Black holes in string theory with higher-derivative corrections

Memoria de Tesis Doctoral realizada por

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Yo, pues, agradecido a la merced que aquí se me ha hecho, no pudiendo corresponder a la misma medida, conteniéndome en los estrechos límites de mi poderío, ofrezco lo que puedo y lo que tengo de mi cosecha...

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List of Publications

The following articles, some of which are unrelated to the content of this thesis, were published by the candidate during the realization of this work:

1. A. Ruipérez, **"Higher-derivative corrections to small black rings"**. arXiv:2003.02269 [hep-th]
2. P. Bueno, P. A. Cano, R. A. Hennigar, V. A. Penas and A. Ruipérez, **"Partition functions on slightly squashed spheres and flux parameters"**, JHEP 04 (2020) 123. arXiv:2001.10020 [hep-th].
3. J. Blabäck, F. F. Gautason, A. Ruipérez and T. Van Riet, **"Anti-brane singularities as red herrings"**, JHEP 1912 (2019) 125. arXiv:1907.05295 [hep-th].
4. T. Ortín and A. Ruipérez, **"Non-Abelian rotating black holes in 4- and 5-dimensional gauged supergravity"**, JHEP 1911 (2019) 167. arXiv:1905.00016 [hep-th].
5. P. A. Cano and A. Ruipérez, **"Leading higher-derivative corrections to the Kerr geometry"**, JHEP 1905 (2019) 189. arXiv:1901.01315 [gr-qc].
6. P. A. Cano, P. F. Ramírez and A. Ruipérez, **"The small black hole illusion"**, JHEP 03 (2020) 115. arXiv:1808.10449 [hep-th].
7. P. A. Cano, P. Meessen, S. Chimento, P. F. Ramírez, T. Ortín and A. Ruipérez, **"Beyond the near-horizon limit: Stringy corrections to Heterotic Black Holes"**, JHEP 1902 (2019) 192, arXiv:1808.10449 [hep-th].
8. P. A. Cano, S. Chimento, T. Ortín and A. Ruipérez, **"Regular Stringy Black Holes?"**, Phys. Rev. D 99 (2019) no.4, 046014. arXiv:1806.08377 [hep-th].
9. S. Chimento, P. Meessen, T. Ortín, P. F. Ramírez and A. Ruipérez, **"On a family of α'-corrected solutions of the Heterotic Superstring effective action"**, JHEP 1807 (2018) 080. arXiv:1803.04463 [hep-th].
10. S. Chimento, T. Ortín and A. Ruipérez, **"Supersymmetric solutions of the cosmological, gauged, $\mathbb{C}$ magic model"**, JHEP 1805 (2018) 107. arXiv:1802.03332 [hep-th].
11. P. Bueno, P. A. Cano and A. Ruipérez, **"Holographic studies of Einsteinian cubic gravity"**, JHEP 1803 (2018) 150. arXiv:1802.00018 [hep-th].
12. S. Chimento, T. Ortín and A. Ruipérez, **"Yang-Mills instantons in Kahler spaces with one holomorphic isometry"**, Phys. Lett. B 778 (2018) 371-376. arXiv:1710.00764 [hep-th].
13. J. Ávila, P. F. Ramírez and A. Ruipérez, **"One thousand and one bubbles"**, JHEP 1801 (2018) 041. arXiv:1709.03985 [hep-th].

Abstract

The low-energy limit of superstring theories admits a description in terms of an effective field theory for its massless modes. The corresponding action is given by a double perturbative expansion in g_s, the string coupling, and in α', the square of the string length. The leading term of this expansion is given by one of the different ten-dimensional supergravity theories, whereas subleading terms involve terms of higher order in derivatives. The work presented in this thesis is the result of a research program that starts with the study of supersymmetric solutions of gauged supergravity and reaches the summit with the understanding of the effects produced by the α' corrections to solutions of the heterotic superstring effective action.

This thesis is divided in two parts. The first one focuses on the supersymmetric solutions of a minimal extension of the STU model of $\mathcal{N} = 1, d = 5$ supergravity whose main interest lies on the fact that it can be obtained as a toroidal compactification of ten-dimensional $\mathcal{N} = 1$ supergravity coupled to a triplet of SU(2) gauge fields. Concretely, we construct and study solutions describing black holes and smooth horizonless geometries with non-trivial Yang-Mills fields.

The understanding of this type of solutions from the framework of string theory serves as a motivation for the work of the second part of the thesis, which is devoted to the study of solutions of the effective action of the heterotic string at first order in α'. The latter does not simply coincide with the action of $\mathcal{N} = 1, d = 10$ supergravity coupled to a Yang-Mills vector multiplet as the Green-Schwarz anomaly cancellation mechanism and supersymmetry enforce us to introduce additional terms in the action. These terms are constructed out of the spin connection with torsion given by the field strength associated to the Kalb-Ramond 2-form and their contributions to the equations of motion are analogous to those of the Yang-Mills fields. This fact is exploited to construct analytic supersymmetric black-hole solutions with α' corrections.

The most important lesson to extract from our results is that the mass and the conserved charges of the black holes do get modified by the α' corrections. This is what one would expect on physical grounds as the corrections act in the string equations of motion as effective sources of energy, momentum and charge. This information is crucial to establish a correspondence between the parameters that characterize the effective or coarse-grained description (the black hole) and those that characterize the microscopic system of string theory that it describes. The effects on the charges introduced by the higher-derivative corrections has a major impact in the understanding of the so-called small black holes, which are an effective description of a fundamental string with winding and momentum charges. Small black holes are singular solutions with vanishing horizon area in the supergravity approximation. It has long been believed that higher-derivative corrections would be able to stretch the horizon, making the solution regular. Our results reveal that this is not the case at first order in α', and that previous regularizations of heterotic small black holes actually describe a different microscopic system which is already regular in the supergravity approximation.

The last chapter of the thesis contains the computation of the most general correction to the four-dimensional Kerr solution when the Einstein-Hilbert term is supplemented with higher-curvature terms up to cubic order, including the possibility of having dynamical couplings. This general set-up includes, as a particular case, the corrections predicted by the heterotic superstring effective action.

Contents

1
Introduction

1.1 Black holes

Black holes, regions of spacetime from where not even light can escape, are one of the most fascinating predictions of Einstein's theory of general relativity. Initially regarded as a mathematical curiosity, it was later understood that they could be created by gravitational collapse of massive stars [1–4], or even in the early universe, where conditions were so extreme that density perturbations might have undergone gravitational collapse [5, 6].[1] Nowadays, there is strong evidence that almost all large galaxies contain a supermassive black hole ($M \sim 10^6 - 10^{10} M_\odot$) at their centers. One such example would be Sagittarius A*, which is believed to be a supermassive black hole of about four million solar masses at the center of our galaxy [8–10]. Another popular example is the supermassive black hole candidate at the center of the galaxy M87, which last year made the headlines after the Event Horizon Telescope obtained an image of its shadow [11].

From an experimental perspective, black holes, together with neutron stars, represent extraordinary laboratories where Einstein's theory can be tested in extreme conditions. In particular, the recent observation of gravitational waves coming from binary black hole and neutron star mergers [12] has provided very valuable information that has been already used to perform precision tests of general relativity [13–17] and also to constrain the parameter space of its possible extensions [18–22]. What is even more exciting is that this is just the dawn of a promising era in which future-planned gravitational wave detectors will play a crucial rôle in deciphering the fundamental nature of the gravitational interaction [23, 24].

On the other hand, black holes have been a central area of research in theoretical physics in the last decades, as they pose profound theoretical puzzles regarding the interplay of gravity and quantum mechanics which have even questioned the solid foundations of the quantum theory. The study of these puzzles is a mandatory task for the theoretical physics community, as their resolution is something that must be achieved by any consistent theory of quantum gravity aimed to describe our real world. From this point of view, black holes can be considered as our guides in the path towards a theory of quantum gravity.

Before considering quantum aspects of black holes, it is convenient first to study them from a classical perspective, as described by general relativity.

[1] Although experimental evidence supporting the existence of primordial black holes is still lacking, they have attracted a great deal of attention over the last fifty years, having even been proposed as possible dark matter candidates [7].

1.1.1 Classical black holes

Preliminar definitions

A black hole region $\mathcal{B}$ in an asymptotically-flat spacetime $(\mathcal{M}, g_{\mu\nu})$ is defined as the set of events from which outgoing null geodesics cannot reach future null infinity, $\mathscr{I}^+$. This can be recasted in mathematical terms as follows:

$$\mathcal{B} = \mathcal{M} - J^-(\mathscr{I}^+)\,, \tag{1.1}$$

where $J^-(\mathscr{I}^+)$ is the chronological or causal past of $\mathscr{I}^+$, i.e. the set of all points that can be reached from future null infinity by means of a past-directed geodesic, either timelike or null. The event horizon, defined as the boundary of the black-hole region,

$$\mathcal{H} = \partial\mathcal{B} = \partial J^-(\mathscr{I}^+)\,, \tag{1.2}$$

can be proven to be a null hypersurface generated by null geodesics that have no future end points [25].

These definitions highlight one of the most intriguing properties of the event horizon, its teleological nature, as the future history of the spacetime must be known before its position can be determined. This shows that the event horizon is a global property of the spacetime. In fact, according to the Equivalence Principle, it has no local relevance at all, which implies that an observer falling into a black hole does not encounter anything special when crossing its event horizon.

In order to further characterize the event horizon, we need to make additional assumptions on the black-hole spacetime. A very powerful one is to assume that the spacetime is *stationary*. This means that the metric $g_{\mu\nu}$ admits a one-parameter family of isometries generated by a Killing vector which is timelike in the asymptotic region. In this case, the rigidity theorems [26, 27] establish that the event horizon is also a *Killing horizon*, i.e. a null hypersurface whose normal vector k^μ is a Killing vector of $g_{\mu\nu}$.[2] As a consequence, the null generators of the horizon are given by the integral curves of k^μ, which satisfy

$$k^\nu \nabla_\nu k^\mu = \kappa\, k^\mu\,, \tag{1.3}$$

when evaluated at the horizon. The function κ is called the *surface gravity* of the Killing horizon, and measures the failure of the integral curves of k^μ to be affinely parametrized. According to the zeroth law of black-hole mechanics [28], which we shall study later, the surface gravity is constant over the horizon. It was proved in [29] that the horizon is *bifurcate* when $\kappa \neq 0$. In contrast, when $\kappa = 0$, the horizon is *degenerate* and the black hole is said to be *extremal*.

The Schwarzshild black hole

The best way to delve into the concept of black hole is to study a particular example. The simplest and most important one is the Schwarzschild black hole [30], which was the first non-trivial exact solution to Einstein's field equations and, in fact, according to Birkhoff's

[2] This means that k^μ satisfies the Killing equation: $\nabla_{(\mu}k_{\nu)} = 0$.

theorem [31], their only spherically-symmetric vacuum solution. The line element of the Schwarzschild solution is given by

$$ds^2 = \left(1 - \frac{R_S}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{R_S}{r}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) , \tag{1.4}$$

where $R_S = 2G_\mathrm{N}M$ is the Schwarzschild radius (the unique parameter of the solution) and M is the mass of the solution that one can obtain by making use of the Arnowitt-Deser-Misner (ADM) prescription [32]

$$M = \frac{1}{16\pi G_\mathrm{N}} \int_{\mathbb{S}^2_\infty} \left(\partial_j h_{ij} - \partial_i h_{jj}\right) \epsilon_{ijk} \, dx^j \wedge dx^k \,, \tag{1.5}$$

where $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the metric perturbation and $\eta_{\mu\nu}$ is the Minkowski metric in Cartesian coordinates, $x^\mu = (x^0 = t, x^i)$.

The ADM mass is a conserved quantity that represents the total mass (or energy) of the spacetime, the only notion of energy that is well defined and conserved in general relativity. This is owed to the fact that the gravitational energy cannot be defined locally. The main obstacle that one finds is the Equivalence Principle itself, which tells us that all the effects of the gravitational field (including the gravitational energy) can always be locally eliminated in a suitable reference frame.[3]

At first sight, there are two special values of the radial coordinate for which the Schwarzschild metric is not well behaved: $r = 0$ and $r = R_S$. This signals possible *spacetime singularities* where the laws that rule the dynamics of the gravitational field —namely, Einstein's equations— break down.

This is in fact what happens at $r = 0$, where there is a curvature singularity. The easiest way to see this is to check that at least one curvature invariant diverges there. In the case of the Schwarzschild solution, the simplest non-trivial curvature invariant is the Kretschmann invariant,

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2\cos^2\theta}{r^6} + \ldots \,, \tag{1.6}$$

which, as we can see, blows up at $r = 0$. One could be tempted to think that the singularity of the Schwarzschild solution is due to the fact that it describes an idealized system with too much symmetry, and that small perturbations around the solution could result into a regular spacetime. After all, this is what occurs in the Newtonian description of gravitational collapse. Nevertheless, the *singularity theorems* [34,35] proven by Hawking and Penrose tell us that singularities in black-hole spacetimes are a generic prediction of general relativity. Concretely, they showed that the presence of *trapped surfaces*[4] inside black holes leads to the formation of singularities if one assumes that some form of energy condition holds. According to the *cosmic censorship conjecture* [37–39], all spacetime singularities in our Universe must be hidden behind event horizons.

[3] A somewhat exotic proposal that avoids the conflict with the Equivalence Principle is the construction of an energy-momentum tensor out of the curvature tensors [33].

[4] A trapped surface T is a spacelike closed two-dimensional surface such that for both congruences of ingoing and outgoing null geodesics orthogonal to T, the cross-sectional area decreases as we proceed into the future [34,36].

It was not until 1933 when Lemaître [40] understood that the singularity at $r = R_S$ was only a coordinate singularity, i.e. a mathematical artefact rather than a physical pathology. In fact, the hypersurface $r = R_S$ is the event horizon of the Schwarzschild black hole. However, in order to see this, one has to extend the solution through $r = R_S$. This can be achieved, for instance, by introducing the ingoing Eddington-Finkelstein coordinate,

$$v = t + r_* \,, \tag{1.7}$$

where r_* is the so-called tortoise coordinate, defined as

$$r_* = r + R_S \log \left| \frac{r}{R_S} - 1 \right| . \tag{1.8}$$

Replacing t for v in the Schwarzschild metric (1.4), one gets the following metric

$$ds^2 = \left(1 - \frac{R_S}{r}\right) dv^2 - 2dvdr - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \,, \tag{1.9}$$

which is now smooth for all $r > 0$. Then, we can use it to study the geodesic motion of a test particle in a neighborhood of the horizon. It is instructive to consider radial null geodesics, which satisfy either

$$\dot{v} = 0 \,, \qquad \text{or} \qquad \left(1 - \frac{R_S}{r}\right) \dot{v} = 2\dot{r} \,, \tag{1.10}$$

corresponding, respectively, to ingoing and outgoing null geodesics. The ingoing ones always end up at the singularity. The outgoing ones can avoid it and escape to the asymptotic region, but only if $r > R_S$. If, instead, $r < R_S$, then r must decrease as v increases. Therefore, when $r < R_S$, both possibilities in (1.10) describe null geodesics that eventually hit the singularity. Let us observe that there are also null geodesics given by $r(v) = R_S$ whose tangent vector, $k = \partial_v$, is a Killing vector of the metric (1.9). These are the null generators of the horizon. It is straightforward to check that the norm of the Killing vector

$$k^\mu k_\mu = g_{vv} = 1 - \frac{R_S}{r} \,, \tag{1.11}$$

vanishes at $r = R_S$, which shows that the event horizon is a Killing horizon, as predicted by the rigidity theorems. These apply because the exterior of the Schwarzschild solution ($r > R_S$) is stationary, as the Killing vector (1.11) becomes timelike in that region. More precisely, the Schwarzschild solution is *static*, which means that there is a family of spacelike hypersurfaces orthogonal to the Killing vector k^μ.

The global structure of the Schwarzschild spacetime can be better appreciated in its Carter-Penrose diagram. In order to construct it, we first need the maximal analytic extension of the Schwarzschild solution. This is given by the Kruskal-Szekeres coordinates (U, V, θ, ϕ) [41, 42], which are defined in terms of the Schwarzschild coordinates as follows

$$\left(\frac{r}{R_S} - 1\right) e^{r/R_S} = -UV \,, \qquad \frac{V}{U} = -e^{t/R_S} \,. \tag{1.12}$$

In these coordinates, the Schwarzschild metric reads

$$ds^2 = \frac{4R_S^3 e^{-r/R_S}}{r} dU dV - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{1.13}$$

The Kruskal-Szekeres coordinates reveal us a larger spacetime than the patch covered by the Eddington-Finkelstein coordinates, which corresponds to regions I and II in the Carter-Penrose diagram of Fig. 1.1. Region I is the exterior of the black hole, the only region that is covered by the Schwarzschild coordinates. Region II is the black-hole interior that we discovered using the Eddington-Finkelstein coordinates. The zigzagging line represents the singularity, which appears in the future of any point that enters region II. In fact, the interior of the black hole (which is non-static) admits the interpretation of an anisotropic and homogeneous universe that collapses into a *big crunch* singularity. Apart from regions I and II, we can see in the diagram that there are two new regions, labeled by III and IV. Region III is an exact copy of region I. Its appearance (as well as that of IV) is owed to the fact that U and V can take values in the real line. Then, for instance, a hypersurface of constant r is now a 2-branch hyperbola. One of the branches falls either in region I (if $r > R_S$) or in region II (if $r < R_S$) and the other in either of the two new regions: in III (if $r > R_S$) or in IV (if $r < R_S$). In particular, the event horizon bifurcates into two null hypersurfaces, $U = 0$ and $V = 0$, which are sometimes called the future and past event horizons.[5] They intersect at $U = V = 0$, the *bifurcation 2-sphere*. The region between the past horizons is IV, the *white hole*, where nothing from the outside can enter and everything inside eventually comes out. As opposed to the singularity in II, the one in IV has the interpretation of a *big bang* singularity.

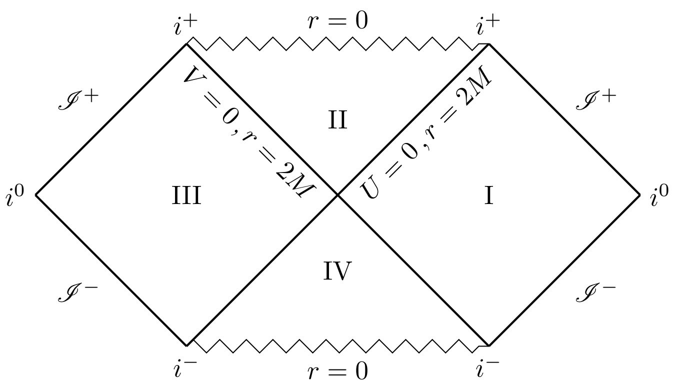

Figure 1.1: Carter-Penrose diagram of the maximally extended Schwarzschild spacetime. $\mathscr{I}^+$ ($\mathscr{I}^-$) and i^+ (i^-) denote respectively the future (past) null and timelike infinities and i^0 the spatial infinity. The rules to interpret this diagram are simple: light rays propagate in straight lines at 45º (each point of the diagram is a 2-sphere) and one must think time increases as we move upwards.

[5]The future (past) horizon for region I is $U = 0$ ($V = 0$), while for region III is $V = 0$ ($U = 0$).

Uniqueness theorems

Astrophysical black holes formed by gravitational collapse eventually settle down into an equilibrium state in which they are expected to be well described by a stationary solution of Einstein's equations. Then, one would expect they admit a cornucopia of solutions describing stationary black holes with a wide variety of shapes and features inherited from their stellar predecessors. Surprisingly, this reasonable expectation was proven to be false by the *uniqueness theorems*.

The first of this class of theorems is due to Israel [43] and states that any static black-hole solution of the vacuum Einstein's equations must be a Schwarzschild black hole. This striking result poses a rather interesting question: what happens with the higher multiple moments of the initial object? The answer to this question was provided by Price [44,45], who showed that these multiple moments are radiated away either to infinity or to the interior of the black hole.

Yet, Israel's theorem is not sufficiently general for astrophysical purposes, as it assumes staticity. The extension of this result to non-static spacetimes was provided by Carter [46] and Robinson [47], who proved that any stationary and axisymmetric vacuum black hole must be described by the Kerr solution [48], whose line element in Boyer-Lindquist coordinates is given by

$$\begin{aligned} ds^2 = &\left(1 - \frac{2G_{\rm N}Mr}{\Sigma}\right)dt^2 + \frac{4G_{\rm N}Mar\sin^2\theta}{\Sigma}dtd\phi - \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) \\ &- \left(r^2 + a^2 + \frac{2G_{\rm N}Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2\,, \end{aligned} \tag{1.14}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \qquad \text{and} \qquad \Delta = r^2 - 2G_{\rm N}Mr + a^2\,. \tag{1.15}$$

The Kerr solution depends on two parameters: M, the ADM mass, and a, which is related to the total angular momentum of the spacetime J by $J = aM$. It is stationary and axisymmetric with respect to the Killing vectors ∂_t and ∂_ϕ and the event horizon is placed at $r = r_+ \equiv M + \sqrt{M^2 - a^2}$, the largest root of Δ. One can check that the norm of the following linear combination of the aforementioned Killing vectors,[6]

$$k = \partial_t + \Omega_H\partial_\phi\,, \tag{1.16}$$

vanishes at $r = r_+$.

These uniqueness theorems were generalized to charged black-hole solutions of the Einstein-Maxwell theory [49]. Concretely, the extension of Israel's theorem states that a static black hole of this theory is either described by the Reissner-Nordström [50, 51] or Majumdar-Papapetrou [52,53] solutions. If, instead, one only assumes axisymmetry, then, the most general black-hole solution is the Kerr-Newmann [54] solution. All these results gave raise to the famous *no-hair conjecture* [55], according to which a black hole would be

[6]This is also a Killing vector since $\Omega_H \equiv \frac{g_{t\phi}}{|g_{\phi\phi}|}\Big|_{r_+} = \frac{a}{2M\left(M+\sqrt{M^2-a^2}\right)}$, the angular velocity at the horizon, is constant.

fully determined by the values of its conserved charges: the mass, M, the total angular momentum, J, and the possible electric and magnetic charges, collectively denoted as Q.

Since charged black holes are probably not relevant for astrophysical purposes, the conclusion that one extracts from these classical results is that all astrophysical black holes in our Universe end up being described by the Kerr solution. It is fair to mention, however, that these are mathematical theorems which are based on certain premises that in a realistic situation might not be a good approximation. For instance, they are derived assuming that the black hole constitutes a perfect isolated system, which in the real world is not completely true. Another premise that can be relaxed is that the theory that describes these black holes might not be general relativity. Even if it has passed a large number of experimental tests, there are very good theoretical motivations to expect that it will be modified at high energies. This is a generic prediction of quantum gravity and, in particular, of string theory, which predicts the appearance of an infinite series of higher-derivative terms correcting the Einstein-Hilbert action. In general, the presence of these terms implies that the vacuum solutions of general relativity, such as the Schwarzschild and Kerr solutions, no longer solve the equations of motion of the corrected theories. It is therefore an interesting task to determine the new black-hole solutions and to study their properties, even if the deviations with respect to the general relativity predictions are still too small to be detected with our current technology.

The laws of black-hole mechanics

In 1973, Bardeen, Carten and Hawking [28] established four laws governing the behaviour of black holes, which, because of their close resemblance with the four laws of thermodynamics, were called *the four laws of black-hole mechanics*.

The first of these laws, known as the zeroth law of black-hole mechanics, establishes that the surface gravity κ of a black hole —which was defined in (1.3)— is constant over the event horizon. This is analogous to the zeroth law of thermodynamics, which states that a thermodynamical system in equilibrium has uniform temperature. However, this analogy must be merely coincidental, as the temperature of a *classical* black hole must be absolute zero. A simple argument that justifies this statement is that a black hole cannot be in equilibrium with black body radiation at any non-zero temperature.

The first law of black-hole mechanics tells us that the changes in the mass M, area of the horizon A_H and angular momentum J of a stationary black hole in a quasi-static process are related by

$$\delta M = \frac{\kappa}{8\pi G_{\rm N}}\,\delta A_H + \Omega_H\,\delta J\,. \tag{1.17}$$

In combination with the zeroth law, the first one suggests that the area of the horizon plays the same rôle as the entropy in a thermodynamical system. This is in agreement with independent work by Bekenstein [56], who had argued just few months before the publication of [28] that black holes should have an entropy proportional to area of the horizon in order to avoid violations of the second law of thermodynamics.

The first law has been extended in many respects to account for the different contributions of the matter fields as well as for different asymptotics [57]. For example, in the context of electrically-charged black holes, it includes an additional contribution in the right-hand side of (1.17) of the form $\Phi_H\delta Q$, Φ_H being the value of the electrostatic potential at the horizon.[7] The reason why it can be so easily extended was understood years later by Wald in [58], where he showed that the first law was actually a consequence of general covariance.

The content of the second law is the same as the *area theorem* proved by Hawking in 1971, [59]. It states that if the null energy condition is satisfied, the horizon area of a black hole never decreases

$$\delta A_H \geq 0\,. \tag{1.18}$$

Finally, a third law was proposed stating that the surface gravity cannot be reduced to zero in a finite time. This was later proved by Israel [60].

1.1.2 Quantum aspects of black holes

Hawking radiation

Just one year after the publication of the four laws of black-hole mechanics, Hawking came up with an astonishing discovery that would revolutionize the field [61,62]. He found that quantum fluctuations of the vacuum in the presence of black holes cause them to create and emit particles as if they were black bodies at temperature

$$T_{\rm H} = \frac{\hbar\kappa}{2\pi}\,. \tag{1.19}$$

This removed any reluctance to a complete identification of black holes as ordinary thermodynamical systems, and also allowed to fix the proportionality constant between the black-hole entropy $S_{\rm BH}$ and the area of the horizon A_H by virtue of the first law (1.17):

$$S_{\rm BH} = \frac{A_H}{4\hbar G_{\rm N}}\,. \tag{1.20}$$

As we have already mentioned, the first law remains valid even when the gravitational dynamics is not dictated by the Einstein-Hilbert term, though in this case the entropy that appears in the first law is not simply given by the Bekenstein-Hawking formula, but by that of Wald [58], which reads[8]

$$S_{\rm W} = -\frac{2\pi}{\hbar}\int_\Sigma d^2x\sqrt{|h|}\,\mathcal{E}_R^{abcd}\epsilon_{ab}\epsilon_{cd}\,, \tag{1.21}$$

where $|h|$ is the determinant of the induced metric at the bifurcation surface Σ, ϵ_{ab} is the binormal at the horizon normalized such that $\epsilon_{ab}\epsilon^{ab} = -2$ and, finally, $\mathcal{E}_R^{abcd}$ is what would be the equation of motion of the Riemann tensor if it were treated as a fundamental field, namely

$$\mathcal{E}_R^{abcd} = \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta R_{abcd}}\,. \tag{1.22}$$

[7]It is assumed we work in a gauge in which the electrostatic potential vanishes asymptotically. Otherwise, the term $\Phi_H\delta Q$ would not be gauge invariant.

[8]Wald's formula will play an important rôle in Chapter 5 of this thesis, where we will use it to compute the entropy of stringy black holes with higher-derivative corrections.

The physical mechanism behind particle creation by black holes is analogous to the Schwinger pair production in strong electric fields [63]. In the case of black holes, pairs of virtual particles are created just outside the event horizon. One member of the pair has positive energy and escapes to infinity to become part of the Hawking radiation, while the other has negative energy and falls into the black-hole interior, to the region where it can exist as a real particle. The net effect is that the mass and the area of the black hole decrease, hence violating the second law of black-hole mechanics. Still, the evaporation process does not violate *the generalized second law of thermodynamics* [64], which states that the total entropy, i.e. the sum of the black-hole entropy and the entropy of the matter fields in the exterior region, never decreases.

The inmediate consequence of the emission of particles is that black holes evaporate. Let us consider a Schwarzschild black hole of mass M. The surface gravity can be readily computed by using the Killing vector given in (1.11), and the result is $\kappa^{-1} = 4G_{\rm N}M$. Substituting this expression into (1.19), one obtains the Hawking temperature of the Schwarzschild black hole:

$$T_{\rm H} = \frac{\hbar}{8\pi G_{\rm N} M} \,. \tag{1.23}$$

As the above equation indicates, the temperature of the black hole is inversely proportional to its mass. Then, the black hole gets hotter and hotter as it evaporates. The evaporation rate can be deduced from Boltzmann's law to be inversely proportional to the square of the mass, $dM/dt \sim -M^{-2}$, which implies that the end of the life of a black hole is a violent explosion [61]. Hawking himself estimated that in the last 0.1 seconds, 10^{23} J would be realeased, which is equivalent to about a million of 1 Mt hydrogen bombs. He further calculated that the lifetime of a black hole would be roughly

$$\tau \approx 10^{71} \left(M/M_{\odot}\right)^3 \quad s\,. \tag{1.24}$$

Hence, the lifetime of a stellar black hole of a few solar masses is more than fifty orders of magnitude greater than the age of our Universe, which means that they hardly evaporate at present. Instead, the effects of Hawking radiation would be significant in primordial black holes that might have been produced in the early universe [5, 6]. In fact, Eq. (1.24) tells us that those with masses smaller than 10^{15} grams would have been evaporated by now.

The information paradox

The application of quantum mechanics to black holes poses very intriguing puzzles. In first place, we have learnt that a black hole has an entropy given by the Bekenstein-Hawking formula. But according to the principles of statistical mechanics, this implies that there are $N \approx e^S$ microstates characterized by the same conserved charges as the black hole $(M, J, Q, \dots)$. However, the classical uniqueness theorems and the no-hair conjecture tell us that a black hole is univocally characterized by its conserved charges so that the entropy should be $S = \log 1 = 0$. Then, this raises the following question: what are the microstates of a black hole?

There is a second puzzle that concerns the evaporation process itself. Let us imagine that the initial matter that is undergoing gravitational collapse has been arranged in a pure quantum state. As time evolves, an event horizon is formed and the evaporation process starts taking place until we end up in a final state with just the Hawking radiation. This final state cannot be described quantum-mechanically by a pure state but by a mixed one, since the radiation is exactly thermal. In Hawking's own words:

> One would hope that, in the spirit of the "no hair" theorems, the rate of emission would not depend on details of the collapse process except through the mass, angular momentum and charge of the resulting black hole. I shall show that this is indeed the case but that, in addition to the emission in the super- radiant modes, there is a steady rate of emission in all modes at the rate one would expect if the black hole were an ordinary body with temperature $\kappa/(2\pi)$.

The controversial point is that a theory that preserves information —namely, a unitary theory— forbids the evolution from a pure to a mixed state. Therefore, the evaporation of a black hole necessarily entails loss of information if the radiation emitted is in a thermal state. This is the essence of the *information paradox*. A convenient diagnostic of the problem is the von Neumann or entanglement entropy of the radiation, given by $S_R = -\mathrm{Tr}\rho_R \log \rho_R$, where ρ_R denotes the density matrix of the Hawking radiation. This entropy is not a "coarse-grained" entropy as the thermodynamical entropy but rather a "fine-grained" entropy that characterizes the fundamental ignorance about the quantum system. Consequently, the total von Neumann entropy must remain constant in a unitary evolution, where there is no information loss. Yet, the entropy of the radiation can grow initially, but it must decrease back to zero at the end of the evaporation process, following the Page curve [65]. However, according to Hawking's computation, the von Neumann entropy of the radiation increases monotonically under time. This led to an intense debate about the reliability of the semiclassical approximation used by Hawking and about whether a consistent theory of quantum gravity would preserve information or not.

The lack of a theory of quantum gravity initially hampered the study of these puzzles until in the mid 1980s, the community started to get convinced that string theory was a consistent theory of quantum gravity. Some years later, it was suggested by Susskind that black holes could represent effective descriptions of ordinary quantum systems constituted by strings and branes [66], which opened up the possibility of reproducing the black-hole entropy by microstate counting. After some initial attempts, this was finally achieved by Strominger and Vafa [67], who were able to show that the Bekenstein-Hawking entropy of a specific type of extremal black hole could be reproduced by counting the degeneracy of BPS states of a system of D-branes wrapped on internal cycles, hence solving the microstate problem at least for this class of black holes.[9] This constitutes one of the major achievements of string theory and suggests, further supported by the AdS/CFT correspondence [68–70], that black-hole evaporation is indeed a unitary process.

However, a complete resolution of the information paradox would imply a precise understanding of the mechanism by which information is recovered. In fact, some authors have pointed out that this would only be possible if there is non-trivial structure at the scale of the horizon [71,72]. A proposal along these lines is the so-called *fuzzball proposal* [73,74], which posits that in the true quantum description of black holes, the event horizon would be replaced by a horizonless geometry with non-trivial structure.

[9] A precise matching between the black-hole entropy and degeneracy of string states is only possible for a certain type of extremal and near-extremal black holes. This is due to the fact that we only have control of the theory in the perturbative limit. We will further comment on this in the next section.

Before closing this section, we would like to draw the attention of the reader to recent developments in the context of the information paradox [75–81], where a new technique to compute the entropy of the Hawking radiation has been developed. The final result, which differs from Hawking's, turns out to be consistent with unitary evolution.

1.2 String theory

String theory departs from the idea that point particles get replaced by strings, one-dimensional objects whose mass and length scale is given by $m_s = \ell_s^{-1} = \alpha'^{-1/2}$, where α' is the so-called Regge slope, the unique dimensionful parameter of the theory. Besides α', there is a dimensionless parameter, the string coupling constant g_s, which naturally arises as the vacuum expectation value of the dilaton field, $g_s = \langle e^{\phi} \rangle$. The spectrum of ordinary particles emerges as the spectrum of the different vibrational modes of strings and its most remarkable aspect is that it always[10] contains a (massless) graviton, the quantum of the gravitational field. Hence, string theory is a quantum theory of the gravitational interaction.

The many successes of the theory, as well as its structural beauty, place string theory in a privileged position with respect to other candidate theories of quantum gravity such as loop quantum gravity [82–84]. For the purposes of this thesis, the most interesting aspect is that it provides a self-consistent ultraviolet completion of Einstein's theory of general relativity which has satisfactorily answered some of the questions raised by black holes.

This section is devoted to review the aspects of string theory that will be more relevant to us later. After a brief description of the worldsheet formulation and of the quantization of the theory, we will study the low-energy string dynamics, which has proved to be a rich source of information about the non-perturbative structure of the theory. Although a complete non-perturbative formulation of string theory is lacking, we know that it has very interesting non-perturbative dynamics, as it contains extended objects (p-branes) which are dynamical in the strong coupling regime $g_s >> 1$. These extended objects have played a prominent rôle in establishing string dualities and, what is more interesting to us, in the study of black holes.

1.2.1 Worldsheet formulation

Bosonic string theories

The dynamics of a free relativistic string moving in a d-dimensional curved background with metric $g_{\mu\nu}$ is encoded in the Nambu-Goto action

$$S_{\text{NG}} = -T \int_W d^2\xi \sqrt{|g_{ij}|}\,, \tag{1.25}$$

where ξ^i, $i = 0, 1$, are the worldsheet coordinates and $|g_{ij}|$ denotes the determinant of the induced metric on the worldsheet,

[10] The graviton arises in the closed string sector, which must always be present in any consistent (unitary) string theory.

$$g_{ij} = g_{\mu\nu}(X)\, \partial_i X^\mu \partial_j X^\nu \,. \tag{1.26}$$

The $X^\mu(\xi)$ coordinates parametrize the position of the string in the ambient spacetime, so we will simply refer to them as the embedding coordinates. The constant T in front of the action is the string tension, related to the Regge slope α' by

$$T = \frac{1}{2\pi\alpha'} \,. \tag{1.27}$$

The Nambu-Goto action is highly non-linear and therefore very difficult to quantize even in flat space. For this purpose, it is convenient to introduce the Polyakov action,

$$S_{\rm P} = -\frac{T}{2} \int_W d^2\xi\, \sqrt{|\gamma|}\, \gamma^{ij} g_{\mu\nu}(X)\, \partial_i X^\mu \partial_j X^\nu \,, \tag{1.28}$$

where now $\gamma_{ij}(\xi)$ plays the rôle of a non-dynamical worldsheet metric. Its equation of motion gives the vanishing of the energy-momentum tensor,

$$g_{\mu\nu}(X)\, \partial_i X^\mu \partial_j X^\nu - \frac{1}{2}\gamma_{ij}\gamma^{kl} g_{\mu\nu}(X)\, \partial_k X^\mu \partial_l X^\nu = 0 \,, \tag{1.29}$$

which can be used to obtain the following (on-shell) relation between the worldsheet metric and the pullback of the background metric:

$$\gamma_{ij} = \frac{2 g_{ij}}{g_k{}^k} \,, \qquad \text{where} \qquad g_k{}^k = \gamma^{kl} g_{kl} \,. \tag{1.30}$$

It is straightforward to check that when the above relation is substituted into the action (1.28), the Nambu-Goto action (1.25) is recovered, which shows that both actions are clasically equivalent.

Let us note, however, that the Polyakov action, apart from being invariant under worldsheet reparametrizations, is also invariant under the following local scale transformations of the worldsheet metric:

$$\gamma_{ij} \to \Omega^2(\xi)\gamma_{ij} \,. \tag{1.31}$$

This invariance is called Weyl or conformal invariance and turns out to have very important physical consequences, specially regarding the quantization of the theory. It has also consequences already at the classical level: it implies that the trace of the energy-momentum tensor vanishes off-shell.

At this stage, we can wonder if the Polyakov action can be generalized to include a dynamical term for the worldsheet metric γ_{ij} while still preserving its symmetries. It turns out that this is not possible at the classical level, since the candidate meeting these requirements, the Einstein-Hilbert worldsheet term,

$$-\frac{\phi_0}{4\pi} \int_W d^2\xi\, \sqrt{|\gamma|}\, R(\gamma) \,, \tag{1.32}$$

is a total derivative and therefore clasically irrelevant. More precisely, it is a topological invariant, the Euler characteristic of the worldsheet (times ϕ_0):

$$-\frac{\phi_0}{4\pi}\int_W d^2\xi\,\sqrt{|\gamma|}\,R(\gamma)\;=\phi_0\chi\,, \tag{1.33}$$

where $\chi = 2-2g-b-c$, being g the genus, b the number of boundaries and c the number of crosscaps, if we consider unoriented strings.

In spite of being classically irrelevant, the Einstein-Hilbert term does play a rôle in the quantization of the Polyakov action, and in fact an important one: the exponential of ϕ_0 has the interpretation of string coupling constant, $g_s \equiv e^{\phi_0}$. This may sound weird. Since the worldsheet theory is non-interacting, one would expect to have no notion of coupling constant whatsoever. However, when computing string amplitudes, one has to sum over all worldsheet geometries with given boundaries. Because of (1.33), each worldsheet will be multiplied by a factor $g_s^{-\chi}$, which provides a notion of string coupling constant in the usual sense and, at the same time, with a well-defined perturbative description of the quantum worldsheet theory. What is even more remarkable is that the free (non-interacting) theory motivates us in a certain sense to include the Einstein-Hilbert term. As we are about to see, the quantization of free strings predicts the appearance of the dilaton, a massless scalar whose natural coupling to the string is described by (1.33). From this point view, the string coupling constant g_s does not appear as a parameter put by hand, but rather as the expectation value of the dilaton field on the vacuum where strings are quantized, $g_s = \langle e^{\phi}\rangle$.

Keeping this in mind, let us briefly discuss the canonical quantization of a free bosonic string. To this aim, it is crucial to first discuss the issue of the boundary conditions, as these will constrain the vibrational modes of strings. In this respect, we must take into account that the variation of the Polyakov action (1.28) yields the following boundary term

$$\int_{\partial W} dW^i \delta X^\mu \partial_i X^\nu g_{\mu\nu}\,, \tag{1.34}$$

which does not vanish for open strings. In order to make it vanish, one can impose either Neumann (N) boundary conditions,

$$n^i\partial_i X^\mu|_{\partial W} = 0\,, \tag{1.35}$$

or Dirichlet (D) boundary conditions,

$$t^i\partial_i X^\mu|_{\partial W} = 0\,, \tag{1.36}$$

where n^i and t^i are, respectively, a normal and a tangent vector to the boundary of the worldsheet ∂W. For a free open string, the Neumann boundary conditions are equivalent to imposing that no momentum is flowing through the endpoints of the strings,

$$\partial_1 X^\mu|_{\xi^1=0,\ell} = 0\,. \tag{1.37}$$

In turn, the imposition of Dirichlet boundary conditions on $(d-1-p)$ spacelike directions is equivalent to restricting the motion of the endpoints of strings to $(p+1)$-dimensional timelike hypersurfaces, explicitly breaking translation invariance in the transverse directions. As we will see later, these hypersurfaces correspond to the worldvolume of dynamical objects of the theory, the Dp-branes [85].

These boundary conditions completely determine the different spectra that can arise. In a relativistic theory, the polarization states must belong to representations of the little group, i.e. the subgroup of the Lorentz group preserving the particle momenta.[11] An analysis of the spectra of closed and open bosonic strings reveals that this only occurs in $d = 26$ spacetime dimensions, which is known as the *anomalous* dimension. We are interested in the lightest states of the spectra, as these are the ones that govern the low-energy dynamics. In the closed string sector, the lightest states are massless, which means that they fit into representations of SO(24). These are a graviton $g_{\mu\nu}$, a 2-form $B_{\mu\nu}$ and a scalar ϕ, the dilaton. In the open-string sector with NN boundary conditions, the lightest state is also massless and corresponds to a vector field A_μ. Finally, for an open string with DD boundary conditions imposed on just one spacelike direction (let us call it Z), the mass of the lightest state depends on the separation between the D24-branes on which the endpoints of the string are allowed to move. Let us consider, for simplicity, that both ends lie on the same hypersurface, $Z|_{\xi_1=0} = Z|_{\xi_1=\ell}$. Then, the spectrum contains a vector field and a scalar, both massless, and they only propagate in the worldvolume of the D24-brane. The scalar corresponds to the Goldstone boson associated to the breaking of translation invariance of the vacuum, owing to the presence of the D-brane. Its vacuum expectation value gives the position of the brane in the z-axis and its non-trivial profile describes fluctuations of the brane around this position.

Besides these massless excitations, there are also tachyonic scalars signaling that the vacuum of bosonic string theories is quantum-mechanically unstable. This pathology can nevertheless be cured. The ingredient that is missing is supersymmetry.[12]

Superstring theories

The generalization of the Polyakov action which is also invariant under local supersymmetry transformations is [86, 87]

$$S = -\frac{T}{2}\int_W d^2\xi\, e\left[\gamma^{ij}\partial_i X^\mu \partial_j X_\mu - i\overline{\psi}^\mu \not{\mathcal{D}}\psi_\mu + \overline{\chi}_i \rho^j \rho^i (2\,\psi^\mu \partial_j X_\mu + \frac{1}{2}\chi_j \overline{\psi}^\mu \psi_\mu)\right], \quad (1.38)$$

where ψ^μ and χ_i are the worldsheet spinors, $e^\alpha{}_i$ is the zweibein and $\rho^i = \rho^\alpha\, e_\alpha{}^i$ are the two-dimensional gamma matrices. The invariance of the above action under super-Weyl transformations can be used to eliminate the zweibein and the gravitino χ_i. This gives raise to the Ramond-Neveu-Schwarz (RNS) action [88, 89],

$$S_{\text{RNS}} = -\frac{T}{2}\int_W d^2\xi \left(\eta^{ij}\partial_i X^\mu \partial_j X_\mu - i\overline{\psi}^\mu \not{\partial}\psi_\mu\right), \quad (1.39)$$

which is subject to the constraints derived from the equations of motion of the zweibein and gravitino. The variation of the RNS action with respect to ψ^μ yields another non-trivial boundary term (even for closed strings) which must be cancelled by imposing the appropiate boundary conditions on the spinors.

[11] The little group is SO($d-1$) for massive particles and SO($d-2$) for massless particles.

[12] It is worth mentioning that worldsheet supersymmetry does not directly lead to spacetime supersymmetry. An example is the Type 0 superstring, which has worldsheet supersymmetry but not spacetime supersymmetry. In fact, it does not have any spacetime fermions at all.

For open strings, there are two possibilities: either Ramond (R) boundary conditions,

$$\psi^\mu_L|_{\xi^1=0} = \psi^\mu_R|_{\xi^1=0}\,, \qquad \psi^\mu_L|_{\xi^1=\ell} = \psi^\mu_R|_{\xi^1=\ell}\,, \tag{1.40}$$

or Neveu-Schwarz (NS) boundary conditions,

$$\psi^\mu_L|_{\xi_1=0} = \psi^\mu_R|_{\xi_1=0}\,, \qquad \psi^\mu_L|_{\xi_1=\ell} = -\psi^\mu_R|_{\xi_1=\ell}\,, \tag{1.41}$$

where $\psi^\mu_{L,R}$ denote the left- and right-moving components of ψ^μ.

For closed strings, we can choose the boundary conditions for the left- and right-moving fields independently. This gives raise to four possibilities: NSNS, NSR, RNS, RR, which correspond to chosing either Ramond (R) boundary conditions

$$\psi^\mu_{L,R}|_{\xi_1=0} = \psi^\mu_{L,R}|_{\xi_1=\ell}\,, \tag{1.42}$$

or Neveu-Schwarz (NS) boundary conditions

$$\psi^\mu_{L,R}|_{\xi_1=0} = -\psi^\mu_{L,R}|_{\xi_1=\ell}\,. \tag{1.43}$$

The quantization of superstring theories proceeds in an analogous fashion as in the bosonic case. In particular, there also exists an anomalous dimension, $d = 10$, where superstrings can be quantized while preserving Lorentz invariance. The physical (quantum) states are constructed by combining the left- and right-moving components in a very precise manner to avoid the appearance of tachyons. This leaves five different possibilities: type IIA, type IIB, type I, heterotic SO(32) (HO) and heterotic $E_8 \times E_8$ (HE). Their massless spectra are presented in Table 1.1.

Type II theories preserve $\mathcal{N} = 2$ (spacetime) supersymmetry and their massless spectra fit into the supergravity multiplets of the two ten-dimensional supergravities with $\mathcal{N} = 2$ supersymmetry: $\mathcal{N} = 2A$ (non-chiral) and $\mathcal{N} = 2B$ (chiral). Instead, heterotic and type I strings preserve only $\mathcal{N} = 1$ supersymmetry. Heterotic strings are constructed by combining the right-moving fields of a type II superstring with the left-moving fields of the closed bosonic string. The compactification of the sixteen extra dimensions leads to the appearance of vector fields A^A and gaugini χ^A, which fill a vector multiplet with $\mathcal{N} = 1$ supersymmetry. The gauge group is fixed by modular invariance to be either SO(32) or $E_8 \times E_8$. The last possibility that is left, the type I superstring, also contains vector multiplets, though their origin is completely different. They arise from the open-string sector, and it turns out that SO(32) is the only anomaly-free gauge group.

Theory	NSNS	RR	Chiral fermions	Non-chiral fermions	Vector multiplets
Type IIA	$g_{\mu\nu}, B_{\mu\nu}, \phi$	$C^{(1)}{}_\mu, C^{(3)}{}_{\mu\nu\rho}$		ψ_μ, λ	
Type IIB	$g_{\mu\nu}, B_{\mu\nu}, \phi$	$C^{(0)}, C^{(2)}{}_{\mu\nu}, C^{(4)}{}_{\mu\nu\rho\sigma}$	ψ^i_μ, λ^i		
Type I	$g_{\mu\nu}, \phi$	$C^{(2)}{}_{\mu\nu}$	ψ_μ, λ		A^A, χ^A
Heterotic	$g_{\mu\nu}, B_{\mu\nu}, \phi$		ψ_μ, λ		A^A, χ^A

Table 1.1: Massless excitations of the different superstring theories.

1.2.2 String effective actions

For each (super)string theory there is an effective field theory that describes its low-energy dynamics. This is very natural from a physical point of view since the low-energy limit corresponds to the limit in which the size of the strings becomes infinitely small, i.e. $\alpha' \to 0$. Hence, a theory of particles —namely, a field theory— must be recovered. The fields of these effective theories correspond exclusively to the massless modes of the strings. The massive ones decouple from the low-energy dynamics since their masses are proportional to $1/\sqrt{\alpha'}$.

The orthodox procedure to find these effective actions would be to construct a field theory reproducing the string amplitudes in the $\alpha' \to 0$ limit. However, there are simpler approaches which at the end yield the same result. A particularly interesting one, which reflects how crucial conformal invariance is in string theory, consists of coupling a string to general background fields and studying which conditions must be satisfied by the latter in order to preserve conformal invariance at the quantum level.

In order to illustrate this method, let us consider the closed bosonic string, whose coupling to the background fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ is given by the following generalization of the Polyakov action:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{|\gamma|}\left\{\left[\gamma^{ij}g_{\mu\nu}(X) - \frac{\epsilon^{ij}B_{\mu\nu}(X)}{\sqrt{|\gamma|}}\right]\partial_i X^\mu \partial_j X^\nu + \alpha'\phi(X)R(\gamma)\right\}. \quad (1.44)$$

The conditions under which conformal invariance is preserved were studied in [90], where it was shown that they boil down to the vanishing of the following β-functionals:[13]

$$\beta^g_{\mu\nu} = \alpha'\left(R_{\mu\nu} - 2\nabla_\mu\partial_\nu\phi + \frac{1}{4}H_\mu{}^{\rho\sigma}H_{\nu\rho\sigma}\right) + \mathcal{O}(\alpha'^2)\,, \quad (1.45)$$

$$\beta^B_{\mu\nu} = \frac{\alpha'}{2}e^{2\phi}\nabla^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right) + \mathcal{O}(\alpha'^2)\,, \quad (1.46)$$

$$\beta^\phi = -\frac{\alpha'}{2}\left(\nabla^2\phi - (\partial\phi)^2 - \frac{1}{4}R - \frac{1}{48}H^2\right) + \mathcal{O}(\alpha'^2)\,, \quad (1.47)$$

where $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ is the 3-form field strength of the Kalb-Ramond 2-form $B_{\mu\nu}$. As we can see, the β-functionals are given by an expansion in α'. Therefore, their vanishing has to be imposed order by order in α'. At leading order, this is equivalent to the equations of motion that can be derived from the following action

$$S = \int d^{26}x\,\sqrt{|g|}\,e^{-2\phi}\left[R - 4\,(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2\right]\,, \quad (1.48)$$

which is nothing but the effective action of the bosonic string at zeroth order in α'. The next-to-leading order was studied in [92], where it was shown that the vanishing of the β-functionals is captured by the following action[14]

[13] As recently showed in [91], these are a set of sufficient but not necessary conditions.
[14] See also [93, 94].

$$S = \int d^{26}x\, \sqrt{|g|}\, e^{-2\phi} \left[R - 4\,(\partial\phi)^2 + \frac{1}{2\cdot 3!} H^2 \right. \tag{1.49}$$
$$\left. - \frac{\alpha'}{8} \left(R_{(-)\mu\nu ab} R_{(-)}{}^{\mu\nu ab} + R_{(+)\mu\nu ab} R_{(+)}{}^{\mu\nu ab} \right) + \mathcal{O}(\alpha'^2) \right] ,$$

where now the 3-form field strength is defined as

$$H = dB + \frac{\alpha'}{4} \left(\omega^{\mathrm{L}}_{(-)} - \omega^{\mathrm{L}}_{(+)} \right) , \tag{1.50}$$

$\omega^{\mathrm{L}}_{(\pm)}$ being the Lorentz Chern-Simons 3-forms of the torsionful spin connections:

$$\omega^{\mathrm{L}}_{(\pm)} = d\Omega_{(\pm)}{}^{a}{}_{b} \wedge \Omega_{(\pm)}{}^{b}{}_{a} - \frac{2}{3} \Omega_{(\pm)}{}^{a}{}_{c} \wedge \Omega_{(\pm)}{}^{c}{}_{b} \wedge \Omega_{(\pm)}{}^{b}{}_{a} , \tag{1.51}$$

$$\Omega_{(\pm)}{}^{a}{}_{b} = \omega^{a}{}_{b} \pm \frac{1}{2} H_{\mu}{}^{a}{}_{b}\, dx^{\mu} . \tag{1.52}$$

The curvature 2-forms $R_{(\pm)}{}^{a}{}_{b}$ that appear in (1.49) are explicitly given by

$$R_{(\pm)}{}^{a}{}_{b} = d\Omega_{(\pm)}{}^{a}{}_{b} - \Omega_{(\pm)}{}^{a}{}_{c} \wedge \Omega_{(\pm)}{}^{c}{}_{b} . \tag{1.53}$$

The implementation of this procedure order by order in α' would lead to the determination of the complete α' expansion of the effective action of the bosonic string, which would contain an infinite series of higher-curvature terms.

In the case of superstring theories, the identification of the corresponding effective action is much simpler, as they are almost fixed by supersymmetry, which strongly constrains the form of the action and the matter field content, specially in higher dimensions. As a matter of fact, there is only one supergravity theory in eleven dimensions. It was constructed by Cremmer, Julia and Scherk in [95], and it is believed to correspond to the low-energy limit of M-theory [96–99], whose connection with the type IIA superstring will be discussed in the next section. In ten dimensions, the catalogue of supergravity theories is a bit more extensive. Firstly, there are two different supergravity theories with $\mathcal{N} = 2$ supersymmetry: the so-called $\mathcal{N} = 2A$ and $\mathcal{N} = 2B$ [100–103].[15] They describe, respectively, the low-energy dynamics of type IIA and type IIB theories at lowest order in the α' expansion. In addition to these, there is a $\mathcal{N} = 1$ (chiral) supergravity in ten dimensions [105, 106] which describes both the low-energy effective actions of heterotic and type I theories. The supergravity multiplet of this theory contains the zehnbein $e^{a}{}_{\mu}$, the dilaton ϕ, the gravitino ψ_{μ}, the dilatino λ and a 2-form, which can be either the KR 2-form $B_{\mu\nu}$ or the RR 2-form $C^{(2)}{}_{\mu\nu}$, whose main difference at the level of the low-energy action lies on the coupling to the dilaton. This supergravity multiplet can be consistently coupled to a Yang-Mills vector multiplet, which contains a vector field $A^{A}{}_{\mu}$ and a gaugino χ^{A}. Focusing on the fields and couplings of the heterotic theory, the bosonic part of the action is given by[16]

[15] $\mathcal{N} = 2A, d = 10$ supergravity admits a massive deformation which was found by Romans in [104].

[16] Since our primary interest in this thesis will be on the solutions of the effective action, we shall be working with a consistent truncation that sets all the fermions to zero.

$$S = \frac{g_s^2}{16\pi G_{\rm N}^{(10)}} \int d^{10}x \sqrt{|g|}\, e^{-2\phi} \left[R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2 - \frac{\alpha'}{8}F^A\cdot F^A\right], \tag{1.54}$$

where

$$H = dB + \frac{\alpha'}{4}\omega^{\rm YM}, \tag{1.55}$$

and $\omega^{\rm YM}$ is the Chern-Simons 3-form associated to the gauge connection, whose explicit expression is

$$\omega^{\rm YM} = dA^A \wedge A^A - \frac{1}{3!}f_{ABC}A^A\wedge A^B\wedge A^C\,. \tag{1.56}$$

The field strength is defined as

$$F^A = dA^A + \frac{1}{2}f_{BC}{}^A A^B \wedge A^C\,. \tag{1.57}$$

It is well known that ten-dimensional $\mathcal{N}=1$ supergravity suffers from both gauge and gravitational anomalies. However, Green and Schwarz showed in [107] that they can be cancelled for special choices of the gauge group —namely, SO(32) and $\mathrm{E}_8\times\mathrm{E}_8$— if suitable local interactions are added. These interactions modify the local definition of the 3-form H as follows

$$H = dB + \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{\rm L}_{(-)}\right), \tag{1.58}$$

where $\omega^{\rm L}_{(-)}$ is the Lorentz Chern-Simons 3-form of the torsionful spin connection 1-form $\Omega_{(-)}{}^a{}_b$, previously defined in (1.51) and (1.52).

The presence of the Lorentz Chern-Simons 3-form spoils the invariance of the original theory under local supersymmetry transformations. It can be restored, but at the cost of adding an infinite series of higher-derivative terms into the action and supersymmetry transformations. These terms were found in [108] up to order eight in derivatives. In our conventions, the bosonic part of the corrected action is given by

$$\begin{aligned} S = \frac{g_s^2}{16\pi G_{\rm N}^{(10)}} \int d^{10}x \sqrt{|g|}\, e^{-2\phi} \Big[&R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2 - \frac{\alpha'}{2}T^{(0)} \\ &-\frac{\alpha'^3}{4}(T^{(2)})^2 - \frac{\alpha'^3}{48}(T^{(4)})^2 + \ldots\Big], \end{aligned} \tag{1.59}$$

where

$$T^{(4)} \equiv \frac{1}{4}\left(F^A\wedge F^A - R_{(-)ab}\wedge R_{(-)}{}^{ab}\right), \tag{1.60}$$

$$T^{(2)}{}_{\mu\nu} \equiv \frac{1}{4}\left(F^A{}_{\mu\rho}F^A{}_\nu{}^\rho - R_{(-)\mu\rho ab}R_{(-)\nu}{}^{\rho ab}\right), \tag{1.61}$$

$$T^{(0)} \equiv T^{(2)\mu}{}_\mu\,, \tag{1.62}$$

are the so-called T-tensors, defined in terms of the curvature 2-forms of the gauge A^A and torsionful spin $\Omega_{(-)}{}^a{}_b$ connections.

The action (1.59) is the quartic effective action of the heterotic string [108, 109]. It will play a fundamental rôle in the second part of this thesis, where we will study solutions to this action (keeping only the first-order α' corrections) describing supersymmetric black holes in five and four dimensions.

1.2.3 Dualities

One of the most fascinating aspects of string theory is that there is a considerable body of evidence on a web of dualities that relates the different string theories and which suggests that they are different limits of the same underlying theory.

Some of these, the so-called S-dualities, are strong-weak coupling dualities. Therefore, they are necessarily non-perturbative and their existence is mostly inferred from properties of the effective actions and of the non-perturbative states. One example is the type IIB self-duality, which is represented at the level of the effective action by a $\mathrm{SL}(2,\mathbb{R})$ global symmetry that is broken to $\mathrm{SL}(2,\mathbb{Z})$ by quantum effects [97]. Some of the $\mathrm{SL}(2,\mathbb{R})$ transformations act on the complex scalar $\tau = C^{(0)} + ie^{-\phi}$ (constructed out of the RR scalar $C^{(0)}$ and the dilaton ϕ) in such a way that the string coupling constant, $g_s = \langle e^{\phi}\rangle$, gets inverted. This strong-weak coupling self-duality implies that there must be one-dimensional objects in the non-perturbative spectrum of type IIB becoming light and governing the dynamics in the strong-coupling limit. These objects are the D1-branes, which are the S-duals of the fundamental strings. Another well-known example of this type of dualities is the strong-weak coupling duality between type IIA and M-theory. In this case, the relation between the corresponding effective field theories occurs via dimensional reduction, as $\mathcal{N} = 2A, d = 10$ supergravity can be obtained by compactifying $\mathcal{N} = 1, d = 11$ supergravity on a circle. The type IIA dilaton emerges as the Kaluza-Klein scalar that measures the radius of the eleventh dimension, clearly suggesting that the strong coupling limit of type IIA (which coincides with the decompactification limit) is M-theory, which is not a string theory [99].

Another type of string dualities are the so-called T-dualities. These are the ones which are better understood since they they are associated to a symmetry of the perturbative spectrum that interchanges winding and momentum (Kaluza-Klein) modes.

Let us consider the simplest set-up consisting of a closed bosonic string in a spacetime where only one spacelike coordinate, $X^{d-1} \equiv Z$, is compact, $Z \sim Z + 2\pi R_z$. This gives raise to two different types of modes: the Kaluza-Klein or momentum modes, which are already present in field theory, and the winding modes. The latter are a purely stringy effect that corresponds to the ability of closed strings to be wrapped on the compact direction. These two new modes modify the mass operator and the level matching constraint as

$$M^2 = \frac{n^2}{R_z^2} + \frac{R_z^2 w^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) , \qquad \text{with} \qquad N = \tilde{N} + nw \, , \tag{1.63}$$

where n and w are integers associated to the momentum and winding modes respectively, and N and $\tilde{N}$ are the *level operators*. It is straightforward to check that (1.63) is invariant under the following transformations

$$n \to n' = w\,, \qquad w \to w' = n\,, \qquad R_z \to R'_z = \frac{\alpha'}{R_z}\,. \tag{1.64}$$

This is actually a symmetry of the full spectrum which, furthermore, has been proven to hold at all orders in perturbation theory [110]. It turns out that it is related to the invariance of the Polyakov action under Poincaré dualization of the embedding coordinate Z, see e.g. [111].

We are more interested in the manifestation of T-duality at the level of the effective action. In order to study it, we first need to introduce the basics of the Kaluza-Klein (KK) dimensional reduction [112, 113]. The original idea, which goes back to the twenties of the past century, was to unify gravity and electromagnetism by assuming that the spacetime has an extra dimension so that both four-dimensional spacetime and gauge symmetries arise from spacetime symmetries in five dimensions. This beautiful idea, although abandoned for its original purpose, became an extraordinarily powerful tool in the general context of theoretical physics and particularly in string theory, where it is crucial in order to make contact with the four-dimensional world that we experience.

We will follow the modern Scherk-Schwarz formalism [114], which makes use of the vielbein and which is therefore well adapted to describe the dimensional reduction of theories with fermionic degrees of freedom, such as supergravity theories. We will always assume that none of the fields depends on the coordinate $z \sim z + 2\pi R_z$ that parametrizes the compact dimension $\mathbb{S}^1_z$. This is equivalent to just keeping the zero mode in the Fourier expansion of the higher-dimensional fields,

$$\Phi(x, z) = \sum_{n \in \mathbb{Z}} \Phi^{(n)}(x) e^{\frac{inz}{R_z}} \approx \Phi^{(0)}(x)\,, \tag{1.65}$$

which is guaranteed to be a good approximation as long as the typical energy scale E of a given physical process is much lower than the KK scale: $E << m_{\text{KK}} \sim R_z^{-1}$. The reasoning is analogous to the one we made before for the massive string modes. The higher modes in the Fourier expansion will generically acquire a mass inversely proportional to the radius of the circle, R_z. Therefore, they decouple from the low-energy dynamics.

Having said this, let us now carry out the dimensional reduction of the Einstein-Hilbert action. We start by decomposing the $(d+1)$-dimensional vielbein, $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$, and its inverse, $\hat{e}_{\hat{a}}{}^{\hat{\mu}}$, in terms of the lower-dimensional fields as follows[17]

$$\begin{aligned} \hat{e}^a &= e^a{}_\mu \, dx^\mu\,, & \hat{e}^z &= k\,(dz + A_\mu \, dx^\mu)\,, \\ \hat{e}_a &= e_a{}^\mu \, \partial_\mu - A_a \, \partial_{\underline{z}}\,, & \hat{e}_z &= k\, \partial_{\underline{z}}\,, \end{aligned} \tag{1.66}$$

where $e^a{}_\mu$ and $e_a{}^\mu$ are the d-dimensional vielbein and its inverse, A_μ is the KK vector and k is the KK scalar. The latter measures the radius of the circle $\mathbb{S}^1_z$ as a function of the non-compact coordinates x^μ:

$$R(x) = \frac{1}{2\pi} \int_0^{2\pi R_z} \sqrt{|g_{\underline{zz}}|} = R_z k(x)\,. \tag{1.67}$$

[17] The $(d+1)$-dimensional fields and indices will be denoted with hats. Then, we have $\hat{a} = (a, z)$ for flat indices and $\hat{\mu} = (\mu, \underline{z})$ for world indices.

This can be better appreciated after writing down the $(d+1)$-dimensional metric,

$$d\hat{s}^2 = \hat{g}_{\hat{\mu}\hat{\nu}}\, dx^{\hat{\mu}}dx^{\hat{\nu}} = g_{\mu\nu}\, dx^{\mu}dx^{\nu} + k^2\left(dz + A_{\mu}\, dx^{\mu}\right)^2 \,, \tag{1.68}$$

where $g_{\mu\nu} = e^a{}_{\mu}e^b{}_{\nu}\eta_{ab}$.

The simplest way of computing the dimensional reduction of the Einstein-Hilbert action is perhaps by using Palatini's identity, which tells us that

$$\int d^{d+1}x\,\sqrt{|\hat{g}|}\,K\hat{R} = \int d^{d+1}x\,\sqrt{|\hat{g}|}\,K\left[-\hat{\omega}_{\hat{b}}{}^{\hat{b}\hat{a}}\hat{\omega}_{\hat{c}}{}^{\hat{c}}{}_{\hat{a}} - \hat{\omega}_{\hat{a}}{}^{\hat{b}\hat{c}}\hat{\omega}_{\hat{b}\hat{c}}{}^{\hat{a}} + 2\hat{\omega}_{\hat{b}}{}^{\hat{b}\hat{a}}\partial_{\hat{a}}\log K\right] \,, \tag{1.69}$$

up to boundary terms. In order to apply it, we must first compute the spin connection. In the vielbein basis we have chosen, one obtains

$$\hat{\omega}_{ab} = \omega_{cab}\,\hat{e}^c - \frac{1}{2}kF_{ab}\,\hat{e}^z \,, \qquad \hat{\omega}_{az} = \frac{1}{2}kF_{ba}\,\hat{e}^b - \partial_a\log k\,\hat{e}^z \,, \tag{1.70}$$

where $F_{ab} = 2\nabla_{[a}A_{b]}$ is the field strength of the KK vector potential.

Then, making use of Palatini's identity (1.69), one obtains

$$\begin{aligned} S_{\rm EH} =& \frac{1}{16\pi G_{\rm N}^{(d+1)}}\int d^{d+1}x\,\sqrt{|\hat{g}|}\,\hat{R} = \frac{1}{16\pi G_{\rm N}^{(d+1)}}\int d^{d+1}x\,\sqrt{|\hat{g}|}\,\left[-\hat{\omega}_{\hat{b}}{}^{\hat{b}\hat{a}}\hat{\omega}_{\hat{c}}{}^{\hat{c}}{}_{\hat{a}} - \hat{\omega}_{\hat{a}}{}^{\hat{b}\hat{c}}\hat{\omega}_{\hat{b}\hat{c}}{}^{\hat{a}}\right] \\ =& \frac{2\pi R_z}{16\pi G_{\rm N}^{(d+1)}}\int d^dx\,\sqrt{|g|}\,k\left[-\omega_b{}^{ba}\omega_c{}^c{}_a - \omega_a{}^{bc}\omega_{bc}{}^a + 2\omega_b{}^{ba}\partial_a\log k - \frac{1}{4}k^2F^2\right] \\ =& \frac{1}{16\pi G_{\rm N}^{(d)}}\int d^dx\sqrt{|g|}\,k\left[R - \frac{1}{4}k^2F^2\right] \,, \end{aligned} \tag{1.71}$$

where in the last step we have defined the d-dimensional Newton's constant as

$$G_{\rm N}^{(d)} = \frac{G_{\rm N}^{(d+1)}}{2\pi R_z} \,. \tag{1.72}$$

In the general case in which a $(d{+}n)$-dimensional manifold $\mathcal{M}^{(d+n)}$ contains a n-dimensional compact space $\mathcal{C}^{(n)}$, the relation between the Newton's constants is

$$G_{\rm N}^{(d)} = \frac{G_{\rm N}^{(d+n)}}{V_n} \,, \tag{1.73}$$

where V_n is the volume of $\mathcal{C}^{(n)}$.

Rewriting the action (1.71) in terms of the metric in the Einstein frame, $g_{{\rm E}\,\mu\nu} = k^{\frac{2}{d-2}}g_{\mu\nu}$,[18]

[18]The Einstein frame is the one in which there is no conformal factor multiplying the Ricci scalar.

$$S_{\rm EH} = \frac{1}{16\pi G_{\rm N}^{(d)}} \int d^dx \sqrt{|g_{\rm E}|}\left[R_{\rm E} + \frac{d-1}{d-2}(\partial \log k)^2 - \frac{1}{4}k^{\frac{2(d-1)}{d-2}}F^2\right], \tag{1.74}$$

we clearly see that the KK scalar is dynamical and cannot be truncated to a fixed value without imposing the corresponding constraint derived from its equation of motion ($F^2 = 0$ in this case).

Once we know how to reduce the Einstein-Hilbert term, the last piece of information that we need is to learn how to reduce p-forms. The dimensional reduction of a p-form $\hat{C}^{(p)}{}_{\hat{\mu}_1\ldots\hat{\mu}_p}$ on a circle gives raise to a p-form $C^{(p)}{}_{\mu_1\ldots\mu_p}$ and to a $(p-1)$-form $C^{(p-1)}{}_{\mu_1\ldots\mu_{p-1}}$ in d dimensions:

$$\begin{aligned} \hat{C}^{(p)}{}_{\mu_1\ldots\mu_p} &= C^{(p)}{}_{\mu_1\ldots\mu_p} + p\, A_{[\mu_1}C^{(p-1)}{}_{\mu_2\ldots\mu_p]}\,, \\ \hat{C}^{(p)}{}_{\mu_1\ldots\mu_{p-1}\underline{z}} &= C^{(p-1)}{}_{\mu_1\ldots\mu_{p-1}}\,. \end{aligned} \tag{1.75}$$

This is, however, subject to field redefinitions. In the case of the Kalb-Ramond 2-form, we find convenient to define

$$\hat{B}_{\mu\nu} = B_{\mu\nu} - A_{[\mu}B_{\nu]}\,, \qquad \text{with} \qquad B_\mu = \hat{B}_{\mu\underline{z}}\,, \tag{1.76}$$

where B_μ is the *winding* vector.

Using Eq. (1.76) and the dimensional reduction of the Einstein-Hilbert term, one finds that the dimensional reduction on a circle of the effective action of the closed bosonic string (1.48) is

$$S \sim \int d^dx\, \sqrt{|g|}\, e^{-2\phi}\left[R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2 + (\partial \log k)^2 - \frac{1}{4}k^2F^2 - \frac{1}{4}k^{-2}G^2\right], \tag{1.77}$$

where $G = dB$ is the field strength of the winding vector and

$$\phi = \hat{\phi} - \frac{1}{2}\log k\,, \tag{1.78}$$

is the lower-dimensional dilaton. As one can easily check, the action is invariant under the following transformations

$$A_\mu \to A'_\mu = B_\mu\,, \qquad B_\mu \to B'_\mu = A_\mu\,, \qquad k \to k' = k^{-1}\,, \tag{1.79}$$

which expressed in terms of the higher-dimensional fields lead to

$$\begin{aligned} \hat{g}'_{\underline{zz}} &= 1/\hat{g}_{\underline{zz}}\,, & \hat{B}'_{\mu\underline{z}} &= \hat{g}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,, \\ \hat{g}'_{\mu\underline{z}} &= \hat{B}_{\mu\underline{z}}/\hat{g}_{\underline{zz}}\,, & \hat{B}'_{\mu\nu} &= \hat{B}_{\mu\nu} + 2\hat{g}_{[\mu|\underline{z}}\hat{B}_{\nu]\underline{z}}/\hat{g}_{\underline{zz}}\,, \\ \hat{g}'_{\mu\nu} &= \hat{g}_{\mu\nu} - (\hat{g}_{\mu\underline{z}}\hat{g}_{\nu\underline{z}} - \hat{B}_{\mu\underline{z}}\hat{B}_{\nu\underline{z}})/\hat{g}_{\underline{zz}}\,, & e^{-2\hat{\phi}'} &= e^{-2\hat{\phi}}|\hat{g}_{\underline{zz}}|\,. \end{aligned} \tag{1.80}$$

This set of rules are known as the *Buscher rules* and tell us that two different backgrounds compactified on circles of inverse radii are equivalent from the perspective of string theory.[19] In the case at hands, the relation is between different backgrounds of the same theory, the bosonic string. However, this is not always the case. The most representative example in which T-duality relates backgrounds of *a priori* different theories is the T-duality between type IIA and type IIB theories. As in the bosonic case, this occurs both at the level of the effective actions [115, 116] and of the perturbative spectra.

We would like now to pay attention to T-duality of heterotic strings. At zeroth order in α', without gauge fields, the bosonic action has exactly the same form as the effective action of the bosonic string with the sole difference that in the heterotic case the dimension of the spacetime is $d = 10$. Therefore, the Buscher rules at zeroth order are just given by the rules presented above for the bosonic string. In chapters 4 and 5, we will be interested in studying the transformations under T-duality of solutions of the α'-corrected action (1.59), including the gauge fields. Hence, we will need to know how the Buscher rules are modified by the gauge fields and by the α' corrections. Fortunately to us, these were derived at first order in α' (which is enough for our purposes) in [117]:[20]

$$
\begin{aligned}
g'_{\underline{zz}} &= g_{\underline{zz}}/G^2_{\underline{zz}}\,, & B'_{\underline{z}\mu} &= -B_{\underline{z}\mu}/G_{\underline{zz}} - G_{\underline{z}\mu}/G_{\underline{zz}}\,,\\
g'_{\underline{z}\mu} &= -g_{\underline{z}\mu}/G_{\underline{zz}} + g_{\underline{zz}}G_{\underline{z}\mu}/G^2_{\underline{zz}}\,, & A'^{A}_{\mu} &= A^{A}_{\mu} - A^{A}_{\underline{z}}G_{\underline{z}\mu}/G_{\underline{zz}}\,,\\
g'_{\mu\nu} &= g_{\mu\nu} + \left[g_{\underline{zz}}G_{\underline{z}\mu}G_{\underline{z}\nu} - 2G_{\underline{zz}}G_{\underline{z}(\mu}g_{\nu)\underline{z}}\right]/G^2_{\underline{zz}}\,, & A'^{A}_{\underline{z}} &= -A^{A}_{\underline{z}}/G_{\underline{zz}}\,,\\
B'_{\mu\nu} &= B_{\mu\nu} - G_{\underline{z}[\mu}G_{\nu]\underline{z}}/G_{\underline{zz}}\,, & e^{-2\phi'} &= e^{-2\phi}|G_{\underline{zz}}|\,,
\end{aligned}
\tag{1.81}
$$

where the tensor $G_{\mu\nu}$ is defined by

$$
G_{\mu\nu} \equiv g_{\mu\nu} - B_{\mu\nu} - \frac{\alpha'}{4}\left(A^{A}{}_{\mu}A^{A}{}_{\nu} + \Omega_{(-)\,\mu}{}^{a}{}_{b}\Omega_{(-)\,\nu}{}^{b}{}_{a}\right)\,. \tag{1.82}
$$

The Buscher rules have very interesting practical applications (e.g., they can be used to generate new solutions) but in order to explore them we need at least a solution to the effective string equations with a spacelike isometry. This is precisely the reason why the α'-corrected Buscher rules of [117] have hardly been exploited. To the best of our knowledge, most part of the known analytic solutions with α' corrections (if not all of them) consisted of the heterotic string solitons of [118–120]. These solutions have spacelike isometries but they act with fixed points (rotations) and, in this case, the equivalence of the two backgrounds related by the Buscher rules needs not to be true [121]. In this thesis, we will construct a family of α'-corrected solutions which will allow us to put these rules into practice. We will show that the aforementioned family is self-T-dual, as it is expected. This is considered to be a non-trivial consistency check of both the solutions and the Buscher rules.

[19]The transformation of the dilaton is a quantum effect. The reason why it is captured by the effective action is because the latter contains information about the quantum theory, see the paragraph below (1.44).

[20]We are omitting the hats for the ten-dimensional fields and $\mu, \nu = 0, \ldots, 8$ and $z \equiv x^9$.

1.2.4 Extended objects and black holes

String theory is not only a theory of strings. It also has other extended objects which are non-perturbative. Among these, the best understood are the Dp-branes (or D-branes) since, in spite of their non-perturbative nature, they can be defined in perturbative string theory as the rigid walls where open strings end. Thereby, the open-string sector encodes very valuable information about the fluctuations of the theory around the D-branes. As we have already mentioned, D-branes are a crucial ingredient to establish string dualities, which often predict the existence of non-perturbative objects that do not admit a description in perturbative string theory and, in consequence, are much less understood than D-branes. For instance, the type IIB spectrum must be enlarged in order to make it consistent with S-duality to include S5-branes (as well as S7- and S9-branes), which couple to the 6-form $\tilde{B}$, the magnetic dual of the Kalb-Ramond 2-form B. One can now go a step further and wonder which type IIA object is the T-dual of the S5-brane. This is the KK monopole or the KK6-brane. There is nothing that prevents us from continuing this procedure until we go back to the original object, completing a U-duality orbit.[21] This is precisely the way in which the extended objects of string/M-theory can be classified into U-duality multiplets [122–127].

All these non-perturbative objects can be thought to be the string theory analogs to solitons in gauge theories, such as magnetic monopoles or instantons. Consequently, we expect that at least some of them can be approximately described as solutions to the string effective equations of motion. This approach has been extraordinarily useful to provide evidence for the existence of the extended objects predicted from dualities, and also to predict the existence of new ones.

Extended objects from supergravity solutions

Among all the solutions to the ten-dimensional supergravity equations describing the leading-order approximation to extended objects of string theory (see [111] for a complete review), the most relevant ones for us here will be those which have unbroken supersymmetries, often referred to as BPS objects. Let us review some of them.

Dp-brane solutions. These solutions describe p-dimensional extended objects that are charged with respect to the RR $(p+1)$-form potential, $C^{(p+1)}$. Therefore, they exist for p even in type IIA, for p odd in type IIB and for $p = 1, 5$ in type I string theories. The non-vanishing fields of the solutions are given by [128]

[21] U duality is the combination of both T- and S-dualities.

$$ds^2 = Z_{\mathrm{D}p}^{-\frac{1}{2}}\left(dt^2 - d\vec{y}_p^{\,2}\right) - Z_{\mathrm{D}p}^{\frac{1}{2}} d\vec{x}_{9-p}^{\,2}\,, \tag{1.83}$$

$$C^{(p+1)} = \pm e^{-\phi_0}\left(Z_{\mathrm{D}p}^{-1} - 1\right) dt\wedge d^p y\,, \tag{1.84}$$

$$e^{2\phi} = e^{2\phi_0} Z_{\mathrm{D}p}^{\frac{3-p}{2}}\,, \tag{1.85}$$

$$Z_{\mathrm{D}p} = 1 + \frac{q_{\mathrm{D}p}}{|\vec{x}_{9-p}|^{7-p}}\,, \qquad q_{\mathrm{D}p} = \frac{(2\pi\ell_s)^{7-p}\, e^{\phi_0} N_{\mathrm{D}p}}{(7-p)\,\Omega_{\mathbb{S}^{8-p}}}\,, \qquad 0 < p < 7\,, \tag{1.86}$$

where $\Omega_{\mathbb{S}^{8-p}}$ is the volume of the unit $(8-p)$-sphere and $N_{\mathrm{D}p}$ is the number of coincident Dp-branes. These solutions are asymptotically flat and have a singular horizon at $|\vec{x}_{9-p}| = 0$ for $p \neq 3$. For $p = 3$, the horizon is completely regular and the near-horizon geometry is $\mathrm{AdS}_5 \times \mathbb{S}^5$, which has played a crucial rôle in the AdS/CFT correspondence [68, 129].

S5-brane solution. This is a solution describing a solitonic 5-brane that is electrically charged under the magnetic dual of the KR 2-form, $\tilde{B}$ (therefore, magnetically charged with respect to B). It exists in type II and heterotic theories but not in type I. The non-vanishing fields are given by:

$$ds^2 = dt^2 - d\vec{y}_5^{\,2} - Z_{\mathrm{S5}} d\vec{x}_4^{\,2}\,, \tag{1.87}$$

$$\tilde{B} = \pm e^{-2\phi_0}\left(Z_{\mathrm{S5}}^{-1} - 1\right) dt\wedge d^5 y\,, \tag{1.88}$$

$$e^{2\phi} = e^{2\phi_0} Z_{\mathrm{S5}}\,, \tag{1.89}$$

$$Z_{\mathrm{S5}} = 1 + \frac{q_{\mathrm{S5}}}{|\vec{x}_4|^2}\,, \qquad q_{\mathrm{S5}} = N\ell_s^2\,, \tag{1.90}$$

where N is the number of 5-branes. The metric of this solution (in the string frame) interpolates between the ten-dimensional Minkowski metric at infinity $|\vec{x}_4| \to \infty$ and the metric of the product $\mathbb{R}^{1,6} \times \mathbb{S}^3$ at the core $|\vec{x}_4| \to 0$.

F1 solution. Finally, we would like to present an interesting solution of both type II and heterotic theories that describes the supergravity background created by a fundamental string at finite coupling [130]

$$
\begin{aligned}
ds^2_{\rm E} &= Z_{\rm F1}^{-\frac{3}{4}}\left(dt^2-dy^2\right)-Z_{\rm F1}^{\frac{1}{4}}d\vec{x}_8^2 && (1.91)\\
ds^2 &= Z_{\rm F1}^{-1}\left(dt^2-dy^2\right)-d\vec{x}_8^2\,, && (1.92)\\
B &= \pm\left(Z_{\rm F1}^{-1}-1\right)dt\wedge dy\,, && (1.93)\\
e^{2\phi} &= e^{2\phi_0}Z_{\rm F1}^{-1}\,, && (1.94)\\
Z_{\rm F1} &= 1+\frac{q_{\rm F1}}{|\vec{x}_8|^6}\,, \qquad q_{\rm F1}=\frac{(2\pi\ell_s)^6 g_s^2 w}{6\Omega_{\mathbb{S}^7}}\,. && (1.95)
\end{aligned}
$$

where w is the total winding number along $y \sim y + 2\pi R_y$. This solution describes a string with a singular horizon at $|\vec{x}_8| = 0$. If we forget for a moment about the singularity and reduce this solution along the compact direction, we find the following metric

$$ds^2_{\rm E} = H_{\rm F1}^{-\frac{6}{7}}dt^2 - H_{\rm F1}^{\frac{1}{7}}d\vec{x}_8^2\,, \tag{1.96}$$

which describes a supersymmetric black hole with a singular horizon of zero size. This was expectable, as the original solution was already singular. It turns out that exactly the same occurs if one tries to make a black hole in $10-p$ dimensions out of a Dp-brane wrapped on a torus $\mathbb{T}^p$.

Despite this first attempt of making a black hole out of a wrapped string/D-brane has failed, it is helpful to illustrate how black holes in string theory can be understood as complementary descriptions at finite string coupling of certain string states, as suggested by Susskind [66]. This offers the possibility of reproducing the Bekenstein-Hawking entropy of a (regular) black hole by counting the degeneracy of string states at weak coupling $g_s << 1$. For that to be possible, however, the degeneracy of states must remain invariant when varying g_s, otherwise it cannot be safely extrapolated from the perturbative regime where it can be computed to the strong coupling regime where the black hole is expected to exist. This is precisely what happens for BPS or supersymmetric string states.

Before discussing how supersymmetric black holes with regular horizons can be constructed in string theory, we would like to consider the possibility of adding a momentum wave to the fundamental string solution. The corresponding supergravity solution was found by Garfinkle [131] and later generalized in [132, 133]. The only modification with respect to the F1 solution occurs in the metric, which now is given by

$$ds^2 = Z_{\rm F1}^{-1}du\left[dv + (1-Z_{\rm P})\,du\right] - d\vec{x}_8^2\,, \tag{1.97}$$

where $v = t + y$, $u = t - y$ and

$$Z_{\rm P} = 1 + \frac{q_{\rm P}}{|\vec{x}_8|^6}\,, \qquad \frac{q_{\rm P}}{(2\pi\ell_s)^6} = \frac{g_s^2\alpha' n}{6\Omega_{\mathbb{S}^7}R_y^2}\,, \tag{1.98}$$

n being the quantized momentum.

This is a supersymmetric solution of both type II and heterotic theories which provides an effective description of the BPS states of a fundamental string with winding and momentum charges, also known as Dabholkar-Harvey (DH) states [134]. For large charges, the degeneracy of these states is given by

$$\log d\,(n,w) \approx c\sqrt{nw}\,, \qquad \text{with} \qquad c = \begin{cases} 2\sqrt{2}\pi & \text{type II} \\ 4\pi & \text{heterotic} \end{cases} . \tag{1.99}$$

Given this, the hope is that a supersymmetric black hole with a regular horizon capable to account for the degeneracy of the DH states would arise upon compactification on $\mathbb{S}^y$. However, as we will see, the resulting horizon is also singular and has vanishing area. The intuitive explanation that it is often given in order to explain this mismatch between the macroscopic and microscopic descriptions is that the event horizon scale of these black holes is so small (roughly of the order of the string length $\sqrt{\alpha'}$) that it is not resolved by the supergravity approximation. For this reason, they where dubbed *small black holes*. This certainly makes sense, as on general grounds one expects that the higher-derivative correcctions to the supergravity action will significantly modify the solution near the singular horizon, where these terms are no longer subleading. This was advocated by Sen in [135], who further showed that the corrected entropy, if finite, would scale correctly with the charges. The study of the α' corrections to the heterotic small black holes and black rings constitutes an important part of this thesis. We will address it in chapters 6 and 7, where we will show that quadratic curvature corrections are not not enough to regularize these solutions.

For the time being, we can consider other types of supersymmetric black holes which are regular already in the supergravity approximation.

Supersymmetric black holes

Supersymmetric (BPS) black holes can be constructed as solutions of the lower-dimensional string equations of motion only in five and four dimensions. Let us focus for definiteness on five-dimensional black holes of type IIB compactified on a torus $\mathbb{T}^5$. The low-energy effective theory is $\mathcal{N}=4, d=5$ supergravity, which has a $E_{6,6}$ U-duality symmetry group. The most general supersymmetric black hole of this theory has 27 different charges (as many as vector fields in the theory) and it can be constructed out of the generating solution (the one with the minimum number of charges) by performing a U-duality transformation that leaves the metric of the solution invariant. Since all the solutions related by a U-duality transformation are equivalent from the point of view of string theory, we will focus on the simplest one, the generating solution, which has three electric charges and is given by

$$ds^2 = (Z_{\text{D1}}Z_{\text{D5}}Z_{\text{P}})^{-\frac{2}{3}}\,dt^2 - (Z_{\text{D1}}Z_{\text{D5}}Z_{\text{P}})^{\frac{1}{3}}\left[d\rho^2 + \rho^2 d\Omega^2_{(3)}\right] , \tag{1.100}$$

$$A_i = -\left(Z_i^{-1} - 1\right)\,dt\,, \qquad Z_i = 1 + \frac{q_i}{\rho^2}\,, \qquad i = \text{D1}, \text{D5}, \text{P}\,, \tag{1.101}$$

$$k_{V^4} = \frac{Z_{\text{D1}}}{Z_{\text{D5}}}\,, \qquad k_1 = k_{1,0}\frac{Z_{\text{P}}^{\frac{1}{2}}}{Z_{\text{D1}}^{\frac{1}{4}}Z_{\text{D5}}^{\frac{1}{4}}}\,, \qquad e^{2\varphi} = e^{2\varphi_0}\frac{Z_{\text{D1}}^{\frac{1}{4}}Z_{\text{D5}}^{\frac{1}{4}}}{Z_{\text{P}}^{\frac{1}{2}}}\,. \tag{1.102}$$

As we will see, this is also a solution of the STU model of $\mathcal{N}=1, d=5$ supergravity that arises as a consistent truncation of $\mathcal{N}=4, d=5$ supergravity. The metric has a spherical horizon at $\rho=0$ with area

$$A_H = 2\pi^2\sqrt{q_{\rm D1}q_{\rm D5}q_{\rm P}}\,. \tag{1.103}$$

The parameters q_i have the physical interpretation of three electric charges associated to each of the three Abelian vector fields present in the STU model. However, we are more interested in understanding the stringy interpretation of these parameters. For this, it is necessary to write this five-dimensional solution as a ten-dimensional string background. There is not a unique way of doing this, though all of them are related by duality transformations. In the frame we are working (type IIB on $\mathbb{T}^5$), we have

$$\begin{aligned} ds^2 &= Z_{\rm D1}^{-1/2}Z_{\rm D5}^{-1/2}Z_{\rm P}^{-1}dt^2 - Z_{\rm D1}^{1/2}Z_{\rm D5}^{1/2}ds^2\left(\mathbb{E}^4\right) - Z_{\rm D1}^{1/2}Z_{\rm D5}^{-1/2}ds^2\left(\mathbb{T}^4\right) \\ &\quad -Z_{\rm D1}^{-1/2}Z_{\rm D5}^{-1/2}Z_{\rm P}\left(dz+A_{\rm P}\right)^2\,, \end{aligned} \tag{1.104}$$

$$C^{(2)} = e^{-\phi_0}A_{\rm D1}\wedge dz\,, \qquad C^{(6)} = e^{-\phi_0}A_{\rm D5}\wedge dz\wedge\omega_{\mathbb{T}^4}\,, \tag{1.105}$$

$$e^{2\phi} = e^{2\phi_0}\,\frac{Z_{\rm D1}}{Z_{\rm D5}}\,, \tag{1.106}$$

where $\omega_{\mathbb{T}^4}$ represents the volume form of $\mathbb{T}^4$, which together with $\mathbb{S}^1_z$ constitute the total compact space $\mathbb{T}^5=\mathbb{T}^4\times\mathbb{S}^1_z$.

This solution describes a bound state of the D1-D5-P system. Concretely, there are $N_{\rm D1}$ D1-branes wrapped on $\mathbb{S}^1_z$, $N_{\rm D5}$ D5-branes wrapped on $\mathbb{T}^5=\mathbb{T}^4\times\mathbb{S}^1_z$ and $N_{\rm P}$ units of KK momentum along z, see Table 1.2.

	t	z	z^1	z^2	z^3	z^4	x^1	x^2	x^3	x^4
D1	$\times$	$\times$	$\sim$	$\sim$	$\sim$	$\sim$	$-$	$-$	$-$	$-$
D5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$-$	$-$	$-$	$-$
P	$\times$	$\times$	$\sim$	$\sim$	$\sim$	$\sim$	$-$	$-$	$-$	$-$

Table 1.2: Sources of the ten-dimensional backgrounds which give raise to five-dimensional black holes after dimensional reduction on $\mathbb{T}^4\times\mathbb{S}^1_z$. $\times$ stands for the worldvolume directions and $-$ for the transverse directions. The symbol $\sim$ stands for the transverse directions over which the corresponding extended object has been smeared.

The number of D-branes and KK momentum are related to the electric charges of the black hole by

$$q_{\rm D5} = g_s\alpha' N_{\rm D5}\,, \qquad q_{\rm D1} = \frac{g_s\alpha'^3 N_{\rm D1}}{V}\,, \qquad q_{\rm P} = \frac{g_s^2\alpha'^4}{R^2V}N_{\rm P}\,, \tag{1.107}$$

where R is the radius of $\mathbb{S}^1_z$ and $V=\int_{\mathbb{T}^4}\omega_{\mathbb{T}^4}$ is the volume of $\mathbb{T}^4$. The five-dimensional Newton's constant can be expressed in terms of these moduli, the string coupling constant g_s and α' as

$$G_{\rm N}^{(5)} = \frac{G_{\rm N}^{(10)}}{(2\pi)^5 RV} = \frac{\pi g_s^2 \alpha'^4}{4RV}\,. \tag{1.108}$$

Using this and (1.103), we find that the Bekenstein-Hawking entropy is given by

$$S_{\rm BH} = 2\pi\sqrt{N_{\rm D1}N_{\rm D5}N_{\rm P}}\,. \tag{1.109}$$

Let us notice that the entropy is given by the product of three integers, which strongly suggests that it is possible to reproduce it from microstate counting. This is precisely what Strominger and Vafa did in [67]. Let us discuss very succinctly how.

Under the assumption that the size of the circle $\mathbb{S}^1_z$ is much larger than the size of the torus $\mathbb{T}^4$, the low-energy dynamics of the D1-D5 system is described by a two-dimensional CFT defined on $\mathbb{S}^1_z$. The fields of this CFT correspond to the zero modes of open strings that connect the D-branes, and its central charge is given by $c = 6N_{\rm D1}N_{\rm D5}$. Hence, we need to count the different possibilities that these states have to carry $N_{\rm P}$ units of momentum or, what is equivalent, a total energy $E = N_{\rm P}/R$. When $N_{\rm P}$ is large, we can use Cardy formula [136], which tells us that the degeneracy of states with energy E is given by

$$S(E) \approx 2\pi\sqrt{\frac{cER}{6}} = 2\pi\sqrt{N_{\rm D1}N_{\rm D5}N_{\rm P}}\,, \tag{1.110}$$

in agreement with the Bekenstein-Hawking entropy (1.109).

However, both the macroscopic (black-hole) and microscopic entropies receive corrections when the charges are large but finite, so it is natural to ask if the agreement also holds beyond the leading order. In the macroscopic side, the corrections arise from the higher-derivative terms in the corresponding effective action, which yield non-negligible contributions. We will compute them in Chapter 5 for both the three-charge black holes considered here and for four-charge black holes in four dimensions. For convenience, we will embed these black-hole solutions in the heterotic theory compactified on a torus, where we can make use of the action (1.59) to compute the higher-derivative corrections.

1.3 Summary of the main results

Let us briefly summarize the content and the main results of each of the chapters, which are based on [137–143].

Part I: Non-Abelian supersymmetric solutions from gauged supergravity

In the first part, which consists of chapters 2 and 3, we study supersymmetric solutions of the SU(2)-gauged ST[2, 6] model of $\mathcal{N} = 1, d = 5$ gauged supergravity:

- In Chapter 2, which is based on [137], we construct novel rotating BPS black holes with non-trivial Yang-Mills fields in five and four dimensions. They can be understood as the distortion caused by a dyonic SU(2) instanton (closely related to the one considered in [144, 145]) on the three- and four-charge supersymmetric black holes of the heterotic theory on a torus [146, 147]. The instanton is characterized by two parameters: its size, denoted as κ or λ, and ξ, the parameter that controls the dyonic deformation, which is related to the electric charge of the instanton. As it occurs in [144, 145], the size of the instanton (which for $\xi = 0$ is a free parameter) gets fixed in terms of the angular momenta of the solution. Then, these black holes do not have non-Abelian hair, contrary to what occurs in most non-Abelian supersymmetric black holes constructed so far, see e.g. [148–154].

- In Chapter 3, based on [138], we propose a systematic method to construct five-dimensional BPS microstate geometries free of closed timelike curves (CTCs). A set of rules to construct this type of solutions in supergravity was discovered in [155–157]. The recipe, however, does not tell one how to choose the parameters of the solution so as to avoid the appearance CTCs. Firstly, not all the parameters are independent since the so-called *bubble equations* (which ensure that the solution is free of Dirac-Misner type singularities) must be imposed. These equations have been traditionally solved taking the centers of the solution as the unknowns, which already represents strenuous task. Still, their resolution is only a necessary condition to avoid CTCs. In fact, they generically appear even after these equations have been solved, which enormously hamper the construction of explicit (physically meaningful) solutions. The method that we propose to tackle these two problems at once can be summarized as follows:

  1. The bubble equations are non-linear and hard to solve if the locations of the centers are taken as the unknowns. However, those can be rewritten as a simple system of linear equations by choosing a different set of variables: the magnetic fluxes. The bubble equations become

  $$\mathcal{M}X = B\,, \tag{1.111}$$

  for some symmetric matrix $\mathcal{M}$.

  2. We show evidence that any solution satisfying the bubble equations is free of CTCs if and only if all the eigenvalues of the matrix $\mathcal{M}$ are positive.

Part II: Black holes with higher-derivative corrections

The second part of the thesis consists of the chapters 4, 5, 6, 7 and 8, and it is devoted to the investigation of the effects produced by the higher-derivative corrections on black-hole solutions, mainly in the context of string theory.

- In Chapter 4, based on [139], we find a family of α'-corrected solutions that can be used to describe BPS bound states of the F1-P-S5 system at first order in α'. The field configuration considered is the following

$$ds^2 = \frac{2}{\mathcal{Z}_-}du\left(dt - \frac{\mathcal{Z}_+}{2}du\right) - \mathcal{Z}_0\, d\sigma^2 - dz^\alpha dz^\alpha\,, \tag{1.112}$$

$$H = \star_\sigma d\mathcal{Z}_0 + d\mathcal{Z}_-^{-1}\wedge du \wedge dt\,, \tag{1.113}$$

$$e^{2\phi} = e^{2\phi_\infty}\frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \tag{1.114}$$

$$A_{\mathfrak{i}} = \mathbb{M}^-_{mn}\,\partial_n \log P_{\mathfrak{i}}\, v^m\,, \qquad \mathfrak{i}=1,\ldots,\mathfrak{n}\,, \tag{1.115}$$

where

$$d\sigma^2 = h_{\underline{mn}}\, dx^m dx^n = v^m v^m\,, \qquad m,n = \sharp,1,2,3\,, \tag{1.116}$$

is the metric of a four-dimensional hyper-Kähler space where the functions $\mathcal{Z}_{0,+,-}$ and $P_{\mathfrak{i}}$ take values. The latter determine the $\mathfrak{n}$ triplets of SU(2) gauge fields through the 't Hooft ansatz. We impose that the $P_{\mathfrak{i}}$ functions are harmonic on the hyper-Kähler space so that the field strengths $F_{\mathfrak{i}}$ obey the self-duality imposed by the Killing spinor equations, namely $F_{\mathfrak{i}} = \star_\sigma F_{\mathfrak{i}}$. The remaining Killing spinor equations are automatically satisfied four our ansatz without imposing further constraints. We impose, for the sake of convenience, that the hyper-Kähler metric admits a triholomorphic isometry. In other words, we assume it is a Gibbons-Hawking space [158, 159].

The main result of the chapter is that the above configuration satisfies the α'-corrected equations of motion if the functions $\mathcal{Z}_0, \mathcal{Z}_+$ and $\mathcal{Z}_-$ are given by[22]

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} - \frac{\alpha'}{2}\left(\frac{\partial_n\mathcal{Z}_+^{(0)}\partial_n\mathcal{Z}_-^{(0)}}{\mathcal{Z}_0^{(0)}\mathcal{Z}_-^{(0)}}\right) + \mathcal{O}(\alpha'^2)\,, \tag{1.117}$$

$$\mathcal{Z}_- = \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2)\,, \tag{1.118}$$

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} - \frac{\alpha'}{4}\left[\sum_{\mathfrak{i}=1}^{2}\frac{(\partial P_{\mathfrak{i}})^2}{P_{\mathfrak{i}}^2} - \frac{(\partial\mathcal{Z}_0^{(0)})^2}{\mathcal{Z}_0^{(0)\,2}} - \frac{(\partial\mathcal{H})^2}{\mathcal{H}^2}\right] + \mathcal{O}(\alpha'^2)\,, \tag{1.119}$$

where $\mathcal{Z}_0^{(0)}, \mathcal{Z}_0^{(0)}$ and $\mathcal{Z}_-^{(0)}$ are three harmonic functions that determine the zeroth order (supergravity) solution and $\mathcal{H}$ is the Gibbons-Hawking function that characterizes the hyper-Kähler metric.

- In Chapter 5, based on [140], we apply the results of the previous chapter to study the higher-derivative corrections to supersymmetric black-hole solutions of the heterotic

[22] The functions $\mathcal{Z}_{0,+,-}$ are not completely determined since we have the freedom of adding a harmonic function to them. This will be done in order to remove the spurious singularities that appear in the correction terms, and which are solely due to the use of a singular gauge to compute them. We will discuss this issue in more detail in chapters 4 and 5.

theory on a torus. We analyze two cases: three-charge black holes in five dimensions and four-charge black holes in four dimensions. The most relevant aspect is that the higher-derivative corrections play the rôle of effective sources of energy, momentum and charge in the equations of motion. As a consequence, the Maxwell (or asymptotic) S5- and momentum charges —which we shall denote as $\mathcal{Q}_{\text{S5}}$ and $\mathcal{Q}_{\text{P}}$, respectively—, no longer coincide with the number of S5-branes, N, and KK momentum, n. This distinction results being crucial for comparing the black-hole entropy with earlier results obtained for the degeneracy of BPS microstates.

Let us then summarize the expressions that we have obtained for the mass M and the black-hole (Wald) entropy S_{W} both in terms of the "source parameters" and the Maxwell charges:

– *3-charge black holes:*

$$\begin{aligned} M &= \frac{R_z}{g_s^2\ell_s^2}(N+\mathfrak{n}-1)+\frac{n}{R_z}\left(1+\frac{2}{N}\right)+\frac{R_z}{\ell_s^2}w \\ &= \frac{R_z}{g_s^2\ell_s^2}\mathcal{Q}_{\text{S5}}+\frac{\mathcal{Q}_{\text{P}}}{R_z}+\frac{R_z}{\ell_s^2}\mathcal{Q}_{\text{F1}}\,, \end{aligned} \tag{1.120}$$

$$\begin{aligned} S_{\text{W}} &= 2\pi\sqrt{nwN}\left(1+\frac{2}{N}\right) \\ &= 2\pi\sqrt{\mathcal{Q}_{\text{F1}}\mathcal{Q}_{\text{P}}\left(\mathcal{Q}_{\text{S5}}+3-\mathfrak{n}\right)}\,, \end{aligned} \tag{1.121}$$

where

$$\mathcal{Q}_{\text{P}}=n\left(1+\frac{2}{N}\right)\,,\qquad \mathcal{Q}_{\text{F1}}=w\,,\qquad \mathcal{Q}_{\text{S5}}=N+\mathfrak{n}-1\,, \tag{1.122}$$

w being the winding number of the fundamental string and R_z the radius of the wrapped circle.

– *4-charge black holes:*

$$\begin{aligned} M &= \frac{R_z}{g_s^2\ell_s^2}\left(N+\frac{\mathfrak{n}-2}{W}\right)+\frac{n}{R_z}\left(1+\frac{2}{NW}\right)+\frac{R_z}{\ell_s^2}w+\frac{R_zR_\eta^2}{g_s^2\alpha'^2}W \\ &= \frac{R_z}{g_s^2\ell_s^2}\mathcal{Q}_{\text{S5}}+\frac{\mathcal{Q}_{\text{P}}}{R_z}+\frac{R_z}{\ell_s^2}\mathcal{Q}_{\text{F1}}+\frac{R_zR_\eta^2}{g_s^2\alpha'^2}\mathcal{Q}_{\text{KK}}\,, \end{aligned} \tag{1.123}$$

$$\begin{aligned} S_{\text{W}} &= 2\pi\sqrt{nwNW}\left(1+\frac{2}{NW}\right) \\ &= 2\pi\sqrt{\mathcal{Q}_{\text{F1}}\,\mathcal{Q}_{\text{P}}\,\left(\mathcal{Q}_{\text{S5}}\,\mathcal{Q}_{\text{KK}}+4-\mathfrak{n}\right)}, \end{aligned} \tag{1.124}$$

where

$$\mathcal{Q}_{\rm P} = n\left(1+\frac{2}{NW}\right)\,, \qquad \mathcal{Q}_{\rm F1} = w\,, \qquad \mathcal{Q}_{\rm S5} = N + \frac{\mathfrak{n}-2}{W}\,, \qquad \mathcal{Q}_{\rm KK} = W\,, \tag{1.125}$$

W being the topological charge of the KK monopole and R_η the asymptotic radius associated to the isometric direction of the Gibbons-Hawking space.

- In Chapter 6, based on [141], we argue that a horizon resolution of small black holes via higher-derivative corrections does not actually occur at least at first order in α'. First, we show that the corrected ten-dimensional metric has a curvature singularity at the would-be horizon —see Eq. (6.11)—, which further implies that the solution cannot be trusted near the singularity, where the perturbative approach ceases to be justified. Second, we argue that, from our perspective, the horizon resolution of the four-dimensional small black hole previously found in the literature actually corresponds to a particular extremal black hole with four near-horizon (or brane-source) charges whose asymptotic (or Maxwell) S5-brane charge vanishes, $\mathcal{Q}_{\rm S5} = 0$, hence matching the microscopic degeneracy of the Dabholkar-Harvey states[23]

$$S_{\rm micro} = 4\pi\sqrt{nw}\,. \tag{1.126}$$

These, however, correspond to the BPS states of a fundamental string with winding and momentum charges, whereas the macroscopic configuration also contains S5-branes and a KK monopole. Therefore, the system described be these "fake" small black holes is not the F1-P system. This can be easily seen from the fact that the vanishing of the S5-charge implies that

$$\mathcal{Q}_{\rm S5} = 0\,, \qquad \overbrace{\Rightarrow}^{(\mathfrak{n}=0)} \qquad NW = 2\,, \tag{1.127}$$

so that both N and W must be non-vanishing. In fact, we check that, whenever these (or any other brane-source charge) are set to zero, a curvature singularity develops at the horizon, putting the solution out of perturbative control.

This leads us to the conclusion that an effective black-hole description of the F1-P system seems unlikely to occur perturbatively in α', even including the higher-order terms that we have not take into account in our analysis.

- In Chapter 7, based on [142], we construct a general family of α'-corrected solutions describing bound states of the rotating F1-P system. We study a particular solution within this family that gives raise to small black rings in five dimensions. We show that both the five- and ten-dimensional metrics have a curvature singularity so that the solution does not get regularized at first order in α'. Static solutions giving rise to small black holes in $4 \leq d \leq 9$ dimensions are also discussed. The results of this chapter lend further support to those of Chapter 6.

- In Chapter 8, based on [143], we construct an effective field theory in four dimensions that parametrizes the most general correction to any vacuum solution of general

[23] If one also identifies the quantized momentum n with the momentum charge $\mathcal{Q}_{\rm P}$.

relativity when the Einstein-Hilbert term is supplemented with higher-curvature terms up to cubic order in the curvature, considering also the possibility of having dynamical couplings controlled by massless scalar fields. The action of such effective theory is

$$S = \int d^4x\sqrt{|g|}\Big\{R + \alpha_1\phi_1\ell^2\mathcal{X}_4 + \alpha_2\left(\phi_2\cos\theta_m + \phi_1\sin\theta_m\right)\ell^2 R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} \\ + \lambda_{\rm ev}\ell^4 R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}R_{\delta\gamma}{}^{\mu\nu} + \lambda_{\rm odd}\ell^4 R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}\tilde{R}_{\delta\gamma}{}^{\mu\nu} - \sum_{i=1}^{2}\frac{1}{2}(\partial\phi_i)^2\Big\}, \tag{1.128}$$

where

$$\mathcal{X}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{1.129}$$

is the Gauss-Bonnet density and ϕ_1, ϕ_2 are scalar fields.

Besides the overall length scale ℓ, there are only five parameters in the theory: α_1, α_2, $\lambda_{\rm ev}$, $\lambda_{\rm odd}$ and θ_m. The parameter $\lambda_{\rm odd}$ violates parity, while the "mixing angle" θ_m represents as well a sort of parity breaking phase. For $\theta_m = 0, \pi$ (no mixing between scalars), ϕ_2 is actually a pseudoscalar and the quadratic sector is parity-invariant. For any other value ($\theta_m \neq 0, \pi$), parity is also violated by this sector.

We investigate the corrections to the Kerr solution predicted by this theory. We work perturbatively in ℓ (the scale associated to the higher-derivative terms) and in the spin parameter χ, although we provide an algorithmic method implemented in Mathematica[24] that computes the solution at the desired order in the spin parameter, which gives a reasonably good approximation for high values of the spin. We study some properties of the corrected solutions such as the surface gravity, the shape of the horizon and ergosphere, photon rings, etc.

[24] https://arxiv.org/src/1901.01315v3/anc.

Part I

Non-Abelian supersymmetric solutions from gauged supergravity

2

Non-Abelian supersymmetric black holes

The study of classical solutions of general relativity and its supersymmetric extensions, supergravity theories, has been one of the major sources of information about the properties of these theories. The main interest of supergravity theories is that many of them describe the low-energy dynamics of the different superstring theories. For this reason, a great deal of work has been devoted to them and their classical solutions[1], specially to those describing (supersymmetric) black holes. The major discovery in this context —and one of the most important breakthroughs of string theory— was achieved by Strominger and Vafa [67], who were able to reproduce the Bekenstein-Hawking entropy of a certain class of supersymmetric black holes by counting the degeneracy of BPS states of a system of D-branes wrapped on internal cycles.

A generic feature of supergravity theories is the presence of vector and scalar fields that give rise to many interesting phenomena and modify the properties of black-hole solutions. The electric and magnetic charges associated to those vectors play a crucial rôle in the stringy interpretation of the black holes that carry them. However, most part of the literature has only dealt with Abelian vector fields even though non-Abelian (Yang-Mills) vector fields play a more relevant rôle in our current description of Nature. Furthermore, most string models, specially the most realistic ones, contain them in their spectra. Thus, it is clearly important to study their interplay between gravity and Yang-Mills fields in this context and to understand how the results obtained in the Abelian case are modified by the presence of the latter.

During the last decade, there has been progress in studying and classifying the supersymmetric solutions of $\mathcal{N} = 1, d = 5$ and $\mathcal{N} = 2, d = 4$ gauged supergravities [162–164]. This has allowed the development of solution-generating techniques which have been used to construct a plethora of analytic solutions with non-trivial Yang-Mills fields [137, 149–154, 157, 165, 166], most of them describing supersymmetric black holes.

In this chapter, we are going to construct novel supersymmetric solutions of a particular model of $\mathcal{N} = 1, d = 5$ gauged supergravity which can be obtained by dimensional reduction of heterotic supergravity[2] [106] on a five-dimensional torus. A summary of the contents of this chapter is the following. In Section 2.1 we review the class of theories of $\mathcal{N} = 1, d = 5$ gauged supergravity we are going to work with as well as their (timelike) supersymmetric solutions. We show how the task of constructing supersymmetric solutions boils down, under certain assumptions, to solving a set of differential equations on $\mathbb{E}^3$. In Section 2.2, we study solutions to the differential equations that arise in the non-Abelian sector of the theory when the gauge group is SU(2). This is then used to construct ro-

[1] See e.g. Refs. [111, 160, 161]

[2] More precisely, ten-dimensional $\mathcal{N} = 1$ supergravity coupled to a triplet of SU(2) fields.

tating black holes in five and four dimensions in Sections 2.3 and 2.4 respectively. These solutions are then uplifted to ten dimensions in Section 2.5. Finally, Section 2.6 contains a summary and a brief discussion of the main results.

2.1 $\mathcal{N}=1, d=5$ Super-Einstein-Yang-Mills theories

The aim of this section is, in first place, to give a brief description of the class of theories we are going to work with, which are called $\mathcal{N}=1, d=5$ Super-Einstein-Yang-Mills (SEYM) theories, and, in second place, to explain a solution-generating technique that we will put in practice to construct supersymmetric black-hole solutions. Our conventions will be those of [162,163], which are based on [167]. We shall describe $\mathcal{N}=1, d=5$ SEYM theories as the result of gauging a subgroup of the isometry group of the scalar manifold of a $\mathcal{N}=1, d=5$ supergravity theory coupled to n_v vector multiplets, whose field content is the following:

- The supergravity multiplet is constituted by the graviton $e^a{}_\mu$, the graviphoton $A^0{}_\mu$ and the gravitino $\psi^i{}_\mu$.
- Each of the n_v vector multiplets —that we label with an index $x=1,\ldots,n_v$— contains a real scalar ϕ^x, a vector $A^x{}_\mu$ and a gaugino λ^{ix}.[3]

In order to describe these theories, it is highly convenient to combine all the vectors into a single object, $A^I{}_\mu = \left(A^0_\mu, A^x{}_\mu\right)$, as well as to introduce n_v+1 functions of the (physical) scalars, $h^I = h^I(\phi)$. These n_v+1 functions of the n_v scalars must satisfy a constraint, which $\mathcal{N}=1, d=5$ supersymmetry determines to be of the form

$$C_{IJK}h^I h^J h^K = 1, \tag{2.1}$$

where C_{IJK} is a constant symmetric tensor which completely characterizes the theory and the real special geometry of the scalar manifold.[4] In particular, the kinetic matrix of the vector fields, $a_{IJ}(\phi)$, and the metric of the scalar manifold, $\mathfrak{g}_{xy}(\phi)$, can be derived from it as follows. First, we define

$$h_I \equiv C_{IJK}h^J h^K, \qquad \Rightarrow \qquad h^I h_I = 1, \tag{2.2}$$

and

$$h^I_x \equiv -\sqrt{3}h^I{}_{,x} \equiv -\sqrt{3}\frac{\partial h^I}{\partial \phi^x}, \qquad h_{Ix} \equiv +\sqrt{3}h_{I,x}, \quad \Rightarrow \quad h_I h^I_x = h^I h_{Ix} = 0. \tag{2.3}$$

Then, a_{IJ} is defined implicitly by the relations

$$h_I = a_{IJ}h^I, \qquad h_{Ix} = a_{IJ}h^J{}_x. \tag{2.4}$$

It can be checked that

[3] The spinors are simplectic Majorana spinors and carry a fundamental SU(2) R-symmetry index.
[4] For more details about real special geometry, see for instance Appendix H of [111].

$$a_{IJ} = -2C_{IJK}h^K + 3h_I h_J. \tag{2.5}$$

The metric of the scalar manifold, $\mathfrak{g}_{xy}$, which is used to raise and lower x, y indices is proportional to the pullback of a_{IJ} onto the scalar manifold

$$\mathfrak{g}_{xy} \equiv a_{IJ}h^I{}_x h^J{}_y = -2C_{IJK}h^I_x h^J_y h^K. \tag{2.6}$$

The functions h^I and their derivatives h^I_x satisfy the following completeness relation:

$$a_{IJ} = h_I h_J + \mathfrak{g}_{xy}h^x_I h^y_J. \tag{2.7}$$

Generically, an ungauged theory of $\mathcal{N} = 1, d = 5$ supergravity coupled to vector multiplets will be invariant under certain group of symmetries acting only on the vector and scalar fields.[5] The action of these symmetries on the scalars has to preserve $\mathfrak{g}_{xy}(\phi)$, the metric of the scalar manifold and, therefore, it will act on them as the isometries generated by the Killing vectors $k_I{}^x(\phi)$,[6]

$$\delta\phi^x = c^I k_I{}^x\,, \tag{2.8}$$

which satisfy the Lie algebra

$$[k_I, k_J] = -f_{IJ}{}^K k_K\,. \tag{2.9}$$

At the same time, because of the non-trivial couplings between scalar and vector fields, the vectors will be rotated by some given matrices. In many cases, it is possible to gauge a non-Abelian subgroup of this symmetry group using as gauge fields a subset of the vector fields of the theory. In the gauging procedure, the constant parameters c^I are promoted to arbitrary spacetime functions $c^I \to -g\epsilon^I(x)$. The gauge transformations of the scalars ϕ^x, the functions h^I and the vector fields A^I are the following:

$$\delta_\epsilon\phi^x = -g\epsilon^I k_I{}^x\,, \tag{2.10}$$

$$\delta_\epsilon h^I = -gf_{JK}{}^I\epsilon^J h^K\,, \tag{2.11}$$

$$\delta_\epsilon A^I{}_\mu = \mathfrak{D}_\mu\epsilon^I \equiv \partial_\mu\epsilon^I + gf_{JK}{}^I A^J{}_\mu\epsilon^K\,, \tag{2.12}$$

where g is the gauge coupling constant. As usual, the derivatives of the scalars are promoted to gauge-covariant derivatives, defined as

$$\mathfrak{D}_\mu\phi^x = \partial_\mu\phi^x + gA^I{}_\mu k_I{}^x\,, \tag{2.13}$$

and also

$$\mathfrak{D}_\mu h^I = \partial_\mu h^I + gf_{JK}{}^I A^J{}_\mu h^K\,, \qquad \mathfrak{D}_\mu h_I = \partial_\mu h_I + gf_{IJ}{}^K A^J{}_\mu h_K\,. \tag{2.14}$$

[5] Here we are ignoring R-symmetry.

[6] Some of these Killing vectors can vanish for some values of I.

Finally, the gauge-covariant field strength has the standard form

$$F^I{}_{\mu\nu} = 2\partial_{[\mu}A^I{}_{\nu]} + gf_{JK}{}^I A^J{}_\mu A^K{}_\nu \,, \tag{2.15}$$

Besides these modifications, gauge symmetry also demands the addition of further terms into the action —apart from the Chern-Simons terms that are already present in the ungauged theory—. However, as different from what happens in the gauging of many other supergravity theories, supersymmetry does not demand the addition of a scalar potential and no effective cosmological constant is present in the theory and its solutions.

All in all, the bosonic action of $\mathcal{N} = 1, d = 5$ SEYM —fully characterized, as we have seen, by the C_{IJK} tensor and the structure constants of the gauge group, $f_{IJ}{}^K$— is given by

$$S = \int d^5x\sqrt{|g|}\Big\{R + \frac{1}{2}\mathfrak{g}_{xy}\mathfrak{D}_\mu\phi^x\mathfrak{D}^\mu\phi^y - \frac{1}{4}a_{IJ}F^{I\,\mu\nu}F^J{}_{\mu\nu} + \frac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{|g|}}\Big[F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\alpha$$
$$-\frac{1}{2}gf_{LM}{}^I F^J{}_{\mu\nu}A^K{}_\rho A^L{}_\sigma A^M{}_\alpha + \frac{1}{10}g^2 f_{LM}{}^I f_{NP}{}^J A^K{}_\mu A^L{}_\nu A^M{}_\rho A^N{}_\sigma A^P{}_\alpha\Big]\Big\}\,. \tag{2.16}$$

The equations of motion that follow from this action are

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &\equiv \frac{1}{2\sqrt{g}}e_{a(\mu}\frac{\delta S}{\delta e_a{}^{\nu)}} \\
&= G_{\mu\nu} - \frac{1}{2}a_{IJ}\left(F^I{}_\mu{}^\rho F^J{}_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}\right) \\
&\quad + \frac{1}{2}\mathfrak{g}_{xy}\left(\mathfrak{D}_\mu\phi^x\mathfrak{D}_\nu\phi^y - \frac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho\phi^x\mathfrak{D}^\rho\phi^y\right)\,, && (2.17) \\
\mathcal{E}_I{}^\mu &\equiv \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A^I{}_\mu} \\
&= \mathfrak{D}_\nu\left(a_{IJ}F^{J\,\nu\mu}\right) + \frac{1}{4\sqrt{3}}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}C_{IJK}F^J{}_{\nu\rho}F^k{}_{\sigma\alpha} + gk_{I\,x}\mathfrak{D}^\mu\phi^x\,, && (2.18) \\
\mathcal{E}^x &\equiv -\frac{\mathfrak{g}^{xy}}{\sqrt{g}}\frac{\delta S}{\delta\phi^y} \\
&= \mathfrak{D}_\mu\mathfrak{D}^\mu\phi^x + \frac{1}{4}\mathfrak{g}^{xy}\partial_y a_{IJ}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}. && (2.19)
\end{aligned}$$

2.1.1 Timelike supersymmetric solutions

Let us now explain a recipe to construct supersymmetric solutions of $\mathcal{N} = 1, d = 5$ SEYM theories which stems from the general characterization of the supersymmetric solutions of $\mathcal{N} = 1, d = 5$ gauged supergravity carried out in [163]. Let us assume that the scalar manifold is symmetric, which in practice means that the tensor C^{IJK} that one obtains by raising the indices with the inverse of a_{IJ} is constant and identical to C_{IJK}. This implies that

$$h^I = 27C^{IJK} h_J h_K \,. \tag{2.20}$$

It was shown in [163] that supersymmetric solutions admit a Killing vector V with non-negative norm, $V_\mu V^\mu = f^2 \geq 0$. We restrict to the timelike case, $V_\mu V^\mu = f^2 > 0$ and work in coordinates adapted to the isometry so that $V = \partial_t$. Then, from the analysis of the Killing spinor equations made in [163], we know that the general form of the bosonic fields in this kind of solutions is the following

$$ds^2 = f^2 (dt + \omega)^2 - f^{-1} d\sigma^2 \,, \tag{2.21}$$

$$A^I = -27\sqrt{3} f^3 C^{IJK} Z_J Z_K (dt + \omega) + \hat{A}^I \,, \tag{2.22}$$

$$\phi^x = \frac{Z_x}{Z_0} \,, \tag{2.23}$$

where we have defined $Z_I \equiv h_I/f$. The four-dimensional metric $d\sigma^2 = h_{\underline{mn}} dx^m dx^n$ is the metric of a four-dimensional hyperKähler space —often referred to as base space— where the 1-form $\omega = \omega_{\underline{m}} dx^m$, the vector fields $\hat{A}^I = \hat{A}^I{}_{\underline{m}} dx^m$ and the scalars Z_I are defined. The metric function f can be obtained in terms of the Z_I functions by using (2.20) and the constraint (2.2):

$$f^{-3} = 27C^{IJK} Z_I Z_J Z_K \,. \tag{2.24}$$

The building blocks in terms of which the solutions are constructed ($\hat{A}^I, Z_I$ and ω) must satisfy the following set of non-linear differential equations

$$\hat{F}^I = \star_\sigma \hat{F}^I \,, \tag{2.25}$$

$$\hat{\mathfrak{D}} \star_\sigma \hat{\mathfrak{D}} Z_I = -\tfrac{1}{3} C_{IJK} \hat{F}^I \wedge \hat{F}^J \,, \tag{2.26}$$

$$d\omega + \star_\sigma d\omega = \sqrt{3} Z_I \hat{F}^I \,, \tag{2.27}$$

where $\star_\sigma$ is the Hodge dual operator associated to the metric $d\sigma^2$, $\hat{\mathfrak{D}}$ is the gauge-covariant derivative associated to $\hat{A}^I$,

$$\hat{\mathfrak{D}} Z_I = dZ_I + g f_{IJ}{}^K \hat{A}^J \wedge Z_K \,, \tag{2.28}$$

and

$$\hat{F}^I = d\hat{A}^I + \frac{g}{2} f_{JK}{}^I \hat{A}^J \wedge \hat{A}^K \,. \tag{2.29}$$

As we see, the demand of unbroken supersymmetry has drastically simplified the task of constructing solutions of the full second-order equations of motion. Still, the system (2.25)-(2.27) is hard to solve and therefore one has to make additional assumptions.

Timelike supersymmetric solutions with one isometry

In order to make further progress, let us assume that the base-space metric, $d\sigma^2$, enjoys a triholomorphic isometry, i.e. an isometry respecting the hyperKähler structure. In this case, it was shown in [159] that the metric, in adapted coordinates,[7] is of the Gibbons-Hawking type [158], namely

$$d\sigma^2 = H^{-1}\,(d\eta + \chi)^2 + H\,dx^i dx^i\,, \qquad i = 1,2,3\,, \tag{2.30}$$

where η is a compact coordinate with period $2\pi\ell$ and where H and χ are a function and a 1-form defined on $\mathbb{E}^3$ satisfying

$$dH = \star_3 d\chi\,, \tag{2.31}$$

being $\star_3$ the Hodge dual on $\mathbb{E}^3$. The integrability condition of this equation tells us that the function H must be harmonic on $\mathbb{E}^3$, namely

$$d\star_3 dH = 0\,. \tag{2.32}$$

Therefore, the choice of the base space metric has boiled down to the choice of a harmonic function on $\mathbb{E}^3$, H, and the subsequent determination of χ through (2.31).

Let us further assume that the connection $\hat{A}^I$ and the 1-form ω can be decomposed in terms of functions (Φ^I, ω_5) and 1-forms $(\breve{A}^I, \breve{\omega})$ defined on $\mathbb{E}^3$ as follows [152]

$$\hat{A}^I = -2\sqrt{6}\left[-H^{-1}\Phi^I\,(d\eta+\chi) + \breve{A}^I\right]\,, \tag{2.33}$$

$$\omega = \omega_5\,(d\eta+\chi) + \breve{\omega}\,. \tag{2.34}$$

As it was shown by Kronheimer [168], when (2.33) is substituted back into the selfduality condition (2.25), one obtains the Bogmol'nyi equations for the Yang-Mills-Higgs system in the BPS limit [169], namely

$$\breve{\mathfrak{D}}\Phi^I = \star_3 \breve{F}^I\,, \tag{2.35}$$

where

$$\breve{\mathfrak{D}}\Phi^I = d\Phi^I + \breve{g} f_{JK}{}^I \breve{A}^J \Phi^K\,, \tag{2.36}$$

$$\breve{F}^I = d\breve{A}^I + \breve{g} f_{JK}{}^I \breve{A}^J \wedge \breve{A}^K\,, \tag{2.37}$$

with $\breve{g} = -2\sqrt{6}g$. The integrability condition of (2.35) is a non-Abelian generalization of Laplace's equation

$$\breve{\mathfrak{D}}\star_3\breve{\mathfrak{D}}\Phi^I = 0\,. \tag{2.38}$$

[7] η is the coordinate adapted to the isometry.

Let us now analyze (2.26), for which it is convenient to introduce a new set of functions L_I as follows

$$L_I = Z_I - 8C_{IJK}\Phi^J\Phi^K H^{-1}\,. \tag{2.39}$$

In terms of these functions, (2.26) reads

$$\breve{\mathfrak{D}}^2 L_I - \breve{g}^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J\Phi^K L_M = 0\,. \tag{2.40}$$

Finally, (2.27) imposes that ω_5 is given by

$$\omega_5 = K + 16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K + 3\sqrt{2}H^{-1}L_I\Phi^I\,, \quad \text{where} \quad d\star_3 dK = 0\,, \tag{2.41}$$

and the following equation for $\breve{\omega}$:

$$\star_3 d\breve{\omega} = HdK - KdH + 3\sqrt{2}\left(\Phi^I\breve{\mathfrak{D}}L_I - L_I\breve{\mathfrak{D}}\Phi^I\right)\,, \tag{2.42}$$

whose integrability condition is satisfied wherever the above equations for H, K, Φ^I, L_I are satisfied.

Let us recap the main results of this section. We have studied the timelike supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM theories under the assumption that the hyperKähler metric $d\sigma^2$ admits a triholomorphic isometry. Then, the system of equations (2.25), (2.26) and (2.27) has been reduced to finding a set of functions H, Φ^I, L_I, K and 1-forms $\chi, \breve{A}^I, \breve{\omega}$ on $\mathbb{E}^3$ satisfying the following equations

$$\star_3 dH - d\chi = 0\,, \tag{2.43}$$

$$\star_3\breve{\mathfrak{D}}\Phi^I - \breve{F}^I = 0\,, \tag{2.44}$$

$$\breve{\mathfrak{D}}^2 L_I - \breve{g}^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J\Phi^K L_M = 0\,, \tag{2.45}$$

$$d\star_3 dK = 0\,, \tag{2.46}$$

$$\star_3 d\breve{\omega} - \left\{HdK - KdH + 3\sqrt{2}(\Phi^I\breve{\mathfrak{D}}L_I - L_I\breve{\mathfrak{D}}\Phi^I)\right\} = 0\,. \tag{2.47}$$

Let us notice that in the Abelian limit, Eqs. (2.43)-(2.46) tell us that all the functions that determine the solutions —namely, H, Φ^I, L_I and K— are harmonic functions on $\mathbb{E}^3$. The 1-forms $\chi, \breve{A}^I$ and $\breve{\omega}$ are then found by solving Eqs. (2.43), (2.44) and (2.47).

In the non-Abelian case, however, one has to face the task of solving Eqs. (2.44) and (2.45), for which one needs to know in first place the gauge group. In the next section, we shall assume the gauge group is SU(2).

2.2 Timelike supersymmetric solutions with SU(2) gaugings

Let us then restrict to models of $\mathcal{N} = 1, d = 5$ supergravity whose scalar manifold has an isometry group G such that SU(2) is a subgroup of G. We assume this is the subgroup that will be gauged so that the non-vanishing structure constants are given by $f_{AB}{}^{C} = \varepsilon_{AB}{}^{C}$ and (2.44) and (2.45) reduce to

$$\star_3 \breve{\mathfrak{D}}\Phi^A = \breve{F}^A \,, \tag{2.48}$$

$$\breve{\mathfrak{D}}^2 L_A = \breve{g}^2 \varepsilon_{AB}{}^{D} \varepsilon_{CD}{}^{F} \Phi^B \Phi^C L_F \,, \tag{2.49}$$

where

$$\breve{\mathfrak{D}}\Phi^A = d\Phi^A + \breve{g}\, \varepsilon_{BC}{}^{A} \breve{A}^B \Phi^C \,, \qquad \breve{\mathfrak{D}} L_A = dL_A + \breve{g}\, \varepsilon_{AB}{}^{C} \breve{A}^B L_C \,, \tag{2.50}$$

and

$$\breve{F}^A = d\breve{A}^A + \breve{g}\, \varepsilon_{BC}{}^{A} \breve{A}^B \wedge \breve{A}^C \,. \tag{2.51}$$

The main advantage of studying SU(2) gaugings is that there is a huge amount of results available in the literature, specially regarding Eq. (2.48). The goal of this section will be to review some of these results, which will be used later to construct non-Abelian black-hole solutions.

2.2.1 SU(2) Bogomol'nyi equations on $\mathbb{E}^3$

Spherically-symmetric solutions

A popular class of solutions of the SU(2) Bogomol'nyi equations (2.48) was found by Protogenov in [170] by making use of the *hedgehog ansatz*:

$$\Phi^A = -\delta^A{}_i f\,(r)\, x^i \,, \tag{2.52}$$

$$\breve{A}^A = \varepsilon^A{}_{jk}\, x^j h(r)\, dx^k \,, \tag{2.53}$$

where $r = \sqrt{x^i x^i}$ is the radial coordinate of $\mathbb{E}^3$. When this ansatz is plugged into (2.48), one obtains a system of first-order ODEs which have two independent families of solutions:

1. The first one corresponds to the so-called *colored monopole* [148, 151, 171] and depends on just one parameter that we called λ

$$f_\lambda(r) = h_\lambda(r) = -\frac{1}{\breve{g} r^2} \frac{1}{1+\lambda^2 r} \,. \tag{2.54}$$

2. The second family depends on two parameters, μ and s, and it is given by

$$f_{\mu,s}(r) = -\frac{1}{\breve{g}r^2}\left[1 - \mu r \coth(\mu r + s)\right], \quad h_{\mu,s}(r) = -\frac{1}{\breve{g}r^2}\left[1 - \frac{\mu r}{\sinh(\mu r + s)}\right]. \tag{2.55}$$

The $s = 0$ member of this family corresponds to the 't Hooft-Polyakov magnetic monopole [172, 173] in the Prasad-Sommerfeld limit [174]. In the $s \to \infty$ limit, we get

$$f_{\mu,\infty}(r) = -\frac{1}{\breve{g}r^2} + \frac{\mu}{\breve{g}r}, \quad h_{\mu,\infty} = -\frac{1}{\breve{g}r^2}, \tag{2.56}$$

a μ-dependent generalization of the Wu-Yang monopole [175]. The $\mu = 0$ case, that corresponds to the Wu-Yang monopole, can also be recovered from the colored monopole (2.54) when $\lambda = 0$.

The magnetic monopole charge is defined as

$$p = \frac{1}{4\pi}\int_{\mathbb{S}^2_\infty} \mathrm{Tr}(\hat{\Phi}\breve{F}), \quad \hat{\Phi} = \frac{\Phi}{\sqrt{|\mathrm{Tr}(\Phi^2)|}}, \tag{2.57}$$

and yields $p = 1/\breve{g}$ for the 2-parameter family and $p = 0$ for the colored monopole [148, 150].

The multi-colored monopole solution

The spherically-symmetric colored monopole solution (2.54) can be generalized to a solution without spherical symmetry, which we call the multi-colored monopole solution. It is given by

$$\breve{g}\Phi^A = -\delta^{Ai}\partial_i \log P, \tag{2.58}$$

$$\breve{g}\breve{A}^A = \varepsilon^{Aj}{}_k\, \partial_i \log P\, dx^k, \tag{2.59}$$

with P is a harmonic function on $\mathbb{E}^3$.

2.2.2 Solving Eq. (2.49)

The last stumbling block that we have is (2.49). Unlike (2.48), this equation has barely received attention in the literature, not even in the supergravity literature. The reason is that there is a very simple (almost trivial) solution which corresponds to taking $L_A \propto \Phi^A$ which in turn has allowed to study black-hole and black-ring solutions in the past, see e.g. [152, 176]. However, there are more sofisticated solutions —such as those describing the microstate geometries of non-Abelian three-charge black holes and black rings— for which this simple solution is not enough. For this kind of solutions, one can make use of the non-trivial solution to (2.49) found in [157]. The solution assumes that Φ^A and $\breve{A}^A$ are given by (2.58) and (2.59) respectively. If this is the case, then

$$\breve{g}L_A = \delta_A{}^i \frac{\partial_i Q}{P}\,, \tag{2.60}$$

solves (2.49) if Q is harmonic on $\mathbb{E}^3$.

2.3 Rotating black holes of $\mathcal{N}=1, d=5$ gauged supergravity

2.3.1 The model

Let us now apply the solution-generating technique described in the previous sections to a particular model in which we are interested, the SU(2)-gauged ST[2, 6] model of $\mathcal{N}=1, d=5$ supergravity.[8] The main motivation to study solutions of this model is that it can be obtained by a truncation of ten-dimensional $\mathcal{N}=1$ supergravity coupled to a triplet of SU(2) gauge fields, see Appendix E. The ungauged model has $n_v = 5$ vector multiplets and is characterized by a C_{IJK} tensor whose only non-vanishing components are

$$C_{0xy} = \tfrac{1}{6}\eta_{xy}\,, \quad \text{where} \quad \eta_{xy} = \text{diag}\,(+----) \quad \text{and} \quad x, y = 1, \ldots, 5\,. \tag{2.61}$$

The Real Special manifold parametrized by the five real scalars ϕ^x can be identified with the Riemannian symmetric space

$$\text{SO}(1,1) \times \frac{\text{SO}(1,4)}{\text{SO}(4)}\,. \tag{2.62}$$

The isometry group of the scalar manifold has a SU(2) subgroup acting in the adjoint on the coordinates 3, 4 and 5, which are the directions that we are going to gauge. Therefore, we find convenient to split the index that label the vector fields into a couple of indices: $I = (a, A)$, where $a = 0, 1, 2$ and $A = 3, 4, 5$ label the Abelian and non-Abelian sectors respectively.

We find convenient to perform the following field redefinitions. In first place, instead of the standard parametrization of the physical scalars given in (2.23), we are going to use the parametrization given in [177]. The new physical scalars, ϕ, k and ℓ^A, are related to those appearing in (2.23) by

$$e^{-2\phi} = \tfrac{1}{2}(\phi^1 - \phi^2)\,, \tag{2.63}$$

$$k^4 = 2\left[\frac{(\phi^1)^2 - (\phi^2)^2 - \phi_A\phi^A}{\phi^1 - \phi^2}\right]^2\,, \tag{2.64}$$

$$\ell^A = \phi^A/(\phi^1 - \phi^2)\,. \tag{2.65}$$

In second place, we introduce the following linear combinations of the Abelian vectors A^1 and A^2,

[8] The name of this model stems from the fact that it is related by dimensional reduction on a circle to the so-called SU(2)-gauged ST[2, 6] model of $\mathcal{N}=2, d=4$ supergravity [152].

$$A^{\pm} \equiv A^1 \pm A^2 \,, \tag{2.66}$$

and

$$Z_{\pm} = Z_1 \pm Z_2 \,, \quad L_{\pm} = L_1 \pm L_2 \quad \text{and} \quad \Phi^{\pm} = \Phi^1 \pm \Phi^2 \,. \tag{2.67}$$

Given this, the general form of the timelike supersymmetric solutions (2.21)-(2.23) reduces, for the model at hands, to

$$ds^2 = f^2 (dt + \omega)^2 - f^{-1} d\sigma^2 \,, \tag{2.68}$$

$$A^0 = -\frac{1}{\sqrt{3} Z_0} (dt + \omega) + \hat{A}^0 \,, \tag{2.69}$$

$$A^{\pm} = -\frac{2Z_+}{\sqrt{3} Z_{\pm} \tilde{Z}_+} (dt + \omega) + \hat{A}^{\pm} \,, \tag{2.70}$$

$$A^A = \frac{2Z_A}{\sqrt{3} Z_- \tilde{Z}_+} (dt + \omega) + \hat{A}^A \,, \tag{2.71}$$

$$e^{2\phi} = \frac{2Z_0}{Z_-} \,, \qquad k = \left(\frac{2\tilde{Z}_+^2}{Z_0 Z_-} \right)^{1/4} \,, \qquad \ell^A = \frac{Z_A}{Z_-} \,. \tag{2.72}$$

where

$$f^{-3} = \frac{27}{2} Z_0 \tilde{Z}_+ Z_- \,, \qquad \text{and} \qquad \tilde{Z}_+ = Z_+ - \frac{Z_A Z_A}{Z_-} \,. \tag{2.73}$$

2.3.2 Seed functions

Let us make a suitable choice of the seed functions H, Φ^I, L_I, K to describe rotating black holes.

Gibbons-Hawking metric

First of all, we take the base space metric to be simply the four-dimensional Euclidean metric, $h_{\underline{mn}} = \delta_{\underline{mn}}$, which corresponds with the following choice of Gibbons-Hawking function

$$H = \frac{\ell}{2r} \,, \qquad \Rightarrow \qquad \chi = \frac{\ell}{2} \cos\theta \, d\phi \,, \tag{2.74}$$

where ℓ is a length scale and where we have introduced the spherical coordinates (r, θ, ϕ), defined as

$$x^1 = r \sin\theta \cos\phi \,, \qquad x^2 = r \sin\theta \sin\phi \,, \qquad x^3 = r \cos\theta \,. \tag{2.75}$$

It is not difficult to see that introducing a new radial coordinate, $\rho^2 = 2\ell r$, and an angular coordinate, $\psi = 2\eta/\ell$[9], the Gibbons-Hawking metric (2.30) can be rewritten as

$$d\sigma^2 = d\rho^2 + \rho^2 d\Omega^2_{(3)}\,, \tag{2.76}$$

where

$$d\Omega^2_{(3)} = \frac{1}{4}\left(d\psi^2 + d\phi^2 + d\theta^2 + 2\cos\theta d\psi d\phi\right)\,, \tag{2.77}$$

is the metric of the round 3-sphere, $\mathbb{S}^3$.

Φ^I and L_I functions

Let us continue with the choice of the Φ^I and L_I functions, which will determine the form of the vector fields and the metric function f. As already discussed, in the Abelian sector these functions must be harmonic on $\mathbb{E}^3$. Our choice is

$$L_{0,+,-} = a_{0,+,-} + \frac{b_{0,+,-}}{r} = a_{0,+,-} + \frac{2\ell b_{0,+,-}}{\rho^2}\,, \tag{2.78}$$

$$\Phi^{0,+,-} = 0\,. \tag{2.79}$$

It follows then from (2.44) that $\breve{F}^{0,+,-} = 0$ so that we can always work in the gauge in which $\breve{A}^{0,+,-}$ vanish, which in turn implies that also $\hat{A}^{0,+,-} = 0$.

In the non-Abelian sector, out of all the solutions to the SU(2) Bogomol'nyi equations discussed in Section 2.2.1, we are going to select the spherically-symmetric colored monopole, namely

$$\Phi^A = \frac{\delta^{A-2}{}_i\, x^i}{\breve{g} r^2\,(1+\lambda^2 r)}\,, \tag{2.80}$$

$$\breve{A}^A = -\varepsilon^{A-2}{}_{jk}\,\frac{x^j}{\breve{g} r^2\,(1+\lambda^2 r)}\,dx^k\,. \tag{2.81}$$

The hatted connection $\hat{A}^A$ is found by using (2.33), and corresponds to the well-known BPST instanton [178],

$$g\hat{A}^A = -\frac{\kappa^2}{\rho^2+\kappa^2}\,v^{A-2}\,, \tag{2.82}$$

where $\kappa^2 \equiv 2\ell\lambda^{-2}$ and the triplet of 1-forms v^{A-2} are a set of Maurer-Cartan 1-forms:

$$v^1 = -\sin\phi\, d\theta + \sin\theta\cos\phi\, d\psi\,,\ v^2 = \cos\phi\, d\theta + \sin\theta\sin\phi\, d\psi\,,\ v^3 = d\phi + \cos\theta\, d\psi\,. \tag{2.83}$$

[9] $\psi \sim \psi + 4\pi$.

The main advantage of having selected a member of the colored monopole family is that we can now make use of the solution to (2.49) given by (2.60). The harmonic function that determines it, Q, is chosen to be

$$Q = -2\sqrt{6}\xi x^3 \,, \tag{2.84}$$

which implies (the function P is given by $P = 1 + \lambda^{-2}/r$)

$$L_3 = L_4 = 0 \,, \qquad L_5 = \frac{\xi \lambda^2 r}{g\,(1+\lambda^2 r)} = \frac{\xi \rho^2}{g\,(\rho^2+\kappa^2)} \,. \tag{2.85}$$

Once the functions Φ^I and L_I have been chosen, we can find the functions Z_I by using (2.39). We get

$$\begin{aligned} Z_0 &= L_0 - \frac{4}{3}\frac{\Phi^A\Phi^A}{H} = a_0\left[1 + \frac{\mathcal{Q}_0}{\rho^2} + \frac{2}{9a_0 g^2}\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2}\right] \,, & (2.86)\\ Z_\pm &= L_\pm = a_\pm\left[1+\frac{\mathcal{Q}_\pm}{\rho^2}\right] \,, & (2.87)\\ Z_A &= L_A = \delta_A{}^3 \frac{\xi\rho^2}{g\,(\rho^2+\kappa^2)} \,, & (2.88)\\ \tilde{Z}_+ &= \tilde{a}_+\left[1+\frac{\tilde{\mathcal{Q}}_+}{\rho^2} + \frac{\xi^2}{\tilde{a}_+ a_- g^2}\frac{\left(\mathcal{Q}_- + 2\kappa^2\right)\rho^4 + \left(2\mathcal{Q}_- + \kappa^2\right)\kappa^2\rho^2 + \mathcal{Q}_-\kappa^4}{\left(\rho^2+\kappa^2\right)^2\left(\rho^2+\mathcal{Q}_-\right)}\right] \,, & (2.89) \end{aligned}$$

where we have defined

$$\mathcal{Q}_0 = \frac{2\ell b_0}{a_0} - \frac{2}{9g^2 a_0} \,, \qquad \mathcal{Q}_\pm = \frac{2\ell b_\pm}{a_\pm} \,, \qquad \tilde{\mathcal{Q}}_+ = \frac{2\ell b_+}{\tilde{a}_+} \,, \tag{2.90}$$

and

$$\tilde{a}_+ = a_+ - \frac{\xi^2}{a_- g^2} \,. \tag{2.91}$$

1-form ω

The final step to determine completely the solution is to figure out what ω is. This is done in two steps. In first place, we have to make a choice for the harmonic function K, which will allow us to find ω_5 through (2.41). Our choice is

$$K = \frac{\mathcal{Q}_K}{2\ell^2 r} = \frac{\mathcal{Q}_K}{\ell\rho^2} \,. \tag{2.92}$$

Then, using (2.41), we find

$$\omega_5 = K + \sqrt{3}2H^{-1}L_A\Phi^A = \frac{2}{\ell}\left[\frac{\mathcal{Q}_K}{2\rho^2} - \frac{\sqrt{3}\xi\kappa^2\rho^2\cos\theta}{2g^2\,(\rho^2+\kappa^2)^2}\right] \,. \tag{2.93}$$

The second step consists of solving (2.47) to find the 1-form $\breve{\omega}$. After a bit of algebra, one finds

$$\breve{\omega} = -\frac{\sqrt{3}\xi\kappa^2\rho^2\sin^2\theta}{2g^2\left(\rho^2+\kappa^2\right)^2}\,d\phi\,. \tag{2.94}$$

Finally, from (2.34), we have that ω is given by

$$\omega = \frac{\mathcal{Q}_K}{2\rho^2}\left(d\psi+\cos\theta d\phi\right) - \frac{\sqrt{3}\xi\kappa^2\rho^2}{2g^2\left(\rho^2+\kappa^2\right)^2}\left(d\phi+\cos\theta d\psi\right)\,. \tag{2.95}$$

2.3.3 The solutions

Before writing down the full solution, let us rewrite the parameters $a_0, \tilde{a}_+, a_-$ and ξ in terms of the asymptotic values of the scalars, ϕ_∞, k_∞ and $\ell_\infty \equiv \lim_{\rho\to\infty}\sqrt{\ell^A\ell^A}$ by fixing the asymptotic value of the metric function to $f_\infty = 1$, as customary. We obtain that

$$a_0 = \frac{1}{3}e^{\phi_\infty}k_\infty^{-2/3}\,, \qquad \tilde{a}_+ = \frac{1}{3}k_\infty^{4/3}\,, \qquad a_- = \frac{2}{3}e^{-\phi_\infty}k_\infty^{-2/3}\,, \qquad \frac{\xi}{g} = \frac{2}{3}e^{-\phi_\infty}k_\infty^{-2/3}\ell_\infty\,. \tag{2.96}$$

Making use of these relations to rewrite the five-dimensional fields, given in (2.68)-(2.72), we get

$$ds^2 = \left(\mathcal{Z}_0\tilde{\mathcal{Z}}_+\mathcal{Z}_-\right)^{-\frac{2}{3}}\left(dt+\omega\right)^2 - \left(\mathcal{Z}_0\tilde{\mathcal{Z}}_+\mathcal{Z}_-\right)^{\frac{1}{3}}\left[d\rho^2+\rho^2 d\Omega^2_{(3)}\right]\,, \tag{2.97}$$

$$A^0 = -\sqrt{3}\,e^{-\phi_\infty}k_\infty^{2/3}\mathcal{Z}_0^{-1}\left(dt+\omega\right)\,, \tag{2.98}$$

$$A^+ = -2\sqrt{3}\,k_\infty^{-4/3}\tilde{\mathcal{Z}}_+^{-1}\left(dt+\omega\right)\,, \tag{2.99}$$

$$A^- = -\sqrt{3}\,e^{\phi_\infty}k_\infty^{2/3}\left(1+\frac{2\ell_\infty^2}{e^{\phi_\infty}k_\infty^2}\right)\frac{\mathcal{Z}_+}{\mathcal{Z}_-\tilde{\mathcal{Z}}_+}\left(dt+\omega\right)\,, \tag{2.100}$$

$$A^A = \frac{2\sqrt{3}\,\ell_\infty}{k_\infty^{4/3}}\frac{\mathcal{Z}_A}{\tilde{\mathcal{Z}}_+\mathcal{Z}_-}\left(dt+\omega\right) - \frac{\kappa^2}{\rho^2+\kappa^2}\,v^{A-2}\,, \tag{2.101}$$

$$e^{2\phi} = e^{2\phi_\infty}\frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \qquad \frac{k}{k_\infty} = \left(\frac{\tilde{\mathcal{Z}}_+^2}{\mathcal{Z}_0\mathcal{Z}_-}\right)^4\,, \qquad \frac{\ell^A}{\ell_\infty} = \frac{\mathcal{Z}_A}{\mathcal{Z}_-}\,, \tag{2.102}$$

with

$$\omega = \frac{\mathcal{Q}_K}{2\rho^2}\left(d\psi+\cos\theta d\phi\right) - \frac{\ell_\infty\kappa^2\rho^2}{\sqrt{3}e^{\phi_\infty}k_\infty^{2/3}g\left(\rho^2+\kappa^2\right)^2}\left(d\phi+\cos\theta d\psi\right)\,, \tag{2.103}$$

and

$$\mathcal{Z}_0 = 1+\frac{\mathcal{Q}_0}{\rho^2}+\frac{2e^{-\phi_\infty}k_\infty^{2/3}}{3g^2}\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2}\,, \qquad \mathcal{Z}_\pm = 1+\frac{\mathcal{Q}_\pm}{\rho^2}\,, \tag{2.104}$$

$$\mathcal{Z}_3 = \mathcal{Z}_4=0\,, \qquad \mathcal{Z}_5=\frac{\rho^2}{\rho^2+\kappa^2}\,, \tag{2.105}$$

$$\tilde{\mathcal{Z}}_+ = 1+\frac{\tilde{\mathcal{Q}}_+}{\rho^2}+\frac{2\ell_\infty^2}{e^{\phi_\infty}k_\infty^2}\frac{\left(\mathcal{Q}_-+2\kappa^2\right)\rho^4+\left(2\mathcal{Q}_-+\kappa^2\right)\kappa^2\rho^2+\mathcal{Q}_-\kappa^4}{(\rho^2+\kappa^2)^2\,(\rho^2+\mathcal{Q}_-)}\,. \tag{2.106}$$

Let us analyze the solution. The first we can already observe is that when the non-Abelian fields are removed (setting $\kappa=\ell_\infty=0$, for instance), the solution reduces to the well-known extremal three-charge black holes of $\mathcal{N}=1, d=5$ ungauge supergravity. These were first obtained in [146] as the BPS limit of a general family of non-extremal rotating black-hole solutions of the heterotic effective action compactified on $\mathbb{T}^5$. Hence, our solutions describe the distorsion on these black holes caused by the presence of the SU(2) Yang-Mills vector and scalar fields, which describe a dyonic deformation of the BPST instanton that is qualitatively identical to the one considered in [144, 145]. The parameter that characterizes the dyonic deformation, ξ (or ℓ_∞), also characterizes the breaking of the SU(2) gauge symmetry in our solution.

Physical properties

The solution is characterized by nine parameters:

- four of them, $\phi_\infty, k_\infty, \ell_\infty, g$, are moduli parameters (asymptotic values of the scalars and gauge coupling constant).
- $\mathcal{Q}_0, \tilde{\mathcal{Q}}_+, \mathcal{Q}_-$ are expected to be related to the electric charges of the solution, as it occurs when the non-Abelian fields are removed. However, we must take into account that the non-Abelian fields introduce delocalized sources of electric charge that can contribute to the different notions of charge that one can define when Chern-Simons terms are present in the action [179, 180]. We will come back to this issue.
- $\mathcal{Q}_K$ is related, as we are going to see, to the sum of the two angular momenta of the solution.
- and, finally, κ is the parameter that characterizes the "size" of the dyonic BPST instanton. It was shown in Ref. [144, 145] to be related to its electric charge by

$$\mathcal{Q}_{\text{dyon}} \sim \frac{\xi\kappa^2}{g}\,. \tag{2.107}$$

This is quite interesting, as it tells us that the size of the instanton does not have the interpretation of non-Abelian hair, as it happens when $\xi=0$, see e.g. [152, 153, 157, 166, 176]. In fact, according to [144], the electric charge is what supports this configuration with broken SU(2) symmetry from collapse.

Horizon. The metric of the solution (2.97) has a regular horizon at $\rho = 0$. The induced metric at $\rho = 0$ is that of a squashed 3-sphere

$$-ds_H^2 = \frac{R_H^2}{4}\left[\frac{1}{1+\beta}\left(d\psi^2 + \cos\theta d\phi\right)^2 + d\theta^2 + \sin^2\theta d\phi^2\right], \tag{2.108}$$

with radius and squashing parameter given by

$$R_H = \left(\mathcal{Q}_0\tilde{\mathcal{Q}}_+\mathcal{Q}_-\right)^{1/6}, \qquad \beta = \frac{\mathcal{Q}_K^2}{\mathcal{Q}_0\tilde{\mathcal{Q}}_+\mathcal{Q}_- - \mathcal{Q}_K^2}. \tag{2.109}$$

The Bekenstein-Hawking entropy is given by

$$S_{\text{BH}} = \frac{A_H}{4G_{\text{N}}^{(5)}} = \frac{\pi^2}{2G_{\text{N}}^{(5)}}\sqrt{\mathcal{Q}_0\tilde{\mathcal{Q}}_+\mathcal{Q}_- - \mathcal{Q}_K^2}. \tag{2.110}$$

Mass and angular momenta. The mass and the angular momenta of the solutions can be easily found by first computing the large ρ-expansion of the metric (2.97) and then comparing the result with the metric of the Myers-Perry black hole [181]. To this aim, it is convenient to introduce a new set of coordinates $(t, \varrho, \Theta, \varphi_+, \varphi_-)$, defined as

$$\varrho = \rho\left(\mathcal{Z}_0\tilde{\mathcal{Z}}_+\mathcal{Z}_-\right)^{1/3}, \qquad \Theta = \frac{\theta}{2}, \qquad \varphi_\pm = \frac{\psi\pm\phi}{2}, \tag{2.111}$$

in terms of which the large ρ-expansion of the metric (2.97) reads

$$\begin{aligned} ds^2 \sim &\left(1 - \frac{8G_{\text{N}}^{(5)}M}{3\pi\varrho^2}\right)dt^2 + \frac{8G_{\text{N}}^{(5)}J_+}{\pi\varrho^2}\cos^2\Theta\, dt\, d\varphi_+ + \frac{8G_{\text{N}}^{(5)}J_-}{\pi\varrho^2}\sin^2\Theta\, dt\, d\varphi_- \\ &- \left(1 + \frac{8G_{\text{N}}^{(5)}M}{3\pi\varrho^2}\right)d\varrho^2 - \varrho^2\left(d\Theta^2 + \cos^2\Theta\, d\varphi_+^2 + \sin^2\Theta\, d\varphi_-^2\right), \end{aligned} \tag{2.112}$$

where M is the ADM mass of the solution

$$M = \frac{\pi}{4G_{\text{N}}^{(5)}}\left(\mathcal{Q}_0^\infty + \tilde{\mathcal{Q}}_+^\infty + \mathcal{Q}_-^\infty\right), \tag{2.113}$$

with

$$\mathcal{Q}_0^\infty \equiv \lim_{\rho\to\infty}\rho^2\left(\mathcal{Z}_0 - 1\right) = \mathcal{Q}_0 + \frac{2e^{-\phi_\infty}k_\infty^{2/3}}{3g^2}, \tag{2.114}$$

$$\tilde{\mathcal{Q}}_+^\infty \equiv \lim_{\rho\to\infty}\rho^2\left(\tilde{\mathcal{Z}}_+ - 1\right) = \tilde{\mathcal{Q}}_+ + \frac{2\ell_\infty^2\left(\mathcal{Q}_- + 2\kappa^2\right)}{e^{\phi_\infty}k_\infty^2}, \tag{2.115}$$

$$\mathcal{Q}_-^\infty \equiv \lim_{\rho\to\infty}\rho^2\left(\mathcal{Z}_- - 1\right) = \mathcal{Q}_-, \tag{2.116}$$

and $J_\pm$ are the two independent angular momenta of the solution

$$J_{\pm} = \frac{\pi}{4G_{\rm N}^{(5)}} \left(\mathcal{Q}_K \mp \frac{\sqrt{3}\kappa^2 \xi}{g^2} \right) . \tag{2.117}$$

Inverting the last expression, we find that $\mathcal{Q}_K$ is given by the sum of the two angular momenta

$$\mathcal{Q}_K = \frac{2G_{\rm N}^{(5)}}{\pi} \left(J_+ + J_- \right) , \tag{2.118}$$

and that the instanton size, κ, gets fixed by its diference

$$\kappa^2 = \frac{2g^2 G_{\rm N}^{(5)}}{\sqrt{3}\pi\xi} \left(J_- - J_+ \right) , \qquad \mathcal{Q}_{\rm dyon} \sim \frac{2g^2 G_{\rm N}^{(5)}}{\sqrt{3}\pi} \left(J_- - J_+ \right) . \tag{2.119}$$

Regularity of the solution. The presence of closed timelike curves (CTCs) is a quite common feature of this kind of metrics. The condition that guarantees the spacetime is free of closed timelike curves is studied in Appendix D, see Eq. (D.36). For the case at hands, it reduces to

$$\mathcal{Z}_0 \tilde{\mathcal{Z}}_+ \mathcal{Z}_- H - (\omega_5 H)^2 - \left(\frac{\breve{\omega}_{\underline{\phi}}}{r \sin\theta} \right)^2 \geq 0 . \tag{2.120}$$

We have checked numerically that this condition is satisfied without apparently imposing constraints on the parameters other than the positivity of the mass and horizon area.

Electric charges of the solution. When a 0-brane is coupled to the vector field A^I, its equation of motion, Eq. (2.18), gets modified by a 1-form current J_I^S as follows

$$\frac{1}{16\pi G_{\rm N}^{(5)}} \left\{ -\mathfrak{D} \left(a_{IJ} \star F^J \right) + \frac{1}{\sqrt{3}} C_{IJK} F^J \wedge F^K + g k_{I\,x} \mathfrak{D}\phi^x \right\} = \star J_I^S . \tag{2.121}$$

Then, following [180], we can define the so-called “brane-source” charges, $\mathcal{Q}_I^S$, by integrating both sides over some spacelike hypersurface, V:

$$\mathcal{Q}_I^S \equiv \int_V \star J_I^S = \frac{1}{16\pi G_{\rm N}^{(5)}} \int_V \left\{ -\mathfrak{D}(a_{IJ} \star F^J) + \frac{1}{\sqrt{3}} C_{IJK} F^J \wedge F^K + g k_{I\,x} \mathfrak{D}\phi^x \right\} . \tag{2.122}$$

In general, this charge is not conserved, $d \star J_I^S \neq 0$, because the left-hand-side of (2.121) is not closed. In the ungauged directions ($I = a$), the Killing vectors $k_I{}^x$ vanish and the gauge-covariant derivative becomes an ordinary exterior derivative. Then, we have that:

$$\mathcal{Q}_a^S = \frac{1}{16\pi G_{\rm N}^{(5)}} \int_V \left\{ -d(a_{aJ} \star F^J) + \frac{1}{\sqrt{3}} C_{aJK} F^J \wedge F^K \right\} , \tag{2.123}$$

is conserved if the $C_{aJK} F^J \wedge F^K$ term is a closed 4-form. This is for instance what occurs in the ungauged case. In the gauged case, however, this will depend on the model.

Alternatively, it is possible to define the so-called Maxwell charges as

$$\mathcal{Q}^M_a \equiv -\frac{1}{16\pi G_N^{(5)}} \int_V d\left(a_{aJ} \star F^J\right) , \tag{2.124}$$

which are always conserved, independently of the model under consideration. Applying Stokes' theorem,

$$\mathcal{Q}^M_a \equiv -\frac{1}{16\pi G_N^{(5)}} \int_{\partial V} a_{aJ} \star F^J , \tag{2.125}$$

where ∂V denotes the boundary of V.

For the SU(2)-gauged ST[2, 6] model, the explicit expressions for these Maxwell charges are

$$\mathcal{Q}^M_0 = -\frac{1}{48\pi G_N^{(5)}} \int_{\partial V} e^{2\phi} k^{-4/3} \star F^0 , \tag{2.126}$$

$$\begin{aligned}\mathcal{Q}^M_+ = & -\frac{1}{48\pi G_N^{(5)}} \int_{\partial V} \left\{ \frac{1}{4} k^{8/3} \left(1 + 2e^{-\phi} k^{-2} \ell^B \ell^B\right)^2 \star F^+ + e^{-2\phi} k^{-4/3} \ell^B \ell^B \star F^- \right. \\ & \left. + e^{-\phi} k^{2/3} \left(1 + 2e^{-\phi} k^{-2} \ell^B \ell^B\right) \ell^A \star F^A \right\} ,\end{aligned} \tag{2.127}$$

$$\mathcal{Q}^M_- = -\frac{1}{48\pi G_N^{(5)}} \int_{\partial V} e^{-2\phi} k^{-4/3} \star \left(F^- + \ell_B \ell^B F^+ + 2\ell^A F^A\right) , \tag{2.128}$$

and the relation between these and the brane-source charges for this model is the following

$$\mathcal{Q}^S_0 = \mathcal{Q}^M_0 + \frac{1}{96\sqrt{3}\pi G_N^{(5)}} \int_V \left(F^+ \wedge F^- - F^A \wedge F^A\right) , \tag{2.129}$$

$$\mathcal{Q}^S_\pm = \mathcal{Q}^M_\pm + \frac{1}{48\sqrt{3}\pi G_N^{(5)}} \int_V F^0 \wedge F^\mp . \tag{2.130}$$

Let us point out that for this model the brane-source charges are also conserved since the non-Abelian contributions to the $F \wedge F$ terms amounts to $F^A \wedge F^A$ which is a closed 4-form.

Let us now compute these charges for our solutions. Let us start with the Maxwell charges, which are given by

$$\mathcal{Q}^M_0 = \frac{\pi}{4\sqrt{3} G_N^{(5)}} e^{\phi_\infty} k_\infty^{-2/3} \mathcal{Q}^\infty_0 , \tag{2.131}$$

$$\mathcal{Q}^M_+ = \frac{\pi}{8\sqrt{3} G_N^{(5)}} k_\infty^{4/3} \tilde{\mathcal{Q}}_+ , \tag{2.132}$$

$$\mathcal{Q}^M_- = \frac{\pi}{4\sqrt{3} G_N^{(5)}} e^{-\phi_\infty} k_\infty^{-2/3} \left(1 + \frac{2\ell_\infty^2}{e^{\phi_\infty} k_\infty^2}\right) \mathcal{Q}^\infty_- . \tag{2.133}$$

For the brane-source charges, we find

$$\mathcal{Q}_0^S = \frac{\pi}{4\sqrt{3}G_{\rm N}^{(5)}} e^{\phi_\infty} k_\infty^{-2/3} \left(1+\frac{\beta}{1+\beta}\right) \mathcal{Q}_0\,, \tag{2.134}$$

$$\mathcal{Q}_+^S = \frac{\pi}{8\sqrt{3}G_{\rm N}^{(5)}} k_\infty^{4/3} \left(1+\frac{\beta}{1+\beta}\right) \tilde{\mathcal{Q}}_+\,, \tag{2.135}$$

$$\mathcal{Q}_-^S = \frac{\pi}{4\sqrt{3}G_{\rm N}^{(5)}} e^{-\phi_\infty} k_\infty^{-2/3} \left(1+\frac{2\ell_\infty^2}{e^{\phi_\infty}k_\infty^2}+\frac{\beta}{1+\beta}\right) \mathcal{Q}_-\,. \tag{2.136}$$

As we can see, the brane-source charges $\mathcal{Q}_0^S, \mathcal{Q}_+^S$ and $\mathcal{Q}_-^S$ are proportional to $\mathcal{Q}_0, \tilde{\mathcal{Q}}_+$ and $\mathcal{Q}_-$, the coefficients of the poles of the functions $\mathcal{Z}_0, \tilde{\mathcal{Z}}_+$ and $\mathcal{Z}_-$. The Maxwell charges, instead, are more difficult to interpret since, on the one hand, $\mathcal{Q}_0^M$ and $\mathcal{Q}_-^M$ are proportional to $\mathcal{Q}_0^\infty$ and $\mathcal{Q}_-^\infty$, the coefficients controlling the large ρ-expansion of the functions $\mathcal{Z}_0$ and $\mathcal{Z}_-$, and, on the other hand, $\mathcal{Q}_+^M$ is proportional to $\tilde{\mathcal{Q}}_+$. Finally, we notice that the mass of the solution can be rewritten as the sum of the Maxwell charges plus an additional term

$$M = \sqrt{3}\left(e^{-\phi_\infty} k_\infty^{2/3}\mathcal{Q}_0^M + 2k_\infty^{-4/3}\mathcal{Q}_+^M + e^{\phi_\infty}k_\infty^{2/3}\mathcal{Q}_-^M\right) + \frac{\pi \ell_\infty^2 \kappa^2}{e^{\phi_\infty}k_\infty^2 G_{\rm N}^{(5)}}\,, \tag{2.137}$$

which is proportional, up to moduli factors, to the electric charge of the dyon (2.107) as computed in [144, 145].

2.4 Rotating black holes of $\mathcal{N}=2, d=4$ gauged supergravity

The solution-generating technique described in Section 2.1 is specially well-adapted to describe four-dimensional solutions of the $\mathcal{N}=2$ SEYM theories that one obtains upon dimensional reduction of five-dimensional $\mathcal{N}=1$ SEYM theories on a circle. In order to exploit the results of Section 2.2, we focus again on solutions of the same supergravity model as in Section 2.3. Its dimensional reduction on a circle gives, as we have explained, the so-called SU(2)-gauged ST[2, 6] model of $\mathcal{N}=2, d=4$ supergravity. Let us give an extremely brief description of the model. For further details, see e.g. [111, 182].

2.4.1 The model

The bosonic field content of the SU(2)-gauged ST[2, 6] model of four-dimensional $\mathcal{N}=2$ supergravity is:[10] the vierbein $\tilde{e}^{\tilde{a}}{}_{\tilde{\mu}}$, six vector fields $\tilde{A}^{\Lambda}{}_{\tilde{\mu}}$ ($\Lambda = 0, 1, \ldots, 5$) and six complex scalars $\tilde{Z}^i$ ($i = 1, \ldots, 6$) that parametrize the coset space

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(2,5)}{\mathrm{SO}(2)\times\mathrm{SO}(5)}\,, \tag{2.138}$$

which is a Kähler-Hodge manifold.

[10]The four-dimensional fields and indices will carry a tilde to distinguish them from the five-dimensional ones.

$\mathcal{N}=2, d=4$ (ungauged) supergravity without hypermultiplets is fully specified by the choice of special Kähler geometry, which in turn is determined by the choice of the prepotential $\mathcal{F}$, from which one can derive the couplings of the theory. For the ST[2, 6] model, this prepotential is given by

$$\mathcal{F} = -\tfrac{1}{3!}\frac{d_{ijk}\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \tag{2.139}$$

where the $\mathcal{X}^\Lambda$ are seven functions of the scalars $\tilde{Z}^i$ and d_{ijk} is a constant, fully-symmetric, tensor that is related to the tensor C_{IJK} of the 5-dimensional theory by

$$d_{ijk} = 6C_{i-1\,j-1\,k-1}\,, \qquad i,j,k = 1,\ldots,6\,. \tag{2.140}$$

Therefore, the index 1 corresponds to the five-dimensional 0 and the four-dimensional 0 is associated to the Kaluza-Klein vector. The SU(2) gauge group acts on the complex scalars and vector fields with indices $4,5,6$. Furthermore, the $+$ and $-$ combinations defined in the five-dimensional case now correspond to

$$\tilde{A}^{\pm}{}_{\tilde{\mu}} \equiv \tilde{A}^2{}_{\tilde{\mu}} \pm \tilde{A}^3{}_{\tilde{\mu}}\,. \tag{2.141}$$

The explicit relation between the four and five-dimensional fields can be found in [152], and it is the following ($\tilde{\mu},\tilde{\nu}=0,\ldots,3$ and $\eta \equiv x^\sharp$):

$$\tilde{g}_{\tilde{\mu}\tilde{\nu}} = |g_{\underline{\eta\eta}}|^{1/2}\left(g_{\tilde{\mu}\tilde{\nu}} - \frac{g_{\tilde{\mu}\underline{\eta}}g_{\tilde{\nu}\underline{\eta}}}{g_{\underline{\eta\eta}}}\right)\,, \tag{2.142}$$

$$\tilde{A}^0{}_{\tilde{\mu}} = \frac{1}{2\sqrt{2}}\frac{g_{\tilde{\mu}\underline{\eta}}}{g_{\underline{\eta\eta}}}\,, \qquad \tilde{A}^i{}_{\tilde{\mu}} = -\frac{1}{2\sqrt{6}}\left(A^{i-1}{}_{\tilde{\mu}} - \frac{A^{i-1}{}_{\underline{\eta}}g_{\tilde{\mu}\underline{\eta}}}{g_{\underline{\eta\eta}}}\right)\,, \tag{2.143}$$

$$\tilde{Z}^i = \frac{1}{\sqrt{3}}A^{i-1}{}_{\underline{\eta}} + i|g_{\underline{\eta\eta}}|^{1/2}h^{i-1}\,. \tag{2.144}$$

Before making use of the above formulae, let us determine the five-dimensional solutions by making a choice for the seed functions Φ^I, L_I, H, M suitable to describe black holes in one dimension less.

2.4.2 Seed functions

Gibbons-Hawking metric

The main difference between the five-dimensional black holes described in the previous section and the four-dimensional black holes that we are going to describe lies in the choice of the Gibbons-Hawking function H, which is now given by

$$H = a_H + \frac{b_H}{r}\,, \qquad \Rightarrow \qquad \chi = b_H\cos\theta d\phi\,. \tag{2.145}$$

As it stands, the Gibbons-Hawking metric (2.30) presents a Dirac-Misner string singularity at $\theta = 0,\pi$ (x^3-axis), where the 1-form χ is not well-defined. To cure this pathology, we must cover the Gibbons-Hawking manifold with two different patches

$$\eta^{\pm} = \eta \pm b_H \phi \,, \tag{2.146}$$

which implies that

$$d\eta + \chi = d\eta^{\pm} + \chi^{\pm} \,, \qquad \text{where} \qquad \chi^{\pm} = b_H \left(\cos\theta \mp 1\right) d\phi \,. \tag{2.147}$$

Now, χ^+ and χ^- are well-defined, respectively, in the positive and negative x^3-axis. Hence, the problem is solved. However, this restricts the possible values that b_H can take since $\eta^{\pm}$ have the same periodicity as η (namely, $\eta^{\pm} \sim \eta^{\pm} + 2\pi\ell$) and the period of ϕ is $\phi \sim \phi + 2\pi$. Then, because of (2.146), the allowed values that b_H can take are

$$b_H = \frac{m\ell}{2} \,, \qquad \text{with} \qquad m \in \mathbb{Z} \,. \tag{2.148}$$

Φ^I and L_I functions

These functions will be the same as for the five-dimensional black holes. Therefore, we have

$$\Phi^{0,+,-} = 0 \,, \qquad L_{0,+,-} = a_{0,+,-} + \frac{b_{0,+,-}}{r} \,, \tag{2.149}$$

and, in the non-Abelian sector,

$$\Phi^A = \frac{\delta_i^{A-2}\, x^i}{\breve{g} r^2 \left(1 + \lambda^2 r\right)} \,, \qquad L_3 = L_4 = 0 \,, \quad L_5 = \frac{\xi \lambda^2 r}{g\left(1 + \lambda^2 r\right)} \,. \tag{2.150}$$

Using (2.39), we find that the Z_I functions are:

$$Z_0 = a_0 \left[1 + \frac{q_0}{r} + \frac{2}{9 a_0 a_H g^2} \frac{1 + (q_H + r)\left(2\lambda^2 + \lambda^4 r\right)}{4 q_H (q_H + r)\left(1 + \lambda^2 r\right)^2}\right] \,, \tag{2.151}$$

$$Z_{\pm} = a_{\pm} \left[1 + \frac{q_{\pm}}{r}\right] \,, \tag{2.152}$$

$$Z_3 = Z_4 = 0 \,, \qquad Z_5 = \frac{\xi \lambda^2 r}{g\left(1 + \lambda^2 r\right)} \,, \tag{2.153}$$

$$\tilde{Z}_+ = \tilde{a}_+ \left[1 + \frac{\tilde{q}_+}{r} + \frac{\xi^2}{\tilde{a}_+ a_- g^2} \frac{(r + q_-)\left(1 + 2\lambda^2 r\right) + q_- \lambda^4 r^2}{(r + q_-)\left(1 + \lambda^2 r\right)^2}\right] \,, \tag{2.154}$$

where we have defined

$$q_0 \equiv \frac{b_0}{a_0} - \frac{1}{18 g^2 a_0 b_H} \,, \qquad q_{\pm} \equiv \frac{b_{\pm}}{a_{\pm}} \,, \qquad q_H \equiv \frac{b_H}{a_H} \,, \tag{2.155}$$

and

$$\tilde{q}_+ \equiv \frac{b_+}{\tilde{a}_+} \,, \qquad \text{where} \qquad \tilde{a}_+ = a_+ - \frac{\xi^2}{g^2 a_-} \,. \tag{2.156}$$

1-form ω

We recall that in order to find ω_5, we have to make a choice of the harmonic function K. For simplicity, we choose $K = 0$. Then, ω_5 is found to be given by

$$\omega_5 = -\frac{\sqrt{3}\,\xi}{2a_H g^2}\frac{\lambda^2 r\cos\theta}{(r+q_H)\,(1+\lambda^2 r)^2}\,. \tag{2.157}$$

Taking into account the above choices, one can find $\breve{\omega}$ by solving (2.47). The result is

$$\breve{\omega} = -\frac{\sqrt{3}\xi}{2g^2}\frac{\lambda^2 r\sin^2\theta}{(1+\lambda^2 r)^2}d\phi\,. \tag{2.158}$$

2.4.3 Four-dimensional solutions

All that is left is to obtain the four-dimensional fields is to apply the formulae (2.142)-(2.144). Before writing down the four-dimensional metric, let us impose that it is in the so-called modified Einstein-frame [183], which is the one that yields the correct results for the masses of asymptotically-flat solutions [111]. We do this by normalizing the compact coordinate η in such a way that ℓ represents the asymptotic radius of the compactification circle $\mathbb{S}^1_\eta$. Then, we have that the KK scalar is given by

$$(k_\eta/k_{\eta,\infty})^2 \equiv |g_{\underline{\eta\eta}}| = f^{-1}H^{-1} - f^2\omega_5^2\,. \tag{2.159}$$

The above equation implies the following relation between the asymptotic values of the functions involved

$$a_H = f_\infty^{-1}\,, \qquad \text{where} \qquad f_\infty^{-3} = \frac{27}{2}a_0\tilde{a}_+a_-\,. \tag{2.160}$$

Keeping this in mind, let us make use of (2.142) to obtain the four-dimensional metric. We get:

$$d\tilde{s}^2 = e^{2U}\,(dt+\breve{\omega})^2 - e^{-2U}\left[dr^2 + r^2 d\Omega^2_{(2)}\right]\,, \tag{2.161}$$

where

$$e^{-2U} = \sqrt{f^{-3}H - (\omega_5 H)^2}\,, \qquad \text{and} \qquad f^{-3} = \frac{27}{2}Z_0\tilde{Z}_+Z_-\,. \tag{2.162}$$

We can now get rid of one of the unphysical parameters by imposing the standard normalization of asymptotically-flat metrics, which amounts to impose that $e^{-2U} \to 1$ at infinity. This implies —see Eq. (2.160)—:

$$a_H = 1\,, \qquad \text{and} \qquad a_0\tilde{a}_+a_- = \frac{2}{27}\,, \tag{2.163}$$

Now, as we did in Section 2.3, we can use (2.163) and (2.72) to write $a_0, \tilde{a}_+$ and a_- in terms of the asymptotic values of the five-dimensional scalars[11]

$$a_0 = \frac{1}{3} e^{\phi_\infty} k_\infty^{-2/3}\,, \qquad a_- = \frac{2}{3} e^{-\phi_\infty} k_\infty^{-2/3}\,, \qquad \tilde{a}_+ = \frac{1}{3} k_\infty^{4/3}\,. \tag{2.164}$$

The four-dimensional complex scalars are found by using (2.144). We get:

$$\tilde{Z}^1 = -\frac{1}{3\, Z_0\, H} \left(\omega_5 H - i e^{-2U}\right)\,, \tag{2.165}$$

$$\tilde{Z}^+ = -\frac{2}{3\, \tilde{Z}_+\, H} \left(\omega_5 H - i e^{-2U}\right)\,, \tag{2.166}$$

$$\tilde{Z}^- = -\frac{2\, Z_+}{3\, \tilde{Z}_+\, Z_-\, H} \left(\omega_5 H - i e^{-2U}\right)\,, \tag{2.167}$$

$$\tilde{Z}^{A+1} = \frac{2\, Z_A}{3\, \tilde{Z}_+\, Z_-\, H} \left(\omega_5 H - i e^{-2U}\right) + 2\sqrt{2} H^{-1} \Phi^A\,. \tag{2.168}$$

Their asymptotic values are determined by e^{ϕ_∞}, k_∞ and ξ:

$$\mathfrak{Im}\tilde{Z}^1_\infty = e^{-\phi_\infty} k_\infty^{2/3}\,, \quad \mathfrak{Im}\tilde{Z}^-_\infty = e^{\phi_\infty} k_\infty^{2/3} \left(1 + \frac{9\, e^{\phi_\infty}\, \xi^2}{2 g^2 k_\infty^{2/3}}\right)\,, \tag{2.169}$$

$$\mathfrak{Im}\tilde{Z}^+_\infty = 2 k_\infty^{-4/3}\,, \quad \mathfrak{Im}\tilde{Z}^4_\infty = \mathfrak{Im}\tilde{Z}^5_\infty = 0\,, \quad \mathfrak{Im}\tilde{Z}^6_\infty = -3\, e^{\phi_\infty} k_\infty^{-2/3} \xi / g\,, \tag{2.170}$$

where $\mathfrak{Im}\,\tilde{Z}^i_\infty$ denotes the imaginary part. The real parts simply vanish, $\mathfrak{Re}\,\tilde{Z}^i_\infty = 0$.

Finally, the vector fields (2.143) are given by

$$\tilde{A}^0 = \frac{1}{2\sqrt{2}} \left[-e^{4U} H^2 \omega_5 \left(dt + \breve{\omega}\right) + \chi\right]\,, \tag{2.171}$$

$$\tilde{A}^1 = \frac{1}{6\sqrt{2}} \frac{e^{4U} H f^{-3}}{Z_0} \left(dt + \breve{\omega}\right)\,, \tag{2.172}$$

$$\tilde{A}^\pm = \frac{1}{3\sqrt{2}} \frac{e^{4U} H f^{-3} Z_+}{Z_\pm \tilde{Z}_+} \left(dt + \breve{\omega}\right)\,, \tag{2.173}$$

$$\tilde{A}^{A+1} = -e^{4U} H \left(\frac{9}{2\sqrt{2}} Z_0 Z_A + \omega_5 \Phi^A\right) \left(dt + \breve{\omega}\right) - \frac{\varepsilon^{A-2}{}_{jk}\, x^j}{\breve{g} r^2 \left(1 + \lambda^2 r\right)}\, dx^k\,. \tag{2.174}$$

The analysis of the solutions is similar to the five-dimensional case, so we shall be brief. They describe a non-Abelian generalization of the four-charge extremal black holes

[11] The reason why it is interesting to write them in terms of ϕ_∞ and k_∞ is that these have a direct "stringy" interpretation: they are the vacuum expectation value of the dilaton and the asymptotic radius of one of the circles (that will be denoted later as $\mathbb{S}^1_z$) which forms part of the total compact space.

studied in [147, 183, 184]. The additional charge with respect to the five-dimensional case is q_H, which represents the magnetic charge of the Kaluza-Klein vector $\tilde{A}^0$. The SU(2) vector fields (2.174) are a dyonic deformation of the so-called *colored* monopoles [170,171], which are characterized by having vanishing magnetic charge (2.57). Contrary to what occurs with the *colored* black holes of Refs. [148,151], the parameter λ, which characterizes the "size" of the colored monopole, does not play the rôle of non-Abelian hair since it is fixed in terms of the total angular momentum of the solution, J. By analogy with the five-dimensional case, we expect $q_{\text{dyon}} \sim \frac{\xi}{g\lambda^2}$ to represent the electric charge of the colored dyon.

It is worth to mention that a different class of colored dyons [157] has also been used in [153] to construct multicenter black-hole solutions. This colored dyon, however, has different properties from the one considered here. In particular, it does not seem that we can assign an electric charge to it, see [137] for a discussion about this.

Physical properties of the solution

Horizon. The metric (2.162) has a regular horizon at $r = 0$, where e^{2U} vanishes. The near-horizon geometry is $\text{AdS}_2 \times \mathbb{S}^2$, exactly as in the Abelian case [147]. The Bekenstein-Hawking entropy is given by

$$S_{\text{BH}} = \frac{\pi}{G_{\text{N}}^{(4)}} \sqrt{q_0 \tilde{q}_+ q_- q_H} \,. \tag{2.175}$$

Then, as in the five-dimensional case, the black-hole entropy is not affected by the non-Abelian fields when the latter is written in terms of the near-horizon charges.

Mass and angular momentum. The ADM mass M and the angular momentum J of the solution can be easily computed by comparing the asymptotic expansion of (2.162) with that of the Kerr solution. We find:

$$4G_{\text{N}}^{(4)} M = q_0 + \tilde{q}_+ + q_- + q_H + \frac{e^{-\phi_\infty} k_\infty^{2/3}}{6g^2 q_H} + \frac{9e^{\phi_\infty} k_\infty^{-2/3} \xi^2}{2g^2} \left(q_- + 2\lambda^{-2}\right) , \tag{2.176}$$

$$J = -\frac{\sqrt{3}\xi}{4g^2\lambda^2 G_{\text{N}}^{(4)}} , \tag{2.177}$$

As we see, the colored dyon contributes positively to the total mass and angular momentum of the solution. Let us observe that the parameter λ is fixed in terms of the angular momentum (and the moduli) by

$$\lambda^2 = -\frac{\sqrt{3}\xi}{4g^2 G_{\text{N}}^{(4)} J} \,. \tag{2.178}$$

Hence, the solution has no hair.

Regularity of the solution. The metric of the solutions would contain naked singularities if the function e^{-2U} vanishes outside the event horizon. To the best of our knowledge, this is a feature that is present so far in all single-center, rotating, supersymmetric black holes in four dimensions, see e.g. [185–187]. We recall that function e^{-2U} is given by

$$e^{-2U} = \sqrt{\frac{27}{2} Z_0 \tilde{Z}_+ Z_- H - (\omega_5 H)^2}\,. \tag{2.179}$$

The first term in the square root is positive for $r \geq 0$ if we take the moduli parameters and the charges to be positive. There is the second term, however, which is always negative except in the static limit ($\xi = 0$), where it simply vanishes, see (2.157). In the rotating case, $\xi \neq 0$, the second term is always subdominant in the near-horizon and asymptotic limits, where we have, respectively, that

$$\omega_5 \sim \mathcal{O}(r)\,, \qquad \text{and} \qquad \omega_5 \sim \mathcal{O}(1/r^2)\,. \tag{2.180}$$

There is, however, the possibility of e^{-2U} having a necessarily even number of roots at the intermediate region. We have studied this numerically for different values of the parameters and we have found that the first term always dominates over the second. This is shown in Fig. 2.1, where we have plotted $e^{-2U}|_{\theta=0}$ as a function of r for different values of ξ.

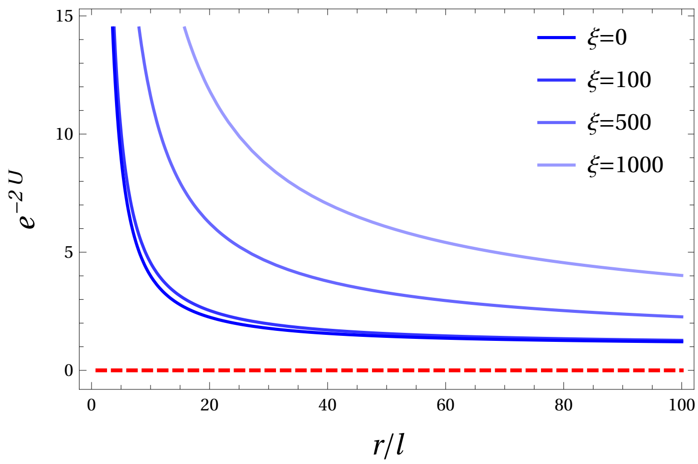

Figure 2.1: Metric function $e^{-2U}|_{\theta=0}$ as a function of the radial coordinate for different values of the parameter that characterizes the dyonic deformation, ξ. The remaining parameters of the solutions are kept fixed to the following values: $e^{\phi_\infty} = 0.1, k_\infty = 10$, $q_0 = q_+ = q_- = q_H = 10\ell$, $\lambda^{-2} = \ell$ and $g = \frac{2k_\infty^{2/3}}{3e^{\phi_\infty}\ell^2}$.

Therefore, we see that the negative term in (2.179) does not represent a real problem in order to construct solutions without naked singularities. Instead, we find that the behaviour of the metric function is improved as we turn on the non-Abelian parameter ξ.

2.5 Uplift to ten dimensions

Let us give a first step to study these non-Abelian black holes in the framework of string theory. As already discussed, the SU(2)-gauged ST[2, 6] model of $\mathcal{N} = 1, d = 5$ supergravity can be obtained by a truncation of ten-dimensional $\mathcal{N} = 1$ supergravity compactified on $\mathbb{T}^4 \times \mathbb{S}^1_z$.[12] The bosonic field content of the ten-dimensional theory is: the graviton $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$, the Kalb-Ramond 2-form $\hat{B}_{\hat{\mu}\hat{\nu}}$, the dilaton $\hat{\phi}$ and the three SU(2) vector fields $\hat{A}^{\hat{A}}$.[13] The bosonic action is

$$\hat{S} = \frac{g_s^2}{16\pi G_{\rm N}^{(10)}} \int d^{10}\hat{x}\, \sqrt{|\hat{g}|}\, e^{-2\hat{\phi}} \left\{ \hat{R} - 4(\partial\hat{\phi})^2 + \frac{1}{12}\hat{H}^2 - \frac{\alpha'}{8}\hat{F}^{\hat{A}} \cdot \hat{F}^{\hat{A}} \right\} , \tag{2.181}$$

where $\alpha' = \ell_s^2$ is the square of the string length and g_s is the string coupling constant (related to the asymptotic value of the dilaton by: $g_s = e^{\hat{\phi}_\infty}$). The ten-dimensional Newton's constant, $G_{\rm N}^{(10)}$, is given in terms of the string moduli by

$$G_{\rm N}^{(10)} = 8\pi^6 g_s^2 \alpha'^4 \,. \tag{2.182}$$

The field strength of the Kalb-Ramond 2-form, $\hat{H}$, is defined as

$$\hat{H} = d\hat{B} + \frac{\alpha'}{4}\omega^{\rm YM} , \tag{2.183}$$

where $\omega^{\rm YM}$ is the Chern-Simons 3-form of the connection $\hat{A}^{\hat{A}}$, namely

$$\omega^{\rm YM} = d\hat{A}^{\hat{A}} \wedge \hat{A}^{\hat{A}} + \frac{1}{3}\epsilon_{\hat{A}\hat{B}\hat{C}}\, \hat{A}^{\hat{A}} \wedge \hat{A}^{\hat{B}} \wedge \hat{A}^{\hat{C}} , \qquad d\omega^{\rm YM} = \hat{F}^{\hat{A}} \wedge \hat{F}^{\hat{A}} . \tag{2.184}$$

The relation between the ten- and five-dimensional fields is given in Appendix E, which is mostly based on the results of [177]. Using it, we find that the ten-dimensional uplift of the class of solutions considered in this chapter (both the five- and four-dimensional) is the following:

$$d\hat{s}^2 = 2\frac{a_-}{Z_-}du \left(dt + \omega - \frac{1}{2}\frac{\tilde{Z}_+}{\tilde{a}_+}du \right) - \frac{Z_0}{a_0}\, d\sigma^2 - ds^2\left(\mathbb{T}^4\right) , \tag{2.185}$$

$$e^{2\hat{\phi}} = g_s^2 \frac{Z_0/a_0}{Z_-/a_-} , \tag{2.186}$$

$$\hat{H} = \star_\sigma d\left(\frac{Z_0}{a_0}\right) + du \wedge \left[(dt+\omega) \wedge d\left(\frac{a_-}{Z_-}\right) + \frac{a_-}{Z_-} \star_\sigma d\omega \right] , \tag{2.187}$$

$$\hat{A}^{\hat{A}} = \frac{3\sqrt{3}\, g g_s}{k_\infty^{2/3}} \frac{a_-}{Z_-}\, Z_A\, du + g\hat{A}^A , \tag{2.188}$$

where $u = t - z$ and $z \sim z + 2\pi k_\infty \ell_s$ is the coordinate parametrizing the circle $\mathbb{S}^1_z$. The gauge coupling constant, g, is related to the ten-dimensional moduli by

[12] The truncation consists of keeping just a triplet of SU(2) gauge filds and setting to zero all the Kaluza-Klein fields (vectors and scalars) associated to the torus $\mathbb{T}^4$. See Appendix E for more details.

[13] We use hats to denote the ten-dimensional world $(\hat{\mu}, \hat{\nu}, \dots)$ and flat indices $(\hat{a}, \hat{b}, \dots)$, as well as to denote the index labelling the triplet of ten-dimensional SU(2) fields, $\hat{A}^{\hat{A}}$, with $\hat{A} = A - 2 = 1, 2, 3$. This also serves as a distinction between these and the base space vector fields defined in (2.22), $\hat{A}^A$, which carry an index $A = 3, 4, 5$.

$$g^2 = \frac{2k_\infty^{2/3}}{3g_s\alpha'}\,. \tag{2.189}$$

We would like to start analyzing these ten-dimensional solutions in order to figure out how the distorsion introduced by the Yang-Mills fields can be understood from the point of view of string theory. In order to do so, however, we have to take into account that the Green-Schwarz anomaly cancellation mechanism and supersymmetry force us to add local interactions which modify the action (2.181) and its equations of motion at first order in α', see (1.59). Consequently, a rigorous analysis of the Yang-Mills fields can only be done if those terms, which are of quadratic order in the curvature, are also considered. We will do this in the second part of the thesis. But now, we can study a class of simple solutions for which the quadratic curvature corrections are subdominant (i.e. of order α'^2).

2.5.1 Heterotic solitons

They arise as a particular case of the five-dimensional black holes of Section 2.3. In order to obtain them, we must set to zero some of the charges of the solution, namely

$$\mathcal{Q}_0 = \tilde{\mathcal{Q}}_+ = \mathcal{Q}_- = \mathcal{Q}_K = 0\,. \tag{2.190}$$

The vanishing of these near-horizon charges has two important consequences. First, that the brane-source charges vanish, so that the equations of motion are solved exactly, i.e. without the need of adding δ-like source terms. And second, that the solution has no horizon nor singularity.

The investigation of this type of *solitonic* solutions has received a great deal of attention, specially in the nineties [118–120, 160, 188–190]. One of the first solutions of this kind was Strominger's *gauge five-brane* [118, 188, 189], whose main ingredient is the well-known BPST instanton [166]. This is precisely the solution that we obtain in the static limit, $\xi = 0$. The rotating case, $\xi \neq 0$, corresponds to the generalization of the gauge five-brane found in [145]. The ten-dimensional solution is given by

$$d\hat{s}^2 = 2du\left(dt + \omega - \frac{\tilde{\mathcal{Z}}_+}{2}du\right) - \mathcal{Z}_0\left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right) - ds^2\left(\mathbb{T}^4\right)\,, \tag{2.191}$$

$$e^{2\hat{\phi}} = g_s^2\mathcal{Z}_0\,, \tag{2.192}$$

$$\hat{H} = \star_4 d\mathcal{Z}_0 + du \wedge \star_4 d\omega\,, \tag{2.193}$$

$$\hat{A}^{\hat{A}} = \delta^A_3 \frac{2\mu\rho^2}{\rho^2+\kappa^2}\,du - \frac{\kappa^2}{\rho^2+\kappa^2}v^A\,, \tag{2.194}$$

where

$$\mathcal{Z}_0 = 1 + \alpha' \frac{\rho^2 + 2\kappa^2}{\left(\rho^2 + \kappa^2\right)^2}, \tag{2.195}$$

$$\tilde{\mathcal{Z}}_+ = 1 + \alpha' \frac{\mu^2\kappa^2\left(2\rho^2 + \kappa^2\right)}{\left(\rho^2 + \kappa^2\right)^2}, \tag{2.196}$$

and

$$\omega = -\frac{\alpha'\kappa^2\mu\rho^2}{2\left(\rho^2 + \kappa^2\right)^2}\left(d\phi + \cos\theta d\psi\right), \tag{2.197}$$

where we have defined $\mu \equiv \frac{3\sqrt{3}g_s\xi}{2k_\infty^{2/3}}$ in order to reabsorb the moduli factors.

This solution describes a rotating superposition of the gauge five-brane and a momentum wave. Down in five dimensions, it describes the backreaction of the dyonic BPST instanton found in Ref. [144]. There is the possibility that this solution could be interpreted as a D1-D5 microstate geometry, though it is not entirely clear to us how the non-Abelian sector arises in this duality frame.

Furthermore, it would be interesting to study if the heterotic solitons constructed in [119, 120][14] admit dyonic deformations such as the one we have just studied for the gauge five-brane.

2.6 Discussion

In this chapter, we have constructed and studied non-Abelian supersymmetric black holes in supergravity models that arise from toroidal compactifications of ten-dimensional $\mathcal{N} = 1$ supergravity coupled to an additional triplet of SU(2) vector fields.

Both the five- and four-dimensional solutions can be understood as non-Abelian generalizations of well-known supersymmetric black holes with three [146] and four charges [147, 183, 184] that arise in various string compactifications (in particular, in the toroidal compactification of the heterotic string). The novel ingredient of the solutions is the addition of a dyonic BPST instanton [144, 145] and its four-dimensional descendant, the dyonic colored monopole. Our current understanding of these objects (see e.g. [154]) is that they contribute to the total mass, charges and angular momenta of the black hole but not to the Bekenstein-Hawking entropy. However, in order to see this, the entropy must be written in terms of the "near-horizon" charges, which are expected to be the ones counting the number of "stringy objects", as it occurs in the $\xi = 0$ case, [140, 154, 193]. In the five-dimensional case, it is possible to switch-off all the near-horizon or brane-source Abelian charges to recover a smooth solitonic solution describing a self-gravitating dyonic BPST instanton, which can be obtained by toroidal compactification of the heterotic dyonic instanton found in [145].

Perhaps the most interesting property of these dyonic deformations is that they give raise to non-static solutions. This is specially relevant in the context of four-dimensional

[14] These solutions are based on the eight-dimensional octonionic instanton of [191] and on the seven-dimensional G_2-instanton of [120]. These solutions have been later generalized to include multicenter instantons in [192].

black holes since the solution presented here is, as far as we are aware, the first non-singular example of a supersymmetric, single-center, asymptotically-flat black hole that is non-static.

3

Closed timelike curves in microstate geometries

Microstate geometries are solitonic solutions of the equations of motion of supergravity theories. Classical results from general relativity established that this type of non-singular solutions cannot be accommodated in a four-dimensional spacetime,[1] at least when non-Abelian matter fields are absent. Actually, asymptotically-flat, spherically-symmetric non-Abelian solitons have been known to exist in four and more dimensions for decades, [118, 149, 150, 166, 196–199], although multicenter solutions have only been discovered very recently [157]. Abelian solitons in supergravity, however, are only possible in five dimensions or more. In that case earlier no-go theorems can be circumvented due to the presence of Chern-Simons topological terms in the action; magnetic fluxes threading non-contractible two-cycles become effective sources of electric charge, mass and angular momentum. It is for this reason that Abelian microstate geometries require the spacetime manifold to have non-trivial topology.

A set of rules to construct Abelian microstate geometries as supersymmetric solutions of five-dimensional supergravity was discovered in [155, 156], where these were conjectured to be related to the classical description of black hole microstates within the context of the *fuzzball proposal* [74]. These works generalized earlier results [73, 200–205] by making use of the solution generating technique of [206, 207]. On the other hand, based on the results of the program for the study of non-Abelian black holes in string theory [148, 152–154, 163, 171, 176, 177], this technique has been extended to include the construction of non-Abelian microstate geometries in [157]. In this article we introduce a unified framework, so our discussion can be applied to all five-dimensional supersymmetric microstate geometries on a Gibbons-Hawking base.[2]

The aforementioned solution generating technique, however, has very limited applications. In few words, this is due to the fact that it is not known how to systematically avoid the presence of Dirac-Misner string singularities or general closed timelike curves (CTCs). We refer to this fact generically as the CTCs problem, since Dirac-Misner strings, if present, can only be resolved if the time coordinate is periodic, which introduces CTCs as well. According to the technique to construct solutions, Dirac-Misner strings are absent when the *bubble equations* are solved, see (3.6), while the geometry is free of general CTCs when the *quartic invariant* function is positive everywhere, (3.15). The problem arises because it is not known how the parameters that specify the solution must be chosen in order to satisfy these two types of constraints. For this reason, the solution generating

[1] See for instance the early work [194]. A more recent article with emphasis in microstate geometries is [195].

[2] The recent solutions of [208] describing supertubes with non-Abelian monodromies are constructed using harmonic functions with codimension-2 singularities and are not included in our framework.

technique of [155, 156] is rarely directly applied to find explicit solutions, being effective only for simplified configurations with few centers or with very special relations among the specifying parameters, see [209, 210] and references therein. More general solutions have been found by making use of more sophisticated tools, like the merging of several few-center solutions, [211, 212], or the link between three-charge supertube configurations and five-dimensional microstate geometries on a Gibbons-Hawking base, [213–215]. Superstrata solutions, which belong to a different class of smooth horizonless solutions of six-dimensional supergravity, deserve a special mention, as they might reproduce the degeneracy of microstates of general three-charge black holes, [216, 217].

In order to understand the situation better, it is convenient to discuss the origin of these pathologies. Timelike supersymmetric solutions of five-dimensional supergravity have a metric of conformastationary form,

$$ds^2 = f^2 \left(dt + \omega\right)^2 - f^{-1} h_{\underline{mn}} dx^m dx^n \, . \tag{3.1}$$

Here $h_{\underline{mn}} dx^m dx^n$ is a four-dimensional hyperKähler metric —usually a Gibbons-Hawking space [158, 159]— known as the *base space*, while f and ω are respectively a function and a 1-form defined on this base space.[3] The 1-form ω must transform as a tensor under coordinate transformations on the base space, since otherwise the hypersurfaces defined by constant values of the coordinate t would not be Cauchy surfaces. As ω is specified by a differential equation, this just means that we need to satisfy the corresponding integrability condition everywhere. This integrability condition becomes a set of algebraic relations known as the bubble equations. From a physical perspective, this phenomena is related to the frame-dragging generated by the interactions between electric and magnetic sources. These sources have Dirac string singularities, and their elimination from any influence in electromagnetic interactions requires imposing charge quantization. In a similar manner, we can think of the bubble equations as the conditions that require the frame-dragging is also invisible to those string singularities.[4] These constraints relate the *charge* parameters with the sizes of the non-contractible 2-cycles, and are typically interpreted as restrictions for the latter. On the other side, the above condition is necessary but not sufficient to ensure t is a global time coordinate. Besides, it is necessary that any hypersurface defined by a constant value of t is a Riemannian manifold with timelike normal vector, so that there are no CTCs. For these solutions, this requires the quartic invariant function to be positive. In the context of BPS microstate geometries, this can be rephrased in terms of the signs of *charge and energy densities* at separated locations. The existence of stationary multicenter supersymmetric solutions is typically related to the cancellation of attractive and repulsive (gravitational and electromagnetic) interactions[5]. If these cancellations cease

[3] This metric can be interpreted as a fibration of an $\mathbb{R}$-bundle over a four-dimensional space; the hypersurfaces with constant t define a local section and $\gamma_{\underline{mn}} = -f^{-1} h_{\underline{mn}}$ is the projection of the spacetime metric orthogonal to the fibres defined by the horizontal connection ω. On the other hand, the metric induced in that hypersurfaces is $g_{\underline{mn}} = f^2 \omega_m \omega_n - f^{-1} h_{\underline{mn}}$. As discussed in [195], for microstate geometries the coordinate t is a global time function and the sections it defines are Cauchy surfaces.

[4] Alternatively, one could get rid of the string singularities without solving the integrability condition by interpreting ω as a connection and solving its defining equation on different patches (as it is done, for instance, for the Dirac monopole). However, the consistency of this construction requires the time coordinate t to be compact, as shown by Misner in [218].

[5] Interestingly, one may notice that the action does not contain any terms introducing "Lorentz-like" electromagnetic forces. However, metric-based theories of gravity "know" about the existence of those interactions through the coupling of the spacetime metric and the electromagnetic energy-momentum

to take place the configuration is not truly supersymmetric, and this is reflected in the form of CTCs when trying to describe the solution as such. Hence, the problem is to find configurations for which all charge densities are of the same sign. However, the relation between the parameters specifying the solution and the charge densities, which arise from the interactions of magnetic fluxes threading non-contractible cycles, is rather complex. This is the reason why the solution of this problem has remained unclear.

Results and plan of the chapter

In this chapter we propose a systematic method to solve the CTCs problem that can be used to find all five-dimensional BPS microstate geometries on a Gibbons-Hawking hyper-Kähler space. It can be summarized as follows:

1. The bubble equations are non-linear and hard to solve if the locations of the centers are taken as the unknowns. However, those can be rewritten as a simple system of linear equations by choosing a different set of unknown variables: the magnetic fluxes. The bubble equations become

$$\mathcal{M}X = B\,, \tag{3.2}$$

for some symmetric matrix $\mathcal{M}$.

2. We conjecture that any solution satisfying the bubble equations is free of CTCs if and only if all the eigenvalues of the matrix $\mathcal{M}$ are positive.

When trying to build generic microstate geometries, all parameters specifying the solution can be treated on an equal footing. Therefore, there is no reason to consider the charge parameters more fundamental than the size of the bubbles. We begin with a description of the parameter space in Section 3.1. Then, we consider the CTCs problem in Section 3.2. In particular, in Section 3.2.1 we rewrite the bubble equations as a linear system with the same number of equations than variables, while in Section 3.2.2 we expose our conjecture and discuss evidence in its support. In Section 3.2.3 we discuss the application of our method to the construction of scaling solutions, describing how the introduction of non-Abelian fields strongly enriches the spectrum of this type of solutions. Afterwards, in Section 3.3, we put our method in practice and describe some solutions with properties and characteristics previously absent in the literature. For instance, as a striking result, we are able to find smooth horizonless five-dimensional solutions with arbitrarily small angular momentum. Those had not been discovered so far and they were even thought to be non-existent. We also describe some solutions with the centers lying on a circle or a line whose parameters can be specified with complete accuracy (to the best of our knowledge, all the explicit microstate geometries with several centers known so far can only be obtained approximately). Last but not least, to show how powerful is our method, we use it to construct a solution with 50 centers that contains more than a thousand 2-cycles. Finally, some final comments are made in Section 3.4.

tensor. See for example [219].

3.1 The parameter space and its restrictions

This section is devoted to the description of the parameter space of supersymmetric five-dimensional microstate geometries and the restrictions that physical configurations must fulfill. As in the previous chapter, we are going to focus on solutions of the SU(2)-gauged ST[2,6]-model of $\mathcal{N}=1, d=5$ supergravity, which is described in detail in Section 2.3.1. When the non-Abelian vector multiplets are truncated, we recover the STU model of supergravity, which is the theory in which five-dimensional Abelian microstate geometries are constructed. Our scheme, therefore, is completely general and accommodates both classes of solutions, Abelian and non-Abelian.

The construction of (non-Abelian) microstate geometries with a Gibbons-Hawking base in the supergravity model under consideration is described in [157] and reviewed in our conventions in Appendix D. These solutions are completely specified by a set of harmonic functions in three-dimensional Euclidean space and some restrictions that strongly constrain the parameter space. Let us first introduce the parameter space and discuss the constraints afterwards. We distinguish the harmonic functions in the Abelian sector

$$H=\sum_{a=1}^{n}\frac{q_a}{r_a}\,,\qquad \Phi^i=\sum_{a=1}^{n}\frac{k^i_a}{r_a}\,,\qquad L_i=l^i_0+\sum_{a=1}^{n}\frac{l^i_a}{r_a}\,,\qquad M=m_0+\sum_{a=1}^{n}\frac{m_a}{r_a}\,,\tag{3.3}$$

(where the index i takes three possible values $i=0,1,2$) and those in the non-Abelian sector

$$P=1+\sum_{a=1}^{n}\frac{\lambda_a}{r_a}\,,\qquad Q=\sum_{a=1}^{n}\frac{\sigma_a\lambda_a}{r_a}\,,\qquad \text{with } \lambda_a\geq 0\,.\tag{3.4}$$

We refer to the n poles of the harmonic functions as *centers*, so $r_a\equiv|\vec{x}-\vec{x}_a|$ is the Euclidean distance from the a^{th} center. The first function, H, plays a special role. It determines the geometry of a four-dimensional Gibbons-Hawking ambipolar space [195, 220],

$$h_{\underline{mn}}dx^m dx^n=H^{-1}(d\varphi+\chi)^2+Hd\vec{x}\cdot d\vec{x}\,,\qquad \star_3 dH=d\chi\,.\tag{3.5}$$

To describe asymptotically-flat solutions we need to recover four-dimensional Euclidean geometry in the base space in the $||\vec{x}||\to\infty$ limit. Hence, we shall impose $\sum_a q_a=1$. On the other hand, regularity at the centers demands that the Gibbons-Hawking charges q_a are integer numbers, and therefore some of them must be negative.

The horizonless condition and the regularity of the metric at the centers require that the parameters l^i_a, m_a and σ_a are given by a certain combination of q_a, k^i_a and other constants, which are specified in Appendix D.3. Moreover, asymptotic flatness also fixes the value of m_0 and imposes one constraint on the product of l^0_0, l^1_0 and l^2_0. The two remaining degrees of freedom in these constants are related to the moduli of the solution; that is, to the asymptotic value of the two Abelian scalars of the theory. Also, as discussed in the appendix, only $(n-1)$ of the k^i_a parameters are *physical*, as there is one redundant degree of freedom associated to gauge transformations of the vectors.

Therefore, asymptotically flat horizonless configurations are specified by $4(n-1)$ *charge* parameters (k^i_a and q_a), the 2 moduli parameters, the n non-Abelian *hair* parameters λ_a and, of course, the coordinates of the centers, which add $3(n-2)$ degrees of freedom. In total, the parameter space of the solutions is $8(n-1)$-dimensional.

Not every point in the parameter space, however, yields a physically sensible solution. Actually, it is most frequent that a random choice of such point gives a solution with closed timelike curves (CTCs) and Dirac-Misner string singularities connecting some of the centers. The absence of Dirac-Misner strings is achieved by imposing the so-called bubble equations,

$$\sum_{b\neq a} \frac{q_a q_b}{r_{ab}} \Pi^0_{ab} \left(\Pi^1_{ab}\Pi^2_{ab} - \frac{1}{2g^2}\mathbb{T}_{ab} \right) = \sum_{b,i} q_a q_b l^i_0 \Pi^i_{ab}. \tag{3.6}$$

where

$$\Pi^i_{ab} = \frac{k^i_b}{q_b} - \frac{k^i_a}{q_a}, \qquad \mathbb{T}_{ab} = \frac{1}{q_a^2} + \frac{1}{q_b^2}, \tag{3.7}$$

the non-Abelian gauge coupling constant is denoted by g and r_{ab} is the distance separating the centers a and b. The combinations Π^i_{ab} are the magnetic fluxes of the i^{th} Abelian vector threading the non-contractible 2-cycle defined by the two centers a and b. Notice that only $(n-1)$ equations are independent, as the sum in the index a that labels the n equations yields a trivial identity. Typically, solving the bubble equations constitutes a very hard step when constructing explicit microstate geometries. This is because, traditionally, those have been understood as equations for the variables r_{ab}, which have to be solved in terms of the independent charge parameters k^i_a and q_a and the moduli.[6] Then, after solving the system, usually by numerical methods, one finds that the obtained values of r_{ab} rarely represent the distances between a collection of points, as they should all be real, positive numbers satisfying the triangle inequalities $r_{ac} \leq r_{ab} + r_{bc}$ for all a, b, c. To construct explicit microstate geometries, one usually relies on further restrictions that reduce the number of independent parameters but make it easier for the bubble equations to admit proper solutions.

However, there seems to be no reason for considering the charge parameters more fundamental than the locations of the centers not only in the bubble equations, but also in the complete description of a particular microstate geometry. On one side, the system *looks asymptotically like a black hole* and its main characteristics are determined by the charge parameters and the moduli. On the other side, the existence of well-separated centers is of the utmost importance for resolving the horizon, and their locations are responsible for the distinction between the different microstates associated to the same black hole. In this chapter we show that the bubble equations can be solved analytically in full generality, with complete access to the whole parameter space of regular solutions, by considering the location of the centers among the independent variables.

As far as general CTCs are concerned, so far there has been no known analytically solvable restriction to guarantee their absence. Usually this has to be checked by evaluating numerically the positivity of the *quartic invariant* of the solution, once the bubble equations have been solved and all parameters are already specified. In the next section we propose an algebraic condition on the space of parameters that allows us to distinguish

[6]The hair parameters λ_a are absent in the bubble equations, although the non-Abelian fields are indirectly present through the term $\frac{1}{2g^2}\mathbb{T}_{ab}$.

whether or not the solution has CTCs, without making use of the numerical analysis of a function.

3.2 The solution of the CTCs problem

3.2.1 Solving the bubble equations analytically

As we outlined in the previous section, the bubble equations have been traditionally solved for the distances between the centers using numerical methods. This, in turn, makes the task of constructing explicit microstate geometries complex and typically limits the regions of the parameter space that can be accessed. Here we use a different approach to address the problem that allows for the analytic resolution of the bubble equations in full generality.

Since the number of independent magnetic fluxes associated to a given vector is the same as the number of independent equations, $(n-1)$, and those appear linearly in the system, it is reasonable to take them as the unknown variables for which the system is solved. As there are three different Abelian vectors, there are three possible ways in which we can write the system. In this section we write the explicit expressions when the 2-fluxes are taken as the unknowns, although equivalent relations can be readily obtained for the 0- and the 1- fluxes. If we define

$$\alpha^2_{ab} = \frac{q_a q_b}{r_{ab}} \left(\Pi^0_{ab}\Pi^1_{ab} - l^2_0 r_{ab} \right) \,, \qquad \text{with } \alpha^2_{aa} = 0\,, \tag{3.8}$$

and

$$\beta^2_a = \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{q_a q_b}{r_{ab}} \left[\frac{1}{2g^2} \mathbb{T}_{ab}\Pi^0_{ab} + \left(l^0_0 \Pi^0_{ab} + l^1_0 \Pi^1_{ab} \right) r_{ab} \right] , \tag{3.9}$$

it is straightforward to see that (3.6) can be rewritten as

$$\sum_{b=1}^{n} \alpha^2_{ab}\Pi^2_{ab} = \beta^2_a . \tag{3.10}$$

This is a system of n equations, but the sum of all of them is trivially satisfied.[7] Therefore, we can directly eliminate one of the equations, which is chosen to be the first one. We define the variables that will play the role of unknowns in the system of equations as follows

$$X^2_{\underline{a}} \equiv \Pi^2_{1(\underline{a}+1)} \,, \qquad \underline{a} = 1, \ldots, n-1\,. \tag{3.11}$$

Then, the rest of the 2-fluxes can be easily written in terms of these quantities as

$$\Pi^2_{(\underline{a}+1)(\underline{b}+1)} \;=\; X^2_{\underline{b}} - X^2_{\underline{a}} \tag{3.12}$$

For this variables, we get a system of $(n-1)$ linear equations

[7]Notice that the fluxes Π^i_{ab} are antisymmetric in their indices, while the matrix α^2_{ab} is defined to be symmetric. Also, we have that $\sum_a \beta^2_a = 0$.

$$\mathcal{M}^2_{\underline{ab}} X^2_{\underline{b}} = B^2_{\underline{a}}\,, \tag{3.13}$$

where the components of the matrix $\mathcal{M}^2$ and the vector B^2 are given by

$$\mathcal{M}^2_{\underline{ab}} = \alpha^2_{(\underline{a}+1)(\underline{b}+1)} - \delta^{\underline{b}}_{\underline{a}} \sum_{c=1}^{n} \alpha^2_{(\underline{a}+1)c}\,, \qquad B^2_{\underline{a}} = \beta^2_{(\underline{a}+1)}\,. \tag{3.14}$$

Thus, in our scheme the bubble equations can be solved by standard linear algebra methods for an arbitrarily large number of centers.[8] This is, of course, if the solution exists. Let us go back some steps to understand this issue. We explained in the previous section that asymptotically flat horizonless configurations are determined by $8(n-1)$ parameters. To become regular microstate geometries, these configurations need to satisfy $(n-1)$ independent algebraic constraints known as bubble equations. This means that there are at most $7(n-1)$ independent parameters. In this section we have shown a way in which the independent parameters can be chosen in order to solve the bubble equations in full generality. But, still, there are special values of the independent parameters for which the bubble equations do not admit a solution: the values for which the determinant of the coefficient matrix $\mathcal{M}^2$ is zero.

To get some intuition about this, let us suppose for a moment that the $7(n-1)$ independent parameters are continuous variables. Then, the condition $|\mathcal{M}| = 0$ defines a codimension one hypersurface in the parameter space, to which we refer as a *wall*. Although walls represent a very small region of the total space, we strongly believe that their presence is highly relevant.

3.2.2 Absence of CTCs: an algebraic criterion

At this stage, there is one last restriction that physically sensible configurations must satisfy: the spacetime cannot contain closed timelike curves. As we already mentioned, this problem is translated to the positivity of a function, the quartic invariant

$$\mathcal{I}_4 \equiv C^{IJK} Z_I Z_J Z_K H - \omega_5^2 H^2 \geq 0\,, \tag{3.15}$$

where we use the combinations (see the appendices for more information)

$$Z_I = L_I + 3C_{IJK}\frac{\Phi^J \Phi^K}{H}\,, \qquad \omega_5 = M + \frac{1}{2} L_I \Phi^I H^{-1} + C_{IJK}\Phi^I \Phi^J \Phi^K H^{-2}\,. \tag{3.16}$$

The parameters are chosen such that Z_I and ω_5 do not diverge at the centers. Moreover, when the bubble equations are satisfied ω_5 vanishes at the centers. Asymptotically, Z_i (just the Abelian sector) go to the positive constant l_0^i while ω_5 goes to zero. In short, this means that $\mathcal{I}_4$ is dominated by the first factor both near the centers and far from them. Motivated by this observation, we claim that the positivity of the quartic invariant is guaranteed if the first term is strictly positive,

$$C^{IJK} Z_I Z_J Z_K H = Z_0 H (Z_1 Z_2 - \frac{1}{2} Z_\alpha Z_\alpha) > 0\,, \tag{3.17}$$

[8] Provided computational resources are unlimited.

which implies that the term with ω_5 is irrelevant for the study of CTCs, even at intermediate regions, unless the first term in $\mathcal{I}_4$ vanishes.[9]

As Z_i changes sign when H does, the inequality (3.17) can be converted into a collection of simpler inequations

$$Z_i H > 0\,, \qquad Z_1 Z_2 - \frac{1}{2} Z_\alpha Z_\alpha > 0\,. \tag{3.18}$$

In terms of the parameters, these combinations of functions can be written as

$$\begin{aligned} Z_i H &= l_0^i \sum_{a=1}^n \frac{q_a}{r_a} - 3C_{ijk} \sum_{\substack{a,b=1\\ a>b}}^n \frac{q_a q_b}{r_a r_b} \Pi^j_{ab} \Pi^k_{ab} + \delta^0_i \frac{1}{2g^2} \sum_{a,b=1}^n \frac{1}{r_a r_b} \left(\frac{q_a}{q_b} - \frac{\lambda_a \lambda_b \vec{n}_a \cdot \vec{n}_b}{r_a r_b P^2} \right)\,, \\ Z_\alpha H &= \sum_{\substack{a,b=1\\ a\neq b}}^n \frac{q_a \lambda_b \Pi^0_{ab}}{g P r_a r_b^2} n_b^{(\alpha-2)}\,, \end{aligned} \tag{3.19}$$

where $n_a^{(\alpha-2)}$ are the coordinates of the unit vector $\vec{n}_a \equiv \frac{\vec{x}-\vec{x}_a}{r_a}$ (recall that $\alpha = 3,4,5$). Evaluating the Abelian functions $Z_i H$ at the centers we obtain

$$\begin{aligned} \lim_{r_a\to 0} Z_0 H &= \frac{1}{r_a} \left[l_0^0 q_a - \sum_{\substack{b=1\\ b\neq a}}^n \frac{q_a q_b}{r_{ab}} \left(\Pi^1_{ab} \Pi^2_{ab} - \frac{1}{2g^2} \mathbb{T}_{ab} \right) + \frac{1}{g^2 \lambda_a} \left(\lambda_0 + \sum_{b\neq a} \frac{\lambda_b}{r_{ab}} \right) \right] + \mathcal{O}(r_a^0)\,, \\ \lim_{r_a\to 0} Z_1 H &= \frac{1}{r_a} \left[l_0^1 q_a - \sum_{\substack{b=1\\ b\neq a}}^n \frac{q_a q_b}{r_{ab}} \Pi^0_{ab} \Pi^2_{ab} \right] + \mathcal{O}(r_a^0)\,, \\ \lim_{r_a\to 0} Z_2 H &= \frac{1}{r_a} \left[l_0^2 q_a - \sum_{\substack{b=1\\ b\neq a}}^n \frac{q_a q_b}{r_{ab}} \Pi^0_{ab} \Pi^1_{ab} \right] + \mathcal{O}(r_a^0)\,, \end{aligned} \tag{3.20}$$

and, from the first set of inequalities in (3.18), we find that the combination of parameters inside the brackets must be positive for all centers. Notice that in these expressions the purely Abelian limit is effectively recovered by taking $g \to \infty$, and that in this limit the last inequality in (3.18) is trivial. At first sight, it is noteworthy that these combinations of parameters look very similar to the elements in the diagonal of the coefficient matrices $\mathcal{M}^0$, $\mathcal{M}^1$ and $\mathcal{M}^2$, which are

$$\mathcal{M}^i_{(a-1)(a-1)} = -\sum_{\substack{b=1\\ b\neq a}}^n \frac{q_a q_b}{r_{ab}} \left(3C_{ijk} \Pi^j_{ab} \Pi^k_{ab} - \delta^i_0 \frac{1}{2g^2} \mathbb{T}_{ab} \right) + l_0^i q_a (1 - q_a)\,. \tag{3.21}$$

[9]While we have not been able to prove this claim in full generality, we have checked its validity in hundreds of thousands of pseudorandom configurations by computer analysis. In all the cases studied, the inequalities (3.15) and (3.17) are both true or both untrue.

But, however, what it is truly remarkable is that the positivity of the elements in the diagonal of those matrices is sufficient to ensure the positivity of the divergences of the functions $Z_i H$ at the centers[10]. This suggests that there might be a relation between the properties of the linear system of bubble equations and the absence of CTCs. To understand this relation better, it is convenient to consider simple configurations.

The study of two-center, purely Abelian microstate geometries provides a great deal of insight in this problem. In this case the bubble equation is

$$-\frac{q_1 q_2}{r_{12}}\left(\Pi^0_{12}\Pi^1_{12} - l^2_0 r_{12}\right) X^2_1 = -q_1 q_2 \left(l^0_0 \Pi^0_{12} + l^1_0 \Pi^1_{12}\right) , \tag{3.22}$$

where $X^2_1 = \Pi^2_{12}$. This configuration will not contain CTCs if

$$Z_i H = \frac{1}{r_1 r_2}\left[l^i_0 \left(q_1 r_2 + q_2 r_1\right) - 3C_{ijk} q_1 q_2 \Pi^j_{12}\Pi^k_{12}\right] > 0 , \tag{3.23}$$

for $i = 0, 1, 2$. Since r_1 and r_2 are positive numbers, we just need to impose the positivity of the function inside the bracket. Without loss of generality, we can take q_1 to be positive and q_2 to be negative, with $q_1 + q_2 = 1$. Then, the function $(q_1 r_2 + q_2 r_1)$ is bounded from below by the number $q_2 r_{12}$, so we can write

$$Z_i H \geq \frac{1}{r_1 r_2}\left(l^i_0 q_2 r_{12} - 3C_{ijk} q_1 q_2 \Pi^j_{12}\Pi^k_{12}\right) . \tag{3.24}$$

In particular, for $Z_2 H$ we need the combination $\left(l^2_0 q_2 r_{12} - q_1 q_2 \Pi^0_{12}\Pi^1_{12}\right)$ to be positive. This combination looks very similar to the eigenvalue of the matrix $\mathcal{M}^2$ (in this case a 1×1 matrix) that defines the bubble equation (3.22). Actually, as $q_1 q_2 < q_2$, if the eigenvalue is positive then the above combination is positive and therefore $Z_2 H > 0$.

In purely Abelian configurations, the three different type of fluxes appear in the bubble equations exactly in the same manner. Then, the bubble equations can be solved for the 0-fluxes or the 1-fluxes, defining two more matrices $\mathcal{M}^0$ and $\mathcal{M}^1$ respectively. Following the same reasoning as in the previous paragraph, we conclude that if the eigenvalues of the three matrices $\mathcal{M}^i$ are positive then $Z_i H > 0$ and therefore the configuration is free of CTCs. Of course, $\mathcal{M}^0$ and $\mathcal{M}^1$ depend on the 2-fluxes, which play the role of unknowns in the bubble equations. So, naively, it might seem that this criterion for identifying CTCs is of little help in practice, as we would like to dispose of a set of conditions on the parameter space only. However, using the bubble equations, it is immediate to prove that the eigenvalues of the three matrices are of the same sign. For example, for $\mathcal{M}^0$ we have that its eigenvalue is positive if $\Pi^1_{12}\Pi^2_{12} - l^0_0 r_{12} > 0$. This combination can be rewritten using the bubble equations (3.22) as

$$\Pi^1_{12}\Pi^2_{12} - l^0_0 r_{12} = \frac{l^1_0 (\Pi^1_{12})^2 r_{12} + l^0_0 l^2_0 (r_{12})^2}{\Pi^0_{12}\Pi^1_{12} - l^2_0 r_{12}} . \tag{3.25}$$

Since the numerator on the right hand side is positive, the left hand side has the same sign as the denominator, which proves our point.

[10] Except, perhaps, at the first center, for which the coefficient of the divergence is not directly related to any diagonal element of the coefficient matrices $\mathcal{M}^i$. This is because in the previous section we decided to eliminate the first of the bubble equations and take Π^i_{1b} as the unknowns. Of course, it is possible to take any other center as reference, obtaining additional conditions to guarantee the positivity of the divergence at the first center.

At this stage we have shown that if the eigenvalue of $\mathcal{M}^2$ is positive, then we have $Z_i H > 0$ and, according to our claim at the beginning of this section, the quartic invariant is positive and the solution does not contain CTCs. But, what would happen if the eigenvalue were negative? According to the preceding discussion it might be possible to have $Z_2 H > 0$ even when the eigenvalue is not positive. That is, it is possible to choose the parameters such that

$$\left(l_0^2 q_2 r_{12} - q_1 q_2 \Pi^0_{12}\Pi^1_{12}\right) > 0 > \left(l_0^2 q_1 q_2 r_{12} - q_1 q_2 \Pi^0_{12}\Pi^1_{12}\right) . \tag{3.26}$$

Remarkably, it turns out that these inequalities imply that $Z_0 H$ and $Z_1 H$ eventually become negative! For instance, we can check it explicitly for the latter (both proofs are identical). In first place, notice that the first inequality requires $\Pi^0_{12}\Pi^1_{12} > 0$. Using the bubble equations we can write

$$Z_1 H = \frac{l_0^0 \left[-q_1 q_2 r_{12}\left(\Pi^0_{12}\right)^2 + \left(\Pi^0_{12}\Pi^1_{12} - l_0^2 r_{12}\right)\left(q_1 r_2 + q_2 r_1\right) - \Pi^0_{12}\Pi^1_{12} q_1 q_2 r_{12}\right]}{r_1 r_2 \left(\Pi^0_{12}\Pi^1_{12} - l_0^2 r_{12}\right)} . \tag{3.27}$$

This function is positive asymptotically, but negative at $r_2 = 0$, $r_1 = r_{12}$ whenever $\left(\Pi^0_{12}\Pi^1_{12} - l_0^2 r_{12}\right) < 0$.

In summary, we have proved that Abelian two-center microstate geometries do not contain CTCs if and only if the eigenvalue of $\mathcal{M}^2$ is positive. The same conclusion can be obtained for non-Abelian two-center microstate geometries. In this case the proof is similar, although it is more technical and not particularly illuminating. In view of this result and based on the observations exposed at the beginning of this section for multicenter configurations, we make the following proposal.

Conjecture: Five-dimensional microstate geometries on a Gibbons-Hawking base, with or without non-Abelian fields, do not contain CTCs if and only if the coefficient matrix of the bubble equations is positive-definite.

If our conjecture is true, the construction of five-dimensional microstate geometries without CTCs will no longer require the numerical evaluation of any function on $\mathbb{R}^3$, but it will be sufficient to check an algebraic property of a matrix. This would extraordinarily simplify the problem of describing and studying this type of supergravity solutions, giving rise to a new plethora of smooth geometries.

To close this section let us mention that, although we have not been able to prove our conjecture in full generality, we have tested its validity with a large number of multi-center configurations. We have analyzed more than 100,000 solutions with pseudo-random parameters, finding a perfect agreement with our proposal.

3.2.3 Contractible clusters and scaling solutions

Scaling microstate geometries can be defined as solutions for which the centers can be brought arbitrarily close without significantly modifying the asymptotic charges [211]. As the centers approach each other, the geometry of the system does not only reproduce the asymptotic charges of an extremal black hole, but also starts to *look like one* at intermediate regions. In the zero-size limit all centers merge, a horizon is developed and the configuration becomes a black hole. But right before reaching the black hole limit, the solution is still horizonless and partially reproduces the *throat* that characterizes the near-horizon geometry of a black hole, capping off smoothly at some finite depth, although arbitrarily large. It is for this reason that scaling microstate geometries are expected to correspond to the classical description of individual microstates of a black hole [210]. We now show that the formalism we have presented is extraordinarily well-suited to describe and study scaling solutions.

We consider scaling solutions that preserve the shape of the distribution.[11] This means that we can write the distances between centers as

$$r_{ab} = \mu d_{ab}\,, \tag{3.28}$$

where d_{ab} remain constant in the scaling process, which is controlled by varying μ to arbitrarily small positive numbers. We can define the following quantities,

$$\bar{\alpha}^2_{ab} = \frac{q_a q_b}{d_{ab}}\Pi^0_{ab}\Pi^1_{ab}\,, \qquad \mathring{\alpha}^2_{ab} = -q_a q_b l_0^2\,, \tag{3.29}$$

and

$$\bar{\beta}^2_a = \sum_{b=1}^{n} \frac{q_a q_b}{d_{ab}}\frac{4}{g^2}\mathbb{T}_{ab}\Pi^0_{ab}\,, \qquad \mathring{\beta}^2_a = \sum_{b=1}^{n} q_a q_b \left(l_0^0\Pi^0_{ab} + l_0^1\Pi^1_{ab}\right)\,, \tag{3.30}$$

which are manifestly invariant during the scaling process. Then, upon substitution of (3.28) in (3.13), the bubble equations can be written as

$$\left(\bar{\mathcal{M}}^2_{\underline{ab}} + \mu\mathring{\mathcal{M}}^2_{\underline{ab}}\right) X^2_{\underline{b}} = \bar{B}^2_{\underline{a}} + \mu\mathring{B}^2_{\underline{a}}\,. \tag{3.31}$$

If compared with the original equation, we have $\mu\mathcal{M}^2 = \bar{\mathcal{M}}^2 + \mu\mathring{\mathcal{M}}^2$ for the coefficient matrix and $\mu B^2 = \bar{B}^2 + \mu\mathring{B}^2$ for the column vector. In terms of the parameters, we have

$$\bar{\mathcal{M}}^2_{\underline{ab}} = \bar{\alpha}^2_{(\underline{a}+1)(\underline{b}+1)} - \delta^{\underline{b}}_{\underline{a}}\sum_{c=1}^{n}\bar{\alpha}^2_{(\underline{a}+1)c}\,, \qquad \bar{B}^2_{\underline{a}} = \bar{\beta}^2_{(\underline{a}+1)}\,, \tag{3.32}$$

$$\mathring{\mathcal{M}}^2_{\underline{ab}} = \mathring{\alpha}^2_{(\underline{a}+1)(\underline{b}+1)} - \delta^{\underline{b}}_{\underline{a}}\sum_{c=1}^{n}\mathring{\alpha}^2_{(\underline{a}+1)c}\,, \qquad \mathring{B}^2_{\underline{a}} = \mathring{\beta}^2_{(\underline{a}+1)}\,. \tag{3.33}$$

The bubble equations as written in (3.31) are well defined even in the zero-size limit $\mu = 0$, where they cease to have a physical meaning. Scaling solutions can be identified as those for which one can take the zero-size limit through a continuous transformation and still obtain a valid solution of the bubble equations, without any of the asymptotic charges becoming zero. The existence of this limit cannot be taken for granted. Actually, for purely Abelian solutions one always has $\bar{B}^2 = 0$ and the bubble equations become a homogeneous system in the zero-size limit. Then, there are non-trivial solutions (that is, solutions with $X^2_{\underline{a}} \neq 0$ for some $\underline{a}$) only if the determinant of the corresponding coefficient

[11] Ideal scaling solutions would preserve the asymptotic charges while slightly modifying the relative distances of the cluster. However these type of scalings are extremely hard to describe and, as we are more interested in the *scaled* configurations than in the *scaling* process itself, we ignore this issue.

matrix, $\bar{\mathcal{M}}^2$, vanishes. In other words, purely Abelian scaling solutions necessarily flow to special points of the parameter space when taking the zero-size limit. It is for this reason that many Abelian solutions cannot be scaled without some of the asymptotic charges becoming zero.

The situation is completely different when non-Abelian fields are also considered. In this situation we have $\bar{B}^2 \neq 0$ and the system is still inhomogeneous in the zero-size limit. This means that non-Abelian microstate geometries can *typically* be scaled.[12] Actually, from our point of view this is the most important contribution that non-Abelian fields bring to the "microstate geometries program". Typically these fields enter the solutions modifying the spacetime metric, the asymptotic charges and the size of the bubbles very softly; in most cases these physical properties are practically preserved after introducing the non-Abelian distortion. However, this distortion becomes critical when we take the zero-size limit, enlarging the spectrum of scaling solutions.

3.3 One Thousand and One Bubbles

3.3.1 Exact solutions on lines and circles

There is one issue that might worry some of the readers: the fluxes that solve the bubble equations are in general irrational numbers. This is because the distances between a collection of points in three dimensions are usually irrational. However, the fluxes and the asymptotic charges, which are directly related through equations (D.41), are expected to be quantized when these solutions are properly interpreted within the context of string theory. More precisely, in the seminal article [156] it was pointed out that the dipole parameters k^i_a satisfy a quantization condition which can be derived from an analysis of the topology of the gauge fields,

$$k^i_a = \frac{(2\pi)^2 \ell^3_{\text{Planck}}}{V_i} \mathfrak{k}^i_a \,, \tag{3.34}$$

where $\mathfrak{k}^i_a$ is a set of integer numbers and V_i is the volume of a compact space that depends on the details of the embedding of the solution in higher-dimensional supergravity.[13] This quantization condition implies that the quotients k^i_a/k^i_b are rational numbers.

The simplest possibility to ensure that the quantization condition can be satisfied is to have the fluxes given by rational quantities. In this manner, it is always possible to find an appropriate embedding of the solution in higher dimensional supergravity satisfying (3.34). This fact can be seen as a motivation for the traditional approach to solve the bubble equations, in which the fluxes are guaranteed to be rational numbers.[14]

As the method we present in this article is completely general, it must also describe all microstate geometries with rational fluxes. These classes of solutions are obtained if the centers are chosen such that all the relative distances between them are rational

[12] The reason why some solutions might not be scaling is because they need to cross a *wall* in the parameter space while being subjected to the scaling process. On the other hand, since non-Abelian configurations can always be truncated to Abelian solutions, we can always recover Abelian scaling configurations if the truncation is implemented at some stage of the scaling process.

[13] The precise expression for the prefactor multiplying $\mathfrak{k}^i_a$ in (3.34) can be vary for different embeddings.

[14] Notice, however, that in the traditional approach the centers are generally separated by irrational distances and their location can only be known approximately.

numbers. An obvious possibility is to take all centers laying on a line, so the solution is axisymmetric. This kind of microstate geometries are easier to build and study, and most of the explicit constructions known are axisymmetric. As a first application of the aforementioned procedure we show in Table 3.1 a first example of a 5-center solution.[15] Motivated by the recent results of [214, 215], the locations of the centers present a *hierarchical structure*; i.e. the values of the relative distances vary between different orders of magnitude. As argued in those articles, such structure potentially favors finding solutions whose angular momentum is far from maximal. The 5-center solution described here has negligible angular momentum, see Table 3.3, and constitutes the first five-dimensional microstate geometry that exhibits this property. As shown in Table 3.1, the solution can be scaled without any problem introducing a conformal factor μ for the coordinates of the centers.

In order to go beyond axisymmetry, we now define a very interesting, arbitrarily large set of points with rational relative distances lying on a circle. The result is based on the original proof of the Erdős-Anning theorem, [221], that states that any infinite collection of points can have integral distances only if these are aligned. However, as we are about to see, it is possible to have an infinite set of points with mutual rational distances. First, pick a circle with unit diameter centered at the origin of coordinates. A primitive Pythagorean triple[16] P_i is composed of three coprime natural numbers a_i, b_i and c_i such that $a_i^2 + b_i^2 = c_i^2$. The triple P_i defines a right triangle whose hypotenuse and catheti lengths are 1, a_i/c_i and b_i/c_i respectively. This triangle can be place such that the hypotenuse lies on the x-axis and the coordinates of the vertices are $(-\frac{1}{2}, 0)$, $(-\frac{1}{2} + l_i, h_i)$ and $(\frac{1}{2}, 0)$, where $l_i = \left(\frac{a_i}{c_i}\right)^2$ and $h_i = \frac{a_i b_i}{c_i^2}$. Then, the triangle defines three points at rational distances on the unit diameter circle.

In virtue of Ptolemy's theorem, any other point with rational distances to the pair of points $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$ is necessarily separated by a rational distance from $(-\frac{1}{2}+l_i, h_i)$ as well. In particular, this means that we can use the same Pythagorean triple P_i to find three more valid points: $(-\frac{1}{2} + l_i, -h_i)$, $(\frac{1}{2} - l_i, h_i)$ and $(\frac{1}{2} - l_i, -h_i)$. Any additional primitive triple can add up to four points more to the set in the obvious manner. Moreover, since the four points associated to a triple define two new diameters of the circle those can also be used as hypotenuses, providing new possibilities to enlarge the collection. The procedure can be prolonged without end, defining a dense set of points on the circle. Finally, the value of the radius can be set to any rational number μ. Therefore, these configurations are very well-suited to build scaling solutions.

Table 3.1 contains a couple of examples of microstate geometries with 6 and 10 centers lying on a circle. Once again, a hierarchic structure has been imposed by making use of Pythagorean triples for which $a_i \ll b_i$.

Some of these solutions have more bubbles than any previously known example and, furthermore, can be specified with exact accuracy. Increasing the number of centers is feasible, although computationally demanding. On one side, solving a linear system of

[15] Our criterion to find a solution free of CTCs is to systematically look for parameters for which the coefficient matrix M^2 is positive definite. In any case, we have also checked numerically the absence of CTCs for all the examples displayed in this article.

[16] Primitive Pythagorean triples are generated through Euclid's formula,

$$a_i = m_i^2 - n_i^2\,, \quad b_i = 2m_i n_i\,, \quad c_i = m_i^2 + n_i^2\,, \tag{3.35}$$

for any pair of coprime integers $m_i > n_i > 0$.

Table 3.1: Input and output parameters of solutions. $l_0^1 = \sqrt{2}$, $l_0^2 = 1/\sqrt{2}$ and $g = 1$ for all the cases. Output parameter values shown are only approximate.

5 centers on a line					
x	-1	-0.999	0	0.999	1
q	1	-1	1	-1	1
k^0	10	-27	-37	17	1
k^1	38	-68	46	14	-11
$k^2(\mu = 1)$	1	-1.03739	0.223899	2.25387	-1.79766
$k^2(\mu = 0.0005)$	1	-1.03707	0.2834	2.10992	-1.65908

10 centers on a circle										
$l_1 = \left(\frac{2001}{2002001}\right)^2$, $h_1 = 2001 \cdot \frac{2002000}{2002001^2}$, $l_2 = \left(\frac{6001}{18006001}\right)^2$, $h_2 = 6001 \cdot \frac{18006000}{18006001^2}$										
x	0.5	$0.5 - l_2$	$0.5 - l_1$	$-0.5 + l_1$	$-0.5 + l_2$	-0.5	-0.5 + l_2	$-0.5 + l_1$	$0.5 - l_1$	$0.5 - l_2$
y	0	h_2	h_1	h_1	h_2	0	$-h_2$	$-h_1$	$-h_1$	$-h_2$
q	2	-1	1	-1	1	-1	1	-1	1	-1
k^0	32	72	12	60	39	30	38	11	9	51
k^1	51	99	32	24	90	11	57	26	9	78
$k^2(\mu = 1)$	1	-0.495548	0.503203	-0.461338	0.467769	-0.456857	0.471085	-0.454882	0.505734	-0.493379
$k^2(\mu = 0.0005)$	1	-0.495561	0.503179	-0.483632	0.490029	-0.479143	0.493333	-0.477239	0.505681	-0.493401

6 centers on a circle						
$l = \left(\frac{2001}{2002001}\right)^2$, $h = 2001 \cdot \frac{2002000}{2002001^2}$						
x	0.5	$0.5 - l$	$l - 0.5$	-0.5	$l - 0.5$	$0.5 - l$
y	0	h	h	0	-h	-h
q	2	-1	1	-1	1	-1
k^0	-100	69	46	-95	-7	73
k^1	-98	56	-15	-68	36	79
$k^2(\mu = 1)$	1	0.133637	11.6034	-11.6405	11.598	-0.421436
$k^2(\mu = 0.0005)$	1	0.102875	10.9491	-10.9852	10.9439	-0.425198

Table 3.2: Input and output parameters of a 50 centre example. $l_0^1 = \sqrt{2}$, $l_0^2 = 1/\sqrt{2}$ and $g = 1$. Output parameter values shown are only approximate.

50 centers on a line				
x	q	k^0	k^1	k^2
0.0330053	2	-20	-55	1
0.0984265	-1	-32	-14	-0.540293
-0.0179676	1	-70	-52	0.510139
0.092019	-1	-33	-2	-0.544646
0.011303	1	-33	-56	0.507942
0.0159932	-1	-42	-97	-0.513755
-0.0419008	1	-59	-83	0.506483
0.00449896	-1	-30	-35	-0.523255
-0.0371543	1	-83	-82	0.520376
0.0249915	-1	-13	-61	-0.523077
0.904343	1	-66	-90	0.51483
0.966033	-1	-100	-27	-0.521793
1.06745	1	-83	-11	0.500632
0.991016	-1	-40	-89	-0.518794
0.918601	1	-79	-38	0.507973
1.09964	-1	-27	-65	-0.515174
0.998465	1	-17	-28	0.502238
0.913144	-1	-41	-12	-0.529991
1.09778	1	-12	-31	0.50078
0.959097	-1	-99	-71	-0.515806
2.04383	1	-74	-77	0.531604
2.03968	-1	-6	-7	-0.561911
1.92718	1	-95	-77	0.531914
1.97688	-1	-23	-78	-0.52514
1.90891	1	-95	-33	0.509497
1.98718	-1	-74	-13	-0.525948
1.95846	1	0	-37	0.446919
2.03144	-1	-46	-7	-0.541284
1.99207	1	-53	-25	0.50149
2.04206	-1	-57	-3	-0.540974
2.96983	1	-9	-84	0.500777
2.92655	-1	-54	-27	-0.515791
2.94343	1	-1	0	0.430675
2.96789	-1	-59	-35	-0.514717
2.96737	1	-55	-7	0.49317
2.97213	-1	-83	-25	-0.511543
2.99724	1	-17	-42	0.49296
2.93693	-1	-1	-61	-0.536688
2.99367	1	-19	-15	0.482956
3.09503	-1	-70	-48	-0.515729
3.97904	1	-35	-38	0.526046
3.98217	-1	-85	-11	-0.540595
4.09649	1	-46	-39	0.491727
3.99963	-1	-17	-2	-1.11207
4.0424	1	-78	-63	0.506147
4.03426	-1	-67	-54	-0.497783
3.99249	1	-51	-20	0.556606
4.01874	-1	-72	-93	-0.452374
4.02437	1	-30	0	0.443444
4.0978	-1	-61	-80	-0.499799

	50-center line	**5-center line**	**10-center circle**	**6-center circle**
$\mathcal{Q}_0$	2378.38	46.7351	48.5981	448.285
$\mathcal{Q}_1$	2749.4	43.2695	37.403	509.907
$\mathcal{Q}_2$	5058525	2723	175063	12273
J_R	$-5.7506 \cdot 10^6$	-1.0708	17689.5	25956.7
$\mathcal{H}$	$2.7 \cdot 10^{-4}$	0.9999998	0.017	0.76

Table 3.3: Asymptotic charges and angular momenta of the solutions for $\mu = 1$.

equations is a problem of complexity P of order $\mathcal{O}(n^3)$. That is, the time required to solve the bubble equations approximately scales with the cube of the number of centers. On the other side, increasing the number of centers seems to favor the appearance of CTCs, so it is more likely that random elections of the parameters yield to unphysical solutions. According to our conjecture, this is just a natural consequence; as the coefficient matrix becomes bigger it is harder and harder to find the parameters such that all its eigenvalues are positive. Nevertheless, we have been able to describe solutions with a very large number of centers by focusing on regions of the parameter space that seem to favor the coefficient matrix is positive definite[17]. A particular example of an axisymmetric 50-center solution is given in Table 3.2.

Another interesting issue is that the parameters λ_a that determine the non-Abelian seed functions seems to play a subleading role in the CTCs problem. In fact, once a solution without CTCs has been found, we can generate as many as we want by modifying the non-Abelian parameters[18], as long as all of them remain positive. This is the reason why we have not specified any particular values in the tables. Therefore, the inclusion of non-Abelian fields not only makes it easier to find scaling solutions, but also enlarges the number of solutions with a given set of asymptotic charges, as expected [157].

3.3.2 General locations

It is comforting that we can use our method to describe, for the first time, many-center five-dimensional microstate geometries with an exact accuracy. Nevertheless, we are also interested in the possibility of describing more general solutions with centers at general locations, which includes the possibility of having irrational distances. In practice, this implies that the bubble equations must be solved approximately. We distinguish two possibilities:

- **Approximate fluxes**. The first possibility is to solve the bubble equations for the fluxes. In this case these will be given by irrational numbers and, as discussed at the beginning of the preceding subsection, this can be considered inconvenient because they are related to the asymptotic charges. Then, one valid option is to round the fluxes such that the charges take valid values, and admit that the solution is only specified approximately. This can be a useful possibility when one is interested in

[17] Our main guides are to impose the presence of hierarchical structures and to take all $k_a^{0,1}$ coefficients of the same sign.

[18] We have checked this by taking arbitrary values of the non-Abelian parameters in a finite range. However, based on how these parameters appear in the non-Abelian seed functions, we think that one can take any positive value for them and CTCs will not appear.

studying generic properties of the solutions, rather than in performing a very precise analysis.

- **Approximate locations**. The procedure that we follow to avoid having approximate fluxes can be summarized as follows. In a first step, we choose our favorite distribution of centers and solve the bubble equations for the fluxes. Then, we round the values and solve again the equations for the distances between the centers, using the fluxes as input data now. We expect the distances not to change too much for small enough changes of the fluxes. Once we know the distances, we have to place the centers in the tridimensional space $\mathbb{R}^3$. Unfortunately, this can only be done in full generality for four centers at most, so in configurations with more centers one has to impose restrictions in the locations when solving the bubble equations the second time (for example, one can consider axisymmetric configurations only).

3.4 Final comments

In this chapter we have presented an efficient method to construct general five-dimensional supersymmetric microstate geometries on a Gibbons-Hawking base. We have conjectured that the CTCs problem can be solved through the evaluation of a simple algebraic relation without the need to numerically evaluate the quartic invariant function. We have accompanied the exposition with a few explicit solutions, which were found making use of our method. These solutions exhibit novel properties in their class, such as arbitrarily small angular momentum or large number of centers, being some of them not axisymmetric distributions. This not only reveals that the spectrum of smooth microstate geometries on a Gibbons-Hawking base is actually very rich, but also that it is possible to find and study this type of solutions.

In particular, this method can be used to describe simple five-dimensional smooth, horizonless scaling solutions with the asymptotic charges of a D1-D5-P black hole without angular momentum. It would be interesting to study general properties of these geometries and compare them with those of a black hole; their geodesics, how they interact with incoming particles or their stability under perturbations. So far, this type of analysis has only been performed for two-charge microstate geometries or three-charge geometries with atypical asymptotic charges and angular momentum [222–225].[19]

As the procedure described is systematic, it would be very interesting to apply the tools developed in [214] to perform macroscopic explorations of the parameter space. For instance, in [215] this type of analysis has been successfully used to study generic four-center axisymmetric configurations, which can be constructed systematically, showing that those can only reproduce solutions with an angular momentum larger than 80% of the cosmic censorship bound when they are smooth in five dimensions, while it is possible to find solutions with arbitrarily small angular momentum if the configuration contains a supertube (which are smooth only in six dimensions or more). Making use of the method that we propose here, we can access the full space of parameters of multicenter, not necessarily axisymmetric, solutions.

[19] See [226] for a first approach to the study of such properties in superstrata microstate geometries, which have arbitrarily small angular momentum but are technically hard to describe and examine.

Part II

Black holes with higher-derivative corrections

4

A family of α'-corrected heterotic backgrounds

In 1984, Green and Schwarz discovered that the gauge and gravitational anomalies of Yang-Mills theory coupled to $\mathcal{N}=1, d=10$ supergravity partially cancel if one introduces suitable local interactions [107]. The remaining pieces automatically vanish when the gauge group is either SO(32) or $E_8 \times E_8$.[1] The aforementioned local interactions modify the definition of the 3-form field strength H associated to the Kalb-Ramond 2-form B as follows

$$H = dB + \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{\rm L}_{(-)}\right) , \tag{4.1}$$

where $\omega^{\rm YM}$ and $\omega^{\rm L}_{(-)}$ are the Chern-Simons 3-forms —see (4.16) and (4.17)— of the Yang-Mills connection, A^A, and of the torsionful spin connection,

$$\Omega_{(-)}{}^{a}{}_{b} = \omega^{a}{}_{b} - \frac{1}{2}H_{\mu}{}^{a}{}_{b}\, dx^{\mu} , \tag{4.2}$$

being $\omega^{a}{}_{b}$ the Levi-Civita spin-connection 1-form.

This implies that the Bianchi identity is modified as follows

$$dH - \frac{\alpha'}{4}\left(F^A \wedge F^A - R_{(-)ab} \wedge R_{(-)}{}^{ab}\right) = 0 , \tag{4.3}$$

where F^A and $R_{(-)}{}^{a}{}_{b}$ denote, respectively, the curvature 2-forms of A^A and $\Omega_{(-)}{}^{a}{}_{b}$.

These new local interactions break the invariance under local supersymmetry transformations. Fortunately, it can be recovered at the prize of introducing additional terms of higher order in derivatives. By dimensional analysis, a term with $2n$ derivatives must be multiplied by a coupling of dimension length^{2n-2}. In the context of superstring theory, such a coupling must necessarily be the string scale $\ell_s = \sqrt{\alpha'}$ since this is the unique dimensionful parameter of the theory. As we will see, these higher-derivative terms are constructed out of contractions of the curvatures of the Yang-Mills and torsionful spin connections and they are less and less relevant as long as these curvatures are small in string units, i.e. as compared to α'. In this limit, the effective action of the heterotic string can be written as a higher-derivative expansion [108, 109], which is known as the α'-expansion.

Our goal in this chapter will be precisely to solve the α'-corrected equations of motion at first order in α'. This is in general a tough task even if we work perturbatively in α',

[1] This discovery gave raise to the first superstring revolution.

specially if one aims to obtain full analytic solutions not restricting just to near-horizon or asymptotic regimes. Nevertheless, the authors of [193], fueled by previous studies on non-Abelian black holes [150, 152, 154, 166, 171, 177], have shown very recently that it is feasible.

Let us first discuss the ten-dimensional background studied in [193]. The metric $g_{\mu\nu}$, the NSNS 3-form H and the dilaton are respectively given by

$$
\begin{aligned}
ds^2 &= \frac{2}{\mathcal{Z}_-}du\left(dt - \frac{\mathcal{Z}_+}{2}du\right) - \mathcal{Z}_0\left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right) - dz^\alpha dz^\alpha\,, && (4.4)\\
H &= \star_4 d\mathcal{Z}_0 + d\mathcal{Z}_-^{-1}\wedge du\wedge dt\,, && (4.5)\\
e^{2\phi} &= g_s^2\,\frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, && (4.6)
\end{aligned}
$$

where $\star_4$ is the Hodge star operator associated to the metric of $\mathbb{E}^4$ and where the functions $\mathcal{Z}_0, \mathcal{Z}_+$ and $\mathcal{Z}_-$ are assumed to depend only on ρ, the radial coordinate of this Euclidean space. The coordinates $z^\alpha \sim z^\alpha + 2\pi\ell_s$ parametrize a four-dimensional torus $\mathbb{T}^4$ with no dynamics and $u = t - z$ is a light-cone coordinate. The coordinate $z \sim z + 2\pi R_z$ parametrize a fifth compact direction.

The zeroth-order equations of motion and Bianchi identity tell us that the three functions that determine the solution are harmonic on $\mathbb{E}^4$, hence

$$
\mathcal{Z}_{0,+,-} = 1 + \frac{\mathcal{Q}_{0,+,-}}{\rho^2}\,, \tag{4.7}
$$

since, by assumption, they only depend on the radial coordinate. This ten-dimensional background describes an intersection of the following extended objects:

- A fundamental string that is wounded around the circle parametrized by the coordinate z. The winding number is related to the charge $\mathcal{Q}_-$.
- A *pp*-wave travelling along z whose momentum is related to $\mathcal{Q}_+$.
- A stack of N solitonic or Neveu-Schwarz 5-branes. The number of branes, N, is related to $\mathcal{Q}_0$.

This ten-dimensional configuration is already familiar to us since it can be obtained from the uplift to ten dimensions of the five-dimensional black holea studied in Section 2.3 by simply switching off the non-Abelian fields and the angular momenta of the solutions.[2] We have now learned that in order to trust the non-Abelian ten-dimensional backgrounds studied in Chapter 2 as genuine heterotic backgrounds, we must also take into account the α' corrections that follow from the supersymmetrization of the Chern-Simons term associated to the torsionful spin connection. This is precisely what it was done in [193].

It turns out that the first obstacle that one faces when trying to find a solution at first order in α' is to solve the Bianchi identity of H, see (4.3). This is due to the

[2]This can be done by setting $\kappa = \xi = \mathcal{Q}_K = 0$.

fact that one typically works with an ansatz for H and then, the Bianchi identity must be imposed. Let us briefly explain how it can be solved analytically for the background under consideration. The first step is to make an ansatz for the gauge fields.[3] As in [118], the authors of [193] considered the addition of a triplet of SU(2) vector fields given by

$$A^A = -\frac{\kappa^2}{\rho^2 + \kappa^2} v^A \,, \tag{4.8}$$

which corresponds to the well-known BPST instanton. It was observed that the instanton number density $F^A \wedge F^A$ takes the form of the Laplacian of a function in $\mathbb{E}^4$ times the volume form. Therefore, if the 3-form H is assumed to be of the form $H = \star_4 d\mathcal{Z}_0$ (up to a closed 3-form on $\mathbb{E}^4$) for some function $\mathcal{Z}_0$ defined on the same space, the first two terms in the above Bianchi identity become the Laplacian of a linear combination of functions with constant coefficients. Almost magically, the third term turns out to be another Laplacian over the same space and the Bianchi identity is solved by equating the argument of the Laplacian to zero, up to a harmonic function on $\mathbb{E}^4$. Furthermore, in this case it is possible to tune the parameter κ associated to the BPST instanton so as to cancel part of the α' corrections, which is indeed a great motivation to consider configurations with non-trivial gauge-fields.

From experience, the simplest generalization one can make to this kind of solutions is to extend the ansatz to multicenter solutions, allowing the functions occurring in the metric to be arbitrary functions of the $\mathbb{E}^4$ coordinates. In the case of the gauge field, this requires the use of the so-called *'t Hooft ansatz* which can describe multicenter BPST instantons. This ansatz is reviewed and generalized in Section 4.2. Perhaps not so surprisingly, allowing the function $\mathcal{Z}_0$ to have arbitrary dependence on the $\mathbb{E}^4$ coordinates automatically forces some components of the torsionful spin connection to take the form of the 't Hooft ansatz too. Then, one can show that the instanton density 4-forms are, once again, Laplacians, and the Bianchi identity can be solved in exactly the same way.

It is natural to wonder if this result can be extended further. An interesting generalization is obtained by replacing $\mathbb{E}^4$ with a four-dimensional hyper-Kähler space that has a curvature with the same self-duality properties as the gauge field. It is well known that the simplest heterotic four-dimensional black holes that one can construct include a Kaluza-Klein monopole, which is a hyper-Kähler space with one additional triholomorphic isometry: a Gibbons-Hawking space [158, 159]. This additional isometry is necessary to obtain a four-dimensional solution by compactification on a 6-torus. Therefore, this generalization could be used to compute α' corrections to four-dimensional black holes.

First of all, one needs to generalize the 't Hooft ansatz to an arbitrary hyper-Kähler space and show that, again, one gets the Laplacian of some function in that space. This is done in Section 4.2. Now, from the torsionful spin connection we get terms with the form of this ansatz, which lead to the same result, and other terms corresponding to the spin connection of the four-dimensional hyper-Kähler manifold. Fortunately, the self-duality properties of these two contributions are opposite and they do to not mix. However, the contribution of the latter to the instanton number density might not necessarily take the form of the Laplacian of some function. At this stage one could try to add a second SU(2) gauge field whose instanton number density cancels that of the hyper-Kähler manifold,

[3] One can simply set them to zero.

as it has been done for this kind of solutions[4] in [227]. However, it turns out that if we restrict ourselves to Gibbons-Hawking spaces, then the instanton number density is also a Laplacian (times the volume form of this space). Funnily enough, we find that the connection can also be written in a 't Hooft ansatz-like form that we have called *twisted* ,'t Hooft ansatz. Therefore, adding a second SU(2) gauge field is optional but convenient if we want to cancel some of the α' corrections, as we will see.

Self-dual connections and the Atiyah-Hitchin-Singer theorem

Before closing this introduction, it is amusing to think about the relation between the 't Hooft ansatz that we use for the Yang-Mills fields and which naturally arises in the torsionful spin connection and the Atiyah-Hitchin-Singer theorem [228] on self-duality in Riemannian geometry.[5] The theorem deals with four-dimensional Riemannian manifolds and the decomposition of the components of their Levi-Civita spin-connection 1-forms into self- and anti-self-dual combinations according to the well-known local isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}^+(2) \oplus \mathfrak{su}^-(2)$. We will denote the two terms corresponding to this decomposition by $\omega^{+\,mn}$, respectively $\omega^{-\,mn}$. On the one hand, the theorem states about $\omega^{+\,mn}$ that:

> The curvature 2-form of $\omega^{+\,mn}$ is self-dual if and only if the manifold is Ricci flat.

This statement applies, in particular, to hyperKähler manifolds, which are Ricci flat and, therefore, for them, $\omega^{+\,mn}$ has self-dual curvature. Moreover, since these have special SU(2) holonomy, $\omega^{-\,mn} = 0$. On the other hand, the theorem also says that:

> The curvature 2-form of $\omega^{-\,mn}$ is self-dual if and only if the Ricci scalar vanishes and the manifold is conformal to another one with self-dual curvature 2-form.

This can be used to construct self-dual SU(2) instantons. Let us consider the metric

$$ds^2 = P^2 d\sigma^2\,, \tag{4.9}$$

where $d\sigma^2$ is a hyperKähler metric and where P is some function defined on it. The Ricci scalar of the full metric is proportional to the Laplacian of P in the hyperKähler space and therefore it vanishes if P is harmonic on the hyperKähler metric, so in this case the second part of the theorem applies. If we choose the vierbein basis $e^m = Pv^m$ where v^m is a Vierbein basis of the hyperKähler manifold, the first Cartan structure equation $de^m + \omega^{mn} \wedge e^n = 0$ leads to

$$d\log P \wedge v^m - \varpi^{mn} \wedge v^n + \omega^{mn} \wedge v^n = 0\,, \quad \Rightarrow \quad \omega^{mn} = \varpi^{mn} - \partial_{[m} \log P \delta_{n]p} v^p\,. \tag{4.10}$$

where we have used the same equation for the hyperKähler spin connection $dv^m + \varpi^{mn} \wedge v^n = 0$. We can now project the above equation onto the anti-self-dual part of $\mathfrak{so}(4)$ with the matrices $(\mathbb{M}^-_{mn})^{pq}$ defined in (4.40), getting

[4] Without the additional two functions that our class of solutions contains.

[5] The theorem is reviewed and applied to the construction of self-dual Yang-Mills instantons on Gibbons-Hawking spaces in [229, 230].

$$\omega^{-\,pq} = (\mathbb{M}^{-}_{nm})^{pq}\,\partial_m \log P\, v^n\,, \tag{4.11}$$

and, then, the theorem tells us that the expression in the right-hand-side is a connection with self-dual curvature 2-form, or, equivalently, a SU(2) gauge connection with self-dual field strength, *i.e.* an instanton. This is explicitly proven in Section 4.2. This provides a justification for the generalized 't Hooft ansatz that we are using, albeit it does not let one suspect that the instanton number density will be proportional to a Laplacian.

On the other hand, the ten-dimensional metric ansatz (4.83) has a four-dimensional piece which is conformal to the four-dimensional hyperKähler manifold, which reads

$$ds^2 = \mathcal{Z}_0\, d\sigma^2\,. \tag{4.12}$$

At zeroth-order in α', $\mathcal{Z}_0$ is a harmonic function in the hyperKähler manifold, as we will see. Now the Ricci scalar does not vanish, because there is a missing factor of 2 in the exponent of $\mathcal{Z}_0$, and the theorem does not apply. This is, nevertheless, the metric associated to solitonic 5-branes, and we cannot change it at will. If we repeat the above calculation we get

$$\omega^{-\,pq} = \frac{1}{2}(\mathbb{M}^{-}_{nm})^{pq}\partial_m \log \mathcal{Z}_0 v^n\,, \tag{4.13}$$

but now the curvature 2-form of this connection will not be self-dual. Moreover, $\omega^{+\,pq}$ contains the spin connection of the hyperKähler manifold ϖ^{mn} and some additional terms, which spoil the self-duality in the $\mathfrak{su}^{+}(2)$ piece as well.

This is where the magic of the heterotic superstring comes to our rescue because, now, the object of interest is not the Levi-Civita connection, but the torsionful spin connection 1-form (4.2) and the contribution of the torsion is such that

$$\Omega^{-}_{(-)}{}^{mn} = (\mathbb{M}^{-}_{pq})^{mn}\,\partial_q \log \mathcal{Z}_0\, v^p\,, \qquad\qquad \Omega^{+}_{(-)}{}^{mn} = \varpi^{mn}\,. \tag{4.14}$$

Therefore, the curvature 2-form of both projections, $\Omega^{\pm}_{(-)}{}^{mn}$, is self-dual.

As we see, in this kind of heterotic backgrounds, the same kind of objects come up naturally in both the Yang-Mills and torsionful spin connections, via the Atiyah-Hitchin-Singer theorem or via a different construction which, perhaps, can be related to a generalization of that theorem. An interesting recent result from [231], which considers the case of compact spaces, sheds light on this direction. It states that given two instantons on a given background that satisfies the equations of motion of the heterotic theory at zeroth order in α', it is always possible to rescale this background to obtain a solution of first order in α'.

4.1 The effective action of the heterotic string

As shallowly discussed at the beginning of this chapter, Green and Schwarz showed that ten-dimensional $\mathcal{N}=1$ supergravity coupled to Yang-Mills fields is free of gauge and gravitational anomalies when the gauge group is either SO(32) or $E_8\times E_8$ if one introduces the following local terms in the definition of the NSNS 3-form:

$$H = dB + \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{\rm L}_{(-)}\right)\,, \tag{4.15}$$

where

$$\omega^{\rm YM} = dA^A \wedge A^A + \frac{1}{3} f_{ABC} A^A \wedge A^B \wedge A^C\,, \tag{4.16}$$

$$\omega^{\rm L}_{(-)} = d\Omega_{(-)}{}^a{}_b \wedge \Omega_{(-)}{}^b{}_a - \frac{2}{3}\Omega_{(-)}{}^a{}_b \wedge \Omega_{(-)}{}^b{}_c \wedge \Omega_{(-)}{}^c{}_a\,. \tag{4.17}$$

We recall that the torsionful spin connection is defined as

$$\Omega_{(-)}{}^a{}_b = \omega^a{}_b - \frac{1}{2} H_\mu{}^a{}_b\, dx^\mu\,, \tag{4.18}$$

where $\omega^a{}_b$ is the Levi-Civita spin connection.

These local terms modify the Bianchi identity, which at zeroth order in α' simply reads

$$dH = 0\,. \tag{4.19}$$

Now, given that

$$d\omega^{\rm YM} = F^A \wedge F^A\,, \qquad \text{and} \qquad d\omega^{\rm L} = R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a\,, \tag{4.20}$$

one has that

$$dH = \frac{\alpha'}{4}\left(F^A \wedge F^A + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a\right)\,. \tag{4.21}$$

The terms that appear due to the presence of the Chern-Simons 3-form of the torsionful spin connection in the definition of the 3-form field strength (4.15) spoil the invariance of the theory under local supersymmetry transformations. In order to recover it, one assumes that the action and supersymmetry transformations can be written in a series expansion in α' and demands invariance under local supersymmetry transformation at each order in α'. It was shown in [108] that this can be done if suitable higher-order derivative terms are added to the supergravity action and this is how one can arrive to the α'-expansion of the heterotic effective action.[6] Although this is an infinite series, only a few terms have been explicitly constructed. Fortunately, this is more than enough for our purposes since we only need the action up to first order in α'. Without any more preambles, the effective action of the heterotic string is given by[7]

$$\begin{aligned} S = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x\sqrt{-g}\, e^{-2\phi} \Big[& R - 4\,\partial_\mu\phi\partial^\mu\phi + \frac{1}{2\cdot 3!} H_{\mu\nu\rho}H^{\mu\nu\rho} \\ & -\frac{\alpha'}{8}\left(F^A{}_{\mu\nu}F^{A\mu\nu} - R_{(-)\mu\nu ab}R_{(-)}{}^{\mu\nu ab}\right) + \mathcal{O}\left(\alpha'^3\right)\Big]\,, \end{aligned} \tag{4.22}$$

[6]There are, of course, alternative ways of arriving to the same action, see e.g. [109].

[7]We adapt the action given in [108] to the conventions of [111].

where R is the Ricci scalar of $g_{\mu\nu}$, the metric in the string frame, and ϕ represents the dilaton. The curvature 2-forms of the Yang-Mills and torsionful spin connections are defined as

$$F^A = dA^A + \frac{1}{2} f_{BC}{}^A \, A^B \wedge A^C \,, \tag{4.23}$$

$$R_{(-)}{}^a{}_b = d\Omega_{(-)}{}^a{}_b - \Omega_{(-)}{}^a{}_c \wedge \Omega_{(-)}{}^c{}_b \,. \tag{4.24}$$

Finally, g_s is the string coupling constant and $G_N^{(10)}$ is the ten-dimensional Newton's constant, whose expression in terms of the string moduli is

$$G_N^{(10)} = 8\pi^6 g_s^2 \alpha'^4 \,. \tag{4.25}$$

The $\mathcal{O}\left(\alpha'^3\right)$ terms are given in [108] and involve contractions of two of the so-called T-tensors which are quadratic in the curvatures. Apart from these, string theory predicts the appearance of a different set of corrections at cubic order in α' which would be unrelated to the supersymmetrization of the Chern-Simons [232]. These are not so well-known in the context of the heterotic string and, in any case, we are going to neglect them since we will be working, by assumption, in the low-curvature regime in which these α'^3 corrections are subleading. In this regard, it is important to notice that the term H^2 in the action (4.22) actually contains an infinite tower of implicit α' corrections since the definition of the 3-form field strength (4.1) is a recursive one and so it has to be implemented order by order in α'. Let us do this. First, at zeroth order in α', we have

$$H^{(0)} \equiv dB \,. \tag{4.26}$$

Next, we use this 3-form, $H^{(0)}$, to construct zeroth-order torsionful spin connection

$$\Omega^{(0)}_{(-)}{}^a{}_b = \omega^a{}_b - \tfrac{1}{2} H^{(0)}{}_\mu{}^a{}_b \, dx^\mu \,, \tag{4.27}$$

and, using it, we define the Chern-Simons 3-form

$$\omega^{\mathrm{L}\,(0)}_{(\pm)} = d\Omega^{(0)}_{(\pm)}{}^a{}_b \wedge \Omega^{(0)}_{(\pm)}{}^b{}_a - \frac{2}{3} \Omega^{(0)}_{(\pm)}{}^a{}_b \wedge \Omega^{(0)}_{(\pm)}{}^b{}_c \wedge \Omega^{(0)}_{(\pm)}{}^c{}_a \,. \tag{4.28}$$

Then, we are ready to define recursively

$$\begin{aligned}
H^{(1)} &= dB + \frac{\alpha'}{4}\left(\omega^{\mathrm{YM}} + \omega^{\mathrm{L}\,(0)}_{(-)}\right) , \\
\Omega^{(1)}_{(-)}{}^a{}_b &= \omega^a{}_b - \frac{1}{2} H^{(1)}{}_\mu{}^a{}_b \, dx^\mu \,, \\
\omega^{\mathrm{L}\,(1)}_{(-)} &= d\Omega^{(1)}_{(-)}{}^a{}_b \wedge \Omega^{(1)}_{(-)}{}^b{}_a - \frac{2}{3} \Omega^{(1)}_{(-)}{}^a{}_b \wedge \Omega^{(1)}_{(-)}{}^b{}_c \wedge \Omega^{(1)}_{(-)}{}^c{}_a \,. \\
H^{(2)} &= dB + \frac{\alpha'}{4}\left(\omega^{\mathrm{YM}} + \omega^{\mathrm{L}\,(1)}_{(-)}\right) ,
\end{aligned} \tag{4.29}$$

and so on.

4.1.1 Equations of motion

The equations of motion that follow from the action (4.22) are quite intricate since they involve terms with higher order derivatives. However, it should be emphasized once more that it only makes sense to work with them perturbatively in α' since the action was derived under this premise. It is in this context where the lemma proven by Bergshoeff and de Roo in [233] acquires its greatest significance. Let us review it here. To this aim, we are going to separate the variations with respect to each field into those corresponding to occurrences via ${\Omega_{(-)}}^a{}_b$, that we will call *implicit*, and the rest, that we will call *explicit*:

$$\begin{aligned}\delta S &= \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta S}{\delta B_{\mu\nu}}\delta B_{\mu\nu} + \frac{\delta S}{\delta {A^A}_\mu}\delta {A^A}_\mu + \frac{\delta S}{\delta\phi}\delta\phi \\ &= \left.\frac{\delta S}{\delta g_{\mu\nu}}\right|_{\text{exp.}}\delta g_{\mu\nu} + \left.\frac{\delta S}{\delta B_{\mu\nu}}\right|_{\text{exp.}}\delta B_{\mu\nu} + \left.\frac{\delta S}{\delta {A^A}_\mu}\right|_{\text{exp.}}\delta {A^A}_\mu + \frac{\delta S}{\delta\phi}\delta\phi \\ &\quad + \frac{\delta S}{\delta {\Omega_{(-)}}^a{}_b}\left(\frac{\delta {\Omega_{(-)}}^a{}_b}{\delta g_{\mu\nu}} + \frac{\delta {\Omega_{(-)}}^a{}_b}{\delta B_{\mu\nu}}\delta B_{\mu\nu} + \frac{\delta {\Omega_{(-)}}^a{}_b}{\delta {A^A}_\mu}\delta {A^A}_\mu\right). \end{aligned} \tag{4.30}$$

Then, the lemma states that $\delta S/\delta {\Omega_{(-)}}^a{}_b$ is proportional to α' and to the zeroth-order equations of motion of $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ plus terms of higher order in α'. This means that the last line of (4.30) can be safely ignored if we work perturbatively in α'[8] since it will only introduce terms of second order in α', which we are going to neglect.

Doing so, we find that, up to $\mathcal{O}\left(\alpha'^2\right)$ terms, the equations of motion reduce to

$$R_{\mu\nu} - 2\nabla_\mu\partial_\nu\phi + \frac{1}{4}H_{\mu\rho\sigma}{H_\nu}^{\rho\sigma} = \frac{\alpha'}{4}\left({F^A}_{\mu\rho}{F^A}_\nu{}^\rho - R_{(-)\,\mu\rho ab}{R_{(-)\,\nu}}^{\rho\, ab}\right), \tag{4.31}$$

$$(\partial\phi)^2 - \frac{1}{2}\nabla^2\phi - \frac{1}{4\cdot 3!}H^2 = -\frac{\alpha'}{32}\left({F^A}_{\mu\nu}F^{A\mu\nu} - R_{(-)\mu\nu ab}{R_{(-)}}^{\mu\nu ab}\right), \tag{4.32}$$

$$d\left(e^{-2\phi}\star H\right) = 0\,, \tag{4.33}$$

$$\alpha' e^{2\phi}\mathfrak{D}_{(+)}\left(e^{-2\phi}\star F^A\right) = 0\,, \tag{4.34}$$

where the covariant dervative $\mathfrak{D}_{(+)}$ is defined as

$$\alpha' e^{2\phi}\mathfrak{D}_{(+)}\left(e^{-2\phi}\star F^A\right) \equiv \alpha' e^{2\phi} d\left(e^{-2\phi}\star F^A\right) + {f_{BC}}^A A^B\wedge\star F^C + \star H\wedge F^A\,. \tag{4.35}$$

4.2 ’t Hooft ansatz in four-dimensional hyperKähler spaces

The 6 generators of the Lie algebra $\mathfrak{so}(4)$ in the defining (vector) representation can be labeled by a pair of antisymmetric indices $m, n = \sharp, 1, 2, 3$. If $(\mathbb{M}_{mn})^{pq}$ denotes the pq matrix element of the generator labeled with the pair mn, we have that[9]

[8] This means that $\Phi = \Phi^{(0)} + \alpha'\Phi^{(1)} + \mathcal{O}\left(\alpha'^2\right)$ where Φ denotes schematically any background field.

[9] There is no difference between upper and lower indices. The position is chosen for the sake of clarity.

$$(\mathbb{M}_{mn})^{pq} \equiv 2\delta_{mn}{}^{pq} = \delta_m{}^p\delta_n{}^q - \delta_m{}^q\delta_n{}^p \,, \tag{4.36}$$

and their commutators are given by

$$[\mathbb{M}_{mn}, \mathbb{M}_{pq}] = -2\mathbb{M}_{[m|r}(\mathbb{M}_{pq})^r{}_{|n]} \,. \tag{4.37}$$

These labels are very convenient but they introduce a twofold redundancy, as each generator appears twice: once as $\mathbb{M}_{\sharp 1}$, for instance, and once as $\mathbb{M}_{1\sharp}$. Thus, if we want to sum once over all the independent generators and we sum over these labels, we must introduce additional factors of 1/2. For instance, the structure constants have to be defined by

$$[\mathbb{M}_{mn}, \mathbb{M}_{pq}] \equiv \tfrac{1}{2} f_{mn\,pq}{}^{rs}\mathbb{M}_{rs} \,, \tag{4.38}$$

and, comparing with the above commutators, we get

$$f_{mn\,pq}{}^{rs} = -4(\mathbb{M}_{pq})^r{}_{[m}\delta_{n]}{}^s \,. \tag{4.39}$$

Let us define the self- and anti-self-dual combinations

$$\mathbb{M}^{\pm}_{mn} \equiv \tfrac{1}{2}\left(\mathbb{M}_{mn} \pm \tfrac{1}{2}\varepsilon_{mn}{}^{pq}\mathbb{M}_{pq}\right) \,, \qquad \tfrac{1}{2}\varepsilon_{mn}{}^{pq}\mathbb{M}^{\pm}{}_{pq} = \pm\mathbb{M}^{\pm}{}_{mn} \,, \tag{4.40}$$

which are explicitly given by[10]

$$(\mathbb{M}^{\pm}_{mn})^{pq} = \delta_{mn}{}^{pq} \pm \tfrac{1}{2}\varepsilon_{mn}{}^{pq} = (\mathbb{M}^{\pm}_{pq})^{mn} \,, \tag{4.41}$$

and which must generate two independent subalgebras because they satisfy the commutation relations

$$[\mathbb{M}^{\pm}_{mn}, \mathbb{M}^{\pm}_{pq}] = -2\mathbb{M}^{\pm}_{[m|r}(\mathbb{M}^{\pm}_{pq})^r{}_{|n]} \,, \tag{4.42}$$

$$\left[\mathbb{M}^{+}_{mn}, \mathbb{M}^{-}_{pq}\right] = 0 \,, \tag{4.43}$$

The (anti-)self-duality properties imply that only three of each kind are independent and we can pick representatives $\mathbb{M}^{\pm}_{\sharp A}$, with $A = 1, 2, 3$, at the expense of losing manifest $\mathfrak{so}(4)$-covariance. When working with an antisymmetric pair of $\mathfrak{so}(4)$ indices, their fourfold redundancy has to be taken into account introducing factors of 1/4:

$$[\mathbb{M}^{\pm}_{mn}, \mathbb{M}^{\pm}_{pq}] \equiv \tfrac{1}{4} f^{\pm}_{mn\,pq}{}^{rs}\mathbb{M}^{\pm}_{rs} \,, \quad \Rightarrow \quad f^{\pm}_{mn\,pq}{}^{rs} = 4\,(\mathbb{M}^{\pm}_{pq})^x{}_{[m}(\mathbb{M}^{\pm}_{n]x})^{rs} \,. \tag{4.44}$$

In order to identify the two three-dimensional Lie subalgebras, it is convenient to use the representatives. From the above commutation relations, and with the convention $\varepsilon_{\sharp 123} = +1$, we find

$$[\mathbb{M}^{\pm}_{\sharp A}, \mathbb{M}^{\pm}_{\sharp B}] = \mp\varepsilon_{ABC}\mathbb{M}^{\pm}_{\sharp C} \,. \tag{4.45}$$

[10]Due to the interchange property, their self-duality properties hold in both sets of indices.

Therefore, they are two $\mathfrak{su}(2)$ subalgebras that we are going to denote by $\mathfrak{su}_\pm(2)$. This corresponds to the well known Lie algebra isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}_+(2) \oplus \mathfrak{su}_-(2)$.

The (anti-)self-dual combinations can be used in different ways. To start with, they can be used as a hypercomplex structure in a hyperKähler space in the basis in which the components are constant.[11] To fix our conventions and get rid of an excess of $\pm$ and $\mp$ symbols, we are only going to use anti-self-dual hypercomplex structures and we are going to define

$$J^A_{mn} \equiv 2(\mathbb{M}^-_{\sharp A})^{mn} \,. \tag{4.46}$$

Then, the preservation of the hypercomplex structure by the hyperKähler space's Levi-Civita connection 1-form ϖ_{mn},[12]

$$\nabla_{\sigma m} J^A{}_{np} = 0 \,, \tag{4.47}$$

implies

$$[\varpi, J^A] = 0 \,, \quad \Rightarrow \quad \varpi = \varpi^+ \in \mathfrak{su}_+(2) \,, \tag{4.48}$$

so the Levi-Civita connection is self-dual in the $\mathfrak{so}(4)$ indices. The integrability condition of the preservation equation

$$[\nabla_{\sigma m}, \nabla_{\sigma n}] J^A{}_{pq} = 0 \,, \tag{4.49}$$

implies

$$[R, J^A] = 0 \,, \quad \Rightarrow \quad R = R^+ \,, \tag{4.50}$$

and the Riemann tensor is also self-dual in the $\mathfrak{so}(4)$ indices. This property combined with the Bianchi identity $\varepsilon^{mnpq} R_{npqr} = 0$ leads to one of the main properties of hyperKähler spaces: their Ricci flatness

$$R_{mn} = R_{mpn}{}^p = 0 \,. \tag{4.51}$$

The second use of the hypercomplex structures we are interested in is the construction of anti-self-dual SU(2) instantons through the so-called *'t Hooft ansatz*, since they can also be seen as generators of the $\mathfrak{su}(2)$ algebra. In this context they are usually called *'t Hooft symbols* and the following notation is commonly used

$$\eta^A{}_{pq} \equiv 2(\mathbb{M}^+_{\sharp A})^{pq} \,, \qquad \bar{\eta}^A{}_{pq} \equiv 2(\mathbb{M}^-_{\sharp A})^{pq} = J^A{}_{pq} \,. \tag{4.52}$$

In this case however, we will stick to the SO(4)-covariant notation, in terms of which the 't Hooft Ansatz for SU(2) connection 1-forms reads

$$A = \frac{1}{2} A^{mn} \mathbb{M}_{mn} \,, \qquad \text{where} \qquad A^{mn} = (\mathbb{M}^\pm_{pq})^{mn} V^q \, v^p \,, \tag{4.53}$$

[11] This basis may not always exists.

[12] ∇_σ denotes the covariant derivative associated to the hyperKähler metric, always denoted in this text by $d\sigma^2$.

for some SO(4) vector field $V^m(x)$ and some basis of 1-forms in the hyperKähler space $v^m = v^m{}_{\underline{n}}\, dx^n$, related to the Levi-Civita 1-form connection by

$$dv^m + \varpi^{mn} \wedge v^n = 0\,, \tag{4.54}$$

in our conventions. Then, as we see, the 't Hooft ansatz projects the $\mathfrak{so}(4)$ connection A into one of the two $\mathfrak{su}_\pm(2)$ subalgebras,

$$A = \mathbb{M}^{\pm}_{mn}\, V^n\, v^m \in \mathfrak{su}_\pm(2)\,. \tag{4.55}$$

To compute the field strength, which will be demanded to be self-dual, we must first compute

$$dA = \nabla_{\sigma m}\left(\mathbb{M}^{\pm}_{np} V_p\right) v^m \wedge v^n\,. \tag{4.56}$$

At this point, the computations drastically simplify if

$$\nabla_{\sigma m} \mathbb{M}^{\pm}_{np} = 0\,, \tag{4.57}$$

where only the lower indices of $\mathbb{M}^{\pm}$ are taken into account in the covariant derivative. This property, however, is only satisfied by either the self- or the anti-self-dual set of $\mathfrak{so}(4)$ generators. In our case, it is by assumption the anti-self-dual set, $\mathbb{M}^{-}_{mn}$, the one that is covariantly conserved, see (4.47).[13]Thus, from now on we shall use only this one, which means that the gauge connection

$$A = \mathbb{M}^{-}_{mp} V_p\, v^m \in \mathfrak{su}_-(2)\,, \tag{4.58}$$

and the spin-connection 1-form $\varpi^{mn} \in \mathfrak{su}_+(2)$ live in orthogonal subspaces, see (4.48). With this ansatz, and taking into account the commutation relations of the representatives $\mathbb{M}^{-}_{\sharp A}$ in (4.45), the definition for the field strength which leads to the standard SU(2) Yang-Mills field strength

$$F^A = dA^A + \frac{1}{2}\varepsilon_{BC}{}^A A^B \wedge A^C\,, \tag{4.59}$$

is

$$F^{mn} = dA^{mn} + A^{mp} \wedge A^{pn}\,, \tag{4.60}$$

and a simple calculation gives

$$F = -\left[\tfrac{1}{2}\mathbb{M}^{-}_{mn} V^p V^p + \mathbb{M}^{-}_{mp}(\nabla_{\sigma n} V^p - V_n V^p)\right]\, v^m \wedge v^n\,. \tag{4.61}$$

Demanding now self-duality

$$F_{mn} = +\tfrac{1}{2}\varepsilon_{mnpq} F_{pq}\,, \quad \Rightarrow \quad \nabla_{\sigma[m} V_{n]} = 0\,, \quad \text{and} \quad \nabla_{\sigma m} V^m + V_m V^m = 0\,, \tag{4.62}$$

[13]There is the trivial exception of the Euclidean space, whose connection is both self- and anti-self-dual simultaneously.

which is solved by

$$V_m = \partial_m \log P\,, \quad \text{where} \quad \nabla^2_\sigma P = 0\,, \tag{4.63}$$

so P is a harmonic function on the hyperKähler space. Observe that the gauge connection and field strengths are both anti-self-dual in the Lie algebra indices, as a consequence of (4.58). However, in the tangent space indices the field strength is self-dual, because of (4.62). There is no chance that the components $F_{mn}{}^{pq}$ can be interpreted as the components of a Riemann curvature tensor because, as we have just remarked, $F_{mn}{}^{pq} \neq F^{pq}{}_{mn}$. We could have made that interpretation if we had demanded anti-self-duality of the field strength, which leads to more complicated equations for V^m.

The Chern-Simons 3-form, defined by

$$\omega^{\rm YM} \equiv -\left(dA^{mn}\wedge A^{nm} + \tfrac{2}{3}A^{mn}\wedge A^{np}\wedge A^{pm}\right)\,, \tag{4.64}$$

takes for this connection the value

$$\omega^{\rm YM} = -\star_\sigma dV^2 = -\star_\sigma d\left[(\partial\log P)^2\right]\,, \tag{4.65}$$

where $V^2 = V^m V^m$. The instanton number density is, then, given by

$$F^A\wedge F^A = d\omega^{\rm YM} = -d\star_\sigma d\left[(\partial\log P)^2\right] = \nabla^2_\sigma\left[(\partial\log P)^2\right]|v|\,d^4x\,, \tag{4.66}$$

where $|v|$ is the determinant of the Vierbein or the square root of the determinant of the metric. In this and other calculations one should be extremely careful to substract, in the end, any spurious, non-physical singularities arising from the singularities of the 't Hooft ansatz, as explained in Section 4.4.

The Lorentz Chern-Simons 3-form of a SO(4) connection Ω_{mn} in a four-dimensional manifold is defined in this case by[14]

$$\omega^{\rm L} \equiv d\Omega^{mn}\wedge\Omega^{nm} + \tfrac{2}{3}\Omega^{mn}\wedge\Omega^{np}\wedge\Omega^{pm}\,. \tag{4.67}$$

If the connection Ω takes the form of the 't Hooft ansatz in a hyperKähler space

$$\Omega = \mathbb{M}^-_{mp}W^p v^m\,, \qquad W_m = \partial_m\log K\,, \quad \text{where} \quad \nabla^2_\sigma K = 0\,, \tag{4.68}$$

then,

$$\omega^{\rm L} = \star dW^2 = \star_\sigma d\left[(\partial\log K)^2\right]\,, \tag{4.69}$$

and

$$R_{mn}\wedge R_{nm} = d\omega^{\rm L} = d\star_\sigma d\left[(\partial\log K)^2\right] = -\nabla^2_\sigma\left[(\partial\log K)^2\right]|v|\,d^4x\,. \tag{4.70}$$

[14] Observe that now the trace directly implies sum over pairs mn, nm, which leads to a different global sign.

4.2.1 The twisted 't Hooft ansatz in Gibbons-Hawking spaces

The metric of hyperKähler spaces admitting a triholomorphic isometry (Gibbons-Hawking spaces) can always be written in the form

$$d\sigma^2 = \mathcal{H}^{-1}(d\eta + \chi)^2 + \mathcal{H}\, dx^i dx^i\,, \quad d\mathcal{H} = \star_3 d\chi \tag{4.71}$$

where $\star_3$ is the Hodge dual in $\mathbb{E}^3$. In the frame

$$\begin{aligned} v^\sharp &= \mathcal{H}^{-\frac{1}{2}}[d\eta + \chi_{\underline{i}} dx^i]\,, & v_\sharp &= \mathcal{H}^{\frac{1}{2}}\partial_\eta \equiv \partial_\sharp\,, \\ v^i &= \mathcal{H}^{\frac{1}{2}} dx^i\,, & v_i &= \mathcal{H}^{-\frac{1}{2}}[\partial_{\underline{i}} - \chi_{\underline{i}}\partial_\eta] = \partial_i\,, \end{aligned} \tag{4.72}$$

the non-vanishing components of the Levi-Civita connection (4.54) are given by

$$\begin{aligned} \varpi_{\sharp\sharp i} &= -\tfrac{1}{2}\partial_i \log \mathcal{H}\,, & \varpi_{i\sharp j} &= -\tfrac{1}{2}\epsilon_{ijk}\partial_k \log \mathcal{H}\,, \\ \varpi_{\sharp ij} &= -\tfrac{1}{2}\epsilon_{ijk}\partial_k \log \mathcal{H}\,, & \varpi_{ijk} &= \delta_{i[j}\partial_{k]} \log \mathcal{H}\,, \end{aligned} \tag{4.73}$$

and they look very similar to those of a $\mathfrak{so}(4)$ connection based on the 't Hooft ansatz (4.53). As we have explained, the 't Hooft ansatz does not give a spin connection that can be associated to a vierbein, or a proper Riemann tensor and a careful inspection indeed shows that not all signs of the above components match with that ansatz.

Nevertheless, it is possible to *twist* the 't Hooft ansatz to adapt it to the above spin connection 1-form, at the expense of breaking the manifest $\mathfrak{so}(4)$ invariance of the ansatz, which is in agreement with the existence of an isometric direction in the space. This requires the introduction of a new set of self- and anti-self-dual $\mathfrak{so}(4)$ generators

$$\mathbb{N}^{\pm}_{mn} = \pm\tfrac{1}{2}\epsilon_{mnpq}\mathbb{N}^{\pm}_{pq}\,, \tag{4.74}$$

whose representation matrices $(\mathbb{N}^{\pm}_{mn})^{pq}$ have the opposite self-duality properties, that is

$$(\mathbb{N}^{\pm}_{mn})^{pq} = \mp\tfrac{1}{2}\epsilon_{pqrs}(\mathbb{N}^{\pm}_{mn})^{rs}\,. \tag{4.75}$$

These matrices can be constructed using the $\mathbb{M}^{\pm}_{mn}$ matrices and a metric $\eta_{mn} = \mathrm{diag}(-+++)$

$$(\mathbb{N}^{\pm}_{mn})^{pq} \equiv \eta_{mr}\eta_{ns}(\mathbb{M}^{\mp}_{rs})^{pq} \qquad \Rightarrow \qquad (\mathbb{N}^{\pm}_{mn})^{pq} = (\mathbb{N}^{\mp}_{pq})^{mn}\,, \tag{4.76}$$

and satisfy the algebra

$$[\mathbb{N}^{\pm}_{mn}, \mathbb{N}^{\pm}_{pq}] = -2\mathbb{N}^{\pm}_{[m|r}(\mathbb{N}^{\pm}_{pq})^{st}\eta_{sr}\eta_{t|n]} = -2\mathbb{N}^{\pm}_{[m|r}(\mathbb{M}^{\pm}_{pq})^{r}{}_{|n]}\,, \tag{4.77}$$

$$[\mathbb{N}^{+}_{mn}, \mathbb{N}^{-}_{pq}] = 0\,, \tag{4.78}$$

Then, in terms of these matrices, the above spin connection can be rewritten in the form

$$\varpi_{mn} = (\mathbb{N}^{+}_{mn})_{pq}\partial_q \log \mathcal{H}\, v^p \equiv (\mathbb{N}^{+}_{mn})_{pq}\, V^q\, v^p\,, \tag{4.79}$$

with curvature

$$R_{mn} = -\left\{\tfrac{1}{2}(\mathbb{N}^{+}_{mn})_{rs}V^{p}V^{p} + (\mathbb{N}^{+}_{mn})_{rp}(\nabla_{s}V^{p} - V_{s}V^{p})\right\} v^{r} \wedge v^{s} \,. \tag{4.80}$$

The Chern-Simons 3-form is given by

$$\omega^{\text{LHK}} = \star_{\sigma} d\left[(\partial \log \mathcal{H})^{2}\right] \,, \tag{4.81}$$

and, therefore

$$R_{mn} \wedge R_{nm} = d\omega^{\text{LHK}} = d \star_{\sigma} d \left[(\partial \log \mathcal{H})^{2}\right] = -\nabla^{2}_{\sigma}\left[(\partial \log \mathcal{H})^{2}\right] |v|\, d^{4}x \,. \tag{4.82}$$

4.3 The ansatz

It is convenient to describe our ansatz for each field separately, starting with the metric, which is assumed to take the general form

$$ds^{2} = \frac{2}{\mathcal{Z}_{-}} du \left(dt - \frac{\mathcal{Z}_{+}}{2} du\right) - \mathcal{Z}_{0}\, d\sigma^{2} - dz^{\alpha} dz^{\alpha} \,, \tag{4.83}$$

where

$$d\sigma^{2} = h_{\underline{mn}}\, dx^{m} dx^{n} \,, \quad m, n = \sharp, 1, 2, 3 \,, \tag{4.84}$$

is the metric of a four-dimensional hyper-Kähler space where the functions $\mathcal{Z}_{0,+,-}$ take values.

Since most of the computations are conveniently performed using flat indices, it is convenient to introduce the following zehnbein basis

$$e^{+} = \frac{du}{\mathcal{Z}_{-}} \,, \quad e^{-} = dt - \tfrac{1}{2}\mathcal{Z}_{+} du \,, \quad e^{m} = \mathcal{Z}_{0}^{1/2}\, v^{m} \,, \quad e^{\alpha} = dz^{\alpha} \,, \tag{4.85}$$

where v^{m} is the vierbein of the hyper-Kähler metric

$$h_{\underline{mn}} = \delta_{pq}\, v^{p}{}_{\underline{m}} v^{q}{}_{\underline{n}} \,, \tag{4.86}$$

which is characterized by the self-duality of its spin-connection 1-form ϖ^{mn} with respect to the orientation $\varepsilon^{\sharp 123} = +1$. As we have already said, in order to be able to solve the Bianchi identity of the 3-form H to first order in α', we will restrict ourselves to Gibbons-Hawking (GH) spaces (4.71).

The 3-form field strength is assumed to take the form

$$H = \star_{\sigma} d\mathcal{Z}_{0} + d\mathcal{Z}_{-}^{-1} \wedge du \wedge dt \,, \tag{4.87}$$

where $\star_{\sigma}$ is the Hodge operator in the four-dimensional hyper-Kähler metric $d\sigma^{2}$ with the above choice of orientation.

The dilaton field is given by

$$e^{-2\phi} = e^{-2\phi_\infty} \frac{\mathcal{Z}_-}{\mathcal{Z}_0}\,, \tag{4.88}$$

where ϕ_∞ is a constant that, in spaces which asymptote to some vacuum solution, can be identified with the vacuum expectation value of the dilaton, i.e. $e^{\phi_\infty} = g_s$.

Finally, we will include two triplets of SU(2) vector fields defined on the hyper-Kähler space and the ansatz for them will be just the 'Hooft ansatz discussed in the previous section. Then,

$$A_{\mathfrak{i}} = \mathbb{M}^-_{mn}\, \partial_n \log P_{\mathfrak{i}}\, v^m\,, \qquad \mathfrak{i} = 1, 2\,, \tag{4.89}$$

with $P_{\mathfrak{i}}$ harmonic so that their field strength are self-dual

$$F_{\mathfrak{i}} = + \star_\sigma F_{\mathfrak{i}}\,. \tag{4.90}$$

This ansatz generalizes the one recently considered in [193] in three respects:

1. No spherical symmetry is assumed: the ansatz can describe multicenter configurations.
2. The $\mathbb{R}^4$ space transverse to the S5-branes has been replaced by an arbitrary hyper-Kähler space.
3. A second SU(2) gauge field has been added. We will show that it can be used to cancel the α' corrections associated to the non-trivial hyper-Kähler space, just as the first SU(2) gauge field can compensate the α' corrections associated to the S5-brane.

4.3.1 Supersymmetry of the ansatz

All the configurations encompassed by our ansatz preserve 1/4 of the 16 possible supersymmetries, no matter whether they solve the equations of motion or not. The Killing spinor equations associated to the local supersymmetry transformations of the gravitino, dilatino and gaugino are, respectively

$$\nabla^{(+)}_\mu \epsilon \equiv \left(\partial_\mu - \tfrac{1}{4}\not{\Omega}_{(+)\,\mu}\right)\epsilon \;=\; 0\,, \tag{4.91}$$

$$\left(\not{\partial}\phi - \tfrac{1}{12}\not{H}\right)\epsilon \;=\; 0\,, \tag{4.92}$$

$$-\tfrac{1}{4}\alpha' \not{F}_{\mathfrak{i}}^{mn}\epsilon \;=\; 0\,. \tag{4.93}$$

and, using the results of Appendix F it is easy to see that the above equations take the same form as in Section 2.1 of [193], except for the m component of the first equation, which receives a contribution from the spin connection of the four-dimensional hyperKähler space and the "doubling" of the last equation, owed to the presence of a second triplet of SU(2) vector fields.

Since the contribution of the spin connection of the four-dimensional hyperKähler space is self-dual, just as the contribution coming from the conformal factor $\mathcal{Z}_0$, the m

component of the equation simply gets another term containing the chirality projector $\frac{1}{2}(1-\tilde{\Gamma})$ where $\tilde{\Gamma} \equiv \Gamma^{2345}$ is the chirality matrix in the four-dimensional hyperKähler space. Since the two SU(2) gauge fields have self-dual field strengths, the two associated equations (4.93) contain the same chirality projector $\frac{1}{2}(1-\tilde{\Gamma})$ acting on ϵ.

In order to make the paper more self-contained, we write below all the components of the Killing spinor equations in the frame specified in (4.85):

$$\left[\partial_+ + \tfrac{1}{4}\frac{\mathcal{Z}_-\partial_m\mathcal{Z}_+}{\mathcal{Z}_0^{1/2}}\Gamma^m\Gamma^+\right]\epsilon = 0\,, \tag{4.94}$$

$$\left[\partial_- + \tfrac{1}{2}\frac{\partial_m\log\mathcal{Z}_-}{\mathcal{Z}_0^{1/2}}\Gamma^m\Gamma^+\right]\epsilon = 0\,, \tag{4.95}$$

$$\left\{\partial_m + \frac{1}{8\mathcal{Z}_0^{1/2}}\left[\partial_q\log H(\mathbb{N}^+_{np})_{qm} + \partial_q\log\mathcal{Z}_0(\mathbb{M}^+_{qm})_{np}\right]\Gamma^{np}(1-\tilde{\Gamma})\right\}\epsilon = 0\,, \tag{4.96}$$

$$\partial_i\epsilon = 0\,, \tag{4.97}$$

$$-\frac{1}{2\mathcal{Z}_0^{1/2}}\Gamma^m\left[\partial_m\log\mathcal{Z}_-\Gamma^-\Gamma^+ - \partial_m\log\mathcal{Z}_0(1-\tilde{\Gamma})\right]\epsilon = 0\,, \tag{4.98}$$

$$-\frac{1}{8}\alpha'\not{F}_{\mathrm{i}}^{A}(1-\tilde{\Gamma})\epsilon = 0\,. \tag{4.99}$$

We conclude that the Killing spinor equations are solved by constant spinors satisfying the constraints

$$\tilde{\Gamma}\epsilon = +\epsilon\,, \qquad \Gamma^+\epsilon = 0\,, \tag{4.100}$$

exactly as in the solution studied in [193].

4.4 Solving the equations of motion

Since our ansatz is given in terms of the 3-form field strength, it is convenient to start by solving its Bianchi identity (4.3). Due to the structure of our ansatz for H, dH is just a Laplacian in the four-dimensional hyper-Kähler space,

$$dH = d\star_\sigma d\mathcal{Z}_0 = -\nabla^2_\sigma\mathcal{Z}_0\,|v|\,d^4x\,. \tag{4.101}$$

For the remaining pieces appearing in the Bianchi identity we can use (4.66) for the contributions coming from the gauge fields and (F.9) for the contribution coming from the torsionful spin connection.[15] Substituting these partial results in (4.3), we get

$$\nabla^2_\sigma\left\{\mathcal{Z}_0 + \frac{\alpha'}{4}\left[(\partial\log P_1)^2 + (\partial\log P_2)^2 - \left(\partial\log\mathcal{Z}_0^{(0)}\right)^2 - (\partial\log\mathcal{H})^2\right]\right\} = \mathcal{O}(\alpha'^2)\,, \tag{4.102}$$

[15] Recall we are assuming the hyper-Kähler space to be a GH space.

which is solved exactly to this order by[16]

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} - \frac{\alpha'}{4}\left[(\partial \log P_1)^2 + (\partial \log P_2)^2 - \left(\partial \log \mathcal{Z}_0^{(0)}\right)^2 - (\partial \log \mathcal{H})^2\right] + \mathcal{O}(\alpha'^2)\,, \quad (4.103)$$

with

$$\nabla^2_\sigma \mathcal{Z}_0^{(0)} = 0\,. \quad (4.104)$$

Some regular gauge fields, when written in the gauge associated to the 't Hooft ansatz, have singularities that can be removed by a gauge transformation. However, these unphysical singularities end up contributing to the instanton number densities $F^A \wedge F^A$ and $R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a$ as δ-functions, basically because one is taking derivatives at points in which the local form of the gauge field we are using becomes singular. In virtue of the removable singularity theorem of Uhlenbeck [234], it is possible to perform a local gauge transformation that precisely removes those singularities from the evaluation of the instanton number densities and, in the preceding expressions this should be carefully done in the terms inside the squared brackets. Thus, if the gauge fields are indeed regular, and one has eliminated those singularies, the only δ-function singularity that remains is the one associated to the harmonic function $\mathcal{Z}_0^{(0)}$ and this singularity is associated to the presence of solitonic 5-branes, as we will see in the next chapter. These delocalized contributions associated to the instantons correspond, precisely, to the non-singular terms in brackets.

The removal of the singularities is a very subtle problem, because, at the end, the hyperKähler space is not part of the physical space, which is the one that dictates where the physical singularities are and we will not deal with it here. However, this is an important issue from the physical point of view which should be discussed in more depth on a case by case basis. We will make some further comments concerning this point in Section 4.5.

Let us then move to the equations of motion (4.31)-(4.34).

In first place, we find that the Yang-Mills equation (4.34) is automatically satisfied by our ansatz.

The Kalb-Ramond field equation (4.33) reduces to the following Laplace equation in the hyperKähler space

$$\nabla^2_\sigma \mathcal{Z}_- = 0\,, \quad (4.105)$$

which means that the function $\mathcal{Z}_-$ is not corrected at first order in α':

$$\mathcal{Z}_- = \mathcal{Z}_-^{(0)} + \mathcal{O}\left(\alpha'^2\right)\,, \qquad \text{with} \qquad \nabla^2_\sigma \mathcal{Z}_-^{(0)} = 0\,. \quad (4.106)$$

It turns out that there is only one more equation of motion giving non-trivial information, which is the ++ component of Einstein equations.[17] Using the results given in Appendix F, we find

[16]The equations are solved everywhere except at the singularities of the harmonic function $\mathcal{Z}_0^{(0)}$, which, in general, will give δ-function singularities that, in general, indicate the presence of solitonic 5-branes.

[17]We use the frame specified in (4.85).

$$\frac{\mathcal{Z}_-}{2\mathcal{Z}_0}\nabla^2_\sigma \mathcal{Z}_+ = -\frac{\alpha'}{4}R^{(0)}_{(-)\,+abc}R^{(0)}_{(-)\,+}{}^{abc} + \mathcal{O}\left(\alpha'^2\right)$$

$$= -\frac{\alpha'}{4}\frac{\mathcal{Z}^{(0)}_-}{\mathcal{Z}^{(0)}_0}\nabla^2_\sigma\left(\frac{\partial_n\mathcal{Z}^{(0)}_+\partial_n\mathcal{Z}^{(0)}_-}{\mathcal{Z}^{(0)}_0\mathcal{Z}^{(0)}_-}\right) + \mathcal{O}(\alpha'^2)\,,$$

whose general solution is

$$\mathcal{Z}_+ = \mathcal{Z}^{(0)}_+ - \frac{\alpha'}{2}\left(\frac{\partial_n\mathcal{Z}^{(0)}_+\partial_n\mathcal{Z}^{(0)}_-}{\mathcal{Z}^{(0)}_0\mathcal{Z}^{(0)}_-}\right) + \mathcal{O}(\alpha'^2)\,, \tag{4.107}$$

with $\mathcal{Z}^{(0)}_+$ being, again, a harmonic function. Obviously, the same comment concerning the removal of spurious singularities applies here.

It is straightforward to check that the remaining Einstein equations and the dilaton equation of motion (4.32) are satisfied.

Let us recap. We have solved α'-corrected equations of motion and Bianchi identity to first order in α' by making use of the ansatz specified in Section 4.3. The solutions are characterized by a hyperKähler space with metric $d\sigma^2$ which is assumed to enjoy a triholomorphic isometry (GH space) and five harmonic functions defined on that space: $\mathcal{Z}^{(0)}_+, \mathcal{Z}^{(0)}_-, \mathcal{Z}^{(0)}_0$ and $P_\mathfrak{i}$. In other words, the α'-corrected solution is determined by the solution to the zeroth order equations of motion with the form that we have assumed for the metric, dilaton and NSNS 3-form and by the choice of harmonic functions $P_\mathfrak{i}$ that determines gauge fields given in (4.89). Given this, the form of the metric, dilaton and NSNS 3-form in the α'-corrected solution is exactly the same as in the zeroth order solution (by assumption) but now with the functions $\mathcal{Z}^{(0)}_+, \mathcal{Z}^{(0)}_-, \mathcal{Z}^{(0)}_0$ replaced by

$$\mathcal{Z}_+ = \mathcal{Z}^{(0)}_+ - \frac{\alpha'}{2}\left(\frac{\partial_n\mathcal{Z}^{(0)}_+\partial_n\mathcal{Z}^{(0)}_-}{\mathcal{Z}^{(0)}_0\mathcal{Z}^{0}_-}\right) + \mathcal{O}(\alpha'^2)\,, \tag{4.108}$$

$$\mathcal{Z}_- = \mathcal{Z}^{(0)}_- + \mathcal{O}(\alpha'^2)\,, \tag{4.109}$$

$$\mathcal{Z}_0 = \mathcal{Z}^{(0)}_0 - \frac{\alpha'}{4}\left[\sum_{\mathfrak{i}=1}^{2}(\partial\log P_\mathfrak{i})^2 - (\partial\log\mathcal{Z}^{(0)}_0)^2 - (\partial\log\mathcal{H})^2\right] + \mathcal{O}(\alpha'^2)\,. \tag{4.110}$$

As we see, the α' corrections to the function $\mathcal{Z}_0$ can be easily cancelled by setting $P_1 = \mathcal{Z}^{(0)}_0$ and $P_2 = \mathcal{H}$. This is however an option and in general the functions $P_\mathfrak{i}$ are arbitrary harmonic functions. In fact, if we do not care about the cancellation of the α' corrections to $\mathcal{Z}_0$, nothing prevents us from adding as many conmuting SU(2) instantons as we want, as long as the gauge group $\mathrm{SU}(2)\times\cdots\times\mathrm{SU}(2)$ fits into one of the two possible heterotic gauge groups: SO(32) or $\mathrm{E}_8\times\mathrm{E}_8$. If that is the case, the only difference would be that now the sum in (4.110) would run from $\mathfrak{i} = 1$ to $\mathfrak{n}$, where $\mathfrak{n}$ is the number of instantons.

4.5 T-duality

As we have discussed in Section 4.3, the solutions we have found are a generalization of those studied in [193] with a very similar structure but more non-trivial harmonic functions that can be interpreted as describing more extended objects. As we have already commented, the functions $\mathcal{Z}_{-,+,0}$ are associated, respectively, to a fundamental string (F1), a momentum wave (P) and Neveu-Schwarz solitonic 5-branes (S5). The functions $P_{1,2}$ are associated to gauge 5-branes sourced by the instantons. The qualitatively new feature is the non-trivial hyperKähler space which, generically, describes gravitational instantons, and the additional (triholomorphic) isometry of this space, which reduces the possible hyperKähler spaces to be of GH type. These are completely determined by a harmonic function, $\mathcal{H}$. The typical choice, $\mathcal{H} = 1 + \frac{q_H}{r}$, corresponds to a Kaluza-Klein (KK) monopole, often called Euclidean Taub-NUT space.

In [193], it was studied how T-duality acts in the direction along which the fundamental string is wounded, z, in the presence of first-order α' corrections which affect $\mathcal{Z}_+$ but not $\mathcal{Z}_-$. At zeroth order in α' the standard Buscher rules would simply interchange $\mathcal{Z}_+ \leftrightarrow \mathcal{Z}_-$. This would be wrong once the corrections are incorporated since only the transformed $\mathcal{Z}'_+$ can receive α' corrections. Somewhat extraordinarily, using the α'-corrected Buscher rules proposed in [117], it was shown in [193] that the α' corrections of the transformed solution only occur where they should and, therefore, the solutions, as a family, are self-T-dual, as it happens at zeroth order in α'. This was a highly non-trivial test for both the solutions and the T-duality rules.

The existence of a second non-trivial isometry in the GH space transverse to the S5-branes provides us with another non-trivial test. At zeroth order in α', the single S5-brane solution and the KK monopole are T-dual, and T-duality simply interchanges their associated harmonic functions $\mathcal{Z}_0$ and H. Now, only the former has α' corrections and T-duality should leave them there since the solutions we have found should be self-T-dual as a family.

Let us write down the α'-corrected T-duality rules proposed in [117]. If x is the direction along which we want to perform the T-duality transformation, they read ($\mu, \nu \neq \underline{x}$)

$$
\begin{aligned}
g'_{\underline{xx}} &= g_{\underline{xx}}/G^2_{\underline{xx}}\,, & B'_{\underline{x}\mu} &= -B_{\underline{x}\mu}/G_{\underline{xx}} - G_{\underline{x}\mu}/G_{\underline{xx}}\,,\\
g'_{\underline{x}\mu} &= -g_{\underline{x}\mu}/G_{\underline{xx}} + g_{\underline{xx}}G_{\underline{x}\mu}/G^2_{\underline{xx}}\,, & A'^A_{\mu} &= A^A_{\mu} - A^A_{\underline{x}}G_{\underline{x}\mu}/G_{\underline{xx}}\,,\\
g'_{\mu\nu} &= g_{\mu\nu} + \left[g_{\underline{xx}}G_{\underline{x}\mu}G_{\underline{x}\nu} - 2G_{\underline{xx}}G_{\underline{x}(\mu}g_{\nu)\underline{x}}\right]/G^2_{\underline{xx}}\,, & A'^A_{\underline{x}} &= -A^A_{\underline{x}}/G_{\underline{xx}}\,,\\
B'_{\mu\nu} &= B_{\mu\nu} - G_{\underline{x}[\mu}G_{\nu]\underline{x}}/G_{\underline{xx}}\,, & e^{-2\phi'} &= e^{-2\phi}|G_{\underline{xx}}|\,,
\end{aligned}
\tag{4.111}
$$

where $G_{\mu\nu}$ (for all the possible values of the indices μ, ν including $\underline{x}$) is defined by

$$
G_{\mu\nu} \equiv g_{\mu\nu} - B_{\mu\nu} - \frac{\alpha'}{4}\left(A^A{}_{\mu}A^A{}_{\nu} + \Omega_{(-)\,\mu}{}^{a}{}_{b}\Omega_{(-)\,\nu}{}^{b}{}_{a}\right)\,. \tag{4.112}
$$

The use of these rules requires the explicit knowledge of the components of the Kalb-Ramond 2-form B, which are gauge-dependent. It is natural to use the gauge of the 't Hooft ansatz in which the Chern-Simons terms take the forms computed in (4.65) and (F.8), which we reproduce here for convenience[18]

$$\omega^{\rm YM} = -\star_\sigma d\left[(\partial\log P_1)^2 + (\partial\log P_2)^2\right] + \mathcal{O}(\alpha')\,, \tag{4.113}$$

$$\omega^{\rm L}_{(-)} = \star_\sigma d\left[(\partial\log\mathcal{H})^2 + \left(\partial\log\mathcal{Z}_0^{(0)}\right)^2\right] + \mathcal{O}(\alpha')\,. \tag{4.114}$$

Then,

$$dB = H - \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{\rm L}_{(-)}\right) = \star_\sigma d\mathcal{Z}_0^{(0)} + d\mathcal{Z}_-{}^{-1}\wedge du\wedge dt + \mathcal{O}(\alpha'^2)\,, \tag{4.115}$$

and

$$B = \xi_0 + \mathcal{Z}_-{}^{-1}\,du\wedge dt + \mathcal{O}(\alpha'^2)\,, \tag{4.116}$$

where $\xi_0 = \frac{1}{2}\xi_{0\,mn}\,v^m\wedge v^n$ is a 2-form on the hyperKähler space such that

$$d\xi_0 = \star_\sigma d\mathcal{Z}_0^{(0)}\,. \tag{4.117}$$

The integrability condition of this equation is the harmonicity of $\mathcal{Z}_0^{(0)}$ in the hyperKähler space, which guarantees the existence of ξ_0.

In order to apply the Buscher T-duality rules, one needs to compute the tensor $G_{\mu\nu}$ defined above in (4.112). In ten-dimensional flat indices, its non-vanishing components are

$$G_{++} = -\alpha'\frac{\partial_m\mathcal{Z}_+\partial_m\mathcal{Z}_-}{\mathcal{Z}_0}\,, \tag{4.118}$$

$$G_{-+} = 2\,, \tag{4.119}$$

$$G_{\alpha\beta} = -\delta_{\alpha\beta}\,, \tag{4.120}$$

$$\begin{aligned} G_{mn} = & -\delta_{mn} - \frac{\xi_{0\,mn}}{\mathcal{Z}_0} - \frac{\alpha'}{4\mathcal{Z}_0}\left\{\delta_{mn}\left[\sum_{\rm i=1}^{2}(\partial\log P_{\rm i})^2 - (\partial\log\mathcal{H})^2 - \left(\partial\log\mathcal{Z}_0^{(0)}\right)^2\right]\right.\\ & -\sum_{\rm i=1}^{2}\partial_m\log P_{\rm i}\,\partial_n\log P_{\rm i} + \partial_m\log\mathcal{H}\,\partial_n\log\mathcal{H} + \partial_m\log\mathcal{Z}_0^{(0)}\,\partial_n\log\mathcal{Z}_0^{(0)}\\ & \left.+2\partial_m\log\mathcal{Z}_-\partial_n\log\mathcal{Z}_-\right\}\,. \end{aligned} \tag{4.121}$$

[18] According to the discussion in the previous section, in certain cases at least, we should eliminate the spurious singularities from these Chern-Simons terms. In general, this should simply result in a shift by a harmonic function of $\mathcal{Z}_0$ that can be absorbed in $\mathcal{Z}_0^{(0)}$.

If we T-dualize along $z = t - u$, we get

$$
\begin{aligned}
ds^{2\prime} &= \frac{2}{\mathcal{Z}'_-}du'\left(dt' - \frac{\mathcal{Z}'_+}{2}du'\right) - \mathcal{Z}_0\, d\sigma^2 - dz^\alpha dz^\alpha\,, && (4.122)\\
A'^{A}_{\mathrm{i}} &= A^{A}_{\mathrm{i}}\,, && (4.123)\\
B' &= \xi_0 + \left(\frac{1}{\mathcal{Z}'_-} - 1\right) du' \wedge dt'\,, && (4.124)\\
e^{2\phi'} &= e^{2\phi_\infty}\frac{\mathcal{Z}_0}{\mathcal{Z}'_-}\,, && (4.125)
\end{aligned}
$$

where $t' = -t$ and $u' = z$ and

$$
\mathcal{Z}'_- = \mathcal{Z}^{(0)}_+ + \mathcal{O}\left(\alpha'^2\right)\,, \tag{4.126}
$$

$$
\mathcal{Z}'_+ = \mathcal{Z}^{(0)}_- - \frac{\alpha'}{2}\frac{\partial_n\mathcal{Z}^{(0)}_-\partial_n\mathcal{Z}^{(0)}_+}{\mathcal{Z}^{(0)}_0\mathcal{Z}^{(0)}_+} + \mathcal{O}\left(\alpha'^2\right) \tag{4.127}
$$

Then, as we can see, the T-dual solution belongs to the same family as the original. This is the result obtained in [193], extended to accomodate a non-trivial hyperKähler transverse space.

Let us assume that the functions $\mathcal{Z}_{0,+,-}$ and P_{i} are independent of the coordinate adapted to the triholomorphic isometry of the GH metric, η, as $\mathcal{H}$ is. Then the isometry of the GH space is also an isometry of the full solution and one can T-dualize it along η. In this case, the harmonic functions are harmonic in $\mathbb{E}^3$ and (4.117) can be rewritten as

$$
d\xi_0 = (d\eta + \chi)\wedge \star_3 d\mathcal{Z}^{(0)}_0 \equiv (d\eta+\chi)\wedge d\chi_0\,, \quad \text{where} \quad \left\{\begin{aligned} d\chi &= \star_3 d\mathcal{H}\\ d\chi_0 &\equiv \star_3 d\mathcal{Z}^{(0)}_0 \end{aligned}\right. \,. \tag{4.128}
$$

This implies that, up to a closed 2-form,

$$
\xi_0 = \chi_0 \wedge (dz + \chi) + \tilde{\xi}_0\,, \tag{4.129}
$$

where $\tilde{\xi}_0$ is a 2-form on $\mathbb{E}^3$ such that

$$
d\tilde{\xi}_0 = d\chi \wedge \chi_0\,. \tag{4.130}
$$

Then, the original solution, written in coordinates adapted to the isometry we want to T-dualize with respect to, is

$$
\begin{aligned}
ds^2 &= \frac{2}{\mathcal{Z}_-}du\left(dt - \frac{\mathcal{Z}_+}{2}du\right) - \mathcal{Z}_0\left[\frac{1}{\mathcal{H}}(d\eta+\chi)^2 + \mathcal{H}\,dx^i dx^i\right] - dz^\alpha dz^\alpha\,, \quad (4.131)\\
A_{\mathfrak{i}} &= \mathbb{M}^-_{mn}\,\partial_n \log P_{\mathfrak{i}}\, v^m \\
&= \mathcal{H}^{-1}\mathbb{M}^-_{\sharp j}\,\partial_{\underline{j}} \log P_{\mathfrak{i}}\,(d\eta+\chi) + \mathbb{M}^-_{jk}\,\partial_{\underline{k}} \log P_{\mathfrak{i}}\, dx^j\,, \quad (4.132)\\
B &= \chi_0 \wedge (d\eta+\chi) + \tilde{\xi}_0 + \frac{1}{\mathcal{Z}_-}du\wedge dt\,, \quad (4.133)\\
e^{2\phi} &= e^{2\phi_\infty}\frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,. \quad (4.134)
\end{aligned}
$$

and the T-dual solution is

$$
\begin{aligned}
ds'^2 &= \frac{2}{\mathcal{Z}_-}du\left(dt - \frac{\mathcal{Z}_+}{2}du\right) - \mathcal{Z}'_0\left[\frac{1}{\mathcal{H}'}(d\eta+\chi_0)^2 + \mathcal{H}' dx^i dx^i\right] - dz^\alpha dz^\alpha, \quad (4.135)\\
A'_{\mathfrak{i}} &= \mathbb{M}^-_{mn}\,\partial'_n \log P_{\mathfrak{i}}\, v'^m \\
&= \mathcal{H}'^{-1}\,\mathbb{M}^-_{\sharp j}\,\partial_{\underline{j}} \log P_{\mathfrak{i}}\,(d\eta+\chi_0) + \mathbb{M}^-_{jk}\,\partial_{\underline{k}} \log P_{\mathfrak{i}}\, dx^j\,, \quad (4.136)\\
B' &= \chi_0 \wedge (d\eta+\chi) + \tilde{\xi}'_0 + \frac{1}{\mathcal{Z}_-}du\wedge dt\,, \quad (4.137)\\
e^{2\phi} &= e^{2\phi_\infty}\frac{\mathcal{Z}'_0}{\mathcal{Z}_-}\,. \quad (4.138)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}' &= \mathcal{Z}_0^{(0)}\,, \quad (4.139)\\
\mathcal{Z}'_0 &= \mathcal{H} - \frac{\alpha'}{4}\left[\sum_{\mathfrak{i}=1}^{2}(\partial'\log P_{\mathfrak{i}})^2 - (\partial'\log \mathcal{Z}_0^{(0)})^2 - (\partial'\log\mathcal{H})^2\right] + \mathcal{O}(\alpha'^2)\,, \quad (4.140)
\end{aligned}
$$

$\tilde{\xi}'_0$ is a 2-form on $\mathbb{E}^3$ such that

$$
d\tilde{\xi}'_0 = d\chi_0 \wedge \chi\,, \quad (4.141)
$$

and $\partial'_m = v'_m{}^{\underline{m}}\,\partial_{\underline{m}}$ and v'^m are derivatives in flat indices and vierbein associated with the new GH space obtained by replacing $\mathcal{H} \to \mathcal{H}' = \mathcal{Z}_0^{(0)}$ and $\chi \to \chi_0$.

The T-dual solution clearly belongs to the same family as the original and the net effect of the T-duality transformation is the interchange between the harmonic functions associated to S5-branes and KK monopoles ($\mathcal{Z}_0^{(0)}$ and $\mathcal{H}$) everywhere, including the α' corrections. This interchange necessarily has to be accompanied by the interchange of associated 1-forms χ_0 and χ.

4.6 Discussion

In this section we have found a wide class of α'-corrected backgrounds that, in general, describes supersymmetric intersections of several extended objects of string theory and which can give raise, upon toroidal compactification, to extremal black holes in five and four dimensions. The $\alpha' \to 0$ limit of some of the heterotic black holes that can be described with our ansatz is well-known since the nineties [146,147] and the corresponding Bekenstein-Hawking entropy has been reproduced by counting BPS states [67, 183, 184, 235]. However, not much is known about the α' corrections to these backgrounds and we believe this is an important aspect that deserves to be studied since it is precisely there where purely stringy effects start taking place. Therefore, this is the ideal arena to start applying the results derived in this chapter.

It would also be interesting to study if our results can be extended further in several directions. For instance, to include hyperKähler spaces which do not enjoy a triholomorphic isometry, such as the Atiyah-Hitchin space [236], which has already been considered in the context of supergravity in [237,238]. This would amount to figure out if the instanton number density of these kind of metrics can also be written as a Laplacian. Another possible direction would be to extend our ansatz to accomodate rotating spacetimes. This, in particular, would allow us to compute the α' corrections to the three-charge black ring of [239]. We will come back to this point in Chapter 7.

5

Stringy corrections to heterotic black holes

In Chapter 4, we have constructed a large family of solutions of the heterotic superstring effective action at first order in α'. Generically, these solutions describe well-known systems consisting of (supersymmetric) intersections of fundamental strings (F1) with momentum flowing along them (P), solitonic 5-branes (NS or S5) and Kaluza-Klein monopoles (KK). The five-dimensional, extremal, three-charge black holes studied in [154, 193] are simple members of this family without KK monopoles. The main goal of this chapter will be to study the case with a KK monopole, although we will also review the five-dimensional case studied in [193] for the sake of completeness. The addition of a KK monopole will allow us to study four-dimensional, extremal, four-charge black holes which will contain the first-order in α' corrections to the heterotic version of the black holes whose microscopic entropy was computed and compared with the supergravity result in [183, 184, 235].[1]

The agreement between the Bekenstein-Hawking entropy and the degeneracy of string microstates for the black holes mentioned above, initially obtained in regimes in which the α' corrections can be safely ignored (large charge regime), is one of the greatest triumphs of string theory. These results have been extended in several directions to include rotation [242], non-trivial horizon topologies (black rings) [239], etc.

A very important question to study is whether this agreement between the values of the Bekenstein-Hawking entropy calculated by macroscopic and microscopic methods still holds when α' corrections —genuinely stringy effects associated to the finite size of the strings— are taken into account.

In the calculation of the entropy by microstate counting, the AdS/CFT correspondence has proven extraordinarily useful, shedding results that account for all the contributions in the α' perturbative expansion. This is due to the fact that the near-horizon geometry of all the black hole solutions we consider is $\mathrm{AdS}_3 \times \mathbb{S}^3/\mathbb{Z}_W \times \mathbb{T}^4$. The AdS_3 and $\mathbb{S}^3$ factors are standard in the three-charge family of extremal black holes. The quotient of the sphere by $\mathbb{Z}_W$ is related to the presence of a KK monopole with topological charge W. Heterotic string theory on this background was studied in [243], identifying the central charges of the dual CFT. Applying the Cardy formula [136] one obtains the following expression for the entropy

$$S_{\mathrm{CFT}} = 2\pi \sqrt{\mathcal{Q}_{\mathrm{F1}} \mathcal{Q}_{\mathrm{P}} \, (k+2)}\,, \tag{5.1}$$

where $\mathcal{Q}_{\mathrm{F1}}$ and $\mathcal{Q}_{\mathrm{P}}$ are the winding and momentum charges and k is the total level of affine algebra $\widehat{\mathrm{SL}(2)}$ in the right-moving sector. This number, minus two units, was identified

[1] See also [240, 241] and references therein. The $\alpha' \to 0$ limits of these solutions are well known and were first obtained in [147] directly in the heterotic frame.

in [243] with the product of the KK monopole charge and S5 charge: $k = \mathcal{Q}_{S5}W + 2$. As we are going to study in this chapter, the presence of Chern-Simons terms in the action allows for more than one notion of charge [179,180]. On the one hand, there is the "near-horizon" or "brane-source" charge, which is proportional to the parameters that characterize the sources of the solution (e.g. the number of S5-branes, N). On the other hand, there is the "Maxwell" or "asymptotic" charge, which receive contributions from the higher-curvature terms. Being able to make this distinction is essential in order to write the entropy in terms of the same variables used in the counting of string microstates.

The macroscopic calculation of the α' corrections to the black-hole entropy faces a number of difficulties:

1. Finding the α'-corrected solutions is a quite complicated task, owing to the higher-order in curvature terms present in the equations of motion and the complicated interactions between them. In the heterotic superstring effective action, there is an infinite series of terms related to the supersymmetrization of the Lorentz Chern-Simons 3-form present in the definition of the NS 3-form H, [108, 244]. Apart from these, there are also terms of higher order in the curvature that seem to be unrelated to them [232]. We will not deal with them here, since they appear at order α'^3 and we will just work at first order in α'.

2. The black-hole entropy of the α'-corrected solutions is no longer simply given by the Bekenstein-Hawking formula. Instead, one has to make use of the Wald formula, valid when the action contains higher-curvature terms [58, 245]. We will discuss, however, that the presence of Chern-Simons terms represents an obstacle for the direct application of the formula which can nevertheless be avoided by rewriting the heterotic effective action in convenient form [246, 247].

In order to circumvent these difficulties one may try to work with the near-horizon limit. The entropy function formalism developed by Sen [241, 248] provides an elegant and powerful tool to find the near-horizon solutions of extremal black holes and to compute their entropy, making a comparison with the microscopic result (5.1) possible. This approach has important drawbacks, though: it is not guaranteed that a solution interpolating between the near-horizon geometry and Minkowski spacetime describing an asymptotically-flat black-hole spacetime exists and, if it does, it does not give any information on how the higher-derivative corrections affect the physical properties of the solution, such as the values of the conserved charges.

Fortunately, the family of solutions constructed in Chapter 4 allows us to study, for the first time, the complete black-hole spacetime without having to restrict to the near-horizon solution.

5.1 Review of the zeroth-order solutions

Let us present the family of heterotic backgrounds whose dimensional reduction to five and four dimensions gives raise to the supersymmetric (and therefore extremal) black holes that we want to study at first order in α'. These are solutions of ten-dimensional $\mathcal{N} = 1$ supergravity without vector multiplets and they preserve one quarter out of the sixteen possible spacetime supersymmetries. The metric $g_{\mu\nu}$, the NSNS 3-form H and the dilaton ϕ are respectively given by

$$ds^2 = \frac{2}{\mathcal{Z}_-^{(0)}} du \left(dt - \frac{\mathcal{Z}_+^{(0)}}{2} du \right) - \mathcal{Z}_0^{(0)} \, d\sigma^2 - dz^\alpha dz^\alpha \, , \tag{5.2}$$

$$H = \star_\sigma d\mathcal{Z}_0^{(0)} + d\left(1/\mathcal{Z}_-^{(0)}\right) \wedge du \wedge dt \, , \tag{5.3}$$

$$e^{2\phi} = e^{2\phi_\infty} \frac{\mathcal{Z}_0^{(0)}}{\mathcal{Z}_-^{(0)}} \, , \tag{5.4}$$

where $d\sigma^2 = h_{\underline{mn}} \, dx^m dx^n$ is the metric of a four-dimensional hyper-Kähler space where the functions $\mathcal{Z}_{+,-,0}^{(0)}$ are defined. Therefore, all the fields of this configuration are independent of the time coordinate t, of the coordinates $z^\alpha \sim z^\alpha + 2\pi\ell_s$ (with $\alpha = 1, \ldots, 4$) and of the light-cone coordinate $u = t - z$. The coordinate $z \sim z + 2\pi R_z$ is the fifth compact direction necessary to reduce the solutions down to, at least, five dimensions.

The zeroth-order equations of motion and the Bianchi identity, $dH = 0$, are satisfied if the functions that determine the configuration are harmonic in the hyper-Kähler space, namely

$$\nabla_\sigma^2 \mathcal{Z}_{0,+,-}^{(0)} = 0 \, . \tag{5.5}$$

It is then clear that the zeroth-order solutions are specified by the choice of an hyper-Kähler metric $h_{\underline{mn}}$ and three harmonic functions on this space: $\mathcal{Z}_0^{(0)}, \mathcal{Z}_+^{(0)}$ and $\mathcal{Z}_-^{(0)}$. Generically, this background corresponds to a superposition of solitonic 5-branes (S5), fundamental string (F1), momentum (P) and Kaluza-Klein monopoles (KK):

- The function $\mathcal{Z}_0^{(0)}$ is associated to S5-branes that wrap the 5-torus $\mathbb{T}^5 = \mathbb{T}^4 \times \mathbb{S}_z^1$, where $\mathbb{T}^4$ is the four-dimensional torus parametrized by the coordinates z^α.
- The function $\mathcal{Z}_-^{(0)}$ is associated to a fundamental string wrapping the circle $\mathbb{S}_z^1$ which has been smeared over the torus $\mathbb{T}^4$.
- The function $\mathcal{Z}_+^{(0)}$ is associated to a pp-wave travelling along z which has also been smeared over the torus $\mathbb{T}^4$.
- Finally, since the hyper-Kähler metric has self-dual curvature, it generically describes gravitational instantons (except when it is trivial). As in the previous chapter, we will restrict to a metric of the Gibbons-Hawking type, which are those that will allow us to study the four-dimensional black holes of [147]. This kind of metrics are often known in the literature as KK monopoles, since the KK vector that one obtains after dimensional reduction over the isometric coordinate of the Gibbons-Hawking space satisfies the Dirac magnetic monopole equation, see (5.10).

5.1.1 Dimensional reduction to five and four dimensions

Let us reduce these ten-dimensional solutions over the compact space $\mathbb{T}^4 \times \mathbb{S}_z^1$. For the purposes of this chapter, it is more than enough with the dimensional reduction of the metric. For further details, we refer to Appendix E.

The relation between the five-dimensional metric in the modified Einstein frame (whose line element is denoted as $ds^2_{\mathrm{E},5}$) and the ten-dimensional string-frame metric is

$$ds^2 = e^{\phi-\phi_\infty}\left[(k/k_\infty)^{-2/3}ds^2_{\mathrm{E},5} - (k/k_\infty)^2\left(dz + A^+\right)^2\right] - dz^\alpha dz^\alpha\,, \tag{5.6}$$

where

$$(k/k_\infty)^2 = \frac{\mathcal{Z}_+^{(0)}}{\sqrt{\mathcal{Z}_0^{(0)}\mathcal{Z}_-^{(0)}}}\,, \qquad \text{and} \qquad A^+ = \left(\mathcal{Z}_+^{(0)^{-1}} - 1\right)dt\,, \tag{5.7}$$

are the Kaluza-Klein scalar and vector respectively. The asymptotic value of the scalar, k_∞, is related to the asymptotic radius of the compact direction z by $R_z = k_\infty \ell_s$. Finally, the five-dimensional metric is found to be

$$ds^2_{\mathrm{E},5} = \left(\mathcal{Z}_0^{(0)}\mathcal{Z}_+^{(0)}\mathcal{Z}_-^{(0)}\right)^{-2/3} dt^2 - \left(\mathcal{Z}_0^{(0)}\mathcal{Z}_+^{(0)}\mathcal{Z}_-^{(0)}\right)^{1/3} d\sigma^2\,. \tag{5.8}$$

Let us now assume that there is an additional spacelike isometry that respects the hyper-Kähler structure. Then, the metric $d\sigma^2$ can be written as a Gibbons-Hawking metric [158, 159]:

$$d\sigma^2 = \mathcal{H}^{-1}\left(d\eta + \chi\right)^2 + \mathcal{H}\, dx^i dx^i\,, \qquad i = 1,2,3.\,, \tag{5.9}$$

where χ and $\mathcal{H}$ are a 1-form and a function on $\mathbb{E}^3$ satisfying

$$d\chi = \star_3 d\mathcal{H}\,, \tag{5.10}$$

which implies that $\mathcal{H}$ is harmonic in $\mathbb{E}^3$. Further assuming that the functions $\mathcal{Z}^{(0)}_{0,+,-}$ are also independent of $\eta \sim \eta + 2\pi R_\eta$, we can compactify the five-dimensional solution along this coordinate, which yields

$$ds^2_{\mathrm{E},5} = (\ell/\ell_\infty)^{-1}\, ds^2_{\mathrm{E},4} - (\ell/\ell_\infty)^2\left(d\eta + \chi\right)^2\,, \tag{5.11}$$

where now the Kaluza-Klein scalar, ℓ, is given by

$$(\ell/\ell_\infty)^2 = \frac{\left(\mathcal{Z}_0^{(0)}\mathcal{Z}_+^{(0)}\mathcal{Z}_-^{(0)}\right)^{1/3}}{\mathcal{H}}\,, \qquad \text{with} \qquad R_\eta = \ell_\infty \ell_s\,. \tag{5.12}$$

Finally, the four-dimensional metric is

$$ds^2_{\mathrm{E},4} = e^{2U}\, dt^2 - e^{-2U}\, dx^i dx^i\,, \quad \text{with} \quad e^{-2U} = \sqrt{\mathcal{Z}_0^{(0)}\mathcal{Z}_+^{(0)}\mathcal{Z}_-^{(0)}\mathcal{H}}\,. \tag{5.13}$$

Let us see now how particular choices of the harmonic functions and of the hyper-Kähler metric allow us to describe five- and four-dimensional supersymmetric black holes.

5.1.2 Five-dimensional black holes

Let us start with the description of the simplest five-dimensional black holes that one can describe with our ansatz. The hyper-Kähler metric is taken to be simply the Euclidean metric, i.e. $h_{\underline{mn}} = \delta_{\underline{mn}}$. In spherical coordinates, $\rho^2 \equiv x^m x^m$, we have

$$d\sigma^2 = d\rho^2 + \rho^2 d\Omega_{(3)}^2 \,, \tag{5.14}$$

where

$$d\Omega_{(3)}^2 = \frac{1}{4}\left(d\Psi^2 + d\theta^2 + d\phi^2 + 2\cos\theta d\Psi d\phi\right) \,, \tag{5.15}$$

is the metric of the round 3-sphere $\mathbb{S}^3$. Then, the five-dimensional metric (5.11) reduces to

$$ds_{\text{E},5}^2 = \left(\mathcal{Z}_0^{(0)}\mathcal{Z}_+^{(0)}\mathcal{Z}_-^{(0)}\right)^{-2/3} dt^2 - \left(\mathcal{Z}_0^{(0)}\mathcal{Z}_+^{(0)}\mathcal{Z}_-^{(0)}\right)^{1/3}\left(d\rho^2 + \rho^2 d\Omega_{(3)}^2\right) \,. \tag{5.16}$$

The simplest possible choice of harmonic functions,

$$\mathcal{Z}_0^{(0)} = 1 + \frac{\mathcal{Q}_0}{\rho^2} \,, \qquad \mathcal{Z}_+^{(0)} = 1 + \frac{\mathcal{Q}_+}{\rho^2} \,, \qquad \mathcal{Z}_-^{(0)} = 1 + \frac{\mathcal{Q}_-}{\rho^2} \,, \tag{5.17}$$

gives raise to an extremal, asymptotically-flat, spherically-symmetric black hole with three electric charges given by $\mathcal{Q}_0, \mathcal{Q}_+$ and $\mathcal{Q}_-$. This solution was first found Cvetič and Youm as a particular case of a more general family of five-dimensional heterotic black holes [146]. The horizon is placed at $\rho = 0$ and the near-horizon geometry is $\text{AdS}_2 \times \mathbb{S}^3$. The ADM mass M and the Bekenstein-Hawking entropy S_{BH} are given by

$$M = \frac{\pi}{4G_{\text{N}}^{(5)}}\left(\mathcal{Q}_0 + \mathcal{Q}_+ + \mathcal{Q}_-\right) \,, \tag{5.18}$$

$$S_{\text{BH}} = \frac{\pi^2}{2G_{\text{N}}^{(5)}}\sqrt{\mathcal{Q}_0\mathcal{Q}_+\mathcal{Q}_-} \,, \tag{5.19}$$

where $G_{\text{N}}^{(5)}$ is the five-dimensional Newton constant, whose expression in terms of the ten-dimensional moduli is

$$G_{\text{N}}^{(5)} = \frac{G_{\text{N}}^{(10)}}{2\pi R_z \left(2\pi \ell_s\right)^4} = \frac{\pi g_s^2 \alpha'^2}{4R_z} \,. \tag{5.20}$$

The electric charges of the black hole ($\mathcal{Q}_0, \mathcal{Q}_+$ and $\mathcal{Q}_-$) can be related to the parameters that characterize the ten-dimensional (string) sources, whose intersection diagram is given in Table 5.1. We will discuss this in detail in Section 5.3 after the computation of the α' corrections. Advancing information, the relation between the electric charges of the black hole and the sources is

$$\mathcal{Q}_0 = \alpha' N \,, \qquad \mathcal{Q}_+ = \frac{g_s^2 \alpha'^2}{R_z^2} n \,, \qquad \mathcal{Q}_- = g_s^2 \alpha' w \,, \tag{5.21}$$

where N is the number of S5-branes, n is the quantized momentum along the z-direction and w is the winding number of the fundamental string. We can now use these relations to rewrite the mass and the entropy in terms of N, w and n as follows

$$M = \frac{R_z}{g_s^2 \ell_s^2} N + \frac{n}{R_z} + \frac{R_z}{\ell_s^2} w \,, \tag{5.22}$$

$$S_{\rm BH} = 2\pi\sqrt{nwN}\,. \tag{5.23}$$

	t	z	z^1	z^2	z^3	z^4	x^1	x^2	x^3	x^4
F1	×	×	∼	∼	∼	∼	−	−	−	−
P	×	×	∼	∼	∼	∼	−	−	−	−
S5	×	×	×	×	×	×	−	−	−	−

Table 5.1: Sources of the ten-dimensional backgrounds which give raise to five-dimensional black holes after dimensional reduction on $\mathbb{T}^4 \times \mathbb{S}^1_z$. $\times$ stands for the worldvolume directions and $-$ for the transverse directions. The symbol $\sim$ stands for the transverse directions over which the corresponding extended object has been smeared.

5.1.3 Four-dimensional black holes

The simplest supersymmetric black holes in four dimensions that one can construct as solutions of the heterotic effective action contain a KK monopole (see Table 5.2). Let us explain in detail how it arises in our set-up. As already discussed in Section 5.1.1, we assume that the isometry that is needed to reduce the solutions to four dimensions is triholomorphic (i.e., that it respects the hyper-Kähler structure). Then, we can write the hyper-Kähler metric as a Gibbons-Hawking metric (5.9) using coordinates adapted to the isometry. This metric is characterized by a harmonic function $\mathcal{H}$ and 1-form χ satisfying (5.10), which is nothing but the Dirac magnetic monopole. The simplest non-trivial choice for $\mathcal{H}$ is

$$\mathcal{H} = 1 + \frac{q_H}{r}\,, \tag{5.24}$$

where $r = \sqrt{x^i x^i}$ is the radial coordinate of $\mathbb{E}^3$. It is then convenient to introduce the angular coordinates θ and ϕ, defined as usual

$$\frac{x^1}{r} = \sin\theta\cos\phi\,, \quad \frac{x^2}{r} = \sin\theta\sin\phi\,, \quad \frac{x^3}{r} = \cos\theta\,, \tag{5.25}$$

so that, locally, the 1-form χ reads

$$\chi = q_H \cos\theta\, d\phi\,. \tag{5.26}$$

This is the Kaluza-Klein vector of the dimensional reduction and, as we can see, it has the same form as the Dirac magnetic monopole with charge q_H. This is the reason why (5.9) is dubbed the KK monopole metric.[2] In the spherical coordinates we have just introduced, the metric (5.9) takes the form

$$d\sigma^2 = \mathcal{H}^{-1}(d\eta + q_H \cos\theta d\phi)^2 + \mathcal{H}\left(dr^2 + r^2 d\Omega^2_{(2)}\right) , \tag{5.27}$$

where

$$d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta d\phi^2 , \tag{5.28}$$

is the metric of the round $\mathbb{S}^2$ of unit radius. Observe that a global description of the solution requires two patches since the 1-form $\chi = q_H \cos\theta d\phi$ contains a Dirac-Misner string at the poles $\theta = 0, \pi$, or equivalently, in the line $x^1 = x^2 = 0$. This can be easily checked by computing the norm of χ:

$$|\cos\theta d\phi|^2 = q_H^2 \frac{\cot^2\theta}{\mathcal{H}r^2} , \tag{5.29}$$

which is divergent at those points. In order to fix this singularity, we work with two different patches,

$$\chi^{\pm} = q_H\left(\cos\theta \mp 1\right) d\phi . \tag{5.30}$$

In this way, χ^+ is regular everywhere except at $\theta = \pi$, and χ^- is regular everywhere except at $\theta = 0$. We also have to use different coordinates η^+ and η^- in every patch, but in the intersection we must have

$$d\eta^+ + \chi^+ = d\eta^- + \chi^- \Rightarrow d\left(\eta^+ - \eta^-\right) = 2q_H d\phi . \tag{5.31}$$

Hence, we conclude that

$$\eta^+ - \eta^- = 2q_H\phi . \tag{5.32}$$

Since ϕ has period 2π and both $\eta^{\pm}$ have period $2\pi R_\eta$ by definition, this relation can only hold if q_H satisfies the quantization condition

$$q_H = \frac{WR_\eta}{2} , \quad W = 1, 2, \dots , \tag{5.33}$$

as $\eta \sim \eta + 2\pi R_\eta$ trivially implies $\eta \sim \eta + 2\pi W R_\eta$ if $W \in \mathbb{Z}$.

Introducing the angular coordinate,

$$\psi = \frac{2\eta}{R_\eta} \quad \Rightarrow \quad \psi \sim \psi + 4\pi , \tag{5.34}$$

and taking into account the quantization of the magnetic charge q_H, we can write the metric (locally) as

$$d\sigma^2 = \mathcal{H}^{-1}\frac{R_\eta^2}{4}(d\psi + W\cos\theta d\phi)^2 + \mathcal{H}\left(dr^2 + r^2 d\Omega^2_{(2)}\right) . \tag{5.35}$$

[2] The KK monopole metric (5.27) coincides with the extreme limit of the Euclidean Taub-NUT solution.

Let us study the $r \to 0$ and $r \to \infty$ limits of this metric.

- In the $r \to 0$ limit we must use that $\mathcal{H} \sim \frac{WR_\eta}{2r}$, and after performing the change of coordinates

$$r = \frac{\rho^2}{2WR_\eta}\,, \tag{5.36}$$

we obtain

$$d\sigma^2(r \to 0) \sim d\rho^2 + \frac{\rho^2}{4}\left[\left(\frac{d\psi}{W} + \cos\theta d\phi\right)^2 + d\theta^2 + \sin^2\theta d\phi^2\right]. \tag{5.37}$$

When $W = 1$, we recognize the factor that ρ^2 multiplies as the metric of the round $\mathbb{S}^3$. However, for $W > 1$ the cyclic coordinate ψ does not cover the full sphere, but only a $1/W$ part of it. This corresponds to the metric of a lens space $\mathbb{S}^3/\mathbb{Z}_W$, and hence the full space near $r = 0$ is the orbifold $\mathbb{E}^4/\mathbb{Z}_W$. Although lens spaces are regular, the full Gibbons-Hawking metric contains a conical singularity at $r = 0$, because at this point the periodicity of ψ is not "the right one". Nevertheless, it is important to point out that the full ten-dimensional metric (5.2) does not have such a conical singularity if the number of solitonic 5-branes is different from zero. If this is the case, the conformal factor in front of $d\sigma^2$ will behaves as

$$\mathcal{Z}_0^{(0)} \sim \frac{2WR_\eta q_0}{\rho^2}\,, \qquad \text{for} \quad q_0 \neq 0\,, \tag{5.38}$$

near $r = 0$, which makes the conical singularity disappear from the ten-dimensional metric.[3]

- In the asymptotic limit $r \to \infty$, we have $\mathcal{H} \to 1$ and the metric becomes the direct product $\mathbb{S}^1_\eta \times \mathbb{E}^3$:

$$d\sigma^2(r \to \infty) = d\eta^2 + dx^i dx^i\,. \tag{5.39}$$

This can be better seen by using Cartesian coordinates (x^i) and the two patches introduced previously. In that case, the 1-forms $\chi^{(\pm)}$ read

$$\chi^{(\pm)} = q_H \frac{x^1 dx^2 - x^2 dx^1}{r(x^3 \pm r)}\,, \quad \text{where } r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\,. \tag{5.40}$$

We use $\chi^{(+)}$ in the upper space $x^3 \geq 0$ and $\chi^{(-)}$ in the lower one $x^3 \leq 0$. In this way, it is explicit that $\chi^{(\pm)}$ are regular in their respective regions, and we observe that $\lim_{r\to\infty} \chi^{(\pm)} = 0$, where the limit is again taken in the respective region. Hence, the metric (5.9) becomes (5.39).

[3] In the near-horizon limit, the space is $\mathrm{AdS}_3 \times \mathbb{S}^3/\mathbb{Z}_W \times \mathbb{T}^4$.

Let us recall that in Section 5.1.1 we assumed no dependence on η in order to perform a standard Kaluza-Klein reduction. This amounts to impose that $\mathcal{Z}_0^{(0)}, \mathcal{Z}_+^{(0)}$ and $\mathcal{Z}_-^{(0)}$ are functions of the x^i coordinates exclusively. In this case, the harmonicity condition on the full hyper-Kähler space translates into the harmonicity condition on $\mathbb{E}^3$. The choice we make to describe spherically-symmetric black holes is

$$\mathcal{Z}_{0,+,-}^{(0)} = 1 + \frac{q_{0,+,-}}{r}\,, \tag{5.41}$$

which is equivalent to keeping the zero mode of the Fourier expansions of the functions $\mathcal{Z}_{0,+,-}^{(0)}$ appearing in (5.17) with $\rho^2 = \eta^2 + x^i x^i$. In other words, we have smeared the solution over η.

The final step is to use (5.13) to obtain the four-dimensional metric:

$$ds^2_{\mathrm{E},4} = \frac{dt^2}{\sqrt{\mathcal{Z}_0^{(0)}\mathcal{Z}_+^{(0)}\mathcal{Z}_-^{(0)}\mathcal{H}}} - \sqrt{\mathcal{Z}_0^{(0)}\mathcal{Z}_+^{(0)}\mathcal{Z}_-^{(0)}\mathcal{H}}\left(dr^2 + r^2 d\Omega^2_{(2)}\right)\,. \tag{5.42}$$

This metric describes a static, spherically-symmetric extremal black hole with four charges [147]: three of them, q_0, q_+ and q_-, are electric and one of them, q_H, magnetic. The horizon is at $r=0$, where g_{tt} vanishes. The near-horizon limit is $\mathrm{AdS}_2 \times \mathbb{S}^2$ and the ADM mass M and entropy S_{BH} are given by

$$M = \frac{1}{4G_{\mathrm{N}}^{(4)}}\left(q_0 + q_+ + q_- + q_H\right)\,, \tag{5.43}$$

$$S_{\mathrm{BH}} = \frac{\pi}{G_{\mathrm{N}}^{(4)}}\sqrt{q_0 q_+ q_- q_H}\,. \tag{5.44}$$

The expression of the four-dimensional Newton constant in terms of the ten-dimensional moduli is

$$G_{\mathrm{N}}^{(4)} = \frac{G_{\mathrm{N}}^{(5)}}{2\pi R_\eta} = \frac{g_s^2 \alpha'^2}{8 R_z R_\eta}\,. \tag{5.45}$$

As in the five-dimensional case, the electric charges of the solution q_0, q_+ and q_- are related to the number of S5-branes N, momentum n and winding w of the fundamental string. As we will see in Section 5.3, the precise relations are

$$q_0 = \frac{\alpha'}{2R_\eta} N\,, \qquad q_+ = \frac{g_s^2\alpha'^2}{2R_z^2 R_\eta} n\,, \qquad q_- = \frac{g_s^2 \alpha'}{2R_\eta} w\,. \tag{5.46}$$

This must be complemented with the quantization condition for the charge of the KK monopole obtained in (5.33), that we repeat here for convenience

$$q_H = \frac{W R_\eta}{2}\,. \tag{5.47}$$

These relations allow us to rewrite the mass and the black-hole entropy as

$$M = \frac{R_z}{g_s^2 \alpha'} N + \frac{n}{R_z} + \frac{R_z}{\alpha'} w + \frac{R_z R_\eta^2}{g_s^2 \alpha'^2}\,, \tag{5.48}$$

$$S_{\rm BH} = 2\pi \sqrt{nwNW}\,. \tag{5.49}$$

	t	z	z^1	z^2	z^3	z^4	η	x^1	x^2	x^3
F1	×	×	∼	∼	∼	∼	∼	−	−	−
P	×	×	∼	∼	∼	∼	∼	−	−	−
S5	×	×	×	×	×	×	∼	−	−	−
KK5	×	×	×	×	×	×	∼	−	−	−

Table 5.2: Sources for the configuration that gives raise to four-dimensional black holes. The main differences with respect to the five-dimensional case are the presence of a KK monopole and the fact that S5-branes, fundamental string and momentum are smeared over the isometric direction of the GH space, η.

5.2 α'-corrected solutions

Let us now make use of the results derived in Chapter 4 in order to find the α' corrections to the black holes considered in the previous section. We recall that the form of the α'-corrected metric, NSNS 3-form and dilaton is exactly the same as in the zeroth-order solution (5.2)-(5.4), namely

$$ds^2 = \frac{2}{\mathcal{Z}_-} du \left(dt - \frac{\mathcal{Z}_+}{2} du \right) - \mathcal{Z}_0\, d\sigma^2 - dz^\alpha dz^\alpha\ , \tag{5.50}$$

$$H = \star_\sigma d\mathcal{Z}_0 + d\left(1/\mathcal{Z}_-\right) \wedge du \wedge dt\ , \tag{5.51}$$

$$e^{2\phi} = e^{2\phi_\infty} \frac{\mathcal{Z}_0}{\mathcal{Z}_-}\ . \tag{5.52}$$

In addition to these background fields, we studied how to include $\mathfrak{n}$ conmuting triplets of SU(2) vector fields, $A^{\mathfrak{i}} = A^{\mathfrak{i}}{}_{\underline{m}}\, dx^m$ $(\mathfrak{i} = 1, \ldots, \mathfrak{n})$, obeying a self-duality condition on the hyper-Kähler space where they are defined:[4]

$$F^{\mathfrak{i}} = + \star_\sigma F^{\mathfrak{i}}\,. \tag{5.53}$$

We know (see Chapter 4) that this self-duality condition is satisfied by[5]

[4]At this point, this is just an assumption on the vector fields. However, it has been recently showed in [249] that this is indeed imposed by the gaugini Killing spinor equations (if we assume that the gauge fields live in the hyper-Kähler space).

[5]Different types of instantonic fields which are not covered by the 't Hooft ansatz have been considered in [140] and they are based on the uplift of the two-parameter family of spherically-symmetric solutions of the SU(2) Bogomol'nyi equations in $\mathbb{E}^3$ found by Protogenov [170], which are reviewed in Section 2.2. It turns out that these instantons also enjoy the "Laplacian property": $F^A \wedge F^A = d \star_4 dF$, for a certain function F.

$$A^{\mathfrak{i}} = \mathbb{M}^{-}_{mn}\partial_n \log P_{\mathfrak{i}}\, v^m\,, \tag{5.54}$$

if the functions $P_{\mathfrak{i}}$ are harmonic with respect to the hyper-Kähler metric $h_{\underline{mn}}$:

$$\nabla^2_{\sigma} P_{\mathfrak{i}} = 0\,. \tag{5.55}$$

Let us recall that in Eq. (5.54), the 1-forms v^m denote the vierbein of the hyper-Kähler space, $d\sigma^2 = v^m v^m$, and that $\mathbb{M}^{-}_{mn}$ are the anti-selfdual combinations of the $\mathfrak{so}(4)$ generators given in (4.40).

Then, the results obtained in Chapter 4 tell us that the α'-corrected equations of motion are solved if the functions $\mathcal{Z}_0, \mathcal{Z}_+$ and $\mathcal{Z}_-$ are given (up to a harmonic function) by

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} - \frac{\alpha'}{2}\left(\frac{\partial_n \mathcal{Z}_+^{(0)}\partial_n \mathcal{Z}_-^{(0)}}{\mathcal{Z}_0^{(0)}\mathcal{Z}_-^{(0)}}\right) + \mathcal{O}(\alpha'^2)\,, \tag{5.56}$$

$$\mathcal{Z}_- = \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2)\,, \tag{5.57}$$

$$\mathcal{Z}_0 = \mathcal{Z}_0^{(0)} - \frac{\alpha'}{4}\left[\sum_{\mathfrak{i}=1}^{\mathfrak{n}}(\partial \log P_{\mathfrak{i}})^2 - (\partial \log \mathcal{Z}_0^{(0)})^2 - W^2\right] + \mathcal{O}(\alpha'^2)\,, \tag{5.58}$$

where W^2 is a function of the x^m coordinates that determines the Chern-Simons 3-form of ϖ_{mn}, the spin connection of the Gibbons-Hawking space, and which depends on the frame we use to compute it. In the frame specified in (4.72), it is given by

$$W^2 = (\partial \log \mathcal{H})^2 \equiv \partial_m \log \mathcal{H}\, \partial_m \log \mathcal{H}\,, \qquad \text{where} \qquad \partial_m \equiv v_m{}^{\underline{m}}\partial_{\underline{m}}\,. \tag{5.59}$$

This frame is not the most suitable one when the Gibbons-Hawking space is the four-dimensional Euclidean space $h_{\underline{mn}} = \delta_{\underline{mn}}$, as it introduces spurious singularities —see the discussion below (4.104)—. In this case, the best frame choice is $v^m = dx^m$, where one simply finds

$$W = 0\,, \tag{5.60}$$

since the spin connection ϖ_{mn} vanishes.

5.2.1 Five-dimensional black holes

Let us now apply the generic formulae (5.56)-(5.58) to the specific ten-dimensional backgrounds of Section 5.1.2 for which $h_{\underline{mn}} = \delta_{\underline{mn}}$ and the functions $\mathcal{Z}^{(0)}_{0,+,-}$ are given by (5.17). Before doing so, we have to select the harmonic functions $P_{\mathfrak{i}}$ that determine the type of instanton configurations that we are going to include. A convenient choice for us is

$$P_{\mathfrak{i}} = 1 + \frac{\kappa_{\mathfrak{i}}^2}{\rho^2}\,. \tag{5.61}$$

Then, the gauge connection (5.54) is given by

$$A^{\mathfrak{i}} = -2\mathbb{M}^{-}_{mn}\frac{\kappa_{\mathfrak{i}}^2}{\rho^2+\kappa_{\mathfrak{i}}^2}\frac{x^n}{\rho}\frac{dx^m}{\rho}\,, \tag{5.62}$$

which is nothing but one of the multiple ways of writing the BPST instanton [178].

After a direct application of (5.56), (5.57) and (5.58), one finds that the corrected functions $\mathcal{Z}_0, \mathcal{Z}_-$ and $\mathcal{Z}_+$ are given by

$$\mathcal{Z}_+ = 1+\frac{\mathcal{Q}_+}{\rho^2} - 2\alpha'\frac{\mathcal{Q}_+\mathcal{Q}_-}{\rho^2\left(\rho^2+\mathcal{Q}_0\right)\left(\rho^2+\mathcal{Q}_-\right)} + \mathcal{O}(\alpha'^2)\,, \tag{5.63}$$

$$\mathcal{Z}_- = 1+\frac{\mathcal{Q}_-}{\rho^2} + \mathcal{O}(\alpha'^2)\,, \tag{5.64}$$

$$\mathcal{Z}_0 = 1+\frac{\mathcal{Q}_0}{\rho^2} - \alpha'\left[\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\frac{\kappa_{\mathfrak{i}}^4}{\rho^2(\rho^2+\kappa_{\mathfrak{i}}^2)^2} - \frac{\mathcal{Q}_0^4}{\rho^2(\rho^2+\mathcal{Q}_0)^2}\right] + \mathcal{O}(\alpha'^2)\,. \tag{5.65}$$

Let us note, however, that these functions are not univocally determined since we are free to add an arbitrary harmonic function to each of them, and the resulting field configuration is still a solution of the equations of motion at first order in α'. This is nothing but the freedom that we have to fix the boundary conditions. We will use it to impose that $\mathcal{Z}_{+,-,0} \to 1$ at infinity and that the $1/\rho^2$ pole of the $\mathcal{Z}_{+,-,0}$ functions is not changed by the α' corrections. There are two reasons why we proceed this way:

1. The α' corrections are associated to the curvatures of the gauge instantons and torsionful spin connection, which are regular. Thus, they should be regular as well. The poles are spurious and their presence is solely due to the fact that we are using a singular gauge to write the different connections.

2. The residues of the poles are associated to the sources of the solution, and these should not be modified by the α' corrections.

For the functions above this amounts to the changes

$$\mathcal{Z}_+ \to \mathcal{Z}_+ + \frac{2\alpha'\mathcal{Q}_+}{\mathcal{Q}_0\rho^2}\,, \qquad \mathcal{Z}_0 \to \mathcal{Z}_0 + \frac{\alpha'\left(\mathfrak{n}-1\right)}{\rho^2}\,, \tag{5.66}$$

after which the functions read

$$\mathcal{Z}_+ = 1+\frac{\mathcal{Q}_+}{\rho^2} + 2\alpha'\frac{\mathcal{Q}_+\left(\rho^2+\mathcal{Q}_0+\mathcal{Q}_-\right)}{\mathcal{Q}_0\left(\rho^2+\mathcal{Q}_0\right)\left(\rho^2+\mathcal{Q}_-\right)} + \mathcal{O}(\alpha'^2)\,, \tag{5.67}$$

$$\mathcal{Z}_- = 1+\frac{\mathcal{Q}_-}{\rho^2} + \mathcal{O}(\alpha'^2)\,, \tag{5.68}$$

$$\mathcal{Z}_0 = 1+\frac{\mathcal{Q}_0}{\rho^2} + \alpha'\left[\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\frac{\rho^2+2\kappa_{\mathfrak{i}}^2}{(\rho^2+\kappa_{\mathfrak{i}}^2)^2} - \frac{\rho^2+2\mathcal{Q}_0}{(\rho^2+\mathcal{Q}_0)^2}\right] + \mathcal{O}(\alpha'^2)\,, \tag{5.69}$$

in agreement with the results of [193].

Let us analyze the corrections to the function $\mathcal{Z}_0$. The first correction is due to the presence of the BPST instantons. This is already familiar to us since it is exactly the same as the one obtained for the five-dimensional non-Abelian black holes studied in Section 2.3. The physical interpretation of this correction is that the SU(2) instanton acts as a (delocalized) source of S5-brane charge. The second correction to $\mathcal{Z}_0$ comes from the contribution of the torsionful spin connection to the Bianchi identity of the 3-form H, and it is identical (up to a global minus sign) to that of the SU(2) connections. Therefore, its interpretation is analogous: there is a sort of "gravitational instanton" sourced by the S5-branes which, in turn, acts as a delocalized source of S5-charge, thus correcting the function $\mathcal{Z}_0$.

At this point of the discussion, it is obvious from (5.69) that we can cancel the first-order α' corrections to the function $\mathcal{Z}_0$ by including just one BPST instanton ($\mathfrak{n} = 1$) with size fixed by $\kappa_1^2 = \mathcal{Q}_0$. This means that

$$\mathcal{Z}_0 = 1 + \frac{\mathcal{Q}_0}{\rho^2} + \mathcal{O}\left(\alpha'^2\right) . \tag{5.70}$$

We will refer to this particular solution as the *symmetric* solution since it is the one that reduces to the symmetric S5-brane of [190] when $\mathcal{Q}_+ = \mathcal{Q}_- = 0$.

Let us observe that the parameters $\mathcal{Q}_0$ and $\mathcal{Q}_+$ controlling the near-horizon limit of the functions $\mathcal{Z}_0$ and $\mathcal{Z}_+$, to which we will refer as "near-horizon" charges, are not equal to those controlling the large ρ-expansion: $\mathcal{Q}_{0,+}^{\infty} \equiv \lim_{\rho\to\infty} \rho^2 \left(\mathcal{Z}_{0,+} - 1\right)$, to which we will refer as "asymptotic" charges.[6] In particular, the expression for $\mathcal{Q}_0^{\infty}$ is

$$\mathcal{Q}_0^{\infty} = \mathcal{Q}_0 + \alpha' \left(\mathfrak{n} - 1\right) . \tag{5.71}$$

The natural question now is which of these charges is the one "counting" the number of S5-branes. In order to answer this question, we have to study the coupling of a stack of N S5-branes to the background fields. Such analysis was performed in [193], and it turns out that it is $\mathcal{Q}_0$, the near-horizon charge, the one that counts the number of S5-branes. We will repeat such analysis, complementing it, in Section 5.3.

Let us now focus on the correction to the function $\mathcal{Z}_+$. This is qualitatively of the same type as the one obtained for the five-dimensional non-Abelian black holes studied in Section 2.3. In both cases, the correction can be understood as a delocalized contribution to the momentum of the wave coming either from the Yang-Mills fields or from the higher-curvature corrections. This is reflected in the asymptotic momentum charge,

$$\mathcal{Q}_+^{\infty} = \mathcal{Q}_+ \left(1 + \frac{2\alpha'}{\mathcal{Q}_0}\right) , \tag{5.72}$$

which no longer coincides with the near-horizon charge $\mathcal{Q}_+$. Again, as it happens for the S5-charge, it is the near-horizon charge $\mathcal{Q}_+$ the one which is proportional to n, the quantized momentum.

[6]The shrewd reader may be worried because we have assigned the names near-horizon and asympotic charges without actually proving that they correspond to well-defined notions of charge. We will clarify this point in Section 5.3.

The five-dimensional metric incorporating the first-order α' corrections is equal to (5.16) with the functions $\mathcal{Z}^{(0)}_{0,+,-}$ replaced by $\mathcal{Z}_{0,+,-}$, i.e.

$$ds^2_{\mathrm{E},5} = (\mathcal{Z}_0\mathcal{Z}_+\mathcal{Z}_-)^{-2/3}\, dt^2 - (\mathcal{Z}_0\mathcal{Z}_+\mathcal{Z}_-)^{1/3}\left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right)\,. \tag{5.73}$$

The near-horizon geometry is left invariant by the α' corrections, so it is again the metric of $\mathrm{AdS}_2 \times \mathbb{S}^3$,

$$ds^2_{\mathrm{E},5}\,(\rho \to 0) = (\rho/R_\mathrm{H})^4\, dt^2 - \frac{d\rho^2}{(\rho/R_\mathrm{H})^2} - R^2_\mathrm{H}\, d\Omega^2_{(3)}\,, \tag{5.74}$$

where $R_\mathrm{H} = (\mathcal{Q}_0\mathcal{Q}_+\mathcal{Q}_-)^{1/3}$ is the horizon radius. Hence, the area of the horizon is

$$A_\mathrm{H} = 2\pi^2 R^3_\mathrm{H} = 2\pi^2\sqrt{\mathcal{Q}_0\mathcal{Q}_+\mathcal{Q}_-}\,. \tag{5.75}$$

Since the action contains higher-curvature terms, the entropy is not simply given by the Bekenstein-Hawking formula and one has to use Wald's entropy formula [58, 245] instead. This is done in Section 5.4.

The ADM mass of the solution is computed as usual from the asymptotic expansion of the metric, obtaining

$$M = \frac{\pi}{4G^{(5)}_N}\left[\mathcal{Q}_0 + \alpha'\,(\mathfrak{n}-1) + \mathcal{Q}_+\left(1 + \frac{2\alpha'}{\mathcal{Q}_0}\right) + \mathcal{Q}_-\right]\,. \tag{5.76}$$

Finally, if we compare this expression with (5.18), we see that the mass is affected by the presence of the instantons and of the higher-curvature corrections. The first of these corrections was already known to us from Chapter 2. The second is a new effect which lies in the fact that the higher-curvature corrections act in the α'-corrected equations of motion as effective sources of energy, momentum and charge.

5.2.2 Four-dimensional black holes

Let us now study the α' corrections to the extremal four-charge black holes of Section 5.1.3. We recall that in this case the zeroth-order solution is specified by four spherically-symmetric harmonic functions on $\mathbb{E}^3$: $\mathcal{Z}^{(0)}_+, \mathcal{Z}^{(0)}_-, \mathcal{Z}^{(0)}_0$ and $\mathcal{H}$, see (5.41) and (5.24).

Before making use of the formulae (5.56)-(5.58), we make the following choice for the harmonic functions $P_\mathfrak{i}$ that determine the gauge fields:

$$P_\mathfrak{i} = 1 + \frac{\lambda_\mathfrak{i}^{-2}}{r}\,, \qquad \mathfrak{i} = 1, \ldots, \mathfrak{n}\,, \tag{5.77}$$

where we recall that $r = \sqrt{x^i x^i}$ is the radial coordinate of $\mathbb{E}^3$. This choice corresponds to keeping the Fourier zero mode of the BPST instanton periodic in the coordinate η. This is just the standard smearing procedure now applied to the functions $P_\mathfrak{i}$. We studied in Chapter 2 that smeared instantons on Gibbons-Hawking spaces are in one-to-one correspondence with SU(2) magnetic monopoles solving the Bogomol'nyi equations [169] in the Prasad-Sommerfeld limit [174] (i.e. with SU(2) BPS magnetic monopoles). Concretely, those characterized by the choice (5.77) correspond to the one-parameter family of solutions (2.54) found by Protogenov [170], which we have dubbed colored monopoles.

Making use of (5.56)-(5.58), one finds that the corrected functions $\mathcal{Z}_+, \mathcal{Z}_-$ and $\mathcal{Z}_0$ are given by

$$\mathcal{Z}_+ = 1 + \frac{q_+}{r} - \frac{\alpha'}{2}\frac{q_+q_-}{r(r+q_H)(r+q_0)(r+q_-)} + \mathcal{O}(\alpha'^2)\,, \tag{5.78}$$

$$\mathcal{Z}_- = 1 + \frac{q_-}{r} + \mathcal{O}(\alpha'^2)\,, \tag{5.79}$$

$$\mathcal{Z}_0 = 1 + \frac{q_0}{r} + \frac{\alpha'}{4r(r+q_H)}\left\{\frac{q_0^2}{(r+q_0)^2} + \frac{q_H^2}{(r+q_H)^2} - \sum_{\mathrm{i}=1}^{\mathfrak{n}} \frac{1}{(1+\lambda_{\mathrm{i}}^2 r)^2}\right\} + \mathcal{O}\left(\alpha'^2\right)\,, \tag{5.80}$$

where we recall that the charge q_H is quantized according to (5.33). As in the five-dimensional case, the functions are not entirely determined since we can always add to them a harmonic function, which at the end of the day amounts to a redefinition of the charges q_0, q_+ and q_-. Let us then redefine the charges q_0 and q_+ in the following way

$$q_0 \to q_0 + \frac{\alpha'}{4q_H}(2-\mathfrak{n})\,, \qquad q_+ \to q_+\left(1 + \frac{\alpha'}{2q_0q_H}\right)\,, \tag{5.81}$$

so that the functions now read

$$\mathcal{Z}_+ = 1 + \frac{q_+}{r} + \frac{\alpha' q_+}{2q_Hq_0}\frac{r^2 + r(q_0+q_-+q_H) + q_Hq_0 + q_Hq_- + q_0q_-}{(r+q_H)(r+q_0)(r+q_-)} + \mathcal{O}(\alpha'^2)\,, \tag{5.82}$$

$$\mathcal{Z}_- = 1 + \frac{q_-}{r} + \mathcal{O}(\alpha'^2)\,, \tag{5.83}$$

$$\mathcal{Z}_0 = 1 + \frac{q_0}{r} + \alpha'\left\{ - F(r;q_0) - F(r;q_H) + \sum_{\mathrm{i}=1}^{\mathfrak{n}} F(r;\lambda_{\mathrm{i}}^{-2})\right\} + \mathcal{O}(\alpha'^2)\,, \tag{5.84}$$

where we have introduced the function

$$F(r;k) \equiv \frac{(r+q_H)(r+2k)+k^2}{4q_H(r+q_H)(r+k)^2}\,, \tag{5.85}$$

Expressed in this way, it is obvious that we can eliminate all the α' corrections to $\mathcal{Z}_0$ if we use $\mathfrak{n}=2$ instantons of sizes $\lambda_1^{-2} = q_0$, $\lambda_2^{-2} = q_H$. We will come back to this point later. This choice is that of the *symmetric* 5-brane of [190] but now adapted to include a KK monopole of charge W.

The analysis of the α' corrections is very similar to the five-dimensional case except for an important difference which lies in the fact that now the torsionful spin connection

$\Omega_{(-)}{}^a{}_b$ gives raise to two different contributions. One of them was already present in the five-dimensional case. It is due to the presence of the stack of S5-branes, which act as a source of a sort of a gravitational instanton. Such gravitational instanton lives in $\mathfrak{so}_-(3)$, being this one of the two $\mathfrak{so}(3)$ subspaces in which $\mathfrak{so}(4) = \mathfrak{so}_+(3) \oplus \mathfrak{so}_-(3)$ can be decomposed. This corresponds to the first correction in (5.84). The second one, however, is new and its presence is caused by the KK monopole, which sources a second gravitational instanton which lives in the other $\mathfrak{so}_+(3)$ subspace, see (F.5). As we see from (5.84), these contributions are identical to those coming from the SU(2) instantons. This was explained in Chapter 4, where it was shown that the gravitational instanton sourced by the S5-branes can be written using the 't Hooft ansatz, exactly as the SU(2) instantons, whereas the spin connection a Gibbons-Hawking space (KK monopole) requires a slight modification of the aforementioned ansatz which nevertheless yields the same result for the Chern-Simons 3-form, see (4.81).

For the rest we find the same qualitative features as the one we found for the five-dimensional black holes. In particular, we find that the higher-curvature corrections introduce delocalized sources of momentum which leave an imprint on the function $\mathcal{Z}_+$, which also receives α' corrections. It is worth to mention that we do not know of any (necessarily dyonic) gauge field that can be introduced to cancel these α' corrections, analogously to what can be done for $\mathcal{Z}_0$. This would be very interesting since it would allow us to describe α'-exact black holes.

After reducing the solution on $\mathbb{T}^4 \times \mathbb{S}^1_z \times \mathbb{S}^1_\eta$, the four-dimensional metric (in the Einstein frame) incorporating the α' corrections to the family of extremal four-charge black holes studied in Section 5.1.3 is found to be

$$ds^2_{\mathrm{E},4} = e^{2U} dt^2 - e^{-2U}\left(dr^2 + r^2 d\Omega^2_{(2)}\right) , \tag{5.86}$$

with

$$e^{-2U} = \sqrt{\mathcal{Z}_0\, \mathcal{Z}_+\, \mathcal{Z}_- \mathcal{H}} \,. \tag{5.87}$$

As it occurs with their five-dimensional counterparts, the near-horizon geometry $r \to 0$ remains unaffected by the α'-corrections:

$$ds^2(r\to 0) = (r/R_\mathrm{H})^2\, dt^2 + \frac{dr^2}{(r/R_\mathrm{H})^2} - R^2_\mathrm{H}\, d\Omega^2_{(2)} \,, \tag{5.88}$$

which corresponds to the metric of $\mathrm{AdS}_2 \times \mathbb{S}^2$. The horizon radius and the area are given by

$$R^2_\mathrm{H} = \sqrt{q_0 q_+ q_- q_H} \,, \qquad \Rightarrow \qquad A_\mathrm{H} = 4\pi\sqrt{q_0 q_+ q_- q_H} \,. \tag{5.89}$$

The mass M of the solution is found, as usual, from the asymptotic behaviour of the metric function:

$$e^{-2U} = 1 + \frac{2G^{(4)}_\mathrm{N} M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \,. \tag{5.90}$$

We get

$$M = \frac{1}{4G_{\rm N}^{(4)}}\left[q_0 + \frac{\alpha'(\mathfrak{n}-2)}{4q_H} + q_- + q_+\left(1+\frac{\alpha'}{2q_H q_0}\right) + q_H\right]. \tag{5.91}$$

Corrections to the extremal Reissner-Nordström black hole

The extremal Reissner-Nordström black hole corresponds to the zeroth-order in α' solution with $q_+ = q_- = q_0 = q_H \equiv q$. This choice of charges gives constant scalars $e^{\phi} = e^{\phi_\infty}, k = k_\infty$ and $\ell = \ell_\infty$ at this order. However, taking into account the constituents of the black hole, we can only take those charges equal at given points in moduli space

$$g_s = e^{\phi_\infty} = \sqrt{\frac{N}{w}}\,, \qquad R_z/\ell_s = k_\infty = \sqrt{\frac{n}{w}}\,, \qquad R_\eta/\ell_s = k_{\eta,\infty} = \sqrt{\frac{N}{W}}\,, \tag{5.92}$$

which fixes the asymptotic values of the scalars to their attractor values.

Applying the general result to this particular case is straightforward. Taking the symmetric case, we find that the corrected metric function e^{-2U} and the scalars take the form

$$e^{-2U} = \left(1+\frac{q}{r}\right)^2 + \frac{\alpha'}{4q}\frac{[r^2+3rq+3q^2]}{r(r+q)^2} + \mathcal{O}(\alpha'^2)\,, \tag{5.93}$$

$$e^{2\phi} = e^{2\phi_\infty}\,, \tag{5.94}$$

$$k = k_\infty + \frac{\alpha' k_\infty}{4q}\frac{r[r^2+3rq+3q^2]}{(r+q)^4} + \mathcal{O}(\alpha'^2)\,, \tag{5.95}$$

$$\ell = \ell_\infty + \frac{\alpha' \ell_\infty}{12q}\frac{r[r^2+3rq+3q^2]}{(r+q)^4} + \mathcal{O}(\alpha'^2)\,, \tag{5.96}$$

with

$$q = \frac{\ell_s}{2}\sqrt{NW}\,. \tag{5.97}$$

We also have to take into account that the 4-dimensional Newton constant given in Eq. (5.45) now has the value

$$G_{\rm N}^{(4)} = \frac{\ell_s^2}{8}\sqrt{\frac{NW}{nw}}\,. \tag{5.98}$$

Then, if we do not want $G_{\rm N}^{(4)}$ to change with q, we must set $nw = \aleph^2 NW$ for some positive dimensionless constant $\aleph$ so that

$$G_{\rm N}^{(4)} = \frac{\ell_s^2}{8\aleph}\,. \tag{5.99}$$

Staying at weak coupling (so the loop corrections can be safely ignored) and away of the self-dual radii at which new massless degrees of freedom arise, demands the following hierarchy

$$n > w > N > W\,, \quad \Rightarrow \quad \aleph >> 1\,. \tag{5.100}$$

5.3 String sources and T-duality

In order to figure out the relation between the charges of the corresponding black holes and the parameters that characterize the sources, we need to know how the latter couple to the background fields and, more concretely, how the equations of motion of these fields change when one takes into account the δ-like contributions of the sources.

5.3.1 Solitonic 5-branes

The coupling of a stack of N S5-branes to $\tilde{B}$, the magnetic dual of the Kalb-Ramond 2-form,

$$d\tilde{B} = e^{-2\phi} \star H\,, \tag{5.101}$$

is described by a Wess-Zumino term of the form

$$S_{\mathrm{WZ}} = g_s^2 T_{\mathrm{S5}} N \int_{W^6} \tilde{B}\,, \tag{5.102}$$

where W^6 stands for the worldvolume of the five-branes and

$$T_{\mathrm{S5}} = \frac{1}{g_s^2\,(2\pi)^5\,\alpha'^3}\,, \tag{5.103}$$

for the brane tension. The term (5.102) modifies the equation of motion of $\tilde{B}$, which in absence of sources is just the Bianchi identity of B, by adding a δ-like source term at the position of the branes:

$$dH - \frac{\alpha'}{4}\left(\sum_{i=1}^{\mathfrak{n}} F^{A_{\mathrm{i}}} \wedge F^{A_{\mathrm{i}}} + R_{(-)}{}^{a}{}_{b} \wedge R_{(-)}{}^{b}{}_{a}\right) = 4\pi^2\alpha' N \star_\sigma \delta\,, \tag{5.104}$$

where $\star_\sigma\delta$ is a top form in the transverse space of the S5-branes whose integral is equal to 1. Therefore, the number of S5-branes N can be computed by integrating (5.104) over the transverse space $\mathcal{M}_4$. Applying Stokes' theorem:

$$N = \frac{1}{4\pi^2\alpha'}\int_{\partial\mathcal{M}_4} H - \frac{1}{16\pi^2}\int_{\mathcal{M}_4}\left\{\sum_{i=1}^{\mathfrak{n}} F^{A_{\mathrm{i}}} \wedge F^{A_{\mathrm{i}}} + R_{(-)}{}^{a}{}_{b} \wedge R_{(-)}{}^{b}{}_{a}\right\}\,, \tag{5.105}$$

where $\partial\mathcal{M}_4$ denotes de boundary of $\mathcal{M}_4$. Let us consider the five- and four-dimensional cases separately:

- For the five-dimensional black holes, we have $\mathcal{M}_4 = \mathbb{E}^4$ and, therefore, $\partial\mathcal{M}_4 = \mathbb{S}^3_\infty$. For the first term, we find

$$\frac{1}{4\pi^2\alpha'}\int_{\mathbb{S}^3_\infty} H = \frac{1}{4\pi^2\alpha'}\int_{\mathbb{S}^3_\infty} \star_\sigma d\mathcal{Z}_0 = \frac{\mathcal{Q}_0}{\alpha'} + \mathfrak{n} - 1\,. \tag{5.106}$$

The second term is just the sum of the instanton numbers of the $\mathfrak{n}$ BPST instantons, which is just $\mathfrak{n}$:

$$\frac{1}{16\pi^2}\sum_{i=1}^{\mathfrak{n}}\int_{\mathbb{E}^4} F^{A_\mathrm{i}} \wedge F^{A_\mathrm{i}} = \mathfrak{n}\,, \tag{5.107}$$

Finally, the last term can be evaluated by using (F.9), and the result is

$$\frac{1}{16\pi^2}\int_{\mathbb{E}^4} R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a = -1\,. \tag{5.108}$$

Plugging these partial results into (5.105), we find that

$$\mathcal{Q}_0 = \alpha' N\,, \tag{5.109}$$

which is exactly what one obtains at zeroth order in α'. This is what one could expect on general grounds since the α' corrections do not introduce localized sources but delocalized ones that contribute, instead, to the S5-charge, which is defined as

$$\mathcal{Q}_{\mathrm{S5}} \equiv \frac{1}{4\pi^2\alpha'}\int_{\mathbb{S}^3_\infty} H = N + \mathfrak{n} - 1\,. \tag{5.110}$$

This formula tells us that each of the BPST instantons carries +1 unit of S5-charge whereas a stack of solitonic 5-branes carries $N-1$ units of S5-charge instead of N, which would be the charge that one would obtain ignoring the higher-curvature terms in (5.104). A nice property of this formula is that it is exact at all orders in α', as it comes from the evaluation of a topological invariant. Therefore, continuous deformations of the torsionful spin connection, such as those introduced by the α'-corrections, do not change this result.

- In the four-dimensional case, $\mathcal{M}_4$ is the Gibbons-Hawking space, whose boundary is $\partial\mathcal{M}_4 = \mathbb{S}^1_\eta \times \mathbb{S}^2_\infty$. Then, the first term now gives

$$\frac{1}{4\pi^2\alpha'}\int_{\mathbb{S}^1_\eta\times\mathbb{S}^2_\infty} H = \frac{1}{4\pi^2\alpha'}\int_{\mathbb{S}^1_\eta\times\mathbb{S}^2_\infty} \star_\sigma d\mathcal{Z}_0 = \frac{2R_\eta q_0}{\alpha'} + \frac{\mathfrak{n}-2}{W}\,. \tag{5.111}$$

The instanton number can be easily computed by making use of (4.66). We obtain:

$$\frac{1}{16\pi^2}\sum_{\mathrm{i}=1}^{\mathfrak{n}}\int_{\mathcal{M}_4} F^{A_\mathrm{i}} \wedge F^{A_\mathrm{i}} = \frac{1}{16\pi^2}\sum_{\mathrm{i}=1}^{\mathfrak{n}}\int_{\mathcal{M}_4} d\star_\sigma dF\left(r;\lambda_\mathrm{i}^{-2}\right) = \frac{\mathfrak{n}}{W}\,. \tag{5.112}$$

As we see, the instanton number is quantized although its value is not necessarily an integer. This is related to the presence of the lens space $\mathbb{S}^3/\mathbb{Z}_W$, and it shows that these instantons, when $W > 1$, are somewhat exotic in a mathematical sense. Other solutions with rational but discrete instanton number are known in the literature, see e.g. [250, 251].

Finally, in order to compute the last integral, we make use again of (F.9), obtaining

$$\frac{1}{16\pi^2}\int_{\mathcal{M}_4} R_{(-)}{}^{a}{}_{b}\wedge R_{(-)}{}^{b}{}_{a} = -\frac{1}{16\pi^2}\int_{\mathcal{M}_4} d\star_\sigma d\left[F\left(r;q_0\right)+F\left(r;q_H\right)\right] = -\frac{2}{W}\,. \tag{5.113}$$

Substituting these results into (5.105), we find that

$$q_0 = \frac{\alpha' N}{2R_\eta}\,, \tag{5.114}$$

and that the S5-charge is in this case given by

$$\mathcal{Q}_{\mathrm{S5}} = N + \frac{\mathfrak{n}-2}{W}\,. \tag{5.115}$$

Comparing this formula with the one obtained in the five-dimensional case, Eq. (5.110), we observe two differences. The first one is the factor of $1/W$ which is due to the presence of the lens space $\mathbb{S}^3/\mathbb{Z}_W$, as we have already explained. The second difference is the extra contribution from the torsionful spin connection (5.113) due to the KK monopole, which carries $-1/W$ units of S5-charge. That a KK monopole of unit charge ($W = 1$) carries -1 unit of S5-charge was already known [252] and, in fact, it has played an important rôle in testing heterotic/type II S-duality.

5.3.2 Fundamental strings

The coupling of a fundamental string with winding number w to the KR 2-form B is given by the following Wess-Zumino term

$$\frac{w}{2\pi\alpha'}\int_{W^2} B\,, \tag{5.116}$$

where W^2 denotes the worldsheet. When this contribution is taken into account, the equation of motion of the Kalb-Ramond 2-form is modified as follows

$$d\left(e^{-2\phi}\star H\right) = w\left(2\pi\ell_s\right)^6\star_8\delta\,, \tag{5.117}$$

where $\star_8\delta$ is by definition an 8-form that is normalized to 1 when integrated over the transverse space to the string: $\mathcal{M}_4\times\mathbb{T}^4$. Applying Stokes' theorem, we find

$$w = \frac{1}{\left(2\pi\ell_s\right)^6}\int_{\partial\mathcal{M}_4\times\mathbb{T}^4} e^{-2\phi}\star H\,. \tag{5.118}$$

Let us point out that this is the same formula that one would use to compute the winding number at zeroth order in α' since the equation of motion of B does not receive α'

corrections (at least at the order we are working). This was also the reason why the function $\mathcal{Z}_-$ did not receive any correction. Then, the relation between the winding number w and the electric charge $\mathcal{Q}_-$ is the same. If we compute this integral in the five- and four-dimensional cases, we get

$$\mathcal{Q}_- = g_s^2\alpha' w\,, \qquad \text{and} \qquad q_- = \frac{g_s^2\alpha'}{2R_\eta}w\,. \tag{5.119}$$

5.3.3 Momentum wave

The simplest way to find the relation between $\mathcal{Q}_+$ and n first at zeroth order in α' is to T-dualize our solutions in the z-direction since we know that this operation, at the microscopic level, interchanges winding w and momentum n

$$n \to n' = w\,, \qquad w \to w' = n\,, \tag{5.120}$$

and,

$$g_s \to g_s' = \frac{g_s \ell_s}{R_z}\,, \qquad R_z \to R_z' = \frac{\alpha'}{R_z}\,. \tag{5.121}$$

As we have seen in Section 4.5, at the macroscopic (supergravity) level, it interchanges the functions $\mathcal{Z}_+^{(0)}$ and $\mathcal{Z}_-^{(0)}$ or, equivalently, the charges $\mathcal{Q}_+$ and $\mathcal{Q}_-$. Therefore, one has

$$\mathcal{Q}_+ = \mathcal{Q}_-' = \alpha' {g_s'}^2 w' = \alpha'\left(\frac{g_s\ell_s}{R_z}\right)^2 n \quad \Rightarrow \quad \mathcal{Q}_+ = \frac{g_s^2\alpha'^2}{R_z^2}n\,. \tag{5.122}$$

The same operation in the four-dimensional case gives

$$q_+ = q_-' = \frac{{g_s'}^2\alpha'}{2R_\eta}w' \quad \Rightarrow \quad q_+ = \frac{g_s^2\alpha'^2}{2R_z^2R_\eta}n\,. \tag{5.123}$$

Since the α' corrections do not introduce localized sources, the relations (5.122) and (5.123) still hold after the α' corrections have been taken into account. This means, in the language used in this chapter, that the near-horizon charges $\mathcal{Q}_+$ and q_+ are the ones which are proportional to n, the quantized momentum.

In order to quantify the effect of the higher-derivative corrections on the momentum (P), we find convenient to define the asymptotic momentum charge $\mathcal{Q}_\text{P}$ as

$$\mathcal{Q}_\text{P} \equiv \frac{R_z^2}{g_s^2\alpha'^2}\mathcal{Q}_+^\infty = n\left(1+\frac{2}{N}\right)\,, \qquad \mathcal{Q}_\text{P} \equiv \frac{2R_z^2R_\eta}{g_s^2\alpha'^2}q_+^\infty = n\left(1+\frac{2}{NW}\right)\,. \tag{5.124}$$

As we see, $\mathcal{Q}_\text{P}$, which in the absence of the α' corrections is just equal to the quantized momentum n, receives now $\mathcal{O}(1/N)$ corrections. This charge ($\mathcal{Q}_\text{P}$) is expected to coincide with the charge carried by the Kaluza-Klein vector associated to the compactification on the circle $\mathbb{S}^1_z$. Nonetheless, in order to prove this rigorously, one needs the compactification of the α'-corrected action (4.22), which is something that has not been available until very recently [253,254]. Due to the presence of Chern-Simons terms in the action, we expect to be able to define more than one notion of charge (as it happens with the S5-brane charge) which would correspond to what we call here the near-horizon and asymptotic charges.

5.3.4 Mass and T-duality

Let us close this section by rewriting the mass of the five- and four-dimensional black holes in terms of the quantities that we have just defined. In the five-dimensional case, we get

$$M = \frac{R_z}{g_s^2\alpha'}\mathcal{Q}_{\mathrm{S5}} + \frac{\mathcal{Q}_{\mathrm{P}}}{R_z} + \frac{R_z}{\alpha'}\,\mathcal{Q}_{\mathrm{F1}}\,, \tag{5.125}$$

where we have introduced $\mathcal{Q}_{\mathrm{F1}} \equiv w$ for convenience. In the four-dimensional case, we get an additional contribution from the KK monopole,

$$M = \frac{R_z}{g_s^2\alpha'}\mathcal{Q}_{\mathrm{S5}} + \frac{\mathcal{Q}_{\mathrm{P}}}{R_z} + \frac{R_z}{\alpha'}\,\mathcal{Q}_{\mathrm{F1}} + \frac{R_z R_\eta^2}{g_s^2\alpha'^2}\,\mathcal{Q}_{\mathrm{KK}}\,, \tag{5.126}$$

where $\mathcal{Q}_{\mathrm{KK}} \equiv W$.

Notice that both expressions are left invariant under the following transformations

$$g_s \to g_s' = \frac{g_s\ell_s}{R_z}\,, \qquad\qquad R_z \to R_z' = \frac{\alpha'}{R_z}\,, \tag{5.127}$$

and

$$\mathcal{Q}_{\mathrm{P}} \to \mathcal{Q}_{\mathrm{P}}' = \mathcal{Q}_{\mathrm{F1}}\,, \qquad\qquad \mathcal{Q}_{\mathrm{F1}} \to \mathcal{Q}_{\mathrm{F1}}' = \mathcal{Q}_{\mathrm{P}}\,, \tag{5.128}$$

which clearly suggest that these would be the corrected version of the microscopic T-duality transformations (5.120). This implies that the transformation of the momentum n and the winding w is

$$n \to n' = \frac{w}{1+\frac{2}{NW}} \approx w\left(1-\frac{2}{NW}\right)\,, \qquad\qquad w \to w' = n\left(1+\frac{2}{NW}\right)\,, \tag{5.129}$$

which reduces to (5.120) in the large-charge regime.

At this level, the proposed modification of the T-duality rules are just based on the observation that the mass remains invariant under these transformations. However, it turns out that these modifications are also predicted by the α'-corrected Buscher rules proposed in [117]. Let us illustrate this with the T-dualization of the four-dimensional solutions along z, which amounts to the following changes in the functions —see Section 4.5—

$$\begin{aligned}
&\mathcal{Z}_- \to \mathcal{Z}_-' = 1 + \frac{1}{r}\left(q_+ + \frac{\alpha' q_+}{2q_H q_0}\right)\,,\\
&\mathcal{Z}_+ \to \mathcal{Z}_+' = 1 + \frac{1}{r}\left(q_- - \frac{\alpha' q_-}{2q_H q_0}\right) + \frac{\alpha' q_-}{2q_H q_0}\frac{r^2 + r(q_0+q_++q_H) + q_H q_0 + q q_+ + q_0 q_+}{(r+q_H)(r+q_0)(r+q_+)}\,.
\end{aligned} \tag{5.130}$$

Hence, we see that the net effect is

$$q_+ \to q'_+ = \left(q_- - \frac{\alpha' q_-}{2q_H q_0}\right)\,, \qquad q_- \to q'_- = \left(q_+ + \frac{\alpha' q_+}{2q_H q_0}\right)\,. \tag{5.131}$$

Now, it is straightforward to check by using the expressions of the charges q_0, q_+, q_- and q_H in terms of N, n, w and W derived before that (5.131) is equivalent to (5.129).

5.4 Black-hole entropy

In this section, we compute the Wald entropy of the α'-corrected black holes. Following [58, 245], the Wald entropy S_{W} in a D-dimensional theory is

$$S_{\mathrm{W}} = -2\pi \int_{\Sigma} d^{D-2}x\sqrt{|h|}\,\mathcal{E}_R^{abcd}\,\epsilon_{ab}\epsilon_{cd}\,, \tag{5.132}$$

where Σ is a cross-section of the horizon, h is the determinant of the metric induced on Σ, ϵ_{ab} is the binormal to Σ,[7] and, finally, $\mathcal{E}_R^{abcd}$ is the equation of motion one would obtain for the Riemann tensor R_{abcd} treating it as an independent field of the theory, i.e.

$$\mathcal{E}_R^{abcd} = \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta R_{abcd}}\,, \tag{5.133}$$

where S is the D-dimensional action.

We find convenient to apply the formula in $D = 6$ dimensions, after performing the trivial compactification on $\mathbb{T}^4$. The six-dimensional action is

$$\begin{aligned} S =& \frac{g_s^2}{16\pi G_{\mathrm{N}}^{(6)}}\int d^6x\sqrt{|g|}e^{-2\phi}\left\{R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!}H_{\mu\nu\rho}H^{\mu\nu\rho}\right.\\ &\left.-\frac{\alpha'}{8}\left[F^A{}_{\mu\nu}F^{A\mu\nu} - R_{(-)}{}_{\mu\nu ab}R_{(-)}{}^{\mu\nu ab}\right] + \mathcal{O}\left(\alpha'^3\right)\right\}\,, \end{aligned} \tag{5.134}$$

where

$$G_N^{(6)} = \frac{G_N^{(10)}}{(2\pi\ell_s)^4}\,. \tag{5.135}$$

In order to apply Wald's formula, it is convenient to recast the six-dimensional string-frame metric as follows

$$ds^2_{\mathrm{s},6} = e^{\phi-\phi_\infty}\left[(k/k_\infty)^{-2/3}ds^2_{\mathrm{E},5} - (k/k_\infty)^2\left(dz + A^+\right)^2\right]\,, \tag{5.136}$$

where $A^+ = \left(\mathcal{Z}_+^{-1} - 1\right)dt$ and

$$ds^2_{\mathrm{E},5} = (\ell/\ell_\infty)^{-1}\,ds^2_{\mathrm{E},4} - (\ell/\ell_\infty)^2\,(d\eta + \chi)^2\,, \tag{5.137}$$

[7] The binormal ϵ_{ab} is normalized so that $\epsilon_{ab}\epsilon^{ab} = -2$.

is the five-dimensional metric in the modified Einstein frame. The four-dimensional one, $ds^2_{\text{E},4}$, the dilaton ϕ and the Kaluza-Klein scalars k and ℓ are given by

$$\begin{aligned} ds^2_{\text{E},4} &= e^{2U}dt^2 - e^{-2U}\left(dr^2 + r^2 d\Omega^2_{(2)}\right) , \\ e^{2\phi} &= e^{2\phi_\infty}\frac{\mathcal{Z}_0}{\mathcal{Z}_-} , \qquad k = k_\infty \frac{\mathcal{Z}_+^{1/2}}{\mathcal{Z}_0^{1/4}\mathcal{Z}_-^{1/4}} , \qquad \ell = \ell_\infty \frac{\mathcal{Z}_0^{1/6}\,\mathcal{Z}_+^{1/6}\,\mathcal{Z}_-^{1/6}}{\mathcal{H}^{1/2}} , \end{aligned} \tag{5.138}$$

where

$$e^{-2U} = \sqrt{\mathcal{Z}_0\mathcal{Z}_+\mathcal{Z}_-\mathcal{H}} \, . \tag{5.139}$$

We define, for later convenience, the following sechsbein basis:

$$e^{0,1,2,3} = e^{\frac{\phi-\phi_\infty}{2}}\left(\frac{k}{k_\infty}\right)^{-\frac{1}{3}}(\ell/\ell_\infty)^{-1/2}\, v^{0,1,2,3}, \tag{5.140}$$

$$e^{4} = e^{\frac{\phi-\phi_\infty}{2}}\left(\frac{k}{k_\infty}\right)^{-\frac{1}{3}}\ell/\ell_\infty\,(d\eta + \chi) \, , \tag{5.141}$$

$$e^{5} = e^{\frac{\phi-\phi_\infty}{2}}\frac{k}{k_\infty}\left(dz + A^+\right) , \tag{5.142}$$

where $v^{0,1,2,3}$ is the vierbein asociated to the four-dimensional metric $ds^2_{\text{E},4}$,

$$v^0 = e^U\,dt\,, \qquad v^1 = e^{-U}\,dr\,, \qquad v^2 = e^{-U} r\,d\theta\,, \qquad v^3 = e^{-U} r\sin\theta\,d\phi\,. \tag{5.143}$$

In this frame, we have that the non-vanishing component of the binormal is $\epsilon_{01} = -\epsilon_{10} = 1$.

The volume element is given by

$$\sqrt{|h|}\,d^4x = e^{2(\phi_\Sigma-\phi_\infty)}\sqrt{q_0 q_+ q_- q_H}\,\sin\theta\,d\theta\,d\phi\,d\eta\,dz\,, \tag{5.144}$$

where ϕ_Σ means the dilaton evaluated at the horizon.

It turns out that the result obtained when Wald's formula is directly applied to the above action is not invariant under local Lorentz transformations. This is due to the presence, once more, of the Chern-Simons term $\omega^{\text{L}}_{(-)}$ in the definition of H. Although it would be desirable to derive from first principles a generalization of Wald's formula which can be applied to actions that contain explicit occurrences of Chern-Simons terms, this seems to be a tough task and we are not going to deal with it here.[8] Instead, it is possible to rewrite the action in a smart way so as to avoid the problem with the Chern-Simons, see e.g. [246, 247].

In first place, let us modify the action (5.134) by adding the following surface term —which does not modify the entropy according to [245]—

[8] Recent progress in this direction has been made in [253–255].

$$\tilde{S} = S + \frac{g_s^2}{16\pi G_{\rm N}^{(6)}} \int \tilde{H} \wedge \left(H - \frac{\alpha'}{4} (\omega^{\rm L}_{(-)} + \omega^{\rm YM}) \right) , \tag{5.145}$$

where

$$\tilde{H} = e^{-2\phi} \star H \equiv d\tilde{B} \,. \tag{5.146}$$

Then, let us take H as an auxiliary field and $\tilde{B}$ as the dynamical one. The equation of motion of $\tilde{B}$ gives the Bianchi identity of H whereas the Bianchi identity, $d\tilde{H} = 0$, is the former equation of motion of the Kalb-Ramond 2-form B. The next step is to eliminate H in terms of $\tilde{H}$ from the action $\tilde{S}$ by using its equation of motion or, equivalently, its definition (5.146). Then, we have

$$\tilde{S} = S' - \frac{g_s^2}{16\pi G_{\rm N}^{(6)}} \frac{\alpha'}{4} \int \tilde{H} \wedge \omega^{\rm L}_{(-)} \,, \tag{5.147}$$

where

$$\begin{aligned} S' = & \frac{g_s^2}{16\pi G_{\rm N}^{(6)}} \int d^6x \sqrt{|g|} \left\{ e^{-2\phi} \left[R - 4\left(\partial\phi\right)^2 \right] + \frac{1}{2\cdot 3!} e^{2\phi} \tilde{H}^{\mu\nu\rho} \tilde{H}_{\mu\nu\rho} \right. \\ & \left. -\frac{\alpha'}{4} \tilde{H}_{\mu\nu\rho} \frac{\epsilon^{\mu\nu\rho\alpha\beta\gamma}\, \omega^{\rm YM}_{\alpha\beta\gamma}}{(3!)^2 \sqrt{|g|}} + \frac{\alpha'}{8} e^{-2\phi} \left(F^{A}{}_{\mu\nu} F^{A\mu\nu} - R_{(-)}{}_{\mu\nu ab} R_{(-)}{}^{\mu\nu ab} \right) \right\} . \end{aligned} \tag{5.148}$$

The unique term of this piece of the action that contributes to the Wald entropy is the Ricci scalar since the curvature of the torsionful spin connection vanishes when evaluated at the horizon: $R_{(-)}{}^{a}{}_{b}|_\Sigma = 0$. Then, the contribution from S' is

$$\frac{1}{\sqrt{|g|}} \frac{\delta S'}{\delta R_{abcd}} = \frac{e^{-2(\phi-\phi_\infty)}}{16\pi G_{\rm N}^{(6)}} \eta^{c[a} \eta^{b]d} \,. \tag{5.149}$$

This gives the leading contribution to the Wald entropy, the Bekenstein-Hawking term.

Let us now analyze the second term in (5.147). First, let us split the Chern-Simons 3-form of the torsionful spin connection in two pieces,

$$\omega^{\rm L}_{(-)} = \omega^{\rm L} + \mathcal{A} \,, \tag{5.150}$$

where $\omega^{\rm L}$ is the Chern-Simons 3-form of the Levi-Civita spin connection $\omega^{a}{}_{b}$ and where the 3-form $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{1}{2} d \left(\omega^{a}{}_{b} \wedge H^{b}{}_{a} \right) + \frac{1}{4} H^{b}{}_{a} \wedge DH^{b}{}_{a} - R^{a}{}_{b} \wedge H^{b}{}_{a} + \frac{1}{12} H^{a}{}_{b} \wedge H^{b}{}_{c} \wedge H^{c}{}_{a} \,, \tag{5.151}$$

where $H^{a}{}_{b} \equiv H_{\mu}{}^{a}{}_{b}\, dx^{\mu}$ and $DH^{a}{}_{b} = dH^{a}{}_{b} + \omega^{a}{}_{c} \wedge H^{c}{}_{a} - \omega^{c}{}_{b} \wedge H^{a}{}_{c}$. The first term can be ignored since it gives a total derivative. Then, we are left with the third term, whose contribution is found to be

$$\frac{1}{\sqrt{|g|}}\frac{\delta}{\delta R_{abcd}}\left\{-\frac{g_s^2}{16\pi G_{\rm N}^{(6)}}\frac{\alpha'}{4}\int \tilde{H}\wedge\mathcal{A}\right\}=\frac{e^{-2(\phi-\phi_\infty)}}{16\pi G_{\rm N}^{(6)}}\frac{\alpha'}{8}H^{abe}H_e{}^{cd}\,. \tag{5.152}$$

The last contribution due to $\omega^{\rm L}$ is also the most subtle. In order to compute it, we are going to take the variation of the corresponding piece of the action with respect to the only component of the Riemann tensor which is relevant for the application of Wald formula, the 0101 component:

$$\begin{aligned}\frac{1}{\sqrt{|g|}}\frac{\delta}{\delta R_{0101}}\left\{-\frac{g_s^2}{16\pi G_{\rm N}^{(6)}}\frac{\alpha'}{4}\int \tilde{H}\wedge\omega^{\rm L}\right\}=&\frac{g_s^2}{16\pi G_{\rm N}^{(6)}}\frac{\alpha'}{4}\frac{\epsilon^{\mu\nu\rho\alpha\beta\gamma}\tilde{H}_{\alpha\beta\gamma}}{(3!)^2\sqrt{|g|}}\frac{\delta\omega^{\rm L}{}_{\mu\nu\rho}}{\delta R_{0101}}\\ =&\frac{e^{-2(\phi-\phi_\infty)}}{16\pi G_{\rm N}^{(6)}}\frac{\alpha'}{4}\frac{H^{\mu\nu\rho}}{3!}\frac{\delta\omega^{\rm L}{}_{\mu\nu\rho}}{\delta R_{0101}}\,,\end{aligned} \tag{5.153}$$

where we have used that[9]

$$H^{\mu\nu\rho}=\frac{1}{3!}e^{2\phi}\frac{\epsilon^{\mu\nu\rho\alpha\beta\gamma}}{\sqrt{|g|}}\tilde{H}_{\alpha\beta\gamma}\,. \tag{5.154}$$

Finally, in order to express the dependence of the Chern-Simons 3-form on the Riemann tensor in a covariant form, we are going to use that in the near horizon limit, the six-dimensional metric is the metric of the direct product $\mathrm{AdS}_3\times\mathbb{S}^3/\mathbb{Z}_{\rm W}$ and that only the AdS part gives a contribution to (5.153). The last piece of information that we need is that for three-dimensional metrics that have a spacelike isometry —such as the metric of AdS_3—,

$$ds^2=e^{2\varphi(y)}\left[\gamma_{\underline{mn}}\,dy^m dy^n-\left(d\xi+V_{\underline{m}}(y)\,dy^m\right)^2\right]\,,\qquad m,n=0,1\,, \tag{5.155}$$

the Chern-Simons 3-form is given by [256]

$$\omega^{\rm L}{}_{\underline{01\xi}}=\frac{\epsilon^{mn}}{2}\left[R(\gamma)\,(dV)_{\underline{mn}}+(dV)_{\underline{mp}}\,(dV)^{\underline{pq}}\,(dV)_{\underline{qn}}-\partial_{\underline{m}}\left(\omega_{\underline{n}}{}^{pq}\,(dV)_{pq}\right)\right]\,, \tag{5.156}$$

where $R(\gamma)$ is the Ricci scalar of the two dimensional metric $\gamma_{\underline{mn}}$ and ω^{mn} the spin-connection 1-form.[10] Particularizing this expression for the case at hands ($y^0=t$, $y^1=r$ and $\xi=z$), we find

$$\frac{\delta\omega^{\rm L}{}_{trz}}{\delta R_{0101}}=-\frac{1}{2}e^{\frac{\phi-\phi_\infty}{2}}\left(\frac{k}{k_\infty}\right)^2\left(dA^+\right)_{tr}\,, \tag{5.157}$$

[9] In the conventions of [111], $\epsilon^{0\ldots d-1}=+1$ and $\epsilon_{0\ldots d-1}=(-1)^{d-1}|g|$, where d is the spacetime dimension and g the metric in mostly minus signature.

[10] Underlined and non-underlined indices are world and flat indices respectively.

where we have used that $e^{2\varphi} = e^{\phi-\phi_\infty}\left(\frac{k}{k_\infty}\right)^2$ and $V = A^+$. Then, coming back to the Wald entropy, we have

$$\begin{aligned} S_{\rm W} =& -2\pi \int_\Sigma d^4x\sqrt{|h|}\,\mathcal{E}_R^{abcd}\,\epsilon_{ab}\epsilon_{cd} \\ =& -8\pi\sqrt{q_0 q_+ q_- q_H}e^{2(\phi_\Sigma-\phi_\infty)}\int \sin\theta\,d\theta\,d\phi\,d\eta\,dz\,\mathcal{E}_R^{0101}\,, \end{aligned} \tag{5.158}$$

where

$$\begin{aligned} \mathcal{E}_R^{0101} =& -\frac{e^{2(\phi_\infty-\phi_\Sigma)}}{32\pi G_{\rm N}^{(6)}}\left[1+\frac{\alpha'}{4}\left(-H^{01d}H_{01d}+e^{(\phi-\phi_\infty)}\left(\frac{k}{k_\infty}\right)^2 H^{trz}F^{+}{}_{tr}\right)\Bigg|_\Sigma\right] \\ =& -\frac{e^{2(\phi_\infty-\phi_\Sigma)}}{32\pi G_N^{(6)}}\left(1+\frac{\alpha'}{2q_0q_H}\right)\,. \end{aligned} \tag{5.159}$$

Finally, substituting in (5.158), we obtain

$$S_{\rm W} = \frac{\pi}{G_{\rm N}^{(4)}}\sqrt{q_0 q_+ q_- q_H}\left(1+\frac{\alpha'}{2q_0q_H}\right)\,, \tag{5.160}$$

which can be rewritten in terms of the parameters characterizing the extended objects by using Eqs. (5.114), (5.119), (5.123), and (5.33), getting:

$$S_{\rm W} = 2\pi\sqrt{nwNW}\left(1+\frac{2}{NW}\right)\,. \tag{5.161}$$

The entropy for the three-charge black holes can be recovered from this expression by just setting $W = 1$, as the near-horizon limit is exactly the same except for the quotient of the 3-sphere $\mathbb{S}^3$ by $\mathbb{Z}_W$, owed to the presence of the KK monopole.

A different derivation of the Wald entropy for this class of black holes has been recently provided in [253, 254], further supporting the robustness of the result presented here.

At this point, we can ask ourselves if the expression (5.158) will suffer from α'^2 corrections, which would be what one should expect on general grounds. The case we are dealing with is quite special because, as we have shown, the near-horizon limit, $\mathrm{AdS}_3 \times \mathbb{S}^3/\mathbb{Z}_W \times \mathbb{T}^4$, remains uncorrected at first order in α'. Furthermore, since the curvature of the torsionful spin connection vanishes for this background, we expect it to remain invariant at all orders in α'. Under this premise, it was shown in [257, 258] that the Wald entropy does not receive any further α' corrections. Hence, the formula (5.161) can be trusted at all orders in the α' expansion.

5.5 Conclusions

In this chapter we have found the first-order α' corrections to two well-known families of heterotic black holes in five and four dimensions [146, 147] whose ten-dimensional description consists of a superposition of S5-branes, fundamental strings, a momentum wave, and, in the four-dimensional case, a KK monopole.

At zeroth order in α', the expressions for the ADM mass M and the Bekenstein-Hawking entropy $S_{\rm BH}$ in terms of the number of S5-branes, N, the momentum of the wave, n, the winding number of the fundamental string, w, the charge of the KK monopole, W, and the moduli parameters R_z, R_η, g_s and $\alpha' = \ell_s^2$ are,

- *3-charge black holes:*

$$M = \frac{R_z}{g_s^2 \ell_s^2} N + \frac{n}{R_z} + \frac{R_z}{\ell_s^2} w\,, \tag{5.162}$$

$$S_{\rm BH} = 2\pi\sqrt{nwN}\,. \tag{5.163}$$

- *4-charge black holes:*

$$M = \frac{R_z}{g_s^2 \ell_s^2} N + \frac{n}{R_z} + \frac{R_z}{\ell_s^2} w + \frac{R_z R_\eta^2}{g_s^2 \alpha'^2} W\,, \tag{5.164}$$

$$S_{\rm BH} = 2\pi\sqrt{nwNW}\,. \tag{5.165}$$

At first order in α', the main lesson to extract from our studies is that one has to properly distinguish between the asymptotic or Maxwell S5-brane and momentum charges and the number of S5-branes, N, and quantized momentum, n. As we have seen, this is due to the fact that the α' corrections introduce delocalized sources of momentum and S5-brane charge. In the latter case, the different contributions to the S5-brane charge are very well understood, thanks in part to the work done in Chapter 4. In first place, we have the leading contribution from the own S5-branes, which is normalized to be just equal to the number of branes, N, and which correspond to the zeroth order result. Then, we have the subleading contributions coming from the instanton number of the gauge fields and of the torsionful spin connection. The latter arises because the S5-branes and the KK monopole act as sources of two instantons that live each in one of the $\mathfrak{su}_\pm(2)$ subspaces in which $\mathfrak{so}(4) \cong \mathfrak{su}_+(2) \oplus \mathfrak{su}_-(2)$ can be decomposed. This reveals, on the one hand, that KK monopoles carry S5-brane charge —something that was well-known [252] and that has indeed played a fundamental rôle in S-duality of heterotic/type IIB—, and, on the other hand, that the S5-brane charge of a stack of N S5-branes is not simply N but gets a (negative) correction of order $\mathcal{O}(1)$, which of course is negligible in the $N \to \infty$ limit.

Given this, one can write the expressions for the mass and the black-hole entropy using both types of variables and this turns out to be crucial for the comparison with the exhisting results in the literature. Let us collect the different expressions here for the sake of completeness,

- *3-charge black holes:*

$$\begin{aligned} M &= \frac{R_z}{g_s^2\ell_s^2}(N+\mathfrak{n}-1)+\frac{n}{R_z}\left(1+\frac{2}{N}\right)+\frac{R_z}{\ell_s^2}w \\ &= \frac{R_z}{g_s^2\ell_s^2}\mathcal{Q}_{\rm S5}+\frac{\mathcal{Q}_{\rm P}}{R_z}+\frac{R_z}{\ell_s^2}\mathcal{Q}_{\rm F1}\,, \end{aligned} \tag{5.166}$$

$$\begin{aligned} S_{\rm W} &= 2\pi\sqrt{nwN}\left(1+\frac{2}{N}\right) \\ &= 2\pi\sqrt{\mathcal{Q}_{\rm F1}\mathcal{Q}_{\rm P}\left(\mathcal{Q}_{\rm S5}+3-\mathfrak{n}\right)}\,, \end{aligned} \tag{5.167}$$

where we have used that the expression of the asymptotic charges in terms of n, w and N —see Eqs. (5.124) and (5.110)— is

$$\mathcal{Q}_{\rm P}=n\left(1+\frac{2}{N}\right)\,,\qquad \mathcal{Q}_{\rm F1}=w\,,\qquad \mathcal{Q}_{\rm S5}=N+\mathfrak{n}-1\,. \tag{5.168}$$

- *4-charge black holes:*

$$M = \frac{R_z}{g_s^2\ell_s^2}\left(N+\frac{\mathfrak{n}-2}{W}\right)+\frac{n}{R_z}\left(1+\frac{2}{NW}\right)+\frac{R_z}{\ell_s^2}w+\frac{R_zR_\eta^2}{g_s^2\alpha'^2}W \tag{5.169}$$

$$= \frac{R_z}{g_s^2\ell_s^2}\mathcal{Q}_{\rm S5}+\frac{\mathcal{Q}_{\rm P}}{R_z}+\frac{R_z}{\ell_s^2}\mathcal{Q}_{\rm F1}+\frac{R_zR_\eta^2}{g_s^2\alpha'^2}\mathcal{Q}_{\rm KK}\,, \tag{5.170}$$

$$\begin{aligned} S_{\rm W} &= 2\pi\sqrt{nwNW}\left(1+\frac{2}{NW}\right) \\ &= 2\pi\sqrt{\mathcal{Q}_{\rm F1}\,\mathcal{Q}_{\rm P}\,\left(\mathcal{Q}_{\rm S5}\,\mathcal{Q}_{\rm KK}+4-\mathfrak{n}\right)}, \end{aligned} \tag{5.171}$$

where

$$\mathcal{Q}_{\rm P}=n\left(1+\frac{2}{NW}\right)\,,\quad \mathcal{Q}_{\rm F1}=w\,,\quad \mathcal{Q}_{\rm S5}=N+\frac{\mathfrak{n}-2}{W}\,,\quad \mathcal{Q}_{\rm KK}=W\,. \tag{5.172}$$

This is the main result of this chapter. We can now compare the expressions obtained here for the black-hole entropy with those obtained in [243] by using AdS/CFT methods. We observe that both agree when the variables chosen to write the black-hole entropy are the asymptotic charges and if the number of instantons, $\mathfrak{n}$, is set to zero. This was expected since the analysis performed in the that reference does not take into account the gauge fields.

Let us emphasize that although our original motivation was simply to study the next-to-leading order in α' corrections to heterotic black holes, we have been able to actually obtain certain expressions that are exact to all orders in the α' expansion, even if to obtain them we have made an explicit use of the first-order α'-corrected solutions. One of these expressions is the black-hole entropy, whose exactness is based on the assumption that the near-horizon geometry is not modified by the α' corrections. In addition to this, we have also argued that the S5-brane charge, which is obtained through the evaluation of a topological invariant, is not expected to be modified by the higher-order α' corrections, see e.g. [259]. Finally, the authors of [246] used these two facts to show that, under very reasonable assumptions, the relations between the asymptotic charges and the parameters that specify the sources of the system N, n, w and W are expected to be exact all orders in the α' expansion.

So far, we have restricted our attention to black-hole solutions which in the $\alpha' \to 0$ limit have a horizon with a radius much larger than the string scale, $R_H >> \ell_s$. At this point, the next question would be to ask ourselves what occurs if, for instance, we set one of the black-hole charges to zero. This would imply that the horizon shrinks to zero size in the $\alpha' \to 0$ limit since the horizon area of these black holes is proportional to the square root of the product of the charges. Then, it would be interesting to study if the α' corrections can stretch the horizon, thus covering the naked singularity that is left behind. This naive question is deeper than it seems since it turns out that some of the corresponding ten-dimensional backgrounds describe very well-known system in string theory whose degeneracy of BPS sates has been found to be non-vanishing, e.g.: [134, 260]. Therefore, an agreement between the macroscopic (black-hole) entropy and the microscopic counting of BPS states is missing in these cases as the Bekenstein-Hawking entropy vanishes. However, it may happen, as suggested in [135], that the Wald entropy for these kind of *small* black holes could give a finite answer in agreement with the microscopic result. This is a subtle issue that we shall study in detail in the next two chapters.

6
The small black hole illusion

Consider a fundamental heterotic string carrying winding number w and momentum n charges along a circle $\mathbb{S}^1_z$ [130, 134], which forms part of the compact space $\mathbb{T}^4 \times \mathbb{S}^1_z \times \mathbb{S}^1_\eta$. For large values of n and w, the entropy associated to the degeneracy of these states is

$$S_{\text{micro}} \approx 4\pi\sqrt{nw}\,, \qquad n, w >> 1\ . \tag{6.1}$$

It was soon suggested that this system, among others, could be also represented as a black hole at strong coupling [66, 261]. In this case the black hole would be *small*, because the event horizon scale would be of the order of the string length.

Working with the heterotic effective action at lowest order in the perturbative expansion, a solution to the equations of motion carrying the same two charges and preserving the same amount of supersymmetry as this configuration was found in [262]. That solution is characterized by a singular horizon of vanishing area and entropy, so higher-curvature terms in the effective expansion cannot be ignored in the near-horizon region. This just means that the effective classical description fails to give a good approximation of the system. In a seminal article [135] it was then conjectured that the higher-curvature corrections might somehow render the horizon regular, and that the Wald entropy of such black hole would match the microscopic value (6.1).

Always assuming the existence of a regular horizon and making use of the associated attractor mechanism, a precise matching of the macroscopic and microscopic computations of the entropy was later reported in [263] —see also [264]. This correlation was widely interpreted as a proof of the resolution of the horizon previously conjectured. However, in order to firmly establish if there is causation, there are certain aspects of these studies that need clarification.

In the first place, the techniques employed in those articles only allow for the study of the near-horizon regime, but the analytic construction of a full black hole solution interpolating between those and asymptotic Minkowski space is missing. According to numerical studies [265, 266], the solution exists and its causal structure is identical to that of four-charge regular black holes.[1] Nevertheless, in order to fully understand the system studied an analytic solution is needed, specially if we take into consideration that higher-curvature corrections introduce significant global interactions, as we have seen in Chapter 5.

[1] In these studies the asymptotic solution presents oscillations, which should be removable by appropriate field redefinitions by consistency of string theory [265, 267]. As a consequence of a lemma proved in [108], the supersymmetric formulation of the heterotic theory that we use is free of this undesired feature and our solutions do not present asymptotic oscillations.

In the second place, it is certainly surprising that the resolution of the small black hole described in [263–266] is achieved with the inclusion of only curvature squared terms. Since the departing system is singular, there is no reason to expect that further higher-derivative corrections can be disregarded for that purpose.

And in the third place, the resolution of similar singular systems, like a Type II string with winding and momentum charges, has not been observed. The different behavior of small black holes in diverse theories raises a puzzle whose resolution has remained unclear so far.

In this chapter we argue that the resolution of the heterotic small black hole via higher-curvature corrections does not actually occur. To do so, we apply the results of the previous chapter, where the analytic construction of general four-charge, supersymmetric black holes including curvature squared terms has been performed. A simple argument illustrates that apparently regular four-dimensional, two-charge black holes contain a curvature singularity when embedded in the heterotic theory.

We claim that the resolution of the horizon previously reported is an illusion; the higher-curvature corrections do not really resolve the singularity of [262]. Instead, we find a special four-charge black hole whose entropy is precisely given by $4\pi\sqrt{nw}$ and whose asymptotic S5-brane charge vanishes, but that does not actually describe the two-charge system. We call this solution a *fake* small black hole and we argue that the resolution of the horizon observed in the literature corresponds to this system.

Although this might seem to represent a step back in the understanding of small black holes, we actually believe that our result clarifies the situation. It puts every small black hole at the same qualitative level; the system is non-perturbative in α' and cannot be properly described incorporating a subgroup of the higher-curvature corrections. Moreover, we recall that the heterotic small black hole can be resolved in the type IIB frame using the uncorrected effective action [73, 200], via smooth geometries whose degeneracy agrees with (6.1), an observation that gave rise to the fuzzball proposal [74, 268].

6.1 The zeroth-order solution

Let us review the singular small black hole solution. We shall ignore, for the time being, the higher-derivative corrections. We find convenient to first present the four-charge regular black holes and then particularize to the two-charge case. The ten-dimensional embedding of the four-charge black holes is given by

$$ds^2 = \frac{2}{\mathcal{Z}_-}du\left(dt - \frac{\mathcal{Z}_+}{2}du\right) - \mathcal{Z}_0\, d\sigma^2 - dz^\alpha dz^\alpha\,, \tag{6.2}$$

$$H = d\mathcal{Z}_-^{-1} \wedge du \wedge dt + \star_\sigma d\mathcal{Z}_0 \tag{6.3}$$

$$e^{2\phi} = g_s^2 \frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \tag{6.4}$$

where ds^2 denotes the ten-dimensional metric in the string frame and $\star_\sigma$ is the Hodge dual associated to the four-dimensional metric $d\sigma^2$, which is a Gibbons-Hawking space:

$$d\sigma^2 = \mathcal{H}^{-1}(d\eta+\chi)^2 + \mathcal{H}\,dx^i dx^i\,, \qquad d\mathcal{H} = \star_3 d\chi\,. \tag{6.5}$$

Six of the coordinates are compact: the coordinates z^α parametrize a four-torus $\mathbb{T}^4$ with no dynamics, while $z = t-u$ and η have respective periods $2\pi R_z$ and $2\pi R_\eta$ and they parametrize two circles that we denote $\mathbb{S}^1_z$ and $\mathbb{S}^1_\eta$.

The functions $\mathcal{Z}_{0,\pm}$ and $\mathcal{H}$ that determine the solution are given by

$$\mathcal{Z}_{0,+,-} = 1 + \frac{q_{0,+,-}}{r}\,, \qquad \text{and} \qquad \mathcal{H} = 1 + \frac{q_H}{r}\,, \tag{6.6}$$

where $r = \sqrt{x^i x^i}$ is the radial coordinate of $\mathbb{E}^3$. Notice that all of these functions are harmonic in this space. This solution represents the superposition of:

- a string wrapping the circle $\mathbb{S}^1_z$ with winding number w and momentum n charges,
- a stack of N solitonic 5-branes (S5) wrapped on $\mathbb{T}^4 \times \mathbb{S}^1_z$,
- and a Kaluza-Klein monopole (KK) of charge W associated with $\mathbb{S}^1_\eta$.

We recall that the charge parameters q_0, q_+, q_- and q_H are given in terms of the integer numbers n, w, N and W according to

$$q_0 = \frac{\alpha'}{2R_\eta} N\,, \qquad q_+ = \frac{g_s^2 \alpha'^2}{2R_z^2 R_\eta} n\,, \qquad q_- = \frac{g_s^2\alpha'}{2R_\eta} w\,, \qquad q_H = \frac{W R_\eta}{2}. \tag{6.7}$$

After compactification in $\mathbb{T}^4 \times \mathbb{S}^1_z \times \mathbb{S}^1_\eta$, the lower-dimensional spacetime metric in the Einstein frame is

$$ds^2_{\mathrm{E},4} \;=\; \frac{dt^2}{\sqrt{\mathcal{Z}_+\mathcal{Z}_-\mathcal{Z}_0\mathcal{H}}} - \sqrt{\mathcal{Z}_+\mathcal{Z}_-\mathcal{Z}_0\mathcal{H}}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{6.8}$$

For non-vanishing charges this geometry represents an extremal black hole whose horizon is placed at $r=0$ and its area is $A_H = 2\pi\sqrt{nwNW}$.

The small black hole described in the introduction is that without KK monopole and S5 brane: $N = W = 0$. In that case, at $r=0$ there is still a horizon because g_{tt} vanishes. However, its area is zero and, even worse, its curvature diverges. Hence, classically, these solutions have singular horizon and vanishing Bekenstein-Hawking entropy.[2] The dilaton e^ϕ vanishes at the horizon, so loop corrections can be neglected in this region, but the singularity in curvature signals that the tree-level supergravity description of this system is not valid for small values of r. When trying to describe the physics near the horizon, one is forced to include the tower of higher-curvature corrections to the heterotic effective action [108]. For quite some time, it has been believed that their inclusion would render the horizon regular and make the value of the Wald entropy of the solution coincide with that of the microscopic entropy (6.1).

Let us discuss how the first set of these corrections, which are quadratic in the curvature, alter relevant aspects of the solutions.

[2]This statement holds when any of the four charges vanishes.

6.2 Two-charge solution at first order in α'

The origin of part of the α' corrections in the context of the heterotic superstring has already been discussed in Chapter 4. They arise from the supersymmetrization of the Lorentz Chern-Simons 3-form of the torsionful spin connection. We are not going to enter into more details here, since all we need is to apply the results derived in that chapter to the solution describing the two-charge small black hole. As we have seen in the previous section, it is obtained by setting $q_0 = q_H = 0$, so that

$$\mathcal{Z}_\pm = 1 + \frac{q_\pm}{r}\,, \qquad \text{and} \qquad \mathcal{Z}_0 = \mathcal{H} = 1\,. \tag{6.9}$$

The corrected solution has exactly the same form as (6.2)-(6.4) but with the function $\mathcal{Z}_+$ now given by

$$\mathcal{Z}_+ = 1 + \frac{q_+ + \alpha'\delta q_+}{r} - \frac{\alpha' q_+ q_-}{2r^3(r+q_-)} + \mathcal{O}(\alpha'^2)\,. \tag{6.10}$$

The remaining functions, $\mathcal{Z}_-, \mathcal{Z}_0, \mathcal{H}$, do not get corrected at first order in α'. Thus, they are given by the expressions appearing in Eq. (6.9). By looking at the expression for $\mathcal{Z}_+$[3], we see that for the two-charge system $\lim_{r\to 0}\mathcal{Z}_+ \sim 1/r^3$, which is just the right behavior to obtain a horizon with non-vanishing area in (6.8). However, this does not fix the singularity problem, since the Kaluza-Klein scalars as well as the curvature of the full ten-dimensional solution are still divergent at $r=0$. In particular, the ten-dimensional Ricci scalar is given by

$$R = \frac{7q_-^2}{2r^2\,(r+q_-)^2}\,. \tag{6.11}$$

This implies that the expression for $\mathcal{Z}_+$ cannot be trusted near $r=0$, since in its derivation it is assumed that the ten-dimensional curvature is regular at several stages. The conclusion is that the perturbative expansion in α' is not valid near $r=0$ when $q_0 = q_H = 0$, and one would need to include the full tower of higher-curvature corrections.

From these observations we doubt there exists a *true* resolution of the heterotic small black hole, as it does not seem that we can modify the structure of the fields[4], and a finite sized horizon built with only two functions active ($\mathcal{Z}_+$ and $\mathcal{Z}_-$) is headed towards a singularity in curvature. A similar analysis can also be performed for other singular solutions, like those containing three type of localized sources (say that we add S5-brane sources), with the same conclusion. Corrections of quadratic order in curvature are not sufficient to resolve the singular black hole.

6.3 Delocalized sources and fake small black holes

This indicates that the four parameters q_0, q_+, q_-, q_H must be non-vanishing if we want to describe a consistent black hole solution with a regular horizon, even if we include

[3]Here $\alpha'\delta q_+ << q_+$ is an integration constant whose relation with $q_\pm$ is undetermined due to the singular behaviour of the system.

[4]The structure of the fields is tightly constrained by supersymmetry, see e.g. [249].

quadratic curvature terms. Let us therefore consider the corrections to the zeroth-order solution with $q_0 q_+ q_- q_H \neq 0$. These were studied in Chapter 5, where we showed that $\mathcal{Z}_-$ and $\mathcal{H}$ are uncorrected while $\mathcal{Z}_0$ and $\mathcal{Z}_+$ were given by[5]

$$\mathcal{Z}_+ = 1 + \frac{q_+}{r} + \frac{\alpha' q_+[r^2 + r(q_0 + q_- + q_H) + q_H q_0 + q_H q_- + q_0 q_-]}{2 q_H q_0 (r + q_H)(r + q_0)(r + q_-)} + \mathcal{O}(\alpha'^2), \tag{6.12}$$

$$\mathcal{Z}_0 = 1 + \frac{q_0}{r} - \alpha' \left[F(r; q_0) + F(r; q_H)\right] + \mathcal{O}\left(\alpha'^2\right), \tag{6.13}$$

where

$$F(r; k) := \frac{(r + q_H)(r + 2k) + k^2}{4 q_H (r + q_H)(r + k)^2}. \tag{6.14}$$

The most important property of the corrections is that they introduce delocalized sources so that the asymptotic charges and the near-horizon charges do not coincide. These charges are effectively defined as the coefficient of the $1/r$ term in the functions $\mathcal{Z}_{0,\pm}$, $\mathcal{H}$ when $r \to \infty$ and when $r \to 0$, respectively. Of course, this poses the question of which of those counts the number of the corresponding stringy objects. It is particularly relevant for our discussion the case of S5-brane charge, codified by $\mathcal{Z}_0$. In the limits $r \to 0$ and $r \to \infty$, this function behaves as

$$\lim_{r\to 0} \mathcal{Z}_0 = \frac{q_0}{r}, \quad \lim_{r\to\infty} \mathcal{Z}_0 = 1 + \frac{q_0 - \alpha'/(2q_H)}{r}, \tag{6.15}$$

so that near-horizon and asymptotic charges do not coincide. In the language of [180], these are respectively the brane-source and Maxwell charges.

The correct way to determine the relation between those and the number of solitonic 5-branes is to couple the theory to a stack of N of these branes. This can be done by dualizing the Kalb-Ramond 3-form into the NSNS 7-form $\tilde{H} = d\tilde{B} = e^{-2\phi} \star H$ and coupling the 6-form $\tilde{B}$ to the worldvolume action of N solitonic 5-branes by means of a Wess-Zumino term, as we did in Section 5.3. The net effect is a localized source at the right-hand-side of the Bianchi identity. Thus, the number of S5-branes in our solution may be computed according to

$$N = \frac{1}{4\pi^2 \alpha'} \int_{\mathbb{S}^1_\eta \times \mathbb{S}^2_\infty} H - \frac{1}{16\pi^2} \int R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a\,. \tag{6.16}$$

In the first term we used Stokes' theorem and the integral is taken on the boundary of the GH space (6.5), while in the second the integral is taken over the full GH space. The result of the integration coincides with the identification in (6.7) —see Section 5.3— and therefore it is the near-horizon charge q_0 the one that counts the number of S5-branes.

On the other hand, the asymptotic charge does have a physical meaning by itself and, moreover, gives the contribution to the mass of the black hole. The negative subtraction in (6.15) is telling us that the higher-curvature terms introduce a screening mechanism such that the charge and mass detected at infinity are smaller than the local charge produced by the 5-branes:

[5] We point out that these expressions are only valid when both q_0 and q_H are non-vanishing.

$$\mathcal{Q}_{\text{S5}} = N - \frac{2}{W} \,. \tag{6.17}$$

The origin of the negative shift can be identified with precision. It is caused by the presence of two self-dual gravitational instantons in the four-dimensional Gibbons-Hawking space, one for each $\mathfrak{so}(3)$ factor in the decomposition of the group of local Lorentz transformations $\mathfrak{so}(4) \cong \mathfrak{so}_+(3) \times \mathfrak{so}_-(3)$. The two instantons are sourced by the KK monopole and by the stack of S5-branes respectively, and each one contributes to the asymptotic charge with a factor of $-1/W$.

We obtained this result from the first-order corrected solution (6.13), which is expected to receive other corrections in the α' expansion. However, one can see that (6.17) is actually α'-exact and receives no further corrections. The way to prove it is to note that the S5-brane charge is given by

$$\begin{aligned} \mathcal{Q}_{\text{S5}} &= \frac{1}{4\pi^2\alpha'} \int_{\mathbb{S}^1_\eta \times \mathbb{S}^2_\infty} H \\ &= N + \frac{1}{16\pi^2} \int R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a \,, \end{aligned} \tag{6.18}$$

where in the second line we used (6.16). But now, the integral in the second line is actually a topological invariant: it is not modified at all by continuous deformations of the connection $\Omega_{(-)}{}^a{}_b$, such as the ones introduced by α' corrections. Hence, the value of that integral is always $-32\pi^2/W$, and the S5-brane charge measured at infinity is exactly given by (6.17).

A very important consequence of this result is that the asymptotic S5-charge vanishes for configurations with $NW = 2$.

6.3.1 Fake small black holes

We are now ready to present a *fake* resolution of the singular small black hole. Let us describe a four-charge black hole of the form (6.2)-(6.4) as a solution of the heterotic effective theory that includes all the relevant terms of quadratic order in curvature. The functions $\mathcal{Z}_-$ and $\mathcal{H}$ remain uncorrected as in (6.6), while $\mathcal{Z}_0$ and $\mathcal{Z}_+$ are given respectively by (6.13) and (6.12). The solution has a regular event horizon at $r = 0$, with area $A_H = 2\pi\sqrt{nwNW}$. The near-horizon geometry is $AdS_3 \times S^3/\mathbb{Z}_W \times \mathbb{T}^4$, and the Wald entropy is

$$S_{\text{W}} = 2\pi\sqrt{\mathcal{Q}_{\text{F1}}\mathcal{Q}_{\text{P}}\left(\mathcal{Q}_{\text{S5}}\mathcal{Q}_{\text{KK}} + 2\right)} \,, \tag{6.19}$$

where

$$\mathcal{Q}_{\text{P}} = n\left(1 + \frac{2}{NW}\right) \,, \qquad \mathcal{Q}_{\text{F1}} = w \,, \qquad \mathcal{Q}_{\text{KK}} = W \,. \tag{6.20}$$

The crucial point is that the shift between asymptotic charge and number of branes remained unnoticed, so in the preceding literature $\mathcal{Q}_{\text{S5}}$ was incorrectly identified with the number of branes. There, the parameters of the near-horizon solution are identified in terms of the asymptotic charges using the zeroth-order solution. However, as we have just seen, the relation between the near-horizon parameters and the asymptotic charges is altered when the higher-curvature corrections are included. It is for this reason that setting $\mathcal{Q}_{\rm S5} = 0$ does not automatically imply the absence of solitonic 5-branes.

The fake small black hole is a very special four-charge solution with $NW = 2$ and arbitrary n, w. It has a regular horizon and its entropy just happens to match the value (6.1). On the other hand, it is clearly not a small black hole; it contains solitonic five-branes and Kaluza-Klein monopole localized sources, and its asymptotic charges are different than those of [130, 134]. One should also notice that fake small black holes are already regular in the zeroth-order supergravity approximation.

6.4 Discussion

The resolution of this system reported in [263–266] considered regular near-horizon solutions in the dual frame of Type IIA on $\mathbb{K}_3 \times \mathbb{T}^2$, using a four-dimensional effective description. This phenomenon was also observed directly in the heterotic string on $\mathbb{T}^4 \times \mathbb{S}^1_z \times \mathbb{S}^1_\eta$ in [259]. In all the cases, the S5-brane asymptotic charge, $\mathcal{Q}_{\rm S5}$, is set to zero under the assumption that this implies the absence of S5-branes. As we just saw, that premise is not true. On the other hand, in those works it is also stated that $W = 0$ but, as we have just argued, we found this fact to be incompatible with the assumption of a regular horizon. This incompatibility remains hidden in effective descriptions that only have access to partial information of the solution. This is a crucial point that, if dismissed, can produce the illusion of a stretched horizon. Then, from all angles, it seems the solution described in these studies corresponds to the *fake* small black hole we presented above.

Our conclusions can be straightforwardly extended to five-dimensional small black holes by using $\mathbb{E}^4$ for the Gibbons-Hawking space $d\sigma^2$. This case is simpler because there is no KK monopole and we get $\mathcal{Q}_{\rm S5} = N - 1$ for the screening effect. A fake resolution of the five-dimensional small black hole is then straightforward. In this case there is just one solitonic 5-brane, whose asymptotic charge and mass vanish. The entropy is then given by $S_{\rm W} = 2\pi\sqrt{\mathcal{Q}_{\rm F1}\mathcal{Q}_{\rm P}\left(\mathcal{Q}_{\rm S5}+3\right)}$,[6] which no longer happens to coincide with (6.1) for $N = 1$. Notice that the resolution of five-dimensional small black holes had so far remained uncertain —see the discussion in [259, 269].

Notice that our results do not apply to the qualitatively different class of solutions that come into existence only after the corrections are included —see [270, 271] for an explicit example. The understanding of these black holes in the context of string theory is still an open issue.

Before closing the chapter, we can very briefly consider the small black hole made of a Type II string carrying winding and momentum charges. We recall that in this theory the Bianchi identity does not receive corrections, so the charges are the same at the horizon and asymptotically. This means that one cannot design a fake resolution of the singularity in the terms we just described, which according to our findings clarifies why no cure for their singularity had been reported.

[6]The momentum charge is given by $\mathcal{Q}_{\rm P} = n\left(1 + \frac{2}{N}\right)$.

7

Higher-derivative corrections to small black rings

In the previous chapter, we have studied if a heterotic fundamental string with winding w and momentum n charges along a circle $\mathbb{S}^1_z$ (usually referred to as the F1-P system) can be described as a black hole with a regular horizon once the first-order α' corrections have been taken into account. We have shown that although the four-dimensional metric has a regular horizon with finite size, the full solution is singular, as the Kaluza-Klein scalar that measures the radius of the circle $\mathbb{S}^1_z$ diverges at the horizon. This is reflected in ten dimensions in a curvature singularity that tell us that the solution cannot be trusted near the horizon, since in its derivation is assumed that the curvature is small with respect to α'. Furthermore, we have argued that previous regularizations of four-dimensional small black holes would correspond to a very special type of extremal black holes with four charges (which have been dubbed *fake* small black holes) whose entropy happens to coincide numerically with the microscopic degeneracy of the F1-P system [134], but which describe a system that also contains S5-branes and KK monopoles.

We would like now to add angular momentum to the F1-P system. The degeneracy of BPS states of this system was computed by Russo and Susskind in [260] —see also [272, 273]. The result is

$$\mathcal{S}_{\text{micro}} = 4\pi\sqrt{nw - JW}\,, \tag{7.1}$$

where J and W are the angular momentum and the winding number along the direction of rotation. Supergravity solutions with the same conserved charges as the rotating F1-P system were constructed in [132,133,274].[1] It was shown in [275,276] that a particular class of them gives raise to supersymmetric two-charge black rings which also have a singular horizon with vanishing area, analogously to what occurs for small black holes. Then, the question is: do higher-derivative corrections stretch the horizon of small black rings? Some indirect evidence in favor was given in [277], where the entropy of five-dimensional small black rings was related to that of static four-dimensional small black holes by making use of the 4d-5d connection [278–281]. Furthermore, by means of a generalization of the scaling analysis presented in [135], it was argued in [276] that the Wald entropy of the α'-corrected small black ring would reproduce (7.1) up to an overall proportionality constant.

However, these arguments are based on certain premises that need not to be true. On the one hand, the evidence presented in [277] is based on the regularization of four-dimensional small black holes, which has been refuted in the previous chapter. On the other, the scaling argument of [135, 276] only works if the correction to the Bekenstein-Hawking entropy is finite. However, we already know —see Eqs. (5.167) and (5.171)—

[1] These are reviewed in Section 7.3.

that this is not the case for small black holes and, by analogy, we expect the same for small black rings.

These aspects could be clarified by a computing the α' corrections to the singular small black ring. This is the main goal of this chapter.

7.1 The effective action of the heterotic superstring

Let us briefly review the relevant information about the effective field theory of the heterotic superstring that we shall need in the remaining of the chapter. We are going to assume that the string coupling constant is small so that loop or quantum corrections can be safely ignored. Even in this limit, the effective action of the heterotic superstring [108, 109] contains an infinite tower of higher-derivative terms, although only a few of them have been explicitly constructed. This higher-derivative expansion is usually referred to as α'-expansion since a term with $2n$ derivatives will be multiplied by α'^{n-1}, where $\alpha' = \ell_s^2$ and ℓ_s is the string scale.[2] For our purposes, however, it is enough to present the action up to second order in α'. Using the conventions of [111], we have[3]

$$S = \frac{g_s^2}{16\pi G_{\rm N}^{(10)}} \int d^{10}x\sqrt{|g|}\, e^{-2\phi} \left\{ R - 4\,(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2 + \frac{\alpha'}{8} R_{(-)\mu\nu a b}R_{(-)}{}^{\mu\nu a b} + \mathcal{O}\left(\alpha'^2\right) \right\}\,, \tag{7.2}$$

where $G_{\rm N}^{(10)}$ is the 10-dimensional Newton constant and g_s is the string coupling constant. The metric $g_{\mu\nu}$ is the string frame metric, ϕ is the dilaton and H is the 3-form field strength of the Kalb-Ramond 2-form B, whose definition is

$$H = dB + \frac{\alpha'}{4}\omega^{\rm L}_{(-)}\,, \tag{7.3}$$

where $\omega^{\rm L}_{(-)}$ is the Lorentz Chern-Simons 3-form associated to the torsionful spin connection

$$\omega^{\rm L}_{(-)} = d\Omega_{(-)}{}^{a}{}_{b} \wedge \Omega_{(-)}{}^{b}{}_{a} - \frac{2}{3}\Omega_{(-)}{}^{a}{}_{b} \wedge \Omega_{(-)}{}^{b}{}_{c} \wedge \Omega_{(-)}{}^{c}{}_{a}\,, \tag{7.4}$$

which in turn is defined as

$$\Omega_{(-)}{}^{a}{}_{b} = \omega^{a}{}_{b} - \frac{1}{2}H_{c}{}^{a}{}_{b}\, e^{c}\,, \tag{7.5}$$

where $\omega^{a}{}_{b}$ represents the Levi-Civita spin connection. Finally,

$$R_{(-)}{}^{a}{}_{b} = d\Omega_{(-)}{}^{a}{}_{b} - \Omega_{(-)}{}^{a}{}_{c} \wedge \Omega_{(-)}{}^{c}{}_{b}\,, \tag{7.6}$$

is the curvature 2-form associated to the torsionful spin connection.

[2] By dimensional analysis, a term with $2n$ derivatives must be multiplied by a coupling of dimension length^{2n-2}. In the case of the superstring theory, these couplings must be fully controlled by α', which is the unique dimensionful parameter of the theory.

[3] We have set to zero the gauge fields that are also active at first order in α', which is always a consistent truncation.

Let us notice two important aspects of the definition (7.3). The first one is that it implies that the Bianchi identity of H gets corrected by

$$dH - \frac{\alpha'}{4} R_{(-)}{}^{a}{}_{b} \wedge R_{(-)}{}^{b}{}_{a} = 0\,. \tag{7.7}$$

The second one is that (7.3) is a recursive definition that one has to implement order by order in α'. Hence, the action (7.2) and the Bianchi identity (7.7) actually contain an infinite tower of implicit α' corrections.

We want to emphasize that the action (7.2) makes sense only in the limit where the higher-order α'-corrections are subleading. This occurs, on general grounds, when the curvature is small as compared to α', namely

$$\alpha' \mathcal{R} << 1\,, \tag{7.8}$$

where $\mathcal{R}$ denotes schematically the curvature. If this is case, then it is justified to ignore terms with increasing number of derivatives since these will be more and more suppressed.

7.1.1 Equations of motion

In order to write the equations of motion derived from (7.2), we shall use a lemma which was proven in [108]. The lemma states that the variation of the action with respect to the torsionful spin connection $\frac{\delta S}{\delta \Omega^{(-)a}{}_{b}}$ is proportional to α' and the zeroth-order equations of motion plus $\mathcal{O}\left(\alpha'^2\right)$ terms. Taking this into account, let us now separate the variation of the action with respect to the fields into explicit and implicit variations occurring through the torsionful spin connection as follows

$$\delta S = \left.\frac{\delta S}{\delta e^{a}{}_{\mu}}\right|_{\text{ex}} \delta e^{a}{}_{\mu} + \frac{\delta S}{\delta \phi}\delta\phi + \left.\frac{\delta S}{\delta B_{\mu\nu}}\right|_{\text{ex}} \delta B_{\mu\nu} + \frac{\delta S}{\delta \Omega_{(-)}{}^{a}{}_{b}} \left[\frac{\delta \Omega_{(-)}{}^{a}{}_{b}}{\delta e^{c}{}_{\rho}} \delta e^{c}{}_{\rho} + \frac{\delta \Omega_{(-)}{}^{a}{}_{b}}{\delta B_{\mu\nu}} \delta B_{\mu\nu} \right]. \tag{7.9}$$

Because of the aforementioned lemma, if we work perturbatively in α' the last term above will yield $\mathcal{O}\left(\alpha'^2\right)$ terms which we shall ignore. Then, taking into account only the explicit variations, one has that the α'-corrected equations of motion are

$$R_{\mu\nu} - 2\nabla_{\mu}\partial_{\nu}\phi + \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} = -\frac{\alpha'}{4} R^{(0)}_{(-)\mu\rho ab} R^{(0)}_{(-)\nu}{}^{\rho ab} + \mathcal{O}\left(\alpha'^2\right), \tag{7.10}$$

$$(\partial\phi)^2 - \frac{1}{2}\nabla^2\phi - \frac{1}{4\cdot 3!} H^2 = \frac{\alpha'}{32} R^{(0)}_{(-)\mu\nu ab} R^{(0)}_{(-)}{}^{\mu\nu ab} + \mathcal{O}\left(\alpha'^2\right), \tag{7.11}$$

$$d\left(e^{-2\phi} \star H\right) = \mathcal{O}\left(\alpha'^2\right), \tag{7.12}$$

where $R^{(0)}_{(-)\mu\nu ab}$ denotes the zeroth-order curvature. Therefore, the equations of motion will be of second-order in derivatives since the quadratic-curvature terms in the action only act as effective "sources" of energy and momentum.

7.2 A family of α'-corrected heterotic backgrounds

7.2.1 The zeroth-order solutions

Let us consider the following field configuration at zeroth order in α'

$$ds^2 = \frac{2}{\mathcal{Z}_-^{(0)}} du \left(dt + \omega^{(0)} - \frac{\mathcal{Z}_+^{(0)}}{2} du \right) - dx^m dx^m - dz^\alpha dz^\alpha \,, \tag{7.13}$$

$$B = \frac{1}{\mathcal{Z}_-^{(0)}} du \wedge \left(dt + \omega^{(0)} \right) \,, \tag{7.14}$$

$$e^{2\phi} = \frac{g_s^2}{\mathcal{Z}_-^{(0)}} \,, \tag{7.15}$$

where x^m, with $m = 1, \ldots, d-1$, are the Cartesian coordinates of the Euclidean space $\mathbb{E}^{d-1}$ where the functions $\mathcal{Z}_\pm^{(0)}$ and the 1-form $\omega^{(0)}$ are defined. The coordinates $z^\alpha \sim z^\alpha + 2\pi \ell_s$, with $\alpha = 1, \ldots, 9-d$, parametrize a torus $\mathbb{T}^{9-d}$ with no dynamics. There is an additional compact direction, $z = t - u$, which parametrizes a circle whose asymptotic radius is denoted by R_z.

This configuration has been extensively studied in the literature, see e.g. [73,130–133, 233, 274, 276, 282–284]. It preserves half of the spacetime supersymmetries and describes, as we will see in the next section, a rotating superposition of a fundamental string and a momentum wave.

The zeroth-order equations of motion can be straightforwardly derived from (7.10), (7.11) and (7.12) by just setting $\alpha' = 0$. One finds that they are satisfied by our configuration if

$$\partial^2 \mathcal{Z}_\pm^{(0)} = 0 \,, \tag{7.16}$$

$$\partial_p {\Omega^{(0)}}_{pm} = 0 \,, \tag{7.17}$$

where $\partial^2 = \partial_m \partial_m$ and $\Omega^{(0)} = d\omega^{(0)}$.

7.2.2 First-order α' corrections

Assuming that we have a solution to the zeroth-order equations of motion[4], let us try to find a solution to the corrected equations of motion. For that, we will assume the same ansatz as before, i.e.:

[4] Namely, two functions $\mathcal{Z}_\pm^{(0)}$ and a 1-form $\omega^{(0)}$ satisfying (7.16) and (7.17), respectively.

$$
\begin{aligned}
ds^2 &= \frac{2}{\mathcal{Z}_-}du\left(dt+\omega-\frac{\mathcal{Z}_+}{2}du\right)-dx^m dx^m - dz^\alpha dz^\alpha\,, && (7.18)\\
B &= \frac{1}{\mathcal{Z}_-}\,du\wedge(dt+\omega)\,, && (7.19)\\
e^{2\phi} &= \frac{g_s^2}{\mathcal{Z}_-}\,, && (7.20)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{Z}_\pm &= \mathcal{Z}_\pm^{(0)}+\alpha'\mathcal{Z}_\pm^{(1)}+\mathcal{O}\left(\alpha'^2\right)\,, && (7.21)\\
\omega &= \omega^{(0)}+\alpha'\omega^{(1)}+\mathcal{O}\left(\alpha'^2\right)\,. && (7.22)
\end{aligned}
$$

For the sake of convenience, we define the following zehbein basis

$$
e^+=\mathcal{Z}_-^{-1}du\,,\qquad e^-=dt-\frac{\mathcal{Z}_+}{2}du+\omega\,,\qquad e^m=dx^m\,,\qquad e^\alpha=dz^\alpha\,. \tag{7.23}
$$

It is not difficult to see by using the results of Appendix F.2 that the Lorentz Chern-Simons 3-form (7.4) vanishes for our ansatz. Then, we find that the form of H is exactly the same as in the zeroth-order case

$$
H=dB=\frac{\partial_m\mathcal{Z}_-}{\mathcal{Z}_-^2}\,dx^m\wedge(dt+\omega)\wedge du-\frac{1}{\mathcal{Z}_-}\Omega\wedge du\,, \tag{7.24}
$$

where $\Omega=d\omega$. This implies that the conditions imposed by the equation of motion of the Kalb-Ramon 2-form, Eq. (7.12), are exactly those that we already found at zeroth order in α', namely

$$
\begin{aligned}
\partial^2\mathcal{Z}_- &= \mathcal{O}\left(\alpha'^2\right)\,, && (7.25)\\
\partial_p\Omega_{pm} &= \mathcal{O}\left(\alpha'^2\right)\,. && (7.26)
\end{aligned}
$$

As a consequence of supersymmetry [249], the equation of motion of the dilaton is also satisfied if (7.25) holds. For the Einstein equations (7.10), we find that the $+-$ and $+m$ components are satisfied if (7.25) and (7.26) hold, whereas the $++$ component gives

$$\frac{\mathcal{Z}_-^{(0)}}{2}\partial^2\mathcal{Z}_+ = -\alpha' R^{(0)}_{(-)+mn+}R^{(0)}_{(-)+mn-} + \frac{\alpha'}{4}R^{(0)}_{(-)+mnp}R^{(0)}_{(-)+mnp} + \mathcal{O}\left(\alpha'^2\right)$$

$$=\alpha'\mathcal{Z}_-^{(0)}\left\{-\frac{1}{2}\left(\partial_m\partial_n\mathcal{Z}_+^{(0)} - \frac{\partial_m\mathcal{Z}_+^{(0)}\partial_n\mathcal{Z}_-^{(0)}}{\mathcal{Z}_-^{(0)}}\right)\left(\frac{\partial_m\partial_n\mathcal{Z}_-^{(0)}}{\mathcal{Z}_-^{(0)}} - \frac{\partial_m\mathcal{Z}_-^{(0)}\partial_n\mathcal{Z}_-^{(0)}}{(\mathcal{Z}_-^{(0)})^2}\right)\right.$$

$$\left.+\frac{1}{4}\mathcal{Z}_-^{(0)}\partial_m\left(\frac{\Omega^{(0)}_{np}}{\mathcal{Z}_-^{(0)}}\right)\partial_m\left(\frac{\Omega^{(0)}_{np}}{\mathcal{Z}_-^{(0)}}\right)\right\} + \mathcal{O}\left(\alpha'^2\right) . \tag{7.27}$$

This equation can be rewritten using that $\mathcal{Z}_\pm^{(0)}$ and $\omega^{(0)}$ satisfy (7.16) and (7.17). We obtain

$$\partial^2\left\{\mathcal{Z}_+ - \alpha'\frac{\Omega^{(0)}{}_{mn}\Omega^{(0)mn} - 2\partial_m\mathcal{Z}_+^{(0)}\partial_m\mathcal{Z}_-^{(0)}}{4\mathcal{Z}_-^{(0)}}\right\} = \mathcal{O}\left(\alpha'^2\right) , \tag{7.28}$$

whose solution is

$$\mathcal{Z}_+ = \mathcal{Z}_+^{(0)} + \alpha'\frac{\Omega^{(0)}{}_{mn}\Omega^{(0)mn} - 2\partial_m\mathcal{Z}_+^{(0)}\partial_m\mathcal{Z}_-^{(0)}}{4\mathcal{Z}_-^{(0)}} + \mathcal{O}\left(\alpha'^2\right) , \tag{7.29}$$

with $\mathcal{Z}_+^{(0)}$ harmonic in $\mathbb{E}^{d-1}$. The remaining components of Einstein's equations are automatically satisfied for our ansatz.

7.3 Small black rings from rotating strings

Let us discuss a particular class of solutions to which the results of Section 7.2 can be applied. These can be derived from the ones obtained originally in [132, 133], where also dependence in u is allowed. The functions $\mathcal{Z}_\pm^{(0)}$ and the 1-form $\omega^{(0)}$ are given by

$$\mathcal{Z}_-^{(0)} = 1 + \frac{q_-}{||x^m - F^m||^{d-3}} , \tag{7.30}$$

$$\mathcal{Z}_+^{(0)} = 1 + \frac{q_+ + q_-\dot{F}^2}{||x^m - F^m||^{d-3}} , \tag{7.31}$$

$$\omega_m^{(0)} = \frac{q_-\dot{F}^m}{||x^m - F^m||^{d-3}} , \tag{7.32}$$

where $F^m = F^m(u)$ are arbitrary functions of $u = t - z$, q_- and q_+ are constants and the dot denotes derivative with respect to u.

In the static limit, which corresponds to $F^m = \text{const}$, one recovers the solutions of Refs. [130, 131]. These describe a fundamental string wrapped along $\mathbb{S}^1_z$ with winding and a momentum charges. The dimensional reduction of these solutions on $\mathbb{T}^{9-d} \times \mathbb{S}^1_z$ yields singular two-charge black holes (namely, small black holes) in $4 \leq d \leq 9$ dimensions.

In the rotating case, $F^m \neq$ const, the string is no longer located at a point in the non-compact space. In turn, its position is parametrically given by

$$x^m = F^m(u)\,. \tag{7.33}$$

From this general family of rotating string backgrounds, one can obtain a class of solutions with no dependence in the internal coordinate z (see [274]) by the usual smearing procedure, which amounts to keeping only the zero mode in the Fourier expansion, namely:

$$\mathcal{Z}_-^{(0)} = 1 + \int_0^\ell \frac{q_-}{||x^m - F^m||^{d-3}}\, du\,, \tag{7.34}$$

$$\mathcal{Z}_+^{(0)} = 1 + \int_0^\ell \frac{q_+ + q_- \dot{F}^2}{||x^m - F^m||^{d-3}}\, du\,, \tag{7.35}$$

$$\omega_m^{(0)} = \int_0^\ell \frac{q_- \dot{F}^m}{||x^m - F^m||^{d-3}}\, du\,, \tag{7.36}$$

where $\ell = 2\pi w R_z$.

Let us check that the smearing procedure yields a solution. In first place, it is clear that $\mathcal{Z}_\pm^{(0)}$ are harmonic functions in $\mathbb{E}^{d-1}$, so that Eqs. (7.16) are satisfied.[5] It remains to check that Eq. (7.17) is also satisfied. Since $\omega_m^{(0)}$ are harmonic in $\mathbb{E}^{d-1}$, we have to show that

$$\partial_m \partial_p \omega_p^{(0)} = 0\,. \tag{7.37}$$

From (7.36), we have that

$$\partial_p \omega_p^{(0)} = \frac{q_-}{||x^m - F^m\,(0)\,||^{d-3}} - \frac{q_-}{||x^m - F^m\,(\ell)\,||^{d-3}}\,, \tag{7.38}$$

which vanishes if $F^m\,(u) = F^m\,(u+\ell)$, in which case (7.17) is satisfied. We shall assume this in what follows. In fact, as in [132, 274], we will restrict to a circular profile of the form

$$F^1 = R\cos\left(\frac{Wu}{wR_z}\right)\,, \qquad F^2 = R\sin\left(\frac{Wu}{wR_z}\right)\,, \qquad F^3 = \cdots = F^{d-1} = 0\,, \tag{7.39}$$

with $W \in \mathbb{Z}$. For this particular choice, the string is wound over a two-dimensional torus parametrized by the coordinates z and ψ, the latter being the angular direction in the $x^1 - x^2$ plane. The integers w and W tell us how many times the string is wound around the $z-$ and $\psi-$directions, respectively. Upon compactification on $\mathbb{T}^{9-d} \times \mathbb{S}^1_z$, the solution gives raise to singular two-charge black rings (small black rings) in $4 \leq d \leq 9$ dimensions [276, 285]. Let us study the five-dimensional case.

[5] Except at the pole of the harmonic function, where one must take into account also the contribution from the sources [132].

7.3.1 Five-dimensional small black rings

Let us consider the ten-dimensional configuration (7.13), (7.14) and(7.15) for $d = 5$. If we reduce it on $\mathbb{T}^4 \times \mathbb{S}^1_z$ —using the results of Appendix E—, we obtain the following five-dimensional configuration[6]

$$ds^2_{\mathrm{E},5} = \left(\mathcal{Z}^{(0)}_+ \mathcal{Z}^{(0)}_-\right)^{-2/3} \left(dt + \omega^{(0)}\right)^2 - \left(\mathcal{Z}^{(0)}_+ \mathcal{Z}^{(0)}_-\right)^{1/3} ds^2\left(\mathbb{E}^4\right) , \tag{7.40}$$

$$A^{\pm} = c^{\pm} \frac{dt + \omega^{(0)}}{\mathcal{Z}^{(0)}_{\pm}} , \qquad A^0 = c^0 \chi , \tag{7.41}$$

$$e^{2\phi} = \frac{g_s^2}{\mathcal{Z}^{(0)}_-} , \qquad k = k_\infty \frac{\mathcal{Z}^{(0)\,1/2}_+}{\mathcal{Z}^{(0)\,1/4}_-} , \tag{7.42}$$

where $c^+ = 2\sqrt{3} k_\infty^{-4/3}$, $c^- = \sqrt{3} e^{\phi_\infty} k_\infty^{2/3}$, $c^0 = -\sqrt{3} e^{-\phi_\infty} k_\infty^{2/3}$ and χ is a 1-form defined on $\mathbb{E}^4$ such that

$$d\chi = \star_4 d\omega , \tag{7.43}$$

where $\star_4$ is the Hodge dual with respect to the Euclidean metric.[7] The functions $\mathcal{Z}^{(0)}_{\pm}$ and the 1-form $\omega^{(0)}$ are given by

$$\mathcal{Z}^{(0)}_- = 1 + \frac{\mathcal{Q}_-}{\ell} \int_0^\ell \frac{du}{||x^m - F^m||^2} , \tag{7.44}$$

$$\mathcal{Z}^{(0)}_+ = 1 + \frac{\mathcal{Q}_+}{\ell} \int_0^\ell \frac{du}{||x^m - F^m||^2} , \tag{7.45}$$

$$\omega^{(0)}_m = \frac{\mathcal{Q}_-}{\ell} \int_0^\ell \frac{\dot{F}^m du}{||x^m - F^m||^2} , \tag{7.46}$$

where we have defined

$$\mathcal{Q}_- = q_- \ell , \qquad \mathcal{Q}_+ = q_+ \ell + \frac{4\pi^2 W^2 R^2 q_-}{\ell} . \tag{7.47}$$

As shown in [275, 276], this five-dimensional solution is a particular case of the supersymmetric three-charge black ring constructed in [207, 239, 286]. In order to see this, we have to perform the integrals appearing in (7.44), (7.45) and (7.46), for which it is convenient to introduce a new set of coordinates (ξ, η, ψ, ϕ) defined as

$$x^1 = \xi \cos\psi , \quad x^2 = \xi \sin\psi , \quad x^3 = \eta \cos\phi , \quad x^4 = \eta \sin\phi . \tag{7.48}$$

[6] This is a solution of the STU model of five-dimensional $\mathcal{N} = 1$ supergravity.

[7] The integrability condition of χ is guaranteed to be satisfied because of the equation of motion of the Kalb-Ramond 2-form (7.17).

Then, after a bit of algebra, one finds

$$\mathcal{Z}_{\pm}^{(0)} = 1+\frac{\mathcal{Q}_{\pm}}{\sqrt{\left(\xi^2+\eta^2+R^2\right)^2-4R^2\xi^2}}\,, \tag{7.49}$$

$$\omega^{(0)} = \frac{\pi\mathcal{Q}_{-}W}{\ell}\left(\frac{\xi^2+\eta^2+R^2}{\sqrt{\left(\xi^2+\eta^2+R^2\right)^2-4R^2\xi^2}}-1\right)d\psi\,, \tag{7.50}$$

where we have used that

$$\int_0^{2\pi}\frac{\cos^n x\,dx}{1+a\cos x}=\frac{2\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n\,. \tag{7.51}$$

The solution can be written in a more recognizable form in terms of the so-called "ring coordinates" [287], which are defined as

$$\xi=\frac{\sqrt{y^2-1}}{x-y}R\,,\quad \eta=\frac{\sqrt{1-x^2}}{x-y}R\,, \tag{7.52}$$

where $-\infty\le y\le -1$ and $-1\le x\le 1$. In term of these coordinates, the four-dimensional Euclidean metric, the functions $\mathcal{Z}_{\pm}^{(0)}$, and the 1-form $\omega^{(0)}$ take the following form

$$ds^2\left(\mathbb{E}^4\right) = \frac{R^2}{(x-y)^2}\left[\frac{dy^2}{y^2-1}+\left(y^2-1\right)d\psi^2+\frac{dx^2}{1-x^2}+\left(1-x^2\right)d\phi^2\right]\,, \tag{7.53}$$

$$\mathcal{Z}_{\pm}^{(0)} = 1+\frac{\mathcal{Q}_{\pm}}{2R^2}\left(x-y\right)\,, \tag{7.54}$$

$$\omega^{(0)} = -\frac{q}{2}\left(1+y\right)d\psi\,, \tag{7.55}$$

where

$$q=\frac{2\pi\mathcal{Q}_{-}W}{\ell}\,. \tag{7.56}$$

Written in this way, it is straightforward to check that our solution can be obtained from the supersymmetric three-charge black ring of [207, 239, 286] by setting to zero one of the monopole charges and two of the dipole charges.

Physical parameters

Let us discuss the physical interpretation of the four parameters, $\mathcal{Q}_+,\mathcal{Q}_-,q$ and R, that, together with the asymptotic values of the scalars, determine the solution. To this aim, it is convenient to introduce a new pair of coordinates ρ and θ such that

$$\rho\sin\theta=R\frac{\sqrt{y^2-1}}{x-y}\,,\qquad \rho\cos\theta=R\frac{\sqrt{1-x^2}}{x-y}\,. \tag{7.57}$$

In terms of these coordinates, the Euclidean metric, the functions $\mathcal{Z}_{\pm}^{(0)}$ and the 1-form $\omega^{(0)}$ are given by

$$
\begin{aligned}
ds^2\left(\mathbb{E}^4\right) &= d\rho^2+\rho^2\left(d\theta^2+\sin^2\theta\, d\psi^2+\cos^2\theta\, d\phi^2\right)\,, &(7.58)\\
\mathcal{Z}_{\pm}^{(0)} &= 1+\frac{\mathcal{Q}_{\pm}}{\Sigma}\,, &(7.59)\\
\omega^{(0)} &= -\frac{q}{2}\left(1-\frac{\rho^2+R^2}{\Sigma}\right)d\psi\,, &(7.60)
\end{aligned}
$$

where

$$
\Sigma=\sqrt{\left(\rho^2-R^2\right)^2+4R^2\rho^2\cos^2\theta}\,. \tag{7.61}
$$

The physical parameters can be easily identified by studying the large ρ-expansion of the fields. In particular, the asymptotic form of the vector fields (7.41), which is given by

$$
\begin{aligned}
A^{\pm}_{\underline{t}} &\sim c_{\pm}\left[1-\frac{\mathcal{Q}_{\pm}}{\rho^2}+\mathcal{O}\left(\frac{1}{\rho^4}\right)\right]\,, &(7.62)\\
A^{\pm}_{\underline{\psi}} &\sim c_{\pm}\left[\frac{qR^2\sin^2\theta}{\rho^2}+\mathcal{O}\left(\frac{1}{\rho^4}\right)\right]\,, &(7.63)\\
A^{0}_{\underline{\phi}} &\sim c_{0}\left[\frac{qR^2\cos^2\theta}{\rho^2}+\mathcal{O}\left(\frac{1}{\rho^4}\right)\right]\,, &(7.64)
\end{aligned}
$$

informs us that the parameters $\mathcal{Q}_+$ and $\mathcal{Q}_-$ are the monopole electric charges, and that the parameter q controls the magnetic dipole charges. The relation between these and the parameters that characterize the sources is: [193, 276]

$$
\mathcal{Q}_-=g_s^2\alpha' w\,,\qquad \mathcal{Q}_+=\frac{g_s^2\alpha'^2}{R_z^2}n\,,\qquad q=\frac{g_s^2\alpha' W}{R_z}\,. \tag{7.65}
$$

The mass and angular momenta can be easily obtained by first computing the large ρ-expansion of the metric (7.40) and then comparing the result with the Myers-Perry solution [181]. We get:

$$
\begin{aligned}
M &= \frac{\pi\left(\mathcal{Q}_++\mathcal{Q}_-\right)}{4G_{\rm N}^{(5)}}=\frac{n}{R_z}+\frac{R_z}{\alpha'}w, &(7.66)\\
J_{\psi} &= \frac{\pi q R^2}{4G_{\rm N}^{(5)}}=W\left(R/\ell_s\right)^2\,, &(7.67)
\end{aligned}
$$

where we have used Eq. (7.65) and that the five-dimensional Newton's constant is given in terms of the string moduli by

$$G_{\mathrm{N}}^{(5)} = \frac{\pi g_s^2 \alpha'^2}{4R_z} \,. \tag{7.68}$$

Singular horizon

The supersymmetric three-charge black ring has a regular horizon at $y \to -\infty$ with topology $\mathbb{S}^1 \times \mathbb{S}^2$, see [239]. However, the area vanishes when we set two dipole charges and one monopole charge to zero. Therefore, at the supergravity level, small black rings are characterized by having a null singularity instead of a regular event horizon. This implies that the solution can only be trusted far away from the singularity, where the higher-derivative corrections to the supergravity action can be safely ignored. As soon as we get close to the singularity, we lose control over the solution, as we expect on general grounds that the higher-derivative corrections will modify it significantly in that region. It may happen, as suggested in [135], that the solution is modified in a way such that a regular horizon with finite size appears. We shall explore this interesting possibility in the next section.

7.3.2 Higher-derivative corrections to small black rings

Let us use the results of Section 7.2 to obtain the first-order α' corrections to the singular small black ring solution. The correction to the function $\mathcal{Z}_+$ is obtained by plugging (7.54) and (7.55) into (7.29). We find:

$$\mathcal{Z}_+ = 1 + \frac{\mathcal{Q}_+}{2R^2}(x-y) + \frac{\alpha'}{4R^4}\frac{(x-y)^3\left[q^2R^2(x-y) + \mathcal{Q}_+\mathcal{Q}_-(x+y)\right]}{2R^2 + \mathcal{Q}_-(x-y)} + \mathcal{O}\left(\alpha'^2\right) \,. \tag{7.69}$$

As shown in Section 7.2, the function $\mathcal{Z}_-$ and the 1-form ω remain uncorrected, hence

$$\mathcal{Z}_- = 1 + \frac{\mathcal{Q}_-}{2R^2}(x-y) + \mathcal{O}\left(\alpha'^2\right) \,, \tag{7.70}$$

$$\omega = -\frac{q}{2}(1+y)\, d\psi + \mathcal{O}\left(\alpha'^2\right) \,. \tag{7.71}$$

The metric of the α'-corrected small black ring is obtained from (7.40) by simply replacing $\mathcal{Z}_+^{(0)}$, $\mathcal{Z}_-^{(0)}$ and $\omega^{(0)}$ by the above expressions, (7.69), (7.70) and (7.71). As it occurs in the zeroth-order solution, the would-be horizon is at $y \to -\infty$, where the product $\mathcal{Z}_+\mathcal{Z}_-$ diverges. In Fig. 7.1, we have represented the Ricci scalar of the five-dimensional metric $R_{\mathrm{E},5}$ as a function of $\log|y|$. We see that the curvature singularity at $y \to -\infty$ persists. In fact, we find that in the $y \to -\infty$ limit, the radius of the two-sphere $\mathbb{S}^2$ goes to zero as $R_2 \sim |y|^{-1/3}$, whereas the radius of the circle $\mathbb{S}^1$ diverges as $R_\psi \sim |y|^{2/3}$.[8]

[8] This implies that the product $R_2^2 R_\psi$, which would be proportional to the area, is finite and indeed proportional to $\sqrt{q^2R^2 - \mathcal{Q}_+\mathcal{Q}_-}$, which would be in agreement with the scaling argument of [135, 276].

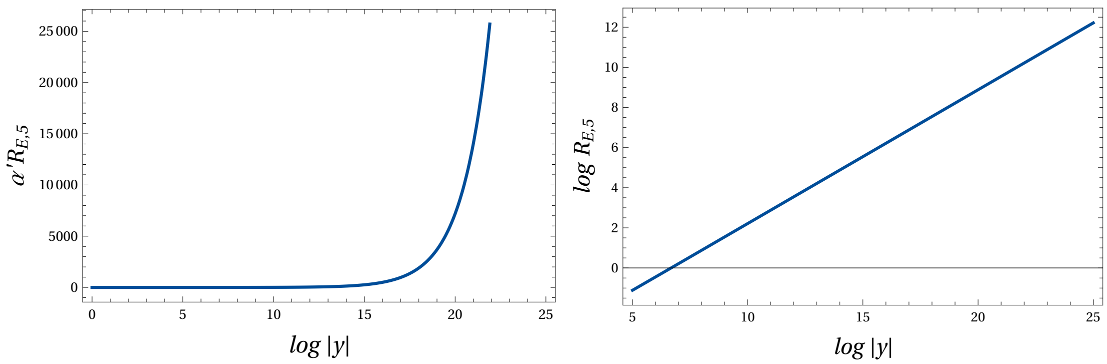

Figure 7.1: In the plot on the left we have represented the five-dimensional Ricci scalar $R_{\mathrm{E},5}$ as a function of $\log|y|$ for $x=-1$ and for the particular values of the parameters: $R=10\sqrt{\alpha'}$, $\mathcal{Q}_+=\mathcal{Q}_-=2\cdot 10^3\alpha'$ and $q=4\cdot 10^2\sqrt{\alpha'}$. We find that the Ricci scalar diverges as $|y|^{2/3}$ as we approach $y\to-\infty$. This behaviour of the Ricci scalar near the singularity can be better appreciated in the plot on the right, where we have represented the logarithm of the Ricci scalar in the vertical axis.

The $R\to 0$ limit: small black holes

The five-dimensional small black holes studied in [193] can be obtained in the $R\to 0$ limit. In order to take this limit, we have to first write the solution using the coordinate system defined in (7.57). Doing so, we find that the 1-form ω vanishes, and that the functions $\mathcal{Z}_\pm$ are given by

$$\mathcal{Z}_+ = 1+\frac{\mathcal{Q}_+}{\rho^2}-\frac{2\alpha'\mathcal{Q}_+\mathcal{Q}_-}{\rho^4\left(\rho^2+\mathcal{Q}_-\right)}+\mathcal{O}\left(\alpha'^2\right), \tag{7.72}$$

$$\mathcal{Z}_- = 1+\frac{\mathcal{Q}_-}{\rho^2}+\mathcal{O}\left(\alpha'^2\right), \tag{7.73}$$

in agreement with Ref. [193]. As we observed in the four-dimensional case, the correction to the function $\mathcal{Z}_+$ is such that $\lim_{\rho\to 0}\mathcal{Z}_+\sim-\frac{2\alpha'\mathcal{Q}_+}{\rho^4}$, just what one would need in order to obtain a regular horizon at $\rho=0$. However, further inspection reveals that the Kaluza-Klein scalar k, given in Eq. (7.42), diverges at the horizon as $k\sim\rho^{-3/2}$, which is clearly not acceptable. This divergence manifests in ten dimensions as a curvature singularity, which implies that our solution cannot be trusted near the singularity, where the low-curvature assumption under which it was derived is not satisfied.

This is in fact what happens for small black holes in any dimension $4\le d\le 9$. The metric that describes them is given by

$$ds^2_{\mathrm{E},d}=\left(\mathcal{Z}_+\mathcal{Z}_-\right)^{\frac{3-d}{d-2}}dt^2-\left(\mathcal{Z}_+\mathcal{Z}_-\right)^{\frac{1}{d-2}}\left[d\rho^2+\rho^2 d\Omega^2_{(d-2)}\right], \tag{7.74}$$

where $\rho=\sqrt{x^m x^m}$ is the radial coordinate of $\mathbb{E}^{d-1}$ and $d\Omega^2_{(d-2)}$ is the metric of the round $(d-2)$-dimensional sphere, $\mathbb{S}^{d-2}$. The functions $\mathcal{Z}_+$ and $\mathcal{Z}_-$ are obtained by plugging the static limit of Eqs. (7.34), (7.35) and (7.36) into (7.29):

$$\mathcal{Z}_+ = 1+\frac{\mathcal{Q}_+}{\rho^{d-3}} - \frac{(3-d)^2\alpha'}{2}\frac{\mathcal{Q}_+\mathcal{Q}_-}{\rho^{d-1}(\rho^{d-3}+\mathcal{Q}_-)} + \mathcal{O}\left(\alpha'^2\right) , \tag{7.75}$$

$$\mathcal{Z}_- = 1+\frac{\mathcal{Q}_-}{\rho^{d-3}} + \mathcal{O}\left(\alpha'^2\right) . \tag{7.76}$$

The area of the apparently regular horizon is now given by

$$A_H = \frac{(d-3)\,\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\sqrt{-2\alpha'\mathcal{Q}_+\mathcal{Q}_-}\,, \qquad \Rightarrow \qquad \frac{A_H}{4G_{\rm N}^{(d)}} = 2\sqrt{2}\pi\sqrt{nw}\,, \tag{7.77}$$

where we have used that the general relation between the charges and the winding and momentum is

$$\frac{|\mathcal{Q}_+|}{16\pi G_{\rm N}^{(d)}} = \frac{\Gamma\left(\frac{d-1}{2}\right)}{2\,(d-3)\,\pi^{\frac{d-1}{2}}}\frac{n}{R_z}\,, \qquad \frac{\mathcal{Q}_-}{16\pi G_{\rm N}^{(d)}} = \frac{\Gamma\left(\frac{d-1}{2}\right)}{2\,(d-3)\,\pi^{\frac{d-1}{2}}}\frac{R_z w}{\alpha'}\,. \tag{7.78}$$

However, as anticipated, the Kaluza-Klein scalar always diverges at the horizon

$$k = k_\infty \frac{\mathcal{Z}_+^{1/2}}{\mathcal{Z}_-^{\frac{d-3}{2(d-1)}}}\,, \qquad \Rightarrow \qquad k(\rho\to 0)\sim \rho^{-\frac{2(d-2)}{d-1}}\,. \tag{7.79}$$

7.4 Discussion

In this chapter, we have computed the first-order α' corrections (which are quadratic in the curvature) to a family of solutions of the low-energy heterotic effective action which describes, after toroidal compactification, small black holes and small black rings in $4 \leq d \leq 9$ dimensions.

We have studied in detail small black rings in $d = 5$ dimensions, finding that this subset of α' corrections is not enough to regularize the solution. In turn, we have found that the would-be horizon is still a null singularity, as in the supergravity (zeroth-order) description. We expect that the same will occur in any dimension, since the torus $\mathbb{T}^{9-d}$ (which forms part of the total compact space $\mathbb{T}^{9-d}\times\mathbb{S}^1_z$) does not have dynamics.

We have also studied small black holes in $4 \leq d \leq 9$ dimensions, finding the same behaviour as in four dimensions. Albeit the α'-corrected metric apparently has a regular horizon with finite area, the Kaluza-Klein scalar that measures the radius of the circle $\mathbb{S}^1_z$ always diverges as $k \sim \rho^{-\frac{2(d-2)}{d-1}}$ at $\rho = 0$, where the would-be horizon is located. This divergence manifests in ten dimensions as a curvature singularity, which tells us that the solution cannot be trusted near $\rho = 0$, where the curvature is no longer small in string units.

8

Leading higher-derivative corrections to the Kerr geometry

General relativity (GR) describes the gravitational interaction as the effect of spacetime curvature. Einstein's field equations, that rule the dynamics of the gravitational field, can be derived from the Einstein-Hilbert (EH) action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}R\,, \tag{8.1}$$

which is essentially the *simplest* non-trivial covariant action one can write for the metric tensor. This beautiful theory has passed a large number of experimental tests —including the recent detection of gravitational waves coming from black hole and neutron star binaries [13, 15, 288–291]— and it is broadly accepted as the correct description of the gravitational interaction.

However, there are good reasons to think that GR should be modified at high energies. One of these reasons is that GR is incompatible with quantum mechanics. Although we still lack a quantum theory of gravity, it is a common prediction of many quantum gravity candidates that the gravitational action (8.1) will be modified when the curvature is large enough. For instance, string theory predicts the appearance of an infinite series of higher-derivative terms [109, 292, 293] correcting the Einstein-Hilbert action. The precise terms and the scale at which they appear depend on the scheme and on the compactification chosen. Nevertheless, whatever the modification of GR is, it should be possible to describe it following the rules of Effective Field Theory (EFT): we add to the action all the possible terms compatible with the symmetries of the theory and we group them following an increasing order of derivatives (or more generally, an increasing energy dimension). In the case of gravity, we would like to preserve diff. invariance and local Lorentz invariance,[1] and this means that the corrections take the form of a higher-curvature, or higher-derivative gravity [298]. A more general possibility —that we will also consider here— is to increase the degrees of freedom in the gravitational sector, by adding other fields that are not active at low energies [299].

Generically, the introduction of higher-derivative interactions means that Ricci-flat metrics no longer solve the gravitational field equations. As a consequence, the Schwarzschild [30] and Kerr [48] metrics, that describe static and rotating black holes (BHs) in GR, are not solutions of the modified theories. One has to solve the modified field equations in order to determine the corrected black hole solutions, and it is an interesting task to look at the properties of these corrected geometries.

[1] See e.g. [294–297] for other possible extensions of GR.

On general grounds, the higher-derivative corrections modify the gravitational interaction when the curvature is large, and they usually improve the UV behaviour of gravity [300]. The effect of the corrections will be drastic precisely in situations where GR fails, such as in the Big-Bang or black hole singularities, and it is expected that higher-derivative terms can resolve these divergencies [301–309]. However, the corrections can also significantly modify the properties of a black hole at the level of the horizon if its mass is small enough. For example, the divergence of Hawking temperature in the limit $M \to 0$ in Einstein gravity (EG) black holes can be cured by higher-derivative interactions [310–312]. In this way, one learns about new high-energy phenomena that might be interpreted as the signature of a UV-complete theory of gravity.

Besides its intrinsic interest, there is another reason why studying higher-derivative-corrected black hole geometries is interesting: they can be used to obtain phenomenological implications of modified gravity. Thanks to the LIGO/VIRGO collaborations [313, 314] and the Event Horizon Telescope [315], amongst other initiatives [316], it will be possible in the next years to test GR with an unprecedented accuracy, and to set bounds on possible modifications of this theory [14, 16, 23, 317–320]. But in order to do so, we first need to derive observational signatures of modified gravity. In order to measure deviations from GR on astrophysical black holes, the corrections should appear at a scale of the order of few kilometers, which is roughly the radius of the horizon for those BHs. Although this seems to be an enormous scale for short-distance modifications of gravity, we should only discard it if there is some fundamental obstruction that forbids unnaturally large couplings in the effective theory [321]. But if that is not the case, the possibility of observing higher-derivative corrections on astrophysical black holes should be considered [322]. Hence, studying in a systematic way black hole solutions of modified gravity and their observational implications is a mandatory task for the black hole community in the coming years.

Black hole solutions in alternative theories of gravity have been largely explored in the literature, but for obvious reasons we will restrict our attention to four-dimensional solutions that modify in a continuous way the Einstein gravity black holes, and that do not include matter. This excludes, for example, solutions of pure quadratic gravity, without a linear R term [323–326]. In the same way, theories such as $f(R)$ gravity are not interesting for us, since they do not modify EG solutions in the vacuum (see e.g. [327]). Some other theories allow for EG solutions, but additionally possess disconnected branches of different solutions, as is the case of black holes in quadratic gravity [328, 329]. We will not consider this case here either, since we are interested in continuous deviations from GR. On the contrary, static black holes correcting Schwarzschild's solution have been studied in the context of Einstein-dilaton-Gauss-Bonnet gravity (EdGB) [299, 330–332], and in other scalar-Gauss-Bonnet theories, e.g. [333–336]. Those theories contain a scalar that is activated due to the higher-curvature terms. In the case of pure-metric theories, spherically symmetric black holes have been constructed, non-perturbatively in the coupling, in Einsteinian cubic gravity [337–339]. Although the profile of the solution has to be determined numerically, this theory has the remarkable property that black hole thermodynamics can be determined analytically. These results have recently been generalized to higher-order versions of the theory [312, 340, 341].

The case of rotating black holes, which is more interesting from an astrophysical perspective, is also more challenging. Obtaining rotating black hole solutions of higher-derivative gravity theories is a very complicated task, and for that reason only approximate solutions or numerical ones are known. One of the most studied theories in this context is EdGB gravity, where rotating black holes have been constructed perturbatively in the spin and in the coupling [322, 342, 343], and numerically [344, 345]. Rotating black holes in dynamical Chern-Simons (dCS) modified gravity[2] [346] have also been studied, both perturbatively [347–349] and numerically [350]. On the other hand, Ref. [351] considers a generalization of EdGB and dCS theories. Finally, for pure-metric theories, the recent work [352] studies rotating black holes in the eight-derivative effective theory introduced in [22].

A usual approximation, that is used by many of the papers above, consists in obtaining the solution perturbatively in the higher-order couplings. For some purposes it is also interesting to obtain non-perturbative solutions —for which one usually needs numerical methods— but, from the perspective of EFT, it does not make any sense to go beyond perturbative level, since the theory will include further corrections at that order. Additionally, the solution is often expanded in a power series of the spin parameter $\chi = a/M$. In most of the literature, only few terms in this expansion are included, so the solutions are only valid for slowly-rotating black holes. However, astrophysical black holes —and in particular those created after the merging of a black hole binary [353]— can have relatively high spin. Moreover, some effects of rotation —such as the deformation of the black hole shadow [315, 354–356]— are barely observable when the spin is low, and other phenomena only happen for rapidly spinning black holes [357–359]. Although numerical solutions are not in principle limited to small values of the spin, analytic solutions are most useful for evident reasons. Hence, it would be interesting to provide analytic solutions valid for high-enough angular momentum. Finally, instead of having a large catalogue of alternative theories of gravity and their black hole solutions, it would be desirable to describe a minimal model that captures all the possible modifications of GR at a given order —probably, up to field redefinitions— and to characterize the black holes of that theory.

The preceding discussion motivates the three main objectives of the present work. First, to establish a general effective theory that can be used to study the leading-order higher-derivative corrections to Einstein gravity vacuum solutions. Second, to obtain the corrections to Kerr black hole in these theories, providing an analytic solution that is accurate for high enough values of the spin. And third, to study in detail some of the properties of these rotating black holes, such as the shape of horizon or the surface gravity, that have often been disregarded in the literature.

8.1 Leading order effective theory

The most general diffeomorphism-invariant and locally Lorentz-invariant metric theory of gravity is given by an action of the form

$$S = \int d^4x\sqrt{|g|}\mathcal{L}\left(g^{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\alpha R_{\mu\nu\rho\sigma}, \nabla_\alpha\nabla_\beta R_{\mu\nu\rho\sigma}, \ldots\right) . \tag{8.2}$$

This is, the most general Lagrangian for such theory will be an invariant formed from contractions and products of the metric, the Riemann tensor, and its derivatives. However,

[2] This theory does not modify spherically symmetric GR solutions, because the corrections are sourced by the Pontryagin density, that vanishes in the presence of spherical symmetry. However, it does modify rotating black holes.

the theory above can be generalized by slightly relaxing some of the postulates. We may construct the Lagrangian using as well the dual Riemann tensor:

$$\tilde{R}_{\mu\nu\alpha\beta} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}\,. \tag{8.3}$$

These terms generically lead to violation of parity, hence the theory is not (locally) invariant under the full Lorentz group, but only under one of its connected components. However, we know that parity is not a symmetry of nature, so in principle there is no reason to discard terms constructed with $\tilde{R}_{\mu\nu\alpha\beta}$. In general, one expands this Lagrangian in terms containing increasing numbers of derivatives, being the first one the Einstein-Hilbert term R, with two derivatives. The rest of the terms can symbolically be written as

$$\nabla^p\mathcal{R}^n\,. \tag{8.4}$$

Since this term contains $2n+p$ derivatives, it should be multiplied by a constant of dimensions of $length^{2n+p-2}$ with respect to the Einstein-Hilbert term. This is the length scale ℓ at which the higher-derivative terms modify the law of gravitation. When the curvature is much smaller than this length scale ($||R_{\mu\nu\rho\sigma}|| << \ell^{-2}$), the effect of the higher-derivative terms can be treated as a perturbative correction, and terms with increasing number of derivatives become more and more irrelevant. Thus, it is an interesting exercise to obtain the most general theory that includes all the possible leading-order corrections. Here we summarize how we construct this theory, but we refer to the Appendix G.1 for the details. The first terms one may introduce in the action are quadratic in the curvature and hence they contain four derivatives. These terms would induce corrections in the metric tensor at order ℓ^2, but in four dimensions it turns out that all of these terms either are topological or do not introduce corrections at all. Thus, the first corrections in a metric theory appear at order ℓ^4 and they are associated to six-derivative terms. As we show in Appendix G.1, it turns out that, up to field redefinitions, there are only two inequivalent six-derivative curvature invariants, one of them parity-even and the other one parity-odd. However, one could consider a more general theory, allowing the coefficients of the higher-derivative terms to be dynamical i.e., controlled by scalars. This is actually a very natural possibility that is predicted, for instance, by string theory [299]. In that case, some of the four-derivative terms do contribute to the equations and they also correct the metric at order ℓ^4. For simplicity, we will restrict ourselves to massless scalars, but we will allow, in principle, to have an undetermined number of them. Within this large family of theories, it is possible to show that the most general leading correction to Einstein's theory is captured by the action

$$\begin{aligned} S =&\frac{1}{16\pi G}\int d^4x\sqrt{|g|}\bigg\{R+\alpha_1\phi_1\ell^2\mathcal{X}_4+\alpha_2\left(\phi_2\cos\theta_m+\phi_1\sin\theta_m\right)\ell^2R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\\ &+\lambda_{\rm ev}\ell^4R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}R_{\delta\gamma}{}^{\mu\nu}+\lambda_{\rm odd}\ell^4R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}\tilde{R}_{\delta\gamma}{}^{\mu\nu}-\frac{1}{2}(\partial\phi_1)^2-\frac{1}{2}(\partial\phi_2)^2\bigg\}\,, \end{aligned} \tag{8.5}$$

where

$$\mathcal{X}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2 \tag{8.6}$$

is the Gauss-Bonnet density and ϕ_1, ϕ_2 are scalar fields. Besides the overall length scale ℓ, there are only five parameters: α_1, α_2, $\lambda_{\rm ev}$, $\lambda_{\rm odd}$ and θ_m. The parameter $\lambda_{\rm odd}$ violates parity, while the "mixing angle" θ_m represents as well a sort of parity breaking phase. For

$\theta_m = 0, \pi$ (no mixing between scalars), ϕ_2 is actually a pseudoscalar and the quadratic sector is parity-invariant. For any other value ($\theta_m \neq 0, \pi$), parity is also violated by this sector.

The theory (8.5) contains, as particular cases, some well-known models that have been frequently used in the literature. The case $\lambda_{\text{ev}} = \lambda_{\text{odd}} = \theta_m = 0$, $\alpha_2 = -\alpha_1 = 1/8$ corresponds to the prediction of string theory, where the length scale of the corrections in that case is the string length $\ell^2 = \ell_s^2 \equiv \alpha'$. As we show in the appendix G.2, the corresponding action can be obtained from direct compactification and truncation of the Heterotic superstring effective action at order α'. In that case, ϕ_1 is identified with the dilaton, while ϕ_2 is the axion, which appears after dualization of the Kalb-Ramond 2-form. Another well-known possibility (which is also claimed to proceed from the low-energy limit of string theory) is $\lambda_{\text{odd}} = \lambda_{\text{ev}} = \alpha_2 = 0$, which corresponds to the Einstein-dilaton-Gauss-Bonnet theory. Rotating black holes in EdGB gravity have been studied both numerically [344, 345] and in the slowly-rotating limit [322, 342, 343]. The case $\theta_m = \pi/2$, which represents an extension of EdGB gravity, has also been considered [351] (note that this case only contains one dynamical scalar and violates parity). On the other hand, the case $\alpha_2 \neq 0$ with the rest of couplings set to zero corresponds to dynamical Chern-Simons gravity, whose rotating black holes were studied in Refs. [347–349] in the slowly-rotating approximation, while Ref. [350] performs a non-perturbative numerical study. As for the cubic theories, the parity-even term (controlled by λ_{ev}) can be mapped (modulo field redefinitions) to the Einsteinian cubic gravity (ECG) term [337], for which static black hole solutions have been constructed non-perturbatively in the coupling [338, 339]. Phenomenological signatures of static black holes in ECG have also been recently studied in [360, 361], where a first bound on the coupling was provided, and the possibility to detect deviations from GR in gravitational lensing observations was discussed. Rotating black holes in ECG have not been studied so far. Lastly, to the best of our knowledge, the parity odd cubic term has never been used in the context of black hole solutions.

The theory (8.5) has been constructed following the sole requirement of diff. invariance, but there are some other constraints that could be imposed on physical grounds. For instance, if one wants to preserve parity, then one should set $\theta_m = \lambda_{\text{odd}} = 0$. Nevertheless, we know that nature is not parity-invariant, so keeping these terms is not unreasonable. If one does not wish to include additional light degrees of freedom the scalars should be removed, which amounts to setting $\alpha_1 = \alpha_2 = 0$ (in that case the scalars are just not activated). On the other hand it is known that higher-derivative terms may break unitarity by introducing ghost modes — non normalizable states. In the case of the scalar fields and the quadratic terms in (8.5) this problem does not exist since the field equations of that sector are actually of second order. The equations of the cubic terms do contain higher-order derivatives —namely of fourth order—, but the mass scale at which we expect the new modes to appear is

$$m^2 \sim \frac{1}{\ell^4 ||R_{\mu\nu\rho\sigma}||} \, . \tag{8.7}$$

This is simply telling us that Effective Field Theory works up to the scale $||R_{\mu\nu\rho\sigma}|| \sim \ell^{-2}$, which is something we already knew. Finally, it is also possible to study causality constraints [362]. In relation to this, the results in [321] impose a severe bound on the coupling constants $\lambda_{\text{ev}}\ell^4$, $\lambda_{\text{odd}}\ell^4$ of the cubic terms. If one wants to observe any effects of higher-derivative corrections on astrophysical black holes, necessarily the corrections should appear at a scale ℓ of the order of few kilometers (otherwise the effect would be too small to be detected). Such large couplings are very unnatural, since the natural scale of (quantum) gravity should be Planck length. According to [321], these large couplings lead to violation of causality, that could only be restored by adding an infinite tower of higher-spin particles of mass $\sim \ell^{-1}$. Since, obviously, this is not observed, it was concluded that the couplings associated to the cubic terms should be of the order of Planck scale, hence those corrections are not viable phenomenologically. However, it was noted in [22] that the analysis of [321] relies on certain assumptions about the UV completion, and that it has not been proven yet that the result applies for any possible UV completion. The conclusion of [22] was that one should cautiously include the cubic terms for phenomenological purposes.

In any case, nothing prevents us from studying the effect of the cubic curvature terms on black holes, no matter the scale at which they appear. These corrections give us valuable information about the effects of modified gravity at high energies, and this is intrinsically interesting, even if those corrections are not viable on an observational basis.

If, for some reason, all the theories in the model (8.5) were discarded, then the leading correction to GR would be given by the quartic-curvature terms introduced in [22]. These terms modify the metric at order $\mathcal{O}(\ell^6)$ hence they are subleading when the couplings in (8.5) are non-vanishing. Rotating black holes in those theories were recently studied in [352] up to order χ^4 in the spin. The methods that we present in this chapter could be applied to the quartic theories as well and could be used in order to extend some of the results in [352]. For instance, one might compute the solution for higher values of the spin or obtain the form of the horizon, as we do in Sec. 8.3.1.

8.1.1 Equations of motion

Our goal is to compute the leading corrections to vacuum solutions of Einstein's theory. Thus, our starting point is a metric $g^{(0)}_{\mu\nu}$ that satisfies vacuum Einstein's equations

$$R^{(0)}_{\mu\nu} = 0 \ , \tag{8.8}$$

while the scalars $\phi_1^{(0)}$, $\phi_2^{(0)}$ take a constant value that can be chosen to be zero without loss of generality.[3] But this field configuration is not a solution when we take into account the higher-derivative terms. First we note that the coupling between scalars and the curvature densities in the action (8.5) induce source terms in the scalar equations of motion, so that they will not be constant anymore. More precisely the first correction is of order ℓ^2,

$$\phi_1 = \ell^2 \phi_1^{(2)} \, , \quad \phi_2 = \ell^2 \phi_2^{(2)} \, , \tag{8.9}$$

and it satisfies

$$\nabla^2 \phi_1^{(2)} = -\alpha_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\Big|_{g=g^{(0)}} - \alpha_2 \sin\theta_m R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\Big|_{g=g^{(0)}} \, , \tag{8.10}$$

$$\nabla^2 \phi_2^{(2)} = -\alpha_2 \cos\theta_m R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\Big|_{g=g^{(0)}} \, . \tag{8.11}$$

On the other hand, the modified Einstein equations, derived from the action (8.5), can be written as

[3] The action 8.5 is invariant (up to a surface term) under constant shifts of the scalars.

$$G_{\mu\nu} = T^{\text{scalars}}_{\mu\nu} + T^{\text{cubic}}_{\mu\nu} \,, \tag{8.12}$$

where we have passed all the corrections to the right-hand-side in the form of some energy-momentum tensors, that read

$$\begin{aligned} T^{\text{scalars}}_{\mu\nu} =& - \alpha_1 \ell^2 g_{\nu\lambda} \delta^{\lambda\sigma\alpha\beta}_{\mu\rho\gamma\delta} R^{\gamma\delta}{}_{\alpha\beta} \nabla^\rho \nabla_\sigma \phi_1 + 4\alpha_2 \ell^2 \nabla^\rho \nabla^\sigma \left[\tilde{R}_{\rho(\mu\nu)\sigma} \, (\cos\theta_m \phi_2 + \sin\theta_m \phi_1) \right] \\ &+ \frac{1}{2}\left[\partial_\mu \phi_1 \partial_\nu \phi_1 - \frac{1}{2} g_{\mu\nu} \left(\partial\phi_1\right)^2\right] + \frac{1}{2}\left[\partial_\mu \phi_2 \partial_\nu \phi_2 - \frac{1}{2} g_{\mu\nu} \left(\partial\phi_2\right)^2\right] \,, \end{aligned} \tag{8.13}$$

and

$$\begin{aligned} T^{\text{cubic}}_{\mu\nu} =& \lambda_{\text{ev}} \ell^4 \left[3 R_{\mu}{}^{\sigma\alpha\beta} R_{\alpha\beta}{}^{\rho\lambda} R_{\rho\lambda\sigma\nu} + \frac{1}{2} g_{\mu\nu} R_{\alpha\beta}{}^{\rho\sigma} R_{\rho\sigma}{}^{\delta\gamma} R_{\delta\gamma}{}^{\alpha\beta} - 6 \nabla^\alpha \nabla^\beta \left(R_{\mu\alpha\rho\lambda} R_{\nu\beta}{}^{\rho\lambda} \right) \right] \\ &+ \lambda_{\text{odd}} \ell^4 \bigg[- \frac{3}{2} R_{\mu}{}^{\rho\alpha\beta} R_{\alpha\beta\sigma\lambda} \tilde{R}_{\nu\rho}{}^{\sigma\lambda} - \frac{3}{2} R_{\mu}{}^{\rho\alpha\beta} R_{\nu\rho\sigma\lambda} \tilde{R}_{\alpha\beta}{}^{\sigma\lambda} + \frac{1}{2} g_{\mu\nu} R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\delta\gamma} \tilde{R}_{\delta\gamma}{}^{\mu\nu} \\ &+ 3 \nabla^\alpha \nabla^\beta \left(R_{\mu\alpha\sigma\lambda} \tilde{R}_{\nu\beta}{}^{\sigma\lambda} + R_{\nu\beta\sigma\lambda} \tilde{R}_{\mu\alpha}{}^{\sigma\lambda} \right) \bigg] \end{aligned} \tag{8.14}$$

Since the scalars are of order $\mathcal{O}(\ell^2)$, we can see that the leading correction to the metric associated to the scalar sector is of order $\mathcal{O}(\ell^4)$, the same order at which cubic curvature terms come into play. Thus, we expand the metric as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \ell^4 g^{(4)}_{\mu\nu} \,, \tag{8.15}$$

where $g^{(4)}_{\mu\nu}$ is a perturbative correction. Now, taking into account that $g^{(0)}_{\mu\nu}$ solves Einstein's equations, we get the value of the Einstein tensor to linear order in $g^{(4)}_{\mu\nu}$:

$$G_{\mu\nu} = \ell^4 \left[-\frac{1}{2} \nabla^2 \hat{g}^{(4)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} \nabla^\alpha \nabla^\beta \hat{g}^{(4)}_{\alpha\beta} + \nabla^\alpha \nabla_{(\mu} \hat{g}^{(4)}_{\nu)\alpha} \right] + \mathcal{O}(\ell^6) \,. \tag{8.16}$$

where ∇ is the covariant derivative associated with the zeroth-order metric, and $\hat{g}^{(4)}_{\mu\nu}$ is the trace-reversed metric perturbation

$$\hat{g}^{(4)}_{\mu\nu} = g^{(4)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} g^{(4)}_{\alpha\beta} g^{(0)\,\alpha\beta} \,. \tag{8.17}$$

Then, $\hat{g}^{(4)}_{\mu\nu}$ satisfies the equation

$$-\frac{1}{2} \nabla^2 \hat{g}^{(4)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} \nabla^\alpha \nabla^\beta \hat{g}^{(4)}_{\alpha\beta} + \nabla^\alpha \nabla_{(\mu} \hat{g}^{(4)}_{\nu)\alpha} = \ell^{-4} \left[T^{\text{scalars}}_{\mu\nu} + T^{\text{cubic}}_{\mu\nu} \right] \Big|_{g=g^{(0)}\,,\, \phi_i = \ell^2 \phi_i^{(2)}} \tag{8.18}$$

8.2 The corrected Kerr metric

After introducing the theory (8.5), here we present the rotating black hole ansatz that we will use in the rest of the text, and in Sec. 8.2.1 we sketch how to solve the equations of motion. From now on we set $G = 1$. Let us first consider Kerr's metric expressed in Boyer-Lindquist coordinates:

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \Sigma\left(\frac{dr^2}{\Delta}+d\theta^2\right)\\ &+\left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2\,, \end{aligned} \tag{8.19}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,, \quad \Delta = r^2 - 2Mr + a^2\,. \tag{8.20}$$

Let us very briefly recall some of the properties of this metric.

- Being a solution of vacuum Einstein's equations, it is Ricci flat: $R_{\mu\nu} = 0$.
- It is stationary and axisymmetric, with related Killing vectors ∂_t and ∂_ϕ respectively.
- It represents an asymptotically flat spacetime with total mass M and total angular momentum $J = aM$.
- When $M > |a|$ the solution represents a black hole, whose (outer) horizon is placed at the largest radius r_+ where Δ vanishes:

$$r_+ = M + \sqrt{M^2 - a^2}\,. \tag{8.21}$$

Since Ricci flat metrics do not solve the modified Einstein's equations, the rotating black holes of the theory (8.5) will not be described by Kerr metric. The search for an appropriate metric ansatz that can be used to parametrize deviations from Kerr metric is a far from trivial problem that has been studied in the literature [363, 364]. However, as long as the mass is much larger than the scale at which the higher-derivative terms appear, $M >> \ell$, the deviation with respect to general relativity will be small —at least outside the horizon. In that case, we can build the rotating black hole solution of (8.5) as a perturbative correction over Kerr metric. Since we want to describe an stationary and axisymmetric solution, the corrected metric has to conserve the Killing vectors ∂_t and ∂_ϕ. On the other hand, we do not expect to "activate" additional components of the metric, so that the corrections appear in the already non-vanishing components. Taking into account these observations, we can write a general ansatz for the corrected Kerr metric

$$\begin{aligned} ds^2 = &-\left(1-\frac{2M\rho}{\Sigma}-H_1\right)dt^2 - (1+H_2)\frac{4Ma\rho(1-x^2)}{\Sigma}dtd\phi + (1+H_3)\frac{\Sigma}{\Delta}d\rho^2\\ &+(1+H_5)\frac{\Sigma dx^2}{1-x^2} + (1+H_4)\left(\rho^2+a^2+\frac{2M\rho a^2(1-x^2)}{\Sigma}\right)(1-x^2)d\phi^2\,, \end{aligned} \tag{8.22}$$

where $H_{1,2,3,4,5}$ are functions of $x = \cos\theta$ and ρ only, and they are assumed to be small $|H_i| << 1$. Note that we have introduced the coordinate ρ in order to distinguish it from the coordinate r in Kerr metric. We have also introduced the functions

$$\Sigma = \rho^2 + a^2x^2\,, \quad \Delta = \rho^2 - 2M\rho + a^2\,. \tag{8.23}$$

However, the ansatz (8.22) is far too general, and it turns out that we can fix some of the functions H_i by performing a change of coordinates. In particular, it can be shown that there exists a (infinitesimal) change of coordinates $(\rho, x) \to (\rho', x')$ that preserves the form of the metric and for which $H_5' = H_3'$. Thus, we are free to choose $H_3 = H_5$, and in that case, the metric reads

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2M\rho}{\Sigma}-H_1\right)dt^2-(1+H_2)\,\frac{4Ma\rho(1-x^2)}{\Sigma}dtd\phi+(1+H_3)\,\Sigma\left(\frac{d\rho^2}{\Delta}+\frac{dx^2}{1-x^2}\right)\\
&+(1+H_4)\left(\rho^2+a^2+\frac{2M\rho a^2(1-x^2)}{\Sigma}\right)(1-x^2)d\phi^2\,.
\end{aligned}
\tag{8.24}
$$

Note that we are choosing the coordinates x and ρ such that the form of the (ρ, x)-metric is respected —up to a conformal factor— when the corrections are included. It is easy to see that this choice of coordinates has a crucial advantage: the horizon of the metric (8.24) will still be placed at the (first) point where Δ vanishes: $\rho_+ = M + \sqrt{M^2 - a^2}$. If we were not careful enough choosing the coordinates, the description of the horizon could be very messy, and this is perhaps the reason why in previous studies the horizon of the corrected solutions has not been studied in depth.

We note that, whenever we consider the corrections, the coordinate ρ does not coincide asymptotically with the usual radial coordinate r. Advancing the results in next subsection, we get that the functions H_i behave asymptotically as

$$
H_i = h_i^{(0)} + \frac{h_i^{(1)}}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right)\,,\ i = 1,2,3,4\,,
\tag{8.25}
$$

where $h_i^{(k)}$ are constant coefficients. Then, we can see that the usual radial coordinate r that asymptotically measures the area of 2-spheres is related to ρ according to

$$
\rho = r\left(1-\frac{h_3^{(0)}}{2}\right)-\frac{h_3^{(1)}}{2}+\mathcal{O}\left(\frac{1}{r}\right)\,.
\tag{8.26}
$$

Using this coordinate, the asymptotic expansion of the metric (8.24) reads

$$
\begin{aligned}
ds^2(r\to\infty) = &-\left(1-h_1^{(0)}-\frac{2M+Mh_3^{(0)}+h_1^{(1)}}{r}\right)dt^2-\left(1+h_2^{(0)}+h_3^{(0)}/2\right)\frac{4Ma\sin^2\theta}{r}dtd\phi\\
&+dr^2\left(1+\frac{2M+Mh_3^{(0)}+h_3^{(1)}}{r}\right)+r^2d\theta^2+\left(1+h_4^{(0)}-h_3^{(0)}\right)r^2\sin^2\theta d\phi^2\,.
\end{aligned}
\tag{8.27}
$$

When we solve the equations, we see that we are free to fix the asymptotic values of the coefficients $h_i^{(0)}$. On the other hand, the metric must be asymptotically flat (with the correct normalization at infinity), and we want the parameters M and a to still represent the mass and the angular momentum per mass of the solution, so the asymptotic expansion should read

$$ds^2(r\to\infty) = -\left(1-\frac{2M}{r}\right)dt^2 - \frac{4Ma\sin^2\theta}{r}dtd\phi + dr^2\left(1+\frac{2M}{r}\right) + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\,. \tag{8.28}$$

From this, we derive the asymptotic conditions that we have to impose on our solution:

$$h_1^{(0)} = 0\,,\quad h_3^{(0)} = h_4^{(0)} = -\frac{h_3^{(1)}}{M}\,,\quad h_2^{(0)} = -\frac{h_3^{(0)}}{2}\,. \tag{8.29}$$

Apparently, the condition $Mh_3^{(0)} + h_1^{(1)} = 0$ is also required, but this is actually imposed by the field equations.

8.2.1 Solving the equations

Once we have found an appropriate ansatz for our metric, Eq. (8.24), we have to solve the equations of the theory (8.5). The first step is to solve the equations for the scalars (8.10), (8.11), from where we obtain ϕ_1 and ϕ_2 at order $\mathcal{O}(\ell^2)$. Using this result we determine the right-hand-side of (8.18), while in the left-hand-side we introduce the metric correction $g_{\mu\nu}^{(4)}$,

$$\begin{aligned}\ell^4 g_{\mu\nu}^{(4)}dx^\mu dx^\nu =& H_1 dt^2 - H_2\frac{4Ma\rho(1-x^2)}{\Sigma}dtd\phi + H_3\Sigma\left(\frac{d\rho^2}{\Delta}+\frac{dx^2}{1-x^2}\right)\\ &+ H_4\left(\rho^2+a^2+\frac{2M\rho a^2(1-x^2)}{\Sigma}\right)(1-x^2)d\phi^2\,,\end{aligned} \tag{8.30}$$

which can be read from (8.24). In this way, we get a (complicated) system of equations for the functions H_i, that we have to solve. Unfortunately, these equations (including the ones for the scalars) are very intricate and we are not able to obtain an exact solution. However, a possible strategy is to expand the solution in powers of the angular momentum a, assuming that it is a small parameter. In previous works [322, 342, 343, 347, 348, 351], this method has been employed in order to obtain a few terms in the expansion, which yields an approximate solution for slowly rotating black holes. Nonetheless, if one includes enough terms in the expansion, the result should give a good approximation to the solution also for high values of the spin. One of the goals of this chapter is precisely to provide a method that allows for the construction of the solution at arbitrarily high-orders in the spin.

For simplicity, let us first introduce the dimensionless parameter

$$\chi = \frac{a}{M}\,, \tag{8.31}$$

that ranges from 0 to 1 in Kerr's solution, $\chi = 0$ corresponding to static black holes and $\chi = 1$ to extremal ones.[4] Then, we expand our unknown functions in a power series in χ

$$\phi_1 = \sum_{n=0}^{\infty}\phi_1^{(n)}\chi^n\,,\quad \phi_2 = \sum_{n=0}^{\infty}\phi_2^{(n)}\chi^n\,,\quad H_i = \sum_{n=0}^{\infty}H_i^{(n)}\chi^n\,,\quad i = 1,2,3,4\,, \tag{8.32}$$

[4]When the corrections are included, we expect that the extremality condition is modified, $\chi_{\text{ext}} \neq 1$, but this is not important for our discussion, since we will not deal with extremal or near-extremal geometries here.

where we recall that all the functions depend on ρ and x. Then, the idea is to plug these expansions in (8.10), (8.11), (8.18), expand the equations in powers of χ, and solve them order by order. The equations satisfied by the n-th components are much simpler than the full equations, and we are indeed able to solve them analytically. These are second-order, linear, inhomogeneous, partial differential equations, so that the general solution can be expressed as the sum of a particular solution plus all the solutions of the homogeneous equation. In general, the "homogeneous part" of the solution represents infinitesimal changes of coordinates, and the physics is contained in the inhomogeneous part, which is the one sourced by the higher-derivative terms. So, we have to find the solution that captures the corrections but does not introduce unnecessary changes of coordinates. We observe that the appropriate solution can always be expressed as a polynomial in x and in $1/\rho$. More precisely, we get[5]

$$\phi_{1,2}^{(n)} = \sum_{p=0}^{n}\sum_{k=0}^{k_{\max}} \phi_{1,2}^{(n,p,k)} x^p \rho^{-k}\,, \qquad H_i^{(n)} = \sum_{p=0}^{n}\sum_{k=0}^{k_{\max}} H_i^{(n,p,k)} x^p \rho^{-k}\,, \tag{8.33}$$

where $\phi_{1,2}^{(n,p,k)}$, $H_i^{(n,p,k)}$ are constant coefficients and in each case the value of $k_{\max}$ depends on n and p. When we solve the equations we also observe that all the terms in these series are determined except the constant ones: those with $p = k = 0$. However, those coefficients are fixed by the boundary conditions. In the case of the scalars, their value at infinity is arbitrary, so we can set it to zero for simplicity (this does not affect the rest of the solution)

$$\phi_1^{(n,0,0)} = \phi_2^{(n,0,0)} = 0\,, \quad n = 0, 1, 2, \ldots\,. \tag{8.34}$$

On the other hand, for the H_i functions we take into account the relations (8.29) that we derived previously, which imply that

$$H_1^{(n,0,0)} = 0\,, \quad H_3^{(n,0,0)} = H_4^{(n,0,0)} = -\frac{H_3^{(n,0,1)}}{M}\,, \quad H_2^{(n,0,0)} = -\frac{H_3^{(n,0,0)}}{2}\,. \tag{8.35}$$

In this way, the solution is completely determined. Since this process is systematic, we can easily program an algorithm that computes the series (8.32) at any (finite) order n. A Mathematica notebook that does the job is provided in `https://arxiv.org/src/1901.01315v3/anc`. Using this code, we have computed the solution up to order χ^{14}. As we show in Appendix G.4, this expansion provides a minimum accuracy of about 1% everywhere outside the horizon for $\chi = 0.7$, and much higher for smaller χ. Thus, we have an analytic solution that works for relatively high values of χ, and we will exploit this fact in next section. Due to the length of the expressions, in Appendix G.3 we show the solution explicitly up to order χ^3, but the full series up to order $n = 14$ is available in the Mathematica notebook.

Before closing this section, we would like to clarify the following point. In the preceding scheme the corrections are expressed as a powers series in the spin, but we are taking the zeroth-order solution to be the exact Kerr's metric, which is non-perturbative in the spin. Thus, for consistency sake, one should imagine that we also expand the zeroth-order solution in the spin up to the same order at which the corrections were computed. However, for evident reasons we do not do this explicitly. Thus, in the next section, we

[5] Equivalently, one may expand these functions using Legendre polynomials $P_p(x)$.

will write the formulas for several quantities as the result for Kerr's metric, exact in the spin, plus linear corrections, perturbative in the spin, but one should bear in mind that the zeroth-order result should also be expanded.

8.3 Properties of the corrected black hole

In this section we analyze some of the most relevant physical properties of the rotating black hole solutions we have found. We study the geometry of the horizon and of the ergosphere, light rings on the equatorial plane, and scalar hair.

8.3.1 Horizon

In order for the metric (8.24) to represent a black hole, we have to show that it contains an event horizon. We have argued that, with the choice of coordinates we have made, the horizon is defined by the equation $\Delta = 0$, whose roots are $\rho = \rho_{\pm}$, where

$$\rho_{\pm} = M\left(1 \pm \sqrt{1-\chi^2}\right) . \tag{8.36}$$

The largest root ρ_+ corresponds to the event horizon, while $\rho = \rho_-$ is in principle an inner horizon.[6] Here we will only deal with the exterior solution $\rho \geq \rho_+$.

Then, let us show that $\rho = \rho_+$ is indeed an event horizon. More precisely, we will show that it is a Killing horizon, i.e. a null hypersurface whose normal is a Killing vector. Let us first check that the hypersurface defined by $\rho = \rho_+$ is null. In order to do so, we consider the induced metric at some constant ρ, which is given by

$$\begin{aligned} ds^2\big|_{\rho=\text{const}} = &-\left(1-\frac{2M\rho}{\Sigma}-H_1\right)dt^2-(1+H_2)\,\frac{4Ma\rho(1-x^2)}{\Sigma}dtd\phi+(1+H_3)\,\frac{\Sigma dx^2}{1-x^2} \\ &+(1+H_4)\left(\rho^2+a^2+\frac{2M\rho a^2(1-x^2)}{\Sigma}\right)(1-x^2)d\phi^2 . \end{aligned} \tag{8.37}$$

Then, we can see that when we evaluate at $\rho = \rho_+$, the previous metric is singular, namely it has rank 2. Evaluating the determinant of the (t, ϕ)-metric at ρ_+, we get

$$\left(g_{tt}g_{\phi\phi}-g_{t\phi}^2\right)\Big|_{\rho=\rho_+} = \frac{4M^2\rho_+^2(1-x^2)}{\rho_+^2+a^2x^2}\left[H_1-\frac{a^2\left(1-x^2\right)}{\rho_+^2+a^2x^2}\left(2H_2-H_4\right)\right]\Bigg|_{\rho=\rho_+} , \tag{8.38}$$

where, for consistency with the perturbative approach, we have expanded linearly in the H_i functions. When we expand the combination between brackets in powers of χ using the solution we found, we see that all the terms vanish. Thus, the determinant vanishes,

$$\left(g_{tt}g_{\phi\phi}-g_{t\phi}^2\right)\Big|_{\rho=\rho_+} = 0 , \tag{8.39}$$

[6] When the corrections are included, most likely the inner horizon of Kerr's black hole becomes singular. For instance, one expects that the scalars diverge there.

which proves that this hypersurface is null. The next step is to show that there exists a Killing vector whose norm vanishes at $\rho = \rho_+$. Such vector is a linear combination of the two Killing vectors ∂_t and ∂_ϕ:

$$\xi = \partial_t + \Omega_H \partial_\phi \ , \tag{8.40}$$

for some constant Ω_H. One can check that the only possible choice of Ω_H for which ξ is null at ρ_+ is

$$\Omega_H = \frac{|g_{t\phi}|}{g_{\phi\phi}}\bigg|_{\rho=\rho_+} = \frac{a}{2M\rho_+}\left(1 + H_2 - H_4\right)\big|_{\rho=\rho_+} \ , \tag{8.41}$$

which represents the angular velocity at the horizon. It is then clear that the norm of the vector ξ vanishes at $\rho = \rho_+$, since

$$\xi^2\Big|_{\rho=\rho_+} = \left(g_{tt} + 2g_{t\phi}\Omega_H + \Omega_H^2 g_{\phi\phi}\right)\Big|_{\rho=\rho_+} = \left(g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}}\right)\Bigg|_{\rho=\rho_+} = 0 \ , \tag{8.42}$$

where in the last step we have used (8.39). However, the crucial point here is whether Ω_H, given by (8.41), is constant. A priori, this quantity could well depend on x, in whose case ξ would not be a Killing vector, and therefore $\rho = \rho_+$ would not be a Killing horizon. Nevertheless, expanding this quantity in powers of χ we do find that it is constant (see (8.43) below), a fact that provides a very strong check on the validity of our results. Thus, we have shown that $\rho = \rho_+$ is a Killing horizon, and hence it should correspond to the event horizon of the black hole.

We can now evaluate the angular velocity in order to study deviations with respect to Kerr's solution. A useful way to express it is the following,

$$\Omega_H = \frac{\chi}{2M\left(1+\sqrt{1-\chi^2}\right)} + \frac{\ell^4}{M^5}\left[\alpha_1^2\,\Delta\Omega_H^{(1)} + \alpha_2^2\,\Delta\Omega_H^{(2)} + \lambda_{\rm ev}\,\Delta\Omega_H^{(\rm ev)}\right] \ , \tag{8.43}$$

where the first term is the value in Kerr black hole and we made explicit the linear corrections related to the different terms in the action. It turns out that the parity breaking terms do not contribute to this quantity —nor to many others, as we will see. The dimensionless coefficients $\Delta\Omega_H^{(i)}$ depend on the spin, and the first terms in the χ-expansion read[7]

$$\Delta\Omega_H^{(1)} = \frac{21\chi}{80} - \frac{21103}{201600}\chi^3 - \frac{1356809}{8870400}\chi^5 - \frac{78288521}{461260800}\chi^7 + \mathcal{O}(\chi^9) \ , \tag{8.44}$$

$$\Delta\Omega_H^{(2)} = -\frac{709\chi}{1792} - \frac{169}{1536}\chi^3 - \frac{254929}{2365440}\chi^5 - \frac{613099}{5271552}\chi^7 + \mathcal{O}(\chi^9) \ , \tag{8.45}$$

$$\Delta\Omega_H^{(\rm ev)} = \frac{5\chi}{32} + \frac{1}{64}\chi^3 + \frac{3}{448}\chi^5 + \frac{11}{1792}\chi^7 + \mathcal{O}(\chi^9) \ . \tag{8.46}$$

[7]The first term in each of the formulas (8.44) and (8.45) reproduces previous results in the cases of EdGB gravity [342] and dCS gravity [349], respectively. The horizon area we obtain (see Eqs. (8.59) and (8.60) below) also agrees with the results in those works, that computed the area at quadratic order in the spin.

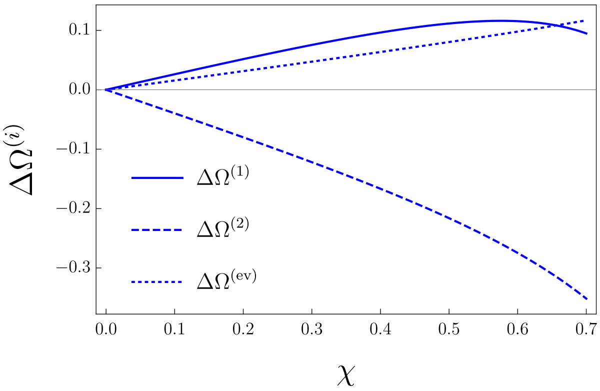

Figure 8.1: Correction to the angular velocity of the black hole associated to every interaction.

The profile of these coefficients is shown in Fig 8.1. This plot was done using the expansion up to order χ^{15}, which provides an accurate result up to $\chi = 0.7$. Interestingly, we observe that the correction related to α_1 increases the angular velocity, while the one related to α_2 decreases it. The one associated to λ_{ev} can have either effect, since the sign of λ_{ev} is in principle arbitrary. We observe that the effect of these terms is larger for smaller masses: the quantity that controls how relevant the corrections are is ℓ^4/M^4 times the corresponding coupling. They become of order 1 when $M \sim \ell$, which marks the limit of validity of the perturbative approach.

Surface gravity

At this stage, the natural step is to compute the surface gravity κ, defined by the relation

$$\xi^\nu \nabla_\nu \xi^\mu = \kappa \xi^\mu \,, \tag{8.47}$$

that the Killing vector (8.40) must satisfy on the horizon. The computation of κ is not straightforward because the coordinates we are using are singular at the horizon. A possibility in order in order to circumvent this problem consists in working in Eddington-Finkelstein coordinates, that cover the horizon. However, there exist a number of alternative methods that can be used in order to obtain the surface gravity even if the coordinates are not well-behaved. Here we will follow a trick proposed in [36]. First, let us rewrite (8.47) as

$$-\partial_\mu \xi^2 = 2\kappa \xi_\mu \,, \tag{8.48}$$

where we made use of the Killing property $\nabla_{(\mu}\xi_{\nu)} = 0$. Then, let us focus on the left hand side of the equation. The norm ξ^2 is a function of x and ρ, so that $\partial_\mu \xi^2$ only has non-vanishing $\mu = x, \rho$ components. However, one can explicitly check that $\lim_{\rho\to\rho_+} \partial_x \xi^2 = 0$, hence the only non-vanishing component is $\mu = \rho$, and it is given by

$$-\partial_\rho \xi^2|_{\rho=\rho_+} = \frac{(\rho_+ - M)}{2M^2\rho_+^2}\left(\rho_+^2 + a^2x^2\right)\Big[1 + 2H_2 - H_4 + 4M^2\rho_+^2 \frac{\partial_\rho\left(-H_1\Sigma + a^2(1-x^2)(2H_2 - H_4)\right) + 2(\rho_+ - M)(H_4 - 2H_2)}{2(\rho_+ - M)\left(\rho_+^2 + a^2x^2\right)^2}\Big]\Bigg|_{\rho=\rho_+}, \tag{8.49}$$

where, as usual, we are expanding linearly in the H_i functions. On the other hand, since ξ is normal to the horizon, we must have $\xi_\mu = C\,\delta_\mu{}^\rho$ for some constant C. Of course, this is not true in general: one should imagine that the previous formula holds only on the horizon, where the coordinate ρ is singular. The exact factor C is computed by taking the norm $\xi^2 = C^2 g^{\rho\rho}$ and evaluating at the horizon, so that we get

$$C = \lim_{\rho\to\rho_+}\sqrt{\frac{\xi^2}{g^{\rho\rho}}} = \frac{\rho_+^2 + a^2x^2}{2M\rho_+}\Big[1 + H_2 + \frac{H_3}{2} - \frac{H_4}{2} + 4M^2\rho_+^2 \frac{\partial_\rho\left(-H_1\Sigma + a^2(1-x^2)(2H_2 - H_4)\right) + 2(\rho_+ - M)(H_4 - 2H_2)}{4(\rho_+ - M)\left(\rho_+^2 + a^2x^2\right)^2}\Big]\Bigg|_{\rho=\rho_+}. \tag{8.50}$$

Then, we can plug (8.49) and (8.50) into (8.48) to find

$$\kappa = -\frac{\partial_\rho \xi^2|_{\rho=\rho_+}}{C} = \frac{(\rho_+ - M)}{2M\rho_+}\Big[1 + H_2 - \frac{H_3}{2} - \frac{H_4}{2} + M^2\rho_+^2 \frac{\partial_\rho\left(-H_1\Sigma + a^2(1-x^2)(2H_2 - H_4)\right) + 2(\rho_+ - M)(H_4 - 2H_2)}{(\rho_+ - M)\left(\rho_+^2 + a^2x^2\right)^2}\Big]\Bigg|_{\rho=\rho_+}. \tag{8.51}$$

Finally, evaluating this expression on the solution and expanding order by order in χ we find

$$\kappa = \frac{\sqrt{1-\chi^2}}{2M\left(1+\sqrt{1-\chi^2}\right)} + \frac{\ell^4}{M^5}\left[\alpha_1^2\Delta\kappa^{(1)} + \alpha_2^2\Delta\kappa^{(2)} + \lambda_{\rm ev}\Delta\kappa^{(\rm ev)}\right], \tag{8.52}$$

where the coefficients $\Delta\kappa^{(i)}$ read

$$\Delta\kappa^{(1)} = \frac{73}{480} - \frac{61}{384}\chi^2 + \frac{3001}{322560}\chi^4 + \frac{5376451}{70963200}\chi^6 + \mathcal{O}\left(\chi^8\right), \tag{8.53}$$

$$\Delta\kappa^{(2)} = \frac{2127}{7168}\chi^2 + \frac{14423}{86016}\chi^4 + \frac{429437}{3153920}\chi^6 + \mathcal{O}\left(\chi^8\right), \tag{8.54}$$

$$\Delta\kappa^{(\rm ev)} = \frac{1}{32} - \frac{7}{64}\chi^2 - \frac{3}{64}\chi^4 - \frac{7}{256}\chi^6 + \mathcal{O}\left(\chi^8\right). \tag{8.55}$$

Again, we observe that parity-breaking terms do not modify this quantity. In addition, the fact that we obtain a constant surface gravity is another strong check of our solution, since this is a general property that any event horizon must satisfy. The profile of these coefficients as functions of χ is shown in Fig 8.2, using an expansion up to order χ^{14}. We see that both quadratic curvature terms controlled by α_1 and α_2 increase the surface gravity, with the difference that the α_2 correction vanishes for static black holes. On the other hand, the contribution from $\lambda_{\rm ev}$ has a different sign depending on χ. For $\chi < 0.5$ the surface gravity is greater than in Kerr black hole, while for $\chi > 0.5$ it is lower, or viceversa, depending on the sign of $\lambda_{\rm ev}$. Another aspect that we can mention is that these contributions do not seem to be vanishing when $\chi \to 1$. This means that the extremal limit will not exactly coincide with $\chi = 1$, so we will have a correction to the extremality condition. However, the series expansion in χ breaks down for $\chi = 1$, so the perturbative approach is not reliable in order to analyze the corrections to the extremal Kerr solution.

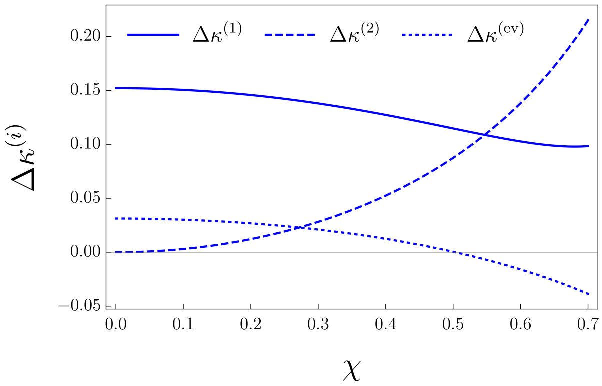

Figure 8.2: Variation of the surface gravity $\Delta\kappa^{(i)}$ due to every correction, as a function of χ. We can observe that contributions coming from curvature-squared terms always increase the temperature, since $\Delta\kappa^{(1)}$ and $\Delta\kappa^{(2)}$ are positive and also the coefficients multiplying them. The contribution from $\lambda_{\rm ev}$ has different sign depending on χ.

Horizon geometry

Let us finally study the size and shape of the horizon, which will be affected by the corrections. The induced metric at the horizon is

$$ds_H^2 = (1+H_3)\,|_{\rho=\rho_+} \frac{\rho_+^2 + a^2x^2}{1-x^2} dx^2 + (1+H_4)\,|_{\rho=\rho_+} \frac{4M^2\rho_+^2\left(1-x^2\right)}{\rho_+^2+a^2x^2} d\phi^2 \,. \tag{8.56}$$

First, we can find the area, which is given by the integral

$$A_H = 4\pi M\rho_+ \int_{-1}^{1} dx \left(1 + \frac{H_3}{2} + \frac{H_4}{2}\right)\Bigg|_{\rho=\rho_+} \,. \tag{8.57}$$

Computing the integral order by order in χ, we can write the area as

$$A_H = 8\pi M^2\left(1+\sqrt{1-\chi^2}\right) + \frac{\pi\ell^4}{M^2}\left(\alpha_1^2\,\Delta A^{(1)} + \alpha_2^2\,\Delta A^{(2)} + \lambda_{\rm ev}\,\Delta A^{(\rm ev)}\right) \,, \tag{8.58}$$

where every contribution $\Delta A^{(i)}$ depends on χ, and the first terms read

$$\Delta A^{(1)} = -\frac{98}{5} + \frac{11\chi^2}{10} + \frac{28267\chi^4}{25200} + \frac{11920241\chi^6}{7761600} + \mathcal{O}\left(\chi^8\right) , \tag{8.59}$$

$$\Delta A^{(2)} = -\frac{915\chi^2}{112} - \frac{25063\chi^4}{6720} - \frac{528793\chi^6}{295680} + \mathcal{O}\left(\chi^8\right) , \tag{8.60}$$

$$\Delta A^{(\text{ev})} = -10 + 4\chi^2 + \frac{69\chi^4}{40} + \frac{263\chi^6}{280} + \mathcal{O}\left(\chi^8\right) . \tag{8.61}$$

In Fig. 8.3 we show the profile of these quantities as functions of χ, using the expansion up to order χ^{14}. We observe that the quadratic corrections always reduce the area (except α_2 in the static case, that does not contribute). On the other hand, the cubic even correction reduces or increases the area depending on whether $\lambda_{\text{ev}} > 0$ or $\lambda_{\text{ev}} < 0$, respectively.

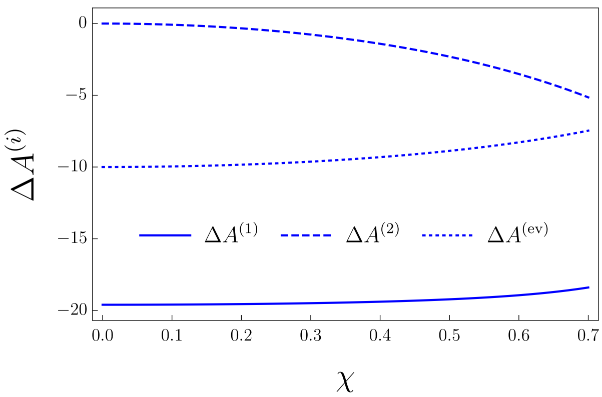

Figure 8.3: Variation of the black hole area $\Delta A^{(i)}$ due to every one of the corrections. The quadratic curvature corrections, controlled by α_1 and α_2 always decrease the area with respect to the result in Einstein gravity, while for the even cubic correction the contribution depends on the sign of λ_{ev}.

So far, we have not observed the effect of the parity-breaking corrections —they do not contribute either to the area, the surface gravity or the angular velocity of the black hole. This is expected on general grounds since these corrections contain only odd powers of x, and it is easy to see that the contribution, for instance, to the area, must vanish. Nevertheless, these terms do change the geometry and they will affect the shape of the horizon. Indeed, these parity-breaking corrections break the $\mathbb{Z}_2$ symmetry of the solution, i.e. the reflection symmetry on the equatorial plane $x \to -x$. It is expected that this loss of symmetry is manifest in the form of the horizon.

In order to visualize the event horizon, we perform an isometric embedding of it in 3-dimensional Euclidean space $\mathbb{E}^3$. In terms of Cartesian coordinates (x^1, x^2, x^3), we can parametrize the most general axisymmetric surface as

$$x^1 = f(x)\sin\phi , \quad x^2 = f(x)\cos\phi , \quad x^3 = g(x) , \tag{8.62}$$

where $f(x)$ and $g(x)$ are some functions that must be determined by imposing that the induced metric on the surface, given by

$$ds^2 = \left[\left(f'\right)^2 + \left(g'\right)^2\right] dx^2 + f^2\, d\phi^2 , \tag{8.63}$$

coincides with (8.56). We get immediately that these functions are given by

$$f(x) = 2M\rho_+\left(1+\frac{H_4}{2}\right)\Bigg|_{\rho=\rho_+}\left(\frac{1-x^2}{\rho_+^2+a^2x^2}\right)^{1/2}, \tag{8.64}$$

$$g(x) = \int dx\left[(1+H_3)\,|_{\rho=\rho_+}\frac{\rho_+^2+a^2x^2}{1-x^2}-\left(f'\right)^2\right]^{1/2}. \tag{8.65}$$

However, it can happen that the solution does not exist if the argument of the square root in the integral becomes negative. In that case, the horizon cannot be embedded completely in $\mathbb{E}^3$. It turns out that this only happens for quite large values of χ (around $\chi \sim 0.9$), and for the values we are considering here, the complete horizon can be embedded. As usual, we expand the expressions (8.64) and (8.65) linearly on H_i and at the desired order in χ and we obtain explicit formulas for f and g that we do not reproduce here for a sake of clarity.

Now we can use the result to visualize the horizon. In Fig. 8.4 we show the horizon for parity-preserving theories. We fix the mass to some constant value and $\chi = 0.65$ and we compare the horizon of Kerr black hole with the one in the corrected solutions for different values of the couplings. In this way, we can observe clearly the change in size and in shape of the horizon. As we already noted, both α_1 and α_2 reduce the area, but it turns out that they deform the horizon in different ways: α_1 squashes it while α_2 squeezes it. We also show the deformation corresponding to the "stringy" prediction $\alpha_1 = \alpha_2$. In that case we observe that the effect of both terms together is to make to horizon rounder than in Einstein gravity. As for the cubic even correction, it mainly changes the size of the black hole while its shape is almost unaffected.

In Fig. 8.5 we present the horizon in the parity-breaking theories (characterized by the two parameters θ_m and λ_{odd}). In the top row we plot the horizon for a fixed choice of higher-order couplings and for various masses, keeping $\chi = 0.65$ constant. The visualization is clearer in this way since these corrections do not change the area. In addition, we can see that for large M the horizon has almost the same form as in EG, but as we decrease the mass the corrections become relevant and it is deformed. We observe in this case that the $\mathbb{Z}_2$ symmetry is manifestly broken. Due to exotic form of these horizons we include as well a 3D plot in which we can appreciate them better. Very recently other works have described black hole solutions that do not possess $\mathbb{Z}_2$ symmetry [352, 365]. However, to the best of our knowledge, these are the first plots of black hole horizons without $\mathbb{Z}_2$ symmetry in purely gravitational theories.

8.3.2 Ergosphere

Another important surface of rotating black holes is the ergosphere, which marks the limit in which an object can remain static outside the black hole. When $g_{tt} < 0$, there are no timelike trajectories with constant (ρ, x, ϕ), so the ergosphere is identified by the condition $g_{tt} = 0$, which for the metric (8.24) can be written as

$$1-\frac{2M\rho}{\Sigma} = H_1\,. \tag{8.66}$$

This equation determines the value of the "ergosphere radius" ρ_{erg}. Unlike the horizon radius ρ_+, that does not receive corrections due to the clever choice of coordinates, the

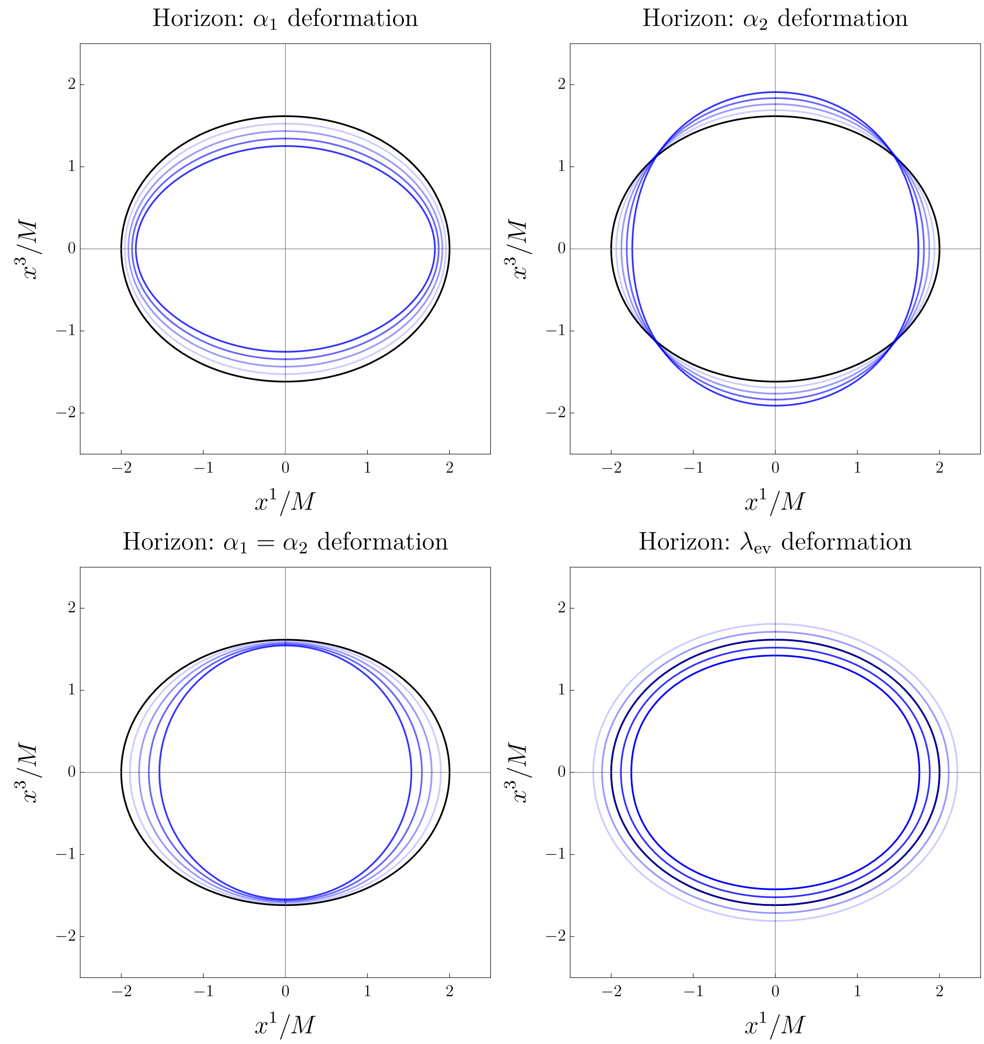

Figure 8.4: Isometric embedding of the horizon in $\mathbb{E}^3$ for different values of the parameters and for $\chi = 0.65$. In black we represent the horizon of Kerr black hole and in blue the horizon of the corrected solution for a fixed mass and different values of the couplings. From light to darker blue we increase the value of the corresponding coupling. In each case, only the indicated couplings are non-vanishing. Top left: $\frac{\ell^4}{M^4}\alpha_1^2 = 0.05, 0.1, 0.15, 0.2$, top right: $\frac{\ell^4}{M^4}\alpha_2^2 = 0.05, 0.1, 0.15, 0.2$, bottom left: $\frac{\ell^4}{M^4}\alpha_1^2 = \frac{\ell^4}{M^4}\alpha_2^2 = 0.05, 0.1, 0.15, 0.2$, bottom right: $\frac{\ell^4}{M^4}\lambda_{\rm ev} = -0.4, -0.2, 0.2, 0.4$.

ergosphere radius is modified with respect to its value in Kerr metric. We may express the corrections to $\rho_{\rm erg}$ as

$$\begin{aligned}\rho_{\rm erg} =& M\left(1+\sqrt{1-\chi^2 x^2}\right) + \frac{\ell^4}{M^3}\Big[\alpha_1^2\Delta\rho^{(1)} + \alpha_2^2\Delta\rho^{(2)} + \alpha_1\alpha_2\sin\theta_m\Delta\rho^{(m)} \\ &+\lambda_{\rm ev}\Delta\rho^{(\rm ev)} + \lambda_{\rm odd}\Delta\rho^{(\rm odd)}\Big]\,,\end{aligned} \tag{8.67}$$

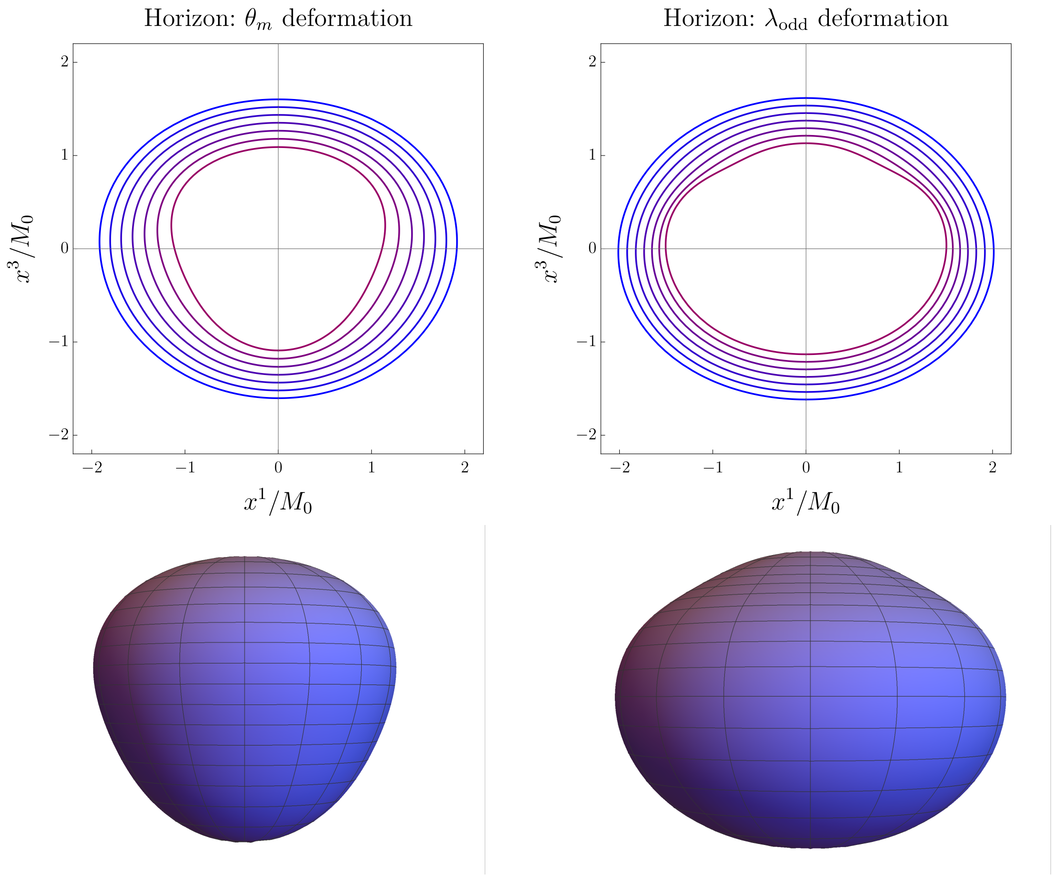

Figure 8.5: Isometric embedding of the horizon in $\mathbb{E}^3$ for parity-breaking theories. For clarity reasons we do not include the comparison with Kerr solution. In the top row we plot the horizon for different masses ($M_0 \geq M \geq 0.7M_0$ for some reference mass M_0) while keeping $\chi = 0.65$ and the couplings constant. In each case, only the indicated couplings are non-vanishing. Left: $\alpha_1 = \alpha_2$, $\theta_m = \pi/2$, $M_0 \approx 2.23\ell\sqrt{|\alpha_1|}$. Right: $\lambda_{\rm odd} > 0$, $M_0 \approx 1.46\ell\lambda_{\rm odd}^{1/4}$. Bottom row: 3D embedding of the horizon for $\frac{\ell^4}{M^4}\alpha_1^2 = \frac{\ell^4}{M^4}\alpha_2^2 = 0.15$, $\theta_m = \pi/2$ (left) and for $\frac{\ell^4}{M^4}\lambda_{\rm odd} = 0.6$ (right). In both cases, the $\mathbb{Z}_2$ symmetry is manifestly broken.

where the first term represents the result in Einstein gravity and we have to determine the value of the coefficients $\Delta\rho^{(i)}$. Plugging this into (8.66), we find these coefficients, whose first terms in the χ-expansion are shown in Eq. (G.36). In this case, we do get a non-vanishing contribution from the parity-breaking terms, though this is not directly relevant, since $\rho_{\rm erg}$ has no physical meaning by itself. However, an interesting property that we note by looking at (G.36) is that all the corrections to $\rho_{\rm erg}$ vanish at $x = \pm 1$, corresponding to the north and south poles of the ergosphere. There is a nice interpretation of this fact: the ergosphere and the horizon overlap at the poles. Indeed, the horizon radius ρ_+ does not have corrections, and the zeroth-order value of the ergosphere radius $\rho^{(0)}_{\rm erg} = M\left(1+\sqrt{1-\chi^2x^2}\right)$ already coincides with ρ_+ at the poles $\rho^{(0)}_{\rm erg}(x = \pm 1) = \rho_+$. Hence, the corrections to $\rho^{(0)}_{\rm erg}$ must vanish at $x = \pm 1$ if we want the horizon and the ergosphere to still overlap.

In order to study the geometry of the ergosphere, we can compute the induced metric for $\rho = \rho_{\rm erg}(x)$ at a constant time $t = t_0$, which reads

$$\begin{aligned} ds^2_{\rm erg} =& (1+H_3)\,\Sigma\left(\frac{1}{\Delta}\left(\frac{d\rho_{\rm erg}}{dx}\right)^2 + \frac{1}{1-x^2}\right)dx^2 \\ &+ (1+H_4)\left(\rho^2 + a^2 + \frac{2M\rho a^2(1-x^2)}{\Sigma}\right)(1-x^2)d\phi^2\Bigg|_{\rho=\rho_{\rm erg}(x)}\,, \end{aligned} \tag{8.68}$$

Using the value of $\rho_{\rm erg}$ that we have found yields a complicated expression that we omit here for clarity sake. The most useful way to visualize the geometric properties of the ergosphere is to find an isometric embedding of the previous metric in Euclidean space, as we have just done with the horizon. The embedding is shown in Fig. 8.6 for parity-preserving theories, and in Fig. 8.7 for parity-breaking ones. In the former case, we plot the ergosphere for a fixed mass and $\chi = 0.65$, and for different values of the couplings, including the GR result. We observe that the corrections change the size and shape of the ergosphere. The quadratic terms α_1 and α_2 both reduce the area of the ergosphere, while the cubic even term reduces its size for $\lambda_{\rm ev} > 0$, and increases it for $\lambda_{\rm ev} < 0$. The characteristic conical singularity at the poles of the ergosphere is also considerable affected by some corrections. In particular, we see that α_2 and $\lambda_{\rm ev} < 0$ have the effect of making the cone less sharp. In the top row of Fig. 8.7 we show instead the embedding of the ergosphere for several values of the mass, while keeping the couplings and $\chi = 0.65$ constant. This helps the visualization since parity-breaking interactions do not change the area of the ergosphere. As the mass decreases, the effect of the corrections becomes relevant and we observe, as in the case of the horizon, that the ergosphere does not possess $\mathbb{Z}_2$ symmetry. This is more explicit for the cubic odd correction $\lambda_{\rm odd}$ that deforms the ergosphere giving it a characteristic "trompo" shape. The effect of $\mathbb{Z}_2$ symmetry breaking is less obvious for the θ_m deformation, but nevertheless it can still be observed. To the best of our knowledge, these are the first examples of ergospheres without $\mathbb{Z}_2$ symmetry.

8.3.3 Photon rings

Another aspect of the modified Kerr black holes we would like to explore is their geodesics. The analysis of geodesics is necessary in order to obtain some observable quantities, such as the form of the black hole shadow [319]. However, a detailed analysis of geodesics will require of an independent study due to their intricate character.[8] For that reason, here we consider only a special type of geodesics that are particularly interesting: circular orbits (ρ = constant) for light rays at the equatorial plane, *i.e.* at $x = 0$, known as the photon rings or light rings of the black hole. However, an appropriate question that we must answer first is whether there are geodesics contained in the equatorial plane at all. In the case of Kerr metric, the reason of their existence is the reflection symmetry $x \to -x$, but we have seen that in our black holes this symmetry does not exist if we include parity-breaking terms. In fact, in those solutions there is no equatorial plane! Therefore, we should not expect the existence of geodesics contained in the plane $x = 0$ if we include those corrections. In order to understand this better, let us examine the

[8] For instance, a preliminary exploration shows that integrability is lost, *i.e.*, there is no Carter constant [366].

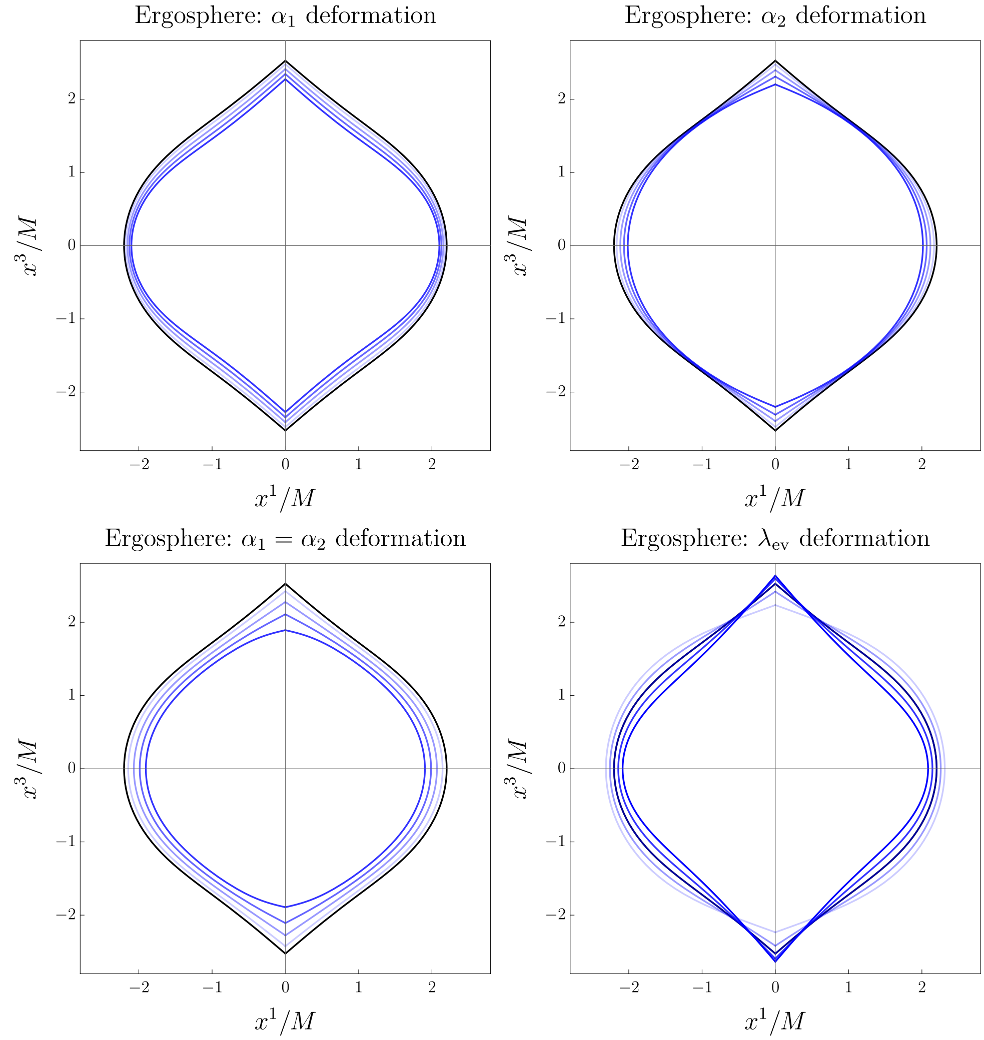

Figure 8.6: Isometric embedding of the ergosphere in $\mathbb{E}^3$ for different values of the parameters and for $\chi = 0.65$. In black we represent the ergosphere of Kerr black hole and in blue the ergosphere of the corrected solution, for a fixed mass and different values of the couplings. From light to darker blue we increase the value of the corresponding coupling. In each case, only the indicated couplings are non-vanishing. From left to right and top to bottom: $\frac{\ell^4}{M^4}\alpha_1^2 = 0.03, 0.07, 0.11, 0.15$, $\frac{\ell^4}{M^4}\alpha_2^2 = 0.03, 0.07, 0.11, 0.15$, $\frac{\ell^4}{M^4}\alpha_1^2 = \frac{\ell^4}{M^4}\alpha_2^2 = 0.03, 0.07, 0.11, 0.15$, $\frac{\ell^4}{M^4}\lambda_{\rm ev} = -0.6, -0.3, 0.3, 0.6$.

geodesic equations:

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0\,, \tag{8.69}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ and λ parametrizes the curve $x^\mu(\lambda)$. Let us evaluate these equations for a trajectory with $\dot{\rho} = 0$ and $x = 0$, which represents a circular orbit. We find that the $\mu = x$ component of (8.69) reads

$$-\frac{\partial_x H_1|_{x=0}}{2\rho_\pm^2}\,\dot{t}^2 + \frac{2M^2\chi}{\rho_\pm^3}\partial_x H_2|_{x=0}\,\dot{t}\,\dot{\phi} - \frac{\rho_\pm^3 + 2M^3\chi^2 + M^2\chi^2\rho_\pm}{2\rho_\pm^3}\partial_x H_4|_{x=0}\,\dot{\phi}^2 = 0\,. \tag{8.70}$$

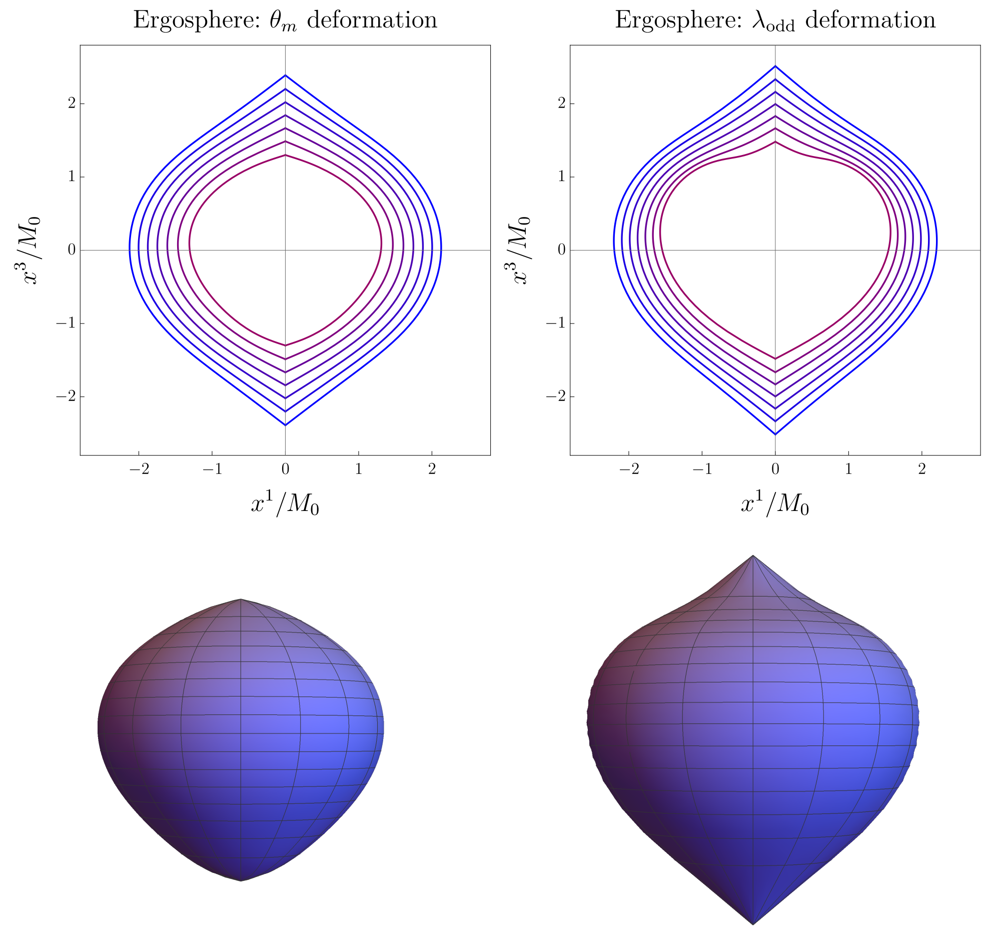

Figure 8.7: Isometric embedding of the ergosphere in $\mathbb{E}^3$ for parity-breaking theories. In the top row we plot the ergosphere for different masses ($M_0 \geq M \geq 0.7M_0$ for some reference mass M_0) while keeping $\chi = 0.65$ and the couplings constant. In each case, only the indicated couplings are non-vanishing. Left: $\alpha_1 = \alpha_2$, $\theta_m = \pi/2$, $M_0 \approx 2.23\ell\sqrt{|\alpha_1|}$. Right: $\lambda_{\rm odd} > 0$, $M_0 \approx 1.35\ell\lambda_{\rm odd}^{1/4}$. In the bottom row we show a 3D embedding of the ergosphere for $\frac{\ell^4}{M^4}\alpha_1^2 = \frac{\ell^4}{M^4}\alpha_2^2 = 0.15$, $\theta_m = \pi/2$ (left) and for $\frac{\ell^4}{M^4}\lambda_{\rm odd} = 0.6$ (right). In the latter case we observe clearly that the $\mathbb{Z}_2$ symmetry is broken and the ergosphere acquires a characteristic "trompo" shape. The effect is more subtle in the left picture, but the $\mathbb{Z}_2$ symmetry is also broken.

In order for the truncation $x = 0$ to be consistent, the left-hand-side should vanish *independently* of the value of $\dot{t}$ and $\dot{\phi}$. This does not always happens, and the reason is precisely the presence of parity-breaking interactions, controlled by $\lambda_{\rm odd}$ and $\sin\theta_m$. Note that all the terms appearing in (8.70) are proportional to $\partial_x H_i|_{x=0}$. When the theory preserves parity, the solution possesses $\mathbb{Z}_2$ symmetry and the functions H_i only contain even powers of x, so that $\partial_x H_i|_{x=0} = 0$. On the contrary, the parity-breaking terms introduce odd powers of x in the H_i functions —in particular terms linear in x— implying that $\partial_x H_i|_{x=0} \neq 0$. Thus, in such theories setting $x = 0$ is not consistent and there are no orbits contained in the plane $x = 0$ (probably there are no orbits contained in a plane at all, besides the radial geodesics at the axes $x = \pm 1$).[9] For simplicity, from now on we set the parity-violating parameters to zero, $\lambda_{\text{odd}} = \theta_m = 0$, so that we can study equatorial geodesics. However, we believe that studying the geodesics in those theories is an interesting problem that should be addressed elsewhere.

Let us then focus on the remaining equations. When they are evaluated on $\dot{\rho} = 0$ and $x = 0$, the $\mu = t$ and $\mu = \phi$ components of the geodesic equations (8.69) tell us that $\dot{t} = const$ and $\dot{\phi} = const$ and, consequently, the angular velocity $\omega \equiv d\phi/dt$ is also constant. On the other hand, the component $\mu = \rho$ gives an equation for ω:

$$\Gamma^{\rho}_{\phi\phi}\omega^2 + 2\Gamma^{\rho}_{t\phi}\,\omega + \Gamma^{\rho}_{tt} = 0 \,, \tag{8.71}$$

where the Christoffel symbols are shown in Eq. (G.41). Finally, we take into account that for massless particles we have $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$, that gives the following equation

$$(1 + H_4)\left(\rho^3 + M^2\chi^2\rho + 2M^3\chi^2\right)\omega^2 - 4M^2\chi\,(1 + H_2)\,\omega = \rho - 2M - \rho H_1 \,. \tag{8.72}$$

Now, using the equations (8.71) and (8.72) we can solve for ρ and ω. We get two solutions that we can express as the result in Einstein gravity plus corrections:

$$\frac{\rho_{\text{ph}\pm}}{M} = 2\left(1 + \cos\left(\frac{2}{3}\arccos\left(\mp\chi\right)\right)\right) + \frac{\ell^4}{M^4}\left[\alpha_1^2\Delta\rho^{(1)}_{\text{ph}\pm} + \alpha_2^2\Delta\rho^{(2)}_{\text{ph}\pm} + \lambda_{\text{ev}}\Delta\rho^{(\text{ev})}_{\text{ph}\pm}\right] \,, \tag{8.73}$$

$$M\omega_{\pm} = \pm\left[\frac{1}{\sqrt{48\cos^4\left(\frac{1}{3}\arccos\left(\mp\chi\right)\right) + \chi^2}} + \frac{\ell^4}{M^4}\left(\alpha_1^2\Delta\omega^{(1)}_{\pm} + \alpha_1^2\Delta\omega^{(1)}_{\pm} + \lambda_{\text{ev}}\Delta\omega^{(\text{ev})}_{\pm}\right)\right] \,, \tag{8.74}$$

The "+" solution corresponds to the prograde photon ring (the photons rotate in the same direction as the black hole), while the "−" solution represents the retrograde photon ring.

[9] In Ref. [352], rotating black holes were studied in the presence of quartic-curvature corrections, including a parity-violating combination, and it was stated that this interaction does not have effects on equatorial geodesics. Apparently, the analysis of geodesics in that paper missed the fact that those geodesics are not permitted if the parity-violating term is activated. On the other hand, that analysis should be perfectly valid if the problematic term is removed.

We reproduce here the values of the coefficients $\Delta\omega_{\pm}^{(i)}$ expanded up to order χ^6 in the spin

$$\begin{aligned}\Delta\omega_{\pm}^{(1)} &= \frac{4397}{65610\sqrt{3}} \pm \frac{20596\chi}{295245} + \frac{1028803\chi^2}{14467005\sqrt{3}} \pm \frac{45262543\chi^3}{3906091350} \\ &\quad - \frac{3685587061\chi^4}{328111673400\sqrt{3}} \mp \frac{110632797883\chi^5}{5413842611100} - \frac{910228742414947\chi^6}{17151053391964800\sqrt{3}} + \mathcal{O}\left(\chi^7\right) \ , \end{aligned} \tag{8.75}$$

$$\begin{aligned}\Delta\omega_{\pm}^{(2)} &= \mp\frac{131\chi}{5103} - \frac{11047\chi^2}{381024\sqrt{3}} \mp \frac{9491513\chi^3}{1388832480} - \frac{19022279\chi^4}{925888320\sqrt{3}} \\ &\quad \mp\frac{353193404087\chi^5}{23099061807360} - \frac{2452581602509\chi^6}{63522419970240\sqrt{3}} + \mathcal{O}\left(\chi^7\right) \ , \end{aligned} \tag{8.76}$$

$$\begin{aligned}\Delta\omega_{\pm}^{(\text{ev})} &= \frac{20}{2187\sqrt{3}} \pm \frac{320\chi}{19683} + \frac{26749\chi^2}{1928934\sqrt{3}} \mp \frac{12967\chi^3}{104162436} - \frac{4415651\chi^4}{1249949232\sqrt{3}} \\ &\quad \mp\frac{3101153\chi^5}{937461924} - \frac{33998483\chi^6}{6629195034\sqrt{3}} + \mathcal{O}\left(\chi^7\right) \ , \end{aligned} \tag{8.77}$$

while the coefficients $\Delta\rho_{\text{ph}\pm}^{(i)}$ are shown in Eq. (G.44) of the Appendix. However, $\rho_{\text{ph}\pm}$ is a meaningless quantity, since ρ does not have a direct interpretation as a radius. What we should really consider as the radius of the light rings is

$$R_{\pm} = \sqrt{g_{\phi\phi}}\Big|_{x=0,\, \rho=\rho_{\text{ph}\pm}} \ . \tag{8.78}$$

Since the light ring (more precisely, the photon sphere) determines the shape of the black hole shadow, this quantity give us information about the deformation of the shadow (near the equator) due to the corrections. On the other hand, $\omega_{\pm}$ is also an interesting quantity, since it is related to the time-scale of the response of the black hole when it is perturbed. In fact, there is a known quantitative relation between the orbital frequency of the light ring and the quasinormal frequencies of static black holes in the eikonal limit [367, 368]. Although the relation probably does not extend to the rotating case, we do expect that $\omega_{\pm}$ captures qualitatively the (real) frequencies of the first quasinormal modes. Hence, we can use $\omega_{\pm}$ in order to perform a first estimation of the effects of the corrections on the black hole quasinormal frequencies.

In Fig. 8.8 we show the frequencies $\omega_{\pm}$ and the radius $R_{\pm}$ for several values of the higher-order couplings and we compare them to the GR values. These plots were computed using an expansion up to order χ^{14} of both quantities. We note some characteristic features for each correction. In the case of the quadratic correction controlled by α_1 we see that both ω_+ and $|\omega_-|$ increase with respect to the Einstein gravity values. On the other hand, for α_2 corrections we observe that ω_+ decreases while $|\omega_-|$ increases so that the difference between the two frequencies is reduced. As for the cubic correction, it increases or decreases ω_+ if $\lambda_{\text{ev}} > 0$ or $\lambda_{\text{ev}} < 0$ respectively. It has little effect on ω_-, but interestingly the sign is different depending on the value of χ. However, in order to characterize deviations from GR it is more useful to look at the ratio of frequencies $\omega_+/|\omega_-|$, that we show for a few cases in Fig. ??. In GR, this quantity is completely determined by the spin parameter χ, but in these theories it also depends on the combination ℓ^4/M^4. Thus, if one is able to determine χ by other means, the ratio $\omega_+/|\omega_-|$ can be used to constrain the higher-order couplings.

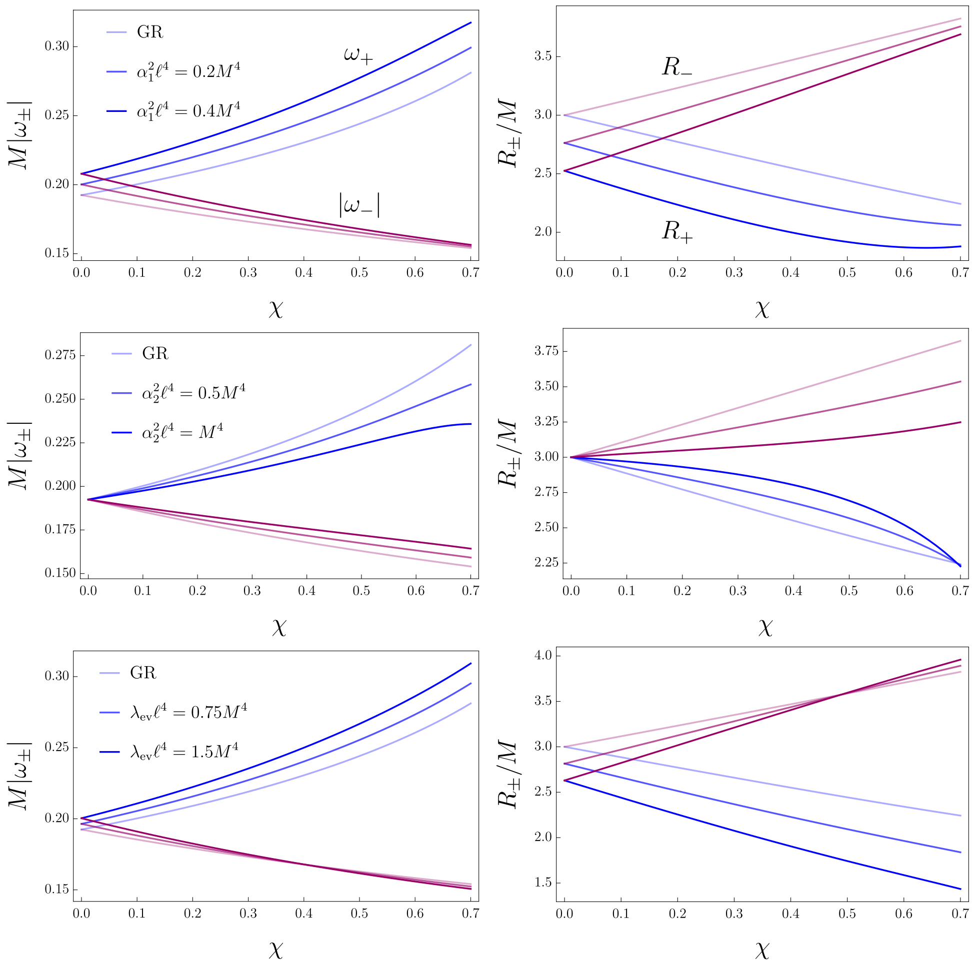

Figure 8.8: Frequencies and radii of the light rings in parity-preserving theories. In blue we plot the quantity corresponding to the prograde orbit and in purple that corresponding to the retrograde one. In the left column we show the frequencies for different values of the couplings and compare them to GR. In the right column we plot the radii $R_\pm$ for the same values of the couplings.

8.3.4 Scalar hair

So far, we have only focused on the geometry, but one of the most remarkable features of the solutions of (8.5) is that the scalar fields acquire a non-trivial profile. In fact, the coupling of the scalars to the quadratic curvature invariants prevent these from being constant whenever the invariants are non-vanishing. A slightly less trivial fact —though also well-known [299, 330–332, 369]— is that the scalars actually get a charge that can be measured at infinity. More precisely, the scalar ϕ_1 gets a charge Q while ϕ_2 gets dipolar

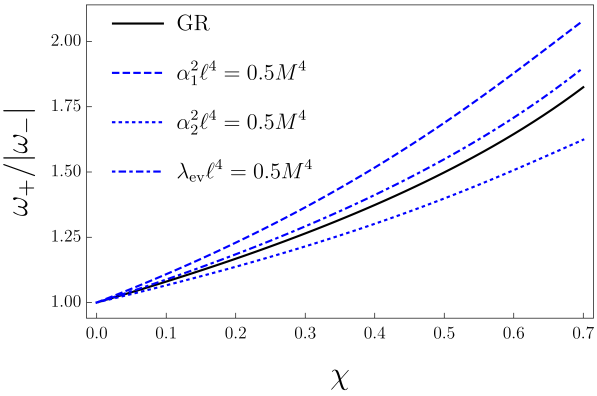

Figure 8.9: Ratio of light ring frequencies $\omega_+/|\omega_-|$ in several theories.

moment P, that can be identified by looking at the asymptotic behaviour[10]

$$\phi_1 \sim -\frac{Q}{\rho}\,, \quad \phi_2 \sim P\frac{x}{\rho^2}\,. \tag{8.80}$$

Using the solution in powers of χ that we have found, we obtain

$$Q = -\frac{\alpha_1\ell^2}{M}\left(2-\frac{\chi^2}{2}-\frac{\chi^4}{4}-\frac{5\chi^6}{32}-\frac{7\chi^8}{64}-\frac{21\chi^{10}}{256}-\frac{33\chi^{12}}{512}+\ldots\right), \tag{8.81}$$

$$P = \alpha_2\ell^2\cos\theta_m\left(\frac{5\chi}{2}-\frac{\chi^3}{4}-\frac{3\chi^5}{32}-\frac{3\chi^7}{64}-\frac{7\chi^9}{256}-\frac{9\chi^{11}}{512}-\frac{99\chi^{13}}{8192}+\ldots\right). \tag{8.82}$$

Remarkably enough, it is possible to guess the general term of these series and to sum them. We find

$$Q = -\frac{4\alpha_1\ell^2}{M}\frac{\sqrt{1-\chi^2}}{1+\sqrt{1-\chi^2}}\,, \tag{8.83}$$

$$P = \alpha_2\ell^2\cos\theta_m\frac{2\chi(5-8\chi^2+4\chi^4)}{2-3\chi^2+2\chi^4+2(1-\chi^2)^{3/2}}\,. \tag{8.84}$$

One can check that the series expansion of these expressions matches those in (8.81) and (8.82), so they are most likely correct, and they give the exact value of the charges as functions of the spin. In the case of the charge Q, we also check that it agrees with previous results [14, 16, 370].

Despite having non-vanishing scalar charge, we note however that the solution has no "hair", because the charge is completely fixed in terms of the mass and the spin. In other words, the charge cannot be arbitrary. The reason is that the previous value of the charge is the only one compatible with the requirement of regularity of the solution at

[10]The reason for the negative sign in front of Q is that the charge is conventionally defined as

$$Q = \frac{1}{4\pi}\int d^2\Sigma^\mu\partial_\mu\phi_1\,, \tag{8.79}$$

where the integral is taken on spatial infinity.

the horizon. If we introduce, by hand, any other value of the scalar charge, the resulting solution would develop a singularity at the horizon.

As we mentioned in Sec. 8.1, in the context of string theory ϕ_1 is related to the dilaton, while ϕ_2 is the axion. In Appendix G.2 we show that the precise identification with the effective action of the heterotic superstring is $\alpha_1 = -\alpha_2 = -1/8$,$\ell^2 = \alpha'$, $\varphi = \varphi_\infty + \frac{\phi_1}{2}$. Then, the dilaton charge D associated to a rotating black hole reads, at leading order in α',

$$D = \frac{\alpha'}{4M} \frac{\sqrt{1-\chi^2}}{1+\sqrt{1-\chi^2}} \,. \tag{8.85}$$

This can be expressed in a very appealing form as $D = \alpha' \pi T$, where $T = \kappa/(2\pi)$ is the Hawking temperature of the black hole. It turns out that this intriguing connection between asymptotic charge and temperature (or surface gravity) is not a coincidence, but a general phenomenon that happens in EdGB theory with linear coupling [370].

The field ϕ_2 gets a dipolar moment instead of charge because it is sourced by the parity-violating Pontryagin density —ϕ_2 is essentially the scalar that appears in dynamical Chern-Simons gravity [346]. When the spin vanishes we get $P = 0$, and in fact, $\phi_2 = 0$, so that this kind of scalar hair is not present in spherically symmetric solutions [371].

Besides the asymptotic behaviour, it is also interesting to study the profile of the scalar fields as a function of x. The field ϕ_2 is odd under the $\mathbb{Z}_2$ transformation $x \to -x$, while ϕ_1 is even only for $\theta_m = n\pi$, $n \in \mathbb{Z}$. For other values of θ_m, ϕ_1 does not have a defined parity, which is a manifestation of the breaking of the $\mathbb{Z}_2$ symmetry. For instance, when evaluated on the horizon, $\rho = \rho_+$, the field ϕ_1 is given by

$$\begin{aligned} \phi_1\Big|_{\rho_+} = \frac{\ell^2}{M^2}\Bigg[&\alpha_1\left(\frac{11}{6}+\left(\frac{5}{16}-\frac{59x^2}{40}\right)\chi^2+\left(\frac{11}{160}-\frac{117x^2}{80}+\frac{167x^4}{224}\right)\chi^4+\dots\right)\\ &+\alpha_2\sin(\theta_m)\left(\frac{29x\chi}{16}+\left(\frac{187x}{160}-\frac{13x^3}{12}\right)\chi^3+\left(\frac{67x}{80}-\frac{629x^3}{448}+\frac{251x^5}{512}\right)\chi^5+\dots\right)\Bigg]\,. \end{aligned} \tag{8.86}$$

We only show here a few terms in the χ-expansion for definiteness, but using the solution up to order χ^{14} we can determine accurately the profile of ϕ_1 on the horizon for high values of χ. In Fig. 8.10 we plot ϕ_1 as a colormap on the horizon for $\chi = 0.65$, and $\ell^2\alpha_1 = \ell^2\alpha_2 = 0.4M^2$. From left to right, the parity-breaking parameter θ_m takes the values $\theta_m = 0, \pi/4, \pi/2$. For $\theta_m = 0$ the profile is $\mathbb{Z}_2$-symmetric and has a mild variation, taking a maximum value at the equator. When $\theta_m \neq 0$, we observe the deformation of the horizon that we reported in Sec. 8.3.1, plus a "polarization" of the scalar field, that develops a maximum at the north pole and a minimum at the south one.

Interestingly enough, the scalar profile provides an intuitive picture of the deformation of the horizon. The northern "hemisphere" grows due to the θ_m correction, while the southern one has a smaller size, and this coincides with the fact that the scalar field is "concentrated" on the northern hemisphere, producing a larger energy density there. Thus, the horizon is enlarged in the region that has a greater scalar energy density.

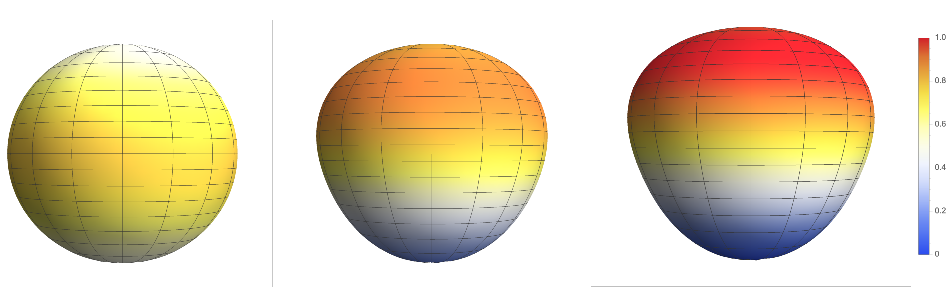

Figure 8.10: Profile of the scalar field ϕ_1 on the horizon. We show here the case for $\ell^2\alpha_1 = \ell^2\alpha_2 = 0.4M^2$ and for a parity-breaking phase $\theta_m = 0, \pi/4, \pi/2$, from left to right.

8.4 Conclusions

In this chapter, we have computed the modified Kerr black hole solution in the effective theory (8.5), which provides a general framework to study the leading-order deviations from GR associated to higher-derivative corrections. We expressed the solution as a power series in the spin parameter χ and we showed that including enough terms we get an accurate result even for large values of χ. In this text we have worked with an expansion up to order χ^{14}, that provides a good approximation for $\chi \leq 0.7$, but with the software we supply it should be possible to compute the series to higher orders in χ and to get a solution valid for $\chi \sim 1$. Although the series expansion involves lengthy expressions, it has obvious advantages with respect to numerical solutions, since it allows for many analytic computations, as we have illustrated in Section 8.3.

We have studied some of the most remarkable properties of these rotating black holes, with special emphasis on the horizon. We have shown that the corrections modify the shape of the horizon, and in particular, that parity-violating interactions break the $\mathbb{Z}_2$ symmetry of Kerr's black hole. We observed the same phenomenon in the case of the ergosphere, and, as far as we know, Fig. 8.7 contains the first example of ergospheres without $\mathbb{Z}_2$ symmetry.

In addition, we have computed some quantities that were disregarded in previous studies on rotating black holes in modified gravity. In particular, we have obtained the surface gravity of these black holes, from which one obtains the Hawking temperature according to $T = \frac{\kappa}{2\pi}$, in natural units. Thus, from the results in Sec. 8.3.1 we conclude that the quadratic curvature terms with non-minimally coupled scalars always increase the temperature of black holes, for any value of the spin. On the other hand, the cubic curvature term raises or lowers the temperature depending on the sign of the coupling $\lambda_{\rm ev}$ and on the value of the spin χ. The modification of Hawking temperature may have important consequences for the evaporation process of black holes [312], and it would be interesting to extend these results by obtaining the value of the temperature non-perturbatively in the coupling and in the spin.

As a first step in analyzing the geodesics of the modified Kerr black holes, we studied the photon rings, i.e. circular light-like geodesics on the equatorial plane. Remarkably, we have found that for parity-breaking theories there are no such orbits: indeed, there are no orbits contained in the equatorial plane because there is no equator at all. Thus, we computed the photon rings for parity-even theories, characterizing the deviations from

GR.

Finally, we also noticed the non-trivial scalar fields, and we were able to obtain exact formulas for the monopole and dipole charges. We also computed the profile of the scalar ϕ_1 on the horizon and we observed how the $\mathbb{Z}_2$ symmetry is broken when the parity-violating phase θ_m is activated.

Let us now comment on some possible extensions and future directions. As we already mentioned, it would be interesting to obtain the solution for even larger values of the angular momentum, since the effects of rotation are more drastic when the spin is close to the extremal value. It would also be more or less straightforward to extend the results of this chapter to other theories that we did not consider here, particularly the quartic ones in [22, 352]. Another possible extension would entail adding a mass term for the scalars in (8.5), though this would considerably increase the difficulty of finding an analytic solution.

We have studied some basic properties of the modified Kerr black holes, but the next natural step is to derive observational signatures of these spacetimes. Analyzing the geodesics of these black holes is an interesting task, as one would potentially observe effects coming from the loss of integrability or from the absence of $\mathbb{Z}_2$ symmetry in parity-violating theories. Once the geodesics are determined, one could study gravitational lensing or the black hole shadow, similarly as done e.g. in [354, 355]. However, the most sensitive quantity to the corrections —and that we expect to measure in the near-future thanks to gravitational wave detectors [17]— is the quasinormal mode spectrum of the black hole. Hence, the determination of the quasinormal modes and frequencies of the rotating black holes presented here is a very relevant task, for which one needs to perform perturbation theory. The analysis of the scalar perturbations has been recently performed in [372]. On the other hand, the study of gravitational perturbations presents a more challenging problem, since one would need to derive the analogous of the Teukolsky equation [373] for the modified Kerr black holes.

The observation of deviations from general relativity in astrophysical black holes would represent a tremendous breakthrough that would revolutionize our current understanding of gravity. But even if this is not the case, the expectation that these corrections appear at some higher energy scale is a realistic one. Studying their effects on black hole geometries provides us with a rich source of new physics, and allows us to learn about new phenomena that could be inherent to an underlying UV-complete theory of gravity.

A
Resumen

El límite de bajas energías de las teorías de supercuerdas admite una descripción en términos de una teoría de campos efectiva para sus modos sin masa. La correspondiente acción efectiva viene dada por una doble expansión perturbativa en g_s, el acoplamiento de la cuerda, y α', el cuadrado de su longitud. El término dominante en esta expansión viene dado por la acción de las diferentes supergravedades en diez dimensiones, mientras que términos subdominantes involucran términos con derivadas de orden superior. El trabajo que se presenta en esta tesis es el resultado de un programa de investigación que empieza con el estudio de las soluciones supersimétricas de supergravedad gaugeada y culmina con el análisis de los efectos producidos por las correcciones en α' en soluciones de la acción efectiva de la supercuerda heterótica.

Esta tesis está dividida en dos partes. La primera se centra en las soluciones supersimétricas de una extensión mínima del modelo STU de la supergravedad $\mathcal{N} = 1$ en cinco dimensiones cuyo principal interés radica en el hecho de que puede obtenerse a partir de la compactificación toroidal de la supergravedad $\mathcal{N} = 1$ en diez dimensiones acoplada a un triplete de campos gauge de SU(2). Concretamente, construimos y estudiamos soluciones que describen agujeros negros y geometrías regulares sin horizonte con campos de Yang-Mills no triviales.

El entendimiento este tipo de soluciones desde el marco de la teoría de cuerdas sirve como motivación para la segunda parte de la tesis, la cual está dedicada a estudiar soluciones de la acción efectiva de la supercuerda heterótica a primer orden en α'. Ésta no coincide simplemente con la acción de la supergravedad $\mathcal{N} = 1$ acoplada a un multiplete vectorial de Yang-Mills en diez dimensiones, ya que el mecanismo de cancelación de anomalías de Green-Schwarz y supersimetría nos obligan a introducir términos adicionales en la acción. Estos términos se construyen a partir de la conexión de espín con torsión dada por la intensidad de campo asociada a la 2-forma de Kalb-Ramond y las contribuciones de éstos a las ecuaciones de movimiento son análogas a las de los campos de Yang-Mills. Este hecho es explotado para construir soluciones analíticas que describen agujeros negros supersimétricos con correcciones en α'.

La lección más importante a extraer de nuestros resultados es que la masa y las cargas conservadas de los agujeros negros se ven modificadas por las correcciones en α'. Esto es lo que cabría esperar desde un punto de vista físico ya que las correcciones aparecen en las ecuaciones de movimiento como términos efectivos de energía, momento y carga. Esta información resulta ser crucial para establecer una correspondencia entre los parámetros que caracterizan la descripción efectiva o "de grano grueso" (el agujero negro) y los parámetros que caracterizan el sistema microscópico de teoría de cuerdas descrito. La mayor relevancia de los efectos producidos por los términos de orden superior en derivadas en las cargas se alcanza en los llamados agujeros negros pequeños, los cuales describen de manera efectiva una cuerda fundamental con cargas de "enrollamiento" y momento. Los agujeros negros pequeños son soluciones singulares cuyo horizonte tiene tamaño nulo en la aproximación de supergravedad. Durante mucho tiempo se ha creído que las correcciones de orden superior en derivadas serían capaces de estirar el horizonte, haciendo la solución regular. Nuestros resultados revelan que éste no es el caso a primer orden en α', y que las regularizaciones existentes de los agujeros negros pequeños heteróticos parecen en realidad describir un sistema microscópico diferente que es regular ya en la aproximación de supergravedad.

El último capítulo de la tesis contiene el cálculo de la corrección más general a la solución de Kerr en cuatro dimensiones cuando al término de Einstein-Hilbert se le añaden términos de orden superior en curvatura hasta orden cúbico, teniendo en cuenta la posibilidad de tener acoplamientos dinámicos. Ésto incluye, como caso particular, las correcciones predichas por la acción efectiva de la supercuerda heterótica.

B

Conclusiones

El objetivo principal de esta tesis ha sido el de profundizar en nuestra comprensión sobre agujeros negros en el contexto de las teorías de supergravedad y de supercuerdas.

La primera parte de esta tesis se enmarca en una línea de investigación cuyo objetivo es el de entender la interacción entre los campos de Yang-Mills y la gravedad a través del estudio y la construcción de nuevas soluciones de supergravedad gaugeada. Concretamente, en esta tesis presentamos nuevas soluciones de tipo agujero negro en cuatro y en cinco dimensiones con propiedades interesantes que no se habían observado antes en la literatura. Además, proponemos un procedimiento sistemático para construir *geometrías de microestado* supersimétricas evitando la aparición de curvas temporales cerradas, lo cual simplifica enormemente la construcción explícita y la exploración de este tipo de soluciones.

El estudio de las correcciones de orden superior en derivadas (o en α') en agujeros negros en el contexto de la teoría de cuerdas constituye la parte más importante de esta tesis. La relevancia que ésto tiene desde un punto de vista teórico radica en que nos permite comprobar la consistencia de teoría de cuerdas más allá del límite de supergravedad, donde los efectos genuinamente *cuerdosos* empiezan a ser relevantes.

Esta tesis contiene el cálculo analítico de las correcciones de orden cuadrático en curvatura a agujeros negros supersimétricos de tres y de cuatro cargas en el marco de la supercuerda heterótica. Ésto nos permite verificar que la entropía de los agujeros negros calculada siguiendo la prescripción de Wald está en consonancia con el cálculo microscópico de la degeneración de estados cuánticos del sistema, lo cual es un test altamente no trivial de la consistencia de la teoría.

Otro problema que atacamos es el estudio de las correcciones en curvatura en los llamados *agujeros negros pequeños*, donde se espera que éstas jueguen un papel fundamental ya que los agujeros negros pequeños tienen un horizonte singular de tamaño nulo en la aproximación de supergravedad. De acuerdo con una propuesta hecha por Sen, las correcciones en curvatura vendrían al rescate para resolver el horizonte de modo que la entropía de la solución corregida estuviese de acuerdo con el contaje de la degeneración de estados microscópicos. Sin embargo, nuestro análisis nos lleva a concluir que las correcciones de segundo orden en curvatura no son suficientes para regularizar la solución, lo cual contradice una creencia ampliamente establecida en la comunidad de teoría de cuerdas. Este resultado se extiende a los *anillos negros pequeños* en cinco dimensiones.

El último capítulo de la tesis es el único en el que abordamos las correcciones de orden superior en derivadas en un contexto más amplio que teoría de cuerdas. En él construimos la teoría efectiva más general que parametriza las correcciones a las soluciones de vacío de la relatividad general cuando al término de Einstein-Hilbert se le añaden términos de orden superior en curvatura hasta orden cúbico, incluyendo la posibilidad de tener acoplos dinámicos controlados por escalares sin masa. Además, estudiamos las correcciones predichas por esta teoría efectiva a la solución de agujero negro de Kerr, analizando en detalle las propiedades del horizonte y de la ergoesfera.

C
Conclusions

The main goal of this thesis has been to delve in our understanding of black holes in the context of supergravity and superstring theories.

The first part of this thesis lies in the context of a research program whose aim is to understand the interplay between Yang-Mills fields and gravity through the study and construction of new solutions of gauged supergravity. Concretely, in this thesis we present novel black-hole solutions in four and five dimensions which exhibit interesting properties which have not been observed in the literature so far. Furthermore, we propose a systematic procedure to construct supersymmetric *microstate geometries* avoiding the appearance of closed timelike curves, which enormously simplifies the explicit construction and exploration of this type of solutions.

The study of higher-derivative (or α') corrections to black holes in the context of string theory constitutes the most important part of this thesis. The relevance that this has from a theoretical point of view lies in the fact that it allows to check the consistency of string theory beyond the supergravity approximation, where genuinely *stringy* effects start becoming relevant.

This thesis contains the analytic computation of the quadratic curvature corrections to supersymmetric black holes with three and four charges in the context of the heterotic superstring. This allows us to verify that the black-hole entropy computed following Wald's prescription is in agreement with the microscopic computation of the degeneracy of quantum states of the system, which is a highly non-trivial test of the consistency of the theory.

Another problem that we address is the study of the higher-curvature corrections in the so-called *small black holes*, where they are expected to play a fundamental rôle since small black holes have a singular horizon with vanishing area in the supergravity approximation. According to a proposal raised by Sen, the higher-curvature corrections would come to the rescue to resolve the horizon in such a way that the entropy of the corrected solution would match the microscopic counting of quantum states. However, our analysis leads us to the conclusion that quadratic curvature corrections are not enough to regularize the solution, which contradicts a widely spread belief in the string theory community. This result is extended to five-dimensional *small black rings*.

The last chapter of the thesis is the only one in which we address the issue of the higher-derivative corrections in a broader context than string theory. We construct the most general effective field theory parametrizing the corrections to vacuum solutions of general relativity when the Einstein-Hilbert term is supplemented with higher-curvature terms up to cubic order, including the possibility of having dynamical couplings controlled by massless scalars. In addition, we study the corrections predicted by this effective theory on the Kerr solution, analyzing in detail the properties of the horizon and ergosphere.

D

Regular, horizonless solutions of SEYM theories

Here we fix the notation used in Chapter 3 and summarize the construction of microstate geometries. We start with a very brief description of SEYM theories and continue in Section D.2 with a summary of the results of [152], but in slightly different conventions. In Section D.3 we describe the construction of microstate geometries, adapting the results of [157] to our current conventions, which have chosen to make contact with most of the literature on five-dimensional microstate geometries. Finally Section D.4 contains the expressions for the asymptotic charges in terms of the parameters of the solutions.

D.1 Theory and conventions

SEYM theories are $\mathcal{N} = 1$, $d = 5$ supergravities in which a non-Abelian subgroup, typically SU(2), of the isometries of the scalar manifold has been gauged. For a thoughtful description of these theories we recommend the magnificent book [111]. We set all fermions to zero and consider the bosonic part of the action,

$$
\begin{aligned}
S \;=\; & \int d^5x\sqrt{|g|}\,\Big\{R+\tfrac{1}{2}g_{xy}\mathfrak{D}_\mu\phi^x\mathfrak{D}^\mu\phi^y-\tfrac{1}{4}a_{IJ}F^{I\,\mu\nu}F^{J}{}_{\mu\nu}-\tfrac{1}{4}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\lambda}}{\sqrt{|g|}}\left[F^{I}{}_{\mu\nu}F^{J}{}_{\rho\sigma}A^{K}{}_{\lambda}\right.\\
& \left.-\tfrac{1}{2}gf_{LM}{}^{I}F^{J}{}_{\mu\nu}A^{K}{}_{\rho}A^{L}{}_{\sigma}A^{M}{}_{\lambda}+\tfrac{1}{10}g^2f_{LM}{}^{I}f_{NP}{}^{J}A^{K}{}_{\mu}A^{L}{}_{\nu}A^{M}{}_{\rho}A^{N}{}_{\sigma}A^{P}{}_{\lambda}\right]\Big\}\,,
\end{aligned}
\tag{D.1}
$$

that describes the coupling of the metric, n_v scalars labeled as $x, y = 1, \ldots, n_v$ and (n_v+1) vector fields labeled with the indices $I, J, \ldots = 0, \ldots, n_v$. The full theory is completely determined by the election of the constant symmetric tensor C_{IJK} and the structure constants of the gauge group $f_{JK}{}^{I}$. We consider the SU(2)-gauged ST[2, 6] model, that contains $n_v = 5$ vector multiplets. This model is characterized by a constant symmetric tensor with the following non-vanishing components

$$
C_{0xy} = \frac{1}{6}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}\,. \tag{D.2}
$$

The first three vectors, A^0, A^1 and A^2 are Abelian, while A^3, A^4 and A^5 correspond to a $SU(2)$ triplet. For convenience, we separate the range of values of the indices I, J in two sectors: the Abelian sector $i, j = 0, 1, 2$ and the non-Abelian sector $\alpha, \beta = 3, 4, 5$. Therefore, if the latter sector is truncated we immediately recover the STU model of supergravity, with $C_{ijk} = |\varepsilon_{ijk}|/6$, the theory in which five-dimensional BPS microstate geometries are naturally described.

It is convenient to introduce $(n_v + 1)$ functions of the physical scalars $h^I(\phi^x)$, which are subjected to the following constraint

$$C_{IJK}h^I h^J h^K = 1\,. \tag{D.3}$$

The functions h^I can be interpreted as coordinates in a (n_v+1)-dimensional ambient space, so the above constraint defines a codimension 1 hypersurface parametrized by the scalars ϕ^x known as the scalar manifold. In the $ST[2,6]$ model, a convenient parametrization is

$$h^0 = e^{-\phi}e^{2k/3}, \quad h^1 = \sqrt{2}e^{-4k/3}, \quad h^2 = \sqrt{2}e^{-4k/3}\left(\vec{l}^2 + \frac{1}{2}e^{\phi}e^{2k}\right), \quad h^{3,4,5} = -2e^{-4k/3}l^{3,4,5}, \tag{D.4}$$

where the physical scalars coincide with the Heterotic dilaton e^{ϕ}, the Kaluza-Klein scalar e^k of the dimensional reduction from six to five dimensions and the non-Abelian scalars l^α appearing in the reduction of the vectors.

We also define

$$h_I \equiv \frac{\partial}{\partial h^I}C_{JKL}h^J h^K h^L = 3C_{IJK}h^J h^K\,, \qquad h_I = a_{IJ}h^J\,. \tag{D.5}$$

The matrix a_{IJ} is the metric in the ambient space, and the σ-model metric g_{xy} in the action is given by the pullback of a_{IJ} on the hypersurface. They are both determined by the election of C_{IJK} as

$$a_{IJ} = -6C_{IJK}h^K + h_I h_J\,, \qquad g_{xy} = a_{IJ}\frac{\partial h^I}{\partial \phi^x}\frac{\partial h^J}{\partial \phi^y}\,. \tag{D.6}$$

We only consider symmetric scalar manifolds, for which

$$C^{IJK}h_I h_J h_K = 1\,, \qquad h^I = 3C^{IJK}h_J h_K\,, \qquad \text{with } C^{IJK} \equiv C_{IJK}\,. \tag{D.7}$$

The field strength and covariant derivatives are defined in the usual manner,

$$F^I{}_{\mu\nu} = 2\partial_{[\mu}A^I{}_{\nu]} + g f_{JK}{}^I A^J{}_\mu A^K{}_\nu\,, \qquad \mathfrak{D}_\mu\phi^x = \partial_\mu\phi^x + gA^\alpha{}_\mu k_\alpha{}^x\,. \tag{D.8}$$

We consider the gauge group $SU(2)$ with structure constants $f_{IJ}{}^K = \varepsilon_{IJ}{}^K$, with the understanding that they vanish whenever any of the indices takes values in the Abelian sector. The covariant derivatives of the functions of the scalars are

$$\mathfrak{D}_\mu h^I = \partial_\mu h^I + g f_{JK}{}^I A^J h^K\,, \qquad \mathfrak{D}_\mu h_I = \partial_\mu h_I + g f_{IJ}{}^K A^J h_K\,. \tag{D.9}$$

D.2 Timelike supersymmetric solutions with one isometry

Supersymmetric solutions of this theory admit a Killing vector of non-negative norm. In adapted coordinates the metric and vectors are independent of the time coordinate, see [163], and can be written as

$$ds^2 = f^2(dt+\omega)^2 - f^{-1}d\hat{s}^2\,, \tag{D.10}$$

$$A^I = h^I f(dt+\omega) + \hat{A}^I\,, \tag{D.11}$$

where $d\hat{s}^2$ is a hyperKähler metric. The equations of motion are reduced to the following BPS system of differential equations on this four-dimensional space,

$$\hat{F}^I = \star_4 \hat{F}^I\,, \tag{D.12}$$

$$\hat{\mathfrak{D}}^2 Z_I = 3\,C_{IJK}\star_4\left(\hat{F}^J\wedge\hat{F}^K\right)\,, \tag{D.13}$$

$$d\omega + \star_4 d\omega = Z_I\hat{F}^I\,, \tag{D.14}$$

where $\star_4$ is the Hodge dual in the hyperKähler space, $\hat{F}^I$ is the field strength of the vector $\hat{A}^I$ and $\hat{\mathfrak{D}}$ is the covariant derivative with connection $\hat{A}$. In these equations we have introduced the functions $Z_I \equiv h_I/f$, so the metric function f is conveniently obtained as

$$f^{-3} = C^{IJK}Z_I Z_J Z_K\,, \tag{D.15}$$

by virtue of equation (D.7).

The system of BPS equations is non-linear due to the presence of non-Abelian fields, although the three equations could be solved independently in the order they have been presented. However, it is possible to further simplify the system under the assumption that the solution admits a spacelike isometry [152], in a way that reduces the problem to a set of equations in three dimensional Euclidean space. First, consider the following decompositions

$$d\hat{s}^2 = H^{-1}(d\psi+\chi)^2 + H dx^s dx^s\,, \tag{D.16}$$

$$\hat{A}^I = -H^{-1}\Phi^I(d\psi+\chi) + \breve{A}^I\,, \tag{D.17}$$

$$Z_I = L_I + 3C_{IJK}\Phi^J\Phi^K H^{-1}\,, \tag{D.18}$$

$$\omega = \omega_5(d\psi+\chi) + \breve{\omega}\,, \tag{D.19}$$

where ψ is the coordinate adapted to the spatial isometry. When these expressions are substituted in the BPS system of equations, we obtain the following simplified system of differential equations and algebraic relations

$$\star_3 dH = d\chi\,, \tag{D.20}$$

$$\star_3 \breve{\mathfrak{D}}\Phi^I = \breve{F}^I\,, \tag{D.21}$$

$$\breve{\mathfrak{D}}^2 L_I = g^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J \Phi^K L_M\,, \tag{D.22}$$

$$\star_3 d\breve{\omega} = HdM - MdH + \frac{1}{2}(\Phi^I \breve{\mathfrak{D}} L_I - L_I \breve{\mathfrak{D}}\Phi^I)\,, \tag{D.23}$$

$$\omega_5 = M + \frac{1}{2} L_I \Phi^I H^{-1} + C_{IJK}\Phi^I \Phi^J \Phi^K H^{-2}\,, \tag{D.24}$$

where $\breve{F}^I$ is the field strength of the vector $\breve{A}^I$ and $\breve{\mathfrak{D}}$ is the covariant derivative with connection $\breve{A}$.

The Abelian functions H, M, Φ^i and L_i are just harmonic functions in $\mathbb{E}^3$, and the 1-forms χ and $\breve{A}^i$ are completely determined from those functions. In the non-Abelian sector, equations (D.21) are non-linear and must be solved simultaneously for Φ^α and $\breve{A}^\alpha$, which make their presence in (D.22). The construction of non-Abelian microstate geometries requires finding a multicenter solution to these equations. The only known example of such solution is the multicolored dyon, found by one of us in [157], which we review in appendix D.3. Last but not least, we have the differential equation (D.23), whose integrability condition will give rise to the bubble equations.

Notice that these solutions are left invariant under the following transformations of the harmonic functions generated by the parameters g^i, whose sole effect is a gauge transformation of the Abelian vectors,

$$\begin{aligned}
&H' = H, \qquad \Phi^{i\,\prime} = \Phi^i + g^i H,\\
&L_i' = L_i - 6C_{ijk}\, g^j \Phi^k - 3C_{ijk}\, g^j g^k H,\\
&M' = M - \frac{1}{2} g^i L_i + \frac{3}{2} C_{ijk}\, g^i g^j \Phi^k + \frac{1}{2} C_{ijk}\, g^i g^j g^k H,
\end{aligned} \tag{D.25}$$

D.3 Microstate geometries in a nutshell

The previous section describes a procedure to find supersymmetric solutions of SEYM theories in terms of a set of three-dimensional *seed functions*: H, M, Φ^I and L_I. As we already commented, those in the Abelian sector are just multicenter harmonic functions with poles in a collection of n points located at (x_a^1, x_a^2, x_a^3) called centers,

$$H = \sum_{a=1}^{n} \frac{q_a}{r_a}\,, \qquad \Phi^i = \sum_{a=1}^{n} \frac{k_a^i}{r_a}\,, \qquad L_i = l_0^i + \sum_{a=1}^{n} \frac{l_a^i}{r_a}\,, \qquad M = m_0 + \sum_{a=1}^{n} \frac{m_a}{r_a}\,, \tag{D.26}$$

with $r_a = |\vec{x} - \vec{x}_a|$. Notice that these functions solve the equations (D.20)-(D.22) in the Abelian sector everywhere except at the locations of the poles. This is the reason why the bubble equations are needed.

In the non-Abelian sector, the Bogomoln'yi equations (D.21) can be readily solved by making use of the following ansatz

$$\Phi^\alpha = -\frac{1}{gP}\frac{\partial P}{\partial x^s}\delta^\alpha_s\,, \qquad \breve{A}^\alpha{}_\mu = -\frac{1}{gP}\frac{\partial P}{\partial x^s}\varepsilon^\alpha{}_{\mu s}\,. \tag{D.27}$$

Obtaining the condition for the function

$$\frac{1}{P}\nabla^2 P = 0\,, \tag{D.28}$$

which is solved again by a harmonic function P, even at the locations of the poles. Equations (D.22) for the non-Abelian sector can also be solved using the ansatz

$$L_\alpha = -\frac{1}{gP}\frac{\partial Q}{\partial x^s}\delta^s_\alpha\,, \tag{D.29}$$

which yields

$$\frac{\partial}{\partial x^s}\left(\frac{1}{P^2}\nabla^2 Q\right) = 0\,. \tag{D.30}$$

This condition is solved everywhere if Q is a harmonic function with the poles at the same locations than P. Therefore, the complete non-Abelian multicolored dyon is specified by two harmonic functions[1]

$$P = 1 + \sum_{a=1}^{n}\frac{\lambda_a}{r_a}\,, \qquad Q = \sum_{a=1}^{n}\frac{\sigma_a\lambda_a}{r_a}\,, \qquad \text{with } \lambda_a > 0\,. \tag{D.31}$$

In order to avoid the presence of event horizons or singularities at the centers, it is necessary to fix the value of some of the parameters,

$$l^0_a = -\frac{1}{q_a}\left(k^1_a k^2_a - \frac{1}{2g^2}\right), \qquad l^{1,2}_a = -\frac{k^0_a k^{2,1}_a}{q_a}\,, \qquad \sigma_a = \frac{k^0_a}{q_a}\,, \qquad m_a = \frac{k^0_a}{2q^2_a}\left(k^1_a k^2_a - \frac{1}{2g^2}\right). \tag{D.32}$$

On its side, asymptotic flatness requires

$$l^0_0 l^1_0 l^2_0 = 1\,, \qquad m_0 = -\frac{1}{2}\sum_{i,a} l^i_0 k^i_a\,. \tag{D.33}$$

The integrability condition of equation (D.23) gives the set of constraints known as bubble equations

$$\sum_{b\neq a}\frac{q_a q_b}{r_{ab}}\Pi^0_{ab}\left(\Pi^1_{ab}\Pi^2_{ab} - \frac{1}{2g^2}\mathbb{T}_{ab}\right) = \sum_{b,i} q_a q_b l^i_0 \Pi^i_{ab}\,. \tag{D.34}$$

where

[1] We assume that the constant term of the function P is non-vanishing, in which case it can always be taken to be 1. From the Bogomol'nyi equation perspective, truncating this constant is equivalent to adding a unit charge monopole at infinity. We leave the study of this possibility for future works.

$$\Pi^i_{ab} \equiv \frac{1}{4\pi}\int_{\Delta_{ab}} F^i = \left(\frac{k^i_b}{q_b} - \frac{k^i_a}{q_a}\right)\,, \qquad \mathbb{T}_{ab} \equiv \left(\frac{1}{q_a^2} + \frac{1}{q_b^2}\right)\,. \tag{D.35}$$

The term $\mathbb{T}_{ab}$ appears due to the presence of non-Abelian fields, that alter the value of the parameters l^0_a when compared with purely Abelian configurations. The i-fluxes threading the non-contractible 2-cycles Δ_{ab} defined by any path connecting two centers $\vec{x}_a$ and $\vec{x}_b$ behave effectively as sources of electric charge and mass. When all the bubble equations are satisfied, the solutions are regular at the centers and do not present Dirac-Misner string singularities, which otherwise could only be removed by compactifying the time direction.

The last restriction for the construction of physically sensible microstate geometries comes from demanding that the solution does not contain closed timelike curves (CTCs). The metric can be rewritten in the following manner

$$ds^2 = f^2 dt^2 + 2f^2 dt\omega - \frac{\mathcal{I}_4}{f^{-2}H^2}\left(d\psi + \chi - \frac{\omega_5 H^2}{\mathcal{I}_4}\breve{\omega}\right)^2 - f^{-1}H\left(d\vec{x}\cdot d\vec{x} - \frac{\breve{\omega}^2}{\mathcal{I}_4}\right)\,, \tag{D.36}$$

where $\mathcal{I}_4$ is the *quartic invariant*, defined as

$$\mathcal{I}_4 \equiv f^{-3}H - \omega_5^2 H^2\,. \tag{D.37}$$

Therefore, a general restriction that must be satisfied in order to avoid CTCs is the positivity of the quartic invariant

$$\mathcal{I}_4 \geq 0\,. \tag{D.38}$$

When studying its positivity numerically, it is sometimes useful to employ the expression for the quartic invariant in terms of the seed functions directly

$$\begin{aligned}\mathcal{I} \;=\; & -M^2H^2 - \tfrac{1}{4}\left(\Phi^I L_I\right)^2 - 2MC_{IJK}\Phi^I\Phi^J\Phi^K - MHL_I\Phi^I \\ & + HC^{IJK}L_IL_JL_K + 9C^{IJK}C_{KLM}L_IL_J\Phi^L\Phi^M \geq 0\,.\end{aligned} \tag{D.39}$$

D.4 Asymptotic charges

The electric asymptotic charge of each Abelian vector can be readily obtained from the asymptotic expansion of the associated warp factor, see [210],

$$Z_{i,\infty} = l^i_0 + \frac{\mathcal{Q}_i}{r} + \mathcal{O}(r^{-2})\,. \tag{D.40}$$

This can be easily seen from the fact that, asymptotically, $A^i{}_{t,\infty} \sim Z_i^{-1}$. The electric charges are

$$\mathcal{Q}_0 = -\sum_{a,b,c} q_a q_b q_c \Pi^1_{ab}\Pi^2_{ac} + \frac{1}{2g^2}\sum_a \frac{1}{q_a}\,, \tag{D.41}$$

$$\mathcal{Q}_1 = -\sum_{a,b,c} q_a q_b q_c \Pi^0_{ab}\Pi^2_{ac}\,, \tag{D.42}$$

$$\mathcal{Q}_2 = -\sum_{a,b,c} q_a q_b q_c \Pi^0_{ab}\Pi^1_{ac}\,. \tag{D.43}$$

In a similar manner, the two angular momenta can be read from the term in $d\psi$ at the asymptotic expansion of ω [210], whose contribution comes entirely from the function ω_5, obtaining

$$J_R = -\frac{1}{2}\sum_{a,b,c,d} q_a q_b q_c q_d \Pi^0_{ab}\Pi^1_{ac}\Pi^2_{ad} + \frac{1}{4g^2}\sum_{a,b}\frac{q_b\Pi^0_{ab}}{q_a}\,, \tag{D.44}$$

$$\vec{J}_L = -\frac{1}{4}\sum_{\substack{a,b\\ a\neq b}} q_a q_b \Pi^0_{ab}\left(\Pi^1_{ab}\Pi^2_{ab} - \frac{1}{2g^2}\mathbb{T}_{ab}\right)\frac{\vec{x}_a - \vec{x}_b}{|\vec{x}_a - \vec{x}_b|}\,. \tag{D.45}$$

such that

$$\omega_{5,\infty} = \frac{1}{r}\left(J_R + J_L\cos\theta_L\right) + \mathcal{O}(r^{-2})\,, \tag{D.46}$$

where θ_L is the angle measured with respect to $\vec{J}_L$, and J_L is the norm of this vector.

The ADM mass is just

$$\mathcal{M} = \frac{\pi}{G_N^{(5)}}\left(\frac{\mathcal{Q}_0}{l_0^0} + \frac{\mathcal{Q}_1}{l_0^1} + \frac{\mathcal{Q}_2}{l_0^2}\right)\,. \tag{D.47}$$

If all the centers were placed at the same location, the solution would describe a black hole with the same asymptotic charges (with $J_L = 0$) and an event horizon whose area would be given by

$$A_H = 2\pi^2\sqrt{\mathcal{Q}_0\mathcal{Q}_1\mathcal{Q}_2 - J_R^2}\,. \tag{D.48}$$

It is convenient to define the *entropy parameter* $\mathcal{H}$ of a microstate geometry as

$$\mathcal{H} \equiv 1 - \frac{J_R^2}{\mathcal{Q}_0\mathcal{Q}_1\mathcal{Q}_2}\,, \tag{D.49}$$

whose value indicates how far from maximal rotation the represented black hole is.

E

Truncation of heterotic supergravity on a 5-torus

The aim of this appendix is to present a consistent truncation of heterotic supergravity[1] compactified on a five-dimensional torus that preserves an SU(2) triplet of vector fields. The resulting five-dimensional theory turns out to be a particular model of gauged $\mathcal{N} = 1, d = 5$ supergravity to which one can apply the solution-generating technique explained in Chapter 2.

E.1 Dimensional reduction and truncation of heterotic supergravity

The action of ten-dimensional $\mathcal{N} = 1$ supergravity coupled to a triplet of SU(2) vector fields is given by

$$\hat{S} = \frac{g_s^2}{16\pi G_{\rm N}^{(10)}} \int d^{10}x \sqrt{|\hat{g}|}\, e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 + \frac{1}{2\cdot 3!}\hat{H}^2 - \frac{\alpha'}{8}\hat{F}^A\hat{F}^A\right], \tag{E.1}$$

where the field strengths are defined as

$$\hat{F}^A = d\hat{A}^A + \tfrac{1}{2}\epsilon^{ABC}\hat{A}^B\wedge\hat{A}^C\,, \tag{E.2}$$

$$\hat{H} = d\hat{B} + \frac{\alpha'}{4}\omega^{\rm YM}\,, \tag{E.3}$$

and $\omega^{\rm YM}$ is the Chern-Simons 3-form

$$\omega^{\rm YM} \equiv \hat{F}^A\wedge\hat{A}^A - \tfrac{1}{3!}\epsilon^{ABC}\hat{A}^A\wedge\hat{A}^B\wedge\hat{A}^C\,, \qquad d\omega^{\rm YM} = \hat{F}^A\wedge\hat{F}^A\,. \tag{E.4}$$

In the above expressions, the Regge slope α' is related to the string length ℓ_s by $\alpha' = \ell_s^2$, and the string coupling g_s is the exponential of the vacuum expectation value of the dilaton, $g_s = e^{\phi_\infty}$. The ten-dimensional Newton's constant is given in terms of the string moduli by the following expression:

$$G_{\rm N}^{(10)} = 8\pi^6 g_s^2 \alpha'^4\,. \tag{E.5}$$

[1] Namely, ten-dimensional $\mathcal{N} = 1$ supergravity coupled to vector multiplets

Let us compactify the action (E.1) on a four-dimensional torus $\mathbb{T}^4$ (that will be later parametrized by the coordinates $z^\alpha \sim z^\alpha + 2\pi\ell_s$) and truncate all the Kaluza-Klein degrees of freedom (scalars and vectors). The six-dimensional action only differs from the ten-dimensional one (E.1) by a global factor given by the volume of the four-dimensional torus $\mathbb{T}^4$ which is reabsorbed into the six-dimensional Newton's constant:

$$\hat{S} = \frac{g_s^2}{16\pi G_{\mathrm{N}}^{(6)}} \int d^6x \sqrt{|g|}\, e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 + \frac{1}{2\cdot 3!}\hat{H}^2 - \frac{\alpha'}{8}\hat{F}^A\hat{F}^A\right] . \tag{E.6}$$

where

$$G_{\mathrm{N}}^{(6)} = \frac{G_{\mathrm{N}}^{(10)}}{(2\pi\ell_s)^4} = \frac{\pi^2}{2} g_s^2 \alpha'^2 . \tag{E.7}$$

The metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ that appears in (E.6) is in the string frame. The relation between this and the metric in the modified Einstein frame $\hat{g}_{\mathrm{E}\,\hat{\mu}\hat{\nu}}$ is

$$\hat{g}_{\hat{\mu}\hat{\nu}} = g_s^{-1} e^{\hat{\phi}} \hat{g}_{\mathrm{E}\,\hat{\mu}\hat{\nu}} . \tag{E.8}$$

Rewriting the action in terms of $\hat{g}_{\mathrm{E}\,\hat{\mu}\hat{\nu}}$, one finds

$$\hat{S} = \frac{(2\pi\ell_s)^4}{16\pi G_{\mathrm{N}}^{(10)}} \int d^6x \sqrt{|g|} \left[\hat{R}_{\mathrm{E}} + (\partial\hat{\phi})^2 + \frac{1}{2\cdot 3!} g_s^2 e^{-2\hat{\phi}}\hat{H}^2 - \frac{\alpha' g_s}{8} e^{-\hat{\phi}}\hat{F}^A\hat{F}^A\right] , \tag{E.9}$$

which is nothing but the action of the theory of gauged $\mathcal{N} = (2,0), d = 6$ supergravity [177]. The relations between the fields in that reference (tilded ones) and ours is

$$\hat{\phi} = -\tilde{\varphi}/\sqrt{2} , \qquad g_s\hat{H}/2 = \tilde{H} , \qquad \sqrt{g_s\alpha'}\hat{F}^A = 2\sqrt{2}\tilde{F}^A . \tag{E.10}$$

The last relation leads to the introduction of the six-dimensional Yang-Mills coupling constant

$$g_6 = \frac{2\sqrt{2}}{\sqrt{g_s\alpha'}} . \tag{E.11}$$

As shown in [177], the compactification of this six-dimensional theory on a circle $\mathbb{S}^1_z$ yields the SU(2)-gauged ST[2, 6] model of $\mathcal{N} = 1, d = 5$ supergravity which will be discussed in detail in the next section. The five-dimensional Newton and Yang-Mills constants are given by

$$G_{\mathrm{N}}^{(5)} = \frac{G_{\mathrm{N}}^{(10)}}{(2\pi)^5\ell_s^4 R_z} = \frac{\pi g_s^2\ell_s^4}{4R_z} , \quad \text{and} \quad g = \frac{g_6 k_\infty^{1/3}}{2\sqrt{3}} = \frac{\sqrt{2}k_\infty^{1/3}}{\sqrt{3g_s\ell_s^2}} , \tag{E.12}$$

where R_z is the radius of the circle $\mathbb{S}^1_z$, parametrized by the coordinate $z \sim z + 2\pi R_z$.

The relation between the six- and five-dimensional fields is given in [177]. We can use it together with (E.10) to find the relation between the ten- and five-dimensional fields, which is all we need here. The latter are: the metric $g_{\mu\nu}$, five scalars ϕ, k, ℓ^A and six vectors $A^{+}_{\mu}, A^{-}{}_{\mu}, A^{0}{}_{\mu}, A^{A}$. Their relation with the ten-dimensional heterotic supergravity fields is:

$$
\begin{aligned}
d\hat{s}^2 &= e^{\phi-\phi_\infty}\left[(k/k_\infty)^{-2/3}ds^2 - (k/k_\infty)^2\,\mathcal{A}^2\right] - dz^\alpha dz^\alpha\,,\\
\hat{\phi} &= \phi\,,\\
\hat{A}^A &= \frac{\sqrt{2}\,k_\infty^{1/3}}{\sqrt{3}\,g_s\alpha'}\left(A^A - \frac{2\sqrt{3}}{k_\infty^{4/3}}\,\ell^A\mathcal{A}\right)\,,\\
\hat{H} &= -\frac{k_\infty^{2/3}}{g_s\sqrt{3}}e^{2\phi}k^{-4/3}\star_5 F^0 + \frac{k_\infty^{-2/3}}{g_s\sqrt{3}}\mathcal{A}\wedge\mathcal{F}\,,
\end{aligned}
\tag{E.13}
$$

where we have introduced the auxiliary fields

$$
\begin{aligned}
\mathcal{A} &\equiv dz + \frac{k_\infty^{4/3}}{\sqrt{12}}A^+\,,\\
\mathcal{F} &\equiv F^- + \ell^2 F^+ - 2\ell^A F^A\,,
\end{aligned}
\tag{E.14}
$$

and $\star_5$ is the Hodge star operator associated to the five-dimensional metric.

E.2 The 5-dimensional theory as a model of $\mathcal{N}=1, d=5$ SEYM

Let us see how the SU(2)-gauged ST[2, 6] model obtained in the previous section fits into the general description of the $\mathcal{N}=1, d=5$ SEYM theories given in Chapter 2. As we have seen, the different models are characterized by the number of vector multiplets n_v, the symmetric C_{IJK} tensor and the structure constants of the gauge group $f_{JK}{}^{I}$.

The SU(2)-gauged ST[2, 6] model has $n_v = 5$ vector multiplets — which are labeled by an index $x = 1, \ldots, n_v$ — each of which contains a scalar ϕ^x, a vector $A^x{}_\mu$ and a gaugino λ^{ix}. In addition to these, the supergravity multiplet contains the fünfbein $e^a{}_\mu$, the graviphoton $A^0{}_\mu$ and the gravitino ψ^i_μ. It is convenient to introduce the indices $I, J = 0, 1, \ldots, n_v$ to label all the vectors of the theory $A^I{}_\mu$ in a unified fashion. Moreover, we introduce $n_v + 1$ functions of the physical scalars $h^I = h^I(\phi)$ satisfying the cubic constraint

$$
C_{IJK}h^I h^J h^K = 1\,, \qquad \Rightarrow \qquad h^I h_I = 1\,,
\tag{E.15}
$$

where we have defined

$$
h_I = C_{IJK}h^J h^K\,.
\tag{E.16}
$$

For the SU(2)-gauged ST[2, 6] model, the non-vanishing components of the C_{IJK} tensor are

$$C_{0xy} = \tfrac{1}{6}\eta_{xy}\,, \qquad \text{where} \quad \eta = \text{diag}(+ - \cdots -)\,. \tag{E.17}$$

The real special manifold parametrized by the physical scalars can be identified with the Riemannian symmetric space

$$\text{SO}(1,1) \times \frac{\text{SO}(1,4)}{\text{SO}(4)}\,, \tag{E.18}$$

and the special parametrization used in the previous section corresponds to

$$h^0 = e^{-\phi}k^{2/3}\,, \quad h^{1,2} = k^{-4/3}\left[1 \pm (\ell^2 + \tfrac{1}{2}e^{\phi}k^2)\right]\,, \quad h^{3,4,5} = -2k^{-4/3}\ell^{3,4,5}\,. \tag{E.19}$$

As we have seen, ϕ is the dilaton field, k is the KK scalar of the dimensional reduction from 6 to 5 dimensions and the triplet $\ell^{3,4,5}$ correspond to the z-component of the gauge fields.

The SU(2) group acts in the adjoint on the coordinates $x = 3, 4, 5$ which we are going to denote by $A, B, \ldots$ and this is the sector that is gauged without the use of Fayet-Iliopoulos terms. The structure constants are $f_{AB}{}^{C} = +\varepsilon_{AB}{}^{C}$.[2] We will denote with $a, b, \ldots = 1, 2$ the ungauged directions. Observe that this sector of the theory corresponds to the so-called STU model: in absence of the h^As we can make the linear redefinitions

$$h^{1\prime} \equiv \tfrac{1}{\sqrt{2}}(h^1 + h^2)\,, \qquad h^{2\prime} \equiv \tfrac{1}{\sqrt{2}}(h^1 - h^2)\,, \;\; \Rightarrow \;\; C_{abc}h^a h^b h^c = h^0 h^{1\prime} h^{2\prime}\,. \tag{E.20}$$

Thus, our model can be also understood as the STU model with an additional SU(2) triplet of vector multiplets. The Kaluza-Klein vector A^+ and the vector A^- correspond to the following linear combinations of the Abelian vectors:

$$A^{\pm} = A^1 \pm A^2\,. \tag{E.21}$$

The bosonic part of the action for this model is given by

$$\begin{aligned} S \;=\; & \int d^5x\sqrt{g}\,\Big\{ R + \partial_\mu\phi\partial^\mu\phi + \tfrac{4}{3}\partial_\mu \log k \partial^\mu \log k + 2e^{-\phi}k^{-2}\mathfrak{D}_\mu \ell^A \mathfrak{D}^\mu \ell^A \\ & -\tfrac{1}{12}e^{2\phi}k^{-4/3}F^0 \cdot F^0 + \tfrac{1}{12}\left(\eta_{xy}e^{-\phi}k^{2/3} - 9h_x h_y\right) F^x \cdot F^y \\ & +\tfrac{1}{24\sqrt{3}}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} A^0{}_\mu \eta_{xy} F^x{}_{\nu\rho} F^y{}_{\sigma\alpha} \Big\}\,, \end{aligned} \tag{E.22}$$

where

[2]The lower and upper indices are identical since they will always be raised and lowered with δ_{AB}.

$$\mathfrak{D}_\mu \ell^A = \partial_\mu \ell^A + g\,\varepsilon^A{}_{BC} A^B{}_\mu \ell^C\,, \tag{E.23}$$

$$F^{0,a}{}_{\mu\nu} = 2\partial_{[\mu} A^{0,a}{}_{\nu]}\,, \tag{E.24}$$

$$F^A{}_{\mu\nu} = 2\partial_{[\mu} A^A{}_{\nu]} + g\,\varepsilon^A{}_{BC} A^B{}_\mu A^C{}_\nu\,. \tag{E.25}$$

E.3 Uplift of the timelike supersymmetric solutions

Let us close this appendix with the uplift to ten dimensions of the timelike supersymmetric solutions of the SU(2)-gauged ST[2,6] model. The general form of the latter is — see Chapter 2 for further details —

$$ds^2 = f^2(dt+\omega)^2 - f^{-1}\,d\sigma^2\,, \tag{E.26}$$

$$A^I = -\sqrt{3} h^I f\,(dt+\omega) + \hat{A}^I\,, \tag{E.27}$$

$$\phi^x = \frac{h_x}{h_0}\,. \tag{E.28}$$

where $d\sigma^2 = h_{\underline{mn}}\,dx^m dx^n$ is the metric of the hyper-Kähler base space where the hatted vector fields $\hat{A}^I$, the 1-form ω and the remaining functions (f, h^I and h_I) are defined. Since the model under consideration is symmetric, we have that

$$h^I = 27 C^{IJK} h_J h_K\,, \qquad \text{and} \qquad C_{IJK} = C^{IJK}\,. \tag{E.29}$$

Defining $Z_I \equiv h_I/f$ and using the cubic constraint (E.15), we find that the metric function f is given by

$$f^{-3} = \frac{27}{2} Z_0 \left(Z_+ Z_- - Z_A Z_A\right)\,, \tag{E.30}$$

where

$$Z_\pm = Z_1 \pm Z_2\,. \tag{E.31}$$

The building blocks $(Z_I, \hat{A}^I, \omega)$ in terms of which the five-dimensional solutions are constructed satisfy the following differential equations:

$$\hat{F}^I = \star_\sigma \hat{F}^I\,, \tag{E.32}$$

$$\hat{\mathfrak{D}} \star_\sigma \hat{\mathfrak{D}} Z_I = \frac{1}{3} C_{IJK}\,\hat{F}^J \wedge \hat{F}^K\,, \tag{E.33}$$

$$d\omega + \star_\sigma d\omega = \sqrt{3} Z_I \hat{F}^I\,, \tag{E.34}$$

where $\hat{\mathfrak{D}}$ is the gauge-covariant derivative associated to the hatted connection $\hat{A}^I$ and $\star_\sigma$ is the Hodge dual associated to the hyper-Kähler metric.

The expressions of the physical scalars e^ϕ, k and ℓ^A in terms of the Z_I functions are given by

$$e^{2\phi} = \frac{2Z_0}{Z_-}\,, \qquad k = \left(\frac{2\tilde{Z}_+^2}{Z_0 Z_-}\right)^{1/4}\,, \qquad \ell^A = \frac{Z_A}{Z_-}\,, \tag{E.35}$$

where

$$\tilde{Z}_+ = Z_+ - \frac{Z_A Z_A}{Z_-}\,. \tag{E.36}$$

Assuming that the asymptotic behaviour of the metric function is $f_\infty \equiv \lim_{|x|\to\infty} f = 1$, we find the following relations between the asymptotic values of the Z_I functions and those of the physical scalars:

$$Z_0^\infty = \frac{1}{3}e^{\phi_\infty}k_\infty^{-2/3}\,, \quad \tilde{Z}_+^\infty = \frac{1}{3}k_\infty^{4/3}\,, \quad Z_-^\infty = \frac{2}{3}e^{-\phi_\infty}k_\infty^{-2/3}\,, \quad Z_A^\infty = \frac{2}{3}e^{-\phi_\infty}k_\infty^{-2/3}\ell_\infty^A\,. \tag{E.37}$$

Using these expressions and the differential equations satisfied by the building blocks (E.32)-(E.34), we obtain that the ten-dimensional uplift of the timelike supersymmetric solutions is the following:

$$d\hat{s}^2 = \frac{2}{\mathcal{Z}_-}\hat{\mathcal{A}}\left(dt + \omega - \frac{\tilde{\mathcal{Z}}_+}{2}\hat{\mathcal{A}}\right) - \mathcal{Z}_0\, d\sigma^2 - dz^\alpha dz^\alpha\,, \tag{E.38}$$

$$e^{2\hat{\phi}} = g_s^2\frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \tag{E.39}$$

$$\hat{A}^{\hat{A}} = g\left(\hat{A}^A + \frac{2\sqrt{3}\ell_\infty^A}{k_\infty^{4/3}}\frac{\mathcal{Z}_A}{\mathcal{Z}_-}\hat{\mathcal{A}}\right)\,. \tag{E.40}$$

$$\begin{aligned}\hat{H} = {}& \star_\sigma d\mathcal{Z}_0 + \frac{k_\infty^{4/3}}{2\sqrt{3}}\mathcal{Z}_-^{-1}(dt+\omega)\wedge\hat{F}^+ + \hat{\mathcal{A}}\wedge\Big[(dt+\omega)\wedge d\mathcal{Z}_-^{-1} \\ & + \frac{\star_\sigma d\omega}{\mathcal{Z}_-} - \frac{k_\infty^{4/3}}{2\sqrt{3}}\frac{\tilde{\mathcal{Z}}_+}{\mathcal{Z}_-}\hat{F}^+ - \frac{1}{\sqrt{3}}g_s k_\infty^{-2/3}\frac{\mathcal{Z}_0}{\mathcal{Z}_-}\hat{F}^0\Big]\,.\end{aligned} \tag{E.41}$$

where

$$\hat{\mathcal{A}} \equiv dz + \frac{k_\infty^{4/3}}{2\sqrt{3}}\hat{A}^+\,, \tag{E.42}$$

and

$$\mathcal{Z}_0 \equiv \frac{Z_0}{Z_0^\infty}\,, \quad \tilde{\mathcal{Z}}_+ \equiv \frac{\tilde{Z}_+}{\tilde{Z}_+^\infty}\,, \quad \mathcal{Z}_- \equiv \frac{Z_-}{Z_-^\infty}\,, \quad \mathcal{Z}_A \equiv \frac{Z_A}{Z_A^\infty}\,. \tag{E.43}$$

F

Connections and curvatures

F.1 F1-P-S5-KK system

Let us compute the Levi-Civita and torsionful spin connections for the ansatz given in Section 4.3. The choice for the zehnbein is

$$e^{+} = \frac{du}{\mathcal{Z}_{-}}\,, \quad e^{-} = dt - \tfrac{1}{2}\mathcal{Z}_{+}du\,, \quad e^{m} = \mathcal{Z}_0^{1/2}\, v^{m}\,, \quad e^{\alpha} = dz^{\alpha}\,, \tag{F.1}$$

where $v^m = v^m{}_{\underline{n}} dx^n$ is a vierbein of the four-dimensional hyperKähler space defined in (4.86). The inverse basis is

$$e_{+} = \mathcal{Z}_{-}(\partial_u + \tfrac{1}{2}\mathcal{Z}_{+}\partial_t)\,, \quad e_{-} = \partial_t\,, \quad e_m = \mathcal{Z}_0^{-1/2}\partial_m\,, \quad e_{\alpha} = \partial_{\alpha}\,, \tag{F.2}$$

where $\partial_m \equiv v_m{}^{\underline{n}}\partial_{\underline{n}}$ is the inverse basis in the hyperKähler space and any other m, n index will be a flat index in the hyperKähler space and will be raised and lowered with $+\delta_{mn}$.

Using the structure equation $de^a = \omega^a{}_b \wedge e^b$ we find that the non-vanishing components of the spin connection are given by

$$\begin{aligned} \omega_{-+m} &= \omega_{+-m} = \omega_{m+-} = \frac{1}{2\mathcal{Z}_0^{1/2}}\partial_m \log \mathcal{Z}_{-}\,, \quad \omega_{++m} = \frac{\mathcal{Z}_{-}}{2\mathcal{Z}_0^{1/2}}\partial_m \mathcal{Z}_{+}\,, \\ \omega_{mnp} &= \mathcal{Z}_0^{-1/2}\left[\varpi_{mnp} + \tfrac{1}{2}\mathbb{M}_{mqnp}\partial_q \log \mathcal{Z}_0\right]\,, \end{aligned} \tag{F.3}$$

where ϖ_{mnp} are the components of the spin connection on the hyperKähler space defined with the convention of (4.54).[1] We assume they satisfy the properties (4.47)-(4.51) with the conventions we use.

In order to compute the components of the torsionful spin connections, we need the components of the 3-form field strength. We find from (6.3) that in the above zehnbein basis they are given by

$$H_{m+-} = -\mathcal{Z}_0^{-1/2}\partial_m \log \mathcal{Z}_{-}\,, \qquad H_{mnp} = \mathcal{Z}_0^{-1/2}\,\varepsilon_{mnpq}\partial_q \log \mathcal{Z}_0\,. \tag{F.4}$$

Then, the non-vanishing flat components of the torsionful spin connection $\Omega_{(-)abc} \equiv \omega_{abc} - \frac{1}{2}H_{abc}$ are

[1] These four-dimensional tangent-space indices are raised and lowered with $+\delta_{mn}$ and there is no difference between them, beyond an esthetic one.

$$\Omega_{(-)+-m} \;=\; \Omega_{(-)m+-} = \mathcal{Z}_0^{-1/2}\partial_m \log \mathcal{Z}_- \,, \qquad \Omega_{(-)++m} \;=\; \tfrac{1}{2}\mathcal{Z}_-\mathcal{Z}_0^{-1/2}\partial_m \mathcal{Z}_+ \,,$$
$$\Omega_{(-)mnp} \;=\; \mathcal{Z}_0^{-1/2}\left[\varpi_{mnp} + \mathbb{M}^-_{mqnp}\partial_q \log \mathcal{Z}_0\right] \,, \tag{F.5}$$

and those of $\Omega_{(+)abc} \equiv \omega_{abc} + \frac{1}{2}H_{3abc}$ are given by

$$\Omega_{(+)-+m} \;=\; \mathcal{Z}_0^{-1/2}\partial_m \log \mathcal{Z}_- \,, \qquad \Omega_{(+)++m} \;=\; \tfrac{1}{2}\mathcal{Z}_-\mathcal{Z}_0^{-1/2}\partial_m \mathcal{Z}_+ \,,$$
$$\Omega_{(+)mnp} \;=\; \mathcal{Z}_0^{-1/2}\left[\varpi_{mnp} + \mathbb{M}^+_{mqnp}\partial_q \log \mathcal{Z}_0\right] \,, \tag{F.6}$$

where the 4×4 matrices $\mathbb{M}^\pm_{np}$ are defined in (4.40).

The Lorentz-Chern-Simons 3-form $\omega^{\rm L}_{(-)}$ reduces to the Chern-Simons 3-form of the connection $\Omega_{(-)mn}$

$$\begin{aligned}\omega^{\rm L}_{(-)} &\equiv d\Omega_{(-)}{}^a{}_b \wedge \Omega_{(-)}{}^b{}_a - \tfrac{2}{3}\Omega_{(-)}{}^a{}_b \wedge \Omega_{(-)}{}^b{}_c \wedge \Omega_{(-)}{}^c{}_a \\ &= d\Omega_{(-)mn} \wedge \Omega_{(-)nm} + \tfrac{2}{3}\Omega_{(-)mn} \wedge \Omega_{(-)np} \wedge \Omega_{(-)pm} \,,\end{aligned} \tag{F.7}$$

which, in its turn, is just the sum of the Chern-Simons 3-forms of the self-dual and anti-self-dual pieces of $\Omega_{(-)mn}$, namley the self-dual spin connection of the hyperKähler manifold ϖ_{mnp} and the anti-self-dual 1-form $\mathbb{M}^-_{mqnp}\partial_q \log \mathcal{Z}_0$. The latter has the form of the 't Hooft ansatz (4.53) discussed in Section 4.2 and, therefore, its Chern-Simons term takes the value computed in (4.69) with K replaced by $\mathcal{Z}_0$. The Chern-Simons 3-form of the spin connection of the hyperKähler manifold has to be computed case by case, except when it is a Gibbons-Hawking space. In that case, there is a general expression for it, see (4.81), and for its exterior derivative which are particularly convenient for us because the Bianchi identity of the 3-form field strength H_3 becomes a linear combination of Laplacians on the Gibbons-Hawking space that can be solved exactly.

Then, in these conditions, we have

$$\omega^{\rm L}_{(-)} = \star_\sigma d\left[(\partial \log H)^2 + (\partial \log \mathcal{Z}_0)^2\right] \,, \tag{F.8}$$

and

$$R_{(-)}{}^a{}_b \wedge R^b_{(-)a} = d\omega^{\rm L}_{(-)} = -\nabla^2_\sigma\left[(\partial \log H)^2 + (\partial \log \mathcal{Z}_0)^2\right] |v|\, d^4x \,. \tag{F.9}$$

Clearly, it would be extremely interesting to find other hyperKähler spaces with no triholomorphic isometry that still enjoy the same property. The Atiyah-Hitchin hyperKähler space [236], which has been considered before in the context of supergravity solutions in [237, 238], might provide an explicit example. We leave this study for future work. Interestingly, for arbitrary self-dual SU(2) instanton fields on $\mathbb{R}^4$, and not just for those in the 't Hooft ansatz, this "Laplacian property" was proven in [374] using the ADHM construction [375, 376]. Our results suggest that this property could also hold in hyperKähler backgrounds and, therefore, for the spin connections of the hyperKähler spaces themselves, as it happens in Gibbons-Hawking spaces.

F.2 Rotating F1-P system

Let us compute the connections and curvatures associated to the field configuration specified in Sec. 7.2. We work with the following zehnbein basis

$$e^+ = \mathcal{Z}_-^{-1}\, du\,, \quad e^- = dt - \frac{\mathcal{Z}_+}{2} du + \omega\,, \quad e^m = dx^m\,, \quad e^\alpha = dz^\alpha\,, \tag{F.10}$$

where $m = 1, \ldots d-1$ and $\alpha = 1, \ldots, 9-d$.

F.2.1 Levi-Civita spin connection

The non-vanishing components of the Levi-Civita spin connection, defined in our conventions as $de^a = \omega^a{}_b \wedge e^b$, are

$$\omega_{+-} = \frac{1}{2}\partial_m \log \mathcal{Z}_-\, e^m\,, \tag{F.11}$$

$$\omega_{+m} = \frac{1}{2}\partial_m \log \mathcal{Z}_-\, e^- - \frac{1}{2}\Omega_{mn}\, e^n + \frac{\mathcal{Z}_-}{2}\partial_m \mathcal{Z}_+\, e^+ \tag{F.12}$$

$$\omega_{-m} = \frac{1}{2}\partial_m \log \mathcal{Z}_-\, e^+ \tag{F.13}$$

$$\omega_{mn} = \frac{1}{2}\Omega_{mn}\, e^+\,. \tag{F.14}$$

The curvature 2-form, defined as $R_{ab} = d\omega_{ab} - \omega_{ac} \wedge \omega^c{}_b$, is given by

$$R_{+-} = e^+ \wedge e^- \left\{ \frac{(\partial \mathcal{Z}_-)^2}{4\mathcal{Z}_-^2} \right\} + e^n \wedge e^+ \left\{ -\frac{1}{4}\Omega_{np}\, \partial_p \log \mathcal{Z}_- \right\}\,, \tag{F.15}$$

$$\begin{aligned} R_{+m} = {} & e^n \wedge e^+ \left\{ \frac{\mathcal{Z}_-}{2}\partial_m \partial_n \mathcal{Z}_+ - \frac{1}{2}\partial_{(m|}\mathcal{Z}_+ \partial_{|n)}\mathcal{Z}_- + \frac{1}{4}\Omega_{np}\Omega_{pm} \right\} \\ & + e^n \wedge e^p \left\{ \partial_p \Omega_{mn} + \Omega_{(n|p}\partial_{|m)} \log \mathcal{Z}_- \right\} + e^+ \wedge e^- \left\{ \frac{1}{4}\Omega_{mp}\frac{\partial_p \mathcal{Z}_-}{\mathcal{Z}_-} \right\} \\ & + e^n \wedge e^- \left\{ \frac{1}{2}\partial_m \partial_n \log \mathcal{Z}_- - \frac{1}{4}\frac{\partial_m \mathcal{Z}_- \partial_n \mathcal{Z}_-}{\mathcal{Z}_-^2} \right\}\,, \end{aligned} \tag{F.16}$$

$$R_{-m} = e^n \wedge e^+ \left\{ \frac{1}{2}\frac{\partial_m \partial_n \mathcal{Z}_-}{\mathcal{Z}_-} - \frac{3}{4}\partial_m \log \mathcal{Z}_- \partial_n \log \mathcal{Z}_- \right\}\,, \tag{F.17}$$

$$R_{mn} = e^p \wedge e^+ \left\{ \frac{\mathcal{Z}_-}{2}\partial_p \left(\mathcal{Z}_-^{-1}\Omega_{mn} \right) - \frac{1}{2}\Omega_{[m|p}\partial_{|n]} \log \mathcal{Z}_- \right\}\,. \tag{F.18}$$

The Ricci tensor is

$$\begin{aligned}
R_{++} &= \frac{\mathcal{Z}_-}{2}\partial^2\mathcal{Z}_+ - \frac{1}{2}\partial_m\mathcal{Z}_+\partial_n\mathcal{Z}_- - \frac{1}{4}\Omega^2\,, && \text{(F.19)}\\
R_{+-} &= \frac{1}{2}\frac{\partial^2\mathcal{Z}_-}{\mathcal{Z}_-} - \frac{(\partial\mathcal{Z}_-)^2}{\mathcal{Z}_-^2}\,, && \text{(F.20)}\\
R_{+m} &= \frac{\mathcal{Z}_-}{2}\partial_n\left(\mathcal{Z}_-^{-1}\Omega_{mn}\right) - \frac{1}{2}\Omega_{mp}\,\partial_p\log\mathcal{Z}_-\,, && \text{(F.21)}\\
R_{mn} &= -\frac{\partial_m\partial_n\mathcal{Z}_-}{\mathcal{Z}_-} + \frac{3}{2}\partial_m\log\mathcal{Z}_-\partial_n\mathcal{Z}_-\,. && \text{(F.22)}
\end{aligned}$$

Finally, the Ricci scalar is

$$R = 2R_{+-} - R_{mm} = 2\frac{\partial^2\mathcal{Z}_-}{\mathcal{Z}_-} - \frac{7}{2}\left(\partial\log\mathcal{Z}_-\right)^2\,. \tag{F.23}$$

F.2.2 Torsionful spin connection $\Omega_{(-)ab}$

The non-vanishing components of the torsionful spin connection are

$$\begin{aligned}
\Omega_{(-)+-} &= \partial_m\log\mathcal{Z}_-\,e^m\,, && \text{(F.24)}\\
\Omega_{(-)+m} &= \frac{\mathcal{Z}_-}{2}\partial_m\mathcal{Z}_+\,e^+\,, && \text{(F.25)}\\
\Omega_{(-)-m} &= \partial_m\log\mathcal{Z}_-\,e^+\,, && \text{(F.26)}\\
\Omega_{(-)mn} &= \Omega_{mn}\,e^+\,. && \text{(F.27)}
\end{aligned}$$

For the curvature 2-form, we get

$$\begin{aligned}
R_{(-)+m} &= e^n\wedge e^+\left\{\frac{\mathcal{Z}_-}{2}\partial_m\partial_n\mathcal{Z}_+ - \frac{1}{2}\partial_m\mathcal{Z}_+\partial_n\mathcal{Z}_-\right\}\,, && \text{(F.28)}\\
R_{(-)-m} &= e^n\wedge e^+\left\{\frac{\partial_m\partial_n\mathcal{Z}_-}{\mathcal{Z}_-} - \frac{\partial_m\mathcal{Z}_-}{\mathcal{Z}_-}\frac{\partial_m\mathcal{Z}_-}{\mathcal{Z}_-}\right\}\,, && \text{(F.29)}\\
R_{(-)mn} &= e^p\wedge e^+\left\{\mathcal{Z}_-\partial_p\left(\frac{\Omega_{mn}}{\mathcal{Z}_-}\right)\right\}\,. && \text{(F.30)}
\end{aligned}$$

G

Leading higher-derivative corrections to Kerr geometry

G.1 Higher-derivative gravity with dynamical couplings

In this appendix, we are going to motivate our choice of effective action (8.5). Since our goal is to parametrize the leading corrections to vacuum solutions, we will start writing down an action including all possible curvature invariants containing at most $2n$ derivatives, and then we will discuss which terms are going to induce corrections. By dimensional analysis, a term with $2n$ derivatives will be multiplied by a factor ℓ^{2n-2}, where ℓ is some length scale that we will assume to be small as compared with the size of the black hole, i.e. $G_{\rm N}M >> \ell$. It is clear then that the effective action can be always written as

$$S = \frac{1}{16\pi G_{\rm N}}\int d^4x\sqrt{-g}R + \sum_{n\geq 2}\frac{\ell^{2n-2}}{16\pi G_{\rm N}}S^{(2n)}\ , \tag{G.1}$$

where in $S^{(2n)}$ we will include the terms with $2n$ derivatives.

Up to four-derivative terms, we can add the following terms to the Einstein-Hilbert action

$$S^{(4)} = \int d^4x\sqrt{-g}\left[\alpha_1\mathcal{X}_4 + \alpha_2 R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \alpha_3 R_{\mu\nu}R^{\mu\nu} + \alpha_4 R^2\right]\ . \tag{G.2}$$

It turns out that, if the coefficients α_i are constants, none of these terms will modify a vacuum solution of GR at $\mathcal{O}\left(\ell^2\right)$. The reasons are the following: both $\mathcal{X}_4$ and $R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$ are topological terms and therefore do not contribute to the equations of motion. The last two terms are quadratic in Ricci curvature, which means that their contributions to the equations of motion will vanish when evaluated on a GR vacuum solution. In other words, Ricci flat metrics are also solutions of EG plus four-derivative terms.

However, we can think of adding dynamical couplings, i.e. promoting $\alpha_i \to \alpha_i f_i\left(\phi^A\right)$, where $\left\{\phi^A\right\}_{A=1,\dots,N}$ is a set of N massless scalars.[1] To this aim, we have to include also a kinetic term for them in the action (G.1) so that it becomes

$$S = \frac{1}{16\pi G_{\rm N}}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}\mathcal{M}_{AB}(\phi)\partial_\mu\phi^A\partial^\mu\phi^B\right] + \sum_{n\geq 2}\frac{\ell^{2n-2}}{16\pi G_{\rm N}}S^{(2n)}\ , \tag{G.3}$$

[1] A natural extension would be to include a non-vanishing scalar potential.

where $\mathcal{M}_{AB}(\phi)$ is the (symmetric) matrix that characterizes the non-linear σ-model. However, as we check *a posteriori*, the scalars will be excited by the higher-derivative terms at order ℓ^2. Then, we only need to include terms that are at most quadratic in the scalars, which contribute to the gravitational equations at order ℓ^4. Thus, we can expand $\mathcal{M}_{AB}$ in a Taylor series and only the constant term will contribute at leading order. By means of a redefinition of the scalar fields, this constant term can always be taken to be the identity matrix: $\mathcal{M}_{AB}|_{\phi^A=0} = \delta_{AB}$. On the other hand, the generalized action for the four-derivative terms, that we denote again by $S^{(4)}$, is

$$S^{(4)} = \int d^4x\sqrt{-g}\left[\alpha_1 f_1(\phi)\,\mathcal{X}_4 + \alpha_2 f_2(\phi)\,R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \alpha_5 f_5(\phi)\,\nabla^2 R\right] , \tag{G.4}$$

where we already neglected the $R_{\mu\nu}R^{\mu\nu}$ and R^2 terms, that do not induce corrections at leading order, and we have now added the term $\alpha_5 f_5(\phi)\nabla^2 R$ that was neglected in (G.2) because in the non-dynamical case it is just a total derivative. In the dynamical case, this term can be written (ignoring total derivatives) as $\alpha_5\nabla^2 f_5(\phi)\,R$ and, it is possible to prove that it can always be eliminated, at leading order, by a field redefinition of the metric, so that we can set $\alpha_5 = 0$.

Indeed it is possible to show that if $K_{\mu\nu}$ is a symmetric tensor and we consider a term $\ell^4\,K_{\mu\nu}R^{\mu\nu}$ in the action, the contribution to the field equations is trivial since it can always be eliminated by a field redefinition: $g_{\mu\nu} \to g_{\mu\nu} - \ell^4\hat{K}_{\mu\nu}$, where $\hat{K}_{\mu\nu} = K_{\mu\nu} - \frac{1}{2}g_{\mu\nu}K^{\alpha}{}_{\alpha}$. To show this, let us compute the contribution of this term to the field equations. Passing this contribution to the right-hand-side of the equations. it can be written as an effective energy-momentum tensor

$$T_{\mu\nu} = \ell^4\left[\nabla^\rho\nabla_{(\mu}K_{\nu)\rho} - \frac{1}{2}\nabla^2 K_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}\nabla_\rho\nabla_\sigma K^{\rho\sigma}\right] + \ldots , \tag{G.5}$$

where the dots indicate other possible contributions that vanish when evaluated on the zeroth-order Ricci-flat metric. Then by comparison with (8.16), it is clear that the corrected Einstein equation is solved by $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \ell^4\hat{K}_{\mu\nu} + \ldots$, being $\hat{K}_{\mu\nu}$ trace-reversed with respect to $K_{\mu\nu}$. Since the equation is integrable, it is equivalent to preforming a field redefinition, so this kind of terms do not really contain new physics. We can use this result to demonstrate that other type of terms such as ϕR or $G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, that appear for instance in Horndeski theories [377], can be also removed by a field redefinition.

Let us now analyze the couplings $f_1(\phi)$ and $f_2(\phi)$. The first we can do is to expand the functions around $\phi^i = 0$ and neglect $\mathcal{O}\left(\phi^2\right)$ terms, which is equivalent to neglect $\mathcal{O}\left(\ell^6\right)$ corrections in the metric. Thus, $f_i = a_i + b_{iB}\phi^B + \mathcal{O}\left(\phi^2\right)$ and, for the same reasons exposed above, the constant coefficients a_i can be neglected. Finally, observe that we still have the freedom to perform a $\mathrm{SO}(N)$ rotation of the scalars that leaves invariant the kinetic terms. Using this freedom, up to global factors that can be reabsorbed in a redefinition of α_1 and α_2, we can always choose

$$f_1 = \phi^1 \ , \qquad f_2 = \phi^2\cos\theta_m + \phi^1\sin\theta_m \ . \tag{G.6}$$

This implies that the theory contains at most two active scalars. In summary, for our purposes the action (G.3) reduces to

$$S = \frac{1}{16\pi G_{\rm N}}\int d^4x\sqrt{-g}\left\{R - \frac{1}{2}\left(\partial\phi^1\right)^2 - \frac{1}{2}\left(\partial\phi^2\right)^2 \right.$$
$$\left. +\ell^2\left[\alpha_1\phi^1\mathcal{X}_4 + \alpha_2\left(\phi^2\cos\theta_m + \phi^1\sin\theta_m\right)R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right]\right\} + \sum_{n\geq 3}\frac{\ell^{2n-2}}{16\pi G_{\rm N}}S^{(2n)} \, . \tag{G.7}$$

Then, corrections to vacuum solutions due to these curvature-squared terms will be parametrized by three parameters: α_1, α_2 and θ_m. These terms will induce $\mathcal{O}\left(\ell^4\right)$ corrections in the metric of the solution, since the scalars will be of order $\mathcal{O}\left(\ell^2\right)$. Therefore, these corrections are equally important to those coming from the six-derivative terms (with constant couplings), which will also induce $\mathcal{O}\left(\ell^4\right)$ corrections in the metric. Since our goal is to parametrize the leading corrections to vacuum solutions in the most general way possible, we shall also include them.

The most general parity-invariant action formed with curvature invariants with six derivatives is

$$S^{(6)} = \int d^4x\sqrt{|g|}\left\{\lambda_1 R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \delta\ \gamma}R_{\delta\ \gamma}^{\ \mu\ \nu} + \lambda_2 R_{\mu\nu}^{\ \ \rho\sigma}R_{\rho\sigma}^{\ \ \delta\gamma}R_{\delta\gamma}^{\ \ \mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\delta}R^{\sigma\delta} \right.$$
$$+\lambda_4 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R + \lambda_5 R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + \lambda_6 R_{\mu}^{\ \nu}R_{\nu}^{\ \rho}R_{\rho}^{\ \mu} \tag{G.8}$$
$$\left. +\lambda_7 R_{\mu\nu}R^{\mu\nu}R + \lambda_8 R^3 + \lambda_9\nabla_\sigma R_{\mu\nu}\nabla^\sigma R^{\mu\nu} + \lambda_{10}\nabla_\mu R\nabla^\mu R\right\} \, .$$

There are other six-derivative terms that could be added, such as $\nabla^\alpha\nabla^\beta R_{\mu\alpha\nu\beta}R^{\mu\nu}$ and $\nabla_\alpha R_{\mu\nu\rho\sigma}\nabla^\alpha R^{\mu\nu\rho\sigma}$, but these can be reduced to a combination of the terms included in the action. In addition, not all the terms in the previous action are linearly independent. In four dimensions we have two constraints that can be expressed as

$$R_{[\mu_1\mu_2}{}^{\mu_1\mu_2}R_{\mu_3\mu_4}{}^{\mu_3\mu_4}R_{\mu_5\mu_6]}{}^{\mu_5\mu_6} = 0 \, , \quad R_{[\mu_1\mu_2}{}^{\mu_1\mu_2}R_{\mu_3\mu_4}{}^{\mu_3\mu_4}R_{\mu_5]}{}^{\mu_5} = 0 \, . \tag{G.9}$$

The first of these constraints actually corresponds to the vanishing of the cubic Lovelock density, $\mathcal{X}_6 = 0$. These relations allow us to express the terms proportional to λ_1 and λ_3 as a combination of the rest of the terms since (G.9) can be rewritten as

$$R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \delta\ \gamma}R_{\delta\ \gamma}^{\ \mu\ \nu} = \frac{1}{2}R_{\mu\nu}^{\ \ \rho\sigma}R_{\rho\sigma}^{\ \ \delta\gamma}R_{\delta\gamma}^{\ \ \mu\nu} - 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\delta}R^{\sigma\delta} + \frac{3}{8}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R$$
$$+3R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + 2R_{\mu}^{\ \nu}R_{\nu}^{\ \rho}R_{\rho}^{\ \mu} - \frac{3}{2}R_{\mu\nu}R^{\mu\nu}R + \frac{1}{8}R^3 \, , \tag{G.10}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\delta}R^{\sigma\delta} = 2\left(R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + R_{\mu}^{\ \nu}R_{\nu}^{\ \rho}R_{\rho}^{\ \mu} - R_{\mu\nu}R^{\mu\nu}R\right)$$
$$+\frac{1}{4}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R + R^3\right) \, . \tag{G.11}$$

Hence, we can always set $\lambda_1 = \lambda_3 = 0$. The remaining terms, except those controlled by λ_2 and λ_4 are at least quadratic in Ricci curvature and do not induce corrections on Ricci-flat metrics, so we can ignore them: $\lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = 0$. As already discussed, the term proportional to λ_4 can be eliminated by a field redefinition, since it is proportional to Ricci curvature. Consequently, we will not take it into account from now on, so we set $\lambda_4 = 0$. Therefore, we are left with only one term out of the initial ten. However, as we did with the four-derivative terms, we can also add parity-breaking densities by using the dual Riemann tensor. One finds again that there is only one independent term, and then the action $S^{(6)}$ reads

$$S^{(6)} = \int d^4x\sqrt{|g|}\left\{\lambda_{\rm ev} R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}R_{\delta\gamma}{}^{\mu\nu} + \lambda_{\rm odd} R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}\tilde{R}_{\delta\gamma}{}^{\mu\nu}\right\}\,, \tag{G.12}$$

where we have renamed the parameter λ_2 for evident reasons. Finally, we combine (G.7) and (G.12) to get the action of the effective field theory considered in the main text (8.5) and that we repeat here for convenience

$$\begin{aligned} S =& \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\bigg\{R - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 + \alpha_2\left(\phi_2\cos\theta_m + \phi_1\sin\theta_m\right)\ell^2 R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} \\ &+ \alpha_1\phi_1\ell^2\mathcal{X}_4 + \lambda_{\rm ev}\ell^4 R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}R_{\delta\gamma}{}^{\mu\nu} + \lambda_{\rm odd}\ell^4 R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}\tilde{R}_{\delta\gamma}{}^{\mu\nu}\bigg\} + \sum_{n\geq 4}\frac{\ell^{2n-2}}{16\pi G_{\rm N}}S^{(2n)}\,. \end{aligned} \tag{G.13}$$

G.2 Compactification and truncation of the heterotic effective action

Let us consider the effective action of the heterotic superstring, at first-order in the α' expansion, without gauge fields. The ten-dimensional action is given by[2]

$$S = \frac{g_s^2}{16\pi G_{\rm N}^{(10)}}\int d^{10}x\sqrt{|g|}\,e^{-2\phi}\left[R + 4\left(\partial\phi\right)^2 - \frac{1}{2\cdot 3!}H^2 + \frac{\alpha'}{8}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right] + \ldots\,, \tag{G.14}$$

where $\alpha' = \ell_s^2$, being ℓ_s the string scale, $G_{\rm N}^{(10)}$ is the ten-dimensional Newton's constant, and g_s is the string coupling constant. The curvature-squared term[3] is needed in order to supersymmetrize the action at first order in α', which otherwise would not be supersymmetric due to the presence of the Chern-Simons terms in the definition of the 3-form field strength H (see [108] for more details). As a consecuence, the Bianchi identity is no longer $dH = 0$ but it is corrected by

[2]With respect to the conventions of [139,193], here we are using mostly plus signature $g_{\mu\nu} \to -g_{\mu\nu}$ and the definition of the Riemann tensor differs by a minus sign, i.e. $R_{\mu\nu\rho}{}^{\sigma} \to -R_{\mu\nu\rho}{}^{\sigma}$.

[3]The curvature-squared term in the Bergshoeff-de Roo scheme [108] is $R_{(-)\mu\nu\rho\sigma}R_{(-)}{}^{\mu\nu\rho\sigma}$, where $R_{(-)}{}^{a}{}_{b}$ is the curvature of the torsionful spin-connection $\Omega_{(-)}{}^{a}{}_{b} = \omega^{a}{}_{b} - \frac{1}{2}H_{\mu}{}^{a}{}_{b}\,dx^{\mu}$. For our purposes, however, $R_{(-)\mu\nu\rho\sigma}R_{(-)}{}^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \ldots$, where the dots are terms that can be ignored.

$$dH = \frac{\alpha'}{4} R^a{}_b \wedge R^b{}_a + \dots \ , \tag{G.15}$$

where $R^a{}_b = \frac{1}{2!} R_{\mu\nu}{}^a{}_b \, dx^\mu \wedge dx^\nu$ is the cuvature 2-form.

Let us now perform the dimensional reduction of (G.14) on a six torus, truncating all the Kaluza-Klein degrees of freedom. We get exactly the same action but now with the indices μ, ν running from 0 to 4

$$S = \frac{1}{16\pi G_\mathrm{N}} \int d^4x \sqrt{|g|}\, e^{-2(\phi-\phi_\infty)} \left[R + 4\,(\partial\phi)^2 - \frac{1}{2\cdot 3!} H^2 + \frac{\alpha'}{8} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right] + \dots \ , \tag{G.16}$$

where G_N is the four-dimensional Newton's constant, related to the ten-dimensional one by

$$G_\mathrm{N}^{(10)} = (2\pi\ell_s)^6 \, G_\mathrm{N} \ , \tag{G.17}$$

and we have also introduced $e^{\phi_\infty} = g_s$. Let us show that, ignoring terms whose contribution to the equations of motion is either zero or trivial, this action can be rewritten in a form such that it is manifestly a particular case of (8.5). First of all, let us rewrite the Bianchi identity (G.15) as

$$\frac{1}{3!} \sqrt{|g|} \nabla_\mu H_{\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = -\frac{\alpha'}{8} \sqrt{|g|} R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} + \dots \ . \tag{G.18}$$

Secondly, we have to dualize the 3-form into a (pseudo)scalar φ. Following the usual procedure, we introduce a Lagrange multiplier into the action (G.16),

$$\begin{aligned} S = \frac{1}{16\pi G_\mathrm{N}} \int d^4x \sqrt{|g|} \Big\{ & e^{-2(\phi-\phi_\infty)} \left(R + 4\,(\partial\phi)^2 - \frac{1}{2\cdot 3!} H^2 + \frac{\alpha'}{8} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \\ & + \varphi \left(\frac{1}{3!} \nabla_\mu H_{\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma} + \frac{\alpha'}{8} R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \right) \Big\} + \dots \ . \end{aligned} \tag{G.19}$$

The relation between the 3-form field strength and the scalar is found by imposing that the variation of the action with respect to H vanishes,

$$\frac{\delta S}{\delta H} = 0 \quad \Rightarrow \quad H^{\mu\nu\rho} = e^{2(\phi-\phi_\infty)} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \varphi \ . \tag{G.20}$$

Now, we rewrite (G.19) in terms of φ, getting

$$\begin{aligned} S = \frac{1}{16\pi G_\mathrm{N}} \int d^4x \sqrt{|g|}\, e^{-2(\phi-\phi_\infty)} \Big[& R + 4\,(\partial\phi)^2 - \frac{e^{4(\phi-\phi_\infty)}}{2} (\partial\varphi)^2 + \frac{\alpha'}{8} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ & + \frac{\alpha' e^{2(\phi-\phi_\infty)}}{8} \varphi R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \Big] + \dots \ . \end{aligned} \tag{G.21}$$

Since this action is not written in the Einstein frame, let us rescale the metric $g_{\mu\nu} \to e^{-2(\phi-\phi_\infty)} g_{\mu\nu}$ in order to eliminate the conformal factor. Expanding in $(\phi - \phi_\infty)$ and keeping only the leading terms, we get

$$S = \frac{1}{16\pi G_{\rm N}} \int d^4x \sqrt{|g|} \left[R - 2\,(\partial\phi)^2 - \frac{1}{2}\,(\partial\varphi)^2 + \frac{\alpha'}{8}(1 - 2\phi + 2\phi_\infty) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right.$$
$$\left. + \frac{\alpha'}{8} \varphi\, R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \right] + \ldots \tag{G.22}$$

where we have dropped some terms that can be removed with a field redefinition. Finally, defining $\phi^1 = 2\phi - 2\phi_\infty$ and $\phi^2 = \varphi$, and ignoring terms that do not contribute to the equations of motion at leading order, we can write the action in the following final form

$$S = \frac{1}{16\pi G_{\rm N}} \int d^4x \sqrt{|g|} \left[R - \frac{1}{2}\left(\partial\phi^1\right)^2 - \frac{1}{2}\left(\partial\phi^2\right)^2 - \frac{\alpha'}{8}\phi^1 \mathcal{X}_4 + \frac{\alpha'}{8}\varphi\, R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \right] . \tag{G.23}$$

We have upgraded the Riemann squared term to the Gauss-Bonnet density $\mathcal{X}_4$ since both give the same contribution at leading order (the Ricci^2 and R^2 terms do not contribute). This can also be done by means of a field redefinition. Then, the choice of parameters that gives us the corrections predicted by the simplest compactification of the effective action of the heterotic superstring is

$$\alpha_1 = -\frac{1}{8}\,, \quad \alpha_2 = \frac{1}{8}\,, \quad \theta_m = 0\,, \quad \lambda_{\rm ev} = \lambda_{\rm odd} = 0\,, \quad \ell = \ell_s\ . \tag{G.24}$$

G.3 The solution

We show the metric functions H_i and the scalars, ϕ_1 and ϕ_2, up to order $\mathcal{O}\left(\chi^3\right)$:

$$
\begin{aligned}
H_1 = \alpha_1^2\ell^4 \Bigg\{ & \frac{416M^3}{11\rho^7} + \frac{112M^2}{165\rho^6} + \frac{428M}{1155\rho^5} - \frac{3202}{385\rho^4} - \frac{122}{385M\rho^3} - \frac{1117}{1155M^2\rho^2} + \frac{1117}{1155M^3\rho} \\
& +\chi^2 \Bigg[x^2 \Bigg(-\frac{87008M^5}{165\rho^9} + \frac{3377728M^4}{35035\rho^8} + \frac{903092M^3}{15015\rho^7} + \frac{493638556M^2}{7882875\rho^6} - \frac{22915196M}{7882875\rho^5} \\
& - \frac{169553}{160875\rho^4} - \frac{7721321}{1126125M\rho^3} \Bigg) - \frac{3635392M^4}{105105\rho^8} - \frac{2245064M^3}{105105\rho^7} - \frac{87538336M^2}{7882875\rho^6} + \frac{995398M}{2627625\rho^5} \\
& + \frac{2988737}{1126125\rho^4} + \frac{736487}{1126125M\rho^3} - \frac{787153}{450450M^2\rho^2} + \frac{787153}{450450M^3\rho} \Bigg] \Bigg\} \\
& + \alpha_2^2\ell^4\chi^2 \Bigg\{ x^2 \Bigg(\frac{342M^5}{\rho^9} - \frac{9279M^4}{637\rho^8} - \frac{19280M^3}{1001\rho^7} - \frac{1094689M^2}{42042\rho^6} + \frac{298393M}{84084\rho^5} + \frac{80291}{24024\rho^4} \\
& + \frac{80291}{24024M\rho^3} \Bigg) - \frac{20268M^4}{637\rho^8} - \frac{11710M^3}{637\rho^7} - \frac{30707M^2}{3234\rho^6} + \frac{1074M}{7007\rho^5} - \frac{271}{12012\rho^4} - \frac{271}{12012M\rho^3} \\
& + \frac{72185}{48048M^2\rho^2} - \frac{72185}{48048M^3\rho} \Bigg\} \\
& + \alpha_1\alpha_2\sin(\theta_m)\ell^4 \Bigg\{ \chi x \Bigg[\frac{21120M^4}{91\rho^8} - \frac{21352M^3}{1001\rho^7} - \frac{43564M^2}{2145\rho^6} - \frac{551776M}{15015\rho^5} - \frac{5618}{15015\rho^4} \\
& + \frac{89989}{30030M^2\rho^2} \Bigg] + \chi^3 \Bigg[x^3 \Bigg(-\frac{11556352M^6}{4641\rho^{10}} + \frac{1402164667M^5}{3828825\rho^9} + \frac{12014583319M^4}{53603550\rho^8} \\
& + \frac{879521737M^3}{4873050\rho^7} - \frac{895892573M^2}{76576500\rho^6} - \frac{611550767M}{153153000\rho^5} - \frac{43683743}{12252240\rho^4} \Bigg) + x \Bigg(-\frac{555211M^5}{49725\rho^9} \\
& - \frac{417419143M^4}{53603550\rho^8} - \frac{278633M^3}{17867850\rho^7} + \frac{2744165393M^2}{536035500\rho^6} + \frac{244492811M}{30630600\rho^5} + \frac{157764391}{306306000\rho^4} \\
& - \frac{75784931}{61261200M^2\rho^2} \Bigg) \Bigg] \Bigg\} \\
& + \lambda_{\rm ev}\ell^4 \Bigg\{ -\frac{48M^3}{11\rho^7} - \frac{8M^2}{33\rho^6} - \frac{40M}{231\rho^5} - \frac{32}{231\rho^4} - \frac{32}{231M\rho^3} - \frac{64}{231M^2\rho^2} + \frac{64}{231M^3\rho} \\
& +\chi^2 \Bigg[x^2 \Bigg(\frac{1728M^5}{11\rho^9} + \frac{1752M^4}{7007\rho^8} - \frac{800M^3}{1001\rho^7} - \frac{8660M^2}{7007\rho^6} - \frac{9518M}{7007\rho^5} - \frac{1005}{1001\rho^4} - \frac{2669}{1001M\rho^3} \Bigg) \\
& - \frac{5952M^4}{7007\rho^8} - \frac{520M^3}{1617\rho^7} - \frac{68M^2}{1911\rho^6} + \frac{830M}{7007\rho^5} + \frac{587}{3003\rho^4} + \frac{587}{3003M\rho^3} - \frac{865}{1001M^2\rho^2} + \frac{865}{1001M^3\rho} \Bigg] \Bigg\} \\
& + \lambda_{\rm odd}\ell^4 \Bigg\{ \chi x \Bigg[-\frac{3456M^4}{91\rho^8} - \frac{1152M^3}{1001\rho^7} - \frac{96M^2}{143\rho^6} - \frac{384M}{1001\rho^5} - \frac{192}{1001\rho^4} + \frac{768}{1001M^2\rho^2} \Bigg] \\
& +\chi^3 \Bigg[x^3 \Bigg(\frac{745344M^6}{1547\rho^{10}} - \frac{86140M^5}{17017\rho^9} - \frac{13190M^4}{2431\rho^8} - \frac{515974M^3}{119119\rho^7} - \frac{47015M^2}{17017\rho^6} - \frac{274763M}{238238\rho^5} \\
& - \frac{511811}{476476\rho^4} \Bigg) + x \Bigg(-\frac{596M^5}{221\rho^9} - \frac{8530M^4}{119119\rho^8} + \frac{100326M^3}{119119\rho^7} + \frac{111563M^2}{119119\rho^6} + \frac{153991M}{238238\rho^5} \\
& + \frac{111151}{476476\rho^4} - \frac{13735}{68068M^2\rho^2} \Bigg) \Bigg] \Bigg\} + \mathcal{O}\left(\chi^4\right) ,
\end{aligned}
\tag{G.25}
$$

$$
\begin{aligned}
H_2 = \alpha_1^2\,\ell^4 \Bigg\{ & \frac{1117}{2310} + \frac{208}{11\rho^6} - \frac{208}{165\rho^5} - \frac{142}{231\rho^4} - \frac{5188}{1155\rho^3} - \frac{337}{1155\rho^2} - \frac{1117}{2310\rho} \\
& + \chi^2 \Bigg[\frac{787153}{900900} - \frac{1817696}{105105\rho^7} - \frac{3258916}{315315\rho^6} - \frac{42983383}{7882875\rho^5} + \frac{93497}{5255250\rho^4} + \frac{36396163}{31531500\rho^3} \\
& + \frac{2033089}{7882875\rho^2} - \frac{787153}{900900\rho} + x^2 \left(-\frac{40384}{165\rho^8} - \frac{4002832}{105105\rho^7} - \frac{1675328}{315315\rho^6} + \frac{190462}{9625\rho^5} + \frac{80274479}{15765750\rho^4} \right. \\
& \left. + \frac{8052437}{6306300\rho^3} - \frac{988269}{350350\rho^2} \right) \Bigg] \Bigg\} \\
& + \alpha_1\alpha_2 \sin\theta_m \ell^4 \Bigg\{ \chi x \left(\frac{10560}{91\rho^7} + \frac{2570584}{45045\rho^6} + \frac{5030294}{225225\rho^5} - \frac{1352913}{175175\rho^4} - \frac{2310579}{350350\rho^3} - \frac{2964341}{750750\rho^2} \right. \\
& \left. - \frac{12761}{17875\rho} \right) + \chi^3 \Bigg[x \left(-\frac{555211}{99450\rho^8} - \frac{33606289}{4123350\rho^7} - \frac{1807455889}{321621300\rho^6} - \frac{185905529}{160810650\rho^5} + \frac{5347197771}{1667666000\rho^4} \right. \\
& \left. + \frac{20139019441}{15008994000\rho^3} + \frac{827344753}{1000599600\rho^2} - \frac{7696421}{25014990\rho} \right) + x^3 \left(-\frac{5239616}{4641\rho^9} - \frac{2886836873}{7657650\rho^8} \right. \\
& \left. - \frac{3454981169}{53603550\rho^7} + \frac{18272087263}{321621300\rho^6} + \frac{3088505012}{134008875\rho^5} + \frac{88918288507}{15008994000\rho^4} + \frac{24278291273}{45026982000\rho^3} \right) \Bigg] \Bigg\} \\
& + \alpha_2^2\,\ell^4 \Bigg\{ -\frac{27}{2\rho^5} - \frac{60}{7\rho^4} - \frac{5}{\rho^3} + \chi^2 \Bigg[-\frac{72185}{96096} - \frac{10134}{637\rho^7} - \frac{447949}{57330\rho^6} - \frac{564161}{194040\rho^5} + \frac{154675}{96096\rho^4} \\
& + \frac{1153277}{1345344\rho^3} + \frac{457841}{3363360\rho^2} + \frac{72185}{96096\rho} + x^2 \left(\frac{171}{\rho^8} + \frac{81219}{637\rho^7} + \frac{4701743}{126126\rho^6} - \frac{32689}{5544\rho^5} \right. \\
& \left. - \frac{1852791}{224224\rho^4} - \frac{3310225}{1345344\rho^3} + \frac{462029}{672672\rho^2} \right) \Bigg] \Bigg\} \\
& + \lambda_{\text{ev}}\,\ell^4 \Bigg\{ \frac{32}{231} - \frac{24}{11\rho^6} - \frac{4}{33\rho^5} - \frac{20}{231\rho^4} - \frac{16}{231\rho^3} - \frac{16}{231\rho^2} - \frac{32}{231\rho} + \chi^2 \Bigg[\frac{865}{2002} - \frac{2976}{7007\rho^7} \\
& - \frac{920}{1617\rho^6} - \frac{853}{1911\rho^5} - \frac{7349}{28028\rho^4} - \frac{15739}{168168\rho^3} + \frac{1783}{84084\rho^2} - \frac{865}{2002\rho} + x^2 \left(\frac{840}{11\rho^8} + \frac{624704}{21021\rho^7} \right. \\
& \left. + \frac{328360}{21021\rho^6} + \frac{1781}{231\rho^5} + \frac{276011}{84084\rho^4} + \frac{156647}{168168\rho^3} - \frac{78439}{84084\rho^2} \right) \Bigg] \Bigg\} \\
& + \lambda_{\text{odd}}\,\ell^4 \Bigg\{ \chi x \left(-\frac{1728}{91\rho^7} - \frac{6560}{1001\rho^6} - \frac{4040}{1001\rho^5} - \frac{17676}{7007\rho^4} - \frac{11112}{7007\rho^3} - \frac{948}{1001\rho^2} - \frac{135}{1001\rho} \right) \\
& + \chi^3 \Bigg[x \left(-\frac{298}{221\rho^8} - \frac{176734}{119119\rho^7} - \frac{255071}{357357\rho^6} - \frac{2333}{51051\rho^5} + \frac{64221}{238238\rho^4} + \frac{257645}{952952\rho^3} \right. \\
& \left. + \frac{10575}{68068\rho^2} - \frac{1185}{34034\rho} \right) + x^3 \left(\frac{343296}{1547\rho^9} + \frac{208174}{2431\rho^8} + \frac{4665742}{119119\rho^7} + \frac{5774105}{357357\rho^6} + \frac{657785}{119119\rho^5} \right. \\
& \left. + \frac{22837}{17017\rho^4} + \frac{12379}{408408\rho^3} \right) \Bigg] \Bigg\} + \mathcal{O}\left(\chi^4\right) \,,
\end{aligned}
\tag{G.26}
$$

$$
\begin{aligned}
H_3 = {} & \alpha_1^2\,\ell^4 \left\{ -\frac{1117}{1155} - \frac{368}{33\rho^6} - \frac{1168}{165\rho^5} - \frac{1102}{231\rho^4} - \frac{404}{1155\rho^3} - \frac{19}{1155\rho^2} + \frac{1117}{1155\rho} \right. \\
& + \chi^2 \left[-\frac{787153}{450450} + \frac{210256}{105105\rho^7} + \frac{358564}{105105\rho^6} + \frac{29284144}{7882875\rho^5} + \frac{2871703}{1126125\rho^4} + \frac{888572}{1126125\rho^3} \right. \\
& + \frac{10139}{150150\rho^2} + \frac{787153}{450450\rho} + x^2 \left(\frac{23488}{165\rho^8} + \frac{6074176}{105105\rho^7} + \frac{1857368}{105105\rho^6} - \frac{64561864}{7882875\rho^5} - \frac{124199}{25025\rho^4} \right. \\
& \left.\left.\left. - \frac{329289}{125125\rho^3} + \frac{43252}{20475\rho^2} \right) \right] \right\} \\
& + \alpha_1\alpha_2 \sin\theta_m\,\ell^4 \left\{ \chi x \left[-\frac{6144}{91\rho^7} - \frac{34044}{1001\rho^6} - \frac{31664}{2145\rho^5} + \frac{16854}{5005\rho^4} + \frac{16241}{5005\rho^3} + \frac{89989}{30030\rho^2} \right] \right. \\
& + \chi^3 \left[x \left(\frac{1399373}{232050\rho^8} + \frac{76931759}{8933925\rho^7} + \frac{237697639}{35735700\rho^6} + \frac{197152489}{76576500\rho^5} - \frac{142049293}{102102000\rho^4} \right. \right. \\
& \left. - \frac{7454231}{5105100\rho^3} - \frac{75784931}{61261200\rho^2} \right) + x^3 \left(\frac{9126080}{13923\rho^9} + \frac{1870089271}{7657650\rho^8} + \frac{1696574476}{26801775\rho^7} \right. \\
& \left.\left.\left. - \frac{320470309}{15315300\rho^6} - \frac{42765071}{4504500\rho^5} - \frac{29731159}{8751600\rho^4} + \frac{132059}{2356200\rho^3} \right) \right] \right\} \\
& + \alpha_2^2\,\ell^4 \left\{ \chi^2 \left[\frac{72185}{48048} + \frac{639}{1274\rho^7} + \frac{2005}{2548\rho^6} + \frac{41549}{42042\rho^5} + \frac{8581}{12012\rho^4} + \frac{887}{1716\rho^3} + \frac{270}{1001\rho^2} \right. \right. \\
& \left.\left. - \frac{72185}{48048\rho} + x^2 \left(-\frac{99}{\rho^8} - \frac{57843}{1274\rho^7} - \frac{455055}{28028\rho^6} + \frac{5891}{3822\rho^5} + \frac{3425}{8008\rho^4} - \frac{2969}{4004\rho^3} - \frac{14015}{6864\rho^2} \right) \right] \right\} \\
& + \lambda_{\text{ev}}\,\ell^4 \left\{ -\frac{64}{231} - \frac{392}{11\rho^6} + \frac{8}{33\rho^5} + \frac{40}{231\rho^4} + \frac{32}{231\rho^3} + \frac{32}{231\rho^2} + \frac{64}{231\rho} + \chi^2 \left[-\frac{865}{1001} \right. \right. \\
& + \frac{3664}{7007\rho^7} + \frac{11380}{21021\rho^6} + \frac{752}{1911\rho^5} + \frac{213}{1001\rho^4} + \frac{139}{3003\rho^3} - \frac{32}{429\rho^2} + \frac{865}{1001\rho} + x^2 \left(\frac{7960}{11\rho^8} \right. \\
& \left.\left.\left. - \frac{372584}{21021\rho^7} - \frac{199660}{21021\rho^6} - \frac{101648}{21021\rho^5} - \frac{6455}{3003\rho^4} - \frac{1921}{3003\rho^3} + \frac{3043}{3003\rho^2} \right) \right] \right\} \\
& + \lambda_{\text{odd}}\,\ell^4 \left\{ \chi x \left(-\frac{19008}{91\rho^7} + \frac{4320}{1001\rho^6} + \frac{384}{143\rho^5} + \frac{1728}{1001\rho^4} + \frac{1152}{1001\rho^3} + \frac{768}{1001\rho^2} \right) \right. \\
& + \chi^3 \left[x \left(\frac{2802}{1547\rho^8} + \frac{167628}{119119\rho^7} + \frac{71475}{119119\rho^6} - \frac{2087}{119119\rho^5} - \frac{148143}{476476\rho^4} - \frac{768}{2431\rho^3} - \frac{13735}{68068\rho^2} \right) \right. \\
& \left.\left. + x^3 \left(\frac{2964736}{1547\rho^9} - \frac{838886}{17017\rho^8} - \frac{2761840}{119119\rho^7} - \frac{1189205}{119119\rho^6} - \frac{3665}{1001\rho^5} - \frac{486569}{476476\rho^4} + \frac{601}{34034\rho^3} \right) \right] \right\} \\
& + \mathcal{O}\left(\chi^4\right) ,
\end{aligned} \tag{G.27}
$$

$$
\begin{aligned}
H_4 = {} & \alpha_1^2 \ell^4 \left\{ -\frac{1117}{1155} - \frac{368}{33\rho^6} - \frac{1168}{165\rho^5} - \frac{1102}{231\rho^4} - \frac{404}{1155\rho^3} - \frac{19}{1155\rho^2} + \frac{1117}{1155\rho} \right. \\
& + \chi^2 \left[-\frac{787153}{450450} + \frac{1984}{33\rho^9} + \frac{3232}{165\rho^8} + \frac{529624}{21021\rho^7} + \frac{92864}{9555\rho^6} + \frac{91595428}{7882875\rho^5} + \frac{489939}{125125\rho^4} \right. \\
& + \frac{5646292}{1126125\rho^3} + \frac{981961}{450450\rho^2} + \frac{787153}{450450\rho} + x^2 \left(-\frac{1984}{33\rho^9} + \frac{6752}{55\rho^8} + \frac{1212104}{35035\rho^7} + \frac{1194428}{105105\rho^6} \right. \\
& \left.\left.\left. - \frac{126873148}{7882875\rho^5} - \frac{7126703}{1126125\rho^4} - \frac{7721321}{1126125\rho^3} \right) \right] \right\} \\
& + \alpha_1 \alpha_2 \sin\theta_m \, \ell^4 \left\{ \chi x \left[-\frac{6144}{91\rho^7} - \frac{34044}{1001\rho^6} - \frac{31664}{2145\rho^5} + \frac{16854}{5005\rho^4} + \frac{16241}{5005\rho^3} + \frac{89989}{30030\rho^2} \right] \right. \\
& + \chi^3 \left[x \left(\frac{33408}{91\rho^{10}} + \frac{17348416}{45045\rho^9} + \frac{736024381}{2552550\rho^8} + \frac{1195689244}{8933925\rho^7} + \frac{5169579091}{107207100\rho^6} + \frac{11447047}{1392300\rho^5} \right. \right. \\
& \left. - \frac{124881623}{102102000\rho^4} - \frac{43008619}{30630600\rho^3} - \frac{75784931}{61261200\rho^2} \right) + x^3 \left(-\frac{33408}{91\rho^{10}} + \frac{69003776}{255255\rho^9} - \frac{291804563}{7657650\rho^8} \right. \\
& \left.\left.\left. - \frac{1659697979}{26801775\rho^7} - \frac{957111191}{15315300\rho^6} - \frac{1159441303}{76576500\rho^5} - \frac{43683743}{12252240\rho^4} \right) \right] \right\} \\
& + \alpha_2^2 \, \ell^4 \left\{ \chi^2 \left[\frac{72185}{48048} - \frac{54}{\rho^8} - \frac{27897}{637\rho^7} - \frac{40995}{1274\rho^6} - \frac{18587}{1617\rho^5} - \frac{12993}{2002\rho^4} - \frac{12241}{3432\rho^3} - \frac{85145}{48048\rho^2} \right. \right. \\
& \left.\left. - \frac{72185}{48048\rho} + x^2 \left(-\frac{45}{\rho^8} - \frac{705}{637\rho^7} + \frac{234445}{14014\rho^6} + \frac{294806}{21021\rho^5} + \frac{183353}{24024\rho^4} + \frac{80291}{24024\rho^3} \right) \right] \right\} \\
& + \lambda_{\rm ev} \, \ell^4 \left\{ -\frac{64}{231} - \frac{392}{11\rho^6} + \frac{8}{33\rho^5} + \frac{40}{231\rho^4} + \frac{32}{231\rho^3} + \frac{32}{231\rho^2} + \frac{64}{231\rho} \right. \\
& \chi^2 \left[-\frac{865}{1001} + \frac{736}{11\rho^9} + \frac{384}{11\rho^8} + \frac{393088}{21021\rho^7} + \frac{5356}{539\rho^6} + \frac{107504}{21021\rho^5} + \frac{6109}{3003\rho^4} + \frac{2075}{1001\rho^3} + \frac{2819}{3003\rho^2} \right. \\
& \left.\left. + \frac{865}{1001\rho} + x^2 \left(-\frac{736}{11\rho^9} + \frac{7576}{11\rho^8} - \frac{251560}{7007\rho^7} - \frac{132388}{7007\rho^6} - \frac{66960}{7007\rho^5} - \frac{3975}{1001\rho^4} - \frac{2669}{1001\rho^3} \right) \right] \right\} \\
& + \lambda_{\rm odd} \, \ell^4 \left\{ \chi x \left(-\frac{19008}{91\rho^7} + \frac{4320}{1001\rho^6} + \frac{384}{143\rho^5} + \frac{1728}{1001\rho^4} + \frac{1152}{1001\rho^3} + \frac{768}{1001\rho^2} \right) \right. \\
& + \chi^3 \left[x \left(\frac{34560}{91\rho^{10}} + \frac{175360}{1001\rho^9} + \frac{1340034}{17017\rho^8} + \frac{3834240}{119119\rho^7} + \frac{25469}{2431\rho^6} + \frac{138241}{119119\rho^5} - \frac{122901}{476476\rho^4} \right. \right. \\
& \left. - \frac{10151}{34034\rho^3} - \frac{13735}{68068\rho^2} \right) + x^3 \left(-\frac{34560}{91\rho^{10}} + \frac{29630976}{17017\rho^9} - \frac{2148098}{17017\rho^8} - \frac{6428452}{119119\rho^7} - \frac{2365711}{119119\rho^6} \right. \\
& \left.\left.\left. - \frac{576463}{119119\rho^5} - \frac{511811}{476476\rho^4} \right) \right] \right\} + \mathcal{O}\left(\chi^4\right) \ .
\end{aligned}
\tag{G.28}
$$

$$
\begin{aligned}
\phi_1 = \alpha_1 \ell^2 &\left\{ \frac{8M}{3\rho^3} + \frac{2}{\rho^2} + \frac{2}{M\rho} \right. \\
&+ \left[-\frac{M^2}{5\rho^4} - \frac{2M}{5\rho^3} - \frac{1}{2\rho^2} - \frac{1}{2M\rho} + \left(-\frac{96M^3}{5\rho^5} - \frac{42M^2}{5\rho^4} - \frac{14M}{5\rho^3} \right) x^2 \right] \chi^2 \\
&+ \left[-\frac{2M^3}{35\rho^5} - \frac{M^2}{7\rho^4} - \frac{3M}{14\rho^3} - \frac{1}{4\rho^2} - \frac{1}{4M\rho} + \left(\frac{4M^4}{7\rho^6} + \frac{24M^3}{35\rho^5} + \frac{3M^2}{7\rho^4} + \frac{M}{7\rho^3} \right) x^2 \right. \\
&\left.\left. + \left(\frac{360M^5}{7\rho^7} + \frac{110M^4}{7\rho^6} + \frac{22M^3}{7\rho^5} \right) x^4 \right] \chi^4 \right\} \\
&+ \alpha_2 \ell^2 \sin\theta_m \left\{ x \left(\frac{9M^2}{\rho^4} + \frac{5M}{\rho^3} + \frac{5}{2\rho^2} \right) \chi + \left[x^3 \left(-\frac{100M^4}{3\rho^6} - \frac{12M^3}{\rho^5} - \frac{3M^2}{\rho^4} \right) \right.\right. \\
&\left.\left. + x \left(-\frac{2M^3}{5\rho^5} - \frac{3M^2}{5\rho^4} - \frac{M}{2\rho^3} - \frac{1}{4\rho^2} \right) \right] \chi^3 \right\} + \mathcal{O}(\chi^5) ,
\end{aligned}
\tag{G.29}
$$

$$
\begin{aligned}
\phi_2 = \alpha_2 \ell^2 \cos\theta_m &\left\{ x \left(\frac{9M^2}{\rho^4} + \frac{5M}{\rho^3} + \frac{5}{2\rho^2} \right) \chi + \left[x^3 \left(-\frac{100M^4}{3\rho^6} - \frac{12M^3}{\rho^5} - \frac{3M^2}{\rho^4} \right) \right.\right. \\
&\left.\left. + x \left(-\frac{2M^3}{5\rho^5} - \frac{3M^2}{5\rho^4} - \frac{M}{2\rho^3} - \frac{1}{4\rho^2} \right) \right] \chi^3 \right\} + \mathcal{O}(\chi^5) .
\end{aligned}
\tag{G.30}
$$

G.4 Convergence of the χ-expansion

In this section we analyze the convergence of the solution presented in Sec. 8.2.1, and whose first terms in the χ-expansion are shown in Appendix G.3. In order to study the convergence, first we must consider the partial sums

$$H_{i,n} = \sum_{k=0}^{n} H_i^{(k)} \chi^k , \quad i = 1, 2, 3, 4 . \tag{G.31}$$

Then, we have to investigate if the sequence of functions $H_{i,n}$ converges to a function H_i, this is,

$$\lim_{n\to\infty} H_{i,n} = H_i , \tag{G.32}$$

and what is the radius of convergence for a given domain $(\rho, x) \in \Omega$. We can study the convergence of the four functions H_i at the same time by introducing the "norm" $\|H\|_n$, as

$$\|H\|_n := \sqrt{H_{1,n}^2 + H_{2,n}^2 + H_{3,n}^2 + H_{4,n}^2} . \tag{G.33}$$

Thus, every $H_{i,n}$ converges if and only if $\|H\|_n$ converges. Since we are only interested in the exterior region of the black hole, it is sufficient to look at the convergence for $\rho \geq \rho_+$, $-1 < x < 1$. Using the terms of the solution up to order $n = 14$, we observe that the sequence $\|H\|_n$ converges in the exterior region if the spin is small enough. We wish to determine the maximum value of χ for which the expansion up to order $n = 14$ —the one we use thorough the text— provides an accurate approximation to the full series. This value of course depends on the point of the spacetime. For instance, far from the black hole the few first terms in the expansion already provide a very precise result, even for

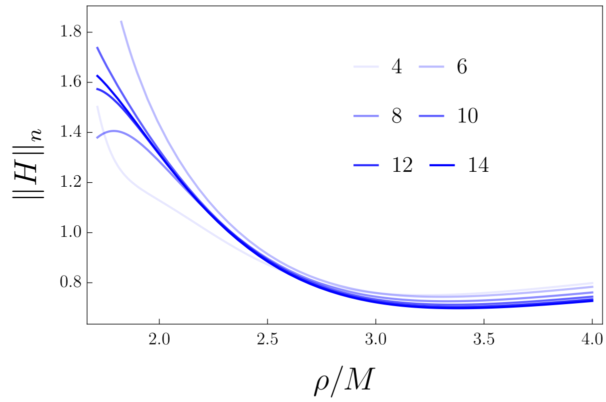

Figure G.1: Convergence of the norm of the H_i functions, $\|H\|_n$. We show the profile of $\|H\|_n$ in the axis $x=1$, $\rho \geq \rho_+$ for the values of n indicated, for spin $\chi = 0.7$, and for couplings $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\theta_m = \pi/4$, $\lambda_{\rm ev} = \lambda_{\rm odd} = 1$.

$\chi \sim 1$. On the contrary, the convergence is worse at the horizon $\rho = \rho_+$, and, specially, at the axes $x = \pm 1$. Thus, we should look at the convergence at those points. It is useful to define the relative differences,

$$d_n = \frac{\|H\|_{n+1} - \|H\|_n}{\|H\|_n}. \tag{G.34}$$

For instance, when we evaluate this for $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\theta_m = \pi/4$, $\lambda_{\rm ev} = \lambda_{\rm odd} = 1$ and $\chi = 0.7$, at the north pole of the horizon, $\rho = \rho_+$, $x = 1$, we get the following sequence (starting at $n = 0$): 1.3, -0.79, 2.7, -0.74, 2.9, -0.61, -0.52, 0.25, 0.72, -0.27, -0.21, 0.14, 0.076, -0.039, ... We see that the sequence starts converging from $n = 8$, and the difference between $\|H\|_{13}$ and $\|H\|_{14}$ is barely a 4%. Since the difference with the term $n = 15$ will be even smaller, we are confident that the series up to $n = 14$ provides a precision of the order of 1%, for $\chi = 0.7$ when evaluated at the north pole of the horizon. In the rest of points, and for smaller values of χ, the accuracy is significantly greater. We illustrate this in Fig. G.1 , where we show the profile of $\|H\|_n$, for several values of n, in the line $x = 1$, $\rho \geq \rho_+$. Also, if for the same values of the couplings we set $\chi = 0.65$, we get $d_{13} = -0.81\%$, so the series up to order $n = 14$ is around five times more accurate than for $\chi = 0.7$.

Finally, one could try to determine what is the maximum value of χ for which the series will converge all the way up to the horizon. In order to find the radius of convergence, one can apply for example the root test to the coefficients $H_i^{(k)}$ in (G.31):

$$\chi_{\rm max} = \inf\left\{ \lim_{k\to\infty} \left|H_i^{(k)}\right|^{-1/k} \;\middle|\; i = 1,2,3,4, \quad \rho \geq \rho_+, \quad -1 < x < 1 \right\}. \tag{G.35}$$

Using the coefficients up to order $n = 14$ it is difficult to provide a definitive answer, but the results seem to be consistent with $\chi_{\rm max} \sim 1$. So, it could be possible to get close to the extremal limit adding enough terms in the series expansion, though the number of terms required to get a good approximation increases quite rapidly as we approach $\chi = 1$.

G.5 Some formulas

Radius of the ergosphere

$$\Delta\rho^{(1)} = \chi^2(1-x^2)\frac{53}{120} + \chi^4(1-x^2)\left(\frac{56750791}{84084000} - \frac{41115397x^2}{63063000}\right) \\ + \chi^6(1-x^2)\left(-\frac{679368329719x^4}{912546835200} + \frac{10245671873x^2}{165917606400} + \frac{336187298257}{1825093670400}\right) + \mathcal{O}(\chi^8)\,, \tag{G.36}$$

$$\Delta\rho^{(2)} = -\chi^2(1-x^2)\frac{709x^2}{448} - \chi^4(1-x^2)\left(\frac{4433503x^2}{2690688} + \frac{504467}{1345344}\right) \\ - \chi^6(1-x^2)\left(\frac{915791950769x^4}{625746401280} + \frac{148163587307x^2}{312873200640} + \frac{8754619243}{18962012160}\right) + \mathcal{O}(\chi^8)\,, \tag{G.37}$$

$$\Delta\rho^{(m)} = \chi^5 x(1-x^2)\left(\frac{401316913x^2}{22870848000} - \frac{401316913}{22870848000}\right) \\ + \chi^7 x(1-x^2)\left(\frac{1222303361x^5}{85085952000} + \frac{116649901427x^3}{12167291136000} - \frac{39651603x}{1655413760}\right) + \mathcal{O}(\chi^9)\,, \tag{G.38}$$

$$\Delta\rho^{(\text{ev})} = \chi^2(1-x^2)\frac{1}{2} + \chi^4(1-x^2)\left(\frac{1245x^2}{5096} + \frac{3243}{10192}\right) \\ + \chi^6(1-x^2)\left(\frac{14596973x^4}{79008384} + \frac{24066599x^2}{158016768} + \frac{1021961}{4051712}\right) + \mathcal{O}(\chi^8)\,, \tag{G.39}$$

$$\Delta\rho^{(\text{odd})} = \chi^5 x(1-x^2)\left(\frac{669}{106624} - \frac{669x^2}{106624}\right) \\ + \chi^7 x(1-x^2)\left(-\frac{109x^4}{14896} + \frac{131x^2}{144704} + \frac{6495}{1012928}\right) + \mathcal{O}(\chi^9)\,. \tag{G.40}$$

Some Christoffel symbols

$$\Gamma^\rho_{tt} = \frac{\rho_\pm^2 - 2M\rho_\pm + M^2\chi^2}{\rho_\pm^4}\left[M + \frac{\ell^4}{2M^4}\left(2MH_3 + \rho^2\partial_\rho H_1\right)\right]\Bigg|_{\rho=\rho_\pm, x=0}\,, \tag{G.41}$$

$$\Gamma^\rho_{t\phi} = -\frac{\left(\rho_\pm^2 - 2M\rho_\pm + M^2\chi^2\right)M^2\chi}{\rho_\pm^4}\left[1 - \frac{\ell^4}{M^4}\left(H_3 - H_2 + \rho\partial_\rho H_2\right)\right]\Bigg|_{\rho=\rho_\pm, x=0}\,, \tag{G.42}$$

$$\Gamma^\rho_{\phi\phi} = -\frac{\rho_\pm^2 - 2M\rho_\pm + M^2\chi^2}{\rho_\pm^4}\left[\left(\rho_\pm^3 - M^3\chi^2\right)\left(1 + \frac{\ell^4}{M^4}\left(H_4 - H_3\right)\right) \right. \\ \left. + \frac{\ell^4}{2M^4}\left(\rho_\pm^4 + 2M^3\chi^2\rho_\pm + M^2\chi^2\rho_\pm^2\right)\partial_\rho H_4\right]\Bigg|_{\rho=\rho_\pm, x=0}\,. \tag{G.43}$$

Photon rings

$$\begin{aligned}\Delta\rho^{(1)}_{\mathrm{ph}\pm} &= -\frac{11833}{280665} \mp \frac{1894454\chi}{841995\sqrt{3}} + \frac{7366829759\chi^2}{3831077250} \mp \frac{63500581373\chi^3}{51719542875\sqrt{3}} \\ &\quad + \frac{4499912684330179\chi^4}{5613018549138000} \mp \frac{2518625711779631\chi^5}{16839055647414000\sqrt{3}} \\ &\quad + \frac{39043683908212961\chi^6}{237415044638772000} + \mathcal{O}\left(\chi^7\right) \ ,\end{aligned} \tag{G.44}$$

$$\begin{aligned}\Delta\rho^{(2)}_{\mathrm{ph}\pm} &= \pm\frac{124\chi}{81\sqrt{3}} - \frac{27237253\chi^2}{27243216} \pm \frac{7143579103\chi^3}{3677834160\sqrt{3}} - \frac{4930918052561\chi^4}{8018597927340} \\ &\quad \pm\frac{941808834424915\chi^5}{538849780717248\sqrt{3}} - \frac{105521162301612787\chi^6}{151476660579404160} + \mathcal{O}\left(\chi^7\right) \ ,\end{aligned} \tag{G.45}$$

$$\begin{aligned}\Delta\rho^{(\mathrm{ev})}_{\mathrm{ph}\pm} &= \frac{424}{6237} \mp \frac{656\chi}{693\sqrt{3}} + \frac{11087308\chi^2}{15324309} \mp \frac{88055819\chi^3}{91945854\sqrt{3}} + \frac{18900112949\chi^4}{44547766263} \\ &\quad \mp\frac{2387981426735\chi^5}{3207439170936\sqrt{3}} + \frac{965001464261\chi^6}{2874198737592} + \mathcal{O}\left(\chi^7\right) \ .\end{aligned} \tag{G.46}$$

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