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Twisted Modules for Vertex Operator Superalgebras and Associative Algebras

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TWISTED MODULES FOR VERTEX OPERATOR SUPERALGEBRAS AND ASSOCIATIVE ALGEBRAS

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Abstract

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by

Charles Petersen

Let V be a vertex operator superalgebra and σ the order 2 automorphism associated with the superstructure of V. For a finite order automorphism g with $o(g\sigma) = T'$, we follow [8] to construct a sequence of associative algebras $A_{g,n}(V)$ for $n \in \frac{1}{T'}\mathbb{Z}_+$ such that $A_{g,n-\frac{1}{T'}}(V)$ is a quotient of $A_{g,n}(V)$. There is a bijection between the irreducible $A_{g,n}(V)$-modules which cannot factor through $A_{g,n-\frac{1}{T'}}(V)$ and the irreducible admissible g-twisted V-modules. These results are then applied to g-rational vertex operator superalgebras. In this case it is shown that V is g-rational if and only if all the $A_{g,n}(V)$ are finite-dimensional. Taking $n = 0$ we obtain the associative algebra $A_g(V)$ constructed in [15]. With $g = 1$ we recover $A_n(V)$ as in [18].

To my wife, Abbey. We did it!

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Chapter 1

Introduction

In order to study the representations of a vertex operator algebra V, the associative algebra $A(V)$ was introduced in [26]. In essence, the representation theory of the vertex operator algebra V was reduced to that of the associative algebra $A(V)$; the inequivalent irreducible admissible V-modules are in one-to-one correspondence with the inequivalent irreducible $A(V)$-modules. It was also shown that if V is rational, then $A(V)$ is finite-dimensional semisimple. Hence, in this case there are only finitely many inequivalent irreducible admissible V-modules up to isomorphism. These results have been used with great success. In its debut ([26]), $A(V)$-theory played a key role in proving modular invariance of the space of characters of irreducible modules for a rational vertex operator algebra. The theory of $A(V)$ has also been used to classify the irreducible modules of many well-known rational vertex operator algebras. These advances led to the extension and generalization of the theory in many directions ([7], [9], [8], [3], [4], [24], [19], [25], [15], [18]). We briefly summarize some of these results which are essential to this thesis.

Given a finite automorphism group G of a vertex operator algebra V, the g-twisted V-modules for $g \in G$ play a prominent role in understanding the representation theory of the orbifold (fixed point) vertex operator algebra V^G. The theory of $A(V)$ was extended in order to study the twisted modules in [7]. The associative algebra $A_g(V)$ was constructed and when $g = 1$, $A(V)$ is recovered. There the arguments of [26] were simplified by way of familiar techniques of highest weight representations of Lie algebras. The irreducible admissible g-twisted V-modules were shown to be in one-to-one correspondence with the irreducible $A_g(V)$-modules. More surprisingly, given a g-rational vertex operator algebra, not only is $A_g(V)$ a finite-dimensional semisimple associative algebra, but each irreducible admissible g-twisted V-module is ordinary. That is, the weight spaces are finite-dimensional.

The theory of $A(V)$ is centered on the fact that the top level of an admissible V-module $M = \bigoplus_{n\in\mathbb{Z}_+} M(n)$ with $M(0) \neq 0$ is an $A(V)$-module. As such, $A(V)$ does not give a full description of the action of V on the remaining homogeneous pieces $M(n)$ for $n \neq 0$. To remedy this, a sequence of associative algebras $A_n(V)$ for n a nonnegative integer was constructed in [9]. Here $A_0(V) = A(V)$ and it was shown that $A_{n-1}(V)$ is a quotient of $A_n(V)$. Each $M(k)$ for $k \leq n$ is an $A_n(V)$-module and in this way the $A_n(V)$ gave a more complete picture of the action of V on an admissible module. The equivalence classes of irreducible $A_n(V)$-modules which cannot factor through $A_{n-1}(V)$ are in one-to-one correspondence with the equivalence classes of irreducible admissible V-modules. Moreover, V is rational if and only if all $A_n(V)$ are finite dimensional semisimple.

To extend [9] to the orbifold theory of a vertex operator algebra V, a sequence of associative algebras $A_{g,n}(V)$ for $n \in \frac{1}{T}\mathbb{Z}_+$ and automorphism g of finite order T was constructed in [8]. In this case, both $A_{g,0}(V) = A_g(V)$ and $A_{1,n}(V) = A_n(V)$. Moreover, $A_{g,n-\frac{1}{T}}(V)$ is a quotient of $A_{g,n}(V)$ and the equivalence classes of irreducible $A_{g,n}(V)$-modules which cannot factor through $A_{g,n-\frac{1}{T'}}(V)$ are in one-to-one-

one correspondence with the equivalence classes of irreducible admissible g-twisted V-modules. Also, V is g-rational if and only if all the $A_{g,n}(V)$ are finite dimensional. Since then, the $A_{g,n}(V)$ have proven to be a powerful tool in the pursuit of the classification of irreducible modules for obifolds of rational vertex operator algebras ([11], [13], [24], [12]).

This brings us to the work of this thesis: the representations of orbifolds of vertex operator superalgebras. The theory of $A(V)$ was extended to vertex operator superalgebras in [19]. The theories of $A_g(V)$ and $A_n(V)$ were extended to vertex operator superalgebras in [15] and in [18], respectively. Motivated by this work and the advances the $A_{g,n}(V)$ have afforded the representation theory of orbifolds of rational vertex operator algebras, in this thesis we introduce the analog of the $A_{g,n}(V)$ for a vertex operator superalgebra V and finite order automorphism g. Our setting follows [8] and we extend the results therein to this case. However, since we are dealing with vertex operator *super*algebras, the differences between our present situation and [8] are enough that we cannot simply quote the results. More so, though the constructions are given in [8], in the name of brevity many arguments were excluded. We provide those omitted details here.

Our general approach is to combine those techniques used in [26], [7], [15] and [9], but with varying degrees of modification. As was pointed out in [15] and [14], the order 2 automorphism σ associated with the superstructure of V plays an important role in the theory. Necessarily the $g\sigma$-eigenspaces of V rather than g-eigenpsaces must be used to define the various structures on V in the construction of $A_{g,n}(V)$. Combined with the nuances of working with superspace and the fact that n is not necessarily an integer, the calculations and technical arguments become slightly more cumbersome in our present situation.

The thesis is organized as follows. In chapter 2 we summarize the requisite background of vertex operator superalgebras and their representation theory needed to develop the results of this work. Chapter 3 is the heart of the thesis. In the sections 3.1 and 3.2 we construct the space $A_{g,n}(V)$ as a quotient of V and gather a handful of relations therein. We verify that $A_{g,n}(V)$ is an associative algebra and that $A_{g,n-\frac{1}{T'}}(V)$ is a quotient of $A_{g,n}(V)$ in section 3.3. In section 3.4 we review the basic properties of the Lie superalgebra $V[g]$ as found in [15]. Section 3.5 is dedicated to the construction the functor Ω_n from the category of admissibe g-twisted V-modules to the category of $A_{g,n}(V)$-modules. In preparation for section 3.7 we gather a few results on $V[g]$-modules in section 3.6. Section 3.7 contains the main results of the thesis. We construct another functor L_n from the category of $A_{g,n}(V)$-modules to the category of $A_{g,n}(V)$-modules and establish the correspondence between the equivalence classes of irreducible $A_{g,n}(V)$-modules which cannot factor through $A_{g,n-\frac{1}{T'}}(V)$ and the equivalence classes of irreducible admissible g-twisted V-modules. We apply these results to g-rational vertex operator superalgebras in section 3.8.

Chapter 2

Vertex Operator Superalgebras and Their Modules

In this chapter we summarize the requisite background information of vertex operator superalgebras and their representation theory which is necessary to develop our results.

2.1 Notation and Formal Series

We will use the convention that $\mathbb{N}$ denotes the positive integers, $\mathbb{Z}$ the integers, $\mathbb{Z}_+$ the nonnegative integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. All vector spaces (super or otherwise) are assumed to be over $\mathbb{C}$.

Underlying vertex operator superalgebras are spaces of formal series in several commuting variables z, z_0, z_1, z_2, etc. The coefficients of these spaces lie in some superspace or else its endomorphism ring. *Formal calculus* makes performing operations on such formal series precise and in turn facilitates calculations on these spaces. A detailed exposition of formal calculus can be found in [17] or [20].

For a superspace V we have the following spaces of formal series in the single variable z:

$$V[z] = \left\{ \sum_{n\in\mathbb{N}} v_n z^n : v_n \in V,\ v_n = 0 \text{ for all but finitely many } n \in \mathbb{N} \right\}$$
$$V[[z]] = \left\{ \sum_{n\in\mathbb{N}} v_n z^n : v_n \in V \right\}$$
$$V[z, z^{-1}] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n : v_n \in V,\ v_n = 0 \text{ for all but finitely many } n \in \mathbb{Z} \right\}$$
$$V((z)) = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n : v_n \in V,\ v_n = 0 \text{ for all } n << 0 \right\}$$
$$V[[z, z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n : v_n \in V \right\}$$
$$V\{z\} = \left\{ \sum_{\lambda\in\mathbb{C}} v_\lambda z^\lambda : v_\lambda \in V \right\}$$

Given a formal series $f(z) = \sum_{\lambda\in\mathbb{C}} v_\lambda z^\lambda \in V\{z\}$, the *formal derivative* and *formal residue* are defined as

$$\frac{d}{dz} f(z) = \sum_{\lambda\in\mathbb{C}} \lambda v_\lambda z^{\lambda-1} \quad \text{and} \quad \mathrm{Res}_z f(z) = v_{-1}.$$

If also $g(z) \in (\mathrm{End}V)\{z\}$ and the product $g(z)f(z) \in V\{z\}$, then we have the *formal residue formula for the product rule*

$$\mathrm{Res}_z \left(\left(\frac{d}{dz} g(z) \right) f(z) \right) = -\mathrm{Res}_z \left(g(z) \left(\frac{d}{dz} f(z) \right) \right). \tag{2.1.1}$$

When dealing with formal series, we will expand all binomials in nonnegative integral powers of the second variable. That is, for any $\alpha \in \mathbb{C}$

$$(z_1 + z_2)^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} z_1^{\alpha-j} z_2^j$$

where $\binom{\alpha}{j}$ is the *generalized binomial coefficient* $\binom{\alpha}{j} = \prod_{k=1}^{j} \frac{\alpha - j + 1}{j}$. Note that

$$(z_1 + z_2)^\alpha \neq (z_2 + z_1)^\alpha$$

unless α is a nonnegative integer.

If T is a linear operator on V, the *formal exponential* $e^{z_0 T} \in (\mathrm{End}V)[[z_0]]$ is defined by

$$e^{z_0 T} = \sum_{n=0}^{\infty} \frac{1}{n!} T^n z_0^n.$$

Applying our binomial convention we easily obtain the *formal Taylor's theorem*

$$e^{z_o \frac{d}{dz}} f(z) = f(z + z_0) \in V\{z\}[[z_0]].$$

As an application of the formal Taylor's theorem, we have the following.

Proposition 2.1.1. *For $\alpha \in \mathbb{C}$ and commuting formal variables z_1, z_2 and z_3*

$$(z_1 + (z_2 + z_3))^\alpha = ((z_1 + z_2) + z_3)^\alpha = ((z_1 + z_3) + z_2)^\alpha = (z_1 + (z_3 + z_2))^\alpha$$

Proof. The outside equality follows since we expand binomials in nonnegative integral powers of the second variable. The equality on the left follows from the formal Taylors theorem. Since z_1, z_2 and z_3, commute we have

$$\begin{aligned}
(z_1 + (z_2 + z_3))^\alpha &= e^{(z_2+z_3)\frac{\partial}{\partial z_1}} z_1^\alpha \\
&= e^{z_2 \frac{\partial}{\partial z_1}} \left(e^{z_3 \frac{\partial}{\partial z_1}} z_1^\alpha \right) \\
&= e^{z_2 \frac{\partial}{\partial z_1}} (z_1 + z_3)^\alpha \\
&= ((z_1 + z_2) + z_3)^\alpha.
\end{aligned}$$

The inside inequality follows from the other two. □

A formal series of particular importance in the theory of vertex operator superalgebras is the *formal delta function*

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n \in \mathbb{C}[[z, z^{-1}]]$$

and its multivariate counterparts. As an example, we have

$$\delta\left(\frac{z_1 - z_2}{z_0}\right) = \sum_{n\in\mathbb{Z}} z_0^{-n}(z_1 - z_2)^n = \sum_{j=0}^{\infty}\sum_{n\in\mathbb{Z}} (-1)^j \binom{n}{j} z_0^{-n} z_1^{n-j} z_2^{j}.$$

The formal delta function satisfies many useful identities. The following is a seemingly innocent property: for any integer k

$$z^k \delta(z) = \delta(z). \tag{2.1.2}$$

Furthermore, as given in [25] (lemma 3.1.3) we have

Proposition 2.1.2. *For any* $\alpha \in \mathbb{C}$

$$z_1^{-1}\left(\frac{z_2 + z_0}{z_1}\right)^{-\alpha} \delta\left(\frac{z_2 + z_0}{z_2}\right) = z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{\alpha} \delta\left(\frac{z_1 - z_0}{z_2}\right). \tag{2.1.3}$$

Under various conditions the formal delta function facilitates formal variable substitutions. We will need the following, given in [22] (lemma 2.1.2).

Proposition 2.1.3. *Let* V *be a superspace.*

1. *If* $f(z_1, z_2) \in (V[[z_1, z_1^{-1}]])((z_2))$, *then*

$$\delta\left(\frac{z_0 + z_2}{z_1}\right) f(z_1, z_2) = \delta\left(\frac{z_0 + z_2}{z_2}\right) f(z_0 + z_2, z_2). \tag{2.1.4}$$

2. If $f(z_1,z_2) \in (V[[z_0,z_0^{-1}]])((z_2))$, then

$$\delta\left(\frac{z_1-z_2}{z_0}\right)f(z_0,z_2) = \delta\left(\frac{z_1-z_2}{z_0}\right)f(z_0,z_1-z_0). \tag{2.1.5}$$

2.2 Vertex Operator Superalgebras

We state the definitions of vertex superalgebra and vertex operator superalgebra and briefly overview their immediate consequences.

Definition 2.2.1. *A **vertex superalgebra** is a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ equipped with a linear map*

$$Y(-,z) : V \to (\mathrm{End}V)[[z,z^{-1}]]$$
$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$$

*and an element $\mathbb{1} \in V$, called the **vacuum vector**, such that the following hold:*

$$Y(u,z)v \in V((z)) \text{ for any } u,v \in V, \tag{2.2.1}$$

$$Y(\mathbb{1},z) = Id_V, \tag{2.2.2}$$

$$Y(u,z)\mathbb{1} \in V[[z]] \quad \text{and} \quad \mathrm{Res}_z Y(u,z)\mathbb{1} = u \text{ for any } u \in V, \tag{2.2.3}$$

and for $\mathbb{Z}_2$-homogeneous $u,v \in V$ the Jacobi identity holds

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) - (-1)^{\tilde{u}\tilde{v}}z_0^{-1}\delta\left(\frac{z_2-z_1}{z_0}\right)Y(v,z_2)Y(u,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2). \tag{2.2.4}$$

*For $n \in \frac{1}{2}\mathbb{Z}$ and $u \in V_n$ we say u has **weight** n and write $\mathrm{wt}u = n$. We may denote such a vertex superalgebra by the triple $(V,Y,\mathbb{1})$.*

Definition 2.2.2. *A **vertex operator superalgebra** is a vertex operator superalgebra* $(V, Y, \mathbb{1})$ *which carries a* $\frac{1}{2}\mathbb{Z}$*-grading*

$$V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n \quad \text{with} \quad V_{\bar{0}} = \bigoplus_{n\in\mathbb{Z}} V_n \quad \text{and} \quad V_{\bar{1}} = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n \tag{2.2.5}$$

where $\dim V_n < \infty$ *for all* $n \in \frac{1}{2}\mathbb{Z}$ *and* $V_n = 0$ *for* n *sufficiently negative. Moreover, there is distinguished vector* $\omega \in V$*, called the **conformal vector**, such that the following Virasoro algebra axioms are satisfied:*

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}c_V \text{ for all } m, n \in \mathbb{Z}, \tag{2.2.6}$$

$$Y(L(-1)u, z) = \frac{d}{dz}Y(u, z) \text{ for all } u \in V, \tag{2.2.7}$$

$$L(0)|_{V_n} = nId_V \text{ for all } n \in \frac{1}{2}\mathbb{Z}, \tag{2.2.8}$$

where $c_V \in \mathbb{C}$ *is the **central charge** of* V *and*

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}. \tag{2.2.9}$$

We may denote such a vertex operator superalgebra by the quadruple $(V, Y, \mathbb{1}, \omega)$.

Remark 2.2.3. *In the case that* $V_{\bar{1}} = 0$*, we have the definitions of a **vertex algebra** and **vertex operator algebra**, respectively.*

For a vertex operator superalgebra $(V, Y, \mathbb{1}, \omega)$ one can readily verify that $\mathbb{1} \in V_0$, $\omega \in V_2$ and for any homogeneous $u, v \in V$ and $n \in \frac{1}{2}\mathbb{Z}$

$$u_n v \in V_{\mathrm{wt}u + \mathrm{wt}v - n - 1}. \tag{2.2.10}$$

In particular if $u \in V_{\bar{i}}$, $v \in V_{\bar{j}}$ and $n \in \mathbb{Z}$, then $u_n v \in V_{\overline{i+j}}$. To denote this we will simply write $V_{\bar{i}}V_{\bar{j}} \subseteq V_{\overline{i+j}}$.

Following directly from the definitions we also have the identities:

$$Y(u,z)\mathbb{1} = e^{zL(-1)}u \text{ for } u \in V, \tag{2.2.11}$$

$$e^{z_0L(-1)}Y(u,z)e^{-z_0L(-1)} = Y(u, z+z_0) \text{ for } u \in V, \tag{2.2.12}$$

$$Y(u,z)v = (-1)^{\tilde{u}\tilde{v}}Y(v,-z)u \text{ for } \mathbb{Z}_2\text{-homogeneous } u, v \in V. \tag{2.2.13}$$

2.3 Automorphisms and Twisted Modules

In this section we summarize the basic representation theory of vertex operator superalgebras.

Definition 2.3.1. *An* ***automorphism*** *of a vertex operator superalgebra V is a linear isomorphism g of V in which $g(\omega) = \omega$ and satisfies*

$$gY(v,z)g^{-1} = Y(gv, z) \tag{2.3.1}$$

for all $v \in V$. Denote the group of automorphisms of V by Aut$V$*.*

It can be shown that any automorphism g of a vertex operator superalgebra V satisfies $g(\mathbb{1}) = \mathbb{1}$. Moreover, since g fixes ω, it must be the case that g commutes with the operator $\omega_1 = L(0)$. In particular g preserves each homogeneous subspace V_n and hence preserves $V_{\bar{0}}$ and $V_{\bar{1}}$.

For a vertex operator superalgebra V we have the linear isomorphism $\sigma : V \to V$ associated to the superstructure on V: for $\mathbb{Z}_2$-homogeneous $v \in V$

$$\sigma v = (-1)^{\tilde{v}}v. \tag{2.3.2}$$

Since $V_{\bar{i}}V_{\bar{j}} \subseteq V_{\overline{i+j}}$ it follows easily that $\sigma \in \mathrm{Aut}V$. Furthermore, it is important to note that since any automorphism g of V preserves $V_{\bar{0}}$ and $V_{\bar{1}}$, the element σ

is central in Aut$V$.

For $g \in \mathrm{Aut}V$ with $o(g) = T$ and $o(g\sigma) = T'$, we denote the decompositions of V into eigenspaces for g and $g\sigma$, respectively, as

$$V = \bigoplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r, \tag{2.3.3}$$

$$V = \bigoplus_{r \in \mathbb{Z}/T'\mathbb{Z}} V^{r*} \tag{2.3.4}$$

where $V^r = \{v \in V : gv = e^{-2\pi i r/T} v\}$ and $V^{r*} = \{v \in V : g\sigma v = e^{-2\pi i r/T'} v\}$.

Definition 2.3.2. *A **weak** g**-twisted** V**-module** is a vector space M equipped with a linear map*

$$Y_M(-, z) : V \to (\mathrm{End}\ M)\,[[z^{1/T}, z^{-1/T}]]$$
$$v \mapsto Y_M(v, z) = \sum_{n \in \frac{1}{T}\mathbb{Z}} v_n z^{-n-1}$$

satisfying the following for all $0 \le r \le T - 1$ and $u \in V^r$ and $v \in V$:

$$Y_M(u, z) = \sum_{n \in \frac{r}{T} + \mathbb{Z}} u_n z^{-n-1}, \tag{2.3.5}$$

$$Y_M(v, z)u \in M((z)), \tag{2.3.6}$$

$$Y_M(\mathbb{1}, z) = Id_M. \tag{2.3.7}$$

For $\mathbb{Z}_2$-homogeneous $u, v \in V$ the twisted Jacobi identity holds

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1) Y_M(v, z_2) - (-1)^{\tilde{u}\tilde{v}} z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y_M(v, z_2) Y_M(u, z_1)$$
$$= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2). \tag{2.3.8}$$

We may denote such a module by the pair (M, Y_M).

Definition 2.3.3. *A* ***submodule*** *of a weak g-twisted V-module M is a subspace W of M such that $Y_M(u,z)w \in W((z))$ for all $u \in V$ and $w \in W$. We say M is* ***irreducible*** *if the only submodules of M are 0 and M itself.*

Definition 2.3.4. *A* ***homomorphism*** *between two weak g-twisted V-modules M and W is a linear map $f : M \to W$ satisfying the following for all $u \in V$*

$$fY_M(u,z) = Y_W(u,z)f. \tag{2.3.9}$$

Remark 2.3.5. *It was shown in [25] (theorem 3.3.2), that for a weak g-twisted V-module M, the twisted Jacobi identity is equivalent to the following associativity formula*

$$\begin{aligned}&(z_0+z_2)^{k+\frac{r}{T}}Y_M(u,z_0+z_2)Y_M(v,z_2)w\\ &= (z_2+z_0)^{k+\frac{r}{T}}Y_M(Y(u,z_0)v,z_2)w\end{aligned} \tag{2.3.10}$$

where $w \in M$, $v \in V^r$ and $k \in \mathbb{Z}_+$ such that $z^{k+\frac{r}{T}}Y_M(u,z) \in M[[z]]$, together with the super-commutator relation

$$\begin{aligned}&[Y_M(u,z_1),Y_M(v,z_2)]\\ &= Res_{z_0} z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-\frac{r}{T}}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M(Y(u,z_0)v,z_2).\end{aligned} \tag{2.3.11}$$

The following easy lemma relates the g-eigenpaces to the $g\sigma$-eigenspaces, and will be extremely useful in sections 3.5-3.7, when we focus on relating the representation theories of V and $A_{g,n}(V)$. As was first pointed out in [15], this relationship will allow us to repurpose arguments from the "non-super" case to our present situation.

Lemma 2.3.6. *Let M be a weak g-twisted V-module. If $u \in V^r \cap V^{s*}$ is homogeneous, then $\mathrm{wt}u + \frac{s}{T'}$ is congruent to $\frac{r}{T}$ modulo $\mathbb{Z}$.*

Proof. We know $T' \mid 2T$ and if T is odd, then $T' = 2T$. In either the case, by comparing gu and $g\sigma u$, it is easy to see that $\frac{s}{T'}$ is equivalent to $\frac{1}{2}\tilde{u} + \frac{r}{T}$ modulo $\mathbb{Z}$. Since $\mathrm{wt}u$ is equivalent to $\frac{1}{2}\tilde{u}$ modulo $\mathbb{Z}$ we are done. □

For $t \in \mathbb{Z}$ and $p, q \in \frac{1}{T}\mathbb{Z}$, by equating the coefficient of $z_0^{-t-1} z_1^{-p-1} z_2^{-q-1}$ on both sides of (2.3.8) we obtain the component form of the twisted Jacobi identity

$$\sum_{j=0}^{\infty} (-1)^j \binom{t}{j} \left(u_{t+p-j} v_{j+q} - (-1)^{\tilde{u}\tilde{v}} (-1)^t v_{t+q-j} u_{j+p} \right) = \sum_{j=0}^{\infty} \binom{p}{j} (u_{t+j} v)_{p+q-j}. \tag{2.3.12}$$

Similarly, equating the coefficient of $z_1^{-p-1} z_2^{-q-1}$ on both sides of (2.3.11) we have the component form of the super-commutativity relation

$$[u_p, v_q] = \sum_{j=0}^{\infty} \binom{p}{j} (u_j v)_{p+q-j}. \tag{2.3.13}$$

Remark 2.3.7. *As shown in [25] (corollary 3.3.3, corollary 3.3.4), the usual Virasoro algebra axioms for the module M hold:*

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12} \delta_{m+n,0} c_V, \tag{2.3.14}$$

$$Y_M(L(-1)v, z) = \frac{d}{dz} Y_M(v, z) \tag{2.3.15}$$

where

$$Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-1}.$$

We will usually drop the subscript on the Virasoro operators $L_M(n)$ associated to the module M.

Definition 2.3.8. *An admissible g-twisted V-module is a weak g-twisted V-module M which carries a* $\frac{1}{T'}\mathbb{Z}_+$*-grading*

$$M = \bigoplus_{n \in \frac{1}{T'}\mathbb{Z}} M(n)$$

satisfying

$$v_m M(n) \subseteq M(wtv + n - m - 1) \tag{2.3.16}$$

for all homogeneous $v \in V$, $m \in \mathbb{Z}$ *and* $n \in \frac{1}{T'}\mathbb{Z}_+$.

Remark 2.3.9. *As developed in [14], an admissible g-twisted V-module M must have a* $\frac{1}{T'}\mathbb{Z}_+$*-grading rather than a* $\frac{1}{T}\mathbb{Z}$*-grading. This is because V is* $\frac{1}{2}\mathbb{Z}$*-graded rather than* $\mathbb{Z}$*-graded.*

Definition 2.3.10. *An ordinary g-twisted V-module is a weak g-twisted V-module M which carries a* $\mathbb{C}$*-grading*

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

such that

$$\dim M_\lambda < \infty \quad \textit{and} \quad M_\lambda = \{w \in M : L(0)w = \lambda w\}$$

and for fixed $\lambda \in \mathbb{C}$ *we have* $M_{\lambda + \frac{n}{T'}} = 0$ *for n sufficiently negative. If* $w \in M_\lambda$ *we say w has* ***weight*** λ *and write* $\mathrm{wt} w = \lambda$.

Remark 2.3.11. *In the case that* $g = 1$ *we drop the "twisted" prefix and have the definitions of* ***weak***, ***admissible*** *and* ***ordinary*** *V-module, respectively.*

The main results of this thesis are concerned with admissible g-twisted V-modules. These modules form a full subcategory of the weak g-twisted V-modules with morphisms as in (2.3.9). By applying the grading restrictions, it is not hard to show that an ordinary g-twisted V-module is admissible. See [7] (lemma 3.4).

Of particular importance are the vertex operator algebras whose admissible g-twisted V-modules are all semisimple.

Definition 2.3.12. *A vertex operator superalgebra V is g-**rational** if the category of admissible g-twisted V-modules is semisimple.*

The next two propositions are very useful in the study of twisted modules and of course have their roots in the representation theory of vertex operator algebras.

As in [21] (lemma 6.1.1) we may apply the associativity formula (2.3.10) to obtain the following useful description of a weak g-twisted V-module. We provide the proof following an argument given in [20] (proposition 4.5.7).

Proposition 2.3.13. *If M is an irreducible weak g-twisted V-module, then for any nonzero $w \in M$*

$$M = span_{\mathbb{C}} \left\{ v_m w : v \in V,\ m \in \frac{1}{T}\mathbb{Z} \right\}.$$

Proof. We only need to show that $X = \mathrm{span}_{\mathbb{C}} \left\{ v_m w : v \in V,\ m \in \frac{1}{T}\mathbb{Z} \right\}$ is a V-submodule of M. By linearity and (2.3.5) it suffices to show that $u_{m+\frac{r}{T}} v_{n+\frac{s}{T}} w \in X$ for all homogeneous $u \in V^r$, $v \in V^s$ and $m, n \in \mathbb{Z}$. First, there exists $k \in \mathbb{Z}_+$ such that $z^{k+\frac{r}{T}} Y_M(u, z)$ contains only non-negative integral powers of z. So

$$\begin{aligned} &(z_0 + z_2)^{k+\frac{r}{T}} Y_M(u, z_0 + z_2) Y_M(v, z_2) w \\ &= (z_2 + z_0)^{k+\frac{r}{T}} Y_M(Y(u, z_0)v, z_2) w. \end{aligned} \tag{2.3.17}$$

We also compute

$$
\begin{aligned}
&u_{m+\frac{r}{T}}v_{n+\frac{s}{T}}w \\
&= \mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}} Y_M(u,z_1)Y_M(v,z_2)w \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) z_1^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}} Y_M(u,z_1)Y_M(v,z_2)w \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right) z_1^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}} Y_M(u,z_1)Y_M(v,z_2)w \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right) (z_0+z_2)^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}} Y_M(u,z_0+z_2)Y_M(v,z_2)w \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_2} (z_0+z_2)^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}} Y_M(u,z_0+z_2)Y_M(v,z_2)w. \qquad (2.3.18)
\end{aligned}
$$

Combining (2.3.17) and (2.3.18), we obtain

$$
\begin{aligned}
&u_{m+\frac{r}{T}}v_{n+\frac{s}{T}}w \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_2} (z_0+z_2)^{m-k+\frac{r}{T}} z_2^{n+\frac{s}{T}} \left((z_2+z_0)^{k+\frac{r}{T}} Y_M(Y(u,z_0)v,z_2)w\right). \qquad (2.3.19)
\end{aligned}
$$

Since the right hand side of (2.3.19) lies in X we are done. □

Analogous to [26] (lemma 1.2.1), it follows that $L(0)$ acts semisimply on any irreducible admissible g-twisted V-module. We give a different proof here using an argument due to C. Dong which relies on the *countably infinite dimensional version of Shur's lemma* originally due to J. Dixmier [2].

Proposition 2.3.14. *If $M = \bigoplus_{m\in\frac{1}{T}\mathbb{Z}_+} M(m)$ is an irreducible admissible g-twisted V-module, then $L(0)$ acts semisimply on M. In particular there exists $h \in \mathbb{C}$ such that*

$$M(m) = \{w \in M : L(0)w = (h+m)w\}.$$

The number h is called the ***conformal weight*** *of M.*

Proof. As $M \neq 0$ we may assume, after a possible grading shift, that $M(0) \neq 0$. Take $w \in M(0)$ to be nonzero. Since M is irreducible, using proposition 2.3.13, we can write

$$M = \mathrm{span}_{\mathbb{C}}\{v_m w : v \in V, m \in \frac{1}{T}\mathbb{Z}\}$$

for any nonzero $w \in W$. Since M is admissible, for any homogeneous $v \in V$ and $n \in \frac{1}{T'}\mathbb{Z}_+$ we have $v_{\mathrm{wt}v-n-1}M(0) \subseteq M(n)$. In particular

$$M(n) = \mathrm{span}_{\mathbb{C}}\{v_{\mathrm{wt}v-n-1}w : v \in V\}. \tag{2.3.20}$$

Let A be the associative subalgebra of $\mathrm{End}_{\mathbb{C}}M(0)$ generated by the set $\{v_{\mathrm{wt}v-1} : v \in V\}$ so that $M(0)$ is an irreducible A-module. As $L(0) = \omega_1 = \omega_{\mathrm{wt}\omega-1}$ it follows that $L(0)$ preserves $M(0)$. Moreover, from (2.3.13) we get $[L(0), v_{\mathrm{wt}v-1}] = 0$ on M. That is, $L(0)$ is an A-module map.

Since V has countable dimension, so does M by proposition 2.3.13. In particular $M(0)$ has countable dimension. Then the countably infinite version of Shur's lemma [2] tells us that $L(0)$ acts on $M(0)$ as a scalar $h \in \mathbb{C}$.

Applying (2.3.13) once more we obtain $[L(0), v_{\mathrm{wt}v-n-1}] = nv_{\mathrm{wt}v-n-1}$. So we have $L(0)v_{\mathrm{wt}v-n-1}w = (h+n)v_{\mathrm{wt}v-n-1}w$. Combined with (2.3.20) we have proven the result. □

Chapter 3

Twisted Modules and Associative Algebras

In this chapter we develop the results of tthisthesis. Given a vertex operator superalgbera V and automorphism g of finite order T, we follow [8] to construct a sequence of associative algebras $A_{g,n}(V)$ for $n \in \frac{1}{T'}\mathbb{Z}_+$ such that $A_{g,n-\frac{1}{T'}}(V)$ is a quotient of $A_{g,n}(V)$. There is a bijection between the irreducible $A_{g,n}(V)$-modules which cannot factor through $A_{g,n-\frac{1}{T'}}(V)$ and the irreducible admissible g-twisted V-modules. These results are then applied to g-rational vertex operator superalgebras where it is shown that V is g-rational if and only if all $A_{g,n}(V)$ are finite-dimensional semisimple.. Taking $n = 0$ we obtain the associative algebra $A_g(V)$ constructed in [15]. With $g = 1$ we recover $A_n(V)$ as in [18].

3.1 The Spaces $O_{g,n}(V)$ and $A_{g,n}(V)$

Let V be a vertex operator superalgebra V and g an automorphism of finite order T. Fix $n \in \frac{1}{T'}\mathbb{Z}_+$ and write $n = \ell + \frac{i}{T'}$ for unique nonnegative integers ℓ and i with $0 \le i \le T' - 1$. In this section we equip V with two bilinear products

$\circ_{g,n} : V \times V \to V$ and $*_{g,n} : V \times V \to V$, then define the subspace $O_{g,n}(V)$ and the quotient $A_{g,n}(V)$ of V, respectively.

For integers $0 \le r \le T' - 1$, define

$$\delta_i(r) = \begin{cases} 1 & \text{if} \quad r \le i, \\ 0 & \text{if} \quad i < r \le T' - 1. \end{cases}$$

Also set $\delta_i(T') = 0$. For $v \in V$ and homogeneous $u \in V^{r*}$, define

$$u \circ_{g,n} v = \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}\,u - 1 + \delta_i(r) + \ell + \frac{r}{T'}}}{z^{2\ell + \delta_i(r) + \delta_i(T'-r) + 1}} \tag{3.1.1}$$

and

$$u *_{g,n} v = \begin{cases} \displaystyle\sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}\,u + \ell}}{z^{\ell+m+1}} & \text{if} \quad r = 0, \\ 0 & \text{if} \quad r > 0. \end{cases} \tag{3.1.2}$$

Extend both $\circ_{g,n}$ and $*_{g,n}$ linearly to obtain bilinear products on all of V. Now define

$$O_{g,n}(V) = \mathrm{span}_{\mathbb{C}}\{u \circ_{g,n} v,\ L(-1)u + L(0)u \ :\ u, v \in V\}. \tag{3.1.3}$$

Define the linear space $A_{g,n}(V)$ to be the quotient

$$A_{g,n}(V) = V/O_{g,n}(V). \tag{3.1.4}$$

3.2 Relations Modulo $O_{g,n}(V)$

In this section we gather a handful of relations modulo $O_{g,n}(V)$ which will be used to facilitate calculations in $A_{g,n}(V)$.

Lemma 3.2.1. *If $r \neq 0$, then $V^{r*} \subseteq O_{g,n}(V)$.*

Proof. Let $0 < r \leq T' - 1$ and $v \in V^{r*}$ be homogeneous. By definition of $O_{g,n}(V)$ we know $v \circ_{g,n} \mathbb{1} \in O_{g,n}(V)$. By definition of $\circ_{g,n}$ we have

$$v \circ_{g,n} \mathbb{1} = \sum_{k=0}^{\infty} \binom{\mathrm{wt} v - 1 + \delta_i(r) + \ell + \frac{r}{T'}}{k} v_{k-2\ell-\delta_i(r)-\delta_i(T'-r)-1}\mathbb{1}.$$

Since $v_j \mathbb{1} = 0$ and $v_{-j-1}\mathbb{1} = \frac{1}{j!}L(-1)^j v$ for $j \geq 0$, we get

$$v \circ_{g,n} \mathbb{1} = \sum_{k=0}^{2\ell+\delta_i(r)+\delta_i(T'-r)} \frac{\binom{\mathrm{wt} v-1+\delta_i(r)+\ell+\frac{r}{T'}}{k}}{(2\ell + \delta_i(r) + \delta_i(T'-r) - k)!} L(-1)^{2\ell+\delta_i(r)+\delta_i(T'-r)-k} v$$

As $\mathrm{wt} L(-1)v = (\mathrm{wt} v + 1)$ and $L(-1)v \equiv -L(0)v \mod O_{g,n}(V)$, it follows that for all $j \geq 0$

$$L(-1)^j v \equiv j! \binom{-\mathrm{wt} v}{j} v \mod O_{g,n}(V).$$

So we obtain

$$\begin{aligned} v \circ_{g,n} \mathbb{1} \equiv & \sum_{k=0}^{2\ell+\delta_i(r)+\delta_i(T'-r)} \binom{\mathrm{wt} v - 1 + \delta_i(r) + \ell + \frac{r}{T'}}{k} \binom{-\mathrm{wt} v}{2\ell + \delta_i(r) + \delta_i(T'-r) - k} \\ & \times v \mod O_{g,n}(V) \\ \equiv & \binom{\delta_i(r) + \ell + \frac{r}{T'} - 1}{2\ell + \delta_i(r) + \delta_i(T'-r)} v \mod O_{g,n}(V). \end{aligned}$$

As $r > 0$, we know $\delta_i(r) + \ell + \frac{r}{T'} - 1$ is not an integer and so $\binom{\delta_i(r)+\ell+\frac{r}{T'}-1}{2\ell+\delta_i(r)+\delta_i(T'-r)} \neq 0$. □

Lemma 3.2.2. *If $u \in V^{r*}$ is homogeneous and $v \in V$, then for all integers $0 \leq k \leq m$*

$$Res_z Y(u,z)v \frac{(1+z)^{wtu-1+\ell+\delta_i(r)+\frac{r}{T'}+k}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+1+m}} \in O_{g,n}(V).$$

Proof. Note that for all integers $0 \leq k \leq m$

$$\begin{aligned}
&\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u-1+\ell+\delta_i(r)+\frac{r}{T'}+k}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+1+m}} \\
&= \sum_{i=0}^{k} \binom{k}{i} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u-1+\ell+\delta_i(r)+\frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+1+m-i}}.
\end{aligned}$$

Thus it suffices to prove the statement for $k = 0$ and all integers $m \geq 0$. We proceed by induction. The case $m = 0$ holds by definition of $O_{g,n}(V)$. Now, assuming the statement holds for some $m \geq 0$ and all homogeneous elements of V, we must show

$$\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u-1+\ell+\delta_i(r)+\frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+2}} \in O_{g,n}(V).$$

Using the $L(-1)$-derivative property (2.2.7) and the residue formula for the formal product formula (2.1.1), we compute

$$\begin{aligned}
&\mathrm{Res}_z \left(Y(L(-1)u,z)v\right) \frac{(1+z)^{\mathrm{wt}u+\ell+\delta_i(r)+\frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+1}} \\
&= \mathrm{Res}_z \left(\frac{d}{dz} Y(u,z)v\right) \frac{(1+z)^{\mathrm{wt}u+\ell+\delta_i(r)+\frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+1}} \\
&= -\mathrm{Res}_z Y(u,z)v \left(\frac{d}{dz}\left(\frac{(1+z)^{\mathrm{wt}u+\ell+\delta_i(r)+\frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+1}}\right)\right) \\
&= -\mathrm{Res}_z Y(u,z)v \left(\left(\mathrm{wt}u+\ell+\delta_i(r)+\frac{r}{T'}\right) \frac{(1+z)^{\mathrm{wt}u-1+\ell+\delta_i(r)+\frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+1}}\right. \\
&\qquad \left. - \left(2\ell+\delta_i(r)+\delta_i(T'-r)+m+1\right) \frac{(1+z)^{\mathrm{wt}u+\ell+\delta_i(r)+\frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+2}}\right)
\end{aligned}$$

$$= (2\ell + \delta_i(r) + \delta_i(T'-r) + m + 1)\,\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u - 1 + \ell + \delta_i(r) + \frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+2}}(1+z)$$
$$- \left(\mathrm{wt}u + \ell + \delta_i(r) + \frac{r}{T'}\right)\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u - 1 + \ell + \delta_i(r) + \frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+1}}$$
$$= (2\ell + \delta_i(r) + \delta_i(T'-r) + m + 1)\,\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u - 1 + \ell + \delta_i(r) + \frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+2}}$$
$$+ (2\ell + \delta_i(r) + \delta_i(T'-r) + m + 1)\,\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u - 1 + \ell + \delta_i(r) + \frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+1}}$$
$$- \left(\mathrm{wt}u + \ell + \delta_i(r) + \frac{r}{T'}\right)\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u - 1 + \ell + \delta_i(r) + \frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+1}}.$$

As $\mathrm{wt}L(-1)u = \mathrm{wt}u + 1$, the induction assumption gives us that each of

$$\mathrm{Res}_z\,(Y(L(-1)u,z)v)\,\frac{(1+z)^{\mathrm{wt}u + \ell + \delta_i(r) + \frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+1+m}},$$
$$(2\ell + \delta_i(r) + \delta_i(T'-r) + 1 + m)\,\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u - 1 + \ell + \delta_i(r) + \frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+1}}, \quad \text{and}$$
$$- \left(\mathrm{wt}u + \ell + \delta_i(r) + \frac{r}{T'}\right)\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u - 1 + \ell + \delta_i(r) + \frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+m+1}}$$

lies in $O_{g,n}(V)$. □

Lemma 3.2.3. *If $u, v \in V^{0*}$ are homogeneous, then*

(i) $u *_{g,n} v - (-1)^{\tilde{u}\tilde{v}} \sum_{m=0}^{\ell} (-1)^{\ell} \binom{m+\ell}{\ell} Res_z Y(v,z)u \frac{(1+z)^{wtv+m-1}}{z^{\ell+m+1}} \in O_{g,n}(V).$

(ii) $u *_{g,n} v - (-1)^{\tilde{u}\tilde{v}} v *_{g,n} u - Res_z Y(u,z)v(1+z)^{wtu-1} \in O_{g,n}(V).$

Proof. For *(i)*, the skew-symmetry (2.2.13) gives us

$$Y(u,z)v = (-1)^{\tilde{u}\tilde{v}} \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}} \frac{(-1)^{-m-1}}{k!} L(-1)^k v_m u z^{k-m-1}.$$

Now, by definition of $O_{g,n}(V)$ we have

$$L(-1)v_m u \equiv -(\mathrm{wt}v + \mathrm{wt}u - m - 1)v_m u \mod O_{g,n}(V).$$

Then for all $k \geq 0$

$$L(-1)^k v_m u \equiv k! \binom{-\mathrm{wt}v - \mathrm{wt}u + m + 1}{k} v_m u \mod O_{g,n}(V).$$

With this we obtain

$$Y(u,z)v \equiv (-1)^{\tilde{u}\tilde{v}} Y\left(v, \frac{-z}{1+z}\right) u (1+z)^{-\mathrm{wt}u - \mathrm{wt}v} \mod O_{g,n}(V).$$

We now compute

$$\begin{aligned}
u *_{g,n} v &= \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u+\ell}}{z^{\ell+m+1}} \\
&\equiv (-1)^{\tilde{u}\tilde{v}} \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_z Y\left(v, \frac{-z}{1+z}\right) u \frac{(1+z)^{-\mathrm{wt}v+\ell}}{z^{\ell+m+1}} \mod O_{g,n}(V) \\
&= (-1)^{\tilde{u}\tilde{v}} \sum_{k=0}^{\infty} \sum_{m=0}^{\ell} (-1)^{\ell} \binom{m+\ell}{\ell} \binom{\mathrm{wt}v + m - 1}{k} v_{k-\ell-m-1} u \\
&= (-1)^{\tilde{u}\tilde{v}} \sum_{m=0}^{\ell} (-1)^{\ell} \binom{m+\ell}{\ell} \mathrm{Res}_z Y(v,z)u \frac{(1+z)^{\mathrm{wt}v+m-1}}{z^{\ell+m+1}}
\end{aligned}$$

which proves *(i)*. For *(ii)*, swapping u and v in *(i)* we get

$$\begin{aligned}
&u *_{g,n} v - (-1)^{\tilde{u}\tilde{v}} v *_{g,n} u \\
&\equiv \Bigg(\sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u+\ell}}{z^{\ell+m+1}} \\
&\qquad - \sum_{m=0}^{m} (-1)^{\ell} \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)\, v \frac{(1+z)^{\mathrm{wt}u+m-1}}{z^{\ell+m+1}} \Bigg) \mod O_{g,n}(V) \\
&= \mathrm{Res}_z \left(Y(u,z)v(1+z)^{\mathrm{wt}u-1} \sum_{m=0}^{\ell} \binom{m+\ell}{\ell} \frac{(-1)^m (1+z)^{\ell+1} - (-1)^{\ell}(1+z)^m}{z^{m+\ell+1}} \right) \\
&= \mathrm{Res}_z Y(u,z)v(1+z)^{\mathrm{wt}u-1}.
\end{aligned}$$

This follows since

$$\sum_{m=0}^{\ell} \binom{m+\ell}{\ell} \frac{(-1)^m(1+z)^{\ell+1} - (-1)^{\ell}(1+z)^m}{z^{m+\ell+1}} = 1$$

by proposition 5.2 of [9]. □

3.3 The Associative Algebra $A_{g,n}(V)$

We verify that $A_{g,n}(V)$ is in fact an associative algebra under the induced product $*_{g,n}$ and that $A_{g,n-\frac{1}{T'}}(V)$ is a quotient of $A_{g,n}(V)$.

Lemma 3.3.1. *$O_{g,n}(V)$ is a two-sided ideal of V under $*_{g,n}$.*

Proof. As $V^{r*} \subseteq O_{g,n}(V)$ if $0 < r < T'$ by lemma 3.2.1, we only need to prove

$$(L(-1)u + L(0)u) *_{g,n} v \in O_{g,n}(V) \quad \text{for } u, v \in V^{0*}, \tag{3.3.1}$$

$$v *_{g,n} (L(-1)u + L(0)u) \in O_{g,n}(V) \quad \text{for } v, u \in V^{0*}, \tag{3.3.2}$$

$$u *_{g,n} (v \circ_{g,n} w) \in O_{g,n}(V) \quad \text{for } u \in V^{0*}, v \in V^{r*}, w \in V^{(T'-r)*}, \tag{3.3.3}$$

$$(u \circ_{g,n} v) *_{g,n} w \in O_{g,n}(V) \quad \text{for } u \in V^{r*}, v \in V^{(T'-r)*}, w \in V^{0*}. \tag{3.3.4}$$

We first prove (3.3.1). For $u, v \in V^{0*}$, the $L(-1)$-derivitive property and the residue formula for the product rule give us

$$\begin{aligned}
&L(-1)u *_{g,n} v \\
&= \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_z Y(L(-1)u, z)v \frac{(1+z)^{\mathrm{wt}u+1+\ell}}{z^{\ell+m+1}} \\
&= \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_z \left(\frac{d}{dz} Y(u, z)v\right) \frac{(1+z)^{\mathrm{wt}u+1+\ell}}{z^{\ell+m+1}}
\end{aligned}$$

$$= \sum_{m=0}^{\ell} (-1)^{m+1} \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)v \left(\frac{d}{dz} \frac{(1+z)^{\mathrm{wt}u+1+\ell}}{z^{\ell+m+1}} \right)$$

$$= \sum_{m=0}^{\ell} (-1)^{m+1} \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)v$$

$$\left(\frac{(-\ell-m-1)(1+z)^{\mathrm{wt}u+1+\ell}}{z^{\ell+m+2}} + \frac{z(\mathrm{wt}u+1+\ell)(1+z)^{\mathrm{wt}u+\ell}}{z^{\ell+m+2}} \right)$$

With this we compute

$$\begin{aligned}
&(L(-1)u + L(0)u) *_{g,n} v \\
&= \sum_{m=0}^{\ell} \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)v(1+z)^{\mathrm{wt}u+\ell} \frac{mz+\ell+m+1}{z^{\ell+m+2}} \\
&= (-1)^{\ell} \binom{2\ell}{2} (2\ell+1) \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u+\ell}}{z^{2\ell+2}} \\
&= (-1)^{\ell} \binom{2\ell}{2} (2\ell+1)(u \circ_{g,n} v) \\
&\in O_{g,n}(V)
\end{aligned}$$

which proves (3.3.1).

Now we verify (3.3.2). Let $u, v \in V^{0*}$. We know $L(-1)u + L(0)u *_{g,n} v \in O_{g,n}(V)$. Applying lemma 3.2.3 *(iii)* we have

$$\begin{aligned}
&v *_{g,n} (L(-1)u + L(0)u) \\
&\equiv -\mathrm{Res}_z \left(Y(L(-1)u,z)v(1+z)^{\mathrm{wt}v} + Y(L(0)u,z)v(1+z)^{\mathrm{wt}u+1} \right) \mod O_{g,n}(V) \\
&= \mathrm{Res}_z \left(Y(u,z)v \frac{d}{dz}(1+z)^{\mathrm{wt}u} - (\mathrm{wt}u) Y(L(0)u,z)v(1+z)^{\mathrm{wt}u-1} \right) \\
&= 0
\end{aligned}$$

and (3.3.2) follows

To prove(3.3.3) take homogeneous $u \in V^{0*}$, $v \in V^{r*}$ and $w \in V^{(T'-r)*}$. We have

$$\begin{aligned}
&u *_{g,n} (v \circ_{g,n} w) \\
&\equiv \left(u *_{g,n} (v \circ_{g,n} w) - (-1)^{\tilde{u}\tilde{v}} v \circ_{g,n} (u *_{g,n} w)\right) \mod O_{g,n}(V) \\
&= \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_{z_1} Y(u, z_1)(v \circ_{g,n} w) \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m+1}} \\
&\quad - (-1)^{\tilde{u}\tilde{v}} \mathrm{Res}_{z_2} Y(v, z_2)(u *_{g,n} w) \frac{(1+z_2)^{\mathrm{wt}v-1+\ell+\delta_i(r)+\frac{r}{T'}}}{z_2^{2\ell+\delta_i(r)+\delta_i(T'-r)+1}} \\
&= \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} Y(u, z_1) Y(v, z_2) w \\
&\quad \times \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m+1}} \frac{(1+z_2)^{\mathrm{wt}v-1+\ell+\delta_i(r)+\frac{r}{T'}}}{z_2^{2\ell+\delta_i(r)+\delta_i(T'-r)+1}} \\
&\quad - (-1)^{\tilde{u}\tilde{v}} \sum_{m=0}^{\ell} (-1)^m \mathrm{Res}_{z_2} \mathrm{Res}_{z_1} Y(v, z_2) Y(u, z_1) w \\
&\quad \times \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m+1}} \frac{(1+z_2)^{\mathrm{wt}v-1+\ell+\delta_i(r)+\frac{r}{T'}}}{z_2^{2\ell+\delta_i(r)+\delta_i(T'-r)+1}} \\
&= \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_{z_2} \mathrm{Res}_{z_1-z_2} Y(Y(u, z_1 - z_2)v, z_2) w \\
&\quad \times \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m+1}} \frac{(1+z_2)^{\mathrm{wt}v-1+\ell+\delta_i(r)+\frac{r}{T'}}}{z_2^{2\ell+\delta_i(r)+\delta_i(T'-r)+1}} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \binom{\mathrm{wt}u+\ell}{j} \binom{-\ell-m-1}{k} \\
&\quad \times \mathrm{Res}_{z_2} Y(u_{j+k} v, z_2) w \frac{(1+z_2)^{\mathrm{wt}u+\mathrm{wt}v-1+\ell+\delta_i(r)-\frac{r}{T'}-j+\ell}}{z_2^{2\ell+\delta_i(r)+\delta_i(T'-r)+1+\ell+m+k+1}} \\
&\equiv 0 \mod O_{g,n}(V)
\end{aligned}$$

by lemma 3.2.2.

Lastly, we show (3.3.4) holds. Again, take homogeneous $u \in V^{r*}$, $v \in V^{(T'-r)*}$ and $w \in V^{0*}$. We have

$$
\begin{aligned}
&(u \circ_{g,n} v) *_{g,n} w \\
&= \left(\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}}{z^{2\ell+\delta_i(r)+\delta_i(T'-r)+1}}\right) *_{g,n} w \\
&= \sum_{j=0}^{\infty}\binom{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}{j}(u_{j-2\ell-\delta_i(r)-\delta_i(T'-r)-1}v *_{g,n} w) \\
&= \sum_{m=0}^{\ell}(-1)^m\binom{m+\ell}{\ell}\mathrm{Res}_{z_2}\sum_{j=0}^{\infty}\binom{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}{j} \\
&\qquad \times Y(u_{j-2\ell-\delta_i(r)-\delta_i(T'-r)-1}v, z_2)w\frac{(1+z_2)^{\mathrm{wt}u+\mathrm{wt}v+2\ell+\delta_i(r)+\delta_i(T'-r)-j+\ell}}{z_2^{\ell+m+1}} \\
&= \sum_{m=0}^{\ell}(-1)^m\binom{m+\ell}{\ell}\mathrm{Res}_{z_2}\mathrm{Res}_{z_1-z_2}Y(Y(u,z_1-z_2)v,z_2)w \\
&\qquad \times \frac{(1+z_1)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}}{(z_1-z_2)^{2\ell+\delta_i(r)+\delta_i(T'-r)+1}}\frac{(1+z_2)^{\mathrm{wt}v+2\ell+\delta_i(T'-r)-\frac{r}{T'}+1}}{z_2^{\ell+m+1}} \\
&= \sum_{m=0}^{\ell}(-1)^m\binom{m+\ell}{\ell}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}Y(u,z_1)Y(v,z_2)w \\
&\qquad \times \frac{(1+z_1)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}}{(z_1-z_2)^{2\ell+\delta_i(r)+\delta_i(T'-r)+1}}\frac{(1+z_2)^{\mathrm{wt}v+2\ell+\delta_i(T'-r)-\frac{r}{T'}+1}}{z_2^{\ell+m+1}} \\
&\qquad - (-1)^{\tilde{u}\tilde{v}}\sum_{m=0}^{\ell}(-1)^m\binom{m+\ell}{\ell}\mathrm{Res}_{z_2}\mathrm{Res}_{z_1}Y(v,z_2)Y(u,z_1)w \\
&\qquad \times \frac{(1+z_1)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}}{(z_1-z_2)^{2\ell+\delta_i(r)+\delta_i(T'-r)+1}}\frac{(1+z_2)^{\mathrm{wt}v+2\ell+\delta_i(T'-r)-\frac{T'-r}{T'}}}{z_2^{\ell+m+1}} \\
&\equiv 0 \mod O_{g,n}(V)
\end{aligned}
$$

by lemma 3.2.2. □

Theorem 3.3.2. *The product $*_{g,n}$ induces the structure of an associative algebra on $A_{g,n}(V)$ with identity element $\mathbb{1}+O_{g,n}(V)$. Moreover, $\omega+O_{g,n}(V)$ is a central element of $A_{g,n}(V)$.*

Proof. According to lemma 3.3.1, to prove $*_{g,n}$ induces an associative algebra structure on $A_{g,n}(V)$ it is enough to show

$$(u *_{g,n} v) *_{g,n} w \equiv u *_{g,n} (v *_{g,n} w) \mod O_{g,n}(V)$$

for all homogeneous $u, v, w \in V^{0*}$. To that end we compute

$$\begin{aligned}
&(u *_{g,n} v) *_{g,n} w \\
&= \sum_{j=0}^{\infty} \sum_{m_1=0}^{\ell} (-1)^{m_1} \binom{m_1+\ell}{\ell} \binom{\mathrm{wt} u+\ell}{j} (u_{j-\ell-m_1-1} v) *_{g,n} w \\
&= \sum_{j=0}^{\infty} \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_1+m_2} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{\mathrm{wt} u+\ell}{j} \\
&\qquad \times Y(u_{j-\ell-m_1-1} v, z_2) w \frac{(1+z_2)^{\mathrm{wt} u+\mathrm{wt} v+m_1+\ell-j+\ell}}{z_2^{\ell+m_2+1}} \\
&= \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_1+m_2} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \\
&\qquad \times \mathrm{Res}_{z_2} \mathrm{Res}_{z_1-z_2} Y(Y(u, z_1-z_2) v, z_2) w \frac{(1+z_1)^{\mathrm{wt} u+\ell}}{(z_1-z_2)^{\ell+m_1+1}} \frac{(1+z_2)^{\mathrm{wt} v+\ell+m_1}}{z_2^{\ell+m_2+1}} \\
&= \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_1+m_2} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \\
&\qquad \times \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} Y(u, z_1) Y(v, z_2) w \frac{(1+z_1)^{\mathrm{wt} u+\ell}}{(z_1-z_2)^{\ell+m_1+1}} \frac{(1+z_2)^{\mathrm{wt} v+\ell+m_1}}{z_2^{\ell+m_2+1}} \\
&\qquad - (-1)^{\tilde{u}\tilde{v}} \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_1+m_2} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \\
&\qquad \times \mathrm{Res}_{z_2} \mathrm{Res}_{z_1} Y(v, z_2) Y(u, z_1) w \frac{(1+z_1)^{\mathrm{wt} u+\ell}}{(z_1-z_2)^{\ell+m_1+1}} \frac{(1+z_2)^{\mathrm{wt} v+\ell+m_1}}{z_2^{\ell+m_2+1}} \\
&= \sum_{j=0}^{\infty} \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_1+m_2+j} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \\
&\qquad \times \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} Y(u, z_1) Y(v, z_2) w \frac{(1+z_1)^{\mathrm{wt} u+\ell}}{z_1^{\ell+m_1+1+j}} \frac{(1+z_2)^{\mathrm{wt} v+\ell+m_1}}{z_2^{\ell+m_2+1-j}} \\
&\qquad - (-1)^{\tilde{u}\tilde{v}} \sum_{j=0}^{\infty} \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_2+\ell+j+1} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \\
&\qquad \times \mathrm{Res}_{z_2} \mathrm{Res}_{z_1} Y(v, z_2) Y(u, z_1) w \frac{(1+z_1)^{\mathrm{wt} u+\ell}}{z_1^{-j}} \frac{(1+z_2)^{\mathrm{wt} v+\ell+m_1}}{z_2^{2\ell+m_2+m_1+j+2}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_1+m_2+j} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \\
&\qquad \times \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} Y(u,z_1) Y(v,z_2) w \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m_1+1+j}} \frac{(1+z_2)^{\mathrm{wt}v+\ell+m_1}}{z_2^{\ell+m_2+1-j}} \\
&\qquad - (-1)^{\tilde{u}\tilde{v}} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_2+\ell+j+1} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \binom{\mathrm{wt}u+\ell}{k} \\
&\qquad \times \operatorname{Res}_{z_2} Y(v,z_2) u_{j+k} w \frac{(1+z_2)^{\mathrm{wt}v-1+\ell+\delta_i(0)+m_1}}{z_2^{2\ell+\delta_i(0)+\delta_i(T')+1+m_1+m_2+j+1}} \\
&\equiv \sum_{j=0}^{\infty} \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_1+m_2+j} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \\
&\qquad \times \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} Y(u,z_1) Y(v,z_2) w \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m_1+1+j}} \frac{(1+z_2)^{\mathrm{wt}v+\ell+m_1}}{z_2^{\ell+m_2+1-j}} \mod O_{g,n}(V)
\end{aligned}
$$

by lemma 3.2.2. Moreover, we have

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_1+m_2+j} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \\
&\qquad \times \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} Y(u,z_1) Y(v,z_2) w \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m_1+1+j}} \frac{(1+z_2)^{\mathrm{wt}v+\ell+m_1}}{z_2^{\ell+m_2+1-j}} \\
&= \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} \sum_{j=0}^{\ell-m_1} (-1)^{m_1+m_2+j} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \\
&\qquad \times \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} Y(u,z_1) Y(v,z_2) w \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m_1+1+j}} \frac{(1+z_2)^{\mathrm{wt}v+\ell+m_1}}{z_2^{\ell+m_2+1-j}} \\
&\qquad + \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} \sum_{j=\ell-m_1+1}^{\infty} (-1)^{m_1+m_2+j} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \\
&\qquad \times \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} Y(u,z_1) Y(v,z_2) w \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m_1+1+j}} \frac{(1+z_2)^{\mathrm{wt}v+\ell+m_1}}{z_2^{\ell+m_2+1-j}} \\
&\equiv \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} \sum_{j=0}^{\ell-m_1} (-1)^{m_1+m_2+j} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \\
&\qquad \times \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} Y(u,z_1) Y(v,z_2) w \frac{(1+z_1)^{\mathrm{wt}u+\ell}}{z_1^{\ell+m_1+1+j}} \frac{(1+z_2)^{\mathrm{wt}v+\ell+m_1}}{z_2^{\ell+m_2+1-j}} \mod O_{g,n}(V).
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
&(u *_{g,n} v) *_{g,n} w \\
&\equiv \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} \sum_{j=0}^{\ell-m_1} (-1)^{m_1+m_2+j} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \binom{-\ell-m_1-1}{j} \\
&\qquad \times \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} Y(u,z_1)Y(v,z_2)w \frac{(1+z_1)^{\mathrm{wt}\,u+\ell}}{z_1^{\ell+m_1+1+j}} \frac{(1+z_2)^{\mathrm{wt}\,v+\ell+m_1}}{z_2^{\ell+m_2+1-j}} \mod O_{g,n}(V) \\
&= u *_{g,n} (v *_{g,n} w) \\
&\qquad + \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{\ell} (-1)^{m_1+m_2} \binom{m_1+\ell}{\ell} \binom{m_2+\ell}{\ell} \\
&\qquad \times \operatorname{Res}_{z_1} \operatorname{Res}_{z_2} Y(u,z_1)Y(v,z_2)w \frac{(1+z_1)^{\mathrm{wt}\,u+\ell}}{z_1^{\ell+m_1+1}} \frac{(1+z_2)^{\mathrm{wt}\,v+\ell}}{z_2^{\ell+m_2+1}} \\
&\qquad \times \left(\sum_{j=0}^{\ell-m_1} \sum_{k=0}^{\infty} \binom{-\ell-m_1-1}{j} \binom{m_1}{k} \frac{z_2^{j+k}}{z_1^{j}} - 1 \right) \\
&= u *_{g,n} (v *_{g,n} w).
\end{aligned}
$$

This follows since

$$
\left(\sum_{j=0}^{\ell-m_1} \sum_{k=0}^{\infty} \binom{-\ell-m_1-1}{j} \binom{m_1}{k} \frac{z_2^{j+k}}{z_1^{j}} - 1 \right) = 0
$$

by proposition 5.3 of [9].

Finally, we note that $\mathbb{1} *_{g,n} u = u$ for any $u \in V$. Furthermore, according to lemma 3.2.3 *(iii)*

$$
\begin{aligned}
\omega *_{g,n} u - u *_{g,n} \omega &\equiv \operatorname{Res}_z Y(\omega,z)u(1+z) \mod O_{g,n}(V) \\
&= L(-1)u + L(0)u \\
&\equiv 0 \mod O_{g,n}(V),
\end{aligned}
$$

and also

$$\mathbb{1} *_{g,n} u - u *_{g,n} \mathbb{1} \equiv \mathrm{Res}_z Y(\mathbb{1}, z)u \mod O_{g,n}(V)$$
$$= 0.$$

□

Proposition 3.3.3. *The identity map on V induces an onto algebra homomorphism from $A_{g,n}(V)$ to $A_{g,n-\frac{1}{T'}}(V)$.*

Proof. Recall that $n = \ell + \frac{i}{T'}$ for nonnegative integers ℓ and i with $0 \le i \le T' - 1$. We consider $i \ge 1$ and $i = 0$ separately. In the case that $i \ge 1$ it follows from 3.2.2 that $u \circ_{g,n} v \in O_{g,n-\frac{1}{T'}}(V)$ for all $u, v \in V$. Hence, $O_{g,n}(V) \subseteq O_{g,n-\frac{1}{T'}}(V)$. Moreover, $u *_{g,n} v = u *_{g,n-\frac{1}{T'}} v$ and we are done.

If $i = 0$, then $n - \frac{1}{T'} = \ell - 1 + \frac{T'-1}{T'}$. Let $v \in V$ and homogeneous $u \in V^{r*}$. If $r = 0$, then $u \circ_{g,n} v \in O_{g,n-\frac{1}{T'}}(V)$ by 3.2.2. If $0 < r \le T' - 1$, then $u \circ_{g,n} v = u \circ_{g,n-\frac{1}{T'}} v$. In any case, $O_{g,n}(V) \subseteq O_{g,n-\frac{1}{T'}}(V)$.

To finish the proof we must show $u *_{g,n} v \equiv u *_{g,n-\frac{1}{T'}} v \mod O_{g,n-\frac{1}{T'}}(V)$ for any $v \in V$ and homogeneous $u \in V^{0*}$. Using lemma 3.2.2 we have

$$\begin{aligned}
&u *_{g,n} v \\
&= \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u+\ell}}{z^{\ell+m+1}} \\
&= \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{\ell+m+1}} \\
&\quad + \sum_{m=0}^{\ell} (-1)^m \binom{m+\ell}{\ell} \mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{\ell+m}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\ell-1}(-1)^m\binom{m+\ell}{\ell}\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{\ell+m}}\\
&\quad + (-1)^\ell\binom{2\ell}{\ell}\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell}}{z^{2\ell}}\\
&\quad + \sum_{m=0}^{\ell-2}(-1)^m\binom{m+\ell}{\ell}\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{\ell+m+1}}\\
&\quad + (-1)^\ell\binom{2\ell}{\ell}\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{2\ell+1}}\\
&\quad + (-1)^{\ell-1}\binom{2\ell-1}{\ell}\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{2\ell}}\\
&\equiv \sum_{m=0}^{\ell-1}(-1)^m\binom{m+\ell}{\ell}\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{\ell+m}}\\
&\quad + \sum_{m=0}^{\ell-2}(-1)^m\binom{m+\ell}{\ell}\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{\ell+m+1}} \quad \mathrm{mod}\ O_{g,n-\frac{1}{T'}}(V)\\
&= \mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{\ell}}\\
&\quad + \sum_{m=1}^{\ell-1}\left((-1)^m\binom{m+\ell}{\ell}+(-1)^{m-1}\binom{m-1+\ell}{\ell}\right)\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{\ell+m}}\\
&= \sum_{m=0}^{\ell-1}(-1)^m\binom{m+\ell-1}{\ell-1}\mathrm{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}u+\ell-1}}{z^{\ell+m}}\\
&= u *_{g,n-\frac{1}{T'}} v.
\end{aligned}
$$

□

3.4 The Lie Superalgebra $V[g]$

Let V be a vertex operator superalgebra and $g \in \mathrm{Aut}V$ of finite order T. In this section we review the construction and basic properties of the Lie superalgebra $V[g]$ as given in [15].

Following [1] the space $\mathbb{C}[t^{1/T}, t^{-1/T}]$ has the structure of vertex algebra with vertex operator

$$Y(f(t),z)g(t) = \left(e^{z\frac{d}{dt}}f(t)\right)g(t) = f(t+z)g(t).$$

So the tensor product

$$\mathcal{L}(V) = \mathbb{C}[t^{1/T}, t^{-1/T}] \otimes V$$

is a vertex superalgebra with obvious $\mathbb{Z}_2$-grading and vertex operator

$$Y(f(t)\otimes u, z)(g(t)\otimes v) = Y(f(t),z)g(t) \otimes Y(u,z)v.$$

We can extend g to an automorphism of $\mathcal{L}(V)$ upon declaring that for all $m \in \frac{1}{T}\mathbb{Z}$ and $v \in V$

$$g(t^m \otimes v) = e^{2\pi i m}(t^m \otimes gv).$$

Denote the g-invariants of $\mathcal{L}(V)$ by $\mathcal{L}(V,g)$. Then $\mathcal{L}(V,g)$ is a vertex operator subsuperalgebra of $\mathcal{L}(V)$. Moreover, it is not hard to see that

$$\mathcal{L}(V,g) = \bigoplus_{r=0}^{T-1} t^{r/T}\mathbb{C}[t,t^{-1}] \otimes V^r. \tag{3.4.1}$$

With $D = \frac{d}{dt}\otimes 1 + 1\otimes L(-1)$, we see that $D\mathcal{L}(V,g) \subseteq \mathcal{L}(V,g)$. Again, following [1] the space

$$V[g] = \mathcal{L}(V,g)/D\mathcal{L}(V,g)$$

has the structure of a Lie superalgebra with bracket given by

$$[a + D\mathcal{L}(V,g), b + D\mathcal{L}(V,g)] = a_0 b + D\mathcal{L}(V,g)$$

for all $a, b \in \mathcal{L}(V,g)$.

For simplicity, denote by $v(m)$ the image of $t^m \otimes v$ in $V[g]$. From the definitions and (2.3.13) one easily computes

$$[u(m), v(k)] = \sum_{j=0}^{\infty} \binom{m}{j} u_j v(m+k-i). \tag{3.4.2}$$

The following is an immediate consequence of 3.4.2.

Lemma 3.4.1. *Let $u \in V^r$, $v \in V^s$ and $k, m \in \mathbb{Z}$. Then*

(i) $\left[\omega(0), u\left(m + \frac{r}{T}\right)\right] = -\left(m + \frac{r}{T}\right) u\left(m - 1 + \frac{r}{T}\right)$

(ii) $\left[u\left(m + \frac{r}{T}\right), v\left(k + \frac{s}{T}\right)\right] = \sum_{j=0}^{\infty} \binom{m+\frac{r}{T}}{j} u_i v\left(m + k + \frac{r+s}{T} - j\right)$

(iii) If $m \neq -1$, then $\mathbb{1}(m) = 0$ and $\mathbb{1}(-1)$ lies in the center of $V[g]$.

Now, according to proposition 2.3.6 we can introduce a $\frac{1}{T'}\mathbb{Z}$-grading on $\mathcal{L}(V)$ by defining

$$\deg(t^m \otimes v) = \mathrm{wt} v - m - 1 \tag{3.4.3}$$

for homogeneous $v \in V$ and $m \in \frac{1}{T}\mathbb{Z}$. Since the map D increases degree by 1 in $\mathcal{L}(V)$ it follows that $D\mathcal{L}(V,g)$ is a graded subspace of $\mathcal{L}(V,g)$. Hence, $V[g]$ inherits the $\frac{1}{T'}\mathbb{Z}$ grading

$$V[g] = \bigoplus_{m \in \frac{1}{T'}\mathbb{Z}} V[g]_m.$$

By lemma 3.4.1 *(ii)*, $V[g]$ is a $\frac{1}{T'}\mathbb{Z}$-graded Lie superalgebra with triangular decomposition

$$V[g] = V[g]_+ \oplus V[g]_0 \oplus V[g]_-$$

where

$$V[g]_\pm = \bigoplus_{m \in \frac{1}{T'}\mathbb{N}} V[g]_{\pm m}.$$

In particular $V[g]_0$ is a subalgebra of $V[g]$.

Lemma 3.4.2. *$V[g]_0$ is spanned by elements of the form $v(\mathrm{wt} v - 1)$ for homogeneous $v \in V^{0*}$.*

Proof. $V[g]_0$ is spanned by elements of the form $v(m)$ for homogeneous $v \in V^r$ and $m \in \frac{1}{T}\mathbb{Z}$ with $m \equiv r \mod T$ satisfying $\mathrm{wt} v - m - 1 = 0$. In the case that $v \in V_{\bar{0}}$, we must have $r \equiv T \mod \mathbb{Z}$. On the other hand, if $v \in V_{\bar{1}}$, then $r \equiv T/2 \mod \mathbb{Z}$. In either case, $g\sigma u = u$ and we are done. □

Consequently the bracket in $V[g]_0$ is given by

$$[u(\mathrm{wt} u - 1), v(\mathrm{wt} v - 1)] = \sum_{j=0}^{\infty} \binom{\mathrm{wt} u - 1}{j} u_j v(\mathrm{wt}(u_j v) - 1).$$

Set $o(v) = v(\mathrm{wt} v - 1)$ for all homogeneous $v \in V^{0*}$ and extend linearly to all of V^*. This gives the surjective linear map

$$\begin{aligned} o : V^{0*} &\to V[g]_0 \\ v &\mapsto v(\mathrm{wt} v - 1). \end{aligned}$$

We see that the kernel of this map is $(L(-1) + L(0))V^{0*}$.

Now, let us impose a Lie superalgebra structure on the space

$$V^{0*}/(L(-1)+L(0))V^{0*} \tag{3.4.4}$$

with bracket given by

$$[u,v] = \sum_{j=0}^{\infty} \binom{\mathrm{wt} u - 1}{j}. \tag{3.4.5}$$

Lemma 3.4.3. *Let $A_{g,n}(V)_{Lie}$ be the Lie superlagebra associated to the associative algebra $A_{g,n}(V)$. Then the map $o(v) \mapsto v + O_{g,n}(V)$ is a surjective Lie superalgebra homomorphism from $V[g]_0$ to $A_{g,n}(V)_{Lie}$.*

Proof. By construction the map $o(u) \mapsto u$ is a Lie superalgebra isomorphism from $V[g]_0$ to $V^{0*}/(L(-1)+L(0))V^{0*}$. According to lemma 3.2.1 and the fact that $(L(-1)+L(0))V^{0*} \subseteq O_{g,n}(V)$ we also have the well-defined surjective linear map $u \mapsto u + O_{g,n}(V)$ from $V^{0*}/(L(-1)+L(0))V^{0*}$ to $A_{g,n}(V)_{Lie}$. Using lemma 3.2.3 *(ii)* we compute

$$\begin{aligned}
&u *_{g,n} v - (-1)^{\tilde{u}\tilde{v}} v *_{g,n} u \\
&\equiv \mathrm{Res}_z Y(u,z) v (1+z)^{\mathrm{wt} u - 1} \mod O_{g,n}(V) \\
&= \sum_{j=0}^{\infty} \binom{\mathrm{wt} u - 1}{j} u_j v
\end{aligned}$$

for all $u, v \in V^{0*}$. □

3.5 The Functor Ω_n

In this section we take the first step in establishing the equivalence between the category admissible g-twisted V-modules and the category of $A_{g,n}(V)$-modules which have no submodule which factors through $A_{g,n-\frac{1}{T'}}(V)$. We construct a covariant functor Ω_n from the category of weak g-twisted V-modules to the category of $A_{g,n}(V)$-modules. As noted in [15], we easily obtain the following upon comparing 2.3.11 and 2.3.16 with 3.4.2 and 3.4.3.

Proposition 3.5.1. *Any weak g-twisted V-module is a module for $V[g]$ under the map $v(m) \mapsto v_m$. Moreover, a weak g-twisted V-module M which carries a $\frac{1}{T'}\mathbb{Z}_+$-grading is an admissible g-twisted V-module if, and only if, M is a $\frac{1}{T'}\mathbb{Z}_+$-graded module for the Lie superalgebra $V[g]$.*

For a $V[g]$-module W and $m \in \frac{1}{T'}\mathbb{Z}_+$ define the space of *"m-th lowest weight vectors"* to be

$$\Omega_m(W) = \{w \in W : V[g]_{-k}w = 0, k > m\}.$$

Given a $V[g]$-module M, homogeneous $v \in V$ and $a \in \frac{1}{T'}\mathbb{Z}$, we set

$$o_a(v) = v(\mathrm{wt}v - a - 1) \tag{3.5.1}$$

on M and extend linearly to all V. Note that $o_0(u) = o(u)$ for all $u \in V$.

The following lemma (see [8] lemma 3.2 or [9] remark 3.3) is key in constructing admissible g-twisted V-modules from $A_n(V)$-modules in section 3.7. As usual, for a $V[g]$-module M and $v \in V$ we aggregate the operators $v(m)$ for $m \in \frac{1}{T}\mathbb{Z}$ on M together in the generating function

$$Y_M(v,z) = \sum_{m \in \frac{1}{T}\mathbb{Z}} v(m) z^{-m-1}. \tag{3.5.2}$$

Lemma 3.5.2. *Let M be a $V[g]$-module and $u \in V^{r*}$ and $v \in V^{s*}$ be homogeneous. Let $a = A - \frac{r}{T'}$ and $b = B - \frac{s}{T'}$ where $A, B \in \mathbb{Z}$, $a \geq b \geq -n$ and $a + b \geq 0$. Then for any integer $K \geq \ell + B$ the element $w_{u,v}^{a,b} \in V^{(r+s)*}$ defined by*

$$w_{u,v}^{a,b} = \sum_{m=0}^{K} (-1)^m \binom{A + \delta_i(r) + \ell + m - 1}{m} Res_z Y(u, z) v \frac{(1+z)^{wtu - 1 + \delta_i(r) + \ell + \frac{r}{T'}}}{z^{A + \delta_i(r) + \ell + m}}$$

satisfies

$$o_a(u) o_b(v) = o_{a+b}(w_{u,v}^{a,b})$$

on $\Omega_n(M)$.

Proof. As $u \in V^{r*}$ and $w \in \Omega_n(M)$ we have $z^{\mathrm{wt}u - 1 + \delta_i(r) + \ell + \frac{r}{T'}} Y_M(u, z) w$ contains only non-negative integral powers of z. So we get

$$\begin{aligned} 0 = \mathrm{Res}_{z_0} \mathrm{Res}_{z_1} z_1^{\mathrm{wt}u - 1 + \delta_i(r) + \ell + \frac{r}{T'}} z_0^{-1} \delta\left(\frac{-z_2 + z_1}{z_0}\right) \\ \times Y_M(v, z_2) Y_W(u, z_1) w. \end{aligned} \tag{3.5.3}$$

Since M is a $V[g]$-module, from lemma (3.4.1)*(ii)* we know

$$\begin{aligned} &[Y_M(u, z_1), Y_M(v, z_2)] \\ &= \mathrm{Res}_{z_0} z_2^{-1} \left(\frac{z_1 - z_0}{z_2}\right)^{-\mathrm{wt}u - \frac{r}{T'}} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0) v, z_2). \end{aligned} \tag{3.5.4}$$

Combining (3.5.3) and (3.5.4), we compute

$$
\begin{aligned}
&\mathrm{Res}_{z_0}(z_0+z_2)^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}Y_M(u,z_0+z_2)Y_M(v,z_2)w\\
&=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}(z_0+z_2)^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right)\\
&\qquad\times Y_M(u,z_0+z_2)Y_W(v,z_2)w\\
&=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}z_1^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right)\\
&\qquad\times Y_M(u,z_1)Y_M(v,z_2)w\\
&=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}z_1^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\\
&\qquad\times Y_M(u,z_1)Y_M(v,z_2)w\\
&\qquad-\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}z_1^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\\
&\qquad\times Y_M(v,z_2)Y_W(u,z_1)w\\
&=\mathrm{Res}_{z_1}z_1^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}[Y_M(u,z_1),Y_M(v,z_2)]w\\
&=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}z_1^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-\mathrm{wt}\,u-\frac{r}{T'}}\delta\left(\frac{z_1-z_0}{z_2}\right)\\
&\quad\times Y_M(Y(u,z_0)v,z_2)w\\
&=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}z_1^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-\mathrm{wt}\,u-\frac{r}{T'}}\delta\left(\frac{z_1-z_0}{z_2}\right)\\
&\qquad\times Y_M(Y(u,z_0)v,z_2)w\\
&=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}z_1^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}z_1^{-1}\left(\frac{z_2+z_0}{z_1}\right)^{\mathrm{wt}\,u+\frac{r}{T'}}\delta\left(\frac{z_2+z_0}{z_1}\right)\\
&\qquad\times Y_M(Y(u,z_0)v,z_2)w\\
&=\mathrm{Res}_{z_0}(z_2+z_0)^{\mathrm{wt}\,u-1+\delta_i(r)+\ell+\frac{r}{T'}}Y_W(Y(u,z_0)v,z_2)w.
\end{aligned}
$$

That is, we have

$$\begin{aligned}&(z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}Y_M(u,z_0+z_2)Y_M(v,z_2)w\\&=(z_2+z_0)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}Y_M(Y(u,z_0)v,z_2)w.\end{aligned}$$

Now, the same argument used in proposition 2.3.13 gives us that

$$\begin{aligned}&u(\mathrm{wt}u-a-1)v(\mathrm{wt}v-b-1)w\\&=\mathrm{Res}_{z_0}\mathrm{Res}_{z_2}(z_0+z_2)^{\mathrm{wt}u-a-1}z_2^{\mathrm{wt}v-b-1}Y_M(u,z_0+z_2)Y_M(v,z_2)w.\end{aligned}$$

Then for any integer K satisfying $K\geq\ell+B$ we compute

$$\begin{aligned}&o_a(u)o_b(v)w\\&=u(\mathrm{wt}u-a-1)v(\mathrm{wt}v-b-1)w\\&=\mathrm{Res}_{z_0}\mathrm{Res}_{z_2}(z_0+z_2)^{\mathrm{wt}u-a-1}z_2^{\mathrm{wt}v-b-1}Y_M(u,z_0+z_2)Y_M(v,z_2)w\\&=\mathrm{Res}_{z_0}\mathrm{Res}_{z_2}(z_0+z_2)^{-a-\delta_i(r)-\ell-\frac{r}{T'}}z_2^{\mathrm{wt}v-b-1}\\&\qquad\times\left((z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}Y_M(u,z_0+z_2)Y_M(v,z_2)w\right)\\&=\sum_{m=0}^{K}(-1)^m\binom{a+\delta_i(r)+\ell+\frac{r}{T'}+m-1}{m}\mathrm{Res}_{z_0}\mathrm{Res}_{z_2}(z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}\\&\qquad\times Y_M(u,z_0+z_2)Y_M(v,z_2)w\frac{z_2^{m+\mathrm{wt}v-b-1}}{z_0^{a+\delta_i(r)+\ell+\frac{r}{T'}+m}}\\&=\sum_{m=0}^{K}(-1)^m\binom{a+\delta_i(r)+\ell+\frac{r}{T'}+m-1}{m}\mathrm{Res}_{z_0}\mathrm{Res}_{z_2}(z_2+z_0)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}\\&\qquad\times Y_M(Y(u,z_0)v,z_2)w\frac{z_2^{m+\mathrm{wt}v-b-1}}{z_0^{a+\delta_i(r)+\ell+\frac{r}{T'}+m}}\\&=\sum_{m=0}^{K}\sum_{j=0}^{\infty}(-1)^m\binom{a+\delta_i(r)+\ell+\frac{r}{T'}+m-1}{m}\binom{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}{j}\\&\qquad\times(u_{j-a-\delta_i(r)-\ell-\frac{r}{T'}-m}v)_{\mathrm{wt}u+\mathrm{wt}v-j+\delta_i(r)+\ell+\frac{r}{T'}+m-b-2}w\end{aligned}$$

$$= \sum_{m=0}^{K} (-1)^m \binom{a+\delta_i(r)+\ell+\frac{r}{T'}+m-1}{m}$$
$$\times\, o_{a+b}\left(\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}}{z^{a+\delta_i(r)+\ell+\frac{r}{T'}+m}}\right) w$$

and we are done. □

Theorem 3.5.3. *Let M be a weak g-twisted V-module. The map $u + O_{g,n}(V) \mapsto o(u)$ gives a representation of the associative algebra $A_{g,n}(V)$ on $\Omega_n(M)$.*

Proof. To show that $A_{g,n}(V)\Omega_n(M) \subseteq \Omega_n(M)$, take $w \in \Omega_n(M)$ and homogeneous $u, v \in V$ with $\deg(v(k)) \le -n$. Using (2.3.13) we have

$$v_k o(u)w = (-1)^{\tilde{u}\tilde{v}} o(u)v(k)w + \sum_{j=0}^{\infty} \binom{k}{j} (v_j u)(\mathrm{wt}u + k - j - 1)w.$$

As $\deg(v(k)) < -n$ we know the first term on the right is zero. Similarily, since $\deg(v_j u(\mathrm{wt}u + k - j - 1)) = \deg(v(k)) < -n$ we have $(v_j u)_{\mathrm{wt}u+k-j-1} w = 0$ for all $j \ge 0$.

Now we need to show:

$$o(u) = 0 \quad \text{on } \Omega_n(M) \text{ for any } u \in O_{g,n}(V), \tag{3.5.5}$$

$$o(u *_{g,n} v) = o(u)o(v) \quad \text{for any } u, v \in V. \tag{3.5.6}$$

We prove (3.5.5) and (3.5.6)simultaneously. By lemma 3.4.2, $V^{r*} \subseteq O_{g,n}(V)$ if $0 < r < T'$. Furthermore, for homogeneous $u \in V$, $\mathrm{wt}L(-1)u = \mathrm{wt}u + 1$. So applying the $L(-1)$-derivative property for the module M we find that

$$o(L(-1)u) = (L(-1)u)\,\mathrm{wt}u = -(\mathrm{wt}u)u_{\mathrm{wt}u-1}.$$

Hence, $o(L(-1)u + L(0)u) = 0$ for any $u \in V$. From the proof of lemma 3.3.1 we know that

$$(L(-1)u + L(0)u) *_{g,n} v = (-1)^{\ell}(2\ell+1)\binom{2\ell+1}{\ell} u \circ_{g,n} v$$

for all $u, v \in V^{0*}$. So, to finish the proof it suffices to show that $o(u)o(v) = o(u *_{g,n} v)$ for any $u, v \in V^{0*}$ and $o(u \circ_{g,n} v) = 0$ on $\Omega_m(M)$ for $u \in V^{r*}$ and $v \in V^{(T'-r)*}$.

For $u, v \in V^{0*}$, we can apply lemma 3.5.2 with $a = b = 0$ to find $o(u)o(v) = o(u *_{g,n} v)$ on $\Omega_n(M)$. On the other hand, for $u \in V^{r*}$ and $v \in V^{(T'-r)*}$ we apply the identity 2.3.12 with

$$\begin{aligned} t &= -2\ell - \delta_i(r) - \delta_i(T'-r) - 1, \\ p &= \mathrm{wt}u - 1 + \ell + \delta_i(r) + r/T,' \\ q &= \mathrm{wt}v - 1 + \ell + \delta_i(T'-r) + (T'-r)/T'. \end{aligned}$$

It is easily checked that the left hand side acts as zero on $\Omega_n(M)$. So we are left with

$$\begin{aligned} 0 &= \sum_{j=0}^{\infty} \binom{\mathrm{wt}u - 1 + \ell + \delta_i(r) + r/T'}{j} \\ &\quad \times \left(u_{j-2\ell-\delta_i(r)-\delta_i(T'-r)-1} v\right)_{\mathrm{wt}u + \mathrm{wt}v + 2\ell + \delta_i(r) + \delta_i(T'-r) - 1} \\ &= o\left(\mathrm{Res}_z Y(u,z) v \frac{(1+z)^{\mathrm{wt}u - 1 + \ell + \delta_i(r) + \frac{r}{T'}}}{z^{2\ell + \delta_i(r) + \delta_i(T'-r) + 1}}\right) \\ &= o(u \circ_{g,n} v) \end{aligned}$$

on $\Omega_n(M)$ and we are done. □

Note that if M and W are weak g-twisted V-modules and $f : M \to W$ is a V-module homomorphism, then f maps $\Omega_n(M)$ to $\Omega_n(W)$ since f commutes with all component operators v_k for $v \in V$ and $k \in \frac{1}{T}\mathbb{Z}$. In this way we obtain an $A_{g,n}(V)$-module map $\Omega_n(f) : \Omega_n(M) \to \Omega_n(W)$. Combining this fact with theorem 3.5.3 shows that Ω_n defines a covariant functor from the category of weak g-twisted V-modules to the category of $A_{g,n}(V)$-modules.

A grading shift applied to any admissible g-twisted V-module M gives an isomorphic admissible g-twisted module. If $M = 0$, then obviously $M(0) = 0$. However, if $M \neq 0$, then some $M(m) \neq 0$ and we can shift the grading so that $M(0) \neq 0$. Under this convention, we have the following.

Proposition 3.5.4. *If $M = \bigoplus_{m \in \frac{1}{T'}\mathbb{Z}_+} M(m)$ is an admissible g-twisted V-module, then the following hold.*

(i) $\Omega_n(M) \supseteq \bigoplus_{\substack{m \in \frac{1}{T'}\mathbb{Z}_+ \\ m \le n}} M(m)$. *If M is an irreducible admissible g-twisted V-module, then $\Omega_n(M) = \bigoplus_{\substack{m \in \frac{1}{T'}\mathbb{Z}_+ \\ m \le n}} M(m)$.*

(ii) Each $M(p)$ is a $V[g]_0$-submodule of $\Omega_n(M)$ for $0 \le p \le n$. If M is an irreducible admissible g-twisted V-module, then each $M(p)$ is an irreducible $V[g]_0$-module and $M(p)$ and $M(q)$ are inequivalent if $p \neq q$.

(iii) Each $M(p)$ is an $A_{g,n}(V)$-submodule of $\Omega_n(M)$ for $0 \le p \le n$. If M is an irreducible admissible g-twisted V-module, then each $M(p)$ is an irreducible $A_{g,n}(V)$-module and $M(p)$ and $M(q)$ are inequivalent if $p \neq q$.

Proof. We first note that $\Omega_n(M)$ is $\frac{1}{T'}\mathbb{Z}_+$-graded

$$\Omega_n(M) = \bigoplus_{m \in \frac{1}{T'}\mathbb{Z}_+} \Omega_n(M)(m) \tag{3.5.7}$$

where $\Omega_n(M)(m) = \Omega_n(M) \cap M(m)$. To see this, assume $w = w_1 + \cdots + w_k \in \Omega_n(M)$ with each w_j having distinct degree $m_j \geq 0$. Let $u \in V$ be homogeneous and $m \in \mathbb{Z}$ such that $\deg u(m) < 0$. So $u(m)w = 0$ and we have $u(m)w_1 + \cdots + u(m)w_k = 0$. Then since each $u(m)w_j$ has distinct degree $\mathrm{wt}\, u + m_j - m - 1$, we must have that $u(m)w_j = 0$ for all $j = 1, \ldots, k$. Consequently, each $w_j \in \Omega(M)$ and we have the grading in (3.5.7).

Now for *(i)*, since $V[g]_{-k}$ acts on $M(m)$ for all $k > n$ and $m \leq n$ it follows that $M(m) \subseteq \Omega_n(M)$ for all $m \leq n$. Hence $\Omega_n(M) \supseteq \bigoplus_{\substack{m \in \frac{1}{T'}\mathbb{Z}_+ \\ m \leq n}} M(m)$. On the other hand, assuming M is irreducible, by proposition 2.3.13 for nonzero $w \in M$ we have

$$M = \mathrm{span}_{\mathbb{C}} \left\{ u_m w : u \in V, m \in \frac{1}{T}\mathbb{Z} \right\}.$$

Assume by way of contradiction that $\Omega_n(M)(m) \neq 0$ for some $m > n$. Take nonzero $w \in \Omega_n(M)(m)$. Since $w \in \Omega_n(M)$ we see that

$$M = \mathrm{span}_{\mathbb{C}} \left\{ u_{\mathrm{wt}u+k-1} w : u \in V,\ k \leq n \right\}.$$

However it is also true that

$$M(0) = \mathrm{span}_{\mathbb{C}} \left\{ u_{\mathrm{wt}u+k-1} w : u \in V,\ k = m \right\}.$$

So $M(0) = 0$, a contradiction. This proves *(i)*.

(iii) follows from *(ii)*. To prove *(ii)* first we note that $V[g]_0 M(p) \subseteq M(p)$ so that each $M(p)$ is a $V[g]_0$-submodule of $\Omega_n(M)$. As before, if M is an irreducible admissible g-twisted V-module, then for any $0 \leq p \leq n$ and nonzero $w \in M(p)$

$$M(p) = \mathrm{span}_{\mathbb{C}} \left\{ u_{\mathrm{wt}u-1} w : u \in V \right\}.$$

Hence, each $M(p)$ is an irreducible $V[g]_0$-module. From proposition 2.3.14 we know that $L(0)$ acts semisimply on M and for any $m \in \frac{1}{T'}\mathbb{Z}_+$

$$M(m) = \{w \in M : L(0)w = (h+m)w\},$$

where h is the conformal weight of M. (See proposition 2.3.14.) So if $p \neq q$ then $L(0) = \omega_{\mathrm{wt}\omega-1}$ must have different eigenvalues on $M(p)$ and $M(q)$. In this case they cannot be isomorphic and the proof is complete. □

3.6 Preliminaries on $V[g]$-modules

Here we gather a few results on $V[g]$-modules which provide the crux of the arguments needed to prove theorems 3.7.6 and 3.7.8, the backbone to the results of section 3.7.

The following is proposition 6.1 of [15]. We provide the proof here since we will need the argument again in propositions 3.6.2 and 3.6.3. We follow the notation established in (3.5.2).

Proposition 3.6.1. *Assume that M is a $V[g]$-module and U is a subspace of M satisfying the following conditions:*

(i) $M = \mathcal{U}(V[g])U$.

(ii) For any $u \in V^{r*}$ and $w \in U$, there exists $k \in wtu + \mathbb{Z}_+$ such that

$$(z_0+z_2)^{k+\frac{r}{T'}} Y_M(u, z_0+z_2) Y_M(v, z_2)w$$
$$= (z_2+z_0)^{k+\frac{r}{T'}} Y_M(Y(u,z_0)v, z_2)w.$$

for any $v \in V$.

Then M is a weak g-twisted V-module generated by U with twisted vertex operator $Y_M(v,z) = \sum_{m\in\frac{1}{T}\mathbb{Z}} v(m)z^{-m-1}$ for all $v \in V$.

Proof. It is easily verified that $Y_M(v,z)$ satisfies conditions (2.3.5)-(2.3.7). By lemma 3.4.1 *(ii)*, the identity (2.3.11) holds. Following remark 2.3.5, to establish the twisted Jacobi identity for the action of V on M it is sufficient to establish the twisted associativity 2.3.10.

Now using lemma 2.3.6 we can shift from the $g\sigma$-eigenspaces of V to the g-eigenspaces. That is, assumption *(ii)* on M is equivalent to the following:

(iii) For any $u \in V^r$ and $w \in U$, there exists $k \in \mathbb{Z}_+$ such that

$$(z_0+z_2)^{k+\frac{r}{T}} Y_M(u, z_0+z_2) Y_M(v, z_2) w$$
$$= (z_2+z_0)^{k+\frac{r}{T}} Y_M(Y(u,z_0)v, z_2) w$$

for any $v \in V$.

By assumption *(i)* we know that M is generated by U as a $V[g]$-module. Hence, to verify that the twisted associativity holds on M, it suffices to show that for any $u \in V^r$, $x \in V^s$, $w \in U$ and $m \in \frac{1}{T}\mathbb{Z}$, there exists a $k \in \mathbb{Z}_+$ such that

$$(z_0+z_2)^{k+\frac{r}{T}} Y_M(u, z_0+z_2) Y_M(v, z_2) x(m) w$$
$$= (z_2+z_0)^{k+\frac{r}{T}} Y_M(Y(u,z_0)v, z_2) x(m) w.$$

for any $v \in V$.

Fix $u \in V^r$, $x \in V^s$, $w \in U$ and $m \in \frac{1}{T}\mathbb{Z}$. By the truncation condition (2.2.1) on V we can choose a positive integer K such that $x_j u = 0$ for all integers $j \geq K$. Since $w \in U$, by *(iii)* we can uniformly choose a positive integer N such that for any $v \in V$ and all integers $j \geq 0$

$$(z_0 + z_2)^{N+\frac{r+s}{T}} Y_M(x_j u, z_0 + z_2) Y_M(v, z_2) w$$
$$= (z_2 + z_0)^{N+\frac{r+s}{T}} Y_M(Y(x_j u, z_0)v, z_2) x w \tag{3.6.1}$$
$$(z_0 + z_2)^{N+\frac{r+s}{T}} Y_M(u, z_0 + z_2) Y_M(x_j v, z_2) w$$
$$= (z_2 + z_0)^{N+\frac{r+s}{T}} Y_M(Y(u, z_0) x_j v, z_2) w. \tag{3.6.2}$$

Now choose a positive integer k such that $k + \frac{r}{T} + m - K > N + \frac{r+s}{T}$.

The bracket relation on the $V[g]$-module M given in lemma 3.4.1 *(ii)* can be rewritten via generating functions as follows:

$$[u(t), Y_M(v, z)] = \sum_{j=0}^{\infty} \binom{t}{j} Y_M(u_j v, z) z^{t-j}. \tag{3.6.3}$$

Applying the commutator (3.6.3) along with (3.6.1) and (3.6.2) we compute

$$\begin{aligned}
&(z_0 + z_2)^{k+\frac{r}{T}} Y_M(u, z_0 + z_2) Y_M(v, z_2) x(m) w \\
&= (-1)^{\tilde{u}\tilde{x}} (-1)^{\tilde{v}\tilde{x}} (z_0 + z_2)^{k+\frac{r}{T}} x(m) Y_M(u, z_0 + z_2) Y_M(v, z_2) w \\
&\quad - (-1)^{\tilde{v}\tilde{x}} \sum_{j=0}^{\infty} \binom{m}{j} z_2^{m-j} (z_0 + z_2)^{k+\frac{r}{T}} Y_M(u, z_0 + z_2) Y_M(x_j v, z) w \\
&\quad - (-1)^{\tilde{u}\tilde{x}} (-1)^{\tilde{v}\tilde{x}} \sum_{j=0}^{\infty} \binom{m}{j} (z_0 + z_2)^{k+\frac{r}{T}+m-j} Y_M(x_j u, z_0 + z_2) Y_M(v, z_2) w \\
&= (-1)^{\tilde{u}\tilde{x}} (-1)^{\tilde{v}\tilde{x}} (z_0 + z_2)^{k+\frac{r}{T}} x(m) Y_M(Y(u, z_0)v, z_2) w \\
&\quad - (-1)^{\tilde{v}\tilde{x}} \sum_{j=0}^{\infty} \binom{m}{j} z_2^{m-j} (z_2 + z_0)^{k+\frac{r}{T}} Y_M(Y(u, z_0) x_j v, z_2) w \\
&\quad - (-1)^{\tilde{u}\tilde{x}} (-1)^{\tilde{v}\tilde{x}} \sum_{j=0}^{\infty} \binom{m}{j} (z_2 + z_0)^{k+\frac{r}{T}+m-j} Y_M(Y(x_j u, z_0)v, z_2) w
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}(z_0+z_2)^{k+\frac{r}{T}}x(m)Y_M(Y(u,z_0)v,z_2)w \\
&\quad - (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}\sum_{j=0}^{\infty}\binom{m}{j}z_2^{m-j}(z_2+z_0)^{k+\frac{r}{T}}Y_M(x_jY(u,z_0)v,z_2)w \\
&\quad - (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}\sum_{j=0}^{\infty}\binom{m}{j}(z_2+z_0)^{k+\frac{r}{T}+m-j}Y_M(Y(x_ju,z_0)v,z_2)w \\
&\quad + (-1)^{\tilde{v}\tilde{x}}(-1)^{\tilde{u}\tilde{x}}\sum_{j=0}^{\infty}\binom{m}{j}z_2^{m-j}(z_2+z_0)^{k+\frac{r}{T}}Y_M([x_j,Y(u,z_0)]v,z_2)w \\
&= (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}(z_0+z_2)^{k+\frac{r}{T}}x(m)Y_M(Y(u,z_0)v,z_2)w \\
&\quad - (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}\sum_{j=0}^{\infty}\binom{m}{j}z_2^{m-j}(z_2+z_0)^{k+\frac{r}{T}}Y_M(x_jY(u,z_0)v,z_2)w \\
&\quad - (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}\sum_{j=0}^{\infty}\binom{m}{j}(z_2+z_0)^{k+\frac{r}{T}+m-j}Y_M(Y(x_ju,z_0)b,z_2)w \\
&\quad + (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}\sum_{j=0}^{\infty}\sum_{t=0}^{\infty}\binom{m}{j}\binom{j}{t}z_0^{j-t}z_2^{m-j}(z_2+z_0)^{k+\frac{r}{T}}Y_M(Y(x_tu,z_0)v,z_2)w \\
&= (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}(z_0+z_2)^{k+\frac{r}{T}}x(m)Y_M(Y(u,z_0)v,z_2)w \\
&\quad - (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}\sum_{j=0}^{\infty}\binom{m}{j}z_2^{m-j}(z_2+z_0)^{k+\frac{r}{T}}Y_M(x_jY(u,z_0)v,z_2)w \\
&\quad - (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}\sum_{j=0}^{\infty}\binom{m}{j}(z_2+z_0)^{k+\frac{r}{T}+m-j}Y_M(Y(x_ju,z_0)b,z_2)w \\
&\quad + (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}\sum_{t=0}^{\infty}\sum_{j=t}^{\infty}\binom{m}{t}\binom{m-t}{j-t}z_0^{j-t}z_2^{m-j}(z_2+z_0)^{k+\frac{r}{T}}Y_M(Y(x_tu,z_0)v,z_2)w \\
&= (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}(z_0+z_2)^{k+\frac{r}{T}}x(m)Y_M(Y(u,z_0)v,z_2)w \\
&\quad - (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}\sum_{j=0}^{\infty}\binom{m}{j}z_2^{m-j}(z_2+z_0)^{k+\frac{r}{T}}Y_M(x_jY(u,z_0)v,z_2)w \\
&= (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}(z_0+z_2)^{k+\frac{r}{T}}x(m)Y_M(Y(u,z_0)v,z_2)w \\
&\quad - (-1)^{\tilde{u}\tilde{x}}(-1)^{\tilde{v}\tilde{x}}z_2^{m-j}(z_2+z_0)^{k+\frac{r}{T}}[x(m),Y_M(Y(u,z_0)v,z_2)]w \\
&= (z_0+z_2)^{k+\frac{r}{T}}Y_M(Y(u,z_0)v,z_2)x(m)w.
\end{aligned}
$$

$\square$

For a general $\frac{1}{T'}\mathbb{Z}_+$-graded $V[g]$-module

$$M = \bigoplus_{m \in \frac{1}{T'}\mathbb{Z}_+} M(m)$$

we extend $M(n)^* = \mathrm{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$ to all of M by declaring that $M(n)^*$ act on $\bigoplus_{m \neq n} M(m)$ as zero.

Using the argument of proposition 3.6.1 with the modification of $< -, - >$, we have the following.

Proposition 3.6.2. *Let M be a $\frac{1}{T'}\mathbb{Z}_+$-graded $V[g]$-module. Let U be a subspace of $M(n)$ and U' be a subspace of $M(n)^*$ such that*

(i) $M = \mathcal{U}(V[g])U$,

(ii) For $u \in V^{r*}$ and $w \in U$ there exists a $k \in wtu + \mathbb{Z}_+$ such that

$$\begin{aligned} &\left\langle f, (z_0 + z_2)^{k + \frac{r}{T'}} Y_M(u, z_0 + z_2) Y_M(v, z_2) w \right\rangle, \\ &= \left\langle f, (z_2 + z_0)^{k + \frac{r}{T'}} Y_M(Y(u, z_0)v, z_2) w \right\rangle \end{aligned} \tag{3.6.4}$$

for any $v \in V$ and $f \in U^*$.

Then condition (ii) holds for any $w \in M$.

Proposition 3.6.3. *Let M be $\frac{1}{T'}\mathbb{Z}_+$-graded $V[g]$-module. Let U be a subspace of $M(n)$ and U' be a subspace of $M(n)^*$ such that*

(i) $M = \mathcal{U}(V[g])U$,

(ii) For homogeneous $u \in V^{r*}$ and $w \in U$ there exists a $k \in wtu + \mathbb{Z}_+$ such that

$$\begin{aligned} &\left\langle f, (z_0 + z_2)^{k + \frac{r}{T'}} Y_M(u, z_0 + z_2) Y_M(v, z_2) w \right\rangle \\ &= \left\langle f, (z_2 + z_0)^{k + \frac{r}{T'}} Y_M(Y(u, z_0)v, z_2) w \right\rangle \end{aligned} \tag{3.6.5}$$

for any $v \in V$ *and* $f \in U^*$.

Then for any $x \in \mathcal{U}(V[g])$, *homogeneous* $u \in V^{r*}$ *and* $w \in M$, *there exists a* $k \in wtu + \mathbb{Z}_+$ *such that*

$$\begin{aligned}&\left\langle f, (z_0+z_2)^{k+\frac{r}{T'}} x Y_M(u, z_0+z_2) Y_M(v, z_2) w \right\rangle \\ &= \left\langle f, (z_2+z_0)^{k+\frac{r}{T'}} x Y_M(Y(u, z_0)v, z_2) w \right\rangle \end{aligned} \tag{3.6.6}$$

for any $v \in V$ *and* $f \in U^*$.

Proof. By lemma 2.3.6 we know that (3.6.6) is equivalent to the following: for $x \in \mathcal{U}(V[g])$, homogeneous $u \in V^r$ and $w \in M$, there exists a $k \in \mathbb{Z}_+$ such that

$$\begin{aligned}&\left\langle f, (z_0+z_2)^{k+\frac{r}{T}} x Y_M(u, z_0+z_2) Y_M(v, z_2) w \right\rangle \\ &= \left\langle f, (z_2+z_0)^{k+\frac{r}{T}} x Y_M(Y(u, z_0)v, z_2) w \right\rangle \end{aligned} \tag{3.6.7}$$

for any $v \in V$ and $f \in U^*$.

We prove that (3.6.7) holds. Let X be the subspace of $\mathcal{U}(V[g])$ consisting of those x for which (3.6.7) holds. Fix $x \in X$ and homogeneous $y \in V^s$. Since M is a $V[g]$-module, it follows from lemma 3.4.1 *(ii)* that

$$[u(m), Y_M(v, z)] = \sum_{j=0}^{\infty} \binom{m}{j} z^{m-j} Y_M(u_j v, z).$$

With this we compute

$$\begin{aligned}
&\left\langle f,(z_0+z_2)^{k+\frac{r}{T}}xy(m)Y_M(u,z_0+z_2)Y_M(v,z_2)w\right\rangle\\
&=\sum_{j=0}^{\infty}\binom{m}{j}\left\langle f,(z_0+z_2)^{k+m-j+\frac{r}{T}}xY_M(y_ju,z_0+z_2)Y_M(v,z_2)w\right\rangle\\
&\qquad+(-1)^{\tilde{c}\tilde{u}}\sum_{j=0}^{\infty}\binom{m}{j}\left\langle f,z_2^{m-j}(z_0+z_2)^{k+\frac{r}{T}}xY_M(u,z_0+z_2)Y_M(y_jv,z_2)w\right\rangle\\
&\qquad+(-1)^{\tilde{c}\tilde{u}}(-1)^{\tilde{c}\tilde{v}}\left\langle f,(z_0+z_2)^{k+\frac{r}{T}}xY_M(u,z_0+z_2)Y_M(v,z_2)y(m)w\right\rangle
\end{aligned}$$

Now by choosing $k\in\mathbb{Z}_+$ appropriately, the same technique used in the proof of proposition 6.1 of [15] shows that $xy(m)\in X$. Since $\mathcal{U}(V[g])$ is generated by 1 and all such $y(m)$'s, and since 3.6.7 holds for $x=1$ by proposition 3.6.2, we conclude that $X=\mathcal{U}(V[g])$. □

3.7 The Functor L_n

In section 3.5 we saw how to obtain an $A_{g,n}(V)$-module from an admissible g-twisted V-module. Conversely, in this section we show that there is a universal way to construct an admissible g-twisted V-module $\overline{M}_n(U)$ from an $A_{g,n}(V)$-module U. A distinct quotient $L_n(U)$ of $\overline{M}_n(U)$ is an admissible g-twisted V-module and in this way we obtain a functor L_n from the category of $A_{g,n}(V)$-modules to the category of weak g-twisted V-modules.

Upon restriction, L_n gives a functor from the category of $A_{g,n}(V)$-modules which have no submodule that factors through $A_{g,n-\frac{1}{T'}}(V)$ to the category of admissible g-twisted V-modules which is right inverse to the functor $\Omega_n/\Omega_{n-\frac{1}{T'}}$. Here $\Omega_n/\Omega_{n-\frac{1}{T'}}$ is the quotient functor $M\to\Omega_n(M)/\Omega_{n-\frac{1}{T'}}(M)$.

Remark 3.7.1. *We note here that in [8] (see also [9]) there is a slight error. For the functor L_n to be right inverse to the functor $\Omega_n/\Omega_{n-\frac{1}{T'}}$ one must restrict to* *the category of $A_{g,n}(V)$-modules which have no **submodule** that factors through $A_{g,n-\frac{1}{T'}}(V)$.*

To begin let us construct the admissible g-twisted V-module $M_n(U)$ associated to the $A_{g,n}(V)$-module U. In the usual way U is an $A_{g,n}(V)_{Lie}$-module. By lemma 3.4.3, we have that U is a $V[g]_0$-module. We lift U to a module for the $V[g]$-subalgebra $P_n = \bigoplus_{p>n} V[g]_{-p} \oplus V[g]_0$ upon declaring that $V[g]_{-p}$ act as zero. Subsequently form the induced $V[g]$-module

$$M_n(U) = \mathcal{U}\left(V[g]\right) \otimes_{\mathcal{U}(P_n)} U.$$

The $\frac{1}{T'}\mathbb{Z}$-graded Lie superalgebra structure on $V[g]$ induces a $\frac{1}{T'}\mathbb{Z}$-graded associative superalgebra structure on the universal enveloping algebra $\mathcal{U}\left(V[g]\right)$. If we declare that U has degree n, then $M_n(U)$ becomes a $\frac{1}{T'}\mathbb{Z}$-graded $V[g]$-module

$$M_n(U) = \bigoplus_{m \in \frac{1}{T'}\mathbb{Z}} M_n(U)(m).$$

By the Poincaré-Birkhoff-Witt theorem we have the linear isomorphism $M_n(U)(m) \cong \mathcal{U}\left(V[g]\right)_{m-n} U$.

For $v \in V$ we have the generating function

$$Y_{M_n(U)}(v,z) = \sum_{m \in \frac{1}{T}\mathbb{Z}} v(m) z^{-m-1} \tag{3.7.1}$$

and hence a linear function from V to $(\mathrm{End}M)[[z^{\frac{1}{T}}, z^{-\frac{1}{T}}]]$. It is easily verified that $Y_{M_n(U)}(v,z)$ satisfies conditions (2.3.5)-(2.3.7). By lemma 3.4.1 *(ii)*, the identity (2.3.11) holds. Following remark 2.3.5, to establish the twisted Jacobi identity for the action of V on $M_n(U)$ it is sufficient to establish the associativity 2.3.10. To do this, one takes a quotient of $M_n(U)$ by the space spanned by the appropriate relations.

Define $W_n(U)$ be the $V[g]$-submodule of $M_n(U)$ generated by the coefficients of

$$(z_0 + z_2)^{\mathrm{wt} u - 1 + \delta_i(r) + \ell + \frac{r}{T'}} Y_{M_n(U)}(u, z_0 + z_2) Y_{M_n(U)}(v, z_2) w$$
$$= (z_2 + z_0)^{\mathrm{wt} u - 1 + \delta_i(r) + \ell + \frac{r}{T'}} Y_{M_n(U)}(Y(u, z_0)v, z_2) w \tag{3.7.2}$$

for homogeneous $u \in V^{r*}$, $v \in V$ and $w \in U$. Let $\overline{M}_n(U)$ be the quotient

$$\overline{M}_n(U) = M_n(U)/W_n(U).$$

One can easily check that $W_n(U)$ is a graded subspace of $M_n(U)$ since the coefficients on each side of 3.7.2 are homogeneous. Hence the quotient $\overline{M}_n(U)$ inherits the $\frac{1}{T'}\mathbb{Z}_+$-grading.

Remark 3.7.2. *Note that proposition 3.6.1 shows that $\overline{M}_n(U)$ is in fact a weak g-twisted V-module.*

Now we use lemma 3.5.2 to extend the natural pairing of the $A_{g,n}(V)$-module U with its dual $U^* = \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})$ to all of $M_n(U)$.

For $s \in \mathbb{Z}_+$ define U_s to be the subspace of $M_n(U)$ spanned by *"length s" vectors*

$$o_{p_1}(v_1) \cdots o_{p_s}(v_s) w$$

where $v_j \in V$, $w \in U$ and $p_j \in \frac{1}{T'}\mathbb{Z}$ are nonzero satisfying $p_1 \geq \cdots \geq p_s \geq -n$ and $p_1 + \cdots + p_s = 0$. Then the Poincaré-Birkhoff-Witt theorem tells us that $M_n(U)(n) = \sum_{s=0}^{\infty} U_s$ with $U_0 = U$ and $U_s \cap U_t = 0$ if $s \neq t$. The results of lemma 3.5.2 allow us to extend U^* to all of $M_n(U)(n)$ inductively by declaring

$$\langle f, o_{p_1}(v_1)\cdots o_{p_s}(v_2)w\rangle = \left\langle f, o_{p_1+p_2}(w^{p_1,p_2}_{v_1,v_2})\cdots o_{p_s}(v_2)w\right\rangle.$$

Finally extend U^* to all of $M_n(U)$ by declaring that U^* acts on $\bigoplus_{m\neq n} M_n(U)$ as zero.

We set

$$J_n(U) = \{a \in M_n(U) : \langle f, xa\rangle = 0 \text{ for all } f \in U^*,\ x \in \mathcal{U}(V[g])\}. \tag{3.7.3}$$

Then it is easy to see that $J_n(U)$ is $V[g]$-submodule of $M_n(U)$ satisfying $J_n(U) \cap U = 0$. Now, define $L_n(U)$ be the quotient

$$L_n(U) = M_n(U)/J_n(U). \tag{3.7.4}$$

In order to prove theorems 3.7.6 and 3.7.8 we need to establish that $W_n(U) \subseteq J_n(U)$. Lemmas 3.7.3, 3.7.4 and 3.7.5 are committed to this goal.

Lemma 3.7.3. *For all homogeneous $u \in V^{r*}$, $v \in V^{(T'-r)*}$, $f \in U^*$, $w \in U$ and $j, k \in \mathbb{Z}_+$*

$$\begin{aligned} &Res_{z_0} z_0^{k-1} \left\langle f, (z_0+z_2)^{wtu-1+\delta_i(r)+\ell+\frac{r}{T'}+j} Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v, z_2)w\right\rangle \\ &= Res_{z_0} z_0^{k-1} \left\langle f, (z_2+z_0)^{wtu-1+\delta_i(r)+\ell+\frac{r}{T'}+j} Y_{M_n(U)}(Y(u,z_0)v, z_2)w\right\rangle. \end{aligned}$$

Proof. Since $j \geq 0$, we have $u(\mathrm{wt}u - 1 + \delta_i(r) + \ell + \frac{r}{T'} + j) \in \bigoplus_{p>n} V[g]_{-p}$ and so acts on w as zero. The following argument is similar to that used in proposition 3.5.2.

For any $k \in \mathbb{Z}_+$ we get

$$0 = \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^k z_1^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)$$
$$\times Y_{M_n(U)}(v,z_2)Y_{M_n(U)}(u,z_1)w. \tag{3.7.5}$$

As $M_n(U)$ is a $V[g]$-module, from lemma (3.4.1)*(ii)* we know

$$[Y_{M_n(U)}(u,z_1), Y_{M_n(U)}(v,z_2)]$$
$$= \mathrm{Res}_{z_0} z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-\mathrm{wt}u-\frac{r}{T'}}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_{M_n(U)}(Y(u,z_0)v,z_2). \tag{3.7.6}$$

Combining (3.7.5) and (3.7.6) we compute

$$\mathrm{Res}_{z_0} z_0^k (z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} Y_{M_n(U)}(u,z_0+z_2)Y_{M_n(U)}(v,z_2)w$$
$$= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^k (z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right)$$
$$\times Y_{M_n(U)}(u,z_0+z_2)Y_{M_n(U)}(v,z_2)w$$
$$= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^k z_1^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right)$$
$$\times Y_{M_n(U)}(u,z_1)Y_{M_n(U)}(v,z_2)w$$
$$= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^k z_1^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)$$
$$\times Y_{M_n(U)}(u,z_1)Y_{M_n(U)}(v,z_2)w$$
$$- \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^k z_1^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)$$
$$\times Y_{M_n(U)}(v,z_2)Y_{M_n(U)}(u,z_1)w$$
$$= \mathrm{Res}_{z_1}(z_1-z_2)^k z_1^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j}[Y_{M_n(U)}(u,z_1), Y_{M_n(U)}(v,z_2)]w$$
$$= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1}(z_1-z_2)^k z_1^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-\mathrm{wt}u-\frac{r}{T'}}\delta\left(\frac{z_1-z_0}{z_2}\right)$$
$$\times Y_{M_n(U)}(Y(u,z_0)v,z_2)w$$

$$= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^k z_1^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-\mathrm{wt}u-\frac{r}{T'}} \delta\left(\frac{z_1-z_0}{z_2}\right)$$
$$\times Y_{M_n(U)}(Y(u,z_0)v,z_2)w$$
$$= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^k z_1^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} z_1^{-1}\left(\frac{z_2+z_0}{z_1}\right)^{\mathrm{wt}u+\frac{r}{T'}} \delta\left(\frac{z_2+z_0}{z_1}\right)$$
$$\times Y_{M_n(U)}(Y(u,z_0)v,z_2)w$$
$$= \mathrm{Res}_{z_0} z_0^k (z_2+z_0)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} Y_{M_n(U)}(Y(u,z_0)v,z_2)w.$$

So the statement of the lemma holds if $k \geq 1$.

We now assume $k = 0$. Then for any non-negative integer j, we have

$$\mathrm{Res}_{z_0} z_0^{-1}(z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j}$$
$$\times \left\langle f, Y_{M_n(U)}(u,z_0+z_2)Y_{M_n(U)}(v,z_2)w\right\rangle$$
$$= \sum_{t=0}^{j}\binom{j}{t}\mathrm{Res}_{z_0} z_0^{t-1} z_2^{j-t}(z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}$$
$$\times \left\langle f, Y_{M_n(U)}(u,z_0+z_2)Y_{M_n(U)}(v,z_2)w\right\rangle$$
$$= \sum_{t=1}^{j}\binom{j}{t}\mathrm{Res}_{z_0} z_0^{t-1} z_2^{j-t}(z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}$$
$$\times \left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle$$
$$+ \mathrm{Res}_{z_0} z_0^{-1} z_2^{j}(z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}$$
$$\times \left\langle f, Y_{M_n(U)}(u,z_0+z_2)Y_{M_n(U)}(v,z_2)w\right\rangle.$$

So to finish the proof we only need to show

$$\mathrm{Res}_{z_0} z_0^{-1}(z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}$$
$$\times \left\langle f, Y_{M_n(U)}(u,z_0+z_2)Y_{M_n(U)}(v,z_2)w\right\rangle$$
$$= \mathrm{Res}_{z_0} z_0^{-1}(z_2+z_0)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}} \left\langle f, Y_{M_n(U)}Y(u,z_0)v,z_2)w\right\rangle. \tag{3.7.7}$$

Since $\langle f, M_n(U)(m)\rangle = 0$ if $m \neq n$, we have

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-1}(z_2+z_0)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}} z_2^{\mathrm{wt}v+1-\delta_i(r)-\ell-\frac{r}{T'}} \\
&\qquad \times \left\langle f, Y_{M_n(U)} Y(u,z_0)v, z_2)w\right\rangle \\
&= \sum_{p=0}^{\infty} \sum_{t\in\frac{1}{T}\mathbb{Z}} \binom{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}{p} \langle f, (u_{p-1}v)(t)w\rangle\, z_2^{\mathrm{wt}u+\mathrm{wt}v-p-t-1} \\
&= \sum_{p=0}^{\infty} \binom{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}{p} \langle f, (u_{p-1}v)(\mathrm{wt}(u_{p-1})-1)w\rangle \\
&= \left\langle f, \sum_{p=0}^{\infty} \binom{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}{p} o(u_{p-1}v)w\right\rangle \\
&= \left\langle f, o\left(\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}}{z}\right)w\right\rangle.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-1}(z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}} z_2^{\mathrm{wt}v+1-\delta_i(r)-\ell-\frac{r}{T'}} \\
&\qquad \times \left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v,z_2)w\right\rangle \\
&= \sum_{j=0}^{\infty} \left\langle f, u_{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}-j-1} Y_{M_n(U)}(v,z_2)w\right\rangle z_2^{\mathrm{wt}v+1-\delta_i(r)-\ell-\frac{r}{T'})+j} \\
&= \sum_{j=0}^{\infty} \left\langle f, u_{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}-j-1} \sum_{\substack{s\geq -n\\ s\in\mathbb{Z}+\frac{r}{T'}}} v_{\mathrm{wt}v-s-1}w\right\rangle z_2^{1-\delta_i(r)-\ell-\frac{r}{T'}+j+s} \\
&= \sum_{j=0}^{\infty} \left\langle f, u_{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}-j-1} v_{\mathrm{wt}v+j+1-\delta_i(r)-\ell-\frac{r}{T'}-1}w\right\rangle \\
&= \sum_{j=0}^{2\ell+\delta_i(r)} \left\langle f, u_{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}-j-1} v_{\mathrm{wt}v+j+1-\delta_i(r)-\ell-\frac{r}{T'}-1}w\right\rangle \\
&= \sum_{j=0}^{2\ell+\delta_i(r)} \left\langle f, u_{\mathrm{wt}u-(j+1-\delta_i(r)-\ell-\frac{r}{T'})-1} v_{\mathrm{wt}v-(-j+\delta_i(r)+\ell-\frac{T-r}{T'})-1}w\right\rangle \\
&= \sum_{j=0}^{2\ell+\delta_i(r)} \left\langle f, o_{j+1-\delta_i(r)-\ell-\frac{r}{T'}}(u) o_{-j+\delta_i(r)+\ell-\frac{T-r}{T'}}(v)w\right\rangle
\end{aligned}$$

$$= \sum_{j=0}^{2\ell+\delta_i(r)} \sum_{m=0}^{2\ell+\delta_i(r)-j} (-1)^m \binom{m+j}{m}$$
$$\times \left\langle f, o\left(\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}}{z^{m+j+1}}\right) w\right\rangle$$
$$= \left\langle f, o\left(\mathrm{Res}_z Y(u,z)v \frac{(1+z)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}}}{z}\right) w\right\rangle.$$

This follows since

$$\sum_{j=0}^{2\ell+\delta_i(r)} \sum_{m=0}^{2\ell+\delta_i(r)-j} (-1)^m \binom{m+j}{m} \frac{1}{z^{m+j+1}} = \frac{1}{z}$$

by [3] proposition 5.3. Therefore the identity in (3.7.7) holds and the lemma follows. □

Lemma 3.7.4. *For all homogeneous $u \in V^{r*}$, $v \in V^{(T'-r)*}$, $f \in U^*$, $w \in U$, $j \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$, we have*

$$Res_{z_0} z_0^m (z_0+z_2)^{wtu-1+\delta_i(r)+\ell+\frac{r}{T'}+j} \left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v, z_2) w\right\rangle$$
$$= Res_{z_0} z_0^m (z_2+z_0)^{wtu-1+\delta_i(r)+\ell+\frac{r}{T'}+j} \left\langle f, Y_{M_n(U)}(Y(u,z_0)v, z_2) w\right\rangle$$

Proof. The statement holds for any integer $m \geq -1$ by lemma (3.7.3). We proceed by induction on $m \geq 1$ to show

$$\mathrm{Res}_{z_0} z_0^{-m} (z_0+z_2)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j}$$
$$\times \left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v, z_2) w\right\rangle$$
$$= \mathrm{Res}_{z_0} z_0^{-m} (z_2+z_0)^{\mathrm{wt}u-1+\delta_i(r)+\ell+\frac{r}{T'}+j} \left\langle f, Y_{M_n(U)}(Yu, z_0)v, z_2) w\right\rangle$$

Let $q = -1 + \delta_i(r) + \ell + \frac{r}{T'}$. As $\mathrm{wt} L(-1)u = \mathrm{wt} u + 1$, the induction assumption gives

$$\begin{aligned}&\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt} u+q+j+1}\\ &\quad\times \left\langle f, Y_{M_n(U)}(L(-1)u, z_0+z_2) Y_{M_n(U)}(v,z_2)w\right\rangle\\ &= \mathrm{Res}_{z_0} z_0^{-m}(z_2+z_0)^{\mathrm{wt} u+q+j+1}\left\langle f, Y_{M_n(U)}(Y(L(-1)u,z_0)v,z_2)w\right\rangle.\end{aligned}$$

Using the $L(-1)$-derivative property, the residue formula for the product rule and the induction assumption, we compute

$$\begin{aligned}&\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt} u+q+j+1}\left\langle f, Y_{M_n(U)}(L(-1)u, z_0+z_2) Y_{M_n(U)}(v,z_2)w\right\rangle\\ &= -\mathrm{Res}_{z_0}\left(\frac{\partial}{\partial z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt} u+q+j+1}\right)\left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v,z_2)w\right\rangle\\ &= m\mathrm{Res}_{z_0} z_0^{-m-1}(z_0+z_2)^{\mathrm{wt} u+q+j+1}\left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v,z_2)w\right\rangle\\ &\quad-(\mathrm{wt} u+q+j+1)\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt} u+q+j}\\ &\quad\times\left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v,z_2)w\right\rangle\\ &= m\mathrm{Res}_{z_0} z_0^{-m-1}z_2(z_0+z_2)^{\mathrm{wt} u+q+j}\left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v,z_2)w\right\rangle\\ &\quad+ m\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt} u+q+j}\left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v,z_2)w\right\rangle\\ &\quad-(\mathrm{wt} u+q+j+1)\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt} u+q+j}\\ &\quad\times\left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M_n(U)}(v,z_2)w\right\rangle\\ &= m\mathrm{Res}_{z_0} z_0^{-m-1}z_2(z_0+z_2)^{\mathrm{wt} u+q+j}\left\langle f, Y_{M_n(U)}(u, z_0+z_2) Y_{M(U)}(v,z_2)w\right\rangle\\ &\quad+ m\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt} u+q+j}\left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle\\ &\quad-(\mathrm{wt} u+q+j+1)\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt} u+q+j}\left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle.\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt}u+q+j+1}\left\langle f, Y_{M_n(U)}(Y(L(-1)u,z_0)v,z_2)w\right\rangle\\
&= -\mathrm{Res}_{z_0}\left(\frac{\partial}{\partial z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt}u+q+j+1}\right)\left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle\\
&= m\mathrm{Res}_{z_0} z_0^{-m-1}(z_0+z_2)^{\mathrm{wt}u+q+j+1}\left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle\\
&\qquad -(\mathrm{wt}u+q+j+1)\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt}u+q+j}\left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle\\
&= m\mathrm{Res}_{z_0} z_0^{-m-1}z_2(z_0+z_2)^{\mathrm{wt}u+q+j}\left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle\\
&\qquad + m\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt}u+q+j}\left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle\\
&\qquad -(\mathrm{wt}u+q+j+1)\mathrm{Res}_{z_0} z_0^{-m}(z_0+z_2)^{\mathrm{wt}u+q+j}\left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle.
\end{aligned}$$

So we have shown that

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^{-(m+1)}(z_0+z_2)^{\mathrm{wt}u+q+j}\left\langle f, Y_{M_n(U)}(u,z_0+z_2)Y_{M(U)}(v,z_2)w\right\rangle\\
&= \mathrm{Res}_{z_0} z_0^{-(m+1)}(z_0+z_2)^{\mathrm{wt}u+q+j}\left\langle f, Y_{M_n(U)}(Y(u,z_0)v,z_2)w\right\rangle
\end{aligned}$$

and we are done. □

Lemma 3.7.5. *For all homogeneous $u\in V^{r*}$, $v\in V$, $f\in U^*$, $w\in U$ and $j\in\mathbb{Z}_+$*

$$\begin{aligned}
&\left\langle f,(z_0+z_2)^{wtu-1+\delta_i(r)+\ell+\frac{r}{T'}+j}Y_{M_n(U)}(u,z_0+z_2)Y_{M_n(U)}(v,z_2)w\right\rangle\\
&= \left\langle f,(z_2+z_0)^{wtu-1+\delta_i(r)+\ell+\frac{r}{T'}+j}Y_{M_n(U)}(Y(u,z_0)v,z_2)w.\right\rangle
\end{aligned}\tag{3.7.8}$$

Proof. Take homogeneous $v\in V^{s*}$ for some $0\le s\le T'-1$. By (2.3.5) and lemma 2.3.6 we have

$$Y_{M_n(U)}(u,z)=\sum_{m\in\mathrm{wt}u+\frac{r}{T'}+\mathbb{Z}}u(m)z^{-m-1}\quad\text{and}\quad Y_{M_n(U)}(v,z)=\sum_{m\in\mathrm{wt}v+\frac{s}{T'}+\mathbb{Z}}v(m)z^{-m-1}.$$

If $r+s \not\equiv 0 \mod T'$, then since U^* acts on $M_n(U)(m)$ as zero for $m \neq n$, we see by inspection that all of the coefficients on both sides of (3.7.8) are zero.

So we may assume $r+s \equiv 0 \mod T'$. In this case, lemma 3.7.5 is a consequence of the lemma 3.7.3, and lemma 3.7.4. □

Theorem 3.7.6. *Let U be an $A_{g,n}(V)$-module. We have*

$$\overline{M}_n(U) = \bigoplus_{m \in \frac{1}{T'}\mathbb{Z}_+} \overline{M}_n(U)(m)$$

is an admissible g-twisted V-module with $\overline{M}_n(U)(n) \cong U$ as $A_{g,n}(V)$-modules. If U does not factor through $A_{g,n-\frac{1}{T'}}(V)$, then also $\overline{M}_n(U)(0) \neq 0$. Moreover, $\overline{M}_n(U)$ is generated by U as a V-module and satisfies the following universal property: for any weak g-twisted V-module M and any $A_{g,n}(V)$-module homomorphism $\phi : U \to \Omega_n(M)$, there is a unique V-module homomorphism $\overline{\phi} : \overline{M}_n(U) \to M$ which extends ϕ.

Proof. Apply proposition 3.6.1 to $\overline{M}_n(U)$ and the subspace $U + W_n(U)$. Subsequently, $\overline{M}_n(U)$ is a weak g-twisted V-module generated by the subspace $U + W_n(U)$. Moreover, the admissible grading restriction 2.3.16 holds on $\overline{M}_n(U)$ since $\overline{M}_n(U)$ is a $\frac{1}{T'}\mathbb{Z}$-graded $V[g]$-module.

Since $\overline{M}_n(U)$ is generated by $U + W_n(U)$ as a V-module, by degree considerations we see that

$$\overline{M}_n(U)(m) = V[g]_{(m-n)} \left(U + W_n(U)\right)$$

for all $m \in \frac{1}{T'}\mathbb{Z}$. In particular, because $V[g]_{-p}$ acts trivially on U for all $p > n$, we have $\overline{M}_n(U)(m) = 0$ if $m < 0$. Therefore $\overline{M}_n(U)$ is an admissible g-twisted V-module. The universal property follows from the construction of $M_n(U)$ as an induced $V[g]$-module. We still need will show that $\overline{M}_n(U)(n) \cong U$ as $A_{g,n}(V)$-modules, and if U does not factor through $A_{g,n-\frac{1}{T'}}(V)$, then also $\overline{M}_n(U)(0) \neq 0$.

First, note that we may take $M = M_n(U)$ in proposition 3.6.3. Then by proposition 3.6.3 and lemma 3.7.5, we see that $W_n(U) \subseteq J_n(U)$. Hence, $\overline{M}_n(U)(n) \cong U$ as $A_{g,n}(V)$-modules.

Now, assuming that U does not factor through $A_{g,n-\frac{1}{T'}}(V)$, say $\overline{M}_n(U)(0) = 0$. After a grading shift, by proposition 3.5.4 *(i)*, we see that U is an $A_{g,n-\frac{1}{T'}}(V)$-submodule of $\Omega_{n-\frac{1}{T'}}(M_n(U))$. A contradiction. □

The following gives a particularly useful description of $J_n(U)$.

Lemma 3.7.7. *We have that $J_n(U)$ is the maximal $\frac{1}{T'}\mathbb{Z}_+$-graded $V[g]$-submodule of $M_n(U)$ subject to $J_n(U) \cap U = 0$.*

Proof. By definition

$$J_n(U) = \{a \in M_n(U) : \langle f, xa\rangle = 0 \text{ for all } f \in U^*,\ x \in \mathcal{U}(V[g])\}.$$

Let X be the maximal $\frac{1}{T'}\mathbb{Z}_+$-graded $V[g]$-submodule of $M_n(U)$ subject to . $X \cap U = 0$. We know need to show that $X = J_n(U)$.

To see that $J_n(U)$ is a graded submodule, assume $m_1 + \cdots + m_k \in J_n(U)$ with each m_j being from a distinct homogeneous space $M_n(U)(t_j)$. Assume without loss of generality that t_1 is the maximum of $t_1, \ldots, t_k$. Then for any homogeneous $u \in V$ and each $j = 1, \ldots, k$ we know $u(\mathrm{wt} u + t_1 - 1)m_j \in M_n(U)(t_j - t_1)$=0.

Then for any $f \in U^*$, homogeneous $u \in V$ and $1 < j \leq k$ we have that $\langle f, u(\mathrm{wt} u + t_1 - 1)m_j\rangle = 0$. Now, since $m_1 + \cdots + m_k \in X$, it then follows that $\langle f, u(\mathrm{wt} u + t_1 - 1)m_1\rangle = 0$. As this is true for any $f \in U^*$ we conclude that $u(\mathrm{wt} u + t_1 - 1)m_1 = 0$ for all homogeneous $u \in V$. This implies $\langle f, u(m)m_1\rangle = 0$ for all homogeneous $u \in V$ and $m \in \frac{1}{T}\mathbb{Z}$. Therefore $m_1 \in X$. Again, without loss of generality we can now assume t_2 is the maximum of $t_2, \ldots, t_k$ and so on. This shows $J_n(U)$ is a graded submodule and in particular $J_n(U) \subseteq X$.

Assume by way of contradiction that $J_n(U) \neq X$. Then there exists homogeneous $w \in X$ with $w \notin J_n(U)$. In turn there exists $f \in U^*$ and homogeneous $x \in \mathcal{U}(V[g])$ such that $\langle f, xw \rangle \neq 0$. As $M_n(U) \cong U$ we have $xw \in X \cap U$ and is nonzero. A contradiction. □

Theorem 3.7.8. *Let U be an $A_{g,n}(V)$-module. The space $L_n(U) = M_n(U)/J_n(U)$ is an admissible g-twisted V-module satisfying $L_n(U)(n) \cong U$ as $A_{g,n}(V)$-modules. If U does not factor through $A_{g,n-\frac{1}{T'}}(V)$, then also $L_n(U)(0) \neq 0$. Moreover, if U has no submodule which factors through $A_{g,n-\frac{1}{T'}}(V)$, then $\Omega_n/\Omega_{n-\frac{1}{T'}}(L_n(U)) \cong U$ as $A_{g,n}(V)$-modules. In particular L_n, defines a functor from the category of $A_{g,n}(V)$-modules which have no submodule that factors through $A_{g,n-\frac{1}{T'}}(V)$ to the category of admissible g-twisted V-modules such that $\Omega_n/\Omega_{n-\frac{1}{T'}} \circ L_n$ is naturally equivalent to the identity.*

Proof. As $W_n(U) \subseteq J_n(U)$ it is clear that $L_n(U)$ is a quotient of $\overline{M}_n(U)$ and hence an admissible g-twisted V-module satisfying $L_n(U)(n) \cong U$ as $A_{g,n}(V)$-modules. Similarly, if U does not factor through $A_{g,n-\frac{1}{T'}}(V)$, then $L_n(U)(0) \neq 0$.

Now assume that U has no submodule which factors through $A_{g,n-\frac{1}{T'}}(V)$. We show that $\Omega_n/\Omega_{n-\frac{1}{T'}}(L_n(U)) \cong U$. As in 3.5.4 we have

$$\Omega_n/\Omega_{n-\frac{1}{T'}}\left(L_n(U)\right) = \bigoplus_{\substack{m \in \frac{1}{T'}\mathbb{Z}_+ \\ m \geq n}} \Omega_n/\Omega_{n-\frac{1}{T'}}\left(L_n(U)\right)(m)$$

where $\Omega_n/\Omega_{n-\frac{1}{T'}}\left(L_n(U)\right)(m) \cong \Omega_n\left(L_n(U)\right)(m)/\Omega_{n-\frac{1}{T'}}\left(L_n(U)\right)(m)$ linearly. If $\Omega_{n-\frac{1}{T'}}\left(L_n(U)\right)(n) \neq 0$, then $\Omega_{n-\frac{1}{T'}}\left(L_n(U)\right)(n)$ is an $A_{g,n}(V)$ submodule of U which factors through $A_{g,n-\frac{1}{T'}}(V)$. A contradiction to our assumption on U. Hence, $\Omega_n/\Omega_{n-\frac{1}{T'}}\left(L_n(U)\right)(n) \cong U$ as $A_{g,n}(V)$-modules.

Assume by way of contradiction that $\Omega_n(L_n(U))(m) \neq 0$ for some $m > n$. In this case $W = \mathcal{U}(V[g])\Omega_n(L_n(U))(m)$ is a nonzero admissible g-twisted V-submodule of $L_n(U)$ with $W(0) = 0$. Moreover, we have $W(n) \neq 0$ by definition of $L_n(U)$. Then $W(n)$ is an $A_{g,n}(V)$-submodule of U which factors through $A_{g,n-\frac{1}{T'}}(V)$, contradiction. So $\Omega_n/\Omega_{n-\frac{1}{T'}}(L_n(U))(m) = 0$ for all $m > n$ and $\Omega_n/\Omega_{n-\frac{1}{T'}}(L_n(U)) \cong U$. □

Lemma 3.7.9. *If U is an irreducible $A_{g,n}(V)$-module, then $L_n(U)$ is an irreducible V-module.*

Proof. Assume N is a non-zero admissible g-twisted V-submodule of $L_n(U)$. By lemma 3.7.7 we know $N(n) \cap L_n(U)(n) \neq 0$. Since $N(n)$ is an $A_{g,n}(V)$-submodule of $L_n(U)(n)$ and $L_n(U)(n) \cong U$ is irreducible as an $A_{g,n}(V)$-module, it must be the case that $N(n) = L_n(U)(n)$. As $L_n(U)$ is generated by $L_n(U)(n)$ as a $V[g]$-module, it follows that $N = L_n(U)$. □

Theorem 3.7.10. *The functors $\Omega_n/\Omega_{n-\frac{1}{T'}}$ and L_n induce mutually inverse categorical equivalences when restricted to the full subcategories of completely reducible $A_{g,n}(V)$-modules whose irreducible components cannot factor through $A_{g,n-\frac{1}{T'}}(V)$ and completely reducible admissible g-twisted V-modules, respectively. In particular, $\Omega_n/\Omega_{n-\frac{1}{T'}}$ and L_n induce mutually inverse bijections on the isomorphism classes of irreducible $A_{g,n}(V)$-modules which cannot factor through $A_{g,n-\frac{1}{T'}}(V)$ and irreducible admissible g-twisted V-modules.*

Proof. By theorem 3.7.8 we have $\Omega_n/\Omega_{n-\frac{1}{T'}}(L_n(U)) \cong U$ for any $A_{g,n}(V)$-module U which has no submodule which factors through $A_{g,n-\frac{1}{T'}}(V)$. On the other hand, if M is a completely reducible admissible g-twisted V-module, we need to show that $L_n\left(\Omega_n/\Omega_{n-\frac{1}{T'}}(M)\right) \cong M$. It suffices to take M irreducible. In this case it follows that $\Omega_n/\Omega_{n-\frac{1}{T'}}(M) \cong M(n)$ is an irreducible $A_{g,n}(V)$-module. Then

$L_n\left(\Omega_n/\Omega_{n-\frac{1}{T'}}(M)\right)$ is an irreducible admissible g-twisted V-module by lemma 3.7.9. In particular by theorem 3.7.6 both M and $L_n\left(\Omega_n/\Omega_{n-\frac{1}{T'}}(M)\right)$ are irreducible quotients of $\overline{M}_n\left(\Omega_n/\Omega_{n-\frac{1}{T'}}(M)\right)$ and so must be isomorphic by the universal property given in theorem 3.7.6. □

3.8 g-Rational Vertex Operator Superalgebras

We now apply the results of the previous section to g-rational vertex operator superalgebras.

Theorem 3.8.1. *If V is a g-rational vertex operator superalgebra, then the following hold.*

(i) *$A_{g,n}(V)$ is a finite-dimensional, semisimple associative algebra.*

(ii) *The functors $\Omega_n/\Omega_{n-\frac{1}{T'}}$ and L_n are mutually inverse categorical equivalences between the category of $A_{g,n}(V)$-modules whose irreducible components cannot factor through $A_{g,n-\frac{1}{T'}}(V)$ and the category of admissible g-twisted V-modules.*

(iii) *The functors $\Omega_n/\Omega_{n-\frac{1}{T'}}$ and L_n induce mutually inverse categorical equivalences between the category of finite dimensional $A_{g,n}(V)$-modules whose irreducible components cannot factor through $A_{g,n-\frac{1}{T'}}(V)$ and the category of ordinary g-twisted V-module.*

Proof. *(ii)* follows from theorem 3.7.10 and *(i)*. Moreover, *(iii)* follows from *(ii)* since V is g-rational every irreducible admissible g-twisted V-module is an irreducible ordinary g-twisted V-module by [15] theorem 6.6.

Thus we need to prove *(i)*. It is enough to show that every $A_{g,n}(V)$ is a direct sum of finite dimensional irreducible $A_{g,n}(V)$-modules. To that end, let W be an

$A_{g,n}(V)$-module. Then $U = W \oplus V_n$ is an $A_{g,n}(V)$-module which does not factor through $A_{g,n-\frac{1}{T'}}(V)$. Then $\overline{M}_n(U)$ is an admissible g-twisted V-module satisfying $\overline{M}_n(U)(0) \neq 0$ and $\overline{M}_n(U)(n) \cong U$. As V is g-rational, $\overline{M}_n(U)$ is a direct sum of irreducible ordinary g-twisted V-modules. So $\overline{M}_n(U)(n)$ is a direct sum of finite dimensional $A_{g,n}(V)$-modules. Then W must also be a direct sum of finite dimensional $A_{g,n}(V)$-modules. □

Theorem 3.8.2. *V is g-rational if and only if all $A_{g,n}(V)$ are finite-dimensional semisimple associative algebras.*

Proof. If V is g-rational, then all $A_{g,n}(V)$ are finite-dimensional semisimple associative algebras by theorem 3.8.1.

Conversely, assume all $A_{g,n}(V)$ are finite dimensional semisimple associative algebras. Since $A_{g,0}(V) = A_g(V)$ is finite-dimensional semisimple, it follows that V has only finitely many irreducible admissible g-twisted V-modules up to isomorphism by [15] theorem 6.5. Moreover, it must be the case that any irreducible admissible g-twisted V-module is an ordinary g-twisted V-module. To see this let $W = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} W(n)$ be an irreducible admissible g-twisted V-module. By proposition 2.3.14 we know that $L(0)$ acts semisimply on W. Furthermore, proposition 3.5.4 *(iii)* tells us that each $W(n)$ is an irreducible $A_{g,n}(V)$ -module and necessarily finite dimensional. Hence, W is an ordinary g-twisted V-module.

Let $\mathcal{M}_g(V)$ be a complete set of inequivalent irreducible admissible g-twisted V-modules. For $\lambda \in \mathbb{C}$ let $\mathcal{M}_\lambda$ be the set consisting of all $W \in \mathcal{M}_g(V)$ whose conformal weight is congruent to λ modulo $\frac{1}{T'}\mathbb{Z}$. Then for each $W \in \mathcal{M}_\lambda$ we can write $W = \bigoplus_{n\in\frac{1}{T'}\mathbb{Z}_+} W_{\lambda+n_W+n}$ where $n_W \in \frac{1}{T'}\mathbb{Z}$ and $W(n) = W_{\lambda+n_W+n}$. As M_λ is a finite set and $L(-1) : W(n) \to W\left(n + \frac{1}{T'}\right)$ is injective for $n >> 0$, we may choose $m_\lambda \in \frac{1}{T'}\mathbb{N}$ such that $W_{\lambda+m} \neq 0$ for any $W \in \mathcal{M}_\lambda$ and $m \geq m_\lambda$. (See [22], the conformal vector ω is *vaccum like*.)

Take any admissible g-twisted V-module $M = \bigoplus_{m \in \frac{1}{T'}\mathbb{Z}_+} M(m)$. We need to show that M is completely reducible. The idea is to show that each $M(m)$ generates a completely reducible admissible g-twisted V-submodule of M. To start we show that $M^0 = \mathcal{U}(V[g])M(0)$ is completely reducible. As $A_g(V)$ is semisimple, we can decompose $M(0)$ into a direct sum of irreducible $A_g(V)$-modules. If U is an irreducible $A_g(V)$-submodule of $M(0)$, then $U \cong W(0) = W_{\lambda+n_W}$ for some $\lambda \in \mathbb{C}$ and $W \in \mathcal{M}_\lambda$.

To show that M^0 is completely reducible it is enough to show that
$N = \mathcal{U}(V[g])U$ is isomorphic to W. First note that the universal property of $\overline{M}_0(U) = \overline{M}(U)$ gives us that N contains an irreducible quotient isomorphic to W. Take $n \in \frac{1}{T'}\mathbb{Z}_+$ such that $n + n_W \geq m_\lambda$. As $W(n) \neq 0$ is an irreducible $A_{g,n}(V)$-module, it follows from lemma 3.7.9 and theorem 3.7.6 that

$$L_n(W(n)) \cong \overline{M}_n(W(n))/\overline{J} \cong W$$

where $\overline{J}$ is the maximal admissible g-twisted V-submodule of $\overline{M}_n(W(n))$ subject to $\overline{J} \cap \overline{M}_n(W(n)) = 0$. Note that $L(0)$ acts semisimply on $\overline{M}_n(W(n))$ with $\overline{M}_n(W(n))(m) = \overline{M}_n(W(n))_{\lambda+n_W+m}$. So $L(0)$ acts semisimply on $\overline{J}$ and
$\overline{J}_{\lambda+n_W+n} = 0$. Since $n + n_W \geq m_\lambda$ it must be the case that $\overline{J} = 0$ and $W \cong \overline{M}_n(W(n))$. To see this let U' be any irreducible $A_g(V)$-submodule of $\overline{J}(0)$. Then $L_0(U') = L(U')$ is an irreducible admissible g-twisted V-module with $L(U') \cong U'$ and $L(U)_{\lambda+m} = 0$. So U'=0 and hence $\overline{J} = 0$. Now, since $A_{g,n}(V)$ is semisimple, we can write $N(n) = Q \oplus N(n)'$ where $Q \cong W(n)$ as $A_{g,n}(V)$-modules and $N(n)'$ is an $A_{g,n}(V)$-submodule of $N(n)$. We must have that the V-submodule of N generated by Q is isomorphic to W. Then $N \cong W$ and so M^0 is completely reducible admissible g-twisted V-submodule of M.

Now $M^0\left(\frac{1}{T'}\right)$ is an $A_{g,\frac{1}{T'}}(V)$-submodule of $M\left(\frac{1}{T'}\right)$. Since $A_{g,\frac{1}{T'}}(V)$ is semisimple, we can decompose $M\left(\frac{1}{T'}\right)$ into a direct sum of $A_{g,\frac{1}{T'}}(V)$-submodules $M^0\left(\frac{1}{T'}\right)\oplus M\left(\frac{1}{T'}\right)'$. The argument used above shows that the admissible g-twisted V-submodule $M^{\frac{1}{T'}} = \mathcal{U}(V[g])M\left(\frac{1}{T'}\right)'$ of M is a completely reducible. Continuing in this way we see that M is a completely reducible admissible g-twisted V-module. $\square$

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