense, as their dynamics is still constrained by the same symmetries, giving rise to strictly periodic orbits in phase space (at least for bounded orbits).

[1] The post-Newtonian expansion is in terms of $\frac{1}{c^2}$, resulting in an expansion in weak gravitational field and low velocity, while the post-Minkowskian is an expansion in gravitational constant G, i.e. a weak gravitational field expansion only [90].

We will study a class of such systems and demonstrate that they are orbitally equivalent to the Kepler system on a levelset of the Hamiltonian in phase space. Their full Hamiltonians are implicitly defined by

$$f(H(q,p)) = \frac{p^2}{2} - \frac{g(H(q,p))}{r(q)}\,, \tag{4.1}$$

for smooth functions $f, g : \mathbb{R} \to \mathbb{R}$. As we will see, such systems give rise to a phase space which can be thought of as being foliated by energy surfaces of Kepler problems where for each value of H the motion is proportional to that of a Kepler problem with a different coefficient for the gravitational potential; in other words, with a different gravitational constant. The global structure of the phase space is therefore tied to the specific properties of the function $g(H)$.

We will show that examples of the above class of Hamiltonians (4.1) naturally arise in Einstein-Maxwell-dilaton (EMD) theory, when one considers two extremal black holes with opposite charge for a specific value of the dilaton coupling (cf. equation (4.74)). For this case, we derive a functional relation of the form (4.1) to first order in the post-Newtonian expansion of the two-body system and all orders in the test-mass limit.

The case of extremal black holes is somewhat special and unlikely to be realised in nature (with all observed black holes approximately neutral), and the dilaton coupling of $a = \sqrt{3}$ (see Section 4.3) is much larger than current experimental bounds [33]. However, as a physical aside, it is an interesting question how one would observationally distinguish the above Hamiltonians from the Kepler one, e.g. in the solar system. When studying planets orbiting a (much heavier) Sun described by $H(q,p)$ as opposed to the ordinary Kepler problem, the first two laws of Kepler still hold: the bounded orbits are ellipses and the trajectories conserve angular momentum. However, the period of an orbit becomes

$$T = 2\pi\sqrt{\frac{s^3}{GMg(E)}}\,, \tag{4.2}$$

with s the semi-major axis of the ellipse, where for the sake of clarity we included the mass of the sun M and the gravitational constant G. This differs from the usual Newtonian period $T_N = 2\pi\sqrt{s^3/(GM)}$. Therefore the third law[2] is violated: different orbits will have different energies, causing the ratio T^2/s^3 to no longer be the same for all orbiting objects.

To what extent, then, are these Hamiltonians equivalent to the Kepler problem? We will prove they describe the same dynamics at least on the energy shell, so for a fixed $H = E$, in the sense that their flows are proportional. Moreover, there can exist a transformation mapping H to the Kepler Hamiltonian where we do not need to restrict to the energy surface (at least locally, in a small neighbourhood of the energy surface). This transformation is shown to exist as an energy redefinition and canonical transformation at least up to and including 5PN. It is worth noting that our local constructions in Theorems 5 and 6 and Theorem 9 show the existence of Kepler dynamics and the conserved LRL vector, but do

[2]For all bodies orbiting the Sun, the square of the period is proportional to the third power of the semi-major axis of the orbit, *with the same proportionality constant* [151].

not necessarily imply global $SO(4)$ symmetry. Proving the transformations generated by the conserved quantities form a group of canonical transformations isomorphic to $SO(4)$ is not trivial [12]. Our construction only shows the local existence of the so(4) Lie algebra, generated by the angular momentum and the appropriately rescaled LRL vector. Whether the approximate transformations to Kepler can be extended to all orders, whether they exists globally and whether the symmetry group is indeed $SO(4)$, remain topics of future research.

Another question is whether relativistic systems canonically conjugate to Kepler up to time reparametrisation are *the only* ones with a conserved LRL-type vector and the corresponding symmetry. We show this is the case at least to 5PN order.

This chapter is organised as follows. In Section 4.2, we discuss the set of Kepler-like Hamiltonians and their relation to the Kepler problem. Here we show the on-shell equivalence to Kepler in Subsection 4.2.1, a construction that yields an explicit, approximate, off-shell transformation to the Kepler problem in Subsection 4.2.2 and the approximate equivalence of all LRL-preserving Hamiltonians of a certain kind to Kepler in Subsection 4.2.3. We show in Section 4.3 that a particular tuning of the parameters in this theory allows one to write the 1PN two body and all-order test-mass limit as a Kepler-type Hamiltonian, providing an interesting example of relativistic systems with classical dynamics.

4.2 Relativistic Systems with Kepler Dynamics

In this section we will discuss Hamiltonians of type (4.1) central to this chapter. Although they appear naturally from relativistic problems, see Section 4.3, they end up being equivalent (in the ways mentioned in the introduction) to the classical Kepler system. We will prove the equivalence on the energy surface (on-shell) of the Kepler-like Hamiltonians to Kepler problems and, later, how these can be explicitly related (up to fifth post-Newtonian order) to the Kepler problem through canonical transformations and a non-linear energy redefinition (off-shell). Lastly, we show that all Hamiltonians preserving a LRL-like vector are related to Kepler in this way, also up to 5PN.

4.2.1 On-shell equivalence to Kepler dynamics

Let us consider the following family of Hamiltonians $H = H_{f,g} : T^*\mathbb{R}^d_0 \simeq (\mathbb{R}^d - \{0\}) \times \mathbb{R}^d \to \mathbb{R}$, implicitly defined by the functional relation

$$f(H(q,p)) = \frac{p^2}{2} - \frac{g(H(q,p))}{r(q)}\,, \tag{4.3}$$

with $f, g : \mathbb{R} \to \mathbb{R}$ smooth functions, that can be written as powers series in the form

$$f(x) = x + f_1 x^2 + f_2 x^3 + \dots\,, \qquad g(x) = 1 + g_1 x + g_2 x^2 + \dots\,, \tag{4.4}$$

where f_i, g_i are real numbers. In other words, we assume that their Taylor-Maclaurin expansions have their first coefficients fixed by $f_{-1} = 0$, $g_0 = 1$ and $f_0 = 1$ (with labels related to PN orders as will become more clear below). The reason the constant term in $f(H(q,p))$ with coefficient f_{-1} is absent, is that we want to disregard rest-mass terms and (4.3) to yield the Kepler system at lowest order in the PN expansion.

As discussed in the introduction, this is directly motivated by the Hamiltonian of a binary Einstein-Maxwell-dilaton system in the test-mass limit. In Section 4.3, we will see this system has a Hamiltonian that is implicitly defined (at all PN orders) by the above relation with

$$f(x) = x + \tfrac{1}{2}x^2, \qquad g(x) = 1 + x + \tfrac{1}{4}x^2 \,. \tag{4.5}$$

Note that we set the test mass to unity in the identifications, to exactly match the description of the above Hamiltonian family.

Since we already know explicit Hamiltonian functions solving (4.3), we will not pursue the question of sufficient conditions for existence.[3] For the time being, we assume that a solution $H(q,p)$ exists for the given functions f, g and describe some of its properties in relation with the Kepler Hamiltonian. Therefore, let $H : T^*\mathbb{R}^d_0 \to \mathbb{R}$ be a smooth Hamiltonian function satisfying the relation (4.3). For convenience, we define the new Hamiltonian function

$$K(q,p) := f(H(q,p)) = \frac{p^2}{2} - \frac{g(H(q,p))}{r(q)} \,. \tag{4.6}$$

Since K is by definition a function of H, it is also an integral of motion of H, that is, K is constant on the flow of H. If $E \in \mathbb{R}$ is a regular value of H, this implies that on the energy levels $H^{-1}(E)$

$$K|_{H^{-1}(E)}(q,p) = \frac{p^2}{2} - \frac{g(E)}{r(q)} \,, \tag{4.7}$$

which is a Kepler-type Hamiltonian with gravitational constant $g(E)$. In fact, the flow generated by K on all its regular energy surfaces turns out to be proportional to the flow of a Kepler Hamiltonian.

Theorem 5. *Consider $M = T^*\mathbb{R}^d_0 \simeq (\mathbb{R}^d - \{0\}) \times \mathbb{R}^d$ with standard symplectic form $\omega = \sum_k \mathrm{d}p_k \wedge \mathrm{d}q_k$. Assume that there are functions $f, g \in C^2(\mathbb{R})$ such that the Hamiltonian $H : M \to \mathbb{R}$ is defined by (4.3) and*

$$K(q,p) := f(H(q,p)) \,.$$

Then, for any regular energy value $H = E$ the vector fields $X_K|_{H^{-1}(E)}$ and $X_H|_{H^{-1}(E)}$ of K and H are proportional. Moreover, if

$$\mathcal{E} := \{E \in \mathbb{R} \mid f'(E) \neq 0 \text{ and } E \text{ is regular value of } H\} \tag{4.8}$$

and

$$J : M \times \mathcal{E} \to \mathbb{R} \,, \qquad J(q,p,E) := J_E(q,p) := \frac{p^2}{2} - \frac{g(E)}{r(q)} \,,$$

then for all $E \in \mathcal{E}$, the Hamiltonian vector fields $X_{J_E}|_{H^{-1}(E)}$ and $X_K|_{H^{-1}(E)}$ are proportional.

[3] This is a rather interesting technical problem on its own, and we refer the interested reader to [85, 19] for the current state of the art.

Proof. For the Hamiltonian vector fields X_K and X_H of K and H, respectively, we have

$$X_K = -\frac{\partial K}{\partial q}\frac{\partial}{\partial p} + \frac{\partial K}{\partial p}\frac{\partial}{\partial q} = f'(H)\left(-\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\right) = f'(H)X_H\,. \tag{4.9}$$

The vector fields are hence proportional.

For the second part, let $E \in \mathcal{E}$. The Hamiltonian vector field of J_E at a point (q, p) is

$$X_{J_E}(q,p) = -\frac{\partial J(q,p,E)}{\partial q}\frac{\partial}{\partial p} + \frac{\partial J(q,p,E)}{\partial p}\frac{\partial}{\partial q}\,.$$

Using $K(H(q,p)) = J(q,p,H(q,p))$ we have

$$\begin{aligned} X_K &= -\left(\frac{\partial J}{\partial q} + \frac{\partial J}{\partial H}\frac{\partial H}{\partial q}\right)\frac{\partial}{\partial p} + \left(\frac{\partial J}{\partial p} + \frac{\partial J}{\partial H}\frac{\partial H}{\partial p}\right)\frac{\partial}{\partial q} \\ &= \left(-\frac{\partial J}{\partial q}\frac{\partial}{\partial p} + \frac{\partial J}{\partial p}\frac{\partial}{\partial q}\right) + \frac{\partial J}{\partial H}\left(-\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\right)\,. \end{aligned}$$

Therefore

$$X_K|_{H^{-1}(E)} = X_{J_E}|_{H^{-1}(E)} + \frac{\frac{\partial J}{\partial E}|_{H^{-1}(E)}}{f'(E)} X_K|_{H^{-1}(E)}\,,$$

where we used (4.9) for the second term. Solving for X_{J_E} we get that on $H^{-1}(E)$

$$\left(1 - \frac{\frac{\partial J}{\partial E}}{f'(H)}\right) X_K = X_{J_E}\,. \tag{4.10}$$

□

This means that the evolution of Hamiltonians satisfying (4.3) is equivalent to the evolution of a classical Kepler problem - more precisely, for each energy, the trajectories are equivalent to that of a Kepler problem with a specific energy-dependent value of the coupling with the potential up to possibly a time-rescaling. In particular, all bounded orbits are ellipses in the configuration space. This relation is somewhat reminiscent of the Maupertuis-Jacobi transformation, in which the trajectories of a natural Hamiltonian are described via a time reparametrisation as geodesics of a metric [210, 68].

The fact that the above Hamiltonians are equivalent to the Kepler problem and, in particular, the fact that all trajectories are closed, hints at the existence of an associated conserved Laplace-Runge-Lenz (LRL) vector on the energy levels. An obvious candidate would be the vector

$$A^i(q,p) = (p \times L)^i(q,p) - g(H(q,p))\frac{q^i}{r(q)}\,, \tag{4.11}$$

since it is simply the classical LRL vector, with an additional coefficient corresponding to the coefficient of the potential energy in (4.6).

Theorem 6. *Let $\mathcal{E}$ be defined by* (4.8) *and $E \in \mathcal{E}$. On the set*

$$\left\{(q,p) \in H^{-1}(E) \,\Big|\, f'(E) - \frac{\partial J(q,p,E)}{\partial E} \neq 0\right\},$$

the Hamiltonian $H(q,p)$ defined in (4.3) *is in involution with all components of the Laplace-Runge-Lenz vector $A^i(q,p)$ defined in* (4.11)*, and hence these are integrals of motion of the dynamics generated by H.*

Proof. Fix $E \in \mathcal{E}$. On $H^{-1}(E)$ the flow of $J_E(q,p)$ is proportional to that of $H(q,p)$, with proportionality

$$\lambda := \left(1 - \frac{\frac{\partial J(q,p,E)}{\partial E}}{f'(E)}\right), \tag{4.12}$$

which we assume to be regular and nonvanishing. we know for the Lie derivatives of the functions $A^i(q,p)$ with respect to X_H

$$\{A^i, H\} = \mathcal{L}_{X_H}(A^i) = \mathcal{L}_{\lambda^{-1}X_{J_E}}(A^i) = \lambda^{-1}\mathcal{L}_{X_{J_E}}(A^i) = \lambda^{-1}\{A^i, J_E\}, \tag{4.13}$$

where all functions are evaluated on $H^{-1}(E)$.

With (4.13), we reduced ourselves to check whether $J_E(q,p)$ commutes with the components of the LRL vector on $H^{-1}(E)$. Namely,

$$\begin{aligned}\{A^i, J_E(q,p)\} &= \frac{\partial A^i}{\partial q}\frac{\partial J_E}{\partial p} - \frac{\partial A^i}{\partial p}\frac{\partial J_E}{\partial q} \\ &= \left[\frac{\partial}{\partial q}(p\times L)^i - \left(\frac{\partial}{\partial q}\frac{q^i}{r}\right)g(H)\right]\frac{\partial J_E}{\partial p} - \left[\frac{\partial}{\partial p}(p\times L)^i - \left(\frac{\partial}{\partial p}\frac{q^i}{r}\right)g(H)\right]\frac{\partial J_E}{\partial q} \\ &\quad + \left(-\frac{q^i}{r}\right)\left[\frac{\partial g(H)}{\partial q}\frac{\partial J_E}{\partial p} - \frac{\partial g(H)}{\partial p}\frac{\partial J_E}{\partial q}\right].\end{aligned} \tag{4.14}$$

Observe now that on $H^{-1}(E)$, $A^i = A^i_E := (p\times L)^i - g(E)\frac{q^i}{r}$. So the first two terms combine into the Poisson bracket

$$\left\{A^i_E(q,p), J_E(q,p)\right\} = \left\{(p\times L)^i - g(E)\frac{q^i}{r}, \frac{p^2}{2} - g(E)\frac{1}{r}\right\} \tag{4.15}$$

which vanishes as the Poisson bracket between a standard Kepler Hamiltonian and its LRL vector.

The square bracket that form the last term in (4.14) amounts to $\{g(H), J_E\}$ evaluated on $H^{-1}(E)$. This also vanishes due to $g(H) = g(f^{-1}(K))$ and application of Theorem 5. □

Theorem 7. *On regular levelsets of H such that $f(H) < 0$, the rescaled LRL vector defined by $\bar{A}^i := -\frac{A^i}{\sqrt{-2f(H)}}$, where A is defined in* (4.11)*, satisfies the following commutation relations*

$$\{L^i, L^j\} = \epsilon_{ijk}L^k, \qquad \{\bar{A}^i, \bar{A}^j\} = \epsilon_{ijk}L^k, \qquad \{L^i, \bar{A}^j\} = \epsilon_{ijk}\bar{A}^k, \tag{4.16}$$

which define a Lie algebra isomorphic to so(4).[4]

[4]Compare this to the simpler form of equation (2.36) for the original Kepler problem.

Remark 2. *To study the global symmetry, one would need to regularise the problem to complete the temporal flow and then consider the global transformations generated by the integrals above [159], which is out of the scope of this chapter.*

Proof. Let E be a regular value for H and such that $f(E) < 0$, in the rest of this proof we assume all the computations restricted to the levelset $H^{-1}(E)$. For the first relation, the calculation is the same as in the usual Kepler case. Writing out the second, we have

$$\begin{aligned}\{\bar{A}^i, \bar{A}^j\} = & \frac{1}{-2f(H)}\{A_E^i, A_E^j\} + \frac{1}{-2f(H)}\left(\frac{f'}{-2f(H)}A^i - \frac{q^i}{r}g'\right)\{H, A_E^j\} \\ & + \frac{1}{-2f(H)}\left(\frac{f'}{-2f(H)}A^j - \frac{q^j}{r}g'\right)\{A_E^i, H\} \\ & + \frac{1}{-2f(H)}\left(\frac{f'}{-2f(H)}A^i - \frac{q^i}{r}g'\right)\left(\frac{f'}{-2f(H)}A^j - \frac{q^j}{r}g'\right)\{H, H\}\,,\end{aligned} \tag{4.17}$$

where $f' := f'(H)$ and $g' := g'(H)$. The second and third term here vanish due to proportionality of H to J_E, which commutes with A_E^i, while the last term vanishes trivially. Since we are on a fixed levelset, we can consider $g(E)$ constant in the first term, and therefore the bracket yields, as for the usual Kepler problem,

$$\{A_E^i, A_E^j\} = -2\left(\frac{p^2}{2} - \frac{g(E)}{r}\right)\epsilon_{ijk}L^k = -2f(E)\,\epsilon_{ijk}L^k\,, \tag{4.18}$$

completing the calculation of the bracket of rescaled LRL vectors. The remaining bracket again reduces to the computation for the classical Kepler system in complete analogy to the above computation.

The commutation relations (4.16) then define a Lie algebra isomorphic to so(4) as shown e.g. in [87, Chapter 3.2, Proof (3.6)]. □

While in this section we proved that for each fixed value of the energy the family of Hamiltonians satisfying (4.3) has a flow which is proportional to the Kepler flow and admits a LRL vector, we do not know the regularity of the dependence of these objects on the energy itself nor how to relate (4.3) and a Kepler Hamiltonian beyond the energy surface. In the following section, we will consider this problem, looking for an energy-independent way to relate Kepler problems and the implicitly defined Hamiltonians (4.3).

What we can immediately observe is that, while the shape of orbits is the same in both (4.3) and in a Kepler problem, the energy levelsets $H_{\mathrm{Kep}}^{-1}(E)$ and $H^{-1}(E)$ foliate the phase space in a different way. The Hamiltonian $K(q,p)$, and therefore also the implicitly defined Hamiltonians (4.3), induce a bundle of non-equivalent Kepler orbits, the global structure of which is determined by $g(H)$.

4.2.2 Off-shell equivalence to Kepler dynamics

While the on-shell equivalence discussed in the previous subsection explains why the Hamiltonians implicitly defined by (4.3) have an additional constant of motion and hence closed orbits, it does not address the violation of Kepler's third law: the fact that Keplerian energy surfaces can be stacked differently in the Kepler bundle. We now turn to this issue, and address the question whether one can also map families of orbits with different energies onto a fixed Kepler system. Since we would like to avoid issues of singularities and/or topology, we will restrict ourselves to a local construction. In other words, we now aim to generalise the on-shell (on a fixed energy surface) orbital equivalence to an approximate off-shell equivalence (for a neighbourhood of orbits of possibly different energies).

The violation of Kepler's third law demonstrates a physical difference between the Kepler problem and the implicitly defined Hamiltonians on the phase space, so it should not come as a surprise that looking for such a relation will involve a transformation of the phase space itself. The mapping we are looking for therefore involves both a time reparametrisation (related to the mapping from H to $K \equiv f(H)$) as well as a canonical transformation, whose composition will (locally) transform the Kepler Hamiltonian to the implicitly defined ones and establish an orbital equivalence in this sense.

We will provide evidence for the existence of such a canonical transformation by explicitly constructing it up to fifth PN order (see Section 3.1.2). Note that the PN expansion differs from the expansion around an energy surface; even when extending the canonical transformation to all PN orders (or having a closed expression for it), this would still only involve a local equivalence as singularities or topological issues might prevent one to extend the mapping to the whole phase space. Addressing the extension to all PN orders and the question of convergence of the series constructed below (even just in an asymptotic sense) is not a trivial endeavour, as is the question of global existence of the phase space transformation. Therefore, we will leave the all-order analysis for the whole phase space for future research.

The goal is to find a solution H to the functional equation (4.3) to any desired PN order from the perturbation of the Kepler system. To find a perturbative solution for H, it is useful to take H itself dimensionless,[5] but explicitly include the PN expansion parameter ϵ, see Section 3.1.2:

$$f(H) = \epsilon\frac{p^2}{2} - g(H)\frac{\epsilon}{r}\,, \tag{4.19}$$

which is solved to lowest order by

$$H = \epsilon H_{\text{Kep}} = \epsilon\left(\frac{p^2}{2} - \frac{1}{r}\right)\,, \tag{4.20}$$

that is, the Kepler Hamiltonian with an extra factor ϵ. More specifically we will show the following.

Theorem 8. *For given C^∞ functions f and g, the relation* (4.19) *can be solved to at least PN order 5 by*

$$H = \Phi^\star\tau(\epsilon H_{Kep})\,, \tag{4.21}$$

where $\tau : \mathbb{R} \to \mathbb{R}$, $E \mapsto \tau(E)$ is C^∞ with $\tau'(0) = 1$ defining a near-identity time reparametrisation and Φ is a near-identity canonical transformation.

[5] We divide out the rest-mass energy mc^2 and set $m = 1$ as previously.

Remark 3. *As we work with dimensionless Hamiltonian H, the lowest order term in H is order ϵ. Therefore, PN order 5 here corresponds to order ϵ^6. When going back to a dimensionfull Hamiltonian, this additional factor of ϵ vanishes. This understanding of PN orders coincides with the 'relative kPN order' of [204].*

The proof will be given by an explicit construction based on Lie transform perturbation theory combined with a rescaling of the energy function.

Let us introduce the real vector spaces

$$W_j = \operatorname{span}\left\{ \frac{(p^2)^l (p\cdot q)^n}{r^m} \;\Big|\; (l,m,n)\in\mathbb{N}^3,\, l+m-\frac{1}{2}n = j+1 \right\}. \tag{4.22}$$

For instance, the Kepler Hamiltonian H_{Kep} is in W_0. We will mainly consider W_j with non-negative integer j resulting from even n in (4.22) such that $F_j \in W_j$ has PN order j (see the definition (7)). But as we will see below, also half-integer j resulting from odd n in (4.22) can be important. Note that for $F_i \in W_i$ and $F_j \in W_j$,

$$\{F_i, F_j\} \in W_{i+j+\frac{3}{2}}\,, \tag{4.23}$$

implying that

$$W = \bigoplus_{k\in\mathbb{N}} W_{k/2}$$

is closed under the Poisson bracket. In particular,

$$\{p\cdot q, F_j\} \in W_j\,.$$

Let us write the energy rescaling τ in (4.21) in a power series as

$$\tau(E) = \sum_{n=0}^{\infty} \delta_n E^{n+1}\,, \tag{4.24}$$

with $\delta_0 = 1$. For counting the PN orders of τ applied to some Hamiltonian function H it is important to note that for $F_i \in W_i$ and $F_j \in W_j$,

$$F_i\, F_j \in W_{i+j+1}\,,$$

which implies that W is also closed under multiplication.

We will consider a succession of near-identity canonical transformations each of which is obtained from the flow of the Hamiltonian vector field generated by a suitable function G, using repeated adjoint operators as introduced in Section 2.3. Under the canonical transformation given by the time-one map of the flow generated by the Hamiltonian G a function F transforms according to

$$F \mapsto \Phi^\star F = \sum_{m=0}^{\infty} \frac{1}{m!}[\mathrm{ad}_G]^m F. \tag{4.25}$$

From (4.23) we get that for $F_i \in W_i$ and $F_j \in W_j$,

$$\mathrm{ad}_{F_i}(F_j) \in W_{i+j+\frac{3}{2}}\,. \tag{4.26}$$

The idea now is to solve the functional relation (4.3) order by order with H_{Kep} through a succession of canonical transformations generated by functions G_i and an energy rescaling of the form (4.24). To this end let us first inspect the functional relation in terms of the power series for f and g in (4.4) which gives

$$H + f_1 H^2 + f_2 H^3 + \dots = \epsilon \frac{p^2}{2} - \left(1 + g_1 H + g_2 H^2 + \dots\right) \frac{\epsilon}{r} \,. \tag{4.27}$$

In order to find solutions for integer PN orders we will need terms $\mathrm{ad}_{G_i}(H_{\text{Kep}})$ in the canonical transformations to yield integer order and hence the G_i to have half-integer order (see (4.26)). It turns out that this can be achieved by the ansatz

$$G_{i-\frac{1}{2}}(q,p) = (p \cdot q) \Lambda_i(a)(q,p) \,, \tag{4.28}$$

where $\Lambda_i(a)$ again denotes a general function of order ϵ^i with coefficients $a_{l,m,n}$ as defined in (3.12).

Each such $G_{i-\frac{1}{2}}$ generates a canonical transformation Φ_i and will be determined such that

$$H := \Phi_n^\star \cdots \Phi_2^\star \Phi_1^\star \tau(\epsilon H_{\text{Kep}}) \,, \tag{4.29}$$

with suitable δ_i in (4.24) defining the energy rescaling τ solves the functional relation (4.27) to order n.

Lemma 1. *For positive integers k and $i_1 \leq i_2 \leq \ldots \leq i_m$, let $I = (i_m, \ldots, i_2, i_1)$*

$$\mathrm{ad}^I_{G_I} := \mathrm{ad}_{G_{i_m - \frac{1}{2}}} \ldots \mathrm{ad}_{G_{i_2 - \frac{1}{2}}} \mathrm{ad}_{G_{i_1 - \frac{1}{2}}} \tag{4.30}$$

with $G_{i_k - \frac{1}{2}} \in W_{i_k - \frac{1}{2}}$, $k = 1, \ldots, m$, and $|I| = \sum_{k=1}^m i_k$. Let H be defined as in (4.29). Then the PN expansion of H to order N is given by

$$\sum_{j=0}^{N} \epsilon^{j+1} H_j \,, \tag{4.31}$$

where

$$H_j := \sum_{n=0}^{j} \sum_{k=0}^{j-n} \sideset{}{'}\sum_{\substack{I \in \mathbb{N}_+^k \\ |I| = j-n}} \frac{\delta_n}{k!} \mathrm{ad}^I_G \left(H^{n+1}_{\text{Kep}}\right) \tag{4.32}$$

is in W_j. Here the prime in the third sum denotes that the summation is restricted to tuples of ordered integers $I = (i_k, \ldots, i_2, i_1) \in \mathbb{N}_+^k$ with $i_1 \leq i_2 \leq \ldots \leq i_k$.

Proof. The result follows immediately from ordering the terms in (4.29) taking into account (4.25) and (4.26). □

We now come to the proof of Theorem 8.

Proof. The proof is done by explicit computation.

From Lemma 1 we get

$$\begin{aligned} H = \quad & \epsilon H_{\text{Kep}} \\ & + \epsilon^2 \Big(\{G_{1-\frac{1}{2}}, H_{\text{Kep}}\} + \delta_1 H_{\text{Kep}}^2 \Big) \\ & + \epsilon^3 \Big(\delta_2 H_{\text{Kep}}^3 + \{G_{2-\frac{1}{2}}, H_{\text{Kep}}\} + \{G_{1-\frac{1}{2}}, \delta_1 H_{\text{Kep}}^2\} + \frac{1}{2}\{G_{1-\frac{1}{2}}, \{G_{1-\frac{1}{2}}, H_{\text{Kep}}\}\} \Big) \\ & + O(\epsilon^4)\,. \end{aligned} \tag{4.33}$$

A fast way to proceed is to rewrite the functional relation (4.27) as

$$H = \epsilon \frac{p^2}{2} - \Big(1 + g_1 H + g_2 H^2 + \dots \Big) \frac{\epsilon}{r} - f_1 H^2 - f_2 H^3 - \dots\,. \tag{4.34}$$

Equating the right hand sides of (4.33) and (4.34) at order ϵ gives

$$H_0 = \left(\frac{p^2}{2} - \frac{1}{r} \right)\,.$$

Plugging this H_0 into the right hand side of (4.34), reading off the terms of order ϵ^2 and equating with the order ϵ^2 in (4.33) gives

$$-f_1 H_{\text{Kep}}^2 - g_1 H_{\text{Kep}} \frac{1}{r} = \delta_1 H_{\text{Kep}}^2 + \{G_{1-\frac{1}{2}}, H_{\text{Kep}}\}\,. \tag{4.35}$$

This is solved by choosing the coefficient of the energy redefinition as

$$\delta_1 = 2g_1 - f_1 \tag{4.36}$$

and the generating function

$$G_{1-\frac{1}{2}} = -g_1 (p \cdot q) H_{\text{Kep}}\,. \tag{4.37}$$

Filling in the $\epsilon H_0 + \epsilon^2 H_1$ into the right hand side of (4.34), reading off the terms of order ϵ^3 and equating with the order ϵ^3 in (4.33) gives

$$\begin{aligned} & (2f_1^2 - f_2) H_{\text{Kep}}^3 + (3 f_1 g_1 - g_2) \frac{H_{\text{Kep}}^2}{r} + g_1^2 \frac{H_{\text{Kep}}}{r^2} \\ & = \delta_2 H_{\text{Kep}}^3 + \{G_{2-\frac{1}{2}}, H_{\text{Kep}}\} + \{G_{1-\frac{1}{2}}, \delta_1 H_{\text{Kep}}^2\} + \frac{1}{2}\{G_{1-\frac{1}{2}}, \{G_{1-\frac{1}{2}}, H_{\text{Kep}}\}\}\,. \end{aligned} \tag{4.38}$$

This can be solved by choosing the next coefficient in the energy rescaling as

$$\delta_2 = 5g_1^2 - 6g_1 f_1 + 2g_2 + 2f_1^2 - f_2 \tag{4.39}$$

and the generating function

$$G_{2-\frac{1}{2}} = (p \cdot q) \left(\frac{1}{2} \Big(-g_1^2 + 2g_1 f_1 - 2g_2 \Big) H_{\text{Kep}}^2 + \frac{\frac{g_1^2}{2} H_{\text{Kep}}}{r} \right)\,. \tag{4.40}$$

We have carried out the computation to order 5PN (ϵ^6) with the help of Mathematica and present the computations and results in ancillary files [181]. □

We note that, assuming the particular form (4.11) of the LRL vector, one can show this is conserved off-shell as well, to at least order 5PN.

4.2.3 Hidden symmetries require Kepler dynamics

In the previous sections we have described a class of relativistic Hamiltonians that turns out to be equivalent to a classical Kepler problem, either being proportional to it on its energy levels or using an approximate canonical transformation and time reparametrisation. In both cases, we also constructed the modified LRL vector.

In this section, we aim to investigate a more general question: what is the largest class of relativistic two-body Hamiltonians (within a certain set of plausible Hamiltonians) that share the symmetries of the Kepler problem? And secondly, is this class related to the Kepler system through canonical transformation and time reparametrisation? This would in effect generalise an aspect of Bertrand's Theorem, discussed in Section 2.4, as all Hamiltonians obeying the symmetries would be equivalent to the Kepler problem - just like in the classical context the only system obeying the symmetries (which requires a vanishing perihelion shift) is Kepler.

Theorem 9. *Let a spherically symmetric class of relativistic two-body Hamiltonians be given by*

$$H = \frac{1}{\epsilon}\left(\epsilon H_0 + \epsilon^2 H_1 + \epsilon^3 H_2 + \dots\right)\,, \tag{4.41}$$

where

$$H_0 = \frac{p^2}{2} - \frac{1}{r} \qquad \text{and, for } j \geq 1, \qquad H_j = \Lambda_j(c)\,, \tag{4.42}$$

with $\Lambda_j(c)$ a function of order j as defined in (3.12)*. At least up to and including 5PN, these Hamiltonians are canonically conjugate to Kepler up to time reparametrisation if and only if they conserve an extra vector, which to leading order is given by*

$$A_0^i(q,p) = (p \times L)^i - \frac{q^i}{r}\,, \tag{4.43}$$

and may contain (and in general will contain) corrections at higher orders.

This additional conserved vector can be seen as a relativistic version of the Laplace-Runge-Lenz vector.

Remark 4. *There are two notions we can use to constrain the number of coefficients $c_{l,m,n}$ in the Hamiltonian* (4.42) *(see also* (3.12)*). Firstly, if a particle is far away from any gravitational body, one expects the attraction to become negligible and the Hamiltonian to approach the special relativistic Hamiltonian, being an expansion only in p^2. Therefore, all momentum-only terms must be solely dependent on the regular momentum and independent of the radial momentum p_r. This removes all terms with coefficients $c_{l,0,n}$ where $n \neq 0$. Secondly, the first order Hamiltonian is known to possess an ambiguity, allowing one to shift the radius such that the term $\sim p_r^2/r$ vanishes [136, 36]. This kind of ambiguity can be expected also at higher orders, but finding these is not a trivial task and not necessary for our analysis.*

Remark 5. *Here we assume the two-body problems can be reduced by translation, which is necessary for the sought-for symmetry to exist. Indeed, we consider these reduced two-body problems, but will continue calling them two-body problems to remind the reader that the physical origin of the Hamiltonians is more general than just one-center systems. Not all* *two-body problems can be reduced by translations, even in classical mechanics, where for example the two-body problem on the sphere does not allow for such reduction, though still being spherically symmetric.*

Proof. While the second part of Theorem 9 of course is a tautology - all systems conjugate to Kepler problems conserve the symmetries of Kepler problems -, the first part is not at all obvious. We have checked the statement up to fifth post-Newtonian order, and will show here the first two.

To prove the (approximate) equivalence, we want to show that the Hamiltonian of a candidate symmetric system can be related through a time reparametrisation and a canonical transformation to the Kepler system. Since the Hamiltonians are divided up in separate orders, we can demand this transformation exists on each order individually. We find the symmetric Hamiltonian and its associated LRL vector by taking an Ansatz for the vector and its corrections and requiring they commute up to an order. That gives a set of relations among both the coefficients $c_{l,m,n}$ of the Hamiltonian (4.42) and $\alpha_{l,m,n}$, $\beta_{l,m,n}$ of the LRL Ansatz, of which the ϵ^j-order term is given by

$$A^i_j = \Lambda_{j-1}(\alpha)(p \cdot q)p^i + \Lambda_j(\beta)q^i \,, \tag{4.44}$$

where we assume (3.12). The Hamiltonian given in terms of the remaining free variables can then be matched to the time-reparametrised, canonically transformed Kepler Hamiltonian, constructed in the same way as in the previous subsection.

At first non-leading order for example, the terms proportional to ϵ take the form

$$\{H, A^i\} = \{H_0, \epsilon A^i_1\} + \{\epsilon H_1, A^i_0\} = 0 \,, \tag{4.45}$$

which results in relations among the 3 coefficients $c_{l,m,n}$ of the Hamiltonian (see Remark 4) and the 9 coefficients of the Ansatz for the LRL vector at first order. The existence of a LRL vector up to this order requires it to take the following form

$$\begin{aligned} \beta_{1,1,0} &= 3\alpha_{1,0,0} + \beta_{1,0,0}c_{1,1,0} + 4\beta_{1,0,0}c_{2,0,0} \,, & \beta_{2,0,0} &= -\alpha_{1,0,0} \,, \\ \beta_{0,2,0} &= -2\left(\alpha_{1,0,0} + \beta_{1,0,0}c_{1,1,0} + 4\beta_{1,0,0}c_{2,0,0}\right) \,, & \alpha_{0,1,0} &= -2\alpha_{1,0,0} \,, \end{aligned} \tag{4.46}$$

with all other coefficients vanishing because of the choices in Remark 4. As $\beta_{1,0,0}$ is the parameter determining the overall size of the vector, the only free parameter that is left over is $\alpha_{1,0,0}$. The term in the LRL corresponding to this parameter turns out to be proportional to $A^i_0 H_0$. These are trivially commuting quantities with H_0, so we can set the coefficient to 0, yielding an expression completely fixed in terms of two of the coefficients of the Hamiltonian. The corrected vector is then a conserved quantity only provided we constrain the Hamiltonian with

$$c_{0,2,0} = -2\left(c_{1,1,0} + 2c_{2,0,0}\right) \,. \tag{4.47}$$

Using the general generating function (4.28) we can then obtain the transformations needed to produce the above Hamiltonian from the Kepler Hamiltonian. These transformations are defined by

$$\begin{aligned} \delta_1 &= -4\left(c_{1,1,0} + 3c_{2,0,0}\right) \,, \quad a_{1,0,0} = c_{1,1,0} + 4c_{2,0,0} \,, \\ a_{0,1,0} &= -2\left(c_{1,1,0} + 4c_{2,0,0}\right) \,, \quad a_{0,0,1} = 0 \,. \end{aligned} \tag{4.48}$$

Following the same procedure at second order, the equation that needs to be satisfied is

$$\{H, A^i\} = \{H_0, \epsilon^2 A_2^i\} + \{\epsilon H_1, \epsilon A_1^i\} + \{\epsilon^2 H_2, A_0^i\} = 0\,. \tag{4.49}$$

Now there are 7 coefficients for the Hamiltonian and 16 for the LRL. This leads to 15 constraints on the coefficients of the LRL vector, given in the ancillary files [181]. Once more, the remaining degree of freedom is proportional to a vector trivially commuting with H_0, that is $A_0^i H_0^2$, and we can set the corresponding coefficient to 0. The transformation then yields a conserved quantity provided we impose the two constraints:

$$\begin{aligned} c_{0,2,1} &= -2c_{1,1,0}^2 - 16c_{2,0,0}c_{1,1,0} - 32c_{2,0,0}^2 - 3c_{0,3,0} - c_{1,1,1} - 5c_{1,2,0} - 8c_{2,1,0} - 12c_{3,0,0}\,, \\ c_{0,1,2} &= 2c_{1,1,0}^2 + 16c_{2,0,0}c_{1,1,0} + 32c_{2,0,0}^2 + c_{0,3,0} - c_{1,1,1} + c_{1,2,0} - 4c_{3,0,0}\,, \end{aligned} \tag{4.50}$$

and thus five free parameters at this order remain in the Hamiltonian. To relate this to Kepler, we need the transformations given by

$$\begin{aligned} \delta_2 &= -4\left(-c_{1,1,0}^2 - 8c_{2,0,0}c_{1,1,0} - 16c_{2,0,0}^2 + 2c_{0,3,0} + 2c_{1,2,0} + 2c_{2,1,0} + 2c_{3,0,0}\right)\,, \\ a_{2,0,0} &= \frac{1}{2}\left(3c_{1,1,0}^2 + 16c_{2,0,0}c_{1,1,0} + 16c_{2,0,0}^2 + 2c_{0,3,0} + 2c_{1,2,0} + 2c_{2,1,0} + 4c_{3,0,0}\right)\,, \\ a_{1,1,0} &= -7c_{1,1,0}^2 - 40c_{2,0,0}c_{1,1,0} - 48c_{2,0,0}^2 - 5c_{0,3,0} - 5c_{1,2,0} - 4c_{2,1,0} - 4c_{3,0,0}\,, \\ a_{0,2,0} &= 8c_{1,1,0}^2 + 48c_{2,0,0}c_{1,1,0} + 64c_{2,0,0}^2 + 7c_{0,3,0} + 8c_{1,2,0} + 8c_{2,1,0} + 8c_{3,0,0}\,, \\ a_{0,1,1} &= \frac{1}{3}\left(-2c_{1,1,0}^2 - 16c_{2,0,0}c_{1,1,0} - 32c_{2,0,0}^2 - c_{0,3,0} + c_{1,1,1} - c_{1,2,0} + 4c_{3,0,0}\right)\,. \end{aligned} \tag{4.51}$$

Similar calculations confirm up to and including fifth PN order that general Hamiltonians of the type described above conserving a LRL vector are related to Kepler via a canonical transformation and time reparametrisation. The ancillary files [181] include a Mathematica notebook with the higher order computations and the resulting Hamiltonians and transformations. □

In other words, this theorem indicates that there are no free lunches: only systems that are canonically conjugate to Kepler up to time reparametrisation have the same extension of the spatial rotation algebra so(3) with hidden symmetries to so(4).

4.3 The Two-Body System of Extremal Black Holes

We now turn to the dynamics of black holes in Einstein-Maxwell-dilaton gravity as an example, as discussed in Section 3.3. We will consider a pair of non-spinning black holes in EMD theory, carrying both electric and dilatonic charge besides their mass.

In the first part of this section, we restrict ourselves to the first order in the post-Newtonian expansion, i.e. at 1PN, and comparable masses. As we will see, for a specific case of the dilaton coupling a and extremal charges, this system coincides with a Kepler-like system. In the second part, we show how the same equivalence to Kepler dynamics arises in a different region in parameter space: instead of 1PN for arbitrary mass ratio, we now focus on the test-mass limit with a vanishing mass ratio, or $m_1 \ll m_2$. This corresponds to the motion of a charged particle in a given background as outlined in Section 3.3, and can be studied at all orders in the post-Newtonian expansion. Prompted by the two-body discussion, we will focus specifically on extremal black holes with opposite charges.

4.3.1 Kepler dynamics at 1PN

For the two-body system with arbitrary masses $m_{1,2}$, electric charges $Q_{1,2}$ and dilaton charges $D_{1,2}$ (subject to the relation (3.60)), the 0PN Hamiltonian in center-of-mass coordinates reads

$$H_{0PN} = \frac{p^2}{2\mu} - \frac{G_{12}M\mu}{r}\,, \tag{4.52}$$

where the effective Newton's constant is given by the interplay between attractive and repulsive forces,

$$G_{12} = \frac{1}{m_1 m_2}(m_1 m_2 + D_1 D_2 - Q_1 Q_2)\,. \tag{4.53}$$

Moreover, we introduce the total mass, reduced mass and symmetric mass ratio given by

$$M = m_1 + m_2\,, \qquad \mu = \frac{m_1 m_2}{M}\,, \qquad \nu = \frac{\mu}{M}\,, \tag{4.54}$$

in the usual way.

The 1PN Hamiltonian can be found in e.g. [152] and can be written in terms of three terms[6]

$$H_{1\mathrm{PN}} = h_1 \frac{p^4}{4\mu^3} + h_2 \frac{\gamma}{\mu^2}\frac{p^2}{r} + h_3 \frac{\gamma^2}{\mu r^2}\,, \tag{4.55}$$

writing $\gamma = G_{12}M\mu$ and with dimensionless coefficients given by

$$h_1 = -\frac{1}{2}(1-3\nu)\,, \quad h_2 = -\frac{1}{2}\left(\frac{3 - D_1 D_2}{G_{12}}\right) - \nu\,,$$
$$h_3 = \frac{\nu}{2} + \frac{1}{2G_{12}^2}\left[(1+D_1D_2)^2 - 2Q_1Q_2 + \left\{\frac{m_1}{M}(D_1^2\beta_2 + Q_1^2(1+aD_2) - 2Q_1Q_2aD_1) + (1\leftrightarrow 2)\right\}\right]. \tag{4.56}$$

All quantities here are asymptotic values, as measured far away from any dilaton charge. Moreover, note that we introduce a slight abuse in notation in the above and hereafter to switch to charges and dilaton charges per unit mass, as in $\tilde{Q}_{1,2} = Q_{1,2}/m_{1,2}$ but dropped the tilde to avoid cumbersome expressions.

A comparison to the 1PN Kepler-type Hamiltonians discussed in Section 4.2 demonstrates that these have two free parameters at every order (including 1PN), while the two-body system here has three terms. For general values, this system will therefore not be related to Kepler via a symplectic transformation. More precisely, the linear combination[7]

$$\begin{aligned}\Delta =& h_1 + 2h_2 + h_3\,,\\ =& -\frac{1}{2G_{12}^2}\Bigg(6(1-Q_1Q_2) + Q_1^2Q_2^2 + 2D_1D_2(2-D_1D_2)\\ &+ \left\{\frac{m_1}{M}\left(-D_1^2\beta_2 - Q_1^2(1+aD_2) + 2Q_1Q_2aD_1\right) + (1\leftrightarrow 2)\right\}\Bigg)\,,\end{aligned} \tag{4.57}$$

quantifies the deviation away from Kepler-like dynamics:

[6] Note that this in general will also have an additional p_r^2/r term, proportional to the radial momentum only. By means of a constant shift of the radial coordinate, one can set the coefficient of this term to zero, see e.g. [36]. We will do so in order to facilitate the comparison to Section 4.2.

[7] This corresponds to the combination $A + 2B + C + D$ in the conventions of [178, 60].

- When Δ vanishes, the Hamiltonian can be written in the form (4.3) (up to 1PN order), identifying
$$f_1 = -h_1, \qquad g_1 = -2(h_1 + h_2)\,. \tag{4.58}$$
In order to see this explicitly, one needs to set $\mu = 1$ and scale the quantity GM with G Newton's constant to $\frac{1}{8}$.[8] Hence there exists a canonical transformation to Kepler and the system has a LRL vector. The form of both the canonical transformation and the conserved charge follow from the discussion in Section 4.2.

- In contrast, when Δ is non-vanishing, the relativistic corrections of this system are not of the Kepler-like form and the corresponding dynamical system differs from Kepler.

The same quantity also determines whether or not bound states have closed orbits: in general they will not, with a perihelion precession given by[9]
$$\delta\phi_{\mathrm{1PN,EMD}} = -\frac{2\pi\gamma^2}{L^2}\Delta\,, \tag{4.59}$$
as also stressed by [60]. As a consistency check, let us point out that the GR limit, where all parameters except m_1, m_2 and L vanish, reduces to
$$\delta\phi_{\mathrm{1PN,GR}} = 6\pi\frac{M^2\mu^2}{L^2}\,, \tag{4.60}$$
as already found by Einstein. Also, the perihelion in Einstein-Maxwell theory, so with dilaton vanishing, becomes
$$\delta\phi_{\mathrm{1PN,EM}} = \pi\frac{M^2\mu^2}{L^2}\left(6(1-Q_1Q_2) + Q_1^2Q_2^2 - \frac{(m_1Q_1^2 + m_2Q_2^2)}{M}\right)\,, \tag{4.61}$$
which in the limit that one mass is much larger than the other agrees with [15].

At this point it might seem that the introduction of the dilaton complicates the expression for the deviation from Kepler enormously. However, there is a massive simplification in the case where the charges are extremal, whose special nature was also highlighted in Section 3.3.2. In the present case of a two-body system, we will have to take both charges extremal and of opposite sign, see (3.64) – when taking the same extremal sign the static forces cancel out and the effective Newton's constant G_{12} vanishes. Instead, when taking opposite signs, all forces are attractive and hence add up in the 0PN Hamiltonian. Furthermore, in the 1PN Hamiltonian, the parameters $\beta_{1,2}$ vanish entirely, leading to the simple result
$$\delta\phi_{\mathrm{1PN,EMD}}\big|_{\mathrm{ext.}} = \pi\frac{4(1+a^2)M^2\mu^2}{L^2}(3-a^2)\,. \tag{4.62}$$
We therefore find that at $a^2 = 3$, this relativistic system of extremal black holes becomes equivalent to Kepler.[10] It has a LRL vector and therefore $SO(4)$ hidden symmetry. Moreover, the orbit closes as the perihelion precession vanishes.

[8] This is because the effective Newton's constant G_{12} is eight times larger than the usual gravitational constant. This corresponds to the findings of [60], who also found this in their supergravity system.

[9] This result has been derived before in [149], though with a different mass function in the sense of Section 3.3.3, such that the results only coincide for extremal black holes.

[10] This value coincides with the Kaluza-Klein reduction of gravity in 5 dimensions [84].

This result is closely related to the findings for extremal black holes in maximal supergravity [60]. The role of the $SU(8)$ charge vector misalignment in maximal supergravity, needed in order to create a nonzero force between the extremal objects other than velocity dependent forces, is played in our case by the opposite nature of the charges.[11] In contrast to the rigid nature of maximal supergravity, enforced by the $N = 8$ supersymmetry, we have the freedom to tune the dilaton coupling, finding that the two-body systems of extremal and anti-extremal black holes always are a special case with a particularly simple expression for Δ, but that this only corresponds to Kepler-dynamics for a particular dilaton coupling.

4.3.2 Kepler dynamics in the test-mass limit

Above, we have shown that the dynamics of the first relativistic correction of a system with comparably-sized masses in EMD theory behaves just like the classical Kepler problem. Now, we wish to extend our analysis to higher orders and will consider another tractable limit: that of the test-mass limit ($m_1 \ll m_2$). Again, we can show the equivalence of this system with opposite and extremal charges in EMD with $a = \sqrt{3}$ to a Kepler-like system. However, in this system we can include all relativistic corrections.

We will focus immediately on the case with extremal charges. The general (scalar) charged black hole metric simplifies significantly in the extremal limit, and it will be convenient to use the Majumdar-Papapetrou solution in the isotropic coordinate system (3.65) [84]

$$\mathrm{d}s^2 = -U^{-2/(1+a^2)}\mathrm{d}t^2 + U^{2/(1+a^2)}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2)\,, \tag{4.63}$$

with a single center:

$$U(r,\theta) = 1 + (1+a^2)\frac{m_2}{r}\,. \tag{4.64}$$

In the extremal case, the scalar field and vector are given by[12]

$$e^{a\phi} = U^{-a^2/(1+a^2)}\,, \qquad Q_1 A_0 = U^{-1}\,, \tag{4.65}$$

where we have chosen static gauge for the latter.

The Lagrangian for a point particle with charge Q_1 reads

$$L_{pp} = m_1 e^{a\phi}\sqrt{-\dot{x}_\mu \dot{x}^\mu} + m_1 Q_1 A_\mu \dot{x}^\mu\,. \tag{4.66}$$

The above is an extended Lagrangian, where the time coordinate can be seen as another dimension in the space; the Lagrangian is defined on the tangent space $T\bar{M}$ of a $d+1$ dimensional manifold $\bar{M} = \mathbb{R} \times M$, the extended configuration manifold. All coordinates and velocities (denoted as a dot) are parametrised by a time-like variable s. Writing the Lagrangian in terms of the harmonic function we have

$$L_{pp} = m_1 U^{-1}\left(\sqrt{1 - U^{4/(1+a^2)}\left|\frac{\mathrm{d}q}{\mathrm{d}t}\right|^2} + 1\right)\dot{t}\,. \tag{4.67}$$

[11] One could further extend our considerations and include magnetic charges as well. We expect the dyonic charges to span a $U(1)$ charge vector playing a completely analogous role to the $SU(8)$ charge vector of [60].

[12] In terms of the Schwarzschild radial coordinate, this choice of gauge corresponds to $A_0 = -\frac{1}{\sqrt{1+a^2}} + \frac{m_2 Q_2}{r}$. After the change $r \to r + r_\pm$, we find the above.

Note that solutions to the Euler-Lagrange equations following from this action will not be unique, as different choices of time parametrisation will correspond to the same physical solution. This fact allows one to reduce the Hamiltonian of the system to an autonomous Hamiltonian on T^*M instead, which is different from the one related by Legendre transform to the Lagrangian above [171], being defined on $T^*\bar{M}$. This, in turns, leads directly to the relation to the classical Kepler problem. Since the theory was written down parametrisation invariant, we can choose any monotonic function as time parameter [207]. We will take the simple time parametrisation $\dot{t} = -1$. The Legendre transform then results in

$$H(q, \frac{\mathrm{d}q}{\mathrm{d}t}) = \frac{\partial L_{pp}}{\partial \frac{\mathrm{d}q}{\mathrm{d}t}} \cdot \frac{\mathrm{d}q}{\mathrm{d}t} - L_{pp} = m_1 U^{-1} \left(\frac{1}{\sqrt{1 - U^{4/(1+a^2)} \left|\frac{\mathrm{d}q}{\mathrm{d}t}\right|^2}} + 1 \right) , \tag{4.68}$$

where $t = x^0$ is the time of the particle seen from the rest frame of the central mass and $q = (x^1, x^2, x^3)$ the position. Solving for the momenta conjugate to the positions,

$$p_i = m_1 \frac{\partial L_{pp}}{\partial \frac{\mathrm{d}q^i}{\mathrm{d}t}} = \frac{U^{(3-a^2)/(1+a^2)} \frac{\mathrm{d}q_i}{\mathrm{d}t}}{\sqrt{1 - U^{4/(1+a^2)} \left|\frac{\mathrm{d}q}{\mathrm{d}t}\right|^2}} , \tag{4.69}$$

then leads to the Hamiltonian in phase space

$$H(q, p) = m_1 U^{-1} \left(\sqrt{1 + U^{2(a^2-1)/(1+a^2)} \frac{|p|^2}{m_1^2}} + 1 \right) . \tag{4.70}$$

Note that the rest-mass energy is equal to $2m_1c^2$; this differs from the usual m_1c^2 due to the specific gauge choice that we have made for the gauge vector.

There is a number of interesting subcases to consider. First of all, the case $a = 1$ leads to a Hamiltonian that is conformal to the special relativistic case,

$$H(q, p) = m_1 U^{-1}(q) \left(\sqrt{1 + \frac{p^2}{m_1^2}} + 1 \right) . \tag{4.71}$$

Instead, our main interest will be the case $a^2 = 3$ again. In this case we have

$$H(q, p) = m_1 U^{-1}(q) \left(\sqrt{1 + U(q) \frac{p^2}{m_1^2}} + 1 \right) . \tag{4.72}$$

Remarkably, this Hamiltonian satisfies the interesting relation

$$\frac{1}{2} \left(\frac{H^2(q,p)}{m_1} - 2H(q,p) \right) = \frac{p^2}{2m_1} - \frac{2m_2 H^2(q,p)}{m_1 r(q)} , \tag{4.73}$$

where $r(q) = |q|$. Shifting the Hamiltonian by the rest-mass energy and rescaling the distance by a factor 8, one obtains (in terms of the new Hamiltonian)

$$H(q,p) + \frac{1}{2} \frac{H^2(q,p)}{m_1} = \frac{p^2}{2m_1} - m_2 \frac{m_1 + H(q,p) + \frac{1}{4m_1} H^2(q,p)}{r(q)} . \tag{4.74}$$

This specific form of the Hamiltonian shows that, following the arguments of Section 4.2, the extremal EMD 1-center system with $a = \sqrt{3}$ is equivalent to the classical Kepler problem. It therefore also has a hidden LRL symmetry as well as closed orbits.

The same special behaviour can also be seen from the perspective of the equations of motion. Adopting the parametrisation $\dot{x}_\mu \dot{x}^\mu = -1$, there are two conserved quantities from the Lagrangian (4.66)

$$L = m_1 U^{(2-a^2)/(1+a^2)} r^2 \dot{\theta}\,, \qquad E = m_1 U^{(-2-a^2)/(1+a^2)} \dot{t} + m_1 Q_1 A_0\,, \tag{4.75}$$

as angular momentum and energy. Using again $\dot{x}^2 = -1$ we can state

$$-U^2 \left(\frac{E}{m_1} - U^{-1}\right)^2 + U^{2/(1+a^2)} \dot{r}^2 + \frac{L^2}{m_1^2 r^2} U^{(2a^2-2)/(1+a^2)} = -1\,. \tag{4.76}$$

It is useful to now take the Binet variable $u \equiv \frac{1}{r}$, with u' as its derivative with respect to θ so that

$$\dot{r} = -u' \frac{L}{m_1} U^{(a^2-2)/(1+a^2)}\,, \tag{4.77}$$

and we find for the equation of motion

$$(u')^2 + u^2 - U^{4/(1+a^2)} \frac{1}{L^2} (E^2 - 2Em_1 U^{-1}) = 0\,. \tag{4.78}$$

The last term here in principle provides an infinite expansion in increasing orders of u (and its accompanying powers of $\frac{1}{c^2}$). However, if we now choose $a^2 = 3$, the powers of the harmonic function simplify and (restoring the gravitational constant) we have

$$(u')^2 + \left(u - 2\frac{Gm_2 E^2}{L^2}\right)^2 = \frac{(E^2 - 2m_1 E)}{L^2} + \frac{4G^2 m_2^2 E^4}{L^4}\,. \tag{4.79}$$

Compare this to the classical equation of motion (see e.g. [207])

$$(u')^2 + \left(u - \frac{Gm_2 m_1^2}{L^2}\right)^2 = \frac{2E_N m_1}{L^2} + \frac{G^2 m_2^2 m_1^4}{L^4}\,, \tag{4.80}$$

where E_N is the Newtonian energy. We see the only difference resides in the modification of the gravitational constant by a function $g(E) = 2\frac{E^2}{m_1^2}$. Accordingly, the Hamiltonian giving the Kepler-like structure in (4.73) here coincides exactly with the role of the Newtonian energy. The orbits will therefore be the same up to the above modification of the gravitational constant.

4.4 Conclusion

This chapter studies relativistic systems of gravitating bodies, with dynamics equivalent to the classical Kepler problem. In particular, we have shown a class of seemingly relativistic

Hamiltonians to have proportional flow to the Keper Hamiltonian on a levelset and we provided the accompanying Laplace-Runge-Lenz vector. Moreover, to fifth order in the PN expansion, we were able to construct the symplectic transformations and energy redefinitions needed to transform the Kepler Hamiltonian into such Kepler-type Hamiltonians explicitly, beyond the levelset equivalence. Additionally, a conjecture was put forth that all relativistic systems of a certain kind, i.e. Kepler at zeroth order and PN corrections of the form

$$c_{n,m,l}\frac{(p^2)^n(p_r^2)^l}{r^m}\,, \tag{4.81}$$

that conserve a (relativistic version of a) Laplace-Runge-Lenz vector are canonically conjugate up to time reparametrisation to the Kepler system. This conjecture was also shown to hold at least to fifth PN order.

Remarkably, this type of Hamiltonians is not merely a mathematical possibility, but it is actually realised in a comparatively simple and interesting physical theory. The Einstein-Maxwell-dilaton theory, when considering two extremal black holes with opposite signs of the charges and dilaton coupling tuned to the Kaluza-Klein reduction value ($a = \sqrt{3}$), has Hamiltonians of exactly this form in both the test-mass limit and the 1PN expansion of the two-body system.

We therefore have established an interesting link between relativistic Hamiltonians, the ordinary Kepler problem and an explicit realisation. Several directions for further exploration present themselves. Firstly, exploring the conditions for local and global existence of the implicit, Kepler-type Hamiltonians and studying the geometry of the corresponding phase space would make for an intriguing investigation.

Secondly, as the equivalence to Kepler for the discussed Hamiltonians is only shown on a levelset, the full phase space will in general look different from the Kepler phase space. Roughly put, the constant energy surfaces are 'stacked' in a different way in the Kepler-type systems as compared to the original Kepler system. This raises the question whether one can always find a symplectic transformation from one to the other, as we have shown explicitly to a limited order. While we expect the normal-form-like construction of canonical transformations to extend to higher orders, perhaps even arbitrarily high orders, there is no guarantee this procedure will converge. However, it would be very appealing, if possible, to construct the asymptotic series of the transformations. Extending our local relations (on or in a neighbourhood of an energy surface) to Kepler to global relations, would also address the question whether the so(4) algebra is indicative of a $SO(4)$ symmetry group.

Thirdly, in the non-relativistic Kepler problem, the geometrical origin of the $SO(4)$ symmetry of 3-dimensional Kepler is known to stem from a mapping to the motion of a free particle on a three-sphere, as derived by Fock [110] in 1935. In the context of the EMD system, we have a natural way of perturbing the Kepler problem, by allowing for example the dilaton coupling to deviate from $a = \sqrt{3}$. This allows one to investigate which elements of this geometric construction would survive such a perturbation in the mapping to the three-sphere. Can the motion still be described by free motion on some hypersurface?

Also related to the larger-than-expected symmetry group of the EMD 1-center system is the Kaluza-Klein reduction of 5-dimensional Einstein-Hilbert gravity, yielding EMD with the special dilaton coupling. Can we understand the origin of the hidden symmetry from the higher-dimensional origin of its theory? After all, while an $SO(4)$ symmetry in 3 dimensions might surprise the reader unfamiliar with the Kepler problem, this is simply the group of spatial rotations in 5D Minkowski spacetime. It would be interesting to investigate this correspondence and possible relation further.

Closely connected to the latter point is the more involved theory of $\mathcal{N} = 8$ supergravity, which can be obtained as the dimensional reduction of supergravity from 11 to 4 dimensions; many of our EMD findings were already highlighted in this setting from the perspective of vanishing periastrion precession [60]. Moreover, extremal black holes in the $\mathcal{N} = 8$ theory have vanishing periastron precession to third post-Minkowskian order [188], at least leaving open the possibility of conserving a LRL vector to higher order and relating to Kepler. It is not clear that this also applies to the higher order two-body Hamiltonians of the extremal EMD with $a = \sqrt{3}$; we leave this interesting question open for future study.

Chapter 5

Bertrand's Theorem and the Double Copy of Relativistic Field Theories

5.1 Introduction

Having seen the example of a relativistic system with Keplerian symmetry in the previous chapter, it is only natural to ask if there exist more such systems, and whether we can write down all of them in a simple way. That will be our aim this chapter.

As discussed previously in Section 2.4, the classical Kepler problem is one of only two central potential problems whose bounded orbits are all closed, the other being the isotropic harmonic oscillator. This is known as Bertrands theorem and goes back to 1873 [200, 125]. It is a consequence of the large, 'hidden' so(4) symmetry of the system, leading to maximally superintegrable dynamics, see Section 2.2.2.

General Relativity breaks this enhanced so(4) symmetry down to so(3) (or even further in case of spinning objects), as do most other relativistic theories. However, there exist relativistic dynamical systems with the same symmetry group as Kepler to all orders. The simplest of these are test-particle limits[1] and include the 'branonium' systems identified in string theory [49, 50]. More recently, the same hidden symmetry was found in $\mathcal{N} = 4$ super-Yang-Mills [58, 6] and $\mathcal{N} = 8$ supergravity [59]. Additionally, in the latter theory, the two-body system is known to possess the symmetry to first order in the post-Newtonian expansion, while it preserves the related non-precession of orbits to third order in the post-Minkowskian expansion [187]. Finally, also the Kaluza-Klein monopole scattering of [120, 121] should be noted.[2]

As we have shown in Section 4.2.3 by explicit construction, Bertrand's theorem can be extended beyond classical central potentials: at least up to and including 5th PN order, there

[1] This is to be expected: while it is far from trivial to write down the Hamiltonian for a multi-worldline system explicitly, the test-particle case can be viewed as a one-body problem, as the large mass is taken to be non-moving.

[2] Another relativistic so(4)-conserving two-body system is the time-asymmetric scalar-vector theory studied by [101].

exists a unique class of relativistic two-body Hamiltonians with the appropriate classical limit that displays additional symmetry. Moreover, all elements of this class can be related to functions of the Kepler Hamiltonian through canonical transformations [180]. An all-order argument was subsequently put forward by Davis and Melville in [93] and will be outlined below.

In the present chapter, adapted from [179], we further investigate this class of relativistic Hamiltonians possessing Keplerian symmetry to all orders, in the form where they have the correct special-relativistic and non-relativistic limits. Our focus will be on three examples of the class, for which we propose a realisation in terms of fundamental forces. We find that all these can be lifted to 5D systems, and subsequently linked to classical theories via the Eisenhart lift. This provides a geometric interpretation of the link[3] with Keplerian dynamics.

The 5D formulation also allows for a connection between the three examples via the double copy. Known primarily as a relation of scattering amplitudes in gravity appearing as the 'square' of gauge theory amplitudes [23], we will instead employ the so-called classical double copy, similarly linking background solutions of these theories [176]. More specifically, we will use it to connect dynamical systems in the sense of [129]. Interestingly, the three examples of relativistic dynamics with Keplerian dynamics are exactly linked in this way. This provides a precise sense in which hidden symmetries and maximal superintegrability[4] carry over under the classical double copy.

5.2 Relativistic Bertrand's Theorem

It is natural to wonder what lies beyond the celebrated Bertrand's theorem of classical mechanics, once moving into the realm of relativistic physics. For reasons that will be outlined in what follows, we will consider an (implicit) definition for our class of relativistic Hamiltonians, much like (4.3) in the last chapter:

$$H^2 = m^2c^4 + p^2c^2 - \left(\frac{1}{r}\right) F\left(mc^2 + H\right) , \tag{5.1}$$

where $F(x)$ is a continuous and smooth function that encodes the nature of the attractive force. Note that this class of Hamiltonians reduces, in the absence of the attractive force ($F \to 0$), to the special relativistic free particle. When perturbatively including the attractive force,

$$F(x) = f_0 + f_1 x + \tfrac{1}{2} f_2 x^2 + \dots , \tag{5.2}$$

we see the coefficients must have dimensions $[f_i] = M^{(2-i)}L^{(3-i)}T^{(i-2)}$. Specialising to $f_2 = 8GM/c^2$ as the only nonzero component for instance, with M some reference mass, the

[3] We generalise this link between classical and relativistic dynamics to multi-center systems in Section 5.6, along with a rigorous mathematical proof of the relation.

[4] Another, less symmetric example is provided by the integrability of the Kerr space-time that also carries over [16].

resulting Hamiltonian reads in a post-Newtonian expansion

$$H = mc^2 + \frac{p^2}{2m} - \frac{8GMm}{r} \\ - \tfrac{1}{8}\frac{p^4}{m^3c^2} + \frac{32G^2M^2m}{r^2c^2} + \mathcal{O}(\frac{1}{c^4})\,. \tag{5.3}$$

In the non-relativistic limit, the system therefore reduces to Kepler (up to a constant rest-mass energy).

The relativistic corrections of this class of Hamiltonians retain the special property that all bounded trajectories are closed orbits. At a heuristic level, this can be seen in the following way. As energy is conserved, the argument of the force term in (5.1) will take a specific, constant value for a given trajectory. For this trajectory, the force is therefore given by some constant F, and the Hamiltonian reads

$$H = mc^2\sqrt{1 + 2H_{\rm Kep}/mc^2}\,, \tag{5.4}$$

for a Kepler Hamiltonian

$$H_{\rm Kep} = \frac{p^2}{2m} - \frac{1}{r}\frac{F}{2mc^2}\,. \tag{5.5}$$

It is therefore natural to expect that this class of systems has closed orbits. This is confirmed by the presence of a relativistic generalisation of the Laplace-Runge-Lenz vector, that takes the form

$$\vec{A} = \vec{p}\times\vec{L} - \frac{1}{2c^2}F\left(mc^2 + H\right)\hat{r}\,, \tag{5.6}$$

in terms of angular momentum $\vec{L} = \vec{r}\times\vec{p}$ and the unit position vector $\hat{r}$. This vector is responsible for the maximal superintegrability and hence closed orbits in the non-relativistic case.

The dynamics of the above theories can be seen as a non-standard way of stacking Kepler energy levels: the effective gravitational coefficient is set by F and hence is orbit-dependent. In a hypothetical solar system governed by such an attractive force, planets would fail to satisfy the universal Kepler's third law relation between period and radius. As discussed in more detail in Section 5.6, a particular time reparametrisation of (5.1) connects it to a new Kepler Hamiltonian, for which Kepler's third law does hold for all orbits with the same universal proportionality constant.

One issue with the above heuristic argument is that it only applies to separate energy levels and not the complete phase space. It would therefore be desirable to have an alternative perspective, that applies to the union of all orbits. Indeed this appears to be possible: we have demonstrated (with an explicit construction) that this class of Hamiltonians is the unique extension of the Kepler system that combines the special relativistic and the non-relativistic limits (when $F \to 0$ and $c \to \infty$, respectively) that has the so(4) hidden symmetry and closed orbits, up to and including 5PN order. It is therefore the natural generalisation of Bertrand's theorem to the relativistic domain (at least for the Kepler system). Moreover, this system is canonically equivalent to Kepler in a neighbourhood around an energy level, at least up to 5PN order [180].

A closely related claim has been put forward by Davis and Melville in [93], who employ a reorganisation of PN corrections into powers of $1/r$ multiplied by functions of the Kepler Hamiltonian. They argue that the function multiplying the $1/r$ part of this expansion can be made vanishing (their 'LRL gauge') by means of canonical transformations. Subsequently, they demonstrate that all higher-order powers of $1/r$ similarly have to vanish for the system to have closed orbits and hidden symmetries. This then implies that the class of Hamiltonians (5.1) is canonically conjugate to Kepler to all PN orders.

Having introduced the class of Hamiltonians as the relativistic generalisation of the Kepler system that has hidden symmetries, in the next sections we turn to their physical interpretation; in other words, which forces generate this kind of dynamics? We will show that there are three cases - where we take F to be constant, linear or quadratic in its argument - where a natural interpretation in terms of 4D relativistic field theories presents itself. Moreover, all three cases allow for an uplift to a 5D system. Different reductions of this 5D perspective allow one to prove the equivalence up to time-reparametrisation between the classical and relativistic systems to all orders in the PN expansion.

5.3 General Relativity

We will first discuss the quadratic case, $F(x) = \frac{1}{2}f_2x^2$, and its interpretation in terms of general relativity. As the constant f_2 must have the dimension of length, it is natural to suppose this is given in terms of a mass scale set by some reference mass M, such that $f_2 = 2r_M$, with $r_M = \frac{4GM}{c^2}$ (while the factor of 4 will be clear in a moment). The Hamiltonian reads

$$H^2 = m^2c^4 + p^2c^2 - \left(\frac{r_M}{r}\right)(mc^2 + H)^2 \,. \tag{5.7}$$

We have shown previously[5] that this system arises when an extremal particle (with equal mass and charge) is orbiting a specific background of the 4D Einstein-Maxwell-dilaton system, with bulk Lagrangian density

$$\mathcal{L} = \frac{\sqrt{-g}}{16\pi}\left(\frac{c^3}{G}R - \frac{2c^3}{G}(\partial\phi)^2 - \frac{e^{-2a\phi}}{c}F^2\right) , \tag{5.8}$$

where we take the units such that the gauge potential A_μ has $[A^2] = MLT^{-2}$. For closed orbits, the dilaton coupling of this system has to take precisely the Kaluza-Klein value $a = \sqrt{3}$ that allows for the bulk Lagrangian to be uplifted to 5D. This suggests that also the particle Hamiltonian (5.7) has a 5D interpretation. Indeed this is the case; upon making the identifications

$$P_0 = \frac{H}{c} \,, \quad P_5 = mc \,, \quad (\textit{space-like}) \,, \tag{5.9}$$

[5]Compare Equation (4.73) with $H \to H + m$.

the Hamiltonian (5.7) corresponds to geodesic motion along a null geodesic,

$$g^{AB}P_A P_B = 0\,. \tag{5.10}$$

The metric is the uplift of the Einstein-Maxwell-dilaton background

$$ds^2 = \eta_{AB}dx^A dx^B + \left(\frac{r_M}{r}\right)(dx^0 - dx^5)^2\,, \tag{5.11}$$

and is a Bargmann space. Because of the harmonicity of the potential $\frac{r_M}{r}$, this space is Ricci-flat, i.e. it solves the vacuum Einstein equations [100]. This spacetime is an example of a pp-wave (plane-fronted wave with parallel rays) due to the presence of the covariantly constant null-vector $\partial_0 + \partial_5$.

Interestingly, the same 5D perspective can also be related to the non-relativistic Kepler case; this is referred to as the Eisenhart-Duval lift [55, 53]. Starting in 5D and making the alternative identifications

$$\frac{P_0 - P_5}{\sqrt{2}} = \frac{\bar{H}_{\text{Kep}}}{c}\,, \qquad \frac{P_0 + P_5}{\sqrt{2}} = mc\,, \quad (null)\,, \tag{5.12}$$

the vanishing norm of the 5D momentum leads to a particular non-relativistic Kepler system

$$\bar{H}_{\text{Kep}} = \frac{p^2}{2m} - \frac{r_M mc^2}{r}\,. \tag{5.13}$$

Though in different guises, the above two reduced systems therefore are one and the same, up to time reparametrisation (or conformal transformations, in the case of null-geodesics [224]). Yet the reductions correspond to what appear to be distinct physical systems, dependent on the different assignments (5.9) or (5.12) for mass and energy, as discussed in generality in Section 5.6.

The 5D formulation allows for a geometric perspective on the so(4) hidden symmetry of the Kepler system, as these can be seen to be generated by an interplay of Killing vectors and tensors [100, 53], discussed in Section 2.2.4. The Bargmann space (5.11) has three Killing vectors associated with angular momentum. In addition, it has three Killing tensors that generate the Laplace-Runge-Lenz vector reading

$$\vec{A} = (\vec{p} \times (\vec{r} \times \vec{p})) - \frac{1}{2}(P_0 + P_5)^2 \frac{r_M c^2}{r}\vec{r}\,, \tag{5.14}$$

which is quadratic in momenta.

Let us conclude this section with two remarks on the 5D symmetry algebra. Firstly, when we take the limit $M \to 0$ toward flat space, the second term in the above vector vanishes, implying the quadratic Killing tensors become reducible, as they are now built completely out of the Killing vectors related to the conserved angular momentum $\vec{L} = \vec{r} \times \vec{p}$ and the momentum $\vec{p}$. The latter in this case is conserved as well, since the space regains translational symmetry in the three spatial directions of $\vec{r}$.

Secondly, note that none of the conserved quantities related through Noether's theorem to the so(4) symmetry of the Kepler problem is generated by a genuinely conformal Killing tensor or vector. All conserved functions commute with the 5D geodesic Hamiltonian regardless of its value. As such, it is clear that the relation between Kaluza-Klein dynamics and the Kepler system still holds for *massive* geodesics, though the effect on the dynamics is of negligible consequence, as it results in the addition of a fixed constant to the Hamiltonians after reduction.

5.4 Electromagnetism

We now turn to the second element of the class of Hamiltonians (5.1) for which we propose an interpretation as coming from a relativistic field theory, being the case with a linear function $F(x) = f_1 x$. In this case the Hamiltonian reads

$$H^2 = m^2c^4 + p^2c^2 - \left(\frac{1}{r}\right) f_1(mc^2 + H)\,. \tag{5.15}$$

Upon making the space-like replacements (5.9), this becomes

$$\eta^{AB} P_A P_B - \left(\frac{1}{r}\right) \frac{f_1}{c}(P_5 + P_0) = 0\,. \tag{5.16}$$

In contrast to the quadratic case, the above cannot be written in terms of a Lorentzian metric solving the Einstein equations.[6] Instead, the natural interpretation of the lifted form is in terms of an electromagnetic force, linear in momenta.

This can be understood in terms of the classical double copy, which maps the specific class of so-called Kerr-Schild backgrounds of General Relativity to solutions of Maxwell's equations in electromagnetism. Among more obvious solutions such as the Kerr and Schwarzschild black holes, also time-dependent pp-waves have been shown to have a classical double copy structure [176]. The spacetime we are interested in is a subset of this, being a time-independent pp-wave.

The class of Kerr-Schild solutions takes the form

$$g_{AB} = \bar{g}_{AB} + r_s \phi\, l_A l_B\,, \tag{5.17}$$

and consist of a background metric $\bar{g}$ (which should separately satisfy Einstein's equations) plus a deformation in terms of a harmonic scalar function, $\Box\phi(r) = 0$, and a vector l_A that is null with respect to $\bar{g}$ and therefore also total metric g. The Schwarzschild radius reads $r_s = 2GM/c^2$. The Kerr-Schild double copy then says that the vector $A_A = \phi\, l_A$ is a solution of the Maxwell equations and $\phi(r)$ is a solution of the scalar field equation.

The Bargmann space introduced above is exactly of Kerr-Schild form, with a flat background metric and SO(3) rotationally symmetric function, $\phi = 2/r$. This gravitational solution therefore generates, via the double copy procedure, the following solution of Maxwell's equations, now including all constants:

$$A_A = \frac{g}{4\pi} \tilde{q} \phi(r) l_A\,, \tag{5.18}$$

[6] It can be written as a momentum norm on a Finsler space, a generalisation of pseudo-Riemannian space [17].

where we introduced g as electromagnetic coupling and $\tilde{q}$ as a charge. This single copy is itself an electromagnetic pp-wave and l_A is its wave vector.

Given the interesting dynamics of geodesics in the Bargmann space-time, can we expect similar behaviour in the Maxwell configuration? In other words, does the hidden symmetry and maximal superintegrability survive the double copy? Interestingly, this turns out indeed to be the case: the Hamiltonian with linear force function (5.15) has an interpretation as a charged particle in the single copy background (5.18). Specifically, after setting $f_1 = \frac{g^2}{4\pi} q\tilde{q}$ and uplifting to 5D via (5.9), the Hamiltonian of the linear case can be written as

$$\eta^{AB}\tilde{P}_A\tilde{P}_B = 0\,, \text{ with } \tilde{P}_A = P_A - \frac{gq}{c}A_A\,, \tag{5.19}$$

in terms of the modified momentum $\tilde{P}$ of charged particles in the vacuum EM background (5.18). This places the linear case on a par with the quadratic one, with interpretations in terms of electromagnetic and gravitational force fields, respectively.

With the 5D interpretation in hand, one can consider its dimensional reductions. The null reduction straightforwardly leads to the usual non-relativistic Hamiltonian, which in this case is interpreted as the Coulomb system. In contrast, the space-like reduction leads to a relativistic field theory with coupled degrees of freedom. In order to see this, separate the 5-dimensional vector field in parts $A_A = (A_\mu, A_5 = \chi)$. The field Lagrangian then reads

$$\mathcal{L} = -\frac{1}{4}F_{AB}F^{AB} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi)\,, \tag{5.20}$$

implying a theory with a scalar χ and vector A_μ. The particle Lagrangian can be found by taking

$$\mathcal{L}_p = -\frac{1}{2h}\eta_{AB}\dot{x}^A\dot{x}^B + \frac{gq}{c}A_A\dot{x}^A\,, \tag{5.21}$$

which can be Legendre transformed to write it in terms of P_5 instead of $\dot{x}_5$. One then has

$$\mathcal{L}_p = \frac{1}{2h}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{h}{2}\left(\frac{gq}{c}\chi - P_5\right)^2 - \frac{gq}{c}A_\mu\dot{x}^\mu\,. \tag{5.22}$$

Solving the equation for the auxiliary variable h, picking the branch giving the correct kinetic term, we find the Lagrangian

$$\mathcal{L}_p|_h = -\sqrt{-\left(\frac{gq}{c}\chi - P_5\right)^2\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} - \frac{gq}{c}A_\mu\dot{x}^\mu\,. \tag{5.23}$$

The solutions of the fields are, as in (5.18), given by

$$\chi = \frac{g}{4\pi}\tilde{q}\frac{2}{r}\,, \qquad A_0 = -\frac{g}{4\pi}\tilde{q}\frac{2}{r}\,. \tag{5.24}$$

This is exactly the theory found by [6], describing a non-minimally coupled scalar-vector theory, having a LRL symmetry. It has its origin in $\mathcal{N} = 4$ supersymmetric Yang-Mills, which similarly enjoys an additional symmetry in the the limit that one of the particles is much larger than the other [58, 199].

Due to the 5D interpretation of this theory, we now have a natural link between the non-relativistic Coulomb system and this specific relativistic Maxwell-axion field theory. The relation between both symplectic reductions implies that the energy-level equivalence between the two theories necessarily extends to all orders in the PN expansion, with the time reparametrisation as discussed in Section 5.6.

5.5 Nordström gravity

The last Hamiltonian in the class (5.1) that we propose a field theoretic interpretation for, is the one with function $F(x) = f_0$ constant:

$$H^2 = m^2c^4 + p^2c^2 - \left(\frac{1}{r}\right) f_0\,. \tag{5.25}$$

The Hamiltonian of this system is a simple function of the Kepler Hamiltonian, and as such naturally has a hidden symmetry and closed orbits. Given the sequence of double and single copy in the previous two sections, however, it is natural to investigate its interpretation as a zeroth copy relativistic field theory. This will feature an attractive scalar field, as first proposed by Nordström in 1912 [183].

The scalar 'zeroth copy' is an Abelian version of the bi-adjoint scalar. The constant f_0 in the potential term now has the dimension of energy squared times length, so we set $f_0 = r_M m_s^2 c^4$, with r_M as before and m_s some mass. Lifting the Hamiltonian to 5D yields

$$\eta_{AB}P^AP^B = m_s^2c^2\frac{r_M}{r}\,, \tag{5.26}$$

and hence a position-dependent mass. Legendre transforming this to the Lagrangian of a geodesic yields

$$\mathcal{L}_p = \frac{\eta_{AB}\dot{x}^A\dot{x}^B}{2h} + m_s^2c^2\frac{h}{2}\frac{r_M}{r}\,, \tag{5.27}$$

where we have included the auxiliary Einbein h. It can be solved for to generate

$$\mathcal{L}_p|_h = m_s c\sqrt{\phi(r)\eta_{AB}\dot{x}^A\dot{x}^B}\,, \tag{5.28}$$

with the scalar field $\phi(r) = \frac{r_M}{r}$ minimally coupled and now dimensionless, and solving the field equation in a vacuum.

This completes the trilogy of double copy related relativistic field theories, at least from the 5D perspective: the Hamiltonians with constant, linear and quadratic functions correspond to the natural sequence of spin-$0, 1, 2$ exchange field theories.

Again, it will be interesting to investigate the dimensional reductions. As before, the null reduction directly generates a non-relativistic system akin to the Kepler and Coulomb ones. The space-like reduction, instead, generates a coupling to a new scalar field profile. After dimensional reduction of (5.27) and subsequent elimination of the Einbein h, we find

$$\mathcal{L}_p|_h = -m_s c\sqrt{\bar{\phi}(r)(-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)}\,, \tag{5.29}$$

with the scalar now given by

$$\bar{\phi}(r) = 1 - \frac{r_M}{r} \,. \tag{5.30}$$

This particular profile for the scalar field then produces exactly closing time-like curves, and it is the Lagrangian giving rise to H in equation (5.25) with proper time parametrisation.

In general, scalar fields generate a perihelion precession that is given to first post-Newtonian order by

$$\Delta\theta = -\frac{1+a_2}{6}\Delta\theta_{\text{GR}} \,, \tag{5.31}$$

with $\Delta\theta_{\text{GR}}$ the perihelion shift in GR and a_2 is a parameter that characterises the coupling of the scalar field (at quadratic level) to the probe particle [94]. How does this and similar results relate to our system displaying no precession? Whereas $a_2 = 0$ in Nordström's final theory (see [203]), $a_2 = 1$ in the only-scalar limit of [93] and Nordström's first theory (see [183, 193]) and indeed, working out the square-root expansion we have $a_2 = -1$ for the scalar (5.30).

At this point it is also interesting to compare to the classification of Bertrand spacetimes, as pioneered by Perlick [190]. These spacetimes are defined by having the special property of having closed orbits; however, Bertrand spacetimes are not required to solve the source-free Einstein's equations and generically need to be supplemented with non-trivial energy-momentum tensors. In contrast, in all three cases that we have discussed, the backgrounds satisfy the source-free bulk equations of motion of the three relativistic field theories. The latter case of the zeroth copy includes the coupling to the harmonic scalar field in [183]. From the particle's perspective, this is equivalent to coupling to a conformally flat metric with overall factor (5.30); this therefore has to be a Bertrand space-time. We have checked that the general class of solutions of [190] indeed includes this as the unique conformally flat possibility.[7]

5.6 Relativistic dynamics as reparametrised classical trajectories

In this more mathematically oriented section we provide further details on the time reparametrisation that relates the (non-)relativistic Hamiltonians, and make this connection rigorous in terms of the Marsden-Weinstein reduction of the higher-dimensional phase space.

To this end, we will start from a slight generalisation of the above, with Hamiltonians $H_2(q,p)$ (introduced in [180]) solving an equation of the form

$$-f(H_2(q,p)) = \frac{p^2}{2C_u} + \frac{g(H_2(q,p))}{C_u}\Phi(q) \,, \tag{5.32}$$

with f and g suitable smooth functions $\mathbb{R} \to \mathbb{R}$ and C_u a nonzero constant. We will show that these are similarly related to classical Hamiltonians using Marsden-Weinstein reduction,

[7] Specifically, the mapping is from the lower signs in equation (13) of [190] with $D \to -2/r_M^2$, $K \to 4/r_M^4$ and $G \to 0$ and rescaling the time $t \to t\, r_M/\sqrt{2}$.

employing a symmetry of a Hamiltonian system to reduce its dimension by an even number. This procedure guarantees that the reduced space remains symplectic, since the symplectic form is given by the restriction of the initial symplectic form to the new space [172].

Assumptions 1. *1. Let $\Phi : \mathbb{R}^d \backslash \Delta \to \mathbb{R}$ be a smooth function, Δ a discrete set of isolated points and $\frac{\partial \Phi(q)}{\partial q} \neq 0$ for all $q \in \mathbb{R}^d \backslash \Delta$.*

2. Let $H_1 : M \to \mathbb{R}$ be a Hamiltonian on $M = T^*(\mathbb{R}^d \backslash \Delta)$ with coordinates $q \in \mathbb{R}^d$ and $p \in \mathbb{R}^d$, defined by

$$H_1(q,p) = \frac{p^2}{2f(C_v)} + \frac{g(C_v)}{f(C_v)}\Phi(q)\,, \tag{5.33}$$

for two smooth functions $f, g : \mathbb{R} \to \mathbb{R}$.

3. Let $C_u, C_v \in \mathbb{R}$ be nonzero constants such that $f(C_v) \neq 0$, $g(C_v) \neq 0$ and

$$g'(C_v)\Phi(q) + f'(C_v)C_u \neq 0\,, \text{ for all } q \in \mathbb{R}^d \backslash \Delta\,. \tag{5.34}$$

Proposition 1. *Given Assumptions 1(1–3), if the Hamiltonian $H_2(q,p)$ implicitly defined by* (5.32) *exists for $H_2(q,p) = C_v$, the trajectories on an energy level $H_1 = -C_u$ of the first Hamiltonian and $H_2 = C_v$ of the second are in one-to-one correspondence, up to the time reparametrisation*

$$\frac{\mathrm{d}t_1}{\mathrm{d}t_2} = \frac{f(C_v)}{g'(C_v)\Phi(q) + f'(C_v)C_u}\,. \tag{5.35}$$

Proof. Consider a manifold $\bar{M} = T^*(\mathbb{R}^2 \times (\mathbb{R}^d \backslash \Delta))$, with canonical coordinates $(u, v, q; P_u, P_v, p)$ and Hamiltonian $\mathcal{H} : \bar{M} \to \mathbb{R}$ given by

$$\mathcal{H}(u, v, q; P_u, P_v, p) = g(P_v)\Phi(q) + f(P_v)P_u + \frac{p^2}{2}\,. \tag{5.36}$$

The proposition is an application of Marsden-Weinstein reduction restricted to the level set $\mathcal{H}^{-1}(0)$. Since u and v are cyclic coordinates by construction and the corresponding actions (by translation) are both free and proper, we can apply the Marsden-Weinstein theorem [172] to symplectically reduce over either of them. We can construct the two possible reductions using the momentum maps $\mu_\star := (\mathcal{H}, P_\star) : \bar{M} \to \mathbb{R}^2$, where $\star$ denotes either u or v, giving two, in principle different base spaces $B_\star = \mu_\star^{-1}(0, C_\star)/\mathbb{R}_\star$ and corresponding projections $\pi_\star : \mu_\star^{-1}(0, C_\star) \to B_\star$. Here $(0, C_\star)$ are assumed to be regular values of $\mu_\star$, for now. Note that the base spaces are odd-dimensional, as the value of $\mathcal{H}$ is fixed, so that the established relation is between energy levels of H_1 and H_2.

Consider the regularity requirements $\mathrm{d}\mathcal{H} \neq 0$ and $\mathrm{d}P_u, \mathrm{d}P_v \neq 0$. The latter are trivially satisfied, while the former gives

$$\begin{aligned}\mathrm{d}\mathcal{H}(u, v, q; P_u, P_v, p) = &\frac{\partial \Phi}{\partial q} g(P_v)\mathrm{d}q + f(P_v)\mathrm{d}P_u \\ &+ (g'(P_v)\Phi(q) + f'(P_v)P_u)\mathrm{d}P_v + p\,\mathrm{d}p\,.\end{aligned} \tag{5.37}$$

By the Assumptions 1(1–3) this is never vanishing on the domain of consideration, nor are $\mathrm{d}\mathcal{H}$ and $\mathrm{d}P_u, \mathrm{d}P_v$ linearly dependent. This shows that the values $(0, C_\star)$ are indeed regular.

Let us now reduce the system over the cyclic coordinates (u, v) to obtain new Hamiltonian functions that are exclusively dependent on the classical variables (q, p). When we reduce by v we have

$$g(C_v)\Phi + f(C_v)P_u + \frac{p^2}{2} = 0\,, \tag{5.38}$$

so that reparametrisation $\lambda \to f(C_v)\lambda$ results in

$$-P_u = \frac{p^2}{2f(C_v)} + \frac{g(C_v)}{f(C_v)}\Phi(q)\,. \tag{5.39}$$

This means that we effectively have a Hamiltonian $-P_u = H_1(q, p)$, which moreover has a time parameter given by $t_1 = u = f(C_v)\lambda$ associated to it, as is apparent from Hamilton's equation $\frac{\mathrm{d}u}{\mathrm{d}\lambda} = \frac{\partial \mathcal{H}}{\partial P_u}$.

If, instead, we reduce phase space $\bar{M}$ over u, we end up with

$$-f(P_v) = \frac{p^2}{2C_u} + \frac{g(P_v)}{C_u}\Phi(q)\,. \tag{5.40}$$

By assumption, this is solved by $H_2(q, p) = C_v$. Hamilton's equation

$$\frac{\mathrm{d}v}{\mathrm{d}\lambda} = g'(P_v)\Phi + f'(P_v)C_u\,, \tag{5.41}$$

then implies the associated time t_2 is given by $\mathrm{d}t_2 = \mathrm{d}v = (g'(P_v)\Phi + f'(P_v)C_u)\mathrm{d}\lambda$. For energy levels $H_1 = -C_u$ of the first Hamiltonian and $H_2 = C_v$ of the second, this means the reparametrisation from the first Hamiltonian to the second is given by

$$\frac{\mathrm{d}t_1}{\mathrm{d}t_2} = \frac{f(C_v)}{g'(C_v)\Phi(q) + f'(C_v)C_u}\,, \tag{5.42}$$

which concludes the proof of the relation between the two. □

For concreteness we now apply this procedure to the relativistic class of Hamiltonians

$$H^2 = m^2 + p^2 - \left(\frac{1}{r}\right) F\left(m + H\right)\,, \tag{5.43}$$

dropping factors of c^2 for legibility. As discussed in the previous sections, under the replacements (5.9) these lift up to the level set $\mathcal{H}^{-1}(0)$ of

$$\mathcal{H} = P_5^2 - P_0^2 + p^2 - \left(\frac{1}{r}\right) F\left(P_5 + P_0\right)\,. \tag{5.44}$$

Hamilton's equation tells us that the relation between the time-parameter λ of this new Hamiltonian is related to the old time parameter t by

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = -2P_0 - \left(\frac{1}{r}\right) F'\left(P_5 + P_0\right)\,, \tag{5.45}$$

where t is conjugate to P_0 (and hence H).

The canonical transformation to null coordinates

$$(x_+, x_-) = \tfrac{1}{\sqrt{2}}\left(x_5 + t, x_5 - t\right) , \tag{5.46}$$

$$(P_+, P_-) = \tfrac{1}{\sqrt{2}}(P_5 + P_0, P_5 - P_0) , \tag{5.47}$$

results in

$$-2P_-P_+ = p^2 - \left(\frac{1}{r}\right) F\left(\sqrt{2}P_+\right) \tag{5.48}$$

on level set $\mathcal{H}^{-1}(0)$. Assuming $P_+ \neq 0$, a time reparametrisation to $\tilde{\lambda} = 2P_+\lambda$ reduces it further to

$$-P_- = \frac{p^2}{2P_+} - \left(\frac{1}{r}\right) \frac{1}{2P_+} F\left(\sqrt{2}P_+\right) , \tag{5.49}$$

so the alternative symplectic reduction over x_+ setting $P_+ = m_+$ results in $-P_- =: H_-$ defining a Kepler Hamiltonian reading

$$H_- = \frac{p^2}{2m_+} - \left(\frac{1}{r}\right) \frac{1}{2m_+} F\left(\sqrt{2}m_+\right) . \tag{5.50}$$

The relation between both time parametrisation therefore reads

$$\frac{\mathrm{d}x_-}{\mathrm{d}\lambda} = \frac{\partial \mathcal{H}}{\partial P_-} = 2P_+ , \tag{5.51}$$

where x_- is conjugate to H_-.

Crucially, the Marsden-Weinstein result guarantees that both reductions preserve the symplectic structure. From the above, it directly follows that the time reparametrisation that links trajectories on energy levels $H_- = \frac{1}{\sqrt{2}}(H_0 - m)$ of the Kepler Hamiltonian to those on energy levels $H = H_0$ of the original is given by

$$\frac{\mathrm{d}x_-}{\mathrm{d}t} = \frac{\sqrt{2}(H_0 + m)}{-2H_0 - \left(\frac{1}{r}\right) F'\left(m + H_0\right)} . \tag{5.52}$$

This transformation maps relativistic systems that violate Kepler's third law to Kepler ones that satisfy it.

Note that only when $F(x)$ is homogeneously quadratic, does each energy level map to the same Kepler background, such that the potential coefficient in the Hamiltonian becomes linear in the mass of the probe. For all other functions, the effective Newton's constant becomes probe-mass dependent and hence would violate the equivalence between gravitational and inertial mass.

5.7 Discussion and conclusion

In this chapter, we have studied the possible interactions giving rise to Keplerian symmetry in field theories. We have presented an extension of the classical Kepler system to a relativistic system with the same so(4) algebra and the correct special relativistic limit, and have checked its uniqueness up to 5PN. This relativistic class of Hamiltonians contains terms naturally interpretable through a lift to 5 dimensions as spin-0,-1 and -2 interaction. We discussed how these project to relativistic theories in 4 dimensions, and their relation to the classical Kepler problem. Moreover, the interactions can be viewed as zeroth, single and double copy in a classical double copy structure.

The 4D systems with so(4) symmetry corresponding to spin-1 and -2 in 5D are known in literature to stem from $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 8$ supergravity [6, 59], which can be truncated to the non-minimally coupled scalar-vector and Einstein-Maxwell-dilaton with $a = \sqrt{3}$ respectively. However, the spin-0 system in terms of a dilaton field theory, whose resulting effective metric coincides with a type of Bertrand spacetime [190], appears to have escaped attention so far.

The trajectories in all systems from a 5D perspective can be seen as reparametrisations of trajectories of a classical Kepler problem. In a similar vein, all one-body dynamics in the particular relativistic systems discussed can be linked to classical dynamics. Though we have focused on the particular case of relativistic 1-center problems with classical symmetry, all the above constructions hold with any harmonic background potential ϕ: the Einstein, Maxwell and scalar field equations are still solved after the 5D lift. The reduced systems are then through time reparametrisation related to their classical counterparts.

For example, a multi-center background in Einstein-Maxwell-dilaton theory with $a = \sqrt{3}$ and extremal objects can be constructed, which is orbited by an anti-extremal test particle. This is the space-like reduction of a Bargmann spacetime of which the null-reduction is the classical Newtonian multi-center system. The relevant harmonic function for n centers (all extremally charged) is given by $-\phi(\vec{r}) = \frac{r_{M_1}}{|\vec{r}-\vec{r}_1|} + \cdots + \frac{r_{M_n}}{|\vec{r}-\vec{r}_n|}$ [117], and the Hamiltonian for an anti-extremal test particle of mass m satisfies

$$H^2 = m^2c^4 + p^2c^2 + \phi(\vec{r})\left(mc^2 + H\right)^2 \,. \tag{5.53}$$

This connection immediately implies these solutions possess the same integrability properties as their equivalents in non-relativistic mechanics, that is, the two-center case is integrable, while the systems with a higher number of centers all display chaotic scattering [42, 155, 156], as discussed for more values of the dilaton coupling in [154].

Some further questions present themselves. Firstly, all mentioned so(4) preserving systems are test-particle limits. It would be interesting to consider the two-body problem beyond the first order in mass ratio or relativistic expansions in these theories, as done for example by [187], who still found no precession of the orbits at third post-Minkowskian order for extremal black holes in $\mathcal{N} = 8$ supergravity (but did not confirm or exclude persistence of so(4) symmetry). Modern scattering amplitude methods[8] seem to be well-suited to this task, especially in the light of the double copy, which might allow one to directly export results from $\mathcal{N} = 4$ super Yang-Mills to $\mathcal{N} = 8$ supergravity.

Secondly, is it possible to extend the idea of relating relativistic systems to geodesics in higher dimensional spaces to multi-body systems? This would need to account for the multi-

[8] For instance, it would be interesting to investigate whether the effective one-body approach in Kerr-Schild formulation [67] allows to capture comparable mass ratio effects in the 5D Kerr-Schild potential.

worldline nature of such systems, possibly by including more time-like dimensions in the higher dimensional system. It would be intriguing to see if this alternative perspective could lead to new or simplifying insights into relativistic dynamics, and whether interesting connections can be made to classical systems.

Chapter 6

Flavour-kinematics duality for Goldstone modes

This chapter studies the double copy in a very different incarnation from that in the previous chapter, instead focusing on scalar effective field theories (EFTs). It follows our paper [182].

6.1 Introduction

General relativity (GR) and Yang-Mills (YM) theory are amongst the central pillars of 20th century physics and describe the gravitational interaction and gauge forces, respectively. Due to different gauge symmetries, spins and quantum (non-)renormalisation properties, these were for a long time thought to have little similarities. However, following the work of *Kawai, Lewellen, and Tye (KLT)* on open-closed string duality, where closed-string amplitudes can be written as a sum over products of open-string amplitudes [150], it has become natural to ask whether (non-)gravitational amplitudes in field theory are similarly related.

A concrete realisation of this duality was proposed by *Bern, Carrasco and Johannsson (BCJ)*, showing that YM and GR tree-level amplitudes can be factorised in a specific way [20]. The resulting colour and kinematic factors (often referred to as BCJ numerators) can be chosen to satisfy group-theoretical constraints corresponding to Jacobi identities. This algebraic correspondence between kinematics and colour, and hence between gauge theory and gravity, is widely referred to as *colour-kinematics (CK) duality* (see e.g. [103, 71, 30] for reviews on amplitudes and CK duality).

Once colour-kinematic duality is satisfied by the *BCJ numerators*, the gravitational tree-level amplitudes can be written as the "square" of their gauge theory counterpart. Following this approach, the YM and GR amplitudes can be written as

$$A_n^{\mathrm{YM}} = \sum_{i \in \text{ cubic}} \frac{C_i T_i}{D_i}\,, \qquad A_n^{\mathrm{GR}} = \sum_{i \in \text{ cubic}} \frac{T_i T_i}{D_i}\,, \tag{6.1}$$

where C_i, T_i and D_i respectively denote colour factors, kinematic numerators and propagator structures. The sums run over a set of inequivalent diagrams i that are purely trivalent, i.e. cubic (and hence differ from the usual Feynman diagrams for these theories). Given a YM amplitude, its gravitational counterpart is constructed by simply interchanging the YM colour factors C_i by another kinematic numerator T_i. This procedure is famously referred to as the *double copy*[1] and it has been proven to hold at tree-level [47, 38, 86, 169, 37, 99, 64]. Furthermore, CK duality was also conjectured to extend to loop level [23] and it has been verified in various non-trivial cases [22, 21, 61, 25]. However, a general loop-level proof is still missing.

Double copy relations are by no means restricted to GR and YM; a natural third theory, completing the triplet in (6.1), consists of two colour factors and corresponds to cubic interactions of *bi-adjoint scalars* (BAS), in the adjoint of both colour factors. Moreover, similar relations have been found for a large web of different theories, including other spin-one theories as well as scalar effective field theories [63, 76]. The latter include the so-called Non-Linear Sigma Model (NLSM) (whose BCJ factorisation was shown in [69]) as well as the special Galileon (SG), and can be built by including a BCJ factor that only depends on momentum and hence describes *scalar-kinematic numerators* (see [98] for explicit expressions). Similar to the triplet BAS-YM-GR for colour and (tensor-)kinematics, this relates the BAS, NLSM and SG [71].

The NLSM and SG are so-called exceptional scalar field theories, with very special properties in terms of non-linearly realised symmetries and Goldstone mode interpretations. In fact, they can be constructed by specifying the amount of derivatives per field while enforcing an *enhanced soft degree* beyond the Adler zero [78, 77].[2] The latter leads to nontrivial cancellations among Feynman diagrams of different topology, thereby completely constraining the interactions of the theories and their symmetry transformations. Due to this structure, their on-shell structure follows from a soft bootstrap approach: all amplitudes[3] can be seen to follow from a single seed interaction, see e.g. [104, 165]. Relatedly, the SG satisfies the equivalence principle and can be phrased in terms of diffeomorphisms [43, 197]. These scalar EFT properties clearly echo the corresponding features of gravity and gauge theories [217, 95].

However, these aspects also raise an interesting question, as there is a third theory with such properties, being *Dirac-Born-Infeld* (DBI) theory with multiple scalars [130]. Inspired by the original colour-kinematics duality relating BAS, YM and GR, is there a similar relation[4] between NLSM, DBI and SG? We will show that this is indeed the case, and that it corresponds to the introduction of a fourth BCJ numerator, that we refer to as *flavour factors*. In the resulting web of dualities, the NLSM then appears twice: either with one colour or with two flavour factors. Are these theories identical? And if not, how do they differ given the strong symmetry constraints on the theory describing pion scattering? The answer to these questions involves graviton exchange in an interesting manner, and is the topic of Section 6.3.

In the same section, we will also demonstrate intriguing relations between the different BCJ

[1]Partial progress on the mechanism underlying the double copy was provided by the identification of the kinematic algebra of the self-dual sector of YM theory [174, 70, 175].

[2]Recently, non-integer soft degrees have been identified in [46].

[3]See [35] for a recent extension of soft limits and recursion to the off-shell wavefunction of these theories.

[4]These three scalar EFTs also allow for a CHY representation of their amplitudes [51].

numerators. Note that this goes beyond the double copy relations of colour-kinematics (or flavour-kinematics) duality: instead of replacing factors to go from one theory to another, these factors themselves also turn out to be related. In particular, we will show how the scalar-kinematic factor follows from the flavour factor (but not vice versa). Although not the focus of this chapter, we also propose a non-invertible mapping from tensor-kinematic numerators to the flavour factors.[5]

Moving from on-shell amplitudes to off-shell aspects, manifestations of the double copy have been found for certain classes of exact classical solutions of GR and YM, being referred to as the *Kerr-Schild double copy*. Initially established as a map from a stationary charge to the Schwarzschild metric [176], this classical double copy was soon afterwards extended to more general stationary and even time-dependent solutions [167, 123, 195, 168, 66, 4, 14]. This construction hinges on the space-time metric admitting so-called *Kerr-Schild coordinates* (see e.g. [202]), leading to the special property that the non-linearities of the Einstein field equations, and consequently the non-linearities of the YM equations, are completely absent. This means that the Kerr-Schild double copy essentially is a mapping between linear solutions of GR and YM.

The existence of these relations raises the question whether there exists an off-shell double copy formulation that takes into account (non-linear) off-shell information. Such a correspondence would be highly non-trivial, since off-shell information, in contrast to amplitudes, depends on the redundancies of the field-theoretic description (such as field basis and symmetry considerations). This redundancy already played an important role for the Kerr-Schild double copies, where the diffeomorphism symmetry was essential to construct the coordinate system in which one recognises the Schwarzschild metric as a double copy of static gauge charge [176].

We will highlight exactly such an off-shell correspondence between the aforementioned triplet of effective scalar theories of NLSM, DBI and SG in Section 6.2. By picking a field basis for which the non-linear symmetries each contain the same type of terms, here chosen to be of the form $\delta\phi = \mathcal{O}(\phi^0) + \mathcal{O}(\phi^2)$, the field equations also take a very similar form, with each theory involving a distinct number of space-time derivatives and flavour structures.

Given these similarities, we show that one can transform kinematic into flavour information (and vice-versa) by expanding the the scalar fields and the parameters of the non-linear symmetries in terms of *auxiliary flavour coordinates* according to $\phi \to \phi_a\theta^a$, where θ^a is the auxiliary coordinate, and where the scalar field on the RHS only depends on the space-time coordinate. Under the assumption that the auxiliary flavour-coordinates are Grassmanian, the transformations of this type constitute invertible mappings between field equations and symmetries of the three Goldstone theories. The existence of these off-shell double copy relations imply that these theories really are different manifestations of the same underlying structure, expressed in different flavour and kinematic spaces.

The picture that emerges from the above considerations is a web of dualities discussed in

[5] Putting these together maps tensor-kinematics directly onto scalar-kinematics. This is closely related to the recent results of [72], that however uses a trace basis instead of the half-ladder basis that will be central in this chapter.

the concluding Section 6.4: the different theories comprising the tensor-kinematic, colour, flavour and scalar-kinematic factors can be graphically represented as the *tetrahedron* in Figure 6.4.[6] The triplet of scalar theories that are investigated in this chapter, related by flavour-kinematics duality, can be found at the right side of the bottom level, thereby lying on the *self-interaction face* - the face GR-SG-NLSM$_g$ defined as those theories that retain non-trivial interactions even when restricted to a single species. We furthermore outline the relations to the other theories including colour.

6.2 Non-linear symmetries and off-shell duality

Our focus in this chapter will be on the three scalar field theories that both allow for a double copy formulation and are determined by a non-linear (NL) symmetry. We will use the form of the latter to single out a field basis for the Goldstone modes in which the double copy is manifest, already at off-shell level (instead of for on-shell amplitudes). We will subsequently explain how this allows for double copy relations between these theories, corresponding to off-shell flavour-kinematics duality.

6.2.1 A triplet of Goldstone theories

We will first focus on the formulation of the three theories, and adapt our field basis for all theories such that the NL symmetries have a similar structure, consisting of two types of terms: $\delta\phi = \mathcal{O}(\phi^0) + \mathcal{O}(\phi^2)$. The first term is a generalised shift term, that is independent of the scalar fields and only depends on the parameters and possibly space-time coordinates. In contrast, the second term is quadratic in the scalar fields, and furthermore depends on the parameters and possibly a space-time derivative. The first term is responsible for the Goldstone interpretation of the scalar fields and induces soft limits such as the (generalised) Adler zero [78, 77]; the second term reflects the non-Abelian nature of these spontaneously broken symmetries [41, 198].[7] Note that all theories are manifestly parity even in this formulation.

The first theory of this form is the *Non-Linear Sigma Model*, corresponding to the breaking of internal symmetries [81, 52]. The corresponding Goldstone modes parametrise a symmetric coset G/H. We will focus on the case

$$\frac{SO(M+N)}{SO(M)\times SO(N)}\,, \tag{6.2}$$

or with isometry group $SO(M,N)$ instead, for the opposite sign choice between the two types of terms in the NL symmetry. The scalars are then fundamental representations of both $SO(M)$ and $SO(N)$, and will be denoted by[8] $\phi^{a\bar{a}}$ - or in matrix notation as ϕ. Note that both flavour parts a and $\bar{a}$ are independent and not necessarily of the same dimension ($M \neq N$).

[6] The author thanks Thomas Flöss for this version of the diagram, based on the one in [182].
[7] A similar structure can be identified in fermion EFTs with non-linear supersymmetry [146].
[8] This may remind the reader of the formulation of GR inspired by double field theory, see e.g. [138, 73].

This coset structure can be realised by the NL symmetry transformation (in matrix notation)

$$\delta\phi = c + \phi c^T \phi \,, \tag{6.3}$$

in terms of a constant matrix c. Including indices, this would correspond to $\delta\phi^{a\bar{b}} = c^{a\bar{b}} + \phi^{a\bar{c}} c^{d\bar{c}} \phi^{d\bar{b}}$. Note that we have suppressed a dimensionful scale in the second term on the RHS that sets the cut-off scale for the EFT, sometimes referred to as the pion decay constant F.

Note that the above coset differs from the chiral NLSM that is often discussed in the literature, based on the symmetry breaking pattern $(G \times G)/G_{\text{diag}}$ with e.g. the pion case having $G_{\text{diag}} = SU(N)$. Our reasons for focusing on the special orthogonal one instead will become clear as we outline the relations to the other Goldstone scalar field theories. Moreover, the cosets are not unrelated: upon identifying the two types of indices, $a = \bar{a}$ (which requires $M = N$), one can specialise to either the symmetric or anti-symmetric case with $\phi = \pm\phi^T$. The former corresponds to the coset $SL(N)/SO(N)$ while the second leads to the chiral one $(SO(N) \times SO(N))/SO(N)_{\text{diag}}$. As $SU(N/2)$ can be embedded in $SO(N)$ this contains the usual pion case. Moreover, the general $SO(M+N)$ case can be specialised to the $SO(M+1)$ case, which is relevant for e.g. the composite Higgs model with $SO(5)/SO(4)$. See [165] for a discussion of the $SO(M+1)$ case from the soft bootstrap perspective.

Returning to the $SO(M+N)$ coset, the lowest order invariant for these Goldstone modes is the two-derivative NLSM Lagrangian, which in terms of the group element g reads

$$\mathcal{L} = \frac{F^2}{4} \left[\partial g \partial g^{-1}\right] , \tag{6.4}$$

where [...] denotes a trace over flavour indices and F is the pion decay constant, which we take to be one for legibility. One representation of an $SO(M+N)$ group element is given by

$$g = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix} , \tag{6.5}$$

where the matrices are written in terms of the $M \times N$ Goldstone modes ϕ as

$$A = \frac{1-\phi\phi^T}{1+\phi\phi^T} \,, \qquad B = \frac{2}{1+\phi\phi^T}\phi \,, \qquad C = \frac{1-\phi^T\phi}{1+\phi^T\phi} \,, \tag{6.6}$$

which have the correct dimensions and properties to make up the $SO(M+N)$ matrix we need. This Lagrangian can be rewritten into

$$\mathcal{L} = -\tfrac{1}{2} \left[\frac{1}{1+\phi\phi^T} \partial\phi \frac{1}{1+\phi^T\phi} \partial\phi^T \right] , \tag{6.7}$$

which is the form we will adhere to in the following. Given the two-derivative nature of this theory, the corresponding field equations are naturally second order. After a number of simple manipulations that amount to solving for the Laplacian of the scalar field in terms of other quantities, these take the form

$$\Box\phi = 2 \sum_{n=1}^{\infty} (-1)^{n-1} [(\partial\phi)\phi^T (\phi\phi^T)^{n-1} (\partial\phi)] \,, \tag{6.8}$$

in $SO(M) \times SO(N)$ matrix notation.

The second theory will involve *Dirac-Born-Infeld (DBI) scalars* that arise from space-time symmetry breaking.[9] We will employ the multi-field generalisation [130] that corresponds to the symmetry breaking pattern

$$\frac{ISO(D+N)}{SO(D) \times SO(N)}\,, \tag{6.9}$$

where D refers to the space-time dimension (and we are cavalier about its signature; one of the D dimensions is actually time-like). Again, a different sign choice will change the isometry group to $ISO(D,N)$ instead. The spontaneous breaking of translations in the internal dimensions results in a number of scalar fields ϕ^a that transform in the fundamental representation of the internal symmetry $SO(N)$. The NL symmetry takes the form

$$\delta\phi^a = c^a + c^a{}_\mu x^\mu + c^b{}_\mu \phi^b \partial^\mu \phi^a\,, \tag{6.10}$$

and consists of a constant and linear shift (corresponding to translations and boosts) as well as a quadratic part (from the non-Abelian nature of boosts).

The invariant Lagrangian can be written in terms of the induced metric $g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \phi_a \partial_\nu \phi^a$. At lowest order in derivatives, this is given by the measure [130],

$$\mathcal{L} = 1 - \sqrt{g} = -\tfrac{1}{2}\partial_\mu\phi \cdot \partial^\mu\phi + \tfrac{1}{4}\left(\partial_\mu\phi \cdot \partial_\nu\phi\right)\left(\partial^\mu\phi \cdot \partial^\nu\phi\right) - \tfrac{1}{8}\left(\partial_\mu\phi \cdot \partial^\mu\phi\right)^2 + \ldots\,, \tag{6.11}$$

where dots indicate flavour contractions. The field equations can be brought to the form

$$\Box\phi^a = \sum_{n=1}^{\infty}(-1)^{n-1}[(\partial\partial\phi^a)(\partial\phi\cdot\partial\phi)^n]\,, \tag{6.12}$$

in matrix notation for the space-time indices (and similar for the trace [...]).

Finally, the *special Galileon theory* [78, 137] involves only a single Goldstone mode, with NL symmetry

$$\delta\phi = c + c_\mu x^\mu + c_{\mu\nu}(x^\mu x^\nu + \partial^\mu\phi\partial^\nu\phi)\,. \tag{6.13}$$

where again the first two parts are Abelian shift symmetries, and only the tensor transformation (with traceless parameter and a field-dependent, quadratic part) corresponds to the non-Abelian part. The latter corresponds to the coset

$$\frac{ISU(D)}{SO(D)}\,, \tag{6.14}$$

(where $ISU(D)$ denotes the semi-direct product $\mathbb{R}^D \rtimes SU(D)$) while the former are central extensions thereof. The opposite sign choice in the NL symmetry modifies the special unitary

[9] Interestingly, in this case the Goldstone theorem [126, 127] that associates a massless mode to every broken generator no longer applies, see e.g. [142, 162, 153].

group to the special linear group (i.e. corresponds to a different real section of the complex group), and can be seen as Goldstone mode for affine coordinate transformations [197].

The invariant Lagrangian is given by a sum (with specific coefficients) of all Galileon terms with an even number of fields. The Lagrangian in D dimensions reads [137]

$$\mathcal{L}_{\text{SG}} = -\frac{1}{2} \sum_{n=1}^{\left\lfloor \frac{D+1}{2} \right\rfloor} \frac{(-1)^{n-1}}{(2n-1)!} (\partial\phi)^2 \mathcal{L}^{TD}_{2n-2} \,. \tag{6.15}$$

The total derivative terms are given by

$$\mathcal{L}^{\text{TD}}_n = \sum_p (-1)^p \eta^{\mu_1 p(\nu_1)} \eta^{\mu_2 p(\nu_2)} \cdots \eta^{\mu_n p(\nu_n)} \left(\Phi_{\mu_1\nu_1} \Phi_{\mu_2\nu_2} \cdots \Phi_{\mu_n\nu_n} \right) \,, \tag{6.16}$$

where the sum is taken over all permutations of the indices ν, with the sign of the permutation given by $(-1)^p$. The three leading terms explicitly read

$$\mathcal{L}^{\text{TD}}_0 = 1 \,, \quad \mathcal{L}^{\text{TD}}_2 = [\Phi]^2 - \left[\Phi^2\right] \,, \quad \mathcal{L}^{\text{TD}}_4 = [\Phi]^4 - 6\left[\Phi^2\right][\Phi]^2 + 8\left[\Phi^3\right][\Phi] + 3\left[\Phi^2\right]^2 - 6\left[\Phi^4\right] \,, \tag{6.17}$$

where $[\Phi \cdots \Phi]$ denotes the trace over a product of matrices $\Phi \equiv \partial\partial\phi$ (all referring to space-time indices). In contrast to the previous two theories, the number of interaction terms in the Lagrangian is finite. As a consequence, the field equations can be written in a form with a finite number of interactions, which is strikingly different from the NLSM and the DBI theory. However, the SG field equations can also be brought to a similar form with an infinite sum of interactions,

$$\Box\phi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1} [\Phi^{2n+1}] \,, \tag{6.18}$$

after isolating the Laplacian on the field on the left hand side.

The above theories all contain a single scale that sets the magnitude of the interaction; we have not explicitly included this scale in the above, but it can be reinstated using dimensional analysis. Taking this scale to be an imaginary parameter effectively takes one from the above compact expressions to their non-compact versions. There are interesting constraints from UV considerations on which of these two constitute EFTs with viable UV completions; for single scalars, only the $ISO(1, D)$ version of DBI satisfies such positivity bounds, and similarly the $ISL(D) = \mathbb{R}^D \rtimes SL(D)$ version of the special Galileon, see e.g. [5, 194].

Finally, we would like to point out that there is a specific freedom in the construction of the NLSM that is absent for the other two theories. This can be seen at the off-shell level from two perspectives. The first one would be to notice that the NLSM is the only symmetry that is internal and has constant (i.e. space-time independent) parameters. Due to this, one can implement this symmetry on arbitrary backgrounds, and moreover it allows for dynamical gravity. In other words, the construction of the NLSM carries over without modification after coupling it to gravity (see the appendix for details on this Lagrangian). Interestingly, this introduces a second coupling constant; in addition to the pion decay constant, we now also have Newton's constant (or the Planck mass) as a free parameter.

A closely related perspective on this freedom is offered by the coset constructions. As the SG and the DBI theories have space-time symmetries, the corresponding cosets (of broken and unbroken symmetries) therefore necessarily include the Lorentz symmetry. In contrast, the NLSM can be seen as a product of two cosets, the one corresponding to the pion sector and the other forming the background; for flat geometries, the latter is simply Poincare over Lorentz as global symmetries:

$$\frac{ISO(D)}{SO(D)} \times \frac{SO(M+N)}{SO(M)\times SO(N)}\,, \tag{6.19}$$

with the coordinates x^μ and the scalar fields $\phi_{a\bar{b}}$ corresponding to the broken generators. Upon gauging the algebra of space-time symmetries, one obtains dynamical gravity and non-linear (and non-Abelian) diffeomorphisms. The product of both cosets thus leads to the possibility to introduce a parameter for each coset.

6.2.2 Flavour-kinematics duality at the off-shell level

The similar natures of the non-linear transformations and the field equations suggest a relation between these theories at the off-shell level. Indeed one can go from theory to theory by replacing flavour with kinematic information, or vice-versa.

We will start with the non-linear tensor transformation of the SG theory, and note that it can be written in the following compact form

$$\delta\phi = p + \tfrac{1}{2}\partial^\mu\phi\partial_{\mu\nu}p\partial^\nu\phi\,, \tag{6.20}$$

where one should interpret the parameter p as a quadratic polynomial in the space-time coordinates x^μ. In order to introduce flavour, one can augment these coordinates with a set of auxiliary coordinates θ^a; these can be taken as an auxiliary construct, introduced to unify the three different theories. The parameter is now taken to be linear in the novel coordinate, $p = p_a\theta^a$, where in turn the p_a are at most linear in space-time coordinates. The above transformation then takes the form (summing over both types of coordinates in (6.20))

$$\delta\phi = p + \partial^\mu\phi\partial_{\mu a}p\partial^a\phi = p + \partial^\mu\phi\partial_\mu p_a\phi^a\,, \tag{6.21}$$

where in the final expression we have similarly expanded the scalar field as $\phi = \phi_a\theta^a$, i.e. as linear in the flavour coordinates. Note that the resulting components ϕ_a only have space-time dependence. In this way, the resulting transformation law is identical to that of DBI (6.10) after expanding the above expression along the flavour coordinates.[10]

Going one step further, one can also replace the remaining dependence on space-time coordinates with another flavour coordinate, $\bar{\theta}^{\bar{a}}$. These are taken as independent from the flavour

[10] In a similar approach, colour information was systematically replaced by kinematic information in [72], only at the level of currents instead of fields as outlined here.

coordinates θ^a (and in particular should not be read as (anti-)holomorphic coordinates). The parameter can then be taken bilinear in both flavour coordinates, $p = p_{a\bar{b}}\theta^a\bar{\theta}^{\bar{b}}$, with constant coefficients $p_{a\bar{b}}$. Similarly, we will take the fields to be bilinear in these, $\phi = \phi_{a\bar{b}}\theta^a\bar{\theta}^{\bar{b}}$. This results in the transformation law

$$\delta\phi = p + \partial^a\phi\partial_{a\bar{b}}p\partial^{\bar{b}}\phi = p + \bar{\theta}^{\bar{a}}\phi_{a\bar{a}}p_{a\bar{b}}\phi^{b\bar{b}}\theta^b\,. \tag{6.22}$$

Again, after expanding along the flavour coordinates, this expression is identical to the NLSM transformation law (6.3). This demonstrates the close relations between the different NL symmetries: the transformation laws are identical upon the appropriate identification of flavour dependencies.

This discussion has a parallel for the field equations - with an interesting twist moreover. We will consider the effect of the flavour Ansatze outlined above. Taking ϕ to be purely space-time dependent corresponds to the SG field equation (6.18). When instead taking it linear in a flavour coordinate, the LHS retains this linearity and hence is proportional to θ^a. On the other hand, the non-linearities corresponding to interactions can yield higher-order expressions.

For simplicity, let's first discuss the cubic term on the RHS of (6.18). Summing over both space-time and flavour coordinates, this expression yields two types of terms: with either three pairs of space-time contractions, or two space-time and one flavour contraction (terms with more than one pair of flavour contractions vanish as they will involve multiple flavour derivatives on a single field, incompatible with the flavour Ansatz). Starting with the latter, it takes the form

$$[\Pi(\partial\phi\cdot\partial\phi)] = \theta^a\partial_{\mu\nu}\phi_a\partial^\mu\phi_b\partial^\nu\phi^b\,, \tag{6.23}$$

where in the compact first expression, the trace and matrix multiplication are for space-time indices μ, and the dot is for flavour indices a. Note that the 1/3 coefficient of this cubic SG Galileon in (6.18) is cancelled by the three-fold choice to distribute the flavour indices over the trace. Stripping off the overall auxiliary flavour coordinate, the LHS and cubic interaction then exactly combine into the corresponding terms for the multi-DBI field equation (6.12). The same also works for higher-order terms.

So far the discussion is completely analogous to that of the NL symmetries. However, in contrast to that case, the field equations also generate higher-order terms. For the cubic term, it takes the form $[\Pi^3]$ and hence is cubic in flavour coordinates. There will be similar contributions from the quintic and higher-order terms in the field equations that also generate a θ^3 term. We therefore conclude that the SG field equation is, under the flavour Ansatz, translated into a set of conditions on the multi-DBI scalar fields. One of these is exactly the DBI field equation, with contributions such as (6.23). Others are higher-order, such as the cubic one - which is cubic in flavour coordinates, and cubic plus higher-order in fields.

Given the exact mapping between the transformation laws and the distinct dependencies on the flavour coordinates, these different equations must be separately invariant. Indeed the lowest-order one is identical to the DBI field equation, and the higher-order ones must be analogous in that they are invariant conditions. In order to get rid of these, one can take different stances. One would be to explicitly truncate the field equation at order θ, thus retaining the lowest-order contribution. Another would be to take the auxiliary flavour dimensions to be Grassmannian; when contracted with the trace, the anti-symmetry then kills this term.

Analogous considerations apply to the transition from DBI to the NL $\sigma-$model. Instead, we now take ϕ to be bilinear in the flavour coordinate $\phi^a = \phi^{a\bar{b}}\bar{\theta}_{\bar{b}}$.

For simplicity, let's consider cubic RHS of the DBI field equation (6.12). The trace over two space-time matrices generates two types of terms: with either zero or one flavour contraction (note that the term involving two flavour contractions again vanishes due to linearity in the flavour coordinate). Like before, the former is proportional to $\bar{\theta}^3$ and therefore vanishes under the assumption that the flavour dimensions are Grassmannian. The latter takes the form

$$2\bar{\theta}_{\bar{d}}\partial_\mu\phi^{a\bar{b}}\phi^{c\bar{b}}\partial^\mu\phi^{c\bar{d}}\,, \tag{6.24}$$

where the factor two follows from the two-fold possibility to distribute the flavour indices within the trace. Stripping off the auxiliary flavour coordinate, the LHS and cubic interaction exactly coincide with the cubic part of the NL $\sigma-$model field equation (6.8). Again, this also works for the higher-order interaction terms, where the two-fold choice remains because the flavour contraction needs to be on one of the partial derivatives on the two-derivative factor and on all contractions not involving it.

The above identifications lead to mappings of the three different theories with identical coupling constants (which we have not explicitly included for the moment). By allowing for numerical coefficients in the mapping, e.g. $\phi = (M_{\text{SG}}/M_{\text{DBI}})\phi_a\theta^a$, one can also introduce arbitrary ratios. This will always map compact onto compact cosets, however. Finally, the flavour-kinematics duality as outlined here, i.e. at the off-shell level, does not imply any restrictions or identifications between the two coupling constants of the NLSM; we will see in what follows that this will be different for the on-shell story.

6.3 Scattering amplitudes and on-shell duality

We now turn to the scattering amplitudes of the triplet of theories that we focus on. We will first discuss the BCJ formulation in terms of different kinematic numerators, and subsequently outline the relations between these numerators.

6.3.1 The BCJ formulation

In the BCJ formulation, amplitudes are generated by a sum over trivalent diagrams. As the first non-trivial example, the four-point amplitude can be written as

$$A_4 = \sum_{\text{exchange}} \frac{N_{ijkl}\tilde{N}_{ijkl}}{s_{ij}}\,, \tag{6.25}$$

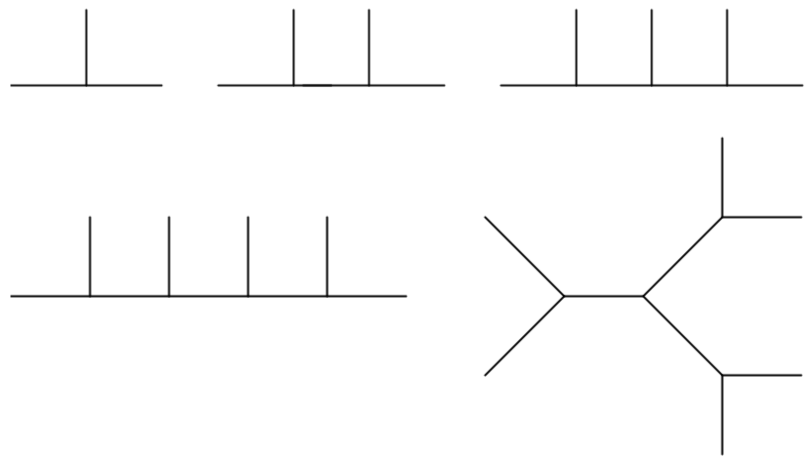


Figure 6.1: *The trivalent diagrams relevant at lower-point scattering; all have half-ladder topology at three-, four- and five-points, while at six-points there are half-ladder and snow-flake topologies.*

where the sum is over the three inequivalent exchange diagrams (see Figure 6.1), corresponding to the permutations $(ijkl) = (1234, 2314, 3124)$ (i.e. the s, t and u exchange channels), and the propagator structure in the denominator is $s_{ij} = (p_i + p_j)^2$. The reduction from in principle 4! permutations down to three is due to three $\mathbb{Z}_2$ symmetries: anti-symmetry in the first pair and in the last pair of indices, as well as order inversion. Note that these are all inequivalent trivalent diagrams when taking order invariance and anti-symmetry into account: using these, particle 4 can always be placed at the final entry and there are three inequivalent options (particle $1, 2, 3$) for the penultimate entry.

More generally, this leads to $(2n-5)!!$ diagrams for general n-point amplitudes, leading to 15 and 105 contributions to five- and six-point amplitudes. The former consists of all halfladder permutations subject to the above $\mathbb{Z}_2$ symmetries (with $5!/2^3 = 15$ inequivalent possibilities). In the latter case, however, there are two inequivalent topologies, half-ladder and snow-flake diagrams, as is shown in Figure 6.1.

There are $6!/2^3 = 90$ independent half-ladder permutations, with kinematic factors N_{ijklmn}. Next to these, there are $6!/(2^3 \cdot 3!)$ snow-flake permutations; there is a 3! reduction due to equivalence of the three legs, and 2^3 due to the anti-symmetry in every leg. The kinematic factor for such diagrams is related via Jacobi identities and given by $N_{ijklmn} - N_{ijlkmn}$. The six-point amplitude then takes the form

$$A_6 = \sum_{\text{half-ladder}} \frac{N_{ijklmn}\tilde{N}_{ijklmn}}{s_{ij}s_{ijk}s_{mn}} + \sum_{\text{snow-flake}} \frac{(N_{ijklmn} - N_{ijlkmn})(\tilde{N}_{ijklmn} - \tilde{N}_{ijlkmn})}{s_{ij}s_{kl}s_{mn}}, \qquad (6.26)$$

in terms of the two numerators N and $\tilde{N}$. Note the two different propagator structures in the denominators (where $s_{ijk} = (p_i + p_j + p_k)^2$).

We now turn to the algebraic conditions on the numerators. In the BCJ formulation, the four-point amplitude factorises into kinematic numerators. These are required to satisfy the following symmetry conditions:

$$N_{ijkl} = -N_{jikl}\,, \quad N_{ijkl} = N_{lkji}\,, \quad N_{ijkl} + N_{jkil} + N_{kijl} = 0\,. \tag{6.27}$$

In addition to anti-symmetry in the first pair of indices and the order invariance, the third condition is often seen as a kinematic version of the Jacobi identity on structure constants.

There is a natural generalisation of this story to higher n-point factors. These are subject to the following Jacobi-like identities,

$$-N_{ijkl...} = N_{jikl...} = N_{k[ij]l...} = N_{l[[ij]k]...} = ... \tag{6.28}$$

involving multiple commutators for the first $n-1$ indices. In total there will be $n-2$ conditions of this form, generalising the two for the 4-point function mentioned above. In addition, one can impose the order reverse condition

$$N_{ijklm...} = (-)^n N_{...mlkji}\,. \tag{6.29}$$

These conditions translate into constraints on the possible representations that the factors can take. The corresponding Young tableaux are illustrated in Figure 6.2 and are given by the following:

- For three-point factors, one can only have the anti-symmetric tensor. Its dimension (as element of the symmetric group) is 1.
- At four-point, the unique representation is the window tensor with dimension 2.
- At five-foint, we find the equal-arms hook tensor with dimension 6.
- At six-point, we find three irreps with different Young tableaux. Their dimensions are 5, 9 and 10, respectively, adding up to 24.

Interestingly, these (ir)reducible representations have dimensions equal to $(n-2)!$ for every n-point factor.

The subsequent question is how to find specific representations that solve these conditions. For colour, these can be represented in terms of the structure constants of the group (where A is the adjoint representation), and read

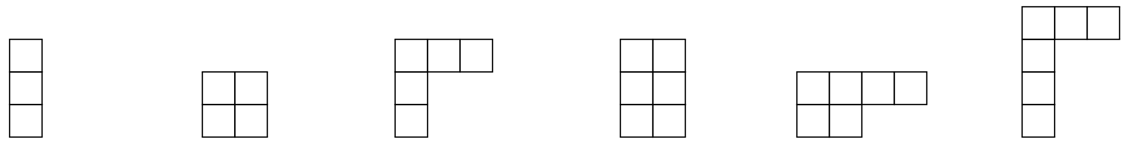
$$N_{1234...} = f_{AB}{}^{P} f_{PC}{}^{Q} f_{QD}{}^{R}....\,, \tag{6.30}$$

involving commutators of the first $n-1$ indices. So for instance, at three and four-point, these are[11]

$$N_{123} = f_{ABC}\,, \quad N_{1234} = f_{AB}{}^{P} f_{PCD}\,. \tag{6.31}$$

Note that we assume the existence of a metric in the adjoint representation to lower the last index. These numerators naturally satisfy the Jacobi identities outlined above, and give rise to unique BCJ representations for colour.

[11] In order not to clutter expressions, we are suppressing the adjoint indices on the LHS here; one should read this as e.g. $(N_{123})_{ABC} = f_{ABC}$, with the understanding that A refers to the adjoint rep of the first particle etc.



(a) (b) (c) (d) (e) (f)

Figure 6.2: *Young tableaux for the BCJ numerators at a) three-, b) four-, c) five-, and d,e,f) six-point.*

6.3.2 Flavour and kinematics numerators

While the original BCJ paper addresses the factorisation of the scattering amplitudes of Yang-Mills and gravity in terms of colour numerators and kinematic numerators that involve momenta and polarisations, there have been many generalisations since. One of these is the identification of a numerator factor that solely depends on the Mandelstam variables, without polarisation. As a consequence, this describes the scattering of scalar particles, without additional structure (such as colour, flavour or spin). We first outline this possibility before considering generalisations that include flavour.

For scalar theories, the kinematic numerators can only depend on Mandelstam variables. The absence of colour structure implies that no anti-symmetric three-point numerators exist in the kinematic case, since all momentum contractions vanish under momentum conservation for massless particles. In this sense there is no kinematic analogue of structure constants.

At four-point, one has to impose the Jacobi identities[12] (6.27). We will be interested in factors that are quadratic in the Mandelstam variables,[13] in order to have the correct number of derivatives to connect to the special Galileon. The most general solution then reads

$$N_{ijkl} = \lambda_4 s_{ij}(s_{jk} - s_{ik}) \,, \tag{6.32}$$

for the s-channel contribution. The resulting amplitude is

$$A_4 = -9{\lambda_4}^2 s_{12} s_{23} s_{13} \tag{6.33}$$

This four-point amplitude indeed coincides with the special Galileon interaction of (6.18). Note that, despite the propagators in the BCJ formulation of the amplitude (6.25), there are no poles over any Mandelstam variables in the result. This coincides of course with the statement that the theory under consideration has no three-point interactions that the four-point amplitude could factorise into.

[12]An alternative to solving Jacobi identities was investigated in [44], by introducing the notion of numerator seeds. Following the rewriting of structure constants as a specific linear combination of traces of generators, their idea is to identify the (simpler) kinematic seeds that are the analogue of traces of generators. Subsequently, the full numerators are constructed by multiplying the seeds with a matrix J encoding the Jacobi-like identities.

[13]There is also a linear solution, with $N_{ijkl} \sim (s_{jk} - s_{ik})$, that will however have a more natural interpretation when including flavour later on. It can also be used as a building block to generate quadratic and higher solutions along the lines of [65]. Monomials of higher order than quadratic could correspond to e.g. higher-derivative corrections to the theories that we consider; however, we will restrict ourselves to the first non-trivial possibilities.

The next case to consider are the 5-point factors. In this case, there are four constraints on the factors: in addition to the anti-symmetry and the order reversion ones, there are two Jacobi-like constraints involving commutators. We have checked that these constraints have no solutions below cubic order in Mandelstam. At cubic order, there is a single parameter. However, it turns out that the full amplitude vanishes in this case; the parameter should therefore correspond to a generalised gauge transformation. One can see this in the following way.

Suppose the numerator would be (or would contain terms) of the following form:

$$N_{ijklm} \sim s_{ij}G_{ijklm} - s_{lm}G_{mlkji} \,. \tag{6.34}$$

When tensored with an arbitrary other numerator $\tilde{N}$, the amplitude receives a contribution

$$\sum \frac{\tilde{N}_{ijklm}G_{ijklm}}{s_{lm}} - \frac{\tilde{N}_{ijklm}G_{mlkji}}{s_{ij}} \,. \tag{6.35}$$

Provided the gauge parameters G_{ijklm} are fully anti-symmetric in the first three parameters (ijk), these 30 terms nicely combine into 10 triplets, where each triplet shares a common denominator. For instance, there will be terms proportional to

$$\frac{(\tilde{N}_{ijklm} + \tilde{N}_{jkilm} + \tilde{N}_{kijlm})G_{ijklm}}{s_{ij}} \,, \tag{6.36}$$

and similar for the other denominators. Of course, these terms nicely combine into a Jacobi identity and therefore cancel. Note that this is independent of the specific form of the second numerator $\tilde{N}$; it only requires that these satisfy the Jacobi identities. It turns out that the most general 5-point factor is exactly of this form, with gauge parameter given by

$$G_{ijklm} = (s_{il}s_{jm} - (l \leftrightarrow m)) + (cyclic) \,, \tag{6.37}$$

where the two cyclic terms refer to cyclicity in (ijk). Therefore, the most general kinematic 5-point numerator is a gauge transformation.

At the six-point level, we will be interested in quartic factors in Mandelstam variables in order to connect to e.g. the special Galileon.[14] Solving the Jacobi identities (6.28) and (6.29) leaves one with 23 free parameters. Additionally, we have to impose factorisation into even amplitudes: the resulting amplitude may not have any poles over single Mandelstam variables, as this would correspond to splitting into a three- and a five-point vertex. This constraint further reduces the free parameters to only six.[15] When calculating the amplitudes, one finds that these only depend on a single linear combination of these coefficients. We therefore find five gauge parameters and one physical one.

Motivated by the overall s_{ij} dependencies of the 4- and 5-point kinematic factors, we consider a similar Ansatz at 6-point:

$$N_{ijklmn} = s_{ij}P_{ijklmn} + \textit{ order reversed} \,. \tag{6.38}$$

[14]There are also quadratic and cubic solutions, that however are again naturally interpreted in the context of flavour.

[15]Operationally, we have first required the correct factorisation for amplitudes with one colour factor, and subsequently checked that kinematic × kinematic amplitudes also factorise correctly.

It turns out this Ansatz contains one gauge parameter and the physical one. The latter corresponds to the explicit expression

$$\begin{aligned} P_{ijklmn} =& - s_{mn}\Big(s_{jk}\Big(- 4s_{in} + 4\Big(s_{jl} + s_{kl} + s_{km}\Big) + 5s_{lm}\Big) + \\ &+ s_{ik}\Big(- 4s_{in} + 4\Big(s_{jl} + s_{kl}\Big) + s_{jk} + 9s_{lm}\Big) + 5s_{ij}s_{ik} + 4s_{ij}^2 + s_{ik}^2\Big) + \\ &+ 4\Big(s_{ij} + s_{ik} + s_{jk}\Big)\Big(\Big(s_{ik} + s_{jk}\Big)\Big(- s_{in} + s_{jk} + s_{jl} + s_{kl}\Big) + s_{jk}s_{km}\Big) \\ &- 4s_{lm}\Big(s_{ik}\Big(s_{jl} + s_{kl}\Big) + s_{jk}s_{jl}\Big) + s_{mn}^2\Big(4s_{ij} + 5s_{ik}\Big)\,, \end{aligned} \tag{6.39}$$

up to a generalised gauge transformation. The resulting amplitude agrees with the special Galileon 6-point interaction.

We expect that this structure of kinematic factors continues at higher points as well, with the relevant non-trivial solution to the Jacobis coming in at order $n-2$ in Mandelstam variables. At odd points, these should be pure gauge (as the NLSM has only even amplitudes), while the even-point factors will be unique up to gauge transformations.[16]

An important aspect of the discussion above is that there are no solutions to the Jacobi identities below a specific order in momenta. However, this assumes a structureless scalar field; new possibilities open up when augmenting the scalar field with additional structure. One example of this is to consider scalars in the biadjoint representation; this leads to the colour factors, starting with the structure constants f_{ABC} as the three-point factor (where A is the adjoint). Instead, we will focus on the fundamental representation, e.g. of the special orthogonal group. Note that this directly eliminates all odd-point factors, as we will only be using the invariant metric δ_{ab}. The remaining even-point expressions will be referred to as flavour factors.

For the flavour factors at four-point, we find the first non-trivial solution[17] at linear order in Mandelstam variables:

$$F_{1234} = f_1(\delta_{ab}\delta_{cd}(s_{23} - s_{13}) - (\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad})s_{12}) + f_2(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{db} + \delta_{ad}\delta_{bc})(s_{13} - s_{23})\,. \tag{6.40}$$

Our notation here is that the first particle has momentum p_1 and flavour index a etc. Note that the interplay between kinematic and flavour allows one to solve the Jacobis in different ways; the above expression is anti-symmetric overall in the exchange of 1 and 2, that arises from anti-symmetry in flavour and symmetry in kinematic or vice versa. One consequence is that, in contrast to the colour and kinematic case, we find a free parameter already at the four-point level. When restricting to a single flavour, both parameters collapse onto the same factor, that was already mentioned in footnote 13, which should therefore really be seen as a special case of the expression above including flavour.

[16]Explicit expressions at six- and higher-point can be found in [98]; however they are constructed in a basis where the order reversion symmetry is not imposed.

[17]The same factors were constructed in [164, 163] with a different interpretation, namely as higher-derivative corrections to a specific NLSM. We will comment on this possibility in the concluding section.

By constructing amplitudes, one finds that both parameters f_1 and f_2 are actually physical instead of pure gauge. For instance, one can build amplitudes from kinematic and flavour factors,

$$A_4 = \sum_{\text{exchange}} \frac{N_{ijkl}F_{ijkl}}{s_{ij}} \,. \tag{6.41}$$

The resulting amplitude splits up into a number of sectors that have different flavour structures, akin to the partial amplitudes of colour structures. For instance, along the $\delta_{ab}\delta_{cd}$ part, one finds

$$A_4^{ab,cd} = -6f_1\lambda_4(s_{23}s_{13}) - 6f_2\lambda_4(s_{23}s_{13} - s_{12}^2) \,. \tag{6.42}$$

Comparing to the explicit Lagrangian of Section 6.2 and its Feynman rules, one concludes that the f_1 part of this expression corresponds to the multi-DBI theory, with the specific relation between the two quartic terms $\mathrm{Tr}[(\partial\phi^a\partial\phi^n)^2]$ and $\mathrm{Tr}[(\partial\phi^a\partial\phi^n)]^2$. The second parameter follows from only the second of these quartic types. We therefore conclude that for generic parameter values this theory has no clear Goldstone interpretation associated with spontaneous symmetry breaking; this is only true for $f_2 = 0$, resulting in the BCJ formulation of multi-DBI.

As a second possibility, it is interesting to consider the product of two flavour numerators. In order to understand these amplitudes, it will be advantageous to introduce trace notation for the flavour structures generated by the δ-functions, for instance

$$[AB] \equiv \delta_{ab}\delta_{\bar{a}\bar{b}} \,, \qquad [ABCD] \equiv \delta_{ab}\delta_{cd}\delta_{\bar{b}\bar{c}}\delta_{\bar{a}\bar{d}} \,, \tag{6.43}$$

where δ_{ab} and $\delta_{\bar{a}\bar{b}}$ generate the flavour structures of the two distinct numerators. In this notation, the amplitude contains two types of contributions:

- Firstly we have single-trace contributions, an example of which is given by

$$\sim f_2^2 \frac{(s_{12}^2 - s_{23}s_{13})^2}{s_{12}s_{23}s_{13}}([ABCD] + [ADCB]) + (\text{cyclic}) \,, \tag{6.44}$$

  where (cyclic) denotes cyclic permutation of three external legs, keeping one fixed. Note that the flavour structure of these terms, with a single trace, would perfectly correspond to the (expected) contributions from four-point operators in the NLSM; however, the kinematic structure includes (single) poles which would be incompatible with this. Therefore, we will henceforth set $f_2 = 0$. Note that this in retrospect justifies the interpretation of the four-point amplitude with one flavour and one kinematic factor as multi-DBI.

  The only remaining single-trace contribution is given by

$$A_4^{\text{contact}} = -\,4f_1^2 s_{13}([ABCD] + [ADCB]) + (\text{cyclic}) \,. \tag{6.45}$$

  These are indeed the amplitudes that would follow as contact diagrams from the Lagrangian (6.7).

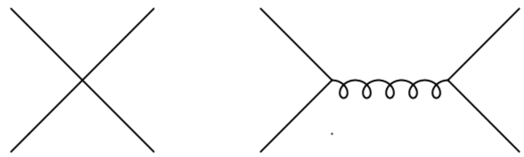


Figure 6.3: *The two types of Feynman diagrams that contribute to the NLSM$_g$ four-point amplitude: contact interactions (left) and graviton exchange (right). Note that the straight lines are NLSM scalars and the curly line represents a (intermediate) graviton.*

- In addition to the single-trace terms, we have double-trace contributions of the form

$$A_4^{\text{exchange}} = -4f_1^2 \frac{s_{23}s_{13}}{s_{12}}[AB][CD] + (\text{cyclic}). \tag{6.46}$$

  Due to the pole structure, these cannot be contact terms; instead they correspond to graviton exchange between the four scalars (see Figure 6.3); as the gravitons do not carry any flavour, this requires the flavours of the four particles to coincide pairwise.[18]

Together, these single- and double-trace amplitudes arise from the scalar-sector of the $SO(M,N)$ non-compact NLSM minimally coupled to Einstein gravity, with the Lagrangian given by

$$\mathcal{L}_{\text{GR+NLSM}} = \sqrt{-g}\left(\frac{1}{2}M_{\text{Pl}}^2 R - \frac{1}{2}[\frac{1}{1-\phi\phi^T/F^2}\nabla^\mu\phi\frac{1}{1-\phi^T\phi/F^2}\nabla_\mu\phi^T]\right), \tag{6.47}$$

where M_{Pl} is the (reduced) Planck mass and F is the pion decay constant. Further details on the amplitudes of this theory are reviewed in the appendix. Given the relative strengths of the exchange and contact diagrams following from the BCJ prescription, we find that this corresponds to the non-compact NLSM coupled to gravity with the two coupling constants identified, $M_{\text{Pl}} = F$, In contrast to the off-shell mapping outlined previously, the BCJ amplitudes therefore lead to a specific identification of the two parameters.

At 6-point, there are 15 possible inequivalent flavour structures of the form $\delta_{..}\delta_{..}\delta_{..}$. On the kinematic side, these would be multiplied by a quadratic expression in one of the 9 Mandelstam variables, leading to 45 different terms (and hence in total 675). Imposing the 6-point Jacobi identities constrain 642 of these parameters, leaving 33 parameters unfixed. These are split up amongst the 3 different irreps in the following way: the 5 contains 9 parameters, the 9 contains 15 parameters and the 10 contains 9 parameters. These then have to be further reduced to satisfy the factorisation constraints. Remarkably, this again leads to six parameters, similar to what was found for the kinematic factors. Moreover, all six parameters have components in all three irreps of Figure 6.2.

Turning to the amplitudes, we find that these also depend on a single linear combination of these six parameters. We therefore again conclude that five correspond to a generalised gauge

[18] It would be interesting to investigate whether the poles introduced by f_2 have a similar interpretation as e.g. gluon exchange; however, as we need to set f_2 to zero to get DBI and NLSM we will not pursue this option here.

transformation, and there is a single physical parameter. The flavour factor F_{123456} in general is given by a rather complicated expression. However, for a specific parameter (that is not equal to pure gauge) it follows from the flavour structure that multiplies the $s_{a,b}^2$ kinematics, which is given by

$$\begin{aligned}&28\delta_{af}\delta_{be}\delta_{cd} - 28\delta_{ae}\delta_{bf}\delta_{cd} + 11\delta_{ab}\delta_{cd}\delta_{ef} + 13\delta_{af}\delta_{bd}\delta_{ce} - 13\delta_{ad}\delta_{bf}\delta_{ce} + ,\\ &- 34\delta_{ae}\delta_{bd}\delta_{cf} - 5\delta_{ad}\delta_{be}\delta_{cf} + 25\delta_{af}\delta_{bc}\delta_{de} - 26\delta_{ac}\delta_{bf}\delta_{de} + 8\delta_{ab}\delta_{cf}\delta_{de} + ,\\ &- 7\delta_{ae}\delta_{bc}\delta_{df} + 26\delta_{ac}\delta_{be}\delta_{df} + 13\delta_{ab}\delta_{ce}\delta_{df} - 9\delta_{ad}\delta_{bc}\delta_{ef} + 7\delta_{ac}\delta_{bd}\delta_{ef} \,.\end{aligned} \tag{6.48}$$

The other terms, that include the dependence on the other eight Mandelstam invariants, then follow from imposing both the Jacobis and the factorisation.

6.3.3 Flavour-kinematics duality at the on-shell level

To complete the discussion of on-shell flavour-kinematics duality, finally we turn to the relations *between* the different factors. Analogous to the mapping of field equations and non-linear symmetries, one can also relate the different factors by a specific operation.

To illustrate this, we consider the four-point flavour factor

$$F_{1234} = \delta_{ab}\delta_{cd}(s_{23} - s_{13}) - (\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad})s_{12} \,. \tag{6.49}$$

Now replace the flavour information (i.e. the $SO(N)$ delta-functions) with kinematic variables according to

$$\delta_{ab} \longmapsto 1 + \lambda\, s_{12} \,, \tag{6.50}$$

where λ is an arbitrary constant, and similar for all other flavour structures. We obtain an expression quadratic in λ, from which we isolate[19] the part that is proportional to λ. Upon using four-point scattering identities (including e.g. $s_{12} + s_{13} + s_{23} = 0$ and similar), the coefficient of this term turns out to be exactly the scalar-kinematic numerator (6.32),

$$N_{1234} = s_{12}(s_{23} - s_{13}) \,, \tag{6.51}$$

after setting $\lambda = 1$. Note that the replacement (6.50) and subsequent restriction to terms linear in λ implies that one goes from an amplitude that is quartic in derivatives to one that is sextic. This is the on-shell counterpart to the off-shell mapping discussed in Section 6.2.2, which also introduces jumps by two derivatives at the four-point level.

The same mapping (6.50) has been verified at six-point scattering, where it relates the full six-point flavour amplitude (partly given by (6.48)) to the kinematic factor (6.38) and (6.39) after retaining the coefficient proportional to λ^2. Similarly, the five gauge parameters on the flavour and the scalar kinematic sides also map onto each other. We conjecture the same relation to extend to higher order as well.

While not the focus of this chapter, it is interesting to note that there is a similar mapping from the tensor kinematic factor (relevant for GR and YM) onto the flavour factor. At

[19] Note that this mapping is not invertible, since we lose information by throwing away terms of highest order in λ.

four-point, the tensor factor is given by (see e.g. [30])

$$T_{1234} = -\Big\{\Big[\,(\varepsilon_1\cdot\varepsilon_2)\,p_1^\mu + 2\,(\varepsilon_1\cdot p_2)\,\varepsilon_2^\mu - (1\leftrightarrow 2)\Big]\Big[\,(\varepsilon_3\cdot\varepsilon_4)\,p_{3\mu} + 2\,(\varepsilon_3\cdot p_4)\,\varepsilon_{4\mu} - (3\leftrightarrow 4)\Big] + 2s_{1,2}\Big[\,(\varepsilon_1\cdot\varepsilon_3)\,(\varepsilon_2\cdot\varepsilon_4) - (\varepsilon_1\cdot\varepsilon_4)\,(\varepsilon_2\cdot\varepsilon_3)\,\Big]\Big\}\,. \tag{6.52}$$

To transform tensor-kinematic information into flavour-kinematics, we map[20]

$$\varepsilon_i\cdot p_j \longmapsto 0\,, \quad \varepsilon_i\cdot\varepsilon_j \longmapsto \delta_{ij}\,, \tag{6.53}$$

which results exactly in the flavour factors (6.40), with $f_1 = 1$ and $f_2 = 0$. Alternatively, one could take the starting point to be the polarisation-stripped version of (6.52), which is given by

$$T_{1234} = \Big[\Big(-\frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\lambda}p_1\cdot p_3 - \eta^{\gamma\lambda}{p_2}^\alpha{p_3}^\beta - \eta^{\alpha\beta}{p_1}^\lambda{p_4}^\gamma - 2\eta^{\beta\lambda}{p_2}^\alpha{p_4}^\gamma\Big) - (1\leftrightarrow 2) - (3\leftrightarrow 4) + (1\leftrightarrow 2, 3\leftrightarrow 4) + \Big(\eta^{\alpha\lambda}\eta^{\beta\gamma} - \eta^{\alpha\gamma}\eta^{\beta\lambda}\Big)\,p_1\cdot p_2\Big]\,, \tag{6.54}$$

where we (analogous to flavour indices) associate the space-time index α with particle 1, β with particle 2, and so forth. For the polarisation-stripped numerator, the transformations (6.53) become

$$\eta^{\alpha\beta} \mapsto \delta^{ab}\,, \tag{6.55}$$

while all terms involving two non-contracted momenta vanish. Note that the indices on the LHS of (6.55) are space-time indices, while the indices on the RHS represent flavour.

This relation between amplitudes spanned by tensor kinematics and flavour factors has a well-known counterpart at the Lagrangian level. The dimensional reduction of general relativity over N dimensions leads to an $SL(N)/SO(N)$ coset, while when instead starting from the common sector $(g_{\mu\nu}, B_{\mu\nu}, \phi)$ in the higher dimensions this leads to an $SO(N,N)$ coset (see e.g. [148]). In view of this, it should not be surprising that the BCJ factors of both theories, being the slightly generalised $SO(M,N)$ with $M \neq N$ and the common extension of GR with dilaton and two-form, are therefore also related.

In a similar vein, the mapping from tensor-kinematics to flavour and finally scalar-kinematics as outlined here is closely related to the operations *dimensional reduction* (or *compactify*) and *generalised dimensional reduction* (or *"compactify"*) of [51]. In the CHY representation, the amplitudes are characterised by integrands that carry all information about the theory; each integrand consists of two building blocks containing polarisations and momenta. The (generalised) dimensional reduction procedure allows the polarisations of one of these building blocks to explore an internal space, thereby mapping e.g. the gravity integrand to its Born-Infeld counterpart. Since gravity and Born-Infeld amplitudes can be constructed out of two tensor-kinematic numerators and a combination of a tensor-kinematic and a scalar-kinematic BCJ numerator respectively, the numerator mapping proposed here mimics this CHY operation at the level of the numerators.

[20] All odd-point tensor factors map onto vanishing flavour factors due to the odd numbers of polarisations and momenta.

	Non-linear symmetry	**Equation of motion**
NLSM	$\delta\phi = c + \phi c^T \phi$	$\Box\phi = \sum_{n=1}^{\infty} (-1)^{n-1} 2\left(\partial_\mu \phi\right)\phi^T \cdot \left(\phi\phi^T\right)^{n-1}\left(\partial^\mu \phi\right)$
DBI	$\delta\phi^a = c^a + c^a{}_\mu x^\mu + c^b{}_\mu \phi^b \partial^\mu \phi^a$	$\Box\phi^a = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \left[\left(\partial\partial\phi^a\right)\left(\partial\phi \cdot \partial\phi\right)^n\right]$
SG	$\delta\phi = c + c_\mu x^\mu + c_{\mu\nu}\left(x^\mu x^\nu + \partial^\mu\phi\partial^\nu\phi\right)$	$\Box\phi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}\left[\Phi^{2n+1}\right]$
	Irrep	**Factor at 4-point**
Colour	adjoint A	$f_{AB}{}^E f_{CDE}$
Flavour	fundamental a	$\delta_{ab}\delta_{cd}(s_{23} - s_{13}) - (\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad})s_{12}$
Kinematics	singlet	$s_{12}(s_{23} - s_{13})$

Table 6.2: *The triplet of exceptional scalar EFTs with their non-linear symmetries and field equations, and the triplet of BCJ numerators into which they can be factorised. The off-shell mapping, as outlined in Section 6.2.2, relates different theories while the on-shell mapping* (6.50) *relates different factors.*

6.4 Conclusion and outlook

This chapter deals with the interrelations between flavour and kinematic aspects of Goldstone theories. We have highlighted novel relations between three cases that display spontaneous symmetry breaking, ranging from internal (the pions of the $SO(M,N)$ NLSM) to space-time symmetry (the $SO(N)$ multi-DBI scalars and the special Galileon). These theories therefore appear as different guises of the same underlying structure, which can be expressed in terms of flavour and/or kinematics. This flavour-kinematics duality results in the three Goldstone theories under study.

At the off-shell level, the field choice ambiguity can be fixed by requiring a common form of the NL symmetry transformation, with a constant and a quadratic part. In this basis, the field equations are seen to take closely related forms. These are summarised in the upper part of Table 6.2. As outlined in Section 6.2.2, the different fields, NL symmetries and field equations are mapped onto each other via an expansion in flavour coordinates θ^a, with e.g. the SG and DBI fields related as $\phi = \phi_a \theta^a$.

At the on-shell level, the amplitude of these theories can be built along the lines of BCJ factorisation from a range of building blocks. In addition to the well-known colour and kinematics factors, we have outlined how to construct flavour factors. These are subject to the same conditions, including group-theoretical (corresponding to the Jacobi-like identities) and physical (ensuring the correct amplitude factorisation) constraints. With the introduction of flavour, these factors employ the simplest $SO(N)$ representations, being scalar, fundamental and adjoint, see the lower part of Table 6.2. As before, the reduction of flavour increases kinematics, as seen from the order in Mandelstam variables, and vice versa. Remarkably, the flavour and kinematics factor are found to be related by the simple substitution (6.50); these therefore really correspond to the same structures expressed in different spaces.

Which picture emerges from these different considerations? Employing the different combinations of factors, one can build a range of double copy theories. In addition to the three scalar factors describing colour, flavour and scalar kinematics, we will also include the possibility that was identified in the original BCJ proposal, including polarisation and thus tensor kinematics. Given these four factors, we find that the most direct graphical representation would be that of a tetrahedron, as illustrated in Figure 6.4. The different nodes of this figure correspond to theories whose amplitudes are the products of the same type of factors, while one finds theories with mixed factors along the edges.

Let us start with discussing the scalar theories at the bottom level of the diagram. At the front of the figure is the special Galileon theory, with two copies of scalar kinematics. When successively replacing scalar kinematics with flavour, one moves via the multi-DBI theory towards the $SO(M, N)$ NLSM minimally coupled to gravity (and with the identification $F = M_{\text{Pl}}$). When instead opting for colour instead of flavour, one encounters the NLSM without gravity, and finally the bi-adjoint scalar theory. The remaining scalar possibility has both a flavour and a colour factor, and corresponds to the scalar sector of the dimensional reduction of Yang-Mills (YMS).[21]

Moving up into the vertical direction, the original colour-kinematics duality corresponds to the edge linking the purely colour-based BAS theory and the tensor kinematics-based GR. Note that this is fully orthogonal to flavour-kinematics duality that we focus on and that interpolates between the SG and the gravity-coupled NLSM. At the spin-1 intermediate level, one also encounters the Born-Infeld vector theory, as well as an $SO(N)$ multiplet of Maxwell vectors that only interact via gravity.

Remarkably, the $SO(M, N)$ NLSM appears twice in the above construction, with and without gravity-mediated interactions: the former consists of purely flavour, while the latter is the product of colour with scalar kinematics. This freedom was also encountered in the off-shell formulation of the theory, and corresponds to the introduction of an additional parameter arising from the direct product structure of the coset. Instead, from the BCJ factorisation, one either finds M_{Pl} infinite or equal to the pion decay constant. Perhaps one can consider combinations of these to interpolate to intermediate values of the Planck mass.

A number of interesting questions appear naturally as a result of our findings. One of these concerns the self-interaction face on the right side of the tetrahedron. These theories retain non-trivial interactions when restricted to a single species (and hence no flavour or colour).

[21] Similar relations between the amplitudes of these sets of theories have been outlined from a different perspective in [51, 76]; our focus is on the BCJ formulation of these theories instead.

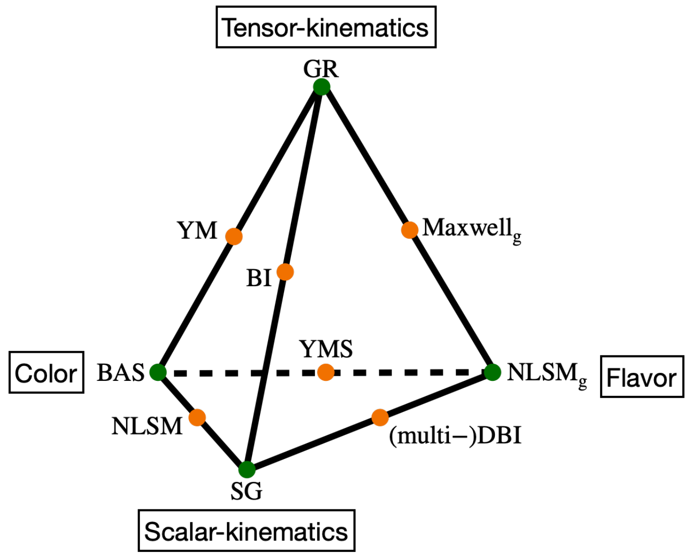


Figure 6.4: *The tetrahedron spanned by the four different BCJ numerators that theories can factorise into. Taking all possible products results in the indicated web of dualities (including colour-(tensor-)kinematic and flavour-(scalar-)kinematic) between the different spin-2, -1, and -0 effective field theories. The right face of the tetrahedron corresponds to EFTs that retain interactions when restricted to a single species.*

As a consequence, these theories therefore will have non-linear responses to a source term, in contrast to e.g. Yang-Mills theory. It would therefore be interesting to investigate the classical solutions of these theories and their possible mappings. Note that this would necessarily differ from the classical double copy as outlined in e.g. [176], as these map the linear Coulomb solution onto the Schwarzschild solution (effectively linearised when written in Kerr-Schild coordinates). Instead, the classical solutions on the self-interaction face would be non-linear, with a simple example provided by the Born-Infeld solution as a non-linear completion of the Coulomb solution [45].

Secondly, the flavour factors as identified in Section 6.3 have already appeared in a different guise, namely as higher-derivative corrections to an $SO(M+1)/SO(M)$ coset. The addition of colour and flavour factors results in the interesting structure of extended DBI theory [164, 163]. Note that this combination is possible due to the special structure of the coset in the case of $N = 1$, with colour and flavour both in the fundamental $SO(M)$ representation. It could be similarly interesting to further specialise to the case $M = 1$, where flavour and scalar kinematics live in the same, trivial representation. The four-point factor for this theory are given by a linear combination of the expressions of Footnote 13 and (6.32). Perhaps this might give rise to a theory that includes DBI and SG as higher-derivative corrections to a free scalar field coupled to gravity.

Finally, it would be interesting to investigate whether the double copy that has been identified for on-shell amplitudes also extends to off-shell aspects such as correlators and wavefunctionals. This issue was addressed recently [35], where it was found that the most straightforward implementation of the double copy structure does not work (however, see also the analysis of [9] employing cosmological scattering equations). Given the off-shell mapping outlined in Section 6.2, it would be interesting to see whether this could be adapted for the gravity-coupled $SO(M, N)$ NLSM, and whether this would allow for a mapping onto multi-DBI and SG.

Chapter 7

Concluding remarks

The main objective of this thesis has been to research to what extent classical dynamical symmetries can persist in relativistic theories. Specifically, attention was paid to celestial-like problems, which are generically known to be much more difficult to solve in relativistic settings.

As the symmetries and consequent integrability of non-relativistic systems are best understood through Hamiltonian mechanics, it makes sense to look for formulations of relativistic theories in terms of Hamiltonians as well. However, because of the lack of a universal time parameter in relativity, it is not trivial to even write down the Hamiltonian for multi-body problems, making classical Hamiltonian mechanics ill-equipped to deal with such systems directly.[1] In asking the question whether there exist relativistic systems preserving classical symmetries, we are therefore naturally led to one-body systems, in which a single moving body is influenced by its static background. While it is possible to describe any two-body system (in a theory with Poincaré symmetry) from the center of mass frame as a spherically symmetric one-body system with a potential, it is far from trivial to write down this potential.

As demonstrated in Chapters 4 and 5, relativistic systems with classical symmetries exist. However, systems exhibiting these symmetries are not separate from the classical ones they share symmetries with, but are in fact, on an energy level-set, the same up to reparametrisation of time. For the two-body system, we have written down the Hamiltonians (reduced to one-body) for all systems displaying both special relativistic limits far away from the center and Keplerian ellipses, conserved quantities and integrability, at least up to 5PN. According to the argument put forth by [93], assuming LRL gauge, any relativistic Hamiltonian with Kepler as the non-relativistic limit and displaying additional symmetry is just a function of Kepler Hamiltonians.

Furthermore, we gave physical, one-center interpretations for some special cases, and shown the relations among them. In principle, relativistic two-body systems with comparable masses, reduced over translations, can also be of the form agreeing with our general, symmetric

[1]The post-Newtonian and post-Minkowskian approximations are ways to systematically account for the non-instantaneous nature of relativistic interactions, and by rephrasing them in terms of instantaneous interactions allow to construct classical potentials.

Hamiltonian (5.1), but so far, only approximations to first PN - or second PM order have been shown to fit the bill [59, 180].

Another aspect of this thesis is the double copy between gauge theory and gravity. The double copy has an extremely wide range of applicability. From relating the scattering amplitudes of scalar theories with soft limits to dynamical systems describing higher-dimensional uplifts of charges or black holes: it is able to relate different physical systems that a priori have no obvious connection. In some cases, this reduces the difficulty of characterising the physics to doing this in the easiest available context, such as in the Yang-Mills amplitude calculations underlying the higher order Hamiltonian calculations for the gravitational two-body problem [24].

In this thesis, we have seen the double copy in two quite different incarnations. First, we discussed how it establishes relations between backgrounds as well as one-body systems in different relativistic theories. Second, it was shown how the flavour-kinematics duality in the amplitudes of a triplet of scalar theories can be used to manifest relations between them, both on-shell in terms of amplitudes and off-shell in terms of equations of motion.

While providing some answers, our exploration has also raised new questions and suggests some possible directions for further research, listed here.

1. It appears that the only relativistic systems displaying the so(4) symmetry of the Kepler problem are those that are related to the Kepler problem through time reparametrisation. This was established to 5th post-Newtonian order in Chapter 4, and it seems possible to extend this to all orders. For this, one would need to make rigorous the claim that all two-body Hamiltonians can be written in LRL gauge.

2. As we have seen multiple theories in which the one-center system has Keplerian symmetries, it would be interesting to consider whether there exist two-body systems displaying these symmetries beyond vanishing mass ratio, with comparable masses or charges, as also discussed at the end of Chapter 5.

3. The 3 instances of the general class of relativistic Kepler systems discussed in Chapter 5 have spin-0,-1, and -2 interpretations in 5 dimensions. Can we similarly take higher-degree monomials to find higher-spin interpretations? Since the spin-0 and -1 theories are linear (i.e. non-self-interacting) to begin with, while for spin-2 the set-up linearises Einsteins equations, it might be expected that a realisation in terms of a free spin-3 theory is possible.

4. Another example of hidden symmetry is the one related to the third constant of motion in the classical two-center problem, next to the symmetries yielding conservation of energy and one angular momentum component. Since these constants are independent and in involution, the system is integrable. As discovered by [221], the two-center system in fact is the only classical axisymmetric system with conserved energy possessing another constant of motion. In a sense, this is Bertrands theorem for the two-center system. In General relativity, a similar system exists, with an analogous constant. It is the system of a geodesic on a Kerr spacetime, and the constant the Carter constant. The generic two-center system in GR, given by Bach and Weyl [11], does not posses a third constant and is not integrable. The question presents itself whether the dynamical system of a

geodesic on a Kerr background is canonically conjugate to (a function of) the classical two-center problem, in much the same way as our relativistic one-center systems appear to be conjugate to the classical Kepler problem. We know that at least all two-center systems in the relativistic theories we have discussed are related to the classical system through time reparametrisation on the energy-level. And can one, similarly to what we have aimed to, attempt to show that all relativistic systems with the classical two-center symmetry can be related through canonical transformations to the classical two-center problem?

5. The interpolation from classical to relativistic systems is generally done through taking the speed of light c from infinity closer to one, its value in natural units. In a similar sense, one can go from flat space to curved space by adjusting Newton's constant G from zero to unity, or from classical physics to quantum physics by 'turning on' Planck's constant h. This three-way approach of a theory of quantum gravity is sometimes summarised in the Bronstein cube [131]. The hope is that each of these approaches can teach us something about the corner theory, uniting relativity, gravity, and quantum physics. In the – less ambitious and far-reaching – context of relativistic binary systems, one can imagine in a similar vein that we might learn from approaching realistic binary systems by departing from a so(4)-preserving relativistic system, such as the one-center system of two extremal Einstein-Maxwell-dilaton black holes with $a = \sqrt{3}$. In this type of perturbation theory, many interesting questions can be asked, in the light of the classically understood dynamics. Does the free flow on the 3-sphere persist, but now on a deformed sphere? Is the flow still on a sphere but no longer free? Or is there another way we can order our understanding of such a system, by assessing the first 'post-symmetric order'? This seems especially interesting in the context of extreme mass-ratio inspirals, as in this context one can conceivably take a one-center system as starting point.

6. The harmonic oscillator (HO) is in many ways the sibling of the Kepler system. It, too, possesses the largest number of independent conserved quantities possible for its dimension, as stipulated by Bertrand's theorem. Moreover, as shown e.g. in [140], by a complex coordinate transformation the planar problems can even be seen to be the same. This suggests there is a natural extension along the lines of Chapter 5 of the HO to relativity
$$H^2 = m^2 + p^2 + x^2 \; G(m + H)\,, \tag{7.1}$$
with $G(y)$ a polynomial. Could there be physical realisations of these Hamiltonians? And does the planar complex coordinate transformation between Kepler and HO also persist in relativity? And, finally, can one argue that these are all relativistic systems with the symmetries of the HO and can these be related through canonical transformations to functions of it?

We will leave the above questions to future research.

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Appendix

NLSM with GR

Here we derive the four-point amplitudes of the compact $SO(M+N)/(SO(M)\times SO(N))$ NLSM coupled to gravity. First, we will use the perturbation theory method as outlined in e.g. [166, 174], after which we apply the *Lehmann, Symanzik and Zimmermann (LSZ) formula* to extract amplitudes.

Recall from section 6.3 that the Lagrangian of the $SO(M+N)$ NLSM minimally coupled to gravity, now including coupling constants, reads

$$\mathcal{L}_{\mathrm{GR+NLSM}} = \sqrt{-g}\left(\frac{2}{\kappa^2}R - \frac{1}{2}[\frac{1}{1+\frac{\phi\phi^T}{F^2}}\nabla^\mu\phi\frac{1}{1+\frac{\phi^T\phi}{F^2}}\nabla_\mu\phi^T]\right)\,, \tag{2}$$

where $\kappa^2 = 4/M_{\mathrm{Pl}}^2$ in terms of the (reduced) Planck mass, and F is the NLSM cut-off scale. Following e.g. [166], we work with the so-called *gothic graviton* $\mathfrak{h}^{\mu\nu}$, such that

$$\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} - \kappa\mathfrak{h}^{\mu\nu}\,. \tag{3}$$

Furthermore, we adopt the *De Donder gauge*, with $\partial_\mu\mathfrak{h}^{\mu\nu} = 0$. These choices lead to the particularly useful properties that the Einstein tensor is given by $G_{\mu\nu} = -\frac{\kappa}{2}\Box\mathfrak{h}_{\mu\nu}$, and the curved space-time d'Alembertian, denoted by $\Box_c \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$, reduces to $\Box_c = g^{\mu\nu}\partial_\mu\partial_\nu$ [96]. The field equation for the graviton and scalar field respectively read

$$\begin{aligned} G_{\mu\nu} =& \frac{\kappa^2}{4}(g_{\rho\mu}g_{\sigma\nu} - \tfrac{1}{2}g_{\mu\nu}g_{\rho\sigma})[\frac{1}{1+\frac{\phi\phi^T}{F^2}}\partial^\rho\phi\frac{1}{1+\frac{\phi^T\phi}{F^2}}\partial^\sigma\phi^T]\,,\\ \Box_c\phi =& 2\sum_{n=1}^{\infty}(-1)^{n-1}\partial_\mu\phi\frac{\phi^T}{F^2}\left(\frac{\phi\phi^T}{F^2}\right)^{n-1}\partial^\mu\phi\,. \end{aligned} \tag{4}$$

By expanding $\mathfrak{h}^{\mu\nu}$ and ϕ in their coupling constants,

$$\mathfrak{h}^{\mu\nu} = \mathfrak{h}^{(0)\mu\nu} + \kappa\mathfrak{h}^{(1)\mu\nu} + \kappa^2\mathfrak{h}^{(2)\mu\nu} + \ldots\,, \qquad \phi = \phi^{(0)} + \frac{\phi^{(1)}}{F^2} + \frac{\phi^{(2)}}{F^4} + \ldots\,, \tag{5}$$

and substituting these expansions into the field equations (4), we obtain a differential equation for each perturbative correction $\mathfrak{h}^{(k)\mu\nu}$ and $\phi^{(k)}$. The relevant equations for four-scalar scattering are given by

$$\begin{aligned}\Box\mathfrak{h}^{(0)\mu\nu} &= -\frac{\kappa}{2}\left(\partial^\mu\phi^{(0)}\partial^\nu\phi^{(0)T} - \frac{1}{2}\eta^{\mu\nu}\partial^\rho\phi^{(0)}\partial_\rho\phi^{(0)T}\right)\,, \\ \Box\phi^{(1)} &= \kappa\partial_\mu\partial_\nu\phi^{(0)}\mathfrak{h}^{(0)\mu\nu} + \frac{2}{F^2}\partial^\mu\phi^{(0)}\phi^{(0)T}\partial_\mu\phi^{(0)}\,.\end{aligned} \tag{6}$$

Fourier transforming the above to momentum space leads to

$$\mathfrak{h}^{(0)\mu\nu}(-p_1) = -\frac{1}{p_1^2}\int đ^4p_2 đ^4p_3 \frac{\kappa}{2}\left\{(p_2^\mu p_3^\nu) - \frac{1}{2}\eta^{\mu\nu}(p_2\cdot p_3)\right\}[CD][\phi^{(0)c\bar{c}}(p_2)\phi^{(0)d\bar{d}}(p_3)]\,, \tag{7}$$

$$\begin{aligned}\phi^{(1)a\bar{a}}(-p_1) = -\frac{1}{p_1^2}\int đ^4p_2 đ^4p_3 đ^4p_4 \Bigg\{&\frac{\kappa^2}{4}\left(\frac{s_{23}s_{24}}{2s_{12}} - \frac{1}{2}(p_2)^2\right)[AB][CD]\phi^{(0)b\bar{b}}(p_2)[\phi^{(0)c\bar{c}}(p_3)\phi^{(0)d\bar{d}}(p_4)] \\ &- \frac{s_{13}}{2F^2}([ABCD]+[ADCB])\phi^{(0)b\bar{b}}(p_2)[\phi^{(0)c\bar{c}}(p_3)\phi^{(0)d\bar{d}}(p_4)]\Bigg\}\,,\end{aligned} \tag{8}$$

where we have explicitly included the flavour indices and suppressed the momentum-conserving delta functions $\bar{\delta}^{(4)}(p_1 + \ldots + p_n)$. Additionally, the common short-hand notation

$$đ^4p \equiv \frac{d^4p}{(2\pi)^4}\,, \quad \bar{\delta}^{(4)}(p) \equiv (2\pi)^4\delta^{(4)}(p)\,, \tag{9}$$

was employed for legibility.

Next, we note that the term proportional to $(p_2)^2$ in (8) vanishes on-shell and use the LSZ formula (see e.g. [174] for similar calculations) in order to extract the four-scalar partial amplitude from $\phi^{(1)}$.[2] The result in terms of $M_{\rm Pl}$ reads

$$\begin{aligned}A_4 &= \lim_{p_1^2\to 0} p_1^2 \frac{\delta^3\phi^{(1)}(-p_1)}{\delta\phi^{(0)}(p_2)\delta\phi^{(0)}(p_3)\delta\phi^{(0)}(p_4)} \\ &= -\frac{1}{2M_{\rm Pl}^2}\frac{s_{14}s_{13}}{s_{12}}[AB][CD] + \frac{s_{13}}{2F^2}([ABCD]+[ADCB]) + (\text{cyclic})\,,\end{aligned} \tag{10}$$

where the first term corresponds to graviton exchange diagrams and the second to contact interactions (see figure 6.3). The structures of these amplitudes coincide with the graviton exchange amplitude in equation (11) of [97] and the NLSM amplitude in equation (1.4) of [62]. Note that we have opposite signs in the above amplitude, in contrast to what we found in section 3; the latter therefore corresponds to a non-compact scalar manifold with non-linear $SO(M,N)$ symmetry.

[2] The LSZ formula extracts n-point amplitudes from n-point connected correlation functions (here corresponding to the perturbative corrections), by functionally differentiating $n-1$ times with respect to the leading order correction, while amputating the off-shell leg.