Propagation of light in Schwarzschild geometry

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ABSTRACT

In this paper, the equivalent medium of Schwarzschild metric is discussed. The corresponding ray-tracing equations are integrated for the equivalent medium of the Schwarzschild geometry, which describes the curved space around a spherically symmetric, irrotational, and uncharged blackhole. We make comparison to the well-known expression by Einstein. While Einstein's estimate is reasonably good for large closest distances of approach to the star, it disregards the optical anisotropy of space. Instead, Virbhadra's estimate which takes the effects of anisotropy of Schwarzschild metric is shown to be more consistent with numerical simulations. Hence, a true physical anisotropy in the velocity of light under gravitational field does exist. We argue that the existence of such an optical anisotropy could be revealed exactly in the same way that the optical interferometry is expected to detect gravitational waves. Therefore, if no optical anisotropy under gravitational fields could be observed, then the possibility of interferometric detection of gravitational waves is automatically ruled out, and vice versa.

Keywords: Black holes, ray-tracing, equivalent medium theory, general relativity

1. INTRODUCTION

It has been known [1,2] that the propagation of electromagnetic waves in curved space, can be described by a mathematically equivalent anisotropic medium in flat geometry. Recently, full theoretical analysis of the propagation in the curved space and in particular for the well-known Schwarzschild metric has been published by the author [12].

Here, the optical anisotropy of curved space is demonstrated by means of a rigorous algebraic analysis. We derive the eigenmodes of propagation and conclude that vacuum exhibits a property very much similar to pseudo-isotropic media [3], but with broken symmetry with respect to the waves travelling forward and backward in time. A simple pseudo-isotropic medium has different refractive indices along all propagation directions, but exhibits no birefringence when standard constitutive relations are used [3,4].

For this purpose, we start by inserting the constitutive relations into the Maxwell wave equations and obtain the governing equation for eigenpolarizations. This is shown to result in a modified normal surface equation for the refractive index eigenvalue. Simplification for the pseudo-isotropic behavior gives rise to two different refractive indices with opposite signs, which are equal in magnitude for a non-rotating spacetime. We are thus led to the conclusion that, the speed of light is dependent on the local geometry of the spacetime, but at the same time, the curved space of a rotating metric is differently seen by photons and anti-photons, which propagate in opposite directions along the time coordinate at velocities slightly below and above c_0. The difference is easily seen to be removed for non-rotating metrics. In otherwords, the time-reversal symmetry of Maxwell's equations breaks down under rotation.

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As examples of applicability of our proposed formulation, we consider Newtonian and Schwarzschild metrics, and derive exact closed forms for refractive index in both cases. It is discussed that the refractive index of the Schwarzschild metric gives a dependence to the radial coordinate as well as the direction of propagation and is explained by a non-homogeneous pseudo-isotropic model. We make a comparison to Einstein's results in 1911 and 1955 and show that the latter is yet subject to an important correction term, which makes the space locally anisotropic.

Ray-tracing equations for Schwarzschild geometry have been recently found [12], and are hereby integrated to verify the bending of light near a massive object. We show quantative agreement with our numerical estimates and the well-known expressions of general relativity. Virbhadra's formula for the deflection of light, which includes the effect of optical anisotropy of Schwarzschild metric, is shown to be a better estimate than that of Einstein's, which disregards the anisotropy. Hence, a true physical anisotropy in the velocity of light as predicted in [12] is found, and therefore, a local optical interferometry experiment could be used as a test for the theory of general relativity.

We furthermore argue that the expected anisotropy resulting from static gravitational fields can be detected by an optical interferometric experiment such as Laser Interferomer Gravitational wave Observatory (LIGO) [13], if and only if, gravitational waves could be detected, too. This is due to the fact that the detection of gravitational waves depends solely on the temporal anisotropy of the metric due to the passing gravitational wave.

2. FORMULATION

An empty curved spacetime may be seen as an equivalent flat spacetime with a nonhomogenous and anisotropic hypothetical dielectric filled everywhere [1], the constitutive relations of which between electric **E** and magnetic **H** fields in SI units take the form [1,5]

$$\mathbf{D} = \varepsilon_0 [\varepsilon] \mathbf{E} + \frac{1}{c} \mathbf{w} \times \mathbf{H} \tag{1a}$$

$$\mathbf{B} = \mu_0 [\mu] \mathbf{H} - \frac{1}{c} \mathbf{w} \times \mathbf{E} \tag{1b}$$

Here, the symmetric dimensionless tensors of relative permittivity $[\varepsilon]$ and permeability $[\mu]$ are given by

$$[\varepsilon] = [\mu] = -\frac{\sqrt{-g}}{g_{00}} [g^{ij}] \tag{2}$$

and the gyration vector **w** is defined as $\mathbf{w} = {g_{00}}^{-1} \{g_{0i}\}$, where g^{ij} and g_{ij} are respectively the contravariant and covariant elements of the metric tensor of space, with g being its determinant. Obviously, most non-rotating metrics have $g_{0i}=0$ for i=1,2,3 so that the gyration vector **w** vanishes [12]. We have previously shown that the electromagnetic curved vacuum behaves as a birefringent pseudo-isotropic medium [12], with the exact refractive indices given by

$$n_1 = +\frac{\sqrt{AC + B^2} - B}{A} = +\frac{\sqrt{(\mathbf{h}\cdot[\xi]\cdot\mathbf{h})(1 - \mathbf{w}\cdot[\xi]\cdot\mathbf{w}) + (\mathbf{h}\cdot[\xi]\cdot\mathbf{w})^2} - \mathbf{h}\cdot[\xi]\cdot\mathbf{w}}{\mathbf{h}\cdot[\xi]\cdot\mathbf{h}} \tag{3a}$$

$$n_2 = -\frac{\sqrt{AC + B^2} + B}{A} = -\frac{\sqrt{(\mathbf{h}\cdot[\xi]\cdot\mathbf{h})(1-\mathbf{w}\cdot[\xi]\cdot\mathbf{w}) + (\mathbf{h}\cdot[\xi]\cdot\mathbf{w})^2} + \mathbf{h}\cdot[\xi]\cdot\mathbf{w}}{\mathbf{h}\cdot[\xi]\cdot\mathbf{h}} \tag{3b}$$

Here, $[\xi]=[\mu]/|\mu|=[\varepsilon]/|\varepsilon|$, and also **h**=**k**/|**k**| is the unit vector along the propagation vector. Hence, $n=|\mathbf{k}|/k_0$ is the index of refraction of vacuum; clearly, we have $\mathbf{k}=nk_0\mathbf{h}$. Furthermore, we have

$$A = \mathbf{h}\cdot[\xi]\cdot\mathbf{h} \tag{4a}$$
$$B = \mathbf{h}\cdot[\xi]\cdot\mathbf{w} \tag{4b}$$
$$C = 1 - \mathbf{w}\cdot[\xi]\cdot\mathbf{w} \tag{4c}$$

If deviation in metric from the flat Minkowskian geometry is not too strong, then the terms being second order in **w** can be dropped and (3) can be approximated as

$$n_1 \approx +(\mathbf{h}\cdot[\xi]\cdot\mathbf{h})^{-\frac{1}{2}} - \frac{\mathbf{h}\cdot[\xi]\cdot\mathbf{w}}{\mathbf{h}\cdot[\xi]\cdot\mathbf{h}} \tag{5a}$$
$$n_2 \approx -(\mathbf{h}\cdot[\xi]\cdot\mathbf{h})^{-\frac{1}{2}} - \frac{\mathbf{h}\cdot[\xi]\cdot\mathbf{w}}{\mathbf{h}\cdot[\xi]\cdot\mathbf{h}} \tag{5b}$$

Note that (3) take on the fairly simple forms when there is no rotation terms in the metric **w**=0, and thus we get the *exact* expressions

$$n_1 = +(\mathbf{h}\cdot[\xi]\cdot\mathbf{h})^{-\frac{1}{2}} \tag{6a}$$
$$n_2 = -(\mathbf{h}\cdot[\xi]\cdot\mathbf{h})^{-\frac{1}{2}} \tag{6b}$$

This shows that birefringence identically vanishes for non-rotating metrics and time-reversal symmetry holds. More discussion on this concept can be found in [12].

For rotating systems we may note that **h** is a unit vector, and thus (5) can be still simplified further in the weak gravitational field limit as

$$n_1 \approx +1 + \tfrac{1}{2}\Delta n \tag{7a}$$
$$n_2 \approx -1 + \tfrac{1}{2}\Delta n \tag{7b}$$

corresponding respectively to photons and anti-photons; here, we define $\Delta n = -2\mathbf{h}\cdot[\xi]\cdot\mathbf{w}$. Firstly, it can be seen that the curved vacuum exhibits a local time-reversal asymmetry given by $|\Delta n| = |n_1 - n_2|$, which is roughly a linear function of gravitational potential (as shown below). Secondly, photons travel at a speed slightly below (above) the speed of light in flat vacuum c_0, while anti-photons travel at a speed slightly above (below) c_0, if the direction of propagation is anti-parallel (along) to the rotation of the universe, where $\Delta n > 0$ ($\Delta n < 0$).

Now as it is shown below for Newtonian and Schwarzschild metrics, the equations (6) turn out to be in complete agreement to the predictions made by Einstein for weak gravitational fields [8,9].

3. CURVED GEOMETRIES

In this section we consider two important cases, which both correspond to spherically symmetric non-rotating universes: Newtonian and Schwarzschild metrics. Rotation can be exactly implemented through (3), however, the resulting expressions are too complicated and hence are not discussed for the sake of convenience.

3.1. Newtonian Metric

The spherically symmetric Newtonian metric is given by the line element

$$ds^2 = -c^2\left(1-2\frac{r_s}{r}\right)dt^2 + \left(dx^2 + dy^2 + dz^2\right) \tag{8}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius of the star with M and G respectively being its mass and gravitational constant, and $dl^2 = dx^2 + dy^2 + dz^2$ is the spacelike path element. This metric is an approximate solution of Einstein field equations, but to high precision most stars are static and spherical [10, p. 446], so that (8) is applicable. The solution (6) can be then used and we readily obtain

$$n_{1,2} = \pm\left(1-2\frac{r}{r_s}\right)^{-\frac{1}{2}} \tag{9}$$

Denoting $\Phi = -r_s/r$, we get

$$n_{1,2} = \pm\left(1+2\Phi\right)^{-\frac{1}{2}} \tag{10}$$

This expression blows up to infinity for $r \to r_s$ as $\left(r-r_s\right)^{-1}$, but approaches in magnitude to unity as $r \to \infty$. In the limit of small normalized gravitational potential $r >> r_s$, however, we get

$$\left|n_{1,2}\right| \approx 1 - \Phi \tag{11}$$

which is actually the Einstein's 1911 result [8]. Later Einstein showed [9] that the correct answer was $\left|n_{1,2}\right| \approx 1 - 2\Phi$. But as it is discussed below, this correction factor of 2 was still inaccurate.

3.2. Schwarzschild Metric

Schwarzschild metric is known by Birkhoff's 1932 theorem [10, p.843], to be the most general solution of Einstein field equations under spherical symmetry and no rotation, given by [10, p. 607]

$$ds^2 = -c^2\left(1-\frac{r_s}{r}\right)dt^2 + \frac{dr^2}{1-\frac{r_s}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{12}$$

where (r,θ,ϕ) constitute the standard spherical polar coordinates. The difficulty in working with this metric arises from the fact that the spacelike path element $dl^2 = dr^2 + r^2 d\Omega^2$ where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, does not appear explicitly in the metric.

Schwarzschild metric is non-rotating so that (6) are exact. Since **h** is a unit vector it can be described by the spherical polar angles (ψ, χ) in the original non-rotated reference frame. But, spherical symmetry makes the absolute choice of the angles (ψ, χ) irrelevant, in the sense that we may set the *z*-axis along the position coordinate **r** and choose $\phi = \theta = 0$. Hence, we get the exact expression [12]

$$\left|n_{1,2}(\mathbf{r})\right| = \frac{\left(\frac{r}{r_s}\right)^2}{\left(\frac{r}{r_s} - 1\right)^{\frac{3}{2}} \sqrt{\frac{r}{r_s} - \sin^2\psi}} \tag{13}$$

where the azimuthal angle of propagation ψ is measured with respect to the position vector **r**. Again for $r \to r_s$ the refractive index blows up respectively as $\left(r - r_s\right)^{-\frac{3}{2}}$ and $\left(r - r_s\right)^{-2}$, if $\psi \neq \pm\frac{\pi}{2}$ and $\psi = \pm\frac{\pi}{2}$. However, it approaches in magnitude to unity as $r \to \infty$, since the metric relaxes to that of the Minkowskian in the limit of infinite radius. We can now rewrite (13) in the limit of small Φ (weak gravitational field) as

$$\left|n_{1,2}(\mathbf{r})\right| \approx 1 - \frac{3 + \sin^2\psi}{2}\Phi \tag{14}$$

The correction factor to the Einstein's 1955 result hence actually varies between 3/2 and 2 depending on the angle of propagation. If the light is passing tangent to the star's equatorial plane, then $\sin^2\psi = 1$ holds only at the nearest point to the center of the star in the light trajectory, while at farther points away from the center of the star, we approach $\sin^2\psi = 0$.

Another conclusion is that this anisotropy is expected to be present everywhere around a massive object, so that the change in refractive index by changing the direction of propagation from $\psi = \frac{\pi}{2}$ to $\psi = 0$, could reach as high as $|\Phi|/2 = \left(GM/c^2\right)/r$. Hence, a *local* interferometry experiment could reveal the existence of Schwarzschild metric. Based on the estimates given in [10, p. 459], this figure should be of the order of 10^{-8} for an experiment done at Earth's distance from Sun, while it would be only about 6×10^{-10} at the surface of Earth when the gravity of Sun is neglected.

4. RAY-TRACING

In the recent paper of the author [12], it has been shown that the ray-tracing equations of Schwarzschild geometry are

$$\begin{aligned}
\frac{d\mathbf{r}}{dl} &\approx \left[1 + (\mathbf{h}\cdot\hat{r})^2\Phi\right]\mathbf{h} - (\mathbf{h}\cdot\hat{r})\Phi\hat{r} \\
\frac{d\mathbf{k}}{dl} &\approx \frac{\Phi}{r}k\left\{\left[2 - \frac{3}{2}(\mathbf{h}\cdot\hat{r})^2\right]\hat{r} + (\mathbf{h}\cdot\hat{r})\mathbf{h}\right\}
\end{aligned} \tag{15}$$

which are correct to $O(\Phi)$. Here, $\hat{r} = \mathbf{r}/r$ is the unit radial vector. These sets of equations may be easily integrated to investigate the effects of Schwarzschild geometry on the propagation of light. It can be easily verified that the ray, despite deflection, sticks to the plane constructed by the location vector and wavevectors $\mathbf{r}$ and $\mathbf{k}$. Hence the problem can be greatly simplified by solving the system of equations (15) on this plane. According to Einstein [9], the angle of deflection for a light ray approaching a massive object is given by

$$\alpha \approx \frac{2r_s}{r_0} \tag{16}$$

where r_0 is the closest distance of approach and α is measured in radians. More accurate result by Virbhadra [11] is

$$\alpha \approx \frac{2r_s}{r_0}\left[1 + 0.972\left(\frac{r_s}{r_0}\right)\right] \tag{17}$$

In order to perform the integration of (15), it is possible to take the initial conditions $\mathbf{r}(0) = -d\hat{y}$ for the position vector and $\mathbf{k}(0) = k\hat{x}$ for the wavevector. Clearly, the closest distance of approach is $d = r_0$. Obviously, meaningful integration of (15) is possible only if the closest distance of approach exceeds the Schwarzschild radius, that is $r_s < d$. This means that the accuracy of (16) is limited to deflection angles roughly less than 114°.

Table 1 summarizes the results of numerical integration of (15) versus the expressions by Einstein (16) and Virbhadra (17). As it can be seen, very good agreement is found. Here, the total number of integration steps was 4×10^4, wavelength was set to one-tenth of the Schwarzschild radius. As it can be seen, Virbhadra's expression (17) which takes the effect of Schwarzschild anisotropy provides much better agreement to the numerical calculations, than that of Einstein's (16). Large deviation for small closest distances of approach can be attributed to the approximate nature of (15), which is correct within the first order of the gravitational potential Φ. But in general, Virbhadra's expression (17), which takes the effect of anisotropy of Schwarzschild metric provides much better agreement to the numerical simulations. This justifies the existence of a true physical optical anisotropy in the curved space, as predicted in the recent work of the author [12].

Finally, traces of light rays passing nearby a massive object are illustrated in Fig. 1. Here, light rays are found by direct integration of (15).

5. LASER INTERFEROMETRY

In order to establish the connection between the gravity-induced optical anisotropy and gravitational waves, we take a look at the metric representing a gravitational wave [14-17]

$$ds^2 = \left(g_{ij} + \gamma_{ij}\right)dx^i dx^j \tag{19}$$

where g_{ij} is the metric of the unperturbed geometry and γ_{ij} is the perturbation metric due to the gravitational wave. For a plane gravitational wave moving along x direction we have [14-17]

$$\left[\gamma_{ij}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{22} & \gamma_{23} \\ 0 & 0 & \gamma_{32} & \gamma_{33} \end{bmatrix} \tag{20}$$

where $\gamma_{22} = -\gamma_{33}$ and $\gamma_{23} = \gamma_{32}$.

The two orthogonal polarizations are here given by $\gamma_{22} \neq 0, \gamma_{23} = 0$ and $\gamma_{22} = 0, \gamma_{23} \neq 0$. Now, for the sake of simplicity we take on the first polarization with $\gamma_{22} = \delta, \gamma_{23} = 0$, to get [14-17]

$$ds^2 = -c^2dt^2 + dx^2 + \left[1 + \delta \sin\left(Kx - \Omega t\right)\right]dy^2 + \left[1 - \delta \sin\left(Kx - \Omega t\right)\right]dz^2 \tag{21}$$

Here, Ω is the angular frequency of the gravitational wave, K is its wavenumber, and δ is its amplitude. The metric in (21) is evidently anisotropic, to which (6) may apply to obtain

$$n = \left(\mathbf{h} \cdot [\zeta] \cdot \mathbf{h} + \mathbf{h} \cdot [\rho] \cdot \mathbf{h}\right)^{-\frac{1}{2}} \tag{22}$$

in which $[\zeta]$ represents the Minkowskian metric with $\mathbf{h} \cdot [\zeta] \cdot \mathbf{h} = 1$, and $[\rho]$ is due to the gravitational wave. If the amplitude of the wave is small, then binomial expansion may be applied to (22) and we arrive at

$$n \approx 1 - \tfrac{1}{2}\mathbf{h} \cdot [\rho] \cdot \mathbf{h} \tag{23}$$

Examining the apparent refractive index along y and z coordinates, respectively with $\mathbf{h} = \hat{y}$ and $\mathbf{h} = \hat{z}$, reveals that we have

$$n_y \approx 1 - \tfrac{1}{2}\hat{y} \cdot [\rho] \cdot \hat{y} = 1 - \tfrac{1}{2}\rho_{22} = 1 - \tfrac{1}{2}\delta \sin\left(Kx - \Omega t\right) \tag{24a}$$
$$n_z \approx 1 - \tfrac{1}{2}\hat{z} \cdot [\rho] \cdot \hat{z} = 1 - \tfrac{1}{2}\rho_{33} = 1 + \tfrac{1}{2}\delta \sin\left(Kx - \Omega t\right) \tag{24b}$$

At $x = 0$, this would be equivalent to an anisotropy $\Delta n = n_y - n_z$ of

$$\Delta n \approx \delta \sin\left(\Omega t\right) \tag{25}$$

which is expected to be on the order of 10^{-24} [14-17]; this is yet to be observed in LIGO.

As it can be seen, the origin of operation of LIGO relies on the optical anisotropy induced in the background geometry due to the passing gravitational field. Since Schwarzschild metric also causes such anisotropy, optical interferometry should be equally applicable to investigate the presence of static gravitational fields. As stated above, the expected order of anisotropy resulting from the gravity of Sun is quite appreciable, being on the order of 10^{-8} [10, p. 459], which is at least 16 orders of magnitude stronger than the anisotropy caused by gravitational waves. This figure is large enough to be easily detectable on a tabletop interferometric setup. If it would, then most normal optical setups had problems working on the Earth, due to interference with the gravity of Sun, Earth, and other massive objects nearby. This is clearly not the case, and as the matter of fact, nobody however has noticed such a large deviation and error in interferometric experiments.

On the other hand, if optical interferometry fails to reveal the existence of Schwarzschild metric (which is the apparent case), then one could expect the gravitational waves not to be detectable via simple interferometry. This explains why the LIGO experiment, despite its extremely high accuracy, has been unsuccessful in the search of gravitational waves.

This conclusion brings up two possibilities:

(a) A uniform compression or stretch in the metric actually does not cause additional phase shifts along the path of propagation, which in turn rules out the possibility of any successful interferometric experiment for detection of plane gravitational wave effects.
(b) The general theory of relativity needs a complete restructuring and revision.

6. CONCLUSIONS

In summary, we have discussed the optical anisotropy of vacuum subject to a gravitational field. It has been shown that as a result of gravity, a strong local anisotropy is predicted to exist, which possibly could be verified by relatively simple interferometry experiments. According to the principle of equivalence [9], this effect could be thought of anisotropy of velocity of light for accelerated (non-inertial) frames [12]. We furthermore solved the ray-tracing equations for Schwarzschild geometry and found quantitative agreement to earlier predictions of Einstein. Numerical simulations and comparison to the improved expression of Virbhadra reveals the existence of a true optical anisotropy in the curved space. We made comparison to the theory behind the operation of Laser Interferometer Gravitational wave Observatory (LIGO) at California Institute of Technology, and discussed that interferometry might have been the inappropriate choice for detection of gravitational waves at all. We argued that if gravitational waves had been observed in LIGO, then static gravitational fields of Sun and Earth could have been equally detected, much easier, via a simple tabletop interferometric setup.

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Table 1. Calculation of the angle of deflection.

d/r_s	α from integration of (15) [12]	α from Einstein's (16) [9,10]	α from Virbhadra's (17) [11]
100	1.12529 °	1.14592 °	1.1571 °
90	1.25925 °	1.27324 °	1.2870 °
80	1.42652 °	1.43239 °	1.4498 °
70	1.64159 °	1.63702 °	1.6598 °
60	1.92882 °	1.90986 °	1.9408 °
50	2.33259 °	2.29183 °	2.3364 °
40	2.94306 °	2.86479 °	2.9344 °
30	3.9758 °	3.81972 °	3.9435 °
20	6.10583 °	5.72958 °	6.0080 °
10	13.0802 °	11.4592 °	12.5730 °
9	14.7615 °	12.7324 °	14.1075 °
8	16.9367 °	14.3239 °	16.0643 °
7	19.861 °	16.3702 °	18.6433 °
6	24.0012 °	19.0986 °	22.1926 °
5	30.3171 °	22.9183 °	27.3736 °
4	41.1634 °	28.6479 °	35.6093 °
3	64.5994 °	38.1972 °	50.5731 °
2	315.8827 °	57.2958 °	85.1415 °

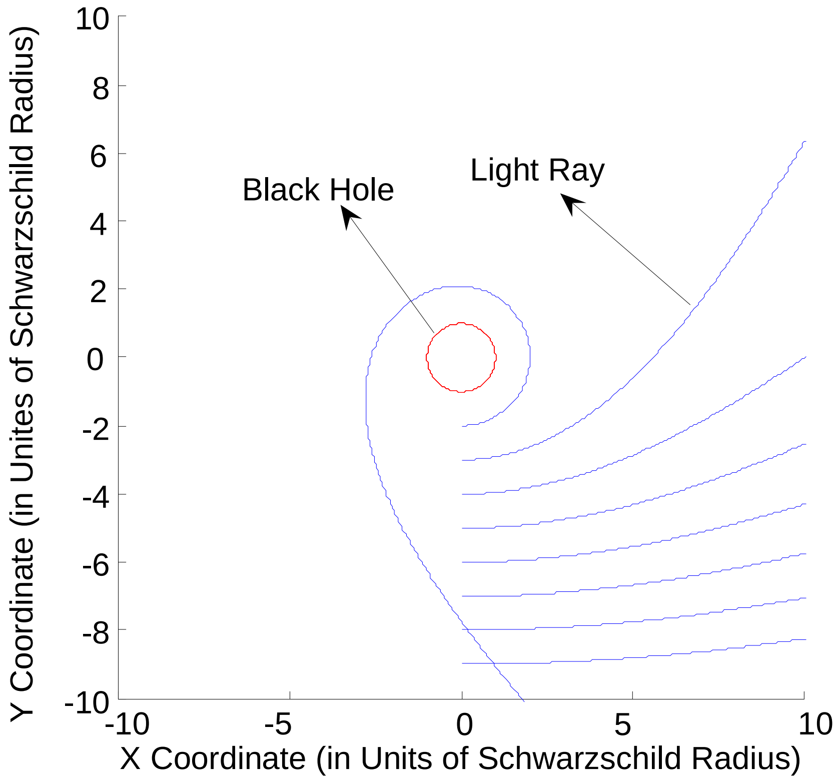

Figure 1. Ray-tracing of light beams moving near a black hole from integration of (15).