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https://doi.org/10.3390/sym16101340

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Finite Mathematics as the Most General (Fundamental) Mathematics

Felix M. Lev

Independent Researcher, San Diego, CA 92101, USA; felixlev314@gmail.com

Abstract: The purpose of this paper is to explain at the simplest possible level why finite mathematics based on a finite ring of characteristic p is more general (fundamental) than standard mathematics. The belief of most mathematicians and physicists that standard mathematics is the most fundamental arose for historical reasons. However, simple *mathematical* arguments show that standard mathematics (involving the concept of infinities) is a degenerate case of finite mathematics in the formal limit $p \to \infty$; standard mathematics arises from finite mathematics in the degenerate case when operations modulo a number are discarded. Quantum theory based on a finite ring of characteristic p is more general than standard quantum theory because the latter is a degenerate case of the former in the formal limit $p \to \infty$.

Keywords: finite mathematics; standard mathematics; finite quantum theory

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Citation: Lev, F.M. Finite Mathematics as the Most General (Fundamental) Mathematics. *Symmetry* **2024**, *16*, 1340. https://doi.org/10.3390/sym16101340

Academic Editors: Christos Volos and Zhibin Du

Received: 14 August 2024
Revised: 9 September 2024
Accepted: 29 September 2024
Published: 10 October 2024

1. The Main Goal of This Paper

In [1,2] and other our publications, we investigated in detail why finite mathematics based on a finite ring of characteristic p is more general (fundamental) than standard mathematics. These publications contain detailed proofs of statements on which our approach is based. The purpose of this paper is to explain the main ideas of our approach at the simplest possible level. Therefore, we do not provide technical details of the proofs, but, for interested readers, we provide links through which those proofs can be found.

SM deals with relations

$$a + b = c, \;\; a \cdot b = c, \;\; etc. \tag{1}$$

On the other hand, FM deals with relations

$$a + b = c \;(mod\, p), \;\; a \cdot b = c \;(mod\, p), \;\; etc. \tag{2}$$

where all the numbers $a, b, c, ...$ can take only values $0, 1, 2, ...p - 1$ and p is called the characteristic of the ring. Therefore, in FM, there are no infinities and all numbers do not exceed p in absolute value.

Before discussing these versions of mathematics, let us discuss the following: Whether we should treat mathematics (i) as a purely abstract science or (ii) as a science that should describe nature. I am a physicist and have worked among physicists for most of my life. For them, only approach (ii) is acceptable. However, when I discussed this issue with mathematicians and philosophers, I discovered that many of them view mathematics only from the point of view of (i) and arguments related to the description of nature are not significant for them. Approach (i) can be called the approach of Hilbert, who was its most famous proponent. There is a great discussion in the literature between him and Gödel about whether Gödel's incompleteness theorems indicate that the approach has foundational problems.

The fact that Hilbert's approach does not raise the question of describing nature does not mean that this approach should be rejected out of hand. For example, Dirac's philosophy is "*I learned to distrust all physical concepts as a basis for a theory. Instead one should* *put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting an interesting mathematics*". Dirac also said that, for him, the most important thing in any physical theory is the beauty of the formulas in this theory. That is, he meant that, sooner or later, in any beautiful mathematical theory, its physical meaning will be found. But even if it is not found, the beauty of the theory itself has aesthetic value. For example, in music, we appreciate its beauty and do not demand that music should somehow describe nature.

Nevertheless, in this paper, we treat mathematics only as a tool for describing nature. In the framework of this approach, most mathematicians and physicists believe that, at the most fundamental level, nature is described by SM, and FM is needed only in some special model problems. This opinion has developed despite the fact that modern quantum theory has known problems, and, despite the numerous efforts of many highly qualified mathematicians and physicists over the years, these problems have not yet been solved.

Modern QFT can calculate observable quantities only within the framework of perturbation theory, and it is not known whether its series is convergent or only asymptotic. However, even within this framework, one of the key problems of QFT (based on SM) is the problem of divergences: the theory gives divergent expressions for the S-matrix. In renormalized theories, the divergences can be eliminated by renormalization where finite observable quantities are formally expressed as products and sums of singularities. From the mathematical point of view, such procedures are not legitimate but, in some cases, they result in impressive agreement with experiments. The most famous case is that the results for the electron and muon magnetic moments obtained at the end of 40th agree with the experiment with the accuracy of eight decimal digits. In view of this and the other successes of QFT, most physicists believe that agreement with the data is much more important than rigorous mathematical substantiation.

At the same time, in non-renormalized QFTs, divergences cannot be eliminated by renormalization, and this is a great obstacle for constructing quantum gravity based on QFT. As the famous Nobel Prize laureate Steven Weinberg wrote in his book [3], "*Disappointingly this problem appeared with even greater severity in the early days of quantum theory, and although greatly ameliorated by subsequent improvements in the theory, it remains with us to the present day*". The title of Weinberg's paper [4] is "Living with infinities".

The main goal of the present paper is to explain at the simplest possible level that, contrary to the belief of most mathematicians and physicists, FM is the most general (fundamental) mathematics, and SM is its degenerate case. For this purpose, it is necessary to give a definition when mathematics A is more general (fundamental) than mathematics B, and mathematics B is a degenerate case of mathematics A. In [1,2], we proposed the following definition:

Definition 1. *Let theory A contain a finite nonzero parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity. Suppose that, with any desired accuracy, A can reproduce any result of B by choosing a value of the parameter. On the contrary, when the limit is already taken, one cannot return to A and reproduce all results of A. Then, A is more general than B and B is a degenerate case of A.*

In this paper, we discuss the result of [1,2] where, contrary to the belief of most mathematicians and physicists, as follows from the **Definition**, the following applies:

Statement: *SM is a degenerate case of FM in the formal limit $p \to \infty$, where p is the characteristic of the ring in FM.*

As explained below, this **Statement** implies that any result of SM can be obtained in FM with a choice of p, and, on the other hand, SM cannot reproduce those results of FM where it is important that p is finite and not infinitely large. As explained below, a consequence of this **statement** is that FM is more general (fundamental) than SM because SM is obtained from FM in the case where all operations modulo a number are discarded. Also, as discussed in [1,2,5] and this paper, a consequence of this **Statement** is that, *for* *describing nature at the most fundamental level, the concepts of infinitesimals, infinitely large, limits, continuity, etc., are not needed; they are needed only for describing nature approximately.*

Kronecker's famous phrase is that God invented integers, and humans invented everything else. However, in view of this **Statement**, this phrase can be reformulated so that God came up with only finite sets of numbers and everything else was invented by people.

As follows from the **Statement**, in QT based on FM (which we call finite quantum theory, FQT), the problem of divergences does not exist in principle because, in FM, there are no infinities. We emphasize that the **Statement** is not only our wish, but a fact proven mathematically in [1,2,5] and Section 2. Therefore, those mathematicians and physicists who insist on their position that SM is more general (fundamental) than FM must either give arguments that the **Definition** is not justified or show that the proof in [1,2,5] and Section 2 is erroneous. In numerous discussions with me, those mathematicians and physicists have presented various arguments that, in their opinion, emphasize the correctness of their position. The typical arguments are as follows:

(a) Formally, you have no divergences, but you introduce the cutoff p which is a huge number. Therefore, in cases where infinities arise in the standard theory, you will obtain a huge number p which is practically infinite.

(b) In your theory, there is only one parameter, p, and it is not clear why this parameter is this and not another. Is it not reasonable to prefer the approach with adeles when there are many characteristics which are on equal footing?

(c) An argument that has some similarities with (b) is the following: When you say that God only invented finite sets of numbers and everything else (infinitesimals, infinitely large, etc.) was invented by people, do you think that he "invented" the biggest (finite) p?

These arguments will be discussed below.

The paper is organized as follows: In Section 2, we explain why a theory proceeding from a finite ring is more general than a theory proceeding from the infinite ring Z. In Section 3, we explain why special relativity where speeds cannot be greater than c is more general than classical mechanics where there is no speed limit. In Section 4, we describe the main ideas of quantum theory based on finite mathematics. In Sections 5 and 6, we explain why quantum theory based on finite mathematics is more general (fundamental) than standard quantum theory. In Section 7, we answer questions that are commonly asked in connection with our approach.

2. Basic Facts about Finite Mathematics

SM starts from the infinite ring of integers $Z = (-\infty, ... - 1, 0, 1, ...\infty)$ but FM can involve only a finite number of elements. FM starts from the ring $R_p = (0, 1, 2, ...p - 1)$, where addition, subtraction and multiplication are defined as usual but modulo p. In our opinion, the notation Z/p for R_p is not adequate because it may give the wrong impression that FM starts from the infinite set Z and that Z is more general than R_p. However, although Z has more elements than R_p, Z cannot be more general than R_p because Z does not contain operations modulo a number. If p is prime, then R_p becomes the Galois field F_p where all the four operations are possible. The number p is called the characteristic of the ring R_p or the field F_p. For example, if $p = 5$, then 3 + 1 = 4 as usual but 3·2 = 1, 4·3 = 2, 4·4 = 1 and 3 + 2 = 0. Therefore, $-2 = 3$, $-4 = 1$, $1/2 = 3$, $1/4 = 4$, etc. The theory of finite rings and fields is described in standard textbooks (see, e.g., [6–8]).

One might say that the above examples have nothing to do with reality since 3 + 2 always equals 5 and not zero. However, since operations in R_p are modulo p, one can represent R_p as a set $\{0, \pm 1, \pm 2, ..., \pm(p-1)/2)\}$ if p is odd or as a set $\{0, \pm 1, \pm 2, ..., \pm(p/2 - 1), p/2\}$ if p is even. Let f be a function from R_p to Z such that $f(a)$ has the same notation in Z as a in R_p. Then, for elements $a \in R_p$ such that $|f(a)| \ll p$, addition, subtraction and multiplication are the same as in Z. In other words, for such elements, we do not notice the existence of p.

One might say that, nevertheless, the field F_p cannot be used in physics since $1/2 = (p+1)/2$, i.e., a very large number, when p is large. However, as explained in [1,2,5] and Section 4, *since quantum states are projective, then, even in SQT, quantum states can be described with any desired accuracy by using only integers and therefore the concepts of rational and real numbers play only an auxiliary role*.

If elements of Z are depicted as integer points on the x axis of the xy plane, then, if p is odd, the elements of R_p can be depicted as points of the circumference in Figure 1 and analogously if p is even. This picture is natural from the following considerations. As explained in textbooks, both R_p and Z are cyclic groups with respect to addition. However, R_p has a higher symmetry because it has a property which we call *strong cyclicity*; if we take any element $a \in R_p$ and sequentially add 1, then, after p steps, we will exhaust the whole set R_p by analogy with the property so that if we move along a circumference in the same direction, then sooner or later we will arrive at the initial point. At the same time, if we take an element $a \in Z$, then the set Z can be exhausted only if we first successively add +1 to a and then -1 to a or *vice versa* and those operations should be performed an infinite number of times. As noted in [1,2], in FQT, strong cyclicity plays an important role. In particular, it explains why one IR of the symmetry algebra describes a particle and its antiparticle simultaneously.

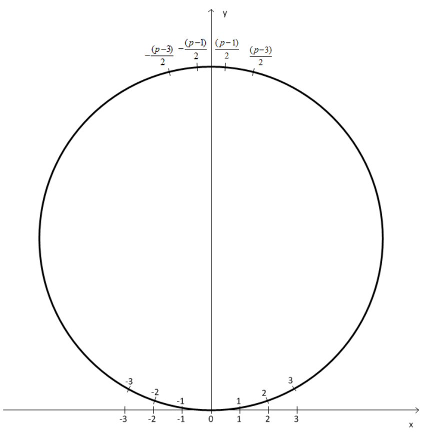

Figure 1. Relation between R_p and Z.

The above construction has a known historical analogy. For many years, people believed that the Earth was flat and infinite, and, only after a long period of time, they realized that it was finite and curved. It is difficult to notice the curvature when we deal only with distances much less than the radius of curvature. Analogously, one might think that the set of numbers describing physics in our universe has a "curvature" defined by a very large number p but we do not notice it when we deal only with numbers much lower than p.

By analogy with SM, one can say that, if $a \in R_p$, then $a > 0$ if $f(a) > 0$ and $a < 0$ if $f(a) < 0$. In other words, "positive" elements of R_p are on the right half-circle of Figure 1 and "negative" elements on the left half-circle. In SM, if $a > 0$ and $b > 0$, then $(a+b) > 0$. However, in FM, this is not necessarily the case because the operations here are modulo p. For example, $(p-1)/2 > 0$ and $1 > 0$ but $(p-1)/2 + 1 = (p+1)/2 = -(p-1)/2$, i.e., $f((p-1)/2+1) < 0$. Therefore, in R_p, the concepts of $>$ and $<$ have the same meaning as in SM only if they apply to numbers a such that $|f(a)|$ is much less than p.

In Section 6.3 of [1,2], the following is proved from the **Definition**:

Statement 1: The ring R_p is more general than the ring Z and the latter is a degenerate case of the former in formal limit $p \to \infty$.

This implies that the ring Z is the limit of the ring R_p when $p \to \infty$. Note that, *in the technique of SM, infinity is understood only as a limit (i.e., as potential infinity) but the basis of SM does involve actual infinity*. SM starts from the infinite ring Z and, even in standard textbooks on mathematics, whether Z can be treated as a limit of finite rings is not even posed as a problem. The problem of actual infinity is discussed in a vast portion of the literature, and, in SM, Z is treated as actual and not potential infinity, i.e., there is no rigorous definition of Z as a limit of finite rings. Moreover, classical set theory considers infinite sets with different cardinalities.

As explained in [1,2,5], *Statement 1* is the basic stage in proving the **Statement**, i.e., that FM is more general than SM. In particular, in approach (ii), this means that FQT is more general (fundamental) than SQT. This issue will be also discussed in Sections 4 and 6. Therefore, *Statement 1* should not be based on the properties of the ring Z derived in SM. The statement should be proved by analogy with the standard proof that a sequence of natural numbers (a_n) goes to infinity if $\forall M > 0\ \exists n_0$ such that $a_n \geq M\ \forall n \geq n_0$. In particular, the proof should involve only potential infinity but not actual infinity.

The meaning of *Statement 1* is that, for any $p_0 > 0$, there exists a set S belonging to all R_p with $p \geq p_0$ and a natural number n such that for any $m \leq n$, the result of any m operations of summation, subtraction or multiplication of elements from S is the same as in R_p for any $p \geq p_0$ and that the cardinality of S and the number n formally go to infinity when $p_0 \to \infty$. This means that, for the set S and number n, there is no manifestation of operations modulo p, i.e., the results of any $m \leq n$ operations of elements from S are formally the same in R_p and Z. This implies that, for experiments involving only sets S and numbers n, it is not possible to conclude whether the experiments are described by a theory involving R_p with a large p or by a theory involving Z.

As noted, e.g., in [1,2], Z can be treated as a limit of R_p when $p \to \infty$ follows from a construction called ultraproducts. However, the theory of ultraproducts is essentially based on classical results involving actual infinity, in particular, on Łoŝ' theorem involving the axiom of choice. Therefore, the theory of ultraproducts cannot be used in proving that FM is more general than SM.

When the radius of the circumference in Figure 1 becomes infinitely large, then a vicinity of zero in R_p becomes the infinite set Z when $p \to \infty$. Therefore, *even from a pure mathematical point of view, the concept of infinity cannot be fundamental because as soon as we involve infinity and replace R_p by Z, we automatically obtain a degenerate theory because, in Z, there are no operations modulo a number*.

In FQT, states are elements of linear spaces over R_p. One might think that SQT is a more general theory than FQT because, in SM, Z is generalized to the case of rational and real numbers. However, as noted in [1,2] and Section 4, since, in SQT, the states are projective, then *even in standard quantum theory*, it suffices to use only integers for describing experimental data with any desired accuracy.

3. Analogy between SR and FM

As noted in Section 1, in the standard physics literature, the fundamental nature of various physical theories is discussed on the basis of physical considerations. However, in Section 4, we will discuss purely mathematical criteria for comparing the fundamentality of various physical theories over SM. Nevertheless, for illustrative purposes, in this section, we consider a comparison of SR and NM from the point of view of a very simple example.

Before the creation of SR, it was believed that NM was the most general (fundamental) mechanics. There are no restrictions on the magnitude of speed there, which can be in the interval $[0, \infty)$. However, in SR, the speed cannot exceed c.

The fact that there is a speed limit greatly changes the standard philosophy of NM. For example, in NM, it seems unnatural that the speed of $0.99c$ is possible but $1.01c$ is not. For this and other reasons, it took a long time for SR to be accepted by the majority of physicists.

Let us consider a simple model example where, in our reference frame, an observer moves with speed v_1 and, in the reference frame of this observer, a particle moves in the same direction with speed v_2. Then, according to the rules of NM, the speed of the particle in our reference frame will be $V = v_1 + v_2$. Therefore, even if $v_1 < c$ and $v_2 < c$, then, in NM, a situation is possible where $V > c$ and this may suggest that the statement of SR about the speed limit is not consistent. However, the result of SR in such a situation is not $V = v_1 + v_2$ but

$$V = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2} \tag{3}$$

and this value cannot exceed c. In particular, if $v_1 = v_2 = 0.6c$, then V is not equal to $1.2c$ as one might think from naive considerations but $V \approx 0.882c$, and, if $v_1 = v_2 = 0.99c$, then V is not equal to $1.98c$ but $V \approx 0.9999495c$. The lesson of this example is that it is not always correct to make judgments proceeding from "common sense".

Here, there is an analogy with FM. For example, if a and b are such natural numbers that $a < p$, $b < p$, and, in SM, there may be a situation when $(a + b) > p$, then, in FM, such a situation cannot exist because the following always applies: $(a + b)\ (mod\ p) < p$.

It is now generally accepted that SR is confirmed experimentally to a greater extent than NM. Also, as explained in Section 4, it follows from the **Definition** that NM is a degenerate case of SR since SR can reproduce any fact of NM with a choice of c, while NM cannot reproduce those facts of SR in which it is essential that c is finite and not infinite. Thus, SR does not disprove NM but shows that it works with high accuracy when speeds are much less than c. There is an analogy here with the fact that, as shown in Section 2, FM does not refute SM but shows that the latter is a good approximation to reality only in situations where the numbers in a given problem are much less than p.

In complete *logical* analogy with the objections to FM in points (a–c) in Section 1, one can put forward similar objections to SR but now the role of p will be played by c. Therefore, I think that, *being completely consistent, if we reject FM, we must also reject SR, and, if we accept SR, then by the same logic we must also accept that FM is more general (fundamental) than SM*.

As follows from the above results, it is not necessary to apply SR in everyday life when speeds are much less than c because, in this case, NM works with a very high accuracy. Analogously, for describing almost all phenomena at the macroscopic level, there is no need to apply QT. For example, there is no need to describe the motion of the Moon by the Schrödinger equation. In principle, this is possible, but results in unnecessary complications. At the same time, microscopic phenomena can be correctly described only in the framework of QT.

4. Quantum Theory Based on Finite Mathematics

In QFT, symmetry at the quantum level is described as follows: First, the existence of a background space-time is postulated, e.g., Galilei, Minkowski, dS, AdS or some other background. This background has a group of motions. It is postulated that the basic operators for the system under consideration commute as required in the Lie algebra representation of this group. That is, these operators form a representation of the Galilei algebra, Poincare algebra, dS algebra, AdS algebra or some other algebra. This approach to symmetry is in the spirit of Felix Klein's Erlangen program.

The Erlangen program was proposed in 1872 when quantum theory did not yet exist. As discussed in detail in [1,2,5], the approach to symmetry at the quantum level should be the opposite. The fact is that background is a purely classical concept. In quantum theory, each physical quantity must have a corresponding operator, but there are no operators for coordinates x of the background. Therefore, the approach to symmetry at the quantum level should be as follows: Each system is described by a set of basic operators and symmetry is determined by how these operators commute with each other. For example, by definition, dS symmetry should not involve the fact that the dS group is the group of motions of dS space. Instead, *the definition* is that the operators M^{ab} ($a, b = 0, 1, 2, 3, 4$, $M^{ab} = -M^{ba}$) describing the system under consideration satisfy the commutation relations

$$[M^{ab}, M^{cd}] = -i(\eta^{ac} M^{bd} + \eta^{bd} M^{ac} - \eta^{ad} M^{bc} - \eta^{bc} M^{ad}) \tag{4}$$

where η^{ab} is the diagonal tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$. The *definition* of AdS symmetry is given by the same equations but $\eta^{44} = 1$.

The concepts (kg, m, s) come from classical theory, so these concepts should not exist in quantum theory. In particular, quantum theory should not contain the parameters $(c, \hbar, R)$ if c is understood as the speed of light in m/s, $\hbar$ is understood as the Planck constant in kg$\cdot$ m^2/s and R is understood as the radius of dS or AdS space in meters. With such a treatment of $(c, \hbar, R)$, these parameters may be different at different stages of the universe. However, as argued by Dyson in his famous paper [9], in quantum theory, $(c, \hbar, R)$ can be treated as contraction parameters from RQT to NQT, from QT to CT and from dSQT or AdSQT to RQT, respectively. Then, the parameters $(c, \hbar, R)$ can be identified with their respected classical values in semiclassical approximation. For the first time, the concept of contraction has been discussed by Inonu and Wigner [10].

The following is argued by Dyson [9] (see also [1,2,5]):

(i) NQT is a degenerate case of RQT in the formal limit $c \to \infty$;
(ii) CT is a degenerate case of QT in the formal limit $\hbar \to 0$;
(iii) RQT is a degenerate case of dSQT and AdSQT in the formal limit $R \to \infty$.

In the literature, those properties are usually discussed from physical considerations. However, as shown in Section 1.3 of [1,2], those properties can be proved purely mathematically taking into account the **Definition** and the fact that symmetry at the quantum level is defined by the corresponding representation of the symmetry algebra.

The above facts prove that R is fundamental to the same extent as $\hbar$ and c (see also [1,2,11] for details). By analogy with the fact that c must be finite, R must be finite too; the formal case $R = \infty$ corresponds to the situation where the dS and AdS algebras do not exist because they become the Poincare algebra. *At the quantum level, R is only the parameter of contraction from dS or AdS algebras to the Poincare one and has nothing to do with the radius of the dS or AdS space*. As shown in [11], the result for the cosmological acceleration obtained in semiclassical approximation to dSQT without any geometry is the same as in GR when the radius of the dS space equals R.

The properties (i)–(iii) have been proved in SQT based on complex numbers. How to generalize these results to the case of FQT is a problem that has arisen. In this theory, the space of states is a linear space over the ring R_{p^2} or the field F_{p^2}, which contain p^2 elements. Any element of R_{p^2} can be represented as $a + bi$ where $a, b \in R_p$ and i is a formal element such that $i^2 = -1$. Then, the definition of addition, subtraction and multiplication in R_{p^2} is obvious and R_{p^2} is a ring regardless of whether p is prime or not.

However, F_{p^2} can be a field only if p is prime and this condition is not sufficient. By analogy with the field of complex numbers, one could define division as $(a + bi)^{-1} = (a - ib)/(a^2 + b^2)$. This definition can be meaningful only if $a^2 + b^2 \neq 0$ in F_p for any $a, b \in F_p$, i.e., $a^2 + b^2$ is not divisible by p. Therefore, the definition is meaningful only if p *cannot* be represented as a sum of two squares. For example, F_{p^2} can be defined as $F_p + iF_p$ if $p = 7$ but cannot be defined in this way if $p = 5$.

We will not consider the case $p = 2$ and therefore p is necessarily odd. Then we have two possibilities; the value of $p \ (mod\, 4)$ is either 1 or 3. The known result of number theory [6–8] is that a prime number p can be represented as a sum of two squares only in the former case and not in the latter one. Therefore, $F_{p^2} = F_p + iF_p$ only if $p \,(mod\, 4) = 3$. Nevertheless, as shown in standard textbooks [6–8], quadratic extensions of F_p exist also in the case $p = 1 (mod\ 4)$.

Every quadratic finite ring or field has only one nontrivial automorphism *. If $R_{p^2} = R_p + iR_p$ or $F_{p^2} = F_p + iF_p$, this automorphism is the complex conjugation $(a + bi)^* = (a - bi)$; however, as shown in standard textbooks (e.g., in [6–8]), the automorphism of F_{p^2} can also be defined if $p = 1 \ (mod\ 4)$.

In spaces over R_{p^2} or F_{p^2}, one can formally define a scalar product (y, x) for the elements x, y belonging to those spaces such that $(y, \lambda x) = \lambda(y, x)$ and $(\lambda y, x) = \lambda^*(y, x)$ where $\lambda \in R_{p^2}$ or $\lambda \in F_{p^2}$, respectively.

In SQT, operators A of physical quantities act in Hilbert spaces $\mathcal{H}$ supplied by a scalar product (...,...), and these operators are selfadjoint: $(Ax, y) = (y, Ax)\ \forall x, y \in \mathcal{H}$ belonging to the domain of A. In particular, the operators in Equation (4) are selfadjoint. By analogy, in FQT, linear operators A of physical quantities act in spaces over R_{p^2} or F_{p^2} and, formally, such operators can be called selfadjoint if $(Ax, y) = (y, Ax)$ for all elements x, y belonging to such spaces.

In SM, scalar products in Hilbert spaces have a property $(x, x) > 0$ if $x \neq 0$, and, in SQT, this property has a known probabilistic interpretation. The physical meaning of probability is such that it is defined by an infinite number of experiments. In nature, there can be no infinite number of experiments and so the concept of probability is based on an idealization. However, as explained in Section 2, in FM, the concepts of $>$ and $<$ have a limited meaning. For example, if $e_1, e_2, , , e_n$ are elements of the basis in a space over R_{p^2} such that $(e_j, e_k) = 0$ if $j \neq k$, and $a_1, a_2, ...a_n$ are elements of R_{p^2}, then

$$(a_1e_1 + ...a_ne_n, a_1e_n + ...a_ne_n) = a_1a_1^*(e_1, e_1) + ...a_na_n^*(e_n, e_n) \tag{5}$$

In SM, the analogous expression will always be positive but, since in FM the operations are performed modulo p, this expression may even be "negative", even if all the quantities $a_ja_j^*$ and (e_j, e_j) are "positive". Therefore, in FQT, the probabilistic interpretation has only a limited meaning when not only $f(a_ja_j^*) > 0$ and $f((e_j, e_j)) > 0\ \forall j$ but also

$$f(a_1a_1^*)f((e_1, e_1)) + ...f(a_na_n^*)f((e_n, e_n)) > 0 \tag{6}$$

It is clear that only those quantum theories over SM can be generalized to theories over FM where all physical quantities are dimensionless and discrete. As shown in [1,2], among the theories considered in this section, only in dSQT and AdSQT are all physical quantities dimensionless and those theories are the most general.

In SQT, IRs of the algebras in Equation (4) when the operators in these expressions are selfadjoint are described in a wide range of the literature. All such IRs are infinite-dimensional. Representations in spaces over a ring or field of nonzero characteristic are called modular representations. According to the Zassenhaus theorem (see, e.g., [12,13]), all modular IRs are finite-dimensional. In [14,15], we constructed modular IRs of the algebras defined by Equation (4).

In SQT, all Hilbert spaces are separable, i.e., they contain a countable dense subset. As shown in standard textbooks on Hilbert spaces (see, e.g., [16]), a Hilbert space is separable if and only if it admits a countable orthonormal basis $(e_1, e_2, ...e_n, ...)$. It is always possible to choose a basis such that the norm of each e_j is an integer. The elements of such spaces can be denoted as $(c_1, c_2, ...c_n, ...)$, where all the coordinates c_j are complex numbers. The known result of the theory of Hilbert spaces is that the set of all points $(c_1, c_2, ...)$ with only finitely many nonzero coordinates, each a rational number, is dense in the separable Hilbert space (see, e.g., [16]). This implies that, *with any desired accuracy*, each element of the Hilbert space can be approximated by a finite linear combination

$$x = \sum_{j=1}^{n} c_je_j \tag{7}$$

where $c_j = a_j + ib_j$ and all the numbers (a_j, b_j) $(j = 1, 2, ...n)$ are rational.

The next observation is that spaces in quantum theory are projective, i.e., for any complex number $c \neq 0$, the elements x and cx describe the same state. The meaning of this statement is that it is not the probability itself but ratios of different probabilities that have a physical meaning. As a consequence, both parts of Equation (7) can be multiplied by a common denominator of all the nonzero numbers a_j and b_j. As a consequence, the following applies:

Statement 2: Each element of a separable Hilbert space can be approximated with any desired accuracy by a finite linear combination (7) where all the numbers a_j and b_j are integers, i.e., belong to Z.

The important consequence for understanding standard quantum theory is that, in this theory, there is a large excess of states. Although, formally, the theory involves Hilbert spaces of states $(c_1, c_2, ...c_n, ...)$ where all the c_j are arbitrary complex numbers and the only limitation is the condition $\sum_{j=1}^{\infty} |c_j|^2 < \infty$, for describing experiments with any desired accuracy, it suffices to involve only states where only a finite number of the coefficients $c_j = a_j + ib_j$ are nonzero and all the numbers (a_j, b_j) are integers.

Now a problem arises regarding how to use the **Definition** for proving that FQT is more general (fundamental) than SQT and the latter is a degenerate case of the former in the formal limit $p \to \infty$. According to this **Definition**, the proof should consist of proving the following two statements:

(A) There exists a value of $p = p_0$ such that any result of SQT can be obtained in FQT for all $p \geq p_0$;
(B) There exist phenomena which FQT can describe while SQT cannot.

Let us first consider property (A).

In SQT, states of a system are described by Equation (7) where the e_j are elements of a basis in a Hilbert space and the $c_j = a_j + ib_j$ are complex numbers. At the same time, in FQT, states of a system are also described by Equation (7) but now the e_j are elements of a basis in a space over R_{p^2} and $c_j = a_j + ib_j$ where the elements a_j and b_j belong to R_p. As explained above, in SQT, it is always possible to find the elements e_j such that their norms are integers and, as noted in *Statement 2*, it suffices to consider such states (7) where only finite numbers of the a_j and b_j are nonzero integers, i.e., they are elements of Z.

As noted above, it follows from (i)–(iii) that, among quantum theories in which the symmetry algebras are ten-parameter, dSQT and AdSQT are the most general. While, in RQT, there are operators having dimensions expressed in terms of (kg, m, s) and containing a continuous spectrum, in dSQT and AdSQT, all the operators in Equation (4) are dimensionless and, as shown in [14,15], it is possible to choose bases in which they have only a discrete spectrum, i.e., the spectrum belonging to Z.

Then, as follows from *Statement 1* in Section 2, if Equations (5) and (6) are satisfied at some $p = p_0$, they will also be satisfied at all $p > p_0$. Therefore, if at some $p = p_0$, FQT gives the same results as SQT, then the same will take place at all $p > p_0$, i.e., property (A) is satisfied.

Property (B) will be demonstrated in Section 6.

5. Why Finite Mathematics Is More Natural than Classical Mathematics

The belief that SM is the most fundamental mathematics arose after Newton and Leibniz proposed the theory of infinitesimals more than 300 years ago. This belief was in the spirit of existing ideas where, when people did not know about the existence of elementary particles, they believed that any object could be divided into arbitrarily large number of arbitrarily small parts. However, the very fact of the existence of elementary particles (which cannot be divided into parts) indicates that, in nature, there are no infinitesimals or continuity. Therefore, theories involving these concepts (including standard geometry) at best can only be a good approximation when the discrete nature of matter is not taken into account.

It seems unnatural that SQT involves SM with differential equations and infinitesimals. Even the name "quantum theory" reflects a belief that nature is quantized, i.e., discrete, and this name has arisen because, in QT, some quantities have a discrete spectrum (e.g., the spectrum of the angular momentum operator, the energy spectrum of the hydrogen atom, etc.). But this discrete spectrum has appeared in the framework of SM.

As a rule, physicists agree that, in nature, there are no infinitesimals. They say that, for example, dx/dt should be understood as $\Delta x/\Delta t$ where Δx and Δt are small but not infinitesimal. I point out that they work with dx/dt not $\Delta x/\Delta t$. They reply that, since mathematics with derivatives works well, then there is no need to philosophize and develop something else (and they are not familiar with FM). So, people invented continuity and infinitesimals which do not exist in nature, created problems for themselves and now apply titanic efforts for solving those problems.

The founders of QT and the scientists who essentially contributed to it were highly educated. But they used only SM, and, even now, FM is not a part of the standard education for physicists. The development of QFT has shown that the theory contains anomalies and divergences. Most physicists considering those problems work in the framework of SM and do not acknowledge that they arise just because this mathematics is applied.

Several famous physicists (e.g., the Nobel Prize laureates Gross, Nambu and Schwinger) discussed approaches where QT involves FM (see, e.g., [17]). A detailed discussion of these approaches is given in book [18], where they are characterized as hybrid quantum systems. The reason is that, here, momenta and coordinates belong to a finite ring or field but wave functions are elements of standard Hilbert spaces. Then, the problem of the foundation of QT is related to the problem of the foundation of SM. On the other hand, in [1,2,5,14,15], we have proposed an approach called finite quantum theory (FQT) where not only physical quantities but also wave functions involve finite rings or fields.

In view of this discussion, a problem arises as to whether it is justified to use mathematics with infinitesimals for describing nature in which infinitesimals do not exist. Although SM describes many physical phenomena with a very high accuracy, a problem arises as to whether there are phenomena which cannot be correctly described by mathematics involving infinitesimals.

Some facts of SM seem to be unnatural. For example, $tg(x)$ is a one-to-one reflection of $(-\pi/2, \pi/2)$ onto $(-\infty, \infty)$, i.e., the impression might arise that both intervals have the same numbers of elements although the first interval is a nontrivial part of the second one. However, Hilbert said, "No one shall expel us from the paradise that Cantor has created for us".

From the point of view of Hilbert's approach (see Section 1), it is not important whether some statements of SM are natural or not since the goal of the approach is to find a complete and consistent set of axioms. In the framework of this approach, the problem of the foundation of SM has been investigated by many great mathematicians (e.g., Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and others). Their philosophy was based on macroscopic experience in which the concepts of infinitesimals, continuity and standard division are natural. However, as noted above, those concepts contradict the existence of elementary particles and are not natural in QT. The illusion of continuity arises when one neglects the discrete structure of matter.

The existence of foundational problems in Hilbert's approach follows, in particular, from Gödel's incompleteness theorems, which state that no system of axioms can ensure that all facts about natural numbers can be proved, and the system of axioms in SM cannot demonstrate its own consistency. The theorems are written in highly technical terms of mathematical logics. As already noted, in this paper, we do not consider Hilbert's approach to mathematics. However, simple arguments in [1,2] show that, if mathematics is treated as a tool for describing nature, then the foundational problems of SM follow from the simple arguments described below.

In the 1920s, the Viennese circle of philosophers developed an approach called logical positivism which contains the following verification principle: *A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false* [19,20]. However, this principle does not work if SM is treated as a tool for describing nature. For example, in Hilbert's approach, one of axioms is that $a + b = b + a$ for all natural numbers a and b, and the question as to whether this is true or false does not arise. However, if mathematics is treated as a tool for describing nature, it cannot be determined whether this statement is true or false.

As noted by Grayling [21], *"The general laws of science are not, even in principle, verifiable, if verifying means furnishing conclusive proof of their truth. They can be strongly supported by repeated experiments and accumulated evidence but they cannot be verified completely"*. So, from the point of view of SM and physics, the verification principle is too strong.

Popper proposed the concept of falsificationism [22]: *If no cases where a claim is false can be found, then the hypothesis is accepted as provisionally true*. In particular, this has been related to the statement that $a + b = b + a$ for all natural numbers a and b can be treated as provisionally true until one has found some numbers a and b for which $a + b \neq b + a$.

According to the philosophy of quantum theory, there should be no statements accepted without proof and based on belief in their correctness (i.e., axioms). The theory should contain only those statements that can be verified where by "verified" physicists mean an experiment involving only a finite number of steps. This philosophy is the result of the fact that quantum theory describes phenomena which, from the point of view of "common sense", seem meaningless but have been experimentally verified. So, the philosophy of QT is similar to verificationism, not falsificationism. Note that Popper was a strong opponent of QT and supported Einstein in his dispute with Bohr.

From the point of view of verificationism and the philosophy of QT, SM is not well defined not only because it contains an infinite number of numbers. Consider, for example, whether the rules of standard arithmetic can be justified.

We can verify that 100 + 100 = 200 and 1000 + 1000 = 2000, but can we verify that, say, $10^{100,000} + 10^{100,000} = 2 \cdot 10^{100,000}$? One might think that this is obvious, and, in Hilbert's approach, this follows from the main axioms. However, if mathematics is treated as a tool for describing nature, then this is only a belief based on extrapolating our everyday experience to numbers where it is not clear whether the experience still works.

In Section 3, we discussed that our life experience works well at speeds that are much less than c, but this experience cannot be extrapolated to situations where speeds are comparable to c. Likewise, our experience with the numbers we deal with in everyday life cannot be extrapolated to situations where the numbers are much greater.

According to verificationism and the principles of quantum theory, whether the statement $10^{100,000} + 10^{100,000} = 2 \cdot 10^{100,000}$ is true or false depends on whether this statement can be verified. Is there a computer which can verify this statement? Any computing device can operate only with a finite number of resources and can perform calculations only modulo some number p. If our universe contains only a finite number of elementary particles, then, in principle, it is not possible to verify that standard rules of arithmetic are valid for any numbers.

That is why the statements in Equation (1) are ambiguous: because they do not contain information on the computing device which verifies those statements. For example, let us ask whether 100 + 200 equals 300. If our computing device is such that $p = 400$, then the experiment will confirm this, while, if $p = 250$, then we will obtain 100 + 200 = 50.

So, the statements that 100 + 200 = 300 and even that $2 \cdot 2 = 4$ are ambiguous because they do not contain information on how they should be verified. On the other hand, the statements

$$100 + 200 = 300\,(mod\,400),\;\; 100 + 200 = 50\,(mod\,250),$$

$$2 \cdot 2 = 4\,(mod\,5),\;\; 2 \cdot 2 = 2\,(mod\,2)$$

are well defined because they do contain such information. Therefore, only operations modulo a number are well defined.

We believe the following observation is very important: Although SM is a part of our everyday life, people typically do not realize that *standard operations with natural numbers are implicitly treated as limits of operations modulo p when $p \to \infty$*. For example, if (a, b, c, p) are natural numbers, then Equation (1) is implicitly treated as

$$\lim_{p \to \infty} [(a + b)\ \ (mod\ \ p)] = c,\;\; \lim_{p \to \infty} [(a \cdot b)\ \ (mod\ \ p)] = c,\;\; etc.$$

As a rule, every limit in mathematics is thoroughly investigated; however, in the case of standard operations with natural numbers, it is not even mentioned that those operations are limits of operations modulo p. In real life, such limits might not even exist if, for example, the universe contains a finite number of elementary particles.

So we can see that the question of what 100 + 200 is equal to is not a question of what some theory says, but a question of how an experiment will be set up to test what this value is equal to. In one experiment, the result may be 300, in another, 50, and there is no theory that says that one experiment is preferable to another.

Now let us discuss the question of what p can be equal to in the theory describing modern physics. Recently, an increasing number of works have appeared that say that the universe works like a computer (see, for example, [23]). From this point of view, the value of p is determined by the state of the universe at a given stage. And, since the state of the universe is changing, it is natural to expect that the number p describing physics at different stages of the evolution of the universe will be different at different stages. Therefore, by analogy with the discussion of what 100 + 200 is equal to, we can say that p is not a number that is determined by some fundamental theory, but a number that depends on the state of the universe at a given stage.

The problem of time is one of the most fundamental problems of quantum theory. Every physical quantity should be described by a selfadjoined operator, but, as noted by Pauli, the existence of the time operator is a problem (see, e.g., the discussion in [1,2]). One of the principles of physics is that the definition of a physical quantity is a description of how this quantity should be measured, and *it is not correct to say that a certain quantity exists but cannot be measured*. The present definition of a second is the time during which 9,192,631,770 transitions in a cesium-133 atom occur. The time cannot be measured with absolute accuracy because the number of transitions is finite. Then, one second is defined with the accuracy 10^{-15} s, and [24] describes efforts to measure time with the accuracy 10^{-19} s. However, a problem arises as to how to define time in the early stages of the universe when atoms did not exist. Therefore, treating time t as a continuous quantity is an approximation which can only work in some conditions. In [1,2], we discussed the conjecture that standard classical time t manifests itself because the value of p changes, i.e., t is a function of p. We do not say that p changes over time because classical time t cannot be present in quantum theory; we say that we feel changing time because p changes. As shown in [11] (see also the subsequent section), with such an approach, the known problem of the baryon asymmetry of the universe does not arise.

6. Examples Where Finite Mathematics Can Solve Problems Which Standard Mathematics Cannot

As noted in Section 4, for proving that FQT is more general (fundamental) than SQT, it is necessary to prove the properties (A) and (B) described at the end of this section. Property (A) has already been demonstrated at the end of Section 4. Property (B) means that there are phenomena that FQT can explain but SQT cannot. In [1,2], we discussed phenomena where it is important that p is finite. They cannot be described in SQT by analogy with the fact that NM cannot describe cases where it is important that c is finite. Below, we describe several such phenomena.

Example 1. Gravity. *The Newton gravitational law cannot be derived in QFT because the theory is not renormalizable. However, the law can be derived from FQT in semiclassical approximation [1,2]. Then, the gravitational constant G is not taken from the outside but depends on p as* $1/ln(p)$. *By comparing this result with the experimental value, one discovers that* $ln(p)$ *is of the order of* 10^{80} *or more, and, therefore, p is a huge number of the order of* $exp(10^{80})$ *or more. One might think that, since p is so huge, then, in practice, p can be treated as an infinite number. However, since* $ln(p)$ *is "only" of the order of* 10^{80}, *gravity is observable. In the formal limit* $p \to \infty$, *G becomes zero and gravity disappears. Therefore, in our approach, gravity is a consequence of the finiteness of nature.*

Example 2. The Dirac vacuum energy problem. *In quantum electrodynamics, the vacuum energy should be zero, but, in QFT, the sum for this energy diverges. This problem was posed by Dirac. To obtain the zero value, the artificial requirement that the operators should be written in the normal order is imposed, but this requirement does not follow from the construction of the theory.*

In Section 8.8 of [1,2], I take the standard expression for this sum and explicitly calculate it in FM without any assumptions. Then, since the calculations are modulo p, I obtain zero as it should be.

Example 3. Equality of masses of particles and their antiparticles. *This is an example demonstrating the power of finite mathematics. A discussion in [1,2,5] and Section 4 shows that, in QT, an elementary particle and its antiparticle should be considered only from the point of view of IRs of the symmetry algebra. In SQT, the algebras are such that their IRs contain either only positive or only negative energies. In the first case, the objects are called particles and, in the second one, antiparticles. Then, the energies of antiparticles become positive after second quantization.*

In QFT, the spectrum of positive energies contains the values $(m_1, m_1 + 1, m_1 + 2, \cdots \infty)$, and, for negative energies, the values $(-m_2, -m_2 - 1, -m_2 - 2, \cdots - \infty)$, where $m_1 > 0$, $m_2 > 0$, m_1 is called the mass of a particle and m_2 is called the mass of the corresponding antiparticle. Experimentally, $m_1 = m_2$, but, in QFT, IRs with positive and negative energies are fully independent of each other. It is claimed that $m_1 = m_2$ because local covariant equations are CPT invariant. However, as explained in [1,2,5], the argument x in local quantized fields does not have a physical meaning because it is not associated with any operator. Therefore, in fact, SQT cannot explain why $m_1 = m_2$.

Consider now what happens in FQT. For definiteness, we consider the case where p is odd and the case where p is even can be considered analogously. One starts constructing the IR with the value m_1, and, by acting on the states by raising operators, one obtains the values $m_1 + 1, m_1 + 2, \cdots$. However, now we are moving not along the x axis but along the circle in Figure 1. When we reach the value $(p - 1)/2$, the next value is $-(p - 1)/2$, i.e., one can say that, by adding 1 to a large positive number $(p - 1)/2$, one obtains a large negative number $-(p - 1)/2$. By continuing this process, one obtains the numbers $-(p - 1)/2 + 1 = -(p - 3)/2$, $-(p - 3)/2 + 1 = -(p - 5)/2$, etc. The explicit calculation [1,2] shows that the procedure ends when the value $-m_1$ is reached.

Therefore, FM gives a clear proof that $m_1 = m_2$ and shows that, instead of two independent IRs in SM, one obtains only one IR describing both a particle and its antiparticle. The case described by SM is degenerate because, in the formal limit $p \to \infty$, one IR in FM splits into two IRs in SM. So, when $p \to \infty$, we obtain symmetry breaking. This example shows that the standard concept of particle–antiparticle is only approximate and is approximately valid only when p is very large. Therefore, constructing a complete QT based on FM should be based on new principles.

Example 4. The problem of baryon asymmetry of the universe. *Modern cosmological theories state that the numbers of baryons and antibaryons in the early stages of the universe were the same. Then, since the baryon number is the conserved quantum number, those numbers should be the same at the present stage. However, at this stage, the number of baryons is much greater than the number of antibaryons.*

To understand this problem, one should understand the concept of particle–antiparticle. In SQT, this concept takes place because IRs describing particles and antiparticles are such that energies in them can be either only positive or only negative but cannot have both signs. However, as explained in **Example 3**, IRs in FQT necessarily contain both positive and negative energies, and, in the formal limit $p \to \infty$, one IR in FQT splits into two IRs in SQT with positive and negative energies.

As noted above, the number p is different at different stages of the universe. As noted in **Example 1**, at the present stage of the universe, this number is huge, and, therefore, the concepts of particles and antiparticles have a physical meaning. However, arguments given in [1,2] indicate that, in the early stages of the universe, the value of p was much less than now. Then, each object described by IR is a superposition of a particle and antiparticle (in SQT, such a situation is prohibited), and the electric charge and baryon quantum number are not conserved. Therefore, in the early stages of the universe, SQT does not work, and the statement asserting that, at such stages, the numbers of baryons and antibaryons were the same does not have a physical meaning. Therefore, the problem of the baryon asymmetry of the universe does not arise.

Example 5. *As argued in Section 6.8 of [1,2], the ultimate QT will be based on a ring not a field, i.e., only addition, subtraction and multiplication are fundamental mathematical operations while division is not.*

The above examples demonstrate that there are phenomena which can be explained only in FQT because, for them, it is important that p is finite and not infinitely large. Therefore, FQT is more general (fundamental) than SQT. Here, we have an analogy with the case where SR can explain phenomena where c is finite while NM cannot explain such phenomena.

7. Answers to Arguments (a–c) in Section 1

To remove divergences, physicists usually carry out the following: In integrals over the absolute values of momenta, the upper limit of integration is taken not as ∞ as it should be, but as the value L, called the Pauli–Villars cutoff. Then, all integrals formally become finite, but they depend on the nonphysical very large quantity L. In renormalizable theories, various contributions to the S-matrix can be arranged in such a way that the contributions with L cancel, but, in non-renormalizable theories, it is not possible to remove L.

The idea of argument (a) is such that, by analogy with SQT, where there are divergent integrals that are cut off by the value of L, in FQT, there are formally no divergences, but there are quantities depending on the enormous value p. However, this analogy does not work for several reasons.

In Section 3, we noted that, from our experience in NM, we think that some of the arguments are based on common sense. But these arguments only work at speeds which are much less than c and often fail at speeds comparable to c. Likewise, some arguments which, from our experience in SM, seem to come from common sense, usually work in FM only for numbers much less than p and often fail for numbers comparable to p.

As noted in Section 2, in FM, there are no strict concepts of positive and negative or concepts of $>$ and $<$. These concepts approximately work for numbers that are much less than p and are in the neighborhood of zero according to Figure 1.

In SM, when we add two positive numbers, we always obtain a positive number that is greater than the original numbers. However, since, in FM, calculations are carried out modulo p, situations are possible where we add two "positive" numbers and obtain a "negative" number. For example, in finite mathematics, $(p-1)/2+1 = -(p-1)/2$, i.e., adding two numbers which in Figure 1 are in the right half-plane, we obtain a number that, in this figure, is in the left half-plane.

In **Example 2** in Section 6, we describe an example where, in SQT, as a result of adding many positive values, a divergent expression is obtained, while, in FQT, the result is 0 because the calculations are carried out modulo p. Thus, argument (a) does not always work in FQT.

Argument (b) is unacceptable because even the theory with adeles is not finite and therefore automatically has foundational problems. Arguments (b) and (c) (that it is not clear from what considerations p is chosen) are not a refutation of FQT for the following reason: As explained in Section 5, the value of p is not a fundamental parameter that follows from some theory. This value is determined by the state of the universe at the given stage of its development, and, at different stages, the values of p are different.

To conclude this section, we note the following: One of the objections to FQT is that the authors of these objections interpret p as the greatest possible number in nature and invoke the argument attributed to Euclid that there can be no greatest number in nature because if p is such a number, then $(p+1) > p$. Similarly, one can say that c cannot be the greatest possible speed because $1.01c > c$. As explained above, these arguments arise because our experience at speeds which are much less than c and numbers which are much less than p is extrapolated to situations where speeds are comparable to c or numbers are comparable to p.

8. Conclusions

The goal of this paper is to explain at the simplest possible level why FM is more general (fundamental) than SM. As noted in Section 5, the belief of most mathematicians and physicists that SM is the most fundamental arose for historical reasons. However, as explained in Section 2, simple mathematical arguments show that SM (involving the concept of infinities) is a degenerate case of FM; SM arises from FM in the degenerate case when operations modulo a number are discarded.

We call FQT a quantum theory based on FM. It is determined by a parameter p which is the characteristic of the ring in finite mathematics describing physics. We note that, in FQT, there are no infinities and that is why divergences are absent in principle. Probabilistic interpretation of FQT is only approximate; it applies only to states described by numbers which are much less than p.

In Section 5, we give arguments that p is not a fundamental quantity that is determined by some theory but depends on the state of the universe at a given stage. Therefore, p is different at different stages of the universe.

The question of why p is this value and not another is similar to the question of why the values of (c,$\hbar$,R) are certain values and not others. As explained in [1,2,11], currently, they are such simply because people want to measure c in m/s, $\hbar$ in kg$\cdot$m^2/s and R in meters, and it is natural to expect that these values at different stages of the universe are different.

As noted in Section 6, at the present stage of the universe, p is an enormous quantity of the order of $exp(10^{80})$. Therefore, at present, SM almost always works with very high accuracy. At the same time, in [1,2,11] and Section 6, we argue that, in the early stages, of the universe, p was much less than now. Therefore, at these stages, the finitude of mathematics played a much greater role than it does now. As a result, the problem of the baryon asymmetry of the universe does not arise.

The famous Kronecker's expression is "God made the natural numbers, all else is the work of man". However, in view of the above discussion, I propose to reformulate this expression as "God made only finite sets of natural numbers, all else is the work of man". For illustration, consider a case where an experiment is conducted N times; the first event happens n_1 times, the second one n_2 times, etc., such that $n_1 + n_2 + ... = N$. Then, the experiment is fully described by a finite set of natural numbers. However, people introduce rational numbers $w_i = w_i(N) = n_i/N$, introduce the concept of limit and define probabilities as limits of the quantities $w_i(N)$ when $N \to \infty$.

The above discussion shows that FM is not only more general (fundamental) than SM but, in addition, in FM, there are no foundational problems because every statement can be explicitly verified by a finite number of steps. The conclusion from the above consideration can be formulated as:

Mathematics describing nature at the most fundamental level involves only a finite number of numbers, while the concepts of limit, infinitesimals and continuity are needed only in calculations describing nature approximately.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: I am grateful to Justin Clarke-Doane who, in numerous letters, explained to me various philosophical issues of mathematics. I am also grateful to Vladimir Karmanov, Teodor Shtilkind and the reviewers of this paper, whose comments were important in preparing the revised version of the paper.

Conflicts of Interest: The author declares no conflicts of interest.

List of Abbreviations

FM	finite mathematics
SM	standard mathematics
SR	special relativity
NM	nonrelativistic mechanics
QT	quantum theory
CT	classical theory
FQT	quantum theory based on finite mathematics
SQT	standard quantum theory
IR	irreducible representation
QFT	quantum field theory
NQT	nonrelativistic quantum theory
RQT	relativistic quantum theory
dS	de Sitter
AdS	anti de Sitter
dSQT	de Sitter quantum theory
AdSQT	anti de Sitter quantum theory

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