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3D bosons, 3-Jack polynomials and affine Yangian of $\mathfrak{gl}(1)$

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Abstract: 3D (3 dimensional) Young diagrams are a generalization of 2D Young diagrams. In this paper, We consider 3D Bosons and 3-Jack polynomials. We associate three parameters h_1, h_2, h_3 to y, x, z-axis respectively. 3-Jack polynomials are polynomials of $P_{n,j}, n \geq j$ with coefficients in $\mathbb{C}(h_1, h_2, h_3)$, which are the generalization of Schur functions and Jack polynomials to 3D case. Similar to Schur functions, 3-Jack polynomials can also be determined by the vertex operators and the Pieri formulas.

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Contents

1 Introduction

2D Young diagrams and symmetric functions are attractive research objects, which were used to determine irreducible characters of highest weight representations of the classical groups [1–3]. Recently they appear in mathematical physics, especially in integrable models. In [4], the group in the Kyoto school uses Schur functions in a remarkable way to understand the KP and KdV hierarchies. In [5, 6], Tsilevich and Sułkowski, respectively, give the realization of the phase model in the algebra of Schur functions and build the relations between the q-boson model and Hall-Littlewood functions. In [7], we build the relations between the statistical models, such as phase model, and KP hierarchy by using 2D Young diagrams and Schur functions. In [8], the authors show that the states in the β-deformed Hurwitz-Kontsevich matrix model can be represented as the Jack polynomials.

3D Young diagrams (plane partitions) are a generalization of 2D Young diagrams, which arose naturally in crystal melting model [9, 10]. 3D Young diagrams also have many applications in many fields of mathematics and physics, such as statistical models, number theory, representations of some algebras (Ding-Iohara-Miki algebras, affine Yangian, etc). In this paper, we consider 3D Bosons and 3D symmetric functions.

The Schur functions S_λ defined on 2D Young diagrams λ can be determined by the vertex operator and the Jacobi-Trudi formula. Let $p = (p_1, p_2, \cdots)$. The operators $S_n(p)$ are determined by the vertex operator:

$$e^{\xi(p,z)} = \sum_{n=0}^{\infty} S_n(p) z^n, \quad \text{with} \quad \xi(p,z) = \sum_{n=1}^{\infty} \frac{p_n}{n} z^n \tag{1.1}$$

and set $S_n(p) = 0$ for $n < 0$. Note that $S_n(p)$ is the complete homogeneous symmetric function by the Miwa transform, i.e., replacing p_i with the power sum $\sum_k x_k^i$. For 2D

Young diagram $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, the Schur function $S_\lambda = S_\lambda(p)$ is a polynomial of variables p in $\mathbb{C}[p]$ defined by the Jacobi-Trudi formula [1, 2]:

$$S_\lambda(p) = \det\left(S_{\lambda_i - i + j}(p)\right)_{1 \le i,j \le l}. \tag{1.2}$$

The Jacobi-Trudi formula can be replaced by the Pieri formula [1]

$$S_n S_\lambda = \sum_\mu C^\mu_{n,\lambda} S_\mu. \tag{1.3}$$

The equivalence between the Jacobi-Trudi formula and the Pieri formula can be proved by induction: the Jacobi-Trudi formula

$$S_{(n,1)} = \det\begin{pmatrix} S_n & S_{n+1} \\ 1 & S_1 \end{pmatrix}$$

is the same with the Pieri formula $S_n S_1 = S_{n+1} + S_{(n,1)}$. The Jacobi-Trudi formula

$$S_{(n,2)} = \det\begin{pmatrix} S_n & S_{n+1} \\ S_1 & S_2 \end{pmatrix} = S_n S_2 - S_{n+1} S_1 = S_n S_2 - (S_{n+2} + S_{(n+1,1)})$$

is equivalent to the Pieri formula $S_n S_2 = S_{n+2} + S_{(n+1,1)} + S_{(n,2)}$. Inductively, The Jacobi-Trudi formula for $S_{(n,m)}$ is equivalent to the Pieri formula for $S_n S_m$. The Jacobi-Trudi formula

$$\begin{aligned} S_{(\lambda_1,\lambda_2,1)} &= \det\begin{pmatrix} S_{\lambda_1} & S_{\lambda_1+1} & S_{\lambda_1+2} \\ S_{\lambda_2-1} & S_{\lambda_2} & S_{\lambda_2+1} \\ 0 & 1 & S_1 \end{pmatrix} \\ &= S_{(\lambda_1,\lambda_2)} S_1 - S_{\lambda_1} S_{\lambda_2+1} + S_{\lambda_1+2} S_{\lambda_2-1}. \end{aligned}$$

If $\lambda_1 > \lambda_2$, we have

$$-S_{\lambda_1} S_{\lambda_2+1} + S_{\lambda_1+2} S_{\lambda_2-1} = -S_{(\lambda_1,\lambda_2+1)} - S_{(\lambda_1+1,\lambda_2)},$$

and if $\lambda_1 = \lambda_2$, we have

$$-S_{\lambda_1} S_{\lambda_2+1} + S_{\lambda_1+2} S_{\lambda_2-1} = -S_{(\lambda_1+1\lambda_1)}.$$

They show that the Jacobi-Trudi formula for $S_{(\lambda_1,\lambda_2,1)}$ is equivalent to the Pieri formula for $S_{(\lambda_1,\lambda_2)} S_1$. The Jacobi-Trudi formula

$$\begin{aligned} S_{(\lambda_1,\lambda_2,2)} &= S_{(\lambda_1,\lambda_2)} S_2 - S_{(\lambda_1,1)} S_{\lambda_2+1} + S_{(\lambda_2-1,1)} S_{\lambda_1+2} \\ &= S_{(\lambda_1,\lambda_2)} S_2 - (S_{\lambda_1} S_{\lambda_2+1} - S_{\lambda_2-1} S_{\lambda_1+2}) S_1 + S_{\lambda_1+1} S_{\lambda_2+1} - S_{\lambda_2} S_{\lambda_1+2} \end{aligned}$$

is equivalent to the Pieri formula for $S_{(\lambda_1,\lambda_2)} S_2$. Inductively, The Jacobi-Trudi formula for $S_{(\lambda_1,\lambda_2,\lambda_3)}$ is equivalent to the Pieri formula for $S_{(\lambda_1,\lambda_2)} S_{\lambda_3}$. For any 2D Young diagrams $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, it can be proved by induction that the Jacobi-Trudi formula for $S_{(\lambda_1,\lambda_2,\cdots,\lambda_l)}$ is equivalent to the Pieri formula for $S_{(\lambda_1,\lambda_2,\cdots,\lambda_{l-1})} S_{\lambda_l}$.

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The equivalence between the Jacobi-Trudi formula and the Pieri formula shows that the Schur functions defined on 2D Young diagrams can be determined by the vertex operator (1.1) and the Pieri formula. The 3-Jack polynomials, which are symmetric functions defined on 3D Young diagrams and the generalization of Schur functions, can also be determined this way. We associate three parameters h_1, h_2, h_3 to y, x, z-axis respectively, where h_1, h_2, h_3 are the parameters appeared in affine Yangian of $\mathfrak{gl}(1)$. 3-Jack polynomials are symmetric about three coordinate axes, which means they are symmetric about h_1, h_2, h_3. Let (n) be the 2D Young diagrams of n box along y-axis, which is treated as 3D Young diagram which have one layer in z-axis direction, then 3-Jack polynomials $\tilde{J}_n$ can be determined by the vertex operator:

$$e^{\xi_y(P,z)} = \sum_{n\geq 0} \frac{1}{\langle \tilde{J}_n, \tilde{J}_n \rangle h_1^n} \tilde{J}_n(P) z^n, \tag{1.4}$$

with

$$\xi_y(P,z) = \sum_{n,j=1}^{\infty} \frac{P_{n,j}}{j! \begin{pmatrix} n+j-1 \\ n-j \end{pmatrix}} \frac{d_{n,j}}{h_1^j} \prod_{k=1}^{j-1} \frac{1}{k+h_2 h_3} z^n, \tag{1.5}$$

where

$$d_{n,j} = \begin{cases} 1 & \text{if } \ n=j, \\ j & \text{if } n>j. \end{cases} \tag{1.6}$$

When $P_{n,1} = p_1,\ P_{n,j\geq 1} = 0$, the vertex operator (1.4) becomes (1.1). The 3-Jack polynomials of n boxes along x-axis or z-axis can also expressed this way from the symmetry. In fact, for 2D Young diagrams λ, which are treated as the 3D Young diagrams who have one layer in z-axis direction, 3-Jack polynomials $\tilde{J}_\lambda$ can also be determined by the vertex operators. 3-Jack polynomials of 3D Young diagrams who have more than one layer can be determined by the "Pieri formula" $\tilde{J}_\lambda \tilde{J}_\pi$ with π being 3D Young diagrams. This Pieri formula can be determined by the representation of affine Yangian of $\mathfrak{gl}(1)$ on 3D Young diagrams.

The paper is organized as follows. In section 2, we recall the definition of affine Yangian of $\mathfrak{gl}(1)$ and its representation on 3D Young diagrams. In section 3, we consider 3D Bosons and the algebra W. In section 4, we discuss the realizations of 3D Bosons and the operators in the algebra W by using affine Yangian of $\mathfrak{gl}(1)$ and its representation on 3D Young diagrams. In section 5, we show the expressions of 3-Jack polynomials by the vertex operators and the Pieri formulas.

2 Affine Yangian of $\mathfrak{gl}(1)$

In this section, we recall the definition of the affine Yangian of $\mathfrak{gl}(1)$ as in papers [11–14] first. Then we calculate some properties of affine Yangian which have relations with 3D Bosons. The affine Yangian $\mathcal{Y}$ of $\mathfrak{gl}(1)$ is an associative algebra with generators e_j, f_j and

ψ_j, $j = 0, 1, \ldots$ and the following relations [13, 14]

$$[\psi_j, \psi_k] = 0, \tag{2.1}$$

$$[e_{j+3}, e_k] - 3[e_{j+2}, e_{k+1}] + 3[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\ +\sigma_2 [e_{j+1}, e_k] - \sigma_2 [e_j, e_{k+1}] - \sigma_3 \{e_j, e_k\} = 0, \tag{2.2}$$

$$[f_{j+3}, f_k] - 3[f_{j+2}, f_{k+1}] + 3[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}] \\ +\sigma_2 [f_{j+1}, f_k] - \sigma_2 [f_j, f_{k+1}] + \sigma_3 \{f_j, f_k\} = 0, \tag{2.3}$$

$$[e_j, f_k] = \psi_{j+k}, \tag{2.4}$$

$$[\psi_{j+3}, e_k] - 3[\psi_{j+2}, e_{k+1}] + 3[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\ +\sigma_2 [\psi_{j+1}, e_k] - \sigma_2 [\psi_j, e_{k+1}] - \sigma_3 \{\psi_j, e_k\} = 0, \tag{2.5}$$

$$[\psi_{j+3}, f_k] - 3[\psi_{j+2}, f_{k+1}] + 3[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}] \\ +\sigma_2 [\psi_{j+1}, f_k] - \sigma_2 [\psi_j, f_{k+1}] + \sigma_3 \{\psi_j, f_k\} = 0, \tag{2.6}$$

together with boundary conditions

$$[\psi_0, e_j] = 0, \qquad [\psi_1, e_j] = 0, \qquad [\psi_2, e_j] = 2e_j, \tag{2.7}$$

$$[\psi_0, f_j] = 0, \qquad [\psi_1, f_j] = 0, \qquad [\psi_2, f_j] = -2f_j, \tag{2.8}$$

and a generalization of Serre relations

$$\mathrm{Sym}_{(j_1,j_2,j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]] = 0, \tag{2.9}$$

$$\mathrm{Sym}_{(j_1,j_2,j_3)} [f_{j_1}, [f_{j_2}, f_{j_3+1}]] = 0, \tag{2.10}$$

where Sym is the complete symmetrization over all indicated indices which include 6 terms. In this paper, we set $\psi_0 = 1$ with no loss of generality.

The notations σ_2, σ_3 in the definition of affine Yangian are functions of three complex numbers h_1, h_2 and h_3:

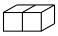
$$\sigma_1 = h_1 + h_2 + h_3 = 0,$$
$$\sigma_2 = h_1 h_2 + h_1 h_3 + h_2 h_3,$$
$$\sigma_3 = h_1 h_2 h_3.$$

This affine Yangian has the representation on 3D Young diagrams or Plane partitions. A plane partition π is a 2D Young diagram in the first quadrant of plane xOy filled with non-negative integers that form nonincreasing rows and columns [9, 15]. The number in the position (i, j) is denoted by $\pi_{i,j}$

$$\begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \cdots \\ \pi_{2,1} & \pi_{2,2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}.$$

The integers $\pi_{i,j}$ satisfy

$$\pi_{i,j} \geq \pi_{i+1,j}, \quad \pi_{i,j} \geq \pi_{i,j+1}, \quad \lim_{i\to\infty} \pi_{i,j} = \lim_{j\to\infty} \pi_{i,j} = 0$$

for all integers $i, j \geq 0$. Piling $\pi_{i,j}$ cubes over position (i, j) gives a 3D Young diagram. 3D Young diagrams arose naturally in the melting crystal model [9, 10]. We always identify 3D Young diagrams with plane partitions as explained above. For example, the 3D Young diagram  can also be denoted by the plane partition $(1, 1)$.

As in our paper [16], we use the following notations. For a 3D Young diagram π, the notation $\square \in \pi^+$ means that this box is not in π and can be added to π. Here "can be added" means that when this box is added, it is still a 3D Young diagram. The notation $\square \in \pi^-$ means that this box is in π and can be removed from π. Here "can be removed" means that when this box is removed, it is still a 3D Young diagram. For a box $\square$, we let

$$h_\square = h_1 y_\square + h_2 x_\square + h_3 z_\square, \tag{2.11}$$

where $(x_\square, y_\square, z_\square)$ is the coordinate of box $\square$ in coordinate system $O - xyz$. Here we use the order $y_\square, x_\square, z_\square$ to match that in paper [13].

Following [13, 14], we introduce the generating functions:

$$\begin{aligned} e(u) &= \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \\ f(u) &= \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \\ \psi(u) &= 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}, \end{aligned} \tag{2.12}$$

where u is a parameter. Introduce

$$\psi_0(u) = \frac{u + \sigma_3 \psi_0}{u} \tag{2.13}$$

and

$$\varphi(u) = \frac{(u + h_1)(u + h_2)(u + h_3)}{(u - h_1)(u - h_2)(u - h_3)}. \tag{2.14}$$

For a 3D Young diagram π, define $\psi_\pi(u)$ by

$$\psi_\pi(u) = \psi_0(u) \prod_{\square \in \pi} \varphi(u - h_\square). \tag{2.15}$$

In the following, we recall the representation of the affine Yangian on 3D Young diagrams as in paper [13] by making a slight change. The representation of affine Yangian on 3D Young diagrams is given by

$$\psi(u)|\pi\rangle = \psi_\pi(u)|\pi\rangle, \tag{2.16}$$

$$e(u)|\pi\rangle = \sum_{\square \in \pi^+} \frac{E(\pi \to \pi + \square)}{u - h_\square} |\pi + \square\rangle, \tag{2.17}$$

$$f(u)|\pi\rangle = \sum_{\square \in \pi^-} \frac{F(\pi \to \pi - \square)}{u - h_\square} |\pi - \square\rangle \tag{2.18}$$

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where $|\pi\rangle$ means the state characterized by the 3D Young diagram π and the coefficients

$$E(\pi \to \pi + \square) = -F(\pi + \square \to \pi) = \sqrt{\frac{1}{\sigma_3} \operatorname{res}_{u \to h_\square} \psi_\pi(u)} \tag{2.19}$$

Equations (2.17) and (2.18) mean generators e_j, f_j acting on the 3D Young diagram π by

$$e_j|\pi\rangle = \sum_{\square \in \pi^+} h_\square^j E(\pi \to \pi + \square)|\pi + \square\rangle, \tag{2.20}$$

$$f_j|\pi\rangle = \sum h_\square^j F(\pi \to \pi - \square)|\pi - \square\rangle. \tag{2.21}$$

The triangular decomposition of affine Yangian $\mathcal{Y}$ is

$$\mathcal{Y} = \mathcal{Y}^+ \oplus \mathcal{B} \oplus \mathcal{Y}^- \tag{2.22}$$

where $\mathcal{Y}^+$ is the subalgebra generated by generators e_j with relations (2.2) and (2.9), $\mathcal{B}$ is the commutative subalgebra with generators ψ_j, $\mathcal{Y}^-$ is the subalgebra generated by generators f_j with relations (2.3) and (2.10).

Define the anti-automorphism $\tilde{a}$ by

$$\tilde{a}(e_j) = -f_j \tag{2.23}$$

The quadratic form on $\mathcal{Y}^+|0\rangle$ is defined by

$$\tilde{B}(x|0\rangle, y|0\rangle) = \langle 0|\tilde{a}(y)x|0\rangle \tag{2.24}$$

where $x, y \in \mathcal{Y}^+$. Note that the quadratic form here is different from the Shapovalov form in [13]. For 3D Young diagrams π, π' and let $\pi = x|0\rangle, \pi' = y|0\rangle$ for $x, y \in \mathcal{Y}^+$, define the orthogonality

$$\langle \pi', \pi\rangle = \langle 0|\tilde{a}(y)x|0\rangle. \tag{2.25}$$

As in paper [13], when $n > 0$, $a_{n,1}$ and $a_{-n,1}$ are defined to be

$$a_{n,1} := -\frac{1}{(n-1)!}\mathrm{ad}_{f_1}^{n-1} f_0, \tag{2.26}$$

$$a_{-n,1} := \frac{1}{(n-1)!}\mathrm{ad}_{e_1}^{n-1} e_0. \tag{2.27}$$

The set

$$\{\mathrm{ad}_{e_1}^{n-1} e_0, n = 1, 2, 3, \cdots\}$$

is denoted by $\diamondsuit$. That the operator A commutes with $\diamondsuit$ means A commute with every element in $\diamondsuit$.

Proposition 2.1. *If the operator A commutes with $\diamondsuit$, so does $[e_1, A]$.*

Proof. That A commutes with $\diamondsuit$ means $[A, \mathrm{ad}_{e_1}^{n-1} e_0] = 0$ for $n = 1, 2, 3, \cdots$. By Jacobi identity,

$$[[e_1, A], \mathrm{ad}_{e_1}^{n-1} e_0] = -[[A, \mathrm{ad}_{e_1}^{n-1} e_0], e_1] - [[\mathrm{ad}_{e_1}^{n-1} e_0, e_1], A] = 0.$$

□

Then we get the following result.

Proposition 2.2. *If the operator A commutes with $\diamondsuit$, then we get that $ad_{e_1}^{n-1}A$ for $n = 1, 2, 3, \cdots$ commute with $\diamondsuit$.*

By similar calculation, we can get that if the operator B commutes with $\mathrm{ad}_{e_1}^{n-1}e_0$ and $\mathrm{ad}_{e_1}^{n-1}A$ for $n = 1, 2, 3, \cdots$, so do $\mathrm{ad}_{e_1}^{n-1}B$ for $n = 1, 2, 3, \cdots$.

3 3D bosons

Introduce the space of all polynomials

$$\mathbb{C}[p] = \mathbb{C}[p_1, p_{2,1}, p_{2,2}, p_{3,1}, p_{3,2}, p_{3,3}, \cdots].$$

Every polynomial is a function of infinitely many variables

$$p = (p_1, p_{2,1}, p_{2,2}, p_{3,1}, p_{3,2}, p_{3,3}, \cdots),$$

but each polynomial itself is a finite sum of monomials, so involves only finitely many of the variables.

Define the weight of $p_{n,k}$ to be n, and the weight of monomial $p_{n_1,k_1}^{l_1} \cdots p_{n_s,k_s}^{l_s}$ to be $l_1 n_1 + \cdots + l_s n_s$, then the 3D Bosonic Fock space is written into

$$\mathbb{C}[p] = \bigoplus_{n=0}^{\infty} \mathbb{C}[p]_n \tag{3.1}$$

where $\mathbb{C}[p]_n$ is the space of polynomials of weight n, which is a subspace of $\mathbb{C}[p]$. The basis of $\mathbb{C}[p]_0$ is 1, the basis of $\mathbb{C}[p]_1$ is p_1, the basis of $\mathbb{C}[p]_2$ is $p_1^2, p_{2,1}, p_{2,2}$, the basis of $\mathbb{C}[p]_3$ is $p_1^3, p_1 p_{2,1}, p_1 p_{2,2}, p_{3,1}, p_{3,2}, p_{3,3}$, we can write the basis of every subspace $\mathbb{C}[p]_n$. By the results in [17], we know that the elements in the basis of $\mathbb{C}[p]_n$ is one to one correspondence with 3D Young diagrams of total box number n. Then we have

$$\sum_{n=0}^{\infty} \dim(\mathbb{C}[p]_n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \tag{3.2}$$

where the notation $\dim(\mathbb{C}[p]_n)$ means the dimension of the vector space $\mathbb{C}[p]_n$.

Define the form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}[p]$ by $\langle p_{n,j}, p_{m,i} \rangle = 0$ unless $n = m,\ i = j$. When $i = j = 1$, $\langle p_{n,1}, p_{m,1} \rangle$ equal $\langle p_n, p_m \rangle = n\delta_{n,m}$ in the 2D Bosons and Schur functions. Define

$$\langle p_{j,j}, p_{j,j} \rangle = (-1)^{j-1} j! \prod_{k=1}^{j-1} (k^3 + k^2\sigma_2 + \sigma_3^2) \tag{3.3}$$

and

$$\langle p_{n,j}, p_{n,j} \rangle = \binom{n+j-1}{n-j} \langle p_{j,j}, p_{j,j} \rangle. \tag{3.4}$$

We give a remark: when $j > 1$, there is a factor $(1 + \sigma_2 + \sigma_3^2)$ in $\langle p_{j,j}, p_{j,j} \rangle$. This shows that when $h_1 = 1, h_2 = -1$, $\langle p_{j,j}, p_{j,j} \rangle = 0$ if $j > 1$. This result matches that $P_{n,j>1}$ become

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zero and 3-Jack polynomials become Schur functions when $h_1 = 1, h_2 = -1$ [11]. When $j = 1$, equation (3.4) becomes $\langle p_{n,1}, p_{n,1}\rangle = \begin{pmatrix} n \\ n-1 \end{pmatrix} \langle p_1, p_1\rangle = n$.

Consider operators $b_{n,k}$, where $n \in \mathbb{Z}, n \neq 0$, and $k \in \mathbb{Z}, 0 < k \leq |n|$, with the commutation relations

$$[b_{m,l}, b_{n,k}] = m\delta_{m+n,0}\delta_{l,k}\langle p_{l,|m|}, p_{l,|m|}\rangle. \tag{3.5}$$

The operators $b_{n,k}$ with the relations above are called 3D Bosons, and the algebra generated by $b_{n,k}$ with relations (3.5) is called 3D Heisenberg algebra, which is denoted by $\mathfrak{B}$. Using the commutation relations (3.5), we can see that any element in 3D Heisenberg algebra can be expressed in a unique way as a linear combination of the following elements:

$$b^{\alpha_1}_{-m_1,l_1} \cdots b^{\alpha_s}_{-m_s,l_s} b^{\beta_1}_{n_1,k_1} \cdots b^{\beta_t}_{n_t,k_t}$$

for

$$(-m_1, l_1) < \cdots < (-m_s, l_s), \quad (n_1, k_1) < \cdots < (n_t, k_t), \quad \alpha_i, \beta_i = 1, 2, \cdots,$$

where the notation $(m, l) < (n, k)$ means $m < n$ or $m = n, l < k$.

Define a linear map $\rho : \mathfrak{B} \to \mathrm{End}(\mathbb{C}[p])$ by

$$\rho(b_{-n,k}) = p_{n,k}, \quad \rho(b_{n,k}) = \langle p_{n,k}, p_{n,k}\rangle n \frac{\partial}{\partial p_{n,k}}, \tag{3.6}$$

for $n > 0$, which gives a representation of the Heisenberg algebra $\mathfrak{B}$ on polynomial space $\mathbb{C}[p]$. The representation space $\mathbb{C}[p]$ is called the 3D Bosonic Fock space. We call the operators $b_{n,k}$ annihilation operators and $b_{-n,k}$ creation operators for $n > 0$. From the commutation relations (3.5), it is easy to find that all the creation operators commute among themselves, so do all the annihilation operators. The element $1 \in \mathbb{C}[p]$ is called the vacuum state. Every annihilation operators kill the vacuum state, that is, $n\frac{\partial}{\partial p_{n,k}} 1 = 0$. The 3D Bosonic Fock space is generated by the vacuum state:

$$\mathbb{C}[p] = \mathfrak{B} \cdot 1 := \{b \cdot 1 | b \in \mathfrak{B}\}. \tag{3.7}$$

We give a remark to explain why we use "3D" in the 3D Boson and 3D Heisenberg algebra. 3D is used to distinguish 3D Boson and ordinary Boson, 3D Heisenberg algebra and ordinary Heisenberg algebra. Similar to that the ordinary Bosonic Fock space is isomorphic to the space of 2D Young diagrams, the 3D Bosonic Fock space is isomorphic to the space of 3D Young diagrams, this is the reason we use the notation 3D.

Since the 3D Bosons $b_{n,k\geq 2}$ can not be represented by the generators of affine Yangian of $\mathfrak{gl}(1)$, instead in the following, we consider the algebra W with the generators $a_{n,k}$, $(n \in \mathbb{Z}, n \neq 0$, and $k \in \mathbb{Z}, 0 < k \leq |n|)$. We let $a_{n,1} = b_{n,1}, (n \in \mathbb{Z}, n \neq 0)$ and $a_{-2,2} = b_{-2,2}$, other relations of $a_{n,j}$ and $b_{n,j}$ have not been found. We think that the 3D Heisenberg algebra $\{b_{n,j}\}$ and the algebra W can be represented by each other, we will discuss these next if we can obtain the results. In the following of this paper, we discuss the algebra W instead of the 3D Heisenberg algebra $\{b_{n,j}\}$. The commutation relations in W are

$$[a_{n,1}, a_{m,1}] = n\delta_{n+m,0}c_1 = \psi_0 n\delta_{n+m,0}, \tag{3.8}$$
$$[a_{n,1}, a_{m,j}] = 0, \quad \text{when} \quad j > 1, \tag{3.9}$$

and when $j>1,\ k>1$,

$$[a_{m,j},a_{n,k}]=\sum_{\substack{0\leq l\leq j+k-2\\ j+k-l\text{even}}}\frac{j!k!}{l!}C^l_{jk}N^l_{jk}(m,n)a_{m+n,l}, \tag{3.10}$$

where the coefficients $N^l_{jk}(m,n)$ are

$$\begin{aligned}N^0_{jk}(m,n)&=\binom{m+j-1}{j+k-1}\delta_{m+n,0}, \qquad (3.11)\\ N^l_{jk}(m,n)&=c^l_{jk}(m,n)\sum_{s=0}^{j+k-l-1}\frac{(-1)^s}{(j+k-l-1)!(2l)_{j+k-l-1}}\binom{j+k-l-1}{s}\\ &\quad\times[j+m-1]_{j+k-l-1-s}[j-m-1]_s[k+n-1]_s[k-n-1]_{j+k-l-1-s},\end{aligned}$$

with $c^l_{jk}(m,n)$ are the functions of σ_2 and σ_3, and the structure constants C^l_{jk} are

$$\begin{aligned}C^0_{jk}&=\frac{(j-1)!^2(2j-1)!}{4^{j-1}(2j-1)!!(2j-3)!!}\delta_{jk}c_j, \qquad (3.12)\\ C^l_{jk}&=\frac{1}{2\times4^{j+k-l-2}}(2l)_{j+k-l-1}\times{}_4F_3\left(\begin{matrix}\frac{1}{2},\frac{1}{2},-\frac{1}{2}(j+k-l-2),-\frac{1}{2}(j+k-l-1)\\ \frac{3}{2}-j,\frac{3}{2}-k,\frac{1}{2}+l\end{matrix};1\right),\end{aligned}$$

with

$$(a)_n=a(a+1)\cdots(a+n-1), \tag{3.13}$$

$$[a]_n=a(a-1)\cdots(a-n+1), \tag{3.14}$$

$${}_mF_n\left(\begin{matrix}a_1,\cdots,a_m\\ b_1,\cdots,b_n\end{matrix};z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_m)_k}{(b_1)_k\cdots(b_n)_k}\frac{z^k}{k!}. \tag{3.15}$$

We give a remark here to explain the central charges c_j. The central charge c_1 of $a_{n,1}$ is ψ_0, we always take $\psi_0=1$ without loss the symmetry of affine Yangian of $\mathfrak{gl}(1)$ about three coordinate axes. The central charge c_j of $a_{n,j}$ is dependent on j. The first few of them are

$$\begin{aligned}c_2&=-2(1+\sigma_2+\sigma_3^2),\\ c_3&=6(1+\sigma_2+\sigma_3^2)(8+4\sigma_2+\sigma_3^2),\\ c_4&=-144\frac{(1+\sigma_2+\sigma_3^2)(8+4\sigma_2+\sigma_3^2)(27+9\sigma_2+\sigma_3^2)(-1+\sigma_2+\sigma_3^2)}{(-17+5\sigma_2+5\sigma_3^2)}.\end{aligned}$$

Define

$$\kappa(n)=-(n+h_1h_2)(n+h_1h_3)(n+h_2h_3)=-(n^3+n^2\sigma_2+\sigma_3^2),$$

then

$$c_2=2\kappa(1),\ c_3=6\kappa(1)\kappa(2),$$

and

$$c_4=144\frac{\kappa(1)\kappa(2)\kappa(3)\kappa(-1)}{5\kappa(1)+22}.$$

For $n>0$, define $P_{n,j}$ be the representation of $a_{-n,j}$ on the vacuum state of the space of 3D Young diagrams, that is, $P_{n,j}=a_{n,j}|0\rangle$, then the 3-Jack polynomials as vectors in the representation space are functions of $P_{n,j}$. Note that the set $\{P_{n,j}\}$ are related with $\{p_{n,j}\}$. For example, $P_{n,1}=p_{n,1}$ and $P_{2,2}=p_{2,2}$. The central charges c_2 and c_3 match $\langle P_{2,2},P_{2,2}\rangle$ and $\langle P_{3,3},P_{3,3}\rangle$ in [11] respectively, and c_4 matches $\langle P_{4,4},P_{4,4}\rangle$ in [18]. Here we choose $P_{4,4}=\overrightarrow{E}_{13}|0\rangle$, where $\overrightarrow{E}_{13}|0\rangle$ is defined in [18]:

$$\begin{aligned}
\overrightarrow{E}_{13}|0\rangle =& -\frac{72(2\sigma_3^2+3\sigma_2-3)}{5\sigma_3^2+5\sigma_2-17}e_0e_0e_0e_0|0\rangle-\frac{36\sigma_3(3\sigma_3^2+7\sigma_2+5)}{5\sigma_3^2+5\sigma_2-17}e_0e_0e_1e_0|0\rangle\\
&+\frac{144(2\sigma_3^2+3\sigma_2-3)}{5\sigma_3^2+5\sigma_2-17}e_0e_0e_2e_0|0\rangle+\frac{6}{5}\frac{5\sigma_3^4+40\sigma_2\sigma_3^2+301\sigma_3^2-25\sigma_2^2+295\sigma_2+150}{5\sigma_3^2+5\sigma_2-17}e_0e_1e_1e_0|0\rangle\\
&+\frac{12\sigma_3(\sigma_3^2-11\sigma_2-49)}{5\sigma_3^2+5\sigma_2-17}e_0e_1e_2e_0|0\rangle-\frac{6(\sigma_3^2-11\sigma_2-49)}{5\sigma_3^2+5\sigma_2-17}e_0e_2e_2e_0|0\rangle\\
&-\frac{18}{5}\frac{5\sigma_3^4+30\sigma_2\sigma_3^2+77\sigma_3^2+5\sigma_2^2+25\sigma_2+30}{5\sigma_3^2+5\sigma_2-17}e_1e_0e_1e_0|0\rangle+\frac{48\sigma_3(\sigma_3^2+4\sigma_2+8)}{5\sigma_3^2+5\sigma_2-17}e_1e_0e_2e_0|0\rangle\\
&-\frac{1}{5}\frac{\sigma_3(5\sigma_3^4+40\sigma_2\sigma_3^2+15\sigma_3^2+35\sigma_2^2-111\sigma_2-200)}{5\sigma_3^2+5\sigma_2-17}e_1e_1e_1e_0|0\rangle\\
&+6\frac{\sigma_3^4+5\sigma_2\sigma_3^2+2\sigma_3^2+4\sigma_2^2-\sigma_2+9}{5\sigma_3^2+5\sigma_2-17}e_1e_1e_2e_0|0\rangle-\frac{18(3\sigma_3^2+7\sigma_2+5)}{5\sigma_3^2+5\sigma_2-17}e_2e_0e_2e_0|0\rangle\\
&-\frac{6}{5}\sigma_3e_2e_1e_2e_0|0\rangle+e_2e_2e_2e_0|0\rangle,
\end{aligned}$$

which is a vector in the 3D Young diagram representation space of affine Yangian of $\mathfrak{gl}(1)$.

Note that the algebra W is closely related to the $W_{1+\infty}$ algebra or W_∞ algebra.

4 3D bosons in affine Yangian of $\mathfrak{gl}(1)$

We denote $[e_i,e_j]$ by $e_{i,j}$. We have the recurrence relation: when $j+k=2n$ in (2.2),

$$\begin{aligned}
e_{n+2,n+1} =& \frac{1}{2n+3}(e_{2n+3,0}+\sigma_2\sum_{j=0}^{n}(j+1)e_{2n+1-j,j}\\
&-\sigma_3\sum_{j=0}^{n-1}(1+2+\cdots+(j+1))\{e_{2n-j},e_j\}\\
&-(1+2+\cdots+(n+1)\sigma_3e_ne_n)),
\end{aligned}$$

the right hand side is denoted by $\blacktriangle$, then

$$\begin{aligned}
e_{n+3+i,n-i} =& (2i+3)\blacktriangle+(-\sigma_2)\sum_{j=0}^{i}(j+1)e_{n+1+i-j,n-i+j}\\
&+\sigma_3\sum_{j=0}^{i-1}(1+2+\cdots+(j+1))\{e_{n+i-j},e_{n-i+j}\}\\
&+(1+2+\cdots+(i+1)\sigma_3e_ne_n))
\end{aligned}$$

where $i = 0, 1, \cdots, n-1$. When $j + k = 2n + 1$ in (2.2),

$$e_{n+3,n+1} = \frac{1}{n+2}(e_{2n+4,0} + \sigma_2 \sum_{j=0}^{n} (j+1) e_{2n+2-j,j} - \sigma_3 \sum_{j=0}^{n} (1+2+\cdots+(j+1))\{e_{2n+1-j}, e_j\},$$

the right hand side is denoted by $\blacktriangledown$, then

$$e_{n+4+i,n-i} = (i+2)\blacktriangledown + (-\sigma_2) \sum_{j=0}^{i} (j+1) e_{n+2+i-j,n-i+j} + \sigma_3 \sum_{j=0}^{i} (1+2+\cdots+(j+1))\{e_{n+1+i-j}, e_{n-i+j}\},$$

where $i = 0, 1, \cdots, n-1$.

We rewrite the recurrence relation in the matrix form

Proposition 4.1. *When $j + k = 2n$ in (2.2),*

$$\begin{pmatrix} e_{2n+2,1} \\ e_{2n+1,2} \\ e_{2n,3} \\ \cdots \\ e_{n+3,n} \\ e_{n+2,n+1} \end{pmatrix} = -\sigma_2 (A_{n+1}^{-1}(I_{n+1} - J_{n+1}))_{\widehat{j_1}} \begin{pmatrix} e_{2n,1} \\ e_{2n-1,2} \\ e_{2n-2,3} \\ \cdots \\ e_{n+2,n-1} \\ e_{n+1,n} \end{pmatrix} + \sigma_3 (A_{n+1}^{-1})_{\widehat{j_1, j_{n+1}}} \begin{pmatrix} e_{2n-1,1} \\ e_{2n-2,2} \\ e_{2n-3,3} \\ \cdots \\ e_{n+2,n-2} \\ e_{n+1,n-1} \end{pmatrix}$$
$$+ \frac{e_{2n+3,0} + \sigma_2 e_{2n+1,0} - \sigma_3 e_{2n,0}}{2n+3} \begin{pmatrix} 2n+1 \\ 2n-1 \\ \cdots \\ 5 \\ 3 \\ 1 \end{pmatrix} + 2\sigma_3 A_{n+1}^{-1} \begin{pmatrix} e_0 e_{2n} \\ e_1 e_{2n-1} \\ e_2 e_{2n-2} \\ \cdots \\ e_{n-1} e_{n+1} \\ \frac{1}{2} e_n e_n \end{pmatrix} \tag{4.1}$$

where

$$A_{n+1}^{-1} = -\frac{1}{2n+1} \begin{pmatrix} 2n+1 \\ \cdots \\ 5 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 6 & \cdots & \frac{(n+1)(n+2)}{2} \end{pmatrix} + \begin{pmatrix} 0 & 1 & 3 & 6 & \cdots & \frac{n(n+1)}{2} \\ 0 & 0 & 1 & 3 & \cdots & \frac{n(n-1)}{2} \\ 0 & 0 & 0 & 1 & \cdots & \frac{(n-2)(n-1)}{2} \\ & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{4.2}$$

the matrix $I_{n+1} = (\delta_{i,j})_{1 \le i,j \le n+1}$ is the $(n+1)$ identity matrix, $J_{n+1} = (\delta_{i+1,j})_{1 \le i,j \le n+1}$, and the notation $\widehat{j_1}$ in the subscript means removing the first column of the matrix, the notation $\widehat{j_1, j_{n+1}}$ means removing the first column and the $(n+1)$th column of the matrix.

When $j+k=2n+1$ in (2.2),

$$\begin{pmatrix} e_{2n+3,1} \\ e_{2n+2,2} \\ e_{2n+1,3} \\ \cdots \\ e_{n+4,n} \\ e_{n+3,n+1} \end{pmatrix} = -\sigma_2(B_{n+1}^{-1}(I_{n+1}-J_{n+1}))_{\widehat{j_1}} \begin{pmatrix} e_{2n,1} \\ e_{2n-1,2} \\ e_{2n-2,3} \\ \cdots \\ e_{n+2,n-1} \\ e_{n+1,n} \end{pmatrix} + \sigma_3(B_{n+1}^{-1})_{\widehat{j_1}} \begin{pmatrix} e_{2n-1,1} \\ e_{2n-2,2} \\ e_{2n-3,3} \\ \cdots \\ e_{n+2,n-2} \\ e_{n+1,n-1} \end{pmatrix}$$
$$+\frac{e_{2n+4,0}+\sigma_2 e_{2n+2,0}-\sigma_3 e_{2n+1,0}}{n+2} \begin{pmatrix} n+1 \\ n \\ \cdots \\ 3 \\ 2 \\ 1 \end{pmatrix} + 2\sigma_3 B_{n+1}^{-1} \begin{pmatrix} e_0 e_{2n+1} \\ e_1 e_{2n} \\ e_2 e_{2n-1} \\ \cdots \\ e_{n-1}e_{n+2} \\ e_n e_{n+1} \end{pmatrix}, \tag{4.3}$$

where

$$B_{n+1}^{-1} = -\frac{1}{n+2} \begin{pmatrix} n+1 \\ \cdots \\ 3 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \; 3 \; 6 \cdots \frac{(n+1)(n+2)}{2} \end{pmatrix} + \begin{pmatrix} 0 & 1 & 3 & 6 & \cdots & \frac{n(n+1)}{2} \\ 0 & 0 & 1 & 3 & \cdots & \frac{n(n-1)}{2} \\ 0 & 0 & 0 & 1 & \cdots & \frac{(n-2)(n-1)}{2} \\ & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{4.4}$$

Note that

$$A_{n+1} = \begin{pmatrix} -3 & 3 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 3 \end{pmatrix}_{(n+1)\times(n+1)} \tag{4.5}$$

and

$$B_{n+1} = \begin{pmatrix} -3 & 3 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix}_{(n+1)\times(n+1)} \tag{4.6}$$

We can see that the numbers in every row are the coefficients in the relation (2.2).

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Corollary 4.2. *Every vector in the vector space spanned by e_ie_j for $i,j=0,1,2,\cdots$, where e_i are the generators of affine Yangian, can be represented as the linear combination of*

$$e_{n,0},\ \ e_me_n\ (m\leq n) \tag{4.7}$$

for $m,n=0,1,2,\cdots$.

Suppose the vector

$$\sum_{n\geq 0}d_ne_{n,0}+\sum_{m\leq n}b_{mn}e_me_n$$

with the coefficients $d_n,\ b_{mn}$, be commutative with Bosons $a_{n,1}$, we only find the zero coefficients, that is, $d_n=0$ and b_{mn}=0. But we found the vector

$$[e_2,e_0]-\sigma_3[e_1,e_0]-e_0^2-2\sum_{n=1}^{\infty}a_{-(n+2),1}a_{n,1} \tag{4.8}$$

commute with Bosons $a_{n,1}$, which is denoted by $a_{-2,2}$. The communication $[a_{-2,2},a_{-n,1}]=0,\ n>0$ holds since

$$\begin{aligned}[][[e_2,e_0]-\sigma_3[e_1,e_0],a_{-n,1}] &= 2na_{-(n+2),1},\\ \left[\sum_{m=1}^{\infty}a_{-(m+2),1}a_{m,1},a_{-n,1}\right] &= na_{-(n+2),1}.\end{aligned}$$

The communication $[a_{-2,2},a_{n,1}]=0,n>0$ holds since

$$\begin{aligned}[][[e_2,e_0]-\sigma_3[e_1,e_0],a_{-n,1}] &= -2na_{(n-2),1},\\ \left[\sum_{m=1}^{\infty}a_{-(m+2),1}a_{m,1},a_{n,1}\right] &= -na_{(n-2),1}.\end{aligned}$$

For $n\geq 2$, define

$$a_{-(n+1),2}=\frac{1}{(n-1)!}\mathrm{ad}_{e_1}^{n-1}\left([e_2,e_0]-\sigma_3[e_1,e_0]\right)-\sum_{i+j=-(n+1)}:a_{i,1}a_{j,1}:. \tag{4.9}$$

Similarly, define

$$a_{n+1,2}=-\frac{1}{(n-1)!}\mathrm{ad}_{f_1}^{n-1}\left([f_2,f_0]-\sigma_3[f_1,f_0]\right)-\sum_{i+j=(n+1)}:a_{i,1}a_{j,1}:. \tag{4.10}$$

We denote the first term in $a_{n,2}$ by $2L_n$, and the second term by $2\bar{L}_n$. Then we have that L_n and $\bar{L}_n$ separately satisfy the Virasoro relations, that is,

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}(-\sigma_2-\sigma_3^2), \tag{4.11}$$

and

$$[\bar{L}_m,\bar{L}_n]=(m-n)\bar{L}_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}. \tag{4.12}$$

By calculation, we can prove the following relations hold

Proposition 4.3. *For $n \geq 2$,*

$$\frac{1}{n-1} e_1 \bar{L}_{-n}|0\rangle = \bar{L}_{-(n+1)}|0\rangle. \tag{4.13}$$

This result means

Proposition 4.4. *For $n \geq 2$,*

$$a_{-n,2}|0\rangle = P_{n,2} = \frac{1}{(n-2)!} \underbrace{e_1 \cdots e_1}_{n-2} a_{-2,2}|0\rangle. \tag{4.14}$$

Note that

$$P_{2,2} = (e_2 e_0 - \sigma_3 e_1 e_0 - e_0 e_0)|0\rangle,$$

which equals $\sqrt{(1+h_1h_2)(1+h_1h_3)(1+h_2h_3)} P_{2,2}$ in [11], and

$$P_{3,2} = e_1(e_2 e_0 - \sigma_3 e_1 e_0 - e_0 e_0)|0\rangle = \frac{1}{2}(e_2 - 2\sigma_3 e_1 - 4e_0)e_1 e_0|0\rangle$$

which equals $2\sqrt{(1+h_1h_2)(1+h_1h_3)(1+h_2h_3)} P_{3,2}$ in [11].

From (2.7) and (2.8), we have

$$[\psi_2, a_{n,1}] = -2n a_{n,1} \tag{4.15}$$

for any n. Then we have

Lemma 4.5. *For any $n \neq 0$,*

$$[e_1, a_{n,1}] = -n a_{n-1,1}. \tag{4.16}$$

Proof. When $n < 0$, it holds clearly. When $n > 0$, it can be proved by induction. □

Proposition 4.6. *For $n \geq 1$,*

$$\frac{1}{(n-1)}[e_1, \bar{L}_{-n}] = \bar{L}_{-(n+1)}. \tag{4.17}$$

Proof. Since the relation (4.13) hold, we only need to prove

$$\frac{1}{(n-1)}[e_1, (\bar{L}_{-n})_+] = (\bar{L}_{-(n+1)})_+,$$

where $(\bar{L}_{-n})_+$ include all terms which have annihilation operators in $\bar{L}_{-n}$, that is,

$$(\bar{L}_{-n})_+ = \sum_{k\geq 1} a_{-(n+k)} a_k.$$

Then

$$\begin{aligned}
[e_1, (\bar{L}_{-n})_+] &= \sum_{k\geq 1}([e_1, a_{-(n+k)}]a_k + a_{-(n+k)}[e_1, a_k]) \\
&= (n-1)\sum_{k\geq 1} a_{-(n+k+1)} a_k = (n-1)(\bar{L}_{-(n+1)})_+,
\end{aligned}$$

which means the result holds. □

Note that here

$$e_1 = L_{-1} \neq \bar{L}_{-1}.$$

Proposition 4.7. *For $n \geq 2$,*

$$a_{-n,2} = \frac{1}{(n-2)!} ad_{e_1}^{n-2} a_{-2,2}. \tag{4.18}$$

In the representation on 3D Young diagrams,

$$P_{n,2} = \frac{1}{(n-2)!} ad_{e_1}^{n-2} P_{2,2}. \tag{4.19}$$

Similarly to the creation operators, the annihilation operators satisfy the following relations.

Proposition 4.8.

$$[f_1, a_{n,1}] = n a_{n+1,1}, \quad \text{for} \quad n \neq 0, \tag{4.20}$$

and

$$a_{n,2} = \frac{1}{(n-2)!} ad_{f_1}^{n-2} a_{2,2}, \tag{4.21}$$

with

$$a_{2,2} = -[f_2, f_0] + \sigma_3 [f_1, f_0] - f_0^2 - 2\sum_{n=1}^{\infty} a_{-n} a_{n+2}. \tag{4.22}$$

Acting on the vacuum state of the dual space of the space spanned by 3D Young diagrams, the representation of $a_{n,2}$ is denoted by $P^{\perp}_{n,2}$,

$$\langle 0| P^{\perp}_{2,2} = \langle 0| a_{2,2} = \langle 0|(f_0 f_2 - \sigma_2 f_0 f_1 - f_1 f_0),$$

which equals the dual state of $P_{2,2}|0\rangle = (e_2 e_0 - \sigma_3 e_1 e_0 - e_0 e_0)|0\rangle$. We have know that [11]

$$\begin{aligned} \langle 0| P^{\perp}_{2,2} P_{2,2} |0\rangle &= \langle 0|(f_0 f_2 - \sigma_2 f_0 f_1 - f_1 f_0)(e_2 e_0 - \sigma_3 e_1 e_0 - e_0 e_0)|0\rangle \\ &= -2(1 + \sigma_2 + \sigma_3^2). \end{aligned}$$

Since

$$f_1 P_{2,2}|0\rangle = 0,$$

we have

$$\begin{aligned} &\langle 0|(f_0 f_2 - \sigma_2 f_0 f_1 - f_1 f_0) \underbrace{f_1 \cdots f_1}_{n+1} \underbrace{e_1 \cdots e_1}_{n+1} (e_2 e_0 - \sigma_3 e_1 e_0 - e_0 e_0)|0\rangle \\ &= \langle 0|(f_0 f_2 - \sigma_2 f_0 f_1 - f_1 f_0) \underbrace{f_1 \cdots f_1}_{n} f_1 e_1 \underbrace{e_1 \cdots e_1}_{n} (e_2 e_0 - \sigma_3 e_1 e_0 - e_0 e_0)|0\rangle \\ &= \langle 0|(f_0 f_2 - \sigma_2 f_0 f_1 - f_1 f_0) \underbrace{f_1 \cdots f_1}_{n} e_1 f_1 \underbrace{e_1 \cdots e_1}_{n} (e_2 e_0 - \sigma_3 e_1 e_0 - e_0 e_0)|0\rangle \\ &\quad -2(n+2)\langle 0|(f_0 f_2 - \sigma_2 f_0 f_1 - f_1 f_0) \underbrace{f_1 \cdots f_1}_{n} \underbrace{e_1 \cdots e_1}_{n} (e_2 e_0 - \sigma_3 e_1 e_0 - e_0 e_0)|0\rangle \\ &= -(n+4)(n+1)\langle 0|(f_0 f_2 - \sigma_2 f_0 f_1 - f_1 f_0) \underbrace{f_1 \cdots f_1}_{n} \underbrace{e_1 \cdots e_1}_{n} (e_2 e_0 - \sigma_3 e_1 e_0 - e_0 e_0)|0\rangle. \end{aligned}$$

From

$$\langle 0|a_{n,2} = \frac{(-1)^n}{(n-2)!}\langle 0|a_{2,2}\underbrace{f_1\cdots f_1}_{n-2},$$

we have

$$\langle 0|a_{n,2}a_{-n,2}|0\rangle = \frac{n+1}{n-2}\langle 0|a_{n-1,2}a_{-(n-1),2}|0\rangle. \tag{4.23}$$

Then

$$\langle 0|a_{n,2}a_{-n,2}|0\rangle = -2\binom{n+1}{n-2}(1+\sigma_2+\sigma_3^2). \tag{4.24}$$

Since

$$L_2 = -\frac{1}{2}\left([f_2,f_0]-\sigma_3[f_1,f_0]\right),\ L_{-2} = \frac{1}{2}\left([e_2,e_0]-\sigma_3[e_1,e_0]\right),$$

we have

$$[L_2,a_{n,1}] = -na_{n+2,1},\quad [L_{-2},a_{n,1}] = -na_{n-2,1}.$$

Then

$$\begin{aligned}
[L_2,\bar{L}_{-2}] &= \frac{1}{2}[L_2,a_{-1,1}^2+2\sum_{n=1}^{\infty}a_{-(n+2),1}a_{n,1}]\\
&= \frac{1}{2}(a_{1,1}a_{-1,1}+a_{-1,1}a_{1,1})+\sum_{n=1}^{\infty}\left((n+2)a_{-n,1}a_{n,1}-na_{-(n+2),1}a_{n+2,1}\right)\\
&= \frac{1}{2}+4\bar{L}_0.
\end{aligned}$$

Similarly, $[\bar{L}_2,L_{-2}] = \frac{1}{2}+4\bar{L}_0$. Therefore,

$$[a_{2,2},a_{-2,2}] = 8a_{0,2}-2(1+\sigma_2+\sigma_3^2), \tag{4.25}$$

where

$$a_{0,2} = \psi_2-\sum_{j=1}^{\infty}a_{-j}a_j.$$

This commutation relation (4.25) is the same with that in (3.10). Other relations $[a_{n,2},a_{-n,2}]$ can be calculate this way. By (4.25), we have

$$\langle P_{2,2}^{n+1},P_{2,2}^{n+1}\rangle = (n+1)\left(-2(1+\sigma_2+\sigma_3^2)+16n\right)\langle P_{2,2}^n,P_{2,2}^n\rangle. \tag{4.26}$$

Then

$$\langle P_{2,2}^n,P_{2,2}^n\rangle = n!\prod_{j=1}^{n}\left(-2(1+\sigma_2+\sigma_3^2)+16(j-1)\right). \tag{4.27}$$

For $n\geq j$, we also have

$$\langle P_{n,j},P_{n,j}\rangle = \binom{n+j-1}{n-j}\langle P_{j,j},P_{j,j}\rangle. \tag{4.28}$$

This equation can be obtained by the similar calculation to get equation (4.23).

5 3-Jack polynomials

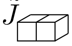
In this section, we want to obtain the expression of 3-Jack polynomials $\tilde{J}_\pi$ for any 3D Young diagram π. It is known that Schur functions S_λ can be determined by

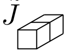
$$e^{\sum_{n=1}^{\infty}\frac{p_n}{n}z^n} = \sum_{n\geq 0} S_n z^n, \tag{5.1}$$

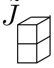
$$S_\lambda = \det(S_{\lambda_i - i + j})_{1\leq i,j\leq l} \tag{5.2}$$

for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_2)$. The formula (5.2) is equivalent to the Pieri formula $S_n S_\lambda = \sum_\mu C^\mu_{n,\lambda} S_\mu$, where the Pieri formula can be found in [2]. Here we treat 2D Young diagrams λ as the special 3D Young diagrams which have one layer in z-axis direction. For 3-Jack polynomials, we need to know the formula for $\tilde{J}_\lambda$ similar to (5.1), and the formula $\tilde{J}_\lambda \tilde{J}_\pi$ similar to the Pieri formula $S_n S_\lambda$.

In [11], we have obtain that

$\tilde{J}$
$$= \frac{1}{(h_1-h_2)(h_1-h_3)}\Big((1+h_2h_3)P_1^2 + (1+h_2h_3)h_1P_{2,1} + P_{2,2}\Big), \tag{5.3}$$

$\tilde{J}$
$$= \frac{1}{(h_2-h_1)(h_2-h_3)}\Big((1+h_1h_3)P_1^2 + (1+h_1h_3)h_2P_{2,1} + P_{2,2}\Big), \tag{5.4}$$

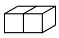
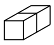
$\tilde{J}$
$$= \frac{1}{(h_3-h_1)(h_3-h_2)}\Big((1+h_1h_2)P_1^2 + (1+h_1h_2)h_3P_{2,1} + P_{2,2}\Big). \tag{5.5}$$

Note that here $P_{2,2}$ equals $\sqrt{1+\sigma_2+\sigma_3^2}P_{2,2}$ in [11] since in this paper we want

$$\langle P_{2,2}, P_{2,2}\rangle = -2(1+\sigma_2+\sigma_3^2).$$

Similarly, we let $P_{3,2}$ here equal $2\sqrt{1+\sigma_2+\sigma_3^2}P_{3,2}$, since we want $P_{3,2} = e_1 P_{2,2}$, which means

$$\langle P_{3,2}, P_{3,2}\rangle = -8(1+\sigma_2+\sigma_3^2).$$

We can see that they are symmetric about three coordinate axes, which means that exchanging  $\leftrightarrow$  corresponds to exchanging $h_1 \leftrightarrow h_2$, others are similar. We associate h_1 to y-axis, h_2 to x-axis, h_3 to z-axis to match the results in [13]. We want this symmetry holds for all 3-Jack polynomials.

We want that 3-Jack polynomials $\tilde{J}_\pi$ behave the same with 3D Young diagrams π in the representation of affine Yangian of $\mathfrak{gl}(1)$. For example,

$$\langle \tilde{J}_\pi, \tilde{J}_{\pi'}\rangle = \langle \pi, \pi'\rangle \tag{5.6}$$

In [11], we show that the 3-Jack polynomials become Jack polynomials defined on 2D Young diagrams when $h_1 = \sqrt{\alpha}$, $h_2 = -1/\sqrt{\alpha}$. In [11], 3-Jack polynomials are obtained under the condition $\psi_0 = 1$ which does not lose the generality. In fact, if we calculate the 3-Jack polynomials for general ψ_0, the 3-Jack polynomials will become the symmetric functions Y_λ when $\psi_0 = -\frac{1}{h_1h_2}$, where Y_λ are defined by us in [19, 20]. Therefore, 3-Jack polynomials are the generalization of the symmetric functions Y_λ to 3D Young diagrams,

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they are also the generalization of Jack polynomials. Different from our previous work, in this paper, we show that 3-Jack polynomials become Y_λ under the following conditions. We see that when

$$P_{2,2} = -(1+h_1h_2)(P_1^2 + h_3P_{2,1}), \tag{5.7}$$

the 3-Jack polynomials of two boxes become

$$\tilde{J}_{\square\square} = \frac{1}{h_1-h_2}\left(P_{2,1} - h_2P_1^2\right) = Y_{\square\square}, \tag{5.8}$$

$$\tilde{J}_{\substack{\square\\\square}} = \frac{1}{h_2-h_1}\left(P_{2,1} - h_1P_1^2\right) = Y_{\substack{\square\\\square}}, \tag{5.9}$$

$$\tilde{J}_{\substack{\square\\\square}} = 0. \tag{5.10}$$

Generally, we take

$$P_{n,2} = -2(1+h_1h_2)(P_1P_{n-1,1} + h_3P_{n,1}), \quad \text{for} \quad n > 2. \tag{5.11}$$

For $j = 3$, we take

$$P_{3,3} = (1+h_1h_2)(2+h_1h_2)(2P_1^3 + 3h_3P_1P_{2,1} + h_3^2P_{3,1}), \tag{5.12}$$

$$P_{n,3} = 3(1+h_1h_2)(2+h_1h_2)(2P_1^2P_{n-2,1} + 3h_3P_1P_{n-1,1} + h_3^2P_{n,1}), \quad \text{for } n > 3.$$

For general j, let

$$(1+x)(2+x)\cdots(j-1+x) = r_0 + r_1x + \cdots + r_{j-1}x^{j-1}$$

with the coefficients $r_0 = (j-1)!, r_1, \cdots, r_{n-2}, r_{j-1} = 1$, we take

$$P_{j,j} = (-1)^{j-1}\prod_{k=1}^{j-1}(k+h_1h_2)(r_0P_1^j + r_1h_3P_1^{j-2}P_{2,1} + \cdots + r_{n-1}h_3^{j-1}P_{j,1}), \tag{5.13}$$

$$P_{n,j} = (-1)^{j-1}j\prod_{k=1}^{j-1}(k+h_1h_2)(r_0P_1^{j-1}P_{n-j+1,1} + r_1h_3P_1^{j-2}P_{n-j+2,1} + \cdots + r_{j-1}h_3^{j-1}P_{n,1}),$$

for $n > j$.

We require the 3-Jack polynomials become the symmetric functions Y_λ under these conditions.

For $n \geq j$, define

$$\xi_y(P,z) = \sum_{n,j=1}^{\infty} \frac{P_{n,j}}{j!\begin{pmatrix} n+j-1 \\ n-j \end{pmatrix}} \frac{d_{n,j}}{h_1^j} \prod_{k=1}^{j-1} \frac{1}{k+h_2h_3} z^n, \tag{5.14}$$

with

$$d_{n,j} = \begin{cases} 1 & \text{if } \; n = j, \\ j & \text{if } n > j. \end{cases} \tag{5.15}$$

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The 3D Young diagram of n boxes along y-axis is denoted by $\underbrace{(1,1,\cdots,1)}_{n}$. For example, when $n=2$, $(1,1)$ is $\square\square$. The 3-Jack polynomials $\tilde{J}_{\underbrace{(1,\cdots,1)}_{n}}$ is determined by

$$e^{\xi_y(P,z)} = \sum_{n\geq 0} \frac{1}{\langle \tilde{J}_{\underbrace{(1,\cdots,1)}_{n}}, \tilde{J}_{\underbrace{(1,\cdots,1)}_{n}} \rangle h_1^n} \tilde{J}_{\underbrace{(1,\cdots,1)}_{n}}(P) z^n. \tag{5.16}$$

Note that when $P_{n,j>1}=0$, the vertex operator above becomes that for the symmetric functions $Y_{(n)}$ [21]. When $P_{n,j>1}=0$ and $h_1=\sqrt{\alpha}, h_2=-1/\sqrt{\alpha}$, the vertex operator above becomes that for the 2D Jack polynomials $\tilde{J}_{(n)}$ [22]. When $P_{n,j>1}=0$ and $h_1=-1, h_2=-1$, the vertex operator above becomes that for the Schur functions $S_{(n)}$ [2, 4].

We list the first few terms of $\tilde{J}_{\underbrace{(1,\cdots,1)}_{n}}(P)$:

$$\begin{aligned}
\tilde{J}_0 &= 1,\\
\frac{1}{\langle \tilde{J}_{\square}, \tilde{J}_{\square}\rangle h_1}\tilde{J}_{\square} &= \frac{1}{h_1}P_1,\\
\frac{1}{\langle \tilde{J}_{\square\square}, \tilde{J}_{\square\square}\rangle h_1^2}\tilde{J}_{\square\square} &= \frac{1}{2(1+h_2h_3)h_1^2}\Big((1+h_2h_3)P_1^2+(1+h_2h_3)h_1P_{2,1}+P_{2,2}\Big),\\
\frac{1}{\langle \tilde{J}_{(1,1,1)}, \tilde{J}_{(1,1,1)}\rangle h_1^3}\tilde{J}_{(1,1,1)} &= \frac{1}{6h_1^3}P_1^3+\frac{1}{2h_1^2}P_1P_{2,1}+\frac{1}{2h_1^3(1+h_2h_3)}P_1P_{2,2}+\frac{1}{3h_1}P_{3,1}\\
&\quad+\frac{1}{4h_1^2(h_2h_3+1)}P_{3,2}+\frac{1}{6(h_2h_3+1)(h_2h_3+2)h_1^3}P_{3,3},\\
\frac{1}{\langle \tilde{J}_{(1,1,1,1)}, \tilde{J}_{(1,1,1,1)}\rangle h_1^3}\tilde{J}_{(1,1,1,1)} &= \frac{1}{24h_1^4}P_1^4+\frac{1}{4h_1^3}P_1^2P_{2,1}+\frac{1}{4h_1^4(1+h_2h_3)}P_1^2P_{2,2}+\frac{1}{3h_1^2}P_1P_{3,1}\\
&\quad+\frac{1}{4h_1^3(h_2h_3+1)}P_1P_{3,2}+\frac{1}{6(h_2h_3+1)(h_2h_3+2)h_1^4}P_1P_{3,3}\\
&+\frac{1}{4h_1}P_{4,1}+\frac{1}{10h_1^2(1+h_2h_3)}P_{4,2}+\frac{1}{12(h_2h_3+1)(h_2h_3+2)h_1^3}P_{4,3}\\
&\quad+\frac{1}{24(h_2h_3+3)(h_2h_3+2)(h_2h_3+1){h_1}^4}P_{4,4}\\
&\quad+\frac{1}{4(1+h_2h_3)h_1^3}P_{2,1}P_{2,2}+\frac{1}{8h_1^2}P_{2,1}^2+\frac{1}{8(1+h_2h_3)^2h_1^4}P_{2,2}^2.
\end{aligned}$$

Since

$$\langle \tilde{J}_{\underbrace{(1,\cdots,1)}_{n+1}}, \tilde{J}_{\underbrace{(1,\cdots,1)}_{n+1}} \rangle = \prod_{j=1}^{n}\frac{(j+1)(j+h_2h_3)}{(jh_1-h_2)(jh_1-h_3)},$$

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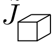
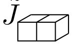
we have

$$\tilde{J}$$  $$= P_1,$$

$$\tilde{J}$$  $$= \frac{1}{(h_1-h_2)(h_1-h_3)}\Big((1+h_2h_3)P_1^2+(1+h_2h_3)h_1P_{2,1}+P_{2,2}\Big),$$

$$\begin{aligned}\tilde{J}_{(1,1,1)} =& \frac{1}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)}\Big((1+h_2h_3)(2+h_2h_3)P_1^3\\ &+3h_1(1+h_2h_3)(2+h_2h_3)P_1P_{2,1}+3(2+h_2h_3)P_1P_{2,2}\\ &+2h_1^2(1+h_2h_3)(2+h_2h_3)P_{3,1}+3h_1(1+\frac{1}{2}h_2h_3)P_{3,2}+P_{3,3}\Big),\end{aligned}$$

which are the same with that in [11].

$$\begin{aligned}\tilde{J}_{(1,1,1,1)} =& \frac{1}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)(3h_1-h_2)(3h_1-h_3)}\\ &\times\Big((1+h_2h_3)(2+h_2h_3)(3+h_2h_3)P_1^4+6h_1(1+h_2h_3)(2+h_2h_3)(3+h_2h_3)P_1^2P_{2,1}\\ &+6h_1(2+h_2h_3)(3+h_2h_3)P_{2,1}P_{2,2}+6(2+h_2h_3)(3+h_2h_3)P_1^2P_{2,2}\\ &+8(1+h_2h_3)(2+h_2h_3)(3+h_2h_3)h_1^2P_1P_{3,1}+6(2+h_2h_3)(3+h_2h_3)h_1P_1P_{3,2}\\ &+4(3+h_2h_3)P_1P_{3,3}+6(1+h_2h_3)(2+h_2h_3)(3+h_2h_3)h_1^3P_{4,1}\\ &+\frac{12}{5}(2+h_2h_3)(3+h_2h_3)h_1^2P_{4,2}+2(3+h_2h_3)h_1P_{4,3}+P_{4,4}\\ &+\frac{3(2+h_2h_3)(3+h_2h_3)}{(1+h_2h_3)}P_{2,2}^2+3(1+h_2h_3)(2+h_2h_3)(3+h_2h_3)h_1^2P_{2,1}^2\Big).\end{aligned}$$

This expression is slightly different from that in [18] since here we choose

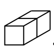
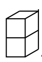
$$\begin{aligned}P_{2,2}^2|0\rangle =& (e_2e_0e_2e_0+2\sigma_3e_0e_1e_2e_0-2\sigma_2e_1e_0e_2e_0|0\rangle-\frac{2}{3}\sigma_3e_1e_1e_1e_0-2e_0e_0e_2e_0\\ &-(2+\sigma_3^2)e_0e_1e_1e_0-e_0e_2e_2e_0+\sigma_3^2e_1e_0e_1e_0+2\sigma_3e_0e_0e_1e_0+e_0e_0e_0e_0)|0\rangle,\end{aligned}$$

which equals $a_{-2,2}^2|0\rangle$.

For $n\geq j$, define

$$\xi_x(P,z) = \sum_{n,j=1}^{\infty}\frac{P_{n,j}}{j!\begin{pmatrix}n+j-1\\ n-j\end{pmatrix}}\frac{d_{n,j}}{h_2^j}\prod_{k=1}^{j-1}\frac{1}{k+h_1h_3}z^n, \tag{5.17}$$

$$\xi_z(P,z) = \sum_{n,j=1}^{\infty}\frac{P_{n,j}}{j!\begin{pmatrix}n+j-1\\ n-j\end{pmatrix}}\frac{d_{n,j}}{h_3^j}\prod_{k=1}^{j-1}\frac{1}{k+h_1h_2}z^n. \tag{5.18}$$

The 3D Young diagram of n boxes along x-axis and z-axis are denoted by $\begin{pmatrix}1\\ \cdots\\ 1\end{pmatrix}$ and (n) respectively. For example, when $n=2$, $\begin{pmatrix}1\\ 1\end{pmatrix}$ is , and (2) is . The 3-Jack polynomials

$\tilde{J}_{\begin{pmatrix}1\\ \cdots\\ 1\end{pmatrix}}$ and $\tilde{J}_{(n)}$ are determined by

$$e^{\xi_x(P,z)} = \sum_{n\geq 0} \frac{1}{\langle \tilde{J}_{\begin{pmatrix}1\\ \cdots\\ 1\end{pmatrix}}, \tilde{J}_{\begin{pmatrix}1\\ \cdots\\ 1\end{pmatrix}} \rangle h_2^n} \tilde{J}_{\begin{pmatrix}1\\ \cdots\\ 1\end{pmatrix}}(P) z^n \tag{5.19}$$

and

$$e^{\xi_z(P,z)} = \sum_{n\geq 0} \frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle h_3^n} \tilde{J}_{(n)}(P) z^n \tag{5.20}$$

respectively. We can see that they are symmetric about the three coordinate axes.

For 2D Young diagrams λ, which are treated as 3D Young diagrams which have one layer in z-axis direction, we define 3-Jack polynomials $\tilde{J}_\lambda$ in the following. Let

$$P_{n,1} = \frac{1}{h_1}(z_1^n + z_2^n + \cdots) = \frac{1}{h_1} \sum_{k=1}^{\infty} z_k^n, \tag{5.21}$$

then $P_{n,j}$ in (5.13) equal

$$\begin{aligned}
P_{j,j} = (-1)^{j-1} \prod_{k=1}^{j-1} (k+h_1h_2)(k+h_1h_3) \frac{1}{h_1^j} \Bigg(& \sum_k z_k^j \\
& + \frac{r_0 C_j^1 + r_1 h_1 h_3 C_{j-2}^1 + r_2 h_1^2 h_3^2 C_{j-3}^1 + \cdots + r_{j-2} h_1^{j-2} h_3^{j-2} C_1^1}{\prod_{k=1}^{j-1}(k+h_1h_3)} \sum_{k,l} z_k^{j-1} z_l \\
& + \frac{r_0 C_j^2 + r_1 h_1 h_3 (C_{j-2}^2 + C_{j-2}^0) + r_2 h_1^2 h_3^2 C_{j-3}^2 + \cdots + r_{j-3} h_1^{j-3} h_3^{j-3} C_2^2}{\prod_{k=1}^{j-1}(k+h_1h_3)} \sum_{k\neq l} z_k^{j-2} z_l^2 \\
& + \frac{r_0 C_j^1 C_{j-1}^1 + r_1 h_1 h_3 (C_{j-2}^1 C_{j-3}^1 + C_{j-2}^0) + r_2 h_1^2 h_3^2 C_{j-3}^1 C_{j-4}^1 + \cdots + r_{j-3} h_1^{j-3} h_3^{j-3} C_2^1 C_1^1}{\prod_{k=1}^{j-1}(k+h_1h_3)} \\
& \cdot \sum_{k_1,k_2,k_3} z_{k_1}^{j-2} z_{k_2} z_{k_3} + \cdots + r_0 C_j^1 C_{j-1}^1 \cdots C_1^1 \sum_{k_1 < \cdots < k_j} z_{k_1} z_{k_2} \cdots z_{k_j} \Bigg),
\end{aligned}$$

and when $n > j$,

$$\begin{aligned}
P_{n,j} = (-1)^{j-1} j \prod_{k=1}^{j-1} (k+h_1h_2)(k+h_1h_3) \frac{1}{h_1^j} \Bigg(& \sum_k z_k^n + \frac{\sum_{k=0}^{j-2} r_k h_1^k h_3^k C_{j-k-1}^1}{\prod_{k=1}^{j-1}(k+h_1h_3)} \sum_{k,l} z_k^{n-1} z_l \\
& + \frac{\sum_{k=0}^{j-3} r_k h_1^k h_3^k C_{j-k-1}^2 + r_0 \delta_{n-j,1}}{\prod_{k=1}^{j-1}(k+h_1h_3)} \sum_{k\neq l} z_k^{n-2} z_l^2 \\
& + \frac{\sum_{k=0}^{j-3} r_k h_1^k h_3^k C_{j-k-1}^1 C_{j-k-2}^1}{\prod_{k=1}^{j-1}(k+h_1h_3)} \sum_{k_1,k_2,k_3} z_{k_1}^{n-2} z_{k_2} z_{k_3} + \cdots \\
& + r_0 C_{j-1}^1 C_{j-2}^1 \cdots C_1^1 \sum_{k_1,\cdots,k_{j+1}} z_{k_1}^{n-j+1} z_{k_2} \cdots z_{k_{j+1}} \Bigg),
\end{aligned}$$

where

$$C_j^k = \begin{pmatrix} j \\ k \end{pmatrix} = \frac{j!}{k!(j-k)!}.$$

Define $\xi_{yx,j,j}$ and $\xi_{yx,n,j}$ by

$$\begin{aligned}
\xi_{yx,j,j} = \frac{P_{j,j}}{j!h_1^j}\Bigg(& \frac{r_0C_j^1+r_1h_1h_3C_{j-2}^1+r_2h_1^2h_3^2C_{j-3}^1+\cdots+r_{j-2}h_1^{j-2}h_3^{j-2}C_1^1}{\prod_{k=1}^{j-1}(k+h_1h_3)}\sum_{k,l} z_k^{j-1}z_l \\
& +\frac{r_0C_j^2+r_1h_1h_3(C_{j-2}^2+C_{j-2}^0)+r_2h_1^2h_3^2C_{j-3}^2+\cdots+r_{j-3}h_1^{j-3}h_3^{j-3}C_2^2}{\prod_{k=1}^{j-1}(k+h_1h_3)}\sum_{k\neq l} z_k^{j-2}z_l^2 \\
& +\frac{r_0C_j^1C_{j-1}^1+r_1h_1h_3(C_{j-2}^1C_{j-3}^1+C_{j-2}^0)+r_2h_1^2h_3^2C_{j-3}^1C_{j-4}^1+\cdots+r_{j-3}h_1^{j-3}h_3^{j-3}C_2^1C_1^1}{\prod_{k=1}^{j-1}(k+h_1h_3)} \\
& \cdot \sum_{k_1,k_2,k_3} z_{k_1}^{j-2}z_{k_2}z_{k_3}+\cdots+r_0C_j^1C_{j-1}^1\cdots C_1^1\sum_{k_1<\cdots<k_j} z_{k_1}z_{k_2}\cdots z_{k_j}\Bigg), \qquad (5.22)
\end{aligned}$$

and for $n > j$,

$$\begin{aligned}
\xi_{yx,n,j} = \frac{P_{n,j}}{j!C_{n+j-1}^{n-j}h_1^j}\Bigg(& \frac{\sum_{k=0}^{j-2} r_k h_1^k h_3^k C_{j-k-1}^1}{\prod_{k=1}^{j-1}(k+h_1h_3)}\sum_{k,l} z_k^{n-1}z_l \\
& +\frac{\sum_{k=0}^{j-3} r_k h_1^k h_3^k C_{j-k-1}^2 + r_0\delta_{n-j,1}}{\prod_{k=1}^{j-1}(k+h_1h_3)}\sum_{k\neq l} z_k^{n-2}z_l^2 \\
& +\frac{\sum_{k=0}^{j-3} r_k h_1^k h_3^k C_{j-k-1}^1C_{j-k-2}^1}{\prod_{k=1}^{j-1}(k+h_1h_3)}\sum_{k_1,k_2,k_3} z_{k_1}^{n-2}z_{k_2}z_{k_3}+\cdots \\
& + r_0C_{j-1}^1C_{j-2}^1\cdots C_1^1\sum_{k_1,\cdots,k_{j+1}} z_{k_1}^{n-j+1}z_{k_2}\cdots z_{k_{j+1}}\Bigg). \qquad (5.23)
\end{aligned}$$

Define

$$T_{yx}(P,z) = \sum_{k=1}^{\infty}\xi_y(P,z_k) + \sum_{j\geq 2}\xi_{yx,j,j} + \sum_{n>j\geq 2}\xi_{yx,n,j}, \qquad (5.24)$$

and let

$$e^{T_{yx}(P,z)} = \sum_{i_1,i_2,\cdots,\geq 0} Q_{i_1,i_2,\cdots}(P)z_1^{i_1}z_2^{i_2}\cdots, \qquad (5.25)$$

where $z = (z_1, z_2, \cdots)$. We can see that $Q_{i_1,i_2,\cdots}$ are polynomials of $P_{n,j}$ which can be determined by this equation. We list the first few of them:

$$\begin{aligned}
Q_1 &= \frac{1}{h_1} P_1, \\
Q_2 &= \frac{1}{2\,(h_1\,h_3+1)\,(h_2\,h_3+1)\,h_1^2} \Big(h_1^2 h_2 h_3^2 P_{2,1} + h_1 h_2 h_3^2 {P_1}^2 + h_1^2 h_3 P_{2,1} + h_1 h_2 h_3 P_{2,1} \\
&\quad + h_1 h_3 {P_1}^2 + h_2 h_3 P_1^2 + h_1 h_3 P_{2,2} + h_1 P_{2,1} + P_1^2 + P_{2,2} \Big), \\
Q_3 &= \frac{1}{12(1+h_1h_3)(1+h_2h_3)(2+h_1h_3)(2+h_2h_3)h_1^3} \Big(4h_1^4 h_2^2 h_3^4 P_{3,1} + 6h_1^3 h_2^2 h_3^4 P_1 P_{2,1} \\
&\quad + 2h_1^2 h_2^2 h_3^4 {P_1}^3 + 12h_1^4 h_2 h_3^3 P_{3,1} + 12h_1^3 h_2^2 h_3^3 P_{3,1} + 18h_1^3 h_2 h_3^3 P_1 P_{2,1} + 18h_1^2 h_2^2 h_3^3 P_1 P_{2,1} \\
&\quad + 6h_1^2 h_2 h_3^3 P_1^3 + 6h_1 h_2^2 h_3^3 P_1^3 + 3h_1^3 h_2 h_3^3 P_{3,2} + 6h_1^2 h_2 h_3^3 P_1 P_{2,2} + 8h_1^4 h_3^2 P_{3,1} \\
&\quad + 36h_1^3 h_2 h_3^2 P_{3,1} + 12h_1^3 h_3^2 P_1 P_{2,1} + 8h_1^2 h_2^2 h_3^2 P_{3,1} + 54h_1^2 h_2 h_3^2 P_1 P_{2,1} + 4h_1^2 h_3^2 P_1^3 \\
&\quad + 12h_1 h_2^2 h_3^2 P_1 P_{2,1} + 18h_1 h_2 h_3^2 P_1^3 + 4h_2^2 h_3^2 P_1^3 + 6h_1^3 h_3^2 P_{3,2} + 9h_1^2 h_2 h_3^2 P_{3,2} \\
&\quad + 12h_1^2 h_3^2 P_1 P_{2,2} + 18h_1 h_2 h_3^2 P_1 P_{2,2} + 24h_1^3 h_3 P_{3,1} + 24h_1^2 h_2 h_3 P_{3,1} + 2h_1^2 h_3^2 P_{3,3} \\
&\quad + 36h_1^2 h_3 P_1 P_{2,1} + 36h_1 h_2 h_3 P_1 P_{2,1} + 12h_1 h_3 P_1^3 + 12h_2 h_3 P_1^3 + 18h_1^2 h_3 P_{3,2} \\
&\quad + 6h_1 h_2 h_3 P_{3,2} + 36h_1 h_3 P_1 P_{2,2} + 12h_2 h_3 P_1 P_{2,2} + 16h_1^2 P_{3,1} + 6h_1 h_3 P_{3,3} \\
&\quad + 24h_1 P_1 P_{2,1} + 8P_1^3 + 12h_1 P_{3,2} + 24P_1 P_{2,2} + 4P_{3,3} \Big),
\end{aligned}$$

and

$$\begin{aligned}
Q_{1,1} &= \frac{1}{(h_1\,h_3+1)\,(h_2\,h_3+1)\,h_1^2} \Big((h_1\,h_3+1)\,(h_2\,h_3+1)\,P_1^2 + P_{2,2} \Big), \\
Q_{2,1} &= \frac{1}{4(1+h_1h_3)(1+h_2h_3)(2+h_1h_3)h_1^3} \Big(2h_1^2 h_2^2 h_3^3 P_1 P_{2,1} + 2h_1 h_2^2 h_3^3 P_1^3 \\
&\quad + 6h_1^2 h_2 h_3^2 P_1 P_{2,1} + 2h_1 h_2^2 h_3^2 P_1 P_{2,1} + 6h_1 h_2 h_3^2 P_1^3 + 2h_2^2 h_3^2 P_1^3 + 2h_1 h_2 h_3^2 P_1 P_{2,2} \\
&\quad + 4h_1^2 h_3 P_1 P_{2,1} + 6h_1 h_2 h_3 P_1 P_{2,1} + 4h_1 h_3 P_1^3 + 6h_2 h_3 P_1^3 + h_1 h_2 h_3 P_{3,2} + 4h_1 h_3 P_1 P_{2,2} \\
&\quad + 6h_2 h_3 P_1 P_{2,2} + 4h_1 P_1 P_{2,1} + 4P_1^3 + 2h_1 P_{3,2} + 12P_1 P_{2,2} + 2P_{3,3} \Big).
\end{aligned}$$

Then we can calculate the 3-Jack polynomials $\tilde{J}_\lambda$ from $Q_{i_1,i_2,\cdots}$. When $(i_1,i_2,i_3,\cdots) = (n,0,0,\cdots)$, we have

$$Q_n(P) = \frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle h_1^n} \tilde{J}_{(n)}(P). \tag{5.26}$$

When $(i_1,i_2,i_3,\cdots) = (n-1,1,0,\cdots)$, we have

$$\begin{aligned}
Q_{n-1,1}(P) = &\frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle h_1^n} \tilde{J}_{(n)}(P) \frac{-nh_2}{(n-1)h_1 - h_2} \\
&+ \frac{1}{\langle \tilde{J}_{(n-1,1)}, \tilde{J}_{(n-1,1)} \rangle h_1^{n-1}} \tilde{J}_{(n-1,1)}(P) \frac{2}{h_1 - h_2},
\end{aligned} \tag{5.27}$$

where $(n-1,1)$ is the 2D Young diagram from $(1,1)$ by adding $n-2$ box in the first row. For example,

$$Q_2 = \frac{1}{\langle \tilde{J}_{(2)}, \tilde{J}_{(2)} \rangle h_1^2} \tilde{J}_{(2)}(P),$$

which means

$$\tilde{J}_{(2)}(P) = \frac{1}{(h_1-h_2)(h_1-h_3)}\left((1+h_2h_3)P_1^2 + (1+h_2h_3)h_1P_{2,1} + P_{2,2}\right),$$

which is the same with (5.3).

$$Q_{1,1} = \frac{1}{\langle \tilde{J}_{(2)}, \tilde{J}_{(2)}\rangle h_1^2}\tilde{J}_{(2)}(P)\frac{-2h_2}{h_1-h_2} + \frac{1}{\langle \tilde{J}_{(1,1)}, \tilde{J}_{(1,1)}\rangle h_1}\tilde{J}_{(1,1)}(P)\frac{2}{h_1-h_2},$$

which means

$$\tilde{J}_{1,1} = \frac{1}{(h_2-h_1)(h_2-h_3)}\left((1+h_1h_3)P_1^2 + (1+h_1h_3)h_2P_{2,1} + P_{2,2}\right),$$

which is the same with (5.4).

In symmetric functions $Y_\lambda(P)$, let $p_n = \frac{1}{h_1}\sum_k z_k^n$, we see that $Y_\lambda(P) = Y_\lambda(z)$ are symmetric about $z_1, z_2, \cdots$. As in [2], we regard 2D Young diagrams arranged in the reverse lexicographical order $\succ$, so that (n) comes first and 1^n comes last. We arrange the terms in $Y_\lambda(z)$ the same as the order of 2D Young diagrams, so that z_i^n comes first and $z_{i_1}z_{i_2}\cdots z_{i_n}$ comes last. We use the notation $c_{i_1,i_2,\cdots}^{Y_\lambda(z)}$ to denote the coefficient of $z_1^{i_1}z_2^{i_2}\cdots$ in $Y_\lambda(z)$. It can be checked that the formulas (5.26) and (5.27) can be written as

$$Q_n(P) = \frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)}\rangle}\tilde{J}_{(n)}(P)c_n^{Y_{(n)}(z)}, \tag{5.28}$$

$$\begin{aligned} Q_{n-1,1}(P) &= \frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)}\rangle}\tilde{J}_{(n)}(P)c_{(n-1,1)}^{Y_{(n)}(z)} \\ &+ \frac{1}{\langle \tilde{J}_{(n-1,1)}, \tilde{J}_{(n-1,1)}\rangle}\tilde{J}_{(n-1,1)}(P)c_{(n-1,1)}^{Y_{(n-1,1)}(z)}. \end{aligned} \tag{5.29}$$

Actually, this formula holds generally, that is, for Young diagram λ, we have

$$Q_\lambda(P) = \sum_{\mu \succ \lambda}\frac{1}{\langle \tilde{J}_\mu, \tilde{J}_\mu\rangle}\tilde{J}_\mu(P)c_{(\lambda_1,\lambda_2,\cdots)}^{Y_\mu(z)}. \tag{5.30}$$

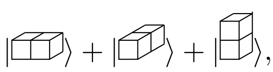
For any 2D Young diagrams μ, which are treated as the 3D Young diagrams having one layer in z-axis direction, the 3-Jack polynomials $\tilde{J}_\mu$ can be obtained from the formula (5.30).

Note that we can similarly define $T_{xy}(P,z)$, $T_{xz}(P,z)$, $T_{zx}(P,z)$, $T_{yz}(P,z)$, $T_{zy}(P,z)$. From them, the 3-Jack polynomials of 3D Young diagrams having one layer in x-axis direction or y-axis direction can be obtained. In fact, the 3-Jack polynomials of 3D Young diagrams having one layer in x-axis direction or y-axis direction can also be obtained from the 3-Jack polynomials of 3D Young diagrams having one layer in z-axis direction by the symmetry of 3-Jack polynomials about three coordinate axes.

To get the expressions of 3-Jack polynomials $\tilde{J}_\pi$ for all 3D Young diagrams π, we need the formula $\tilde{J}_\lambda\tilde{J}_\pi$. Define

$$\tilde{J}_\lambda\tilde{J}_\pi = \hat{\tilde{J}}_\lambda \cdot \tilde{J}_\pi, \tag{5.31}$$

where $\hat{\tilde{J}}_\lambda$ are the functions of operators $a_{-n,j}$ with $n>0$, and the actions of $\hat{\tilde{J}}_\lambda$ on $\tilde{J}_\pi$ are the same with that of affine Yangian of $\mathfrak{gl}(1)$ on 3D Young diagrams. For example, since

$e_0|\square\rangle =$ 

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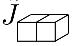
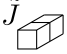
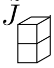
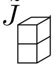
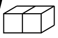
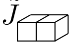
we have

$$\tilde{J}_{\square}\tilde{J}_{\square} = e_0\tilde{J}_{\square} = \tilde{J}$$  $$+\ \tilde{J}$$  $$+\ \tilde{J}$$  $$, \qquad (5.32)$$

then $\tilde{J}$  is obtained, which is the same with (5.5). From

$$\tilde{J}_{\square}\tilde{J}$$  $$= \tilde{J}$$  $$\tilde{J}_{\square} = \tilde{J}_{(1,1,1)} + \tilde{J}_{\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}_{h_1,h_2}} + \tilde{J}_{(2,1)_{h_1,h_3}}, \qquad (5.33)$$

$\tilde{J}_{(2,1)_{h_1,h_3}}$ is obtained. Other 3-Jack polynomials of 3D Young diagrams which have more than one layer in z-axis direction can be obtained this way.

6 Concluding remarks

In this paper, all results are obtained by requiring $\psi_0 = 1$. If interested, one can calculate the results for general ψ_0, which should be similar to that in this paper. For example,

$$1 + \sigma_2 + \sigma_3^2 = (1 + h_1h_2)(1 + h_1h_3)(1 + h_2h_3)$$

in this paper should be

$$1 + \psi_0\sigma_2 + \psi_0^3\sigma_3^2 = (1 + \psi_0h_1h_2)(1 + \psi_0h_1h_3)(1 + \psi_0h_2h_3).$$

for general ψ_0.

$$\langle \tilde{J}_{\underbrace{(1,\cdots,1)}_{n+1}}, \tilde{J}_{\underbrace{(1,\cdots,1)}_{n+1}} \rangle = \prod_{j=1}^{n} \frac{(j+1)(j+h_2h_3)}{(jh_1-h_2)(jh_1-h_3)}$$

in this paper should be

$$\langle \tilde{J}_{\underbrace{(1,\cdots,1)}_{n+1}}, \tilde{J}_{\underbrace{(1,\cdots,1)}_{n+1}} \rangle = \psi_0 \prod_{j=1}^{n} \frac{(j+1)(j+h_2h_3\psi_0)}{(jh_1-h_2)(jh_1-h_3)}.$$

This holds since there is the scaling symmetries in the affine Yangian of $\mathfrak{gl}(1)$ [13]. The scaling symmetries say that changing the value of ψ_0 is equivalent to rescaling parameters $h_j, j = 1, 2, 3$. Next, we will consider the slice of 3-Jack polynomials similar to that the slices of 3D Young diagrams are 2D Young diagrams.

Data availability statement. The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declaration of interest statement. The authors declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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